



University of Pennsylvania
ScholarlyCommons

Statistics Papers

Wharton Faculty Research

1970

On the Supremum of S_n/n

B.J. McCabe

Larry A. Shepp
University of Pennsylvania

Follow this and additional works at: http://repository.upenn.edu/statistics_papers

 Part of the [Applied Statistics Commons](#)

Recommended Citation

McCabe, B. J., & Shepp, L. A. (1970). On the Supremum of S_n/n . *The Annals of Mathematical Statistics*, 41 (6), 2166-2168.
<http://dx.doi.org/10.1214/aoms/1177696723>

This paper is posted at ScholarlyCommons. http://repository.upenn.edu/statistics_papers/349
For more information, please contact repository@pobox.upenn.edu.

On the Supremum of S_n/n

Abstract

Let X_1, X_2, \dots be independent and identically distributed. We give a simple proof based on stopping times of the known result that $\sup (|X_1 + \dots + X_n|/n)$ has a finite expected value if and only if $E|X| \log |X|$ is finite.

Whenever $E|X| \log |X| = \infty$, a simple nonanticipating stopping rule τ , not depending on X , yields $E(|X_1 + \dots + X_\tau|/\tau) = \infty$.

Disciplines

Applied Statistics

ON THE SUPREMUM OF S_n/n

BY B. J. MCCABE¹ AND L. A. SHEPP

Bellcomm Incorporated and Bell Telephone Laboratories, Incorporated

Let X_1, X_2, \dots be independent and identically distributed. We give a simple proof based on stopping times of the known result that $\sup(|X_1 + \dots + X_n|/n)$ has a finite expected value if and only if $E|X| \log |X|$ is finite. Whenever $E|X| \log |X| = \infty$, a simple nonanticipating stopping rule τ , not depending on X , yields $E(|X_1 + \dots + X_\tau|/\tau) = \infty$.

Let X_1, X_2, \dots be an i.i.d. sequence and set $S_n = X_1 + \dots + X_n$. Marcinkiewicz and Zygmund [2] proved that if

- (1) $E|X_1| \log |X_1| < \infty$ then
- (2) $E \sup_n |S_n/n| < \infty$.

D. L. Burkholder [1] proved the converse and showed further that (1), (2), and (3) are equivalent, where

- (3) $E \sup_n |X_n/n| < \infty$.

We give a simple proof of Burkholder's results by using stopping times. We extend his results by proving that (1)–(5) are equivalent, where

- (4) $\sup_{\text{rule } N} E|S_N/N| < \infty$
- (5) $\sup_{\text{rule } N} E|X_N/N| < \infty$,

the sup in (4) and (5) being taken over all nonanticipative stopping rules (times) N . As an immediate corollary we see that whenever anticipative stopping with reward $\sup |S_n/n|$ gives infinite expected reward, then there is a (nonanticipative) stopping rule which also has infinite expected reward. In fact, this rule is very simple: stop the first time that $|X_n| > cn$; the rule thus does not depend on the distribution of X (except for the constant c).

Finally, we give a one-sided version of the equivalence of (1)–(5) in terms of S_n/n rather than $|S_n/n|$.

We learned from D. L. Burkholder after writing this paper that Burgess Davis [4] and Richard F. Gundy [5] have also obtained stopping time proofs of Burkholder's theorem [1], among other results. We have decided to publish our proof because of its simplicity.

A universal stopping time. We begin the proof of the equivalence of (1)–(5) with the implication (5) \Rightarrow (1). It is clearly no loss of generality to suppose that $p_1 = P(|X_1| < 1) > 0$ and we do this for convenience. Taking $N \equiv 1$ in (5) shows that

Received December 5, 1969.

¹ Now at Daniel H. Wagner Assocs., Paoli, Pa.



$E|X_1| < \infty$. To prove that $E|X_1| \log |X_1| < \infty$ as well, take $N =$ the first $n \geq 1$ for which $|X_n| > n$; $N = \infty$ if there is no such n . We then have from (5)

$$(6) \quad \infty > E_{[N < \infty]} |X_N/N| = \sum_{n=1}^{\infty} P(N = n)n^{-1} E[|X_n| \mid N = n] \\ = \sum_{n=1}^{\infty} P(N = n)n^{-1} E[|X_n| \mid |X_n| \geq n]$$

since the X 's are independent. If F denotes the common distribution function of the X 's (6) yields

$$(7) \quad \infty > \sum_{n=1}^{\infty} P(N \geq n)n^{-1} \int_{|x| \geq n} |x| dF(x) \geq P(N = \infty) \sum_{n=1}^{\infty} n^{-1} \int_{|x| \geq n} |x| dF(x).$$

But $P(N = \infty) > 0$ because $P(N = \infty) = \prod_{n=1}^{\infty} P(|X_n| \leq n)$ and $\sum_{n=1}^{\infty} [1 - P(|X_n| \leq n)] = \sum_{n=1}^{\infty} \int_{|x| > n} dF(x) = \int_{-\infty}^{\infty} (\sum_{n < |x|} 1) dF(x) \leq \int_{-\infty}^{\infty} |x| dF(x) = E|X_1| < \infty$. Thus from (7),

$$\sum_{n=1}^{\infty} n^{-1} \int_{|x| \geq n} |x| dF(x) = \int_{-\infty}^{\infty} |x| (\sum_{n \leq |x|} n^{-1}) dF(x) < \infty$$

and so $\int_{-\infty}^{\infty} |x| \log |x| dF(x) < \infty$ and (1) is proved.

