



University of Pennsylvania
ScholarlyCommons

Statistics Papers

Wharton Faculty Research

1977

Notes: Correction to "Radon-Nikodym Derivatives of Gaussian Measures"

Larry A. Shepp
University of Pennsylvania

Follow this and additional works at: http://repository.upenn.edu/statistics_papers

 Part of the [Probability Commons](#)

Recommended Citation

Shepp, L. A. (1977). Notes: Correction to "Radon-Nikodym Derivatives of Gaussian Measures". *The Annals of Probability*, 5 (2), 315-317. <http://dx.doi.org/10.1214/aop/1176995858>

This paper is posted at ScholarlyCommons. http://repository.upenn.edu/statistics_papers/331
For more information, please contact repository@pobox.upenn.edu.

Notes: Correction to "Radon-Nikodym Derivatives of Gaussian Measures"

Disciplines
Probability

NOTES

CORRECTION TO

“RADON-NIKODYM DERIVATIVES OF GAUSSIAN MEASURES”

BY L. A. SHEPP

Bell Laboratories

Introduction. J. R. Klauder kindly pointed out that the first statement of Theorem 11 of my paper [2] is incorrect. It was claimed incorrectly that if $h = h(t)$, $0 \leq t \leq T$ is a (strictly) increasing absolutely continuous function with $h(0) = 0$, then a necessary and sufficient condition that the Gauss-Markov process

$$(1) \quad X(t) = \frac{1}{(h'(t))^{\frac{1}{2}}} W(h(t)), \quad 0 \leq t \leq T$$

is equivalent to the Wiener process W , $X \sim W$, is that

$$(2) \quad \int_0^T \left[\frac{d}{dt} (1/(h'(t))^{\frac{1}{2}}) \right]^2 dt < \infty.$$

The case

$$(3) \quad h(t) = t + t^{\frac{3}{2}}, \quad 0 \leq t \leq T = 1$$

gives an example where (2) fails although $X \sim W$. We will prove that the condition

$$(4) \quad \int_0^T h(t) \left[\frac{d}{dt} (1/(h'(t))^{\frac{1}{2}}) \right]^2 dt < \infty$$

is necessary and sufficient for $X \sim W$. Note that (3) satisfies (4) but not (2). Theorem 1 of [2] gives a general condition for a Gaussian process to be equivalent to W but the condition is difficult to apply in this case. Instead we use the elegant results of M. Hitsuda [1]. Note that [4] gives necessary and sufficient conditions among a restricted class of h for $X \sim W$. Of course the exact scale normalization $1/(h'(t))^{\frac{1}{2}}$ in (1) is necessary for $X \sim W$ (e.g., note that $cW \sim W$ only for $c = 1$).

The error in the argument in [2] that $X \sim W$ implies (2) occurs in the ninth line from the bottom of page 344 where it is incorrectly claimed that $v' \in L^2[0, T]$ if $u'(\min(s, t))v'(\max(s, t)) \in L^2[0, T] \times [0, T]$.

The argument given for the converse assertion, that (2) implies $X \sim W$, tacitly assumes that h is bounded and under this assumption is correct since then (2) implies (4) which implies that $X \sim W$. However for unbounded h , i.e., $h(T) = \infty$, e.g.,

$$(5) \quad h(t) = t/(1 - t), \quad 0 \leq t \leq T = 1,$$

if (1) is defined by continuity at $t = 1$ so that X is the pinned Wiener process with $X(1) = 0$, then (2) holds but $X \sim W$ is false since $W(1) \neq 0$ w.p.1. Thus the assertion " $1 \notin sp(K)$ holds automatically" on page 344 of [2] tacitly assumes bounded h . Of course, Hitsuda's method avoids the spectral condition altogether and has other advantages [1, page 299].

Proof that (4) is necessary and sufficient that $X \sim W$. If (4) holds then

$$(6) \quad \begin{aligned} l(s, u) &= -(h'(u))^{\frac{1}{2}}(1/(h'(s))^{\frac{1}{2}})' ; & s > u \\ &= 0 ; & s \leq u \end{aligned}$$

is a Volterra kernel in $L^2[0, T] \times [0, T]$ the primes denoting differentiation with respect to s or u as indicated in each term by the variable in parentheses. By Theorem 2 of [1], $Y \sim \mathbf{W}$ where Y is defined in terms of a Wiener process \mathbf{W} by

$$(7) \quad \begin{aligned} Y(t) &= \mathbf{W}(t) - \int_0^t \int_0^s l(s, u) d\mathbf{W}(u) ds \\ &= \mathbf{W}(t) - \int_0^t \int_u^t l(s, u) ds d\mathbf{W}(u) \\ &= \mathbf{W}(t) - \int_0^t (h'(u))^{\frac{1}{2}}((h'(u))^{-\frac{1}{2}} - (h'(t))^{-\frac{1}{2}}) d\mathbf{W}(u) \\ &= \frac{1}{(h'(t))^{\frac{1}{2}}} \int_0^t (h'(u))^{\frac{1}{2}} d\mathbf{W}(u) \end{aligned}$$

where we have used the argument on the top of page 306 of [1] to interchange the integrals in the second line of (7), and (6) in the third line. Since the last line of (7) is a Gaussian process with the same covariance as X in (1), it follows that X and Y are the same process (induce the same measure). Since $Y \sim \mathbf{W}$ and \mathbf{W} is a Wiener process we have proved that (4) implies $X \sim W$.

