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Abstract
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Keywords
influences, product space, Kahn-Kalai-Linial influence bound, Gaussian measure, isoperimetric inequality

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GEOMETRIC INFLUENCES

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We present a new definition of influences in product spaces of continuous distributions. Our definition is geometric, and for monotone sets it is identical with the measure of the boundary with respect to uniform enlargement. We prove analogs of the Kahn–Kalai–Linial (KKL) and Talagrand's influence sum bounds for the new definition. We further prove an analog of a result of Friedgut showing that sets with small "influence sum" are essentially determined by a small number of coordinates. In particular, we establish the following tight analog of the KKL bound: for any set in $\mathbb{R}^n$ of Gaussian measure t, there exists a coordinate i such that the ith geometric influence of the set is at least $ct(1-t)\sqrt{\log n}/n$, where c is a universal constant. This result is then used to obtain an isoperimetric inequality for the Gaussian measure on $\mathbb{R}^n$ and the class of sets invariant under transitive permutation group of the coordinates.

1. Introduction.

DEFINITION 1.1. Let $f:\{0,1\}^n \to \{0,1\}$ be a Boolean function. The influence of the ith coordinate on f is

$$I_i(f) := \mathbb{P}_{x\sim\mu}[f(x) \neq f(x \oplus e_i)],$$

where $\mathbb{P}_{x\sim\mu}$ denotes probability when x is chosen at random according to a probability measure μ, and $x \oplus e_i$ denotes the point obtained from x by replacing x_i by $1 - x_i$ and leaving the other coordinates unchanged.

The notion of influences of variables on Boolean functions is one of the central concepts in the theory of discrete harmonic analysis. In the last two decades it

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found several applications in diverse fields, including combinatorics, theoretical computer science, statistical physics, social choice theory, etc. (see, e.g., the survey article [13]). The influences have numerous properties that allow us to use them in applications. The following three properties are among the most fundamental ones:

(1) *Geometric meaning.* The influences on the discrete cube $\{0,1\}^n$ have a clear geometric meaning. $I_i(f)$ is the measure of the *edge boundary in the ith direction* of the set $A = \{x \in \{0,1\}^n : f(x) = 1\}$.

(2) *The KKL theorem.* In the remarkable paper [12], Kahn, Kalai and Linial proved that for any Boolean function $f : \{0,1\}^n \to \{0,1\}$, there exists a variable i whose influence is at least $ct(1-t)\log n/n$, where $t = \mathbb{E}[f]$ is the expectation of f, and c is a universal constant. Many applications of influences make use of the KKL theorem or of related results such as [8, 21] in one way or another.

(3) *The Russo lemma.* Let μ_p denote the Bernoulli measure where 0 is given weight $1-p$ and 1 is given weight p. Clearly if $A \subseteq \{0,1\}^n$ is monotone increasing [i.e., satisfies the condition that if $(x_1,\dots,x_n) \in A$ and $y_j \geq x_j$ for all j, then $y = (y_1,\dots,y_n) \in A$], then $\mu_p^{\otimes n}(A)$ is monotone increasing as function of p. The question of understanding how $\mu_p^{\otimes n}(A)$ varies with p has important applications in the theory of random graphs and in percolation theory. Russo's lemma [15, 18] asserts that the derivative of $\mu_p^{\otimes n}(A)$ with respect to p is the sum of influences of $f = 1_A$.

The basic results on influences were obtained for functions on the discrete cube, but some applications required generalization of the results to more general product spaces. Unlike the discrete case, where there exists a single natural definition of influence, for general product spaces several definitions were presented in different papers (see, e.g., [6, 11, 14, 16]). While each of these definitions has its advantages, in general all of them lack geometric interpretation for continuous probability spaces.

In this paper we present a new definition of the influences in product spaces of continuous random variables, that has a clear geometric meaning. We show that for the Gaussian measure and for a more general class of log-concave product measures called Boltzmann measures (see Definition 3.8), our definition allows us to obtain analogs of the KKL theorem and Russo-type formulas.

