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# First Passage Time for a Particular Gaussian Process

## Abstract

We find an explicit formula for the first passage probability,  $Q_a(T|x) = P_r(S(t) < a, 0 \leq t \leq T | S(0) = x)$ , for all  $T > 0$ , where  $S$  is the Gaussian process with mean zero and covariance  $ES(\tau)S(t) = \max(1 - |t - \tau|, 0)$ .

Previously,  $Q_a(T|x)$  was known only for  $T \leq 1$ .

In particular for  $T = n$  an integer and  $-\infty < x < a < \infty$ ,

$$Q_a(T|x) = 1/\varphi(x) \int_D \dots \int \det \varphi(y_i - y_{j+1} + a) dy_2 \dots dy_{n+1},$$

where the integral is a  $n$ -fold integral of  $y_2, \dots, y_{n+1}$  over the region  $D$  given by

$$D = \{a - x < y_2 < y_1 < \dots < n+1\}$$

and the determinant is of size  $(n+1) \times (n+1)$ ,  $0 < i, j \leq n$ , with  $y_0 \equiv 0$ ,  $y_1 \equiv a - x$ .

## Disciplines

Applied Statistics

## FIRST PASSAGE TIME FOR A PARTICULAR GAUSSIAN PROCESS

BY L. A. SHEPP

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We find an explicit formula for the first passage probability,  $Q_a(T|x) = P_r(S(t) < a, 0 \leq t \leq T | S(0) = x)$ , for all  $T > 0$ , where  $S$  is the Gaussian process with mean zero and covariance  $ES(\tau)S(t) = \max(1 - |t - \tau|, 0)$ . Previously,  $Q_a(T|x)$  was known only for  $T \leq 1$ .

In particular for  $T = n$  an integer and  $-\infty < x < a < \infty$ ,

$$Q_a(T|x) = \frac{1}{\varphi(x)} \int_D \cdots \int \det \varphi(y_i - y_{j+1} + a) dy_2 \cdots dy_{n+1},$$

where the integral is an  $n$ -fold integral on  $y_2, \cdots, y_{n+1}$  over the region  $D$  given by

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and the determinant is of size  $(n+1) \times (n+1)$ ,  $0 < i, j \leq n$ , with  $y_0 \equiv 0$ ,  $y_1 \equiv a - x$ .

**1. Introduction.** Let  $S = S(t)$ ,  $0 \leq t \leq T$  be the Gaussian process with mean zero and covariance

$$(1.1) \quad \begin{aligned} ES(\tau)S(t) &= 1 - |t - \tau|, & |t - \tau| \leq 1 \\ &= 0, & |t - \tau| > 1. \end{aligned}$$

As observed in [5],  $S$  can be represented in terms of the standard Wiener process  $W$  by

$$(1.2) \quad S(t) = W(t) - W(t+1), \quad t \geq 0.$$

The first passage probability

$$(1.3) \quad Q_a(T|x) = P_r(S(t) < a, 0 \leq t \leq T | S(0) = x)$$

was studied by Slepian (1961), Mehr and McFadden (1965), and Shepp (1966). Application to a signal shape problem in radar was found by Zakai and Ziv (1969). We give an explicit formula for  $Q_a(T|x)$  as an integral ((2.15) below) in  $T$ -dimensional space when  $T$  is an integer, and an integral ((2.25) below) in  $2[T] + 2$  dimensional space when  $T$  is not an integer.

Slepian found  $Q_a(T|x)$  for  $T \leq 1$  by deriving a recurrence equation from a certain Markov-like property of  $S$  which was later called the reciprocal property by Jamison (1970). Shepp found an equivalent form of Slepian's result by using the Radon-Nikodym derivative of  $S$  with respect to the Wiener process and integrating in function space. Both of the above methods break down for  $T > 1$ : (a) The reciprocal property is not valid for  $T > 1$ ; (b)  $S$  is not absolutely continuous with respect to the Wiener process for  $T > 1$ . The present method relies instead on an identity of Karlin and McGregor (1959).