To prove that $X \sim W$ implies (4), note that the process

$$(8) \quad \mathbf{X}(t) = \frac{1}{(h'(t))^{\frac{1}{2}}} \int_0^t (h'(u))^{\frac{1}{2}} dW(u)$$

is the same process as X in (1) as observed above. Since \mathbf{X} is equivalent to a Wiener process, by Theorem 1 of [1] there exists on the same space as \mathbf{X} and W in (8), another Wiener process \mathbf{W} for which

$$(9) \quad \mathbf{X}(t) = \mathbf{W}(t) - \int_0^t (\int_0^s \mathbf{I}(s, u) d\mathbf{W}(u)) ds$$

where \mathbf{I} is a (unique) L^2 Volterra kernel. Moreover \mathbf{W} is a Wiener process with respect to the same σ -fields \mathcal{F}_t as W .

Since $(h'(t))^{\frac{1}{2}}\mathbf{X}(t) = \int_0^t (h'(u))^{\frac{1}{2}} d\mathbf{W}(u)$ is a martingale with respect to \mathcal{F}_t , we have for any $\tau < t$

$$(10) \quad E[X(t)(h'(t))^{\frac{1}{2}} | \mathcal{F}_\tau] = X(\tau)(h'(\tau))^{\frac{1}{2}}.$$

From (9) and (10) with $s \wedge \tau = \min(s, \tau)$, for $\tau < t$

$$(11) \quad \begin{aligned} (h'(t))^{\frac{1}{2}}\mathbf{W}(\tau) - (h'(t))^{\frac{1}{2}} \int_0^t (\int_0^{s \wedge \tau} \mathbf{I}(s, u) d\mathbf{W}(u)) ds \\ = (h'(\tau))^{\frac{1}{2}}\mathbf{W}(\tau) - (h'(\tau))^{\frac{1}{2}} \int_0^\tau (\int_0^s \mathbf{I}(s, u) d\mathbf{W}(u)) ds. \end{aligned}$$

Interchanging integrals as before since $\mathbf{l} \in L^2 [0, T] \times [0, T]$ we obtain

$$(12) \quad \mathbf{W}(\tau)((h'(t))^{\frac{1}{2}} - (h'(\tau))^{\frac{1}{2}}) \\ = \int_0^\tau ((h'(t))^{\frac{1}{2}} \int_u^t \mathbf{l}(s, u) ds - (h'(\tau))^{\frac{1}{2}} \int_u^\tau \mathbf{l}(s, u) ds) d\mathbf{W}(u).$$

Considering τ and t as fixed and noting that $\int_a^b \varphi d\mathbf{W} = 0$ for an L^2 function φ implies $\varphi \equiv 0$ a.e., we obtain that for each $0 < u < \tau < t$, a.e.

$$(13) \quad (h'(t))^{\frac{1}{2}} - (h'(\tau))^{\frac{1}{2}} = (h'(t))^{\frac{1}{2}} \int_u^t \mathbf{l}(s, u) ds - (h'(\tau))^{\frac{1}{2}} \int_u^\tau \mathbf{l}(s, u) ds.$$

Setting $\tau = u$ we obtain easily that h is twice differentiable and $\mathbf{l} = l$ in (6). Thus $l \in L^2 [0, T] \times [0, T]$, and since $\int_0^T \int_0^T l^2(s, u) ds du$ is the left side of (4), we have shown that (4) holds.

We remark that since $X \sim W$ implies the scale changed processes \mathbf{X} and \mathbf{W} where, for any Y ,

$$(14) \quad \mathbf{Y}(t) = \frac{1}{(g'(t))^{\frac{1}{2}}} Y(g(t))$$

are also equivalent, we have $\mathbf{X} \sim \mathbf{W}$, for any increasing differentiable function g with $g(0) = 0$. Taking g to be h^{-1} and noting that $\mathbf{X} = W$ in this case we see that $X \sim W$ and only if $\mathbf{X} \sim W$, i.e., the condition (4) must be invariant under the change from h to h^{-1} . A direct proof of this fact is given in [3].

Other corrections in [2].

1. Israel Bar-David pointed out that (16.2), page 347, should include the additional term:

$$-\frac{1}{2}X^2(0)[R(0, 0)]^{-1}$$

on the right-hand side.

- 2. In footnote 3, page 332, the name referred to should be I. M. Golosov.
- 3. (18.19), page 352: change X_j to x_j .
- 4. First line of display below (18.19), page 352: change T to T^2 .
- 5. Change (18.21), page 352 to read

$$(18.21) \quad \Delta^2 g_k = \frac{T^2}{n^2} f_k g_{k+1}.$$

REFERENCES

- [1] HITSUDA, M. (1968). Representation of Gaussian processes equivalent to Wiener process. *Osaka J. Math* 5 299-312.
- [2] SHEPP, L. A. (1966). Radon-Nikodym derivatives of Gaussian measures. *Ann. Math. Statist.* 37 321-354.
- [3] SHEPP, L. A. *SIAM Review* problem section. To appear.
- [4] VARBERG, D. E. (1964). On Gaussian measures equivalent to Wiener measure. *Ann. Math. Statist.* 35 262-273.

BELL LABORATORIES
2C-354
MURRAY HILL, NEW JERSEY 07974