DEFINITION 1.2. Let ν_i be a probability measure on $\mathbb{R}$. Given a Borel-measurable set $A \subseteq \mathbb{R}$, its lower Minkowski content $\nu_i^+(A)$ is defined as

$$\nu_i^+(A) := \liminf_{r \downarrow 0} \frac{\nu_i(A + [-r,r]) - \nu_i(A)}{r}.$$

Consider the product measure $\nu = \nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n$ on $\mathbb{R}^n$. Then for any Borel-measurable set $A \subseteq \mathbb{R}^n$, for each $1 \leq i \leq n$ and an element $x = (x_1, x_2, \dots, x_n) \in$

$\mathbb{R}^n$, the restriction of A along the fiber of x in the ith direction is given by

$$A_i^x := \{y \in \mathbb{R} : (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \in A\}.$$

The *geometric influence* of the ith coordinate on A is

$$I_i^{\mathcal{G}}(A) := \mathbb{E}_x[\nu_i^+(A_i^x)],$$

that is, the expectation of $\nu_i^+(A_i^x)$ when x is chosen according to the measure ν. For sake of clarity, we sometimes denote the influence as $I_i^{\mathcal{G}}(A)|_\nu$.

The geometric meaning of the influence is that for a monotone (either increasing or decreasing) set A, the sum of influences of A is equal to the size of its boundary with respect to a uniform enlargement as shown in the following proposition.

PROPOSITION 1.3. *Let ν be a probability measure on $\mathbb{R}$ with C^1 density λ and cumulative distribution function Λ. Assume further that $\lambda(z) > 0$ for all $z \in \mathbb{R}$, that $\lim_{|z|\to\infty} \lambda(z) = 0$ and that λ' is bounded. Let $A \subset \mathbb{R}^n$ be a monotone set. Then*

$$\lim_{r \downarrow 0} \frac{\nu^{\otimes n}(A + [-r, r]^n) - \nu^{\otimes n}(A)}{r} = \sum_{i=1}^n I_i^{\mathcal{G}}(A).$$

Though the boundary under uniform enlargement is perhaps not as prevalent in the literature as the usual L^2 boundary (where we fatten a set by adding a L^2-ball of radius r instead of a L^∞-ball), the isoperimetric problem for the boundary under uniform enlargement, especially in the context of log-concave measures, was studied, for example, in [2–4]. Note that for the Gaussian measure on $\mathbb{R}^n$, unlike the usual L^2 boundary, the boundary under uniform enlargement of a set is not invariant under rotation.

We show that for the Gaussian measure on $\mathbb{R}^n$, the geometric influences satisfy the following analog of the KKL theorem:

THEOREM 1.4. *Consider the product spaces $\mathbb{R}^n$ endowed with the product Gaussian measure $\mu^{\otimes n}$. Then for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\mu^{\otimes n}(A) = t$ there exists $1 \le i \le n$ such that*

$$I_i^{\mathcal{G}}(A) \ge ct(1-t)\frac{\sqrt{\log n}}{n},$$

where $c > 0$ is a universal constant.

The result extends to the family of Boltzmann measures (see Definition 3.8), and is tight up to the constant factor. The proof uses the relation between geometric influences and the h-influences defined in [14], combined with isoperimetric estimates for the underlying probability measures.

Using the same methods, we obtain analogs of Talagrand's bound on the vector of influences [21] and of Friedgut's theorem stating that a function with a low sum of influences essentially depends on a few coordinates [8].

THEOREM 1.5. *Consider the product spaces* $\mathbb{R}^n$ *endowed with the product Gaussian measure* $\mu^{\otimes n}$. *For any Borel-measurable set* $A \subset \mathbb{R}^n$, *we have*:

(1) *if* $\mu^{\otimes n}(A) = t$, *then*

$$\sum_{i=1}^{n} \frac{I_i^{\mathcal{G}}(A)}{\sqrt{-\log I_i^{\mathcal{G}}(A)}} \geq c_1 t(1-t);$$

(2) *if* A *is monotone and* $\sum_{i=1}^{n} I_i^{\mathcal{G}}(A)\sqrt{-\log I_i^{\mathcal{G}}(A)} = s$, *then there exists a set* $B \subset \mathbb{R}^n$ *such that* 1_B *is determined by at most* $\exp(c_2 s/\epsilon)$ *coordinates and* $\mu^{\otimes n}(A \triangle B) \leq \epsilon$,

where c_1 *and* c_2 *are universal constants.*

We also show that the geometric influences can be used in Russo-type formulas for location families.

PROPOSITION 1.6. *Let* ν *be a probability measure on* $\mathbb{R}$ *with continuous density* λ *and cumulative distribution function* Λ. *Let* $\{\nu_\alpha : \alpha \in \mathbb{R}\}$ *denote a family of probability measures which is obtained by translating* ν; *that is*, ν_α *has a density* λ_α *satisfying* $\lambda_\alpha(x) = \lambda(x-\alpha)$.

