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Larry A. Shepp
University of Pennsylvania

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A Limit Law Concerning Moving Averages

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A LIMIT LAW CONCERNING MOVING AVERAGES[1]

By L. A. Shepp

University of California, Berkeley and Bell Telephone Laboratories

0. Introduction. This note deals with the limiting values of the averages, $T_n = (S_{n+f(n)} - S_n)/f(n)$ where $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$, is a sum of mutually independent and identically distributed random variables. The function f takes positive integer values and nondecreasingly tends to infinity. We shall determine the almost everywhere constant lim sup T_n in terms of the moment generating function of X_1 and the radius of convergence of $\sum x^{f(n)}$, denoted r.

If we define N_n as the number of *consecutive* successes beginning at trial $n + 1$ in a sequence of Bernoulli trials with success probability p, we let $U_n = N_n/f(n)$. It is shown that, almost surely, lim sup $U_n = \log_p r$. This result will be compared with lim sup T_n for the case of Bernoulli trials. The author is indebted to Professor D. J. Newman for drawing his attention to this problem.

1. Statement of results. The events $\{T_n > c$ infinitely often (i.o.)$\}$ are tail events, $-\infty < c < \infty$, and so $T = \limsup T_n$ is constant with probability one ([3], p. 229).

We form the series $\sum x^{f(n)} = \sum A_k x^k$ and determine its radius of convergence r, $0 \leqq r \leqq 1$. The constant $T = T(f)$ will depend only on r, and when this is to be emphasized, will be denoted T_r.

Let $M(t) = Ee^{X_1 t} \leqq +\infty$ be defined for all t, and set [1], for any a, $m(a) = \min M(t)e^{-at} \leqq M(0) = 1$. If $M(t) < \infty$ for some $t > 0$, then $EX_1^+ < \infty$, and so EX_1 is well defined, $-\infty \leqq EX_1 < \infty$. Let ess sup $X_1 = A \leqq +\infty$. We assume X_1 is nonconstant a.s., so $EX_1 < A$.

Theorem 1. *If $M(t) < \infty$ for some $t > 0$, then*

$$T_r = A \quad \text{for} \quad 0 \leqq r \leqq P\{X_1 = A\}. \tag{1.1}$$

$$T_r = a_r \quad \text{for} \quad P\{X_1 = A\} < r < 1 \tag{1.2}$$

where $a = a_r$ is the unique solution (inverse) of $m(a) = r$, $EX_1 < a < A$.

$$T_r = EX_1 \quad \text{for} \quad r = 1. \tag{1.3}$$

Moreover, T_r decreases strictly and continuously for $P\{X_1 = A\} \leqq r \leqq 1$.

The consideration of the sequence $-X_i$ leads to a similar theorem for lim inf T_n. If f is strictly increasing then $r = 1$. The existence of lim T_n is assured if and only if $r = 1$, whence $\lim T_n = EX_1$ a.s., $-\infty < EX_1 < \infty$. Proofs of these assertions are in Section 3. Our methods break down in case $M(t) = \infty$ for all $t > 0$, and this case is not discussed.

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An immediate corollary of this theorem is that if f_1 and f_2 are two functions, as above, and $f_1(n) \leqq f_2(n)$, $n = 1, 2, \cdots$, then $T(f_2) \leqq T(f_1)$. A class of f in central position are $f_c(n) = 1 + [c \log n]$, $c > 0$, $n = 1, 2, \cdots$, whence $r = \exp(-1/c)$.

2. Chernoff's theorem. It is easily seen that $\log P\{S_n \geqq na\}$ is subadditive in n for fixed a, and hence $\lim -(1/n) \log P\{S_n \geqq na\}$ exists. Chernoff [1], has found this limit in terms of $m(a)$. We suppress the subscript of X_1 in this discussion.