We next prove (4) \Rightarrow (1). We define N exactly as before and again we have that $E|X_1| < \infty$. Observing that (4) gives

$$(8) \quad \infty > E_{[N < \infty]} |S_N/N| \geq E_{[N < \infty]} |X_N/N| - E_{[N < \infty]} [(|X_1| + \dots + |X_{N-1}|)/N]$$

we see that (1) follows exactly as before if we can prove that the last term on the right of (8) is finite. This term is

$$(9) \quad \sum_{n=1}^{\infty} P(N = n)n^{-1} \sum_{k < n} E[|X_k| \mid N = n] \\ = \sum_{n=1}^{\infty} P(N = n)n^{-1} \sum_{k < n} E[|X_k| \mid |X_k| < k]$$

by independence of the X 's again. We have

$$E[|X_k| \mid |X_k| < k] = \frac{\int_{|x| < k} |x| dF(x)}{\int_{|x| < k} dF(x)} \leq \frac{E|X_1|}{p_1} < \infty$$

and so (9) and the last term on the right of (8) are finite. Thus (4) \Rightarrow (1).

The implication (2) \Rightarrow (4) is trivial because for every rule N , $|S_N/N| \leq \sup |S_n/n|$. Similarly, (3) \Rightarrow (5). Since $\dots, (|X_1| + |X_2|)/2, |X_1|/1$ is a martingale, ([3] page 317) shows immediately that (1) \Rightarrow (2) and consequently (1) \Rightarrow (3). Thus (1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (1) and the equivalence is proved.

Burkholder also gives a one-sided version of the equivalence of (1)–(3) replacing (1)–(3) by

- (1') $EX_1 \log X_1^+ < \infty$
- (2') $E \sup_n (S_n/n) < \infty$
- (3') $E \sup_n (X_n/n) < \infty$.

He proves that (1')–(3') are equivalent (and each is equivalent to an additional statement about X_1 in terms of conditional expectation operators) *provided that* $E|X_1| < \infty$. We can easily modify our proof to obtain the one-sided version and to prove that (1')–(5') are equivalent, where

$$(4') \quad \sup_{\text{rule } N} E(S_N/N) < \infty$$

$$(5') \quad \sup_{\text{rule } N} E(X_N/N) < \infty,$$

provided that $E|X_1| < \infty$. Indeed the only change required in the proof of (5') \Rightarrow (1) to prove (5') \Rightarrow (1') is to let $N' =$ the first $n \geq 1$ for which $X_n > n$; $N' = \infty$ otherwise. The proof (5') \Rightarrow (1') then proceeds exactly as before. The proof of (4') \Rightarrow (1') is also exactly as before. Again (2') \Rightarrow (4'), (3') \Rightarrow (5') trivially and the final implications (1') \Rightarrow (2') and (1') \Rightarrow (3') are proved by observing that $\dots, (X_1^+ + X_2^+)/2, X_1^+/1$ is a martingale and again applying the martingale theorem ([3] page 317).

We should point out that of course the one-sided version becomes false if $EX_1^- = \infty$ since the negative side of X_1 could then overwhelm the positive side and hence the implication (4') \Rightarrow (1') would break down.

We remark that in case (1) fails, there is a stopping rule N^* which makes the left side of (4) or (5) infinite and satisfies $P(N^* < \infty) = 1$ as well. Indeed, if N is the stopping rule we used above (with $P(N = \infty) > 0$) we can easily find an integer valued random variable T independent of N and having a sufficiently long tail for which $N^* = \min(N, T)$ does the trick. By shifting the sequence X_1, X_2, \dots and defining T as a function of X_1 , we can even take N^* to be defined on the original sample space.

REFERENCES

- [1] BURKHOLDER, D. L. (1962). Successive conditional expectation of an integrable function. *Ann. Math. Statist.* **33** 887–893.
- [2] MARCINKIEWICZ, J. and ZYGMUND, A. (1937). Sur les fonctions indépendantes. *Fund. Math.* **29** Ann. 60–90.
- [3] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.
- [4] DAVIS, BURGESS (1969). Stopping rules for S_n/n , and the class $L \log L$. *Zeit. für Wahr.* (to appear).
- [5] GUNDY, RICHARD F. (1969). On the class $L \log L$, martingales, and singular integrals. *Studia Math.* **33** 109–18.