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In Section 2 we derive the formula for  $Q_a(T | x)$ . When  $T$  is an integer the formula is seen to be very similar to the Fredholm formula for the resolvent kernel. We study this similarity in Section 3, showing that the generating function of  $EQ_a(n | S(0))$ ,  $n = 0, 1, 2, \dots$  can be given in terms of a resolvent kernel. Unfortunately the resolvent kernel does not seem to be easily obtainable and all attempts to find the generating function in simple form have so far been unsuccessful.

**2. Derivation of the formulas.** Let  $X(t)$ ,  $0 \leq t \leq J$ , be a real-valued Markov process with continuous sample paths and let  $X_0, \dots, X_n$  be independent copies of  $X$ . Suppose  $a_0 < \dots < a_n$  and  $b_0 < \dots < b_n$  and let  $db_0, \dots, db_n$  be infinitesimal intervals about  $b_0, \dots, b_n$  respectively. The result of Karlin and McGregor ([2] page 1149) becomes

$$(2.1) \quad P_r(X_0(t) < \dots < X_n(t), 0 \leq t \leq \tau, \text{ and } X_i(\tau) \in db_i, i = 0, \dots, n | X_i(0) = a_i, i = 0, \dots, n) = \det p_\tau(a_i, b_j) \cdot db_0, \dots, db_n$$

where  $p_\tau(a, b) db = P_r(X(\tau) \in db | X(0) = a)$  and  $\det$  stands for the determinant of the  $(n+1) \times (n+1)$  transition probability matrix. Specializing (2.1) by taking  $X$  = the Wiener process, dividing both sides of (2.1) by  $P_r(X_i(\tau) \in db_i, i = 0, \dots, n | X_i(0) = a_i, i = 0, \dots, n)$  we obtain

$$(2.2) \quad P_r(W_0(t) < \dots < W_n(t), 0 \leq t \leq \tau | W_i(0) = a_i, W_i(\tau) = b_i, i = 0, \dots, n) = (\det p_\tau(a_i, b_j)) / \prod_{i=0}^n p_\tau(a_i, b_i),$$

where  $W_0, \dots, W_n$  are independent Wiener processes. The transition probabilities  $p_\tau(a, b)$  are given by the well-known formula

$$(2.3) \quad p_\tau(a, b) = \frac{1}{(2\pi\tau)^{-\frac{1}{2}}} \exp \left[ -\frac{1}{2} \frac{(a-b)^2}{\tau} \right] \equiv \varphi_\tau(a-b).$$

For simplicity we first consider the case when  $T$  is an integer,  $T = n$ , and argue as follows. From (1.2) and (1.3),

$$(2.4) \quad \begin{aligned} Q_a(T | x) &= P_r(W(t) - W(t+1) < a, 0 \leq t \leq n | W(0) = 0, W(0) - W(1) = x) \\ &= P_r(W(t) < W(t+1) + a < W(t+2) + 2a < \dots < W(t+n) + na, 0 \leq t \leq 1 | W(0) = 0, W(0) - W(1) = x). \end{aligned}$$

Integrating out over the values  $x_i$  of  $W$  at times  $i = 0, \dots, n+1$ , and letting  $\Omega$  denote the event of the last term in (2.4) we have,

$$(2.5) \quad \begin{aligned} Q_a(T | x) &= \int \dots \int P_r(\Omega, W(0) \in dx_0, \dots, W(n+1) \in dx_{n+1} | W(0) = 0, W(0) - W(1) = x). \end{aligned}$$

Restating (2.5) in terms of conditional probabilities, and noting that in (2.5) we must have  $x_0 = 0, x_1 = -x$  because of the conditioning, we get

$$(2.6) \quad Q_a(T | x) = \int \cdots \int P_r(\Omega | W(0) = x_0, \dots, W(n+1) = x_{n+1}, W(0) = 0, W(0) - W(1) = x) P_r(W(0) \in dx_0, \dots, W(n+1) \in dx_{n+1} | W(0) = 0, W(0) - W(1) = x).$$