Assume that λ *is bounded and satisfies* $\lambda(z) > 0$ *on* (κ_L, κ_R), *the interior of the support of* ν. *Let* A *be a monotone increasing subset of* $\mathbb{R}^n$. *Then the function* $\alpha \to {\nu_\alpha}^{\otimes n}(A)$ *is differentiable, and its derivative is given by*

$$\frac{d{\nu_\alpha}^{\otimes n}(A)}{d\alpha} = \sum_{i=1}^{n} I_i^{\mathcal{G}}(A),$$

where the influences are taken with respect to the measure $\nu_\alpha^{\otimes n}$.

Theorem 1.4 and Proposition 1.6 can be combined to get the following corollary which is the Gaussian analog of the sharp threshold result obtained by Friedgut and Kalai [9] for the product Bernoulli measure on the hypercube. We call a set transitive if its characteristic function is invariant under the action of some transitive subgroup of the permutation group S_n, where S_n acts by permutation of the n coordinates.

COROLLARY 1.7. *Let* μ_α *denote the Gaussian measure on the real line with mean* α *and variance* 1. *Let* $A \subset \mathbb{R}^n$ *be a monotone increasing transitive set. For any* $\delta > 0$, *denote by* $\alpha_A(\delta)$ *the unique value of* α *such that* $\mu_\alpha^{\otimes n}(A) = \delta$. *Then for any* $0 < \epsilon < 1/2$,

$$\alpha_A(1-\epsilon) - \alpha_A(\epsilon) \leq c \log(1/2\epsilon)/\sqrt{\log n},$$

where c *is a universal constant.*

We use the geometric influences to obtain an isoperimetric result for the Gaussian measure on $\mathbb{R}^n$.

THEOREM 1.8. *Consider the product spaces $\mathbb{R}^n$ endowed with the product Gaussian measure $\mu^{\otimes n}$. Then for any transitive Borel-measurable set $A \subset \mathbb{R}^n$ we have*

$$\liminf_{r \downarrow 0} \frac{\mu^{\otimes n}(A + [-r, r]^n) - \mu^{\otimes n}(A)}{r} \geq ct(1-t)\sqrt{\log n},$$

where $t = \mu^{\otimes n}(A)$ and $c > 0$ is a universal constant.

This result also extends to all Boltzmann measures.

Since the Gaussian measure is rotation invariant, it is natural to consider the influence sum of rotations of sets. Of particular interest are families of sets that are closed under rotations. In Section 5 we study the effect of *rotations* on the geometric influences, and show that under mild regularity condition of being in a certain class $\mathcal{J}_n$ (see Definition 5.1), the sum of geometric influences of a convex set can be increased up to $\Omega(\sqrt{n})$ by (a random) orthogonal rotation.

THEOREM 1.9. *Consider the product Gaussian measure $\mu^{\otimes n}$ on $\mathbb{R}^n$. For any convex set $A \in \mathcal{J}_n$ with $\mu^{\otimes n}(A) = t$, we have*

$$\mathbb{E}_{M\sim\pi}\left[\sum_{i=1}^{n} I_i^{\mathcal{G}}(M(A))\right] \geq ct(1-t)\sqrt{-\log\bigl(t(1-t)\bigr)} \times \sqrt{n},$$

where $\mathbb{E}_{M\sim\pi}$ denotes the expectation when M is drawn according to the Haar measure π over the orthogonal group of rotations, and c is a universal positive constant. In particular, there exists an orthogonal transformation g on $\mathbb{R}^n$ such that

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(g(A)) \geq ct(1-t)\sqrt{-\log\bigl(t(1-t)\bigr)} \times \sqrt{n}.$$

The results presented in this paper lead us to the questions regarding the extension in several directions:

- *Nonproduct measures.* The most challenging direction of extending our results is to consider nonproduct probability measures. The first problem in such generalization is that it is not clear at all what is the natural definition of influences for such measures. The second difficulty is that the techniques used in KKL-type results rely quite heavily on properties of product measures, and it is not clear how they can be extended to more general measures. We note that in a recent paper [10], Graham and Grimmett obtained a variant of the KKL theorem for measures satisfying certain FKG lattice conditions using reduction to the uniform measure on the continuous cube $[0, 1]^n$. Their results hold both for FKG measures on $\{0, 1\}^n$ and for FKG measures on $[0, 1]^n$ which are absolutely continuous with respect to the Lebesgue measure. However, it is not clear whether the definition of influences they consider in the continuous case has an "interesting" geometric interpretation.