THEOREM 2. (Chernoff) *We suppose* $M(t) < \infty$ *for some* $t > 0$. *We have* $-\infty \leqq EX < \infty$, *and* $m(a) = \min_{t \geqq 0} Ee^{(X-a)t} < 1$, *if* $a > EX$. *Moreover, for* $n = 1, 2, \cdots$.

$$P\{S_n \geqq na\} \leqq (m(a))^n, \tag{2.1}$$

and for $0 < \epsilon < m(a)$, *and* $n > n_0(\epsilon)$,

$$P\{S_n \geqq na\} > (m(a) - \epsilon)^n. \tag{2.2}$$

This is a slight variant of [1], and we should complete the proof in [1] by noting that for $0 < t < \delta$, some $\delta > 0$, $r(t) = M(t)e^{-at}$ is convex, and so $r'(t) = E(X - a)e^{(X-a)t}$ decreases to $E(X - a) < 0$ as t *decreases* to zero (*dominated convergence*). It follows that $r'(t) < 0$, $0 < t < \delta_1$, some $\delta_1 > 0$, and so $r(t) = r(0^+) + \int_0^t r'(u)\, du < r(0^+) = 1, 0 < t < \delta_1$, whence $m(a) < 1$. To prove the validity of the statement $m(a) = \min_{t \geqq 0} Ee^{(X-a)t}$ note that this is trivial if $M(t) = \infty$ for all $t < 0$, and if $M(t) < \infty$ for some $t < 0$ then $r'(0) = EX - a < 0$ and so the minimum occurs to the right of zero, by convexity.

If $A = \text{ess sup } X = \infty$, set $m(\infty) = 0$; if $EX = -\infty$, set $m(-\infty) = 1$. The function $m(a)$ is then defined for extended real numbers in the interval $EX \leqq a \leqq A$.

(2.3) LEMMA. *If* $M(t) < \infty$ *for some* $t > 0$, *then for* $EX \leqq a \leqq A$, $m(a)$ *is continuous and strictly monotonic. Moreover,* $m(EX) = 1$.

PROOF. For $h > 0$ and $EX < a < a + h < A$, we have by the first part of Theorem 2,

$$m(a + h) = \min_{t \geqq 0} Ee^{(X-a)t}e^{-ht} < m(a). \tag{2.4}$$

Continuity of $m(a)$ at finite a, $EX \leqq a < A$ is easily shown using the fact that $m(a) = Ee^{(X-a)t_1}$ for $0 \leqq t_1 < \infty$. If $A < \infty$, $\epsilon > 0$, we set $t = \epsilon^{-\frac{1}{2}}$, obtaining

$$m(A - 0) = \lim_{\epsilon \downarrow 0} \min_t Ee^{(X-A+\epsilon)t} \leqq \lim_{\epsilon \downarrow 0} E \exp[(X - A)\epsilon^{-\frac{1}{2}} + \epsilon^{\frac{1}{2}}]. \tag{2.5}$$

It follows that $m(A - 0) \leqq P\{X = A\} = m(A)$. However, for $\epsilon > 0$,

$$m(A - \epsilon) \geqq \min_{t \geqq 0} Ee^{(X-A)t} \min_{t \geqq 0} e^{\epsilon t} = m(A). \tag{2.6}$$

Thus m is continuous at A, if $A < \infty$. If $A = \infty$, then for some fixed $t > 0$,

$$0 \leqq \lim_{a \to \infty} m(a) \leqq \lim_{a \to \infty} Ee^{(X-a)t} = 0 = m(\infty). \tag{2.7}$$

If $EX = -\infty$, we have $1 \geqq \lim_{a\to-\infty} m(a) \geqq \lim_{a\to-\infty} P\{X \geqq a\} = 1 = m(-\infty)$. This proves the continuity assertions. If $EX > -\infty$, then $M'(0^+) = EX$, and $m(EX) = 1$ follows. This proves (2.3).

3. Proof of Theorem 1. We shall need the following simple lemma for power series. Here f is a function of the type described in the introduction.