We introduce the processes  $W_i, i = 0, 1, \dots, n$

$$(2.7) \quad W_i(t) = W(t+i) + ia, \quad 0 \leq t \leq 1.$$

We have

$$(2.8) \quad \Omega = \{W_0(t) < W_1(t) < \cdots < W_n(t), 0 \leq t \leq 1\}$$

and under the conditioning involved in the first probability on the right side of (2.6),

$$(2.9) \quad W_i(0) = W(i) + ia = x_i + ia, \quad W_i(1) = W(i+1) + ia = x_{i+1} + ia.$$

Thus

$$(2.10) \quad Q_a(T | x) = \int \cdots \int P_r(\Omega | W_i(0) = x_i + ia, W_i(1) = x_{i+1} + ia, i = 0, 1, \dots, n, W_0(0) = 0, W_0(0) - W_0(1) = x) \times P_r(W(0) \in dx_0, \dots, W(n+1) \in dx_{n+1} | W(0) = 0, W(0) - W(1) = x).$$

The range of integration is the set where the first probability under the integral is nonzero, that is where the inequalities in (2.8) hold for  $t = 0, t = 1$  and  $W_i(0) = x_i + ia, W_i(1) = x_{i+1} + ia$ . The range is therefore the set where  $x_i + ia < x_{i+1} + (i+1)a, i = 0, \dots, n$ . Since  $W(0) = 0$  and  $W(1) = W(0) - (W(0) - W(1)) = 0 - (x) = -x$  we must have

$$(2.11) \quad x_0 = 0, \quad x_1 = -x.$$

The first probability under the integral in (2.10) is given by (2.2) since  $x_0 = 0, x_1 = -x$ , and the conditioned Wiener processes  $W_i$  are independent. Thus

$$(2.12) \quad P_r(\Omega | W_i(0) = x_i + ia, W_i(1) = x_{i+1} + ia, i = 0, \dots, n) = (\det \varphi(x_i + ia - x_{j+1} - ja)) / \prod_{i=0}^n \varphi(x_i + ia - x_{i+1} - ia),$$

where from (2.3)

$$(2.13) \quad \varphi(u) = \frac{1}{(2\pi)^{-\frac{1}{2}}} \exp[-\frac{1}{2}u^2].$$

The second probability under the integral in (2.10) is simply

$$(2.14) \quad \prod_{i=1}^n \varphi(x_i - x_{i+1}) / \varphi(x_0 - x_1), \quad x_0 = 0, x_1 = -x.$$

Putting (2.12) and (2.14) into (2.10) we obtain after the change of variables  $y_i = x_i + ia, i = 0, \dots, n + 1$ , the following formula for  $Q_a(T | x), -\infty < x < a < \infty, T = n$  an integer.

$$(2.15) \quad Q_a(T | x) = \frac{1}{\varphi(x)} \int_D \dots \int \det \varphi(y_i - y_{j+1} + a) dy_2 \dots dy_{n+1},$$

where the integral is an  $n$ -fold integral on  $y_2, \dots, y_{n+1}$  over the region  $D$  given by

$$(2.16) \quad D = \{a - x < y_2 < y_3 < \dots < y_{n+1}\}$$

and the determinant is of size  $(n + 1) \times (n + 1), 0 \leq i, j \leq n$ , with  $y_0 \equiv 0, y_1 \equiv a - x$ .

Of course,  $Q_a(t | x) = 0$  for  $a < x$ .

It is easily verified that for  $T = 1$  we have

$$(2.17) \quad Q_a(1 | x) = \Phi(a) - \frac{\varphi(a)}{\varphi(x)} \Phi(x)$$

agreeing with ([4] page 349). For  $T \geq 2$ , the integral does not seem to be simply expressible.