- *Geometric meaning of nonmonotone sets.* The geometric meaning of our definition of the influences (i.e., the relation to the size of the boundary with respect to uniform enlargement) holds for all monotone sets (as shown in Proposition 1.3), and even for all convex sets, but not for general sets (see Remark 2.2). While restriction to monotone sets is quite standard in the study of influences (since monotonization arguments similar to Lemma 3.2 show that it is sufficient to prove all the lower bounds on influences in the case of monotone sets), it is interesting to find out whether our definition of influences has other geometric interpretation for general sets. On the other hand, it is interesting to determine exactly the families of sets to which Proposition 1.3 applies.
- *Other continuous measures.* The main results of our paper apply to the Gaussian measure in $\mathbb{R}^n$, and more generally, to the family of Boltzmann measures. Moreover, as shown in Section 3.4, the results extend to a broader class of measures whose isoperimetric function satisfies some specific condition. It seems interesting to extend the results to broader classes of measures, and on the other hand, to determine measures for which such results cannot be obtained.

The paper is organized as follows: in Section 2 we prove Propositions 1.3 and 1.6, thus establishing the geometric meaning of the new definition. In Section 3 we discuss the relation between the geometric influences and the h-influences, and prove Theorem 1.4. In Section 4 we apply Theorem 1.4 to establish a lower bound on the size of the boundary of transitive sets with respect to uniform enlargement, proving Theorem 1.8. Finally, in Section 5 we study the effect of *rotations* on the geometric influences. We conclude the introduction with a brief statistical application of the results established here.

1.1. *A statistical application.* Let $Z_1, Z_2, \dots, Z_n$ be i.i.d. $\mathrm{N}(\theta, 1)$. Suppose we want to test the hypothesis $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ $(\theta_1 > \theta_0)$ with level of significance at most β (for some $0 < \beta < 1/2$).

The remarkable classical result by Neyman and Pearson [17] says that the most powerful test for the above problem is based on the sample average $\bar{Z}_n = n^{-1} \sum_{i=1}^n Z_i$, and the critical region of the test is given by $\mathcal{C}_{\mathrm{mp}} = \{\bar{Z}_n > K\}$ where the constant K is chosen is such that $\mathbb{P}_{\theta_0}\{\mathcal{C}_{\mathrm{mp}}\} = \beta$. It can be easily checked that to achieve power at least $1 - \beta$ for this test, we need the parameters θ_0 and θ_1 to be separated by at least $|\theta_1 - \theta_0| > C(\beta)/\sqrt{n}$ for some appropriate constant $C(\beta)$.

Consider the following setup where the test statistics is given by $f(Z_1, \dots, Z_n)$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function which is nondegenerate, transitive and monotone increasing in each of its coordinates. The transitivity of f ensures equal weight is given to each data point while constructing the test and the monotonicity of f implies that the distribution of f depends on θ in a monotone fashion. Note that we do not assume any smoothness property of f. In general the test statistics $f(Z_1, \dots, Z_n)$, in contrast to the sample average which is a sufficient statistics for this problem, may be a result of an "inefficient compression" of the data, and we have only access to the compressed data.

In this case the critical region would be of the form $\mathcal{C} = \{f(Z_1, \dots, Z_n) > K\}$ where K is chosen so that $\mathbb{P}_{\theta_0}\{\mathcal{C}\} = \beta$.

Note that regions $\mathcal{C}$ satisfy:

(i) $\mathbb{P}_{\theta_0}\{\mathcal{C}\} = \beta$;
(ii) $\mathcal{C}$ is transitive;
(iii) $\mathcal{C}$ is an increasing set.

Clearly, the most powerful test belongs to this class, but, in general, a test of the above type can be of much less power. An interesting open question will be to find the worst test (i.e., having lowest power) among all tests satisfying (i), (ii) and (iii). Intuitively if θ_1 and θ_0 are far apart, even a very weak test can detect the difference between the null and the alternative. Corollary 1.7 gives us a quantitative estimate of how far apart the parameters need to be so that we can safely distinguish them no matter what test we use. Indeed any test satisfying (i), (ii) and (iii) still has power of at least $1 - \beta$ as long as $|\theta_1 - \theta_0| > c \log(1/2\beta)/\sqrt{\log n}$ for some absolute constant c.

For the test $\{\max_i Z_i > K\}$, the dependence on n in the above bound is tight up to constant factors.