(3.1) LEMMA. *If* $0 < x < y < 1$, $b > 0$ *and* $\sum x^{f(n)} < \infty$, *then* $\sum y^{f(n_k)} = \infty$ *where* $[bf(n_1)] \geqq 1$ *and for* $k = 1, 2, \cdots$,

$$(3.2) \qquad n_{k+1} = n_k + [bf(n_k)].$$

PROOF. Choose K so that for $n \geqq n_K$,

$$(3.3) \qquad (x/y)^{f(n)} f(n) < 1.$$

Since f is *nondecreasing*, we have for $k = 1, 2, \cdots$,

$$(3.4) \qquad \sum_{j=0}^{n_{k+1}-n_k-1} y^{f(n_k+j)}/f(n_k + j) \leqq y^{f(n_k)}(n_{k+1} - n_k)/f(n_k) \leqq by^{f(n_k)}.$$

Using (3.3) and the fact that $n_k \to \infty$ (since $[bf(n_k)] \geqq 1$ in (3.2)), we get

$$(3.5) \qquad \sum_{k=K}^{\infty} y^{f(n_k)} \geqq b^{-1} \sum_{k=K}^{\infty} \sum_{j=n_k}^{n_{k+1}-1} y^{f(j)}/f(j) \geqq b^{-1} \sum_{n \geqq n_K} x^{f(n)} = \infty,$$

proving (3.1).

To prove Theorem 1, let $A_n(a)$ be the event $\{T_n \geqq a\}$. Since we are dealing with identically distributed random variables, we have

$$(3.6) \qquad P\{A_n(a)\} = P\{S_{f(n)} \geqq af(n)\}.$$

In proving (1.1), we assume first $A < \infty$. If $r \leqq m(A)$, then for $h > 0$, sufficiently small, $m(A - h) > r$ and we choose $\epsilon > 0$ so that $\alpha = m(A - h) - 2\epsilon > r$. Since $\sum \alpha^{f(n)} = \infty$, we may apply (3.1) with $b = 1, x = \alpha, y = \alpha + \epsilon$. For the sequence n_k of (3.1), we have $\sum y^{f(n_k)} = \infty$. By (2.2) and (3.6), $P\{A_n(A - h)\} \geqq y^{f(n)}$, for n sufficiently large. It follows that $\sum P\{A_{n_k}(A - h)\} = \infty$. Since A_{n_k} depends only on that segment of $\{X_n\}$ for which $n_k < n \leqq n_k + f(n_k) = n_{k+1}$, and these segments are disjoint, the events A_{n_k} are independent, $k = 1, 2, \cdots$. By Borel's lemma ([3], p. 228), $P\{A_{n_k}(A - h)\text{i.o.}\} = 1$ and so $T \geqq A - h$. Since $T_n \leqq A$, a.s. for each n, we obtain $T = A$. If now $A = \infty$ and $r \leqq m(A)$, then $r = 0$. Since $m(a) > 0$ for finite $a > EX_1$ we have for $2\epsilon < m(a)$, $\sum P\{A_n(a)\} \geqq \sum (m(a) - \epsilon)^{f(n)} = \infty$ and arguing as above, $P\{A_{n_k}(a) \text{ i.o.}\} = 1$. Hence $T \geqq a$ for every a, and $T = \infty = A$. This proves (1.1).

If $m(A) < r < 1$, then there is a unique a, $EX_1 < a < A$, for which $m(a) = r$ by (2.3). We shall show $T_r = a$. Let $h > 0$ be sufficiently small in the following. Using (2.1) we have, since $m(a + h) < m(a) = r$,

$$(3.7) \qquad \sum P\{A_n(a + h)\} \leqq \sum (m(a + h))^{f(n)} < \infty.$$

By Cantelli's lemma ([3], p. 228) we have $T_r \leqq a + h$ so $T_r \leqq a$. On the other hand, by (2.2), $P\{A_n(a - h)\} \geqq (m(a - h) - \epsilon)^{f(n)}$ for $0 < \epsilon < m(a - h)$. If ϵ is so small that $m(a - h) - 2\epsilon > m(a) = r$, then $\sum (m(a - h) - 2\epsilon)^{f(n)} = \infty$. By (3.1), again with $b = 1$, we obtain a sequence n_k for which $\sum P\{A_{n_k}(A - h)\} \geqq \sum (m(a - h) - \epsilon)^{f(n_k)} = \infty$. As before we conclude that $T_r \geqq a - h$ and so $T_r = a$ and (1.2) is proved. To prove (1.3) assume that $y > EX_1$. By Theorem 2, $m(y) < 1$ and so