Next we derive the formula for  $Q_a(T | x)$  in case  $T$  is not an integer say  $T = n + \theta, 0 < \theta < 1$ , and integer  $n \geq 0$ . We have

$$(2.18) \quad Q_a(T | x) = P_r(W(t) - W(t+1) < a, 0 \leq t \leq n + \theta | W(0) = 0, W(0) - W(1) = x) = P_r(W(t) < W(t+1) + a < \dots < W(t+n+1) + (n+1)a, 0 \leq t \leq \theta, \text{ and } W(\tau + \theta) < W(\tau + \theta + 1) + a < \dots < W(\tau + \theta + n) + na, 0 \leq \tau \leq 1 - \theta | W(0) = 0, W(0) - W(1) = x).$$

Integrating out over the values  $u_i$  and  $v_i$  of  $W$  at times  $i$  and  $i + \theta, i = 0, 1, 2, \dots, n + 1$ , we have, letting  $\Omega'$  denote the event of the last term of (2.18),

$$(2.19) \quad Q_a(T | x) = \int \dots \int P_r(\Omega', W(0) \in du_0, \dots, W(n+1) \in du_{n+1}, W(\theta) \in dv_0, \dots, W(n+1+\theta) \in dv_{n+1} | W(0) = 0, W(0) - W(1) = x).$$

Restating (2.19) in terms of conditional probabilities, we get

$$(2.20) \quad Q_a(T | x) = \int \dots \int P_r(\Omega' | W(0) = u'_0, \dots, W(n+1) = u_{n+1}, W(\theta) = v_0, \dots, W(n+1+\theta) = v_{n+1}, W(0) = 0, W(0) - W(1) = x) P_r(W(0) \in du_0, \dots, W(n+1) \in du_{n+1}, W(\theta) \in dv_0, \dots, W(n+1+\theta) \in dv_{n+1} | W(0) = 0, W(0) - W(1) = x).$$

We introduce the processes  $W_i, i = 0, \dots, n+1; W'_j, j = 0, \dots, n$

$$(2.21) \quad \begin{aligned} W_i(t) &= W(t+i) + ia, & 0 \leq t \leq \theta \\ W'_j(\tau) &= W(\tau+\theta+j) + ja, & 0 \leq \tau \leq 1-\theta. \end{aligned}$$

We have  $\Omega' = \Omega_1 \cap \Omega_2$  where

$$(2.22) \quad \begin{aligned} \Omega_1 &= \{W_0(t) < \dots < W_{n+1}(t), 0 \leq t \leq \theta\} \\ \Omega_2 &= \{W'_0(\tau) < \dots < W'_n(\tau), 0 \leq \tau \leq 1-\theta\} \end{aligned}$$

and under the conditioning involved in the first probability on the right side of (2.20) we have for  $0 \leq i \leq n+1, 0 \leq j \leq n$ ,

$$(2.23) \quad \begin{aligned} W_i(0) &= W(i) + ia = u_i + ia \\ W_i(\theta) &= W(\theta+i) + ia = v_i + ia \\ W'_j(0) &= W(\theta+j) + ja = v_j + ja \\ W'_j(1-\theta) &= W(j+1) + ja = u_{j+1} + ja. \end{aligned}$$

The processes  $W_i(t)$  and  $W'_j(\tau)$  conditioned to satisfy (2.23) are independent and so the conditional probability of  $\Omega'$  in (2.20) is the product of the conditional probabilities of  $\Omega_1$  and  $\Omega_2$ . Thus with  $u_0 = W(0) = 0, u_1 = W(0) - (W(1) - W(0)) = -x$ , (2.20) becomes

$$(2.24) \quad \begin{aligned} Q_a(T | x) &= \int \dots \int P_r(\Omega_1 | W_i(0) = u_i + ia, W_i(\theta) = v_i + ia, i = 0, \dots, n+1) \\ &\quad \times P_r(\Omega_2 | W'_j(0) = v_j + ja, W'_j(1-\theta) = u_{j+1} \\ &\quad + ja, j = 0, 1, \dots, n) P_r(W(0) \in du_0, \dots, \\ &\quad W(n+1) \in du_{n+1}, W(\theta) \in dv_0, \dots, \\ &\quad W(n+1+\theta) \in dv_{n+1} | W(0) = 0, W(0) - W(1) = x). \end{aligned}$$