We briefly note that the statistical reasoning introduced here may be combined with Theorem 2.1 in [9]. Thus a similar statement holds when $Z_1, Z_2, \dots, Z_n$ are i.i.d. Bernoulli(p), and we want to test the hypothesis $H_0 : p = p_0$ vs. $H_1 : p = p_1$ $(1 > p_1 > p_0 > 0)$. In this case, the power of any test satisfying (i), (ii) and (iii) is at least $1 - \beta$ as long as $|p_1 - p_0| > c \log(1/2\beta)/\log n$ for some absolute constant c.

2. Boundary under uniform enlargement and derivatives.

In this section we provide the geometric interpretation of the influence. We begin by proving Proposition 1.3.

2.1. *Proof of Proposition* 1.3.

In our proof we use the following simple lemma:

LEMMA 2.1. *Let λ be as given in Proposition* 1.3. *Given $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for all $x, y \in \mathbb{R}$,*

$$|\lambda(y) - \lambda(x)| \le C_\varepsilon |\Lambda(y) - \Lambda(x)| + \varepsilon/4.$$

PROOF. Since $\lim_{|z|\to\infty}\lambda(z)=0$, there exists $z_0>0$ such that

$$\sup_{|z|\ge z_0}\lambda(z)\le\varepsilon/8.$$

Fix $x,y\in\mathbb{R}$, and assume without loss of generality that $x\le y$. We have

$$\begin{aligned}|\lambda(y)-\lambda(x)|\le{}&\left|\int_x^{\max(x,-z_0)}\lambda'(z)\,dz\right|+\left|\int_{\max(x,-z_0)}^{\min(z_0,y)}\lambda'(z)\,dz\right|\\&+\left|\int_{\min(z_0,y)}^{y}\lambda'(z)\,dz\right|.\end{aligned}\tag{2.1}$$

By the choice of z_0,

$$\left|\int_{\min(z_0,y)}^{y}\lambda'(z)\,dz\right|\le|\lambda(y)-\lambda(z_0)|\le\varepsilon/8,\tag{2.2}$$

and similarly,

$$\left|\int_x^{\max(x,-z_0)}\lambda'(z)\,dz\right|\le\varepsilon/8.\tag{2.3}$$

On the other hand, since the function λ'/λ is continuous, there exists C_ε such that $|\lambda'(z)|/\lambda(z)\le C_\varepsilon$ for all $|z|\le z_0$. Hence,

$$\begin{aligned}\left|\int_{\max(x,-z_0)}^{\min(z_0,y)}\lambda'(z)\,dz\right|&\le C_\varepsilon\int_{\max(x,-z_0)}^{\min(z_0,y)}\lambda(z)\,dz\\&=C_\varepsilon\bigl(\Lambda(\min(z_0,y))-\Lambda\bigl(\max(x,-z_0)\bigr)\bigr)\\&\le C_\varepsilon\bigl(\Lambda(y)-\Lambda(x)\bigr)=C_\varepsilon|\Lambda(y)-\Lambda(x)|.\end{aligned}\tag{2.4}$$

Substitution of (2.2), (2.3) and (2.4) into (2.1) yields the assertion. $\square$

Now we are ready to present the proof of Proposition 1.3.

PROOF OF PROPOSITION 1.3. Without loss of generality, assume that A is decreasing. Thus, $\nu^{\otimes n}(A+[-r,r]^n)=\nu^{\otimes n}(A+[0,r]^n)$. We decompose $\nu^{\otimes n}(A+[0,r]^n)-\nu^{\otimes n}(A)$ as a telescoping sum

$$\sum_{i=1}^{n}\nu^{\otimes n}(A+[0,r]^i\times\{0\}^{n-i})-\nu^{\otimes n}(A+[0,r]^{i-1}\times\{0\}^{n-i+1}).\tag{2.5}$$

It follows immediately from (2.5) that it is sufficient to show that given $\varepsilon>0$, there exists $\delta>0$ such that for all $1\le i\le n$ and for all $0<r<\delta$,

$$\begin{aligned}&\left|\frac{\nu^{\otimes n}(A+[0,r]^{i-1}\times[0,r]\times\{0\}^{n-i})-\nu^{\otimes n}(A+[0,r]^{i-1}\times\{0\}^{n-i+1})}{r}\right.\\&\qquad\left.-I_i^{\mathcal{G}}(A)\right|\\&\quad\le\varepsilon.\end{aligned}\tag{2.6}$$