$$\sum P\{A_n(y)\} \leqq m(y)^{f(n)} < \infty, \tag{3.8}$$

since $r = 1$. It follows that $T_1 \leqq EX_1$. The weak law of large numbers, ([2], p. 228) shows that for $E|X_1| < \infty$ and $\epsilon > 0$, $P\{S_n > (EX_1 - \epsilon)n\} \to 1$. Thus $P\{A_n(EX_1 - \epsilon)\} \to 1$ and so for $0 \leqq r \leqq 1$, $T_r \geqq EX_1$. We have $T_1 = EX_1$ in any case and (1.3) is proved. In this last case we have lim T_n existing, since

$$\liminf T_n = -\limsup - T_n = -E(-X_1) = EX_1 = \limsup T_n \tag{3.9}$$

almost surely. In (3.9) we have tacitly assumed that $M(t) < \infty$ also for some $t < 0$ and so $E|X_1| < \infty$ here. Thus Theorem 1 and the assertions following are proved.

4. An example. As an illustration of Theorem 1 we consider the Bernoulli case, $P\{X_1 = 1\} = p = 1 - P\{X_1 = 0\}$ for $0 < p < 1$. We have for $p \leqq a \leqq 1$,

$$m(a) = (p/a)^a((1 - p)/(1 - a))^{1-a} \tag{4.1}$$

and thus, for $p \leqq r \leqq 1$, $T_r = a$ where $m(a) = r$, and for $0 \leqq r \leqq p$, $T_r = 1$.

5. A related problem on success runs. If we set $N_n = k$ if $X_{n+1} = \cdots = X_{n+k-1} = 1$ and $X_{n+k} = 0$; $k = 1, 2, \cdots$, in the example, then N_n counts the waiting time until the first failure following the nth Bernoulli trial (success runs [2] p. 197, Example 5, D. J. Newman). If f is a function of the type considered above, we set $U_n = N_n/f(n)$, $n = 1, 2, \cdots$ and $U = \limsup U_n$. It will be shown that U is a constant a.s., depending on f only through r, the radius of convergence of $\sum x^{f(n)}$.

(5.1) COROLLARY. *For* $0 \leqq r \leqq 1$, $U = \limsup U_n = \log_p r$.

PROOF. Let $B_n(b)$ denote the event $\{U_n > b\}$, $0 < b$ and $n = 1, 2, \cdots$. We have $P\{B_n(b)\} = p^{[bf(n)]+1}$, and if $p^b < r$,

$$\sum P\{B_n(b)\} \leqq \sum p^{bf(n)} < \infty. \tag{5.2}$$

It follows that $P\{B_n(b) \text{ i.o.}\} = 0$ and that $U \leqq \log_p r$. If $p^b > r$, then finding an $\epsilon > 0$ for which $p^{b+\epsilon} > r$, $\sum p^{(b+\epsilon)f(n)} = \infty$. Using (3.1) with $x = p^{b+\epsilon}$, $y = p^b$ we obtain a sequence n_k for which

$$\sum p^{bf(n_k)} = \infty \tag{5.3}$$

$$n_{k+1} = n_k + [bf(n_k)]. \tag{5.4}$$

The event $B_{n_k}(b)$ refers to the section of $\{X_n\}$ for $n_k < n \leqq n_k + [bf(n_k)] = n_{k+1}$

and thus $B_{n_k}(b)$ are independent, $k = 1, 2, \cdots$. Moreover, $\sum P\{B_{n_k}(b)\} \geqq p\sum p^{bf(n_k)} = \infty$ by (5.3). It follows that $U \geqq b$ and so $U \geqq \log_p r$. This proves (5.1).

Comparing T and U in the Bernoulli case, it is easily seen that whenever $U < 1$, we have $U_n \leqq T_n$ *eventually* and so $U \leqq T \leqq 1$. If $T = a$ and $U = b$ we have the relationship, from (4.1) and (5.1)

$$(p/a)^a((1-p)/(1-a))^{1-a} = p^b = r \tag{5.5}$$

for $p \leqq a \leqq 1$ and $0 \leqq b \leqq 1$.

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