Using (2.2) to express the first two probabilities under the integral in (2.24) and letting  $x_i = u_i + ia, y_i = v_i + ia, i = 0, \dots, n+1$  we obtain the final result for  $T = n+\theta, 0 < \theta < 1, n$  an integer as

$$(2.25) \quad Q_a(T | x) = \frac{1}{\varphi(x)} \int_{D'} \dots \int (\det \varphi_\theta(x_i - y_j)) (\det \varphi_{1-\theta}(y_i - x_{j+1} + a)) \\ \times dx_2 \dots dx_{n+1} dy_0 \dots dy_{n+1}$$

where the integral is a  $2n+2$ -fold integral over the region  $D'$  given by

$$(2.26) \quad D' = \{a-x < x_2 < \dots < x_{n+1} \text{ and } y_0 < y_1 < \dots < y_{n+1}\}.$$

The first determinant in (2.25) is of size  $(n+2) \times (n+2), 0 \leq i, j \leq n+1$  while the second is of size  $(n+1) \times (n+1), 0 \leq i, j \leq n$ . In each,  $x_0 = 0, x_1 = a-x$ .

One may verify that for  $T < 1, Q_a(T | x)$  agrees with the previous results found in [4] and [5].

**3. Remarks on the similarity with Fredholm theory.** For large  $T$  the expressions (2.15) and (2.25) are unwieldy and apparently not suited for either numerical calculation or asymptotic estimation. For simplicity we restrict attention here to integral  $T$  and to the unconditional probabilities,

$$(3.1) \quad F_n(a) = P_r(S(t) < a, 0 \leq t \leq n).$$

D. Slepian pointed out the strong similarity between (2.15) and the formulas involved in the Fredholm resolvent. Indeed, if we define

$$(3.2) \quad K(s, t) = \varphi(s-t+a)$$

then it can be seen from (2.15) that

$$(3.3) \quad F^n(a) = \frac{(-1)^n}{n!} \int_0^\infty \int_0^y \cdots \int_0^y K \left( \begin{matrix} 0, u_1, \dots, u_n \\ y, u_1, \dots, u_n \end{matrix} \right) du_1 \cdots du_n$$

in the notation of ([6] page 70). Applying Fredholm theory [6] we find that the generating function

$$(3.4) \quad F(\lambda, a) = \sum_{n=0}^{\infty} \lambda^n F_n(a)$$

is given as

$$(3.5) \quad F(\lambda, a) = \int_0^\infty \exp \left[ -\lambda \int_0^y H(\lambda, u, u, u) du \right] H(\lambda, 0, y, y) dy$$

where  $H = H(\lambda, s, t, y)$  is the resolvent kernel of  $K$ , determined uniquely by the resolvent equation

$$(3.6) \quad H(\lambda, s, t, y) = K(s, t) + \lambda \int_0^y H(\lambda, s, u, y) K(u, t) du, \quad 0 \leq s \leq y,$$

the parameter  $a$  being suppressed in both  $H$  and  $K$ .

We have included this section in the hope that (3.5) could be used to obtain bounds on the radius of convergence of (3.4) or equivalently to find bounds on

$$(3.7) \quad \lim_{n \rightarrow \infty} n^{-1} \cdot \log F_n(a),$$

assuming the limit exists. Unfortunately, we were unable to complete this approach because of the difficulty of estimating  $H$  sufficiently closely.

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