For a fixed i, define

$$B_r^i = A + [0,r]^{i-1} \times \{0\}^{n-i+1}.$$

Obviously, B_r^i is a decreasing set. Note that $A + [0,r]^{i-1} \times [0,r] \times \{0\}^{n-i} = B_r^i + \{0\}^{i-1} \times [0,r] \times \{0\}^{n-i}$. Hence, equation (2.6) can be rewritten as

$$\left| \frac{\nu^{\otimes n}(B_r^i + \{0\}^{i-1} \times [0,r] \times \{0\}^{n-i}) - \nu^{\otimes n}(B_r^i)}{r} - I_i^{\mathcal{G}}(A) \right| \le \varepsilon. \tag{2.7}$$

For any decreasing set $D \subset \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, define

$$t_i(D; x) := \sup\{y : y \in D_i^x\} \in [-\infty, \infty]$$

with the convention that the supremum of the empty set is $-\infty$. We use two simple observations:

(1) For any decreasing set D (and in particular, for A and for B_r^i), it is clear that $\nu^{\otimes n}(D) = \mathbb{E}_x \Lambda(t_i(D; x))$.

(2) For a decreasing set D, we have $I_i^{\mathcal{G}}(D) = \mathbb{E}_x \lambda(t_i(D; x))$. This follows from a known property of the lower Minkowski content: in the case when ν has a continuous density λ, and L is a semi-infinite ray, that is, $L = [\ell, \infty)$ or $L = (-\infty, \ell]$, we have $\nu^+(L) = \lambda(\ell)$.

We further observe that

$$\begin{aligned} &\left| \frac{\nu^{\otimes n}(B_r^i + \{0\}^{i-1} \times [0,r] \times \{0\}^{n-i}) - \nu^{\otimes n}(B_r^i)}{r} - \mathbb{E}_x \lambda(t_i(B_r^i; x)) \right| \\ &\quad \le r \|\lambda'\|_\infty. \end{aligned} \tag{2.8}$$

Indeed, by observation (1), the left-hand side of (2.8) is equal to

$$\left| \mathbb{E}_x \left[\frac{\Lambda(t_i(B_r^i; x) + r) - \Lambda(t_i(B_r^i; x))}{r} - \lambda(t_i(B_r^i; x)) \right] \right|. \tag{2.9}$$

By the mean value theorem, there exists $h \in [0, r]$ such that

$$\frac{\Lambda(t_i(B_r^i; x) + r) - \Lambda(t_i(B_r^i; x))}{r} = \lambda\big(t_i(B_r^i; x) + h\big),$$

and thus

$$(2.9) = \big|\mathbb{E}_x\big[\lambda\big(t_i(B_r^i; x) + h\big) - \lambda(t_i(B_r^i; x))\big]\big| \le r \|\lambda'\|_\infty.$$

Combining (2.7) and (2.8), and ensuring that $r < \varepsilon/(2\|\lambda'\|_\infty)$, it is sufficient to show that

$$|\mathbb{E}_x \lambda(t_i(B_r^i; x)) - I_i^{\mathcal{G}}(A)| \le \varepsilon/2,$$

and by observation (2), this is equivalent to

$$|\mathbb{E}_x \lambda(t_i(B_r^i; x)) - \mathbb{E}_x \lambda(t_i(A; x))| \le \varepsilon/2. \tag{2.10}$$

By Lemma 2.1 and observation (1), we have

$$\begin{aligned}|\mathbb{E}_x\lambda(t_i(B_r^i;x)) - \mathbb{E}_x\lambda(t_i(A;x))| &\le C_\varepsilon\mathbb{E}_x|\Lambda(t_i(B_r^i;x)) - \Lambda(t_i(A;x))| + \varepsilon/4\\ &= C_\varepsilon\mathbb{E}_x\big(\Lambda(t_i(B_r^i;x)) - \Lambda(t_i(A;x))\big) + \varepsilon/4\\ &= C_\varepsilon\big(\nu^{\otimes n}(B_r^i) - \nu^{\otimes n}(A)\big) + \varepsilon/4.\end{aligned}$$

It thus remains to show that there exists $\delta > 0$ sufficiently small such that for all $0 < r < \delta$,

$$\nu^{\otimes n}(B_r^i) - \nu^{\otimes n}(A) \le \frac{\varepsilon}{4C_\varepsilon}. \tag{2.11}$$

We can write

$$\begin{aligned}&\nu^{\otimes n}(B_r^i) - \nu^{\otimes n}(A)\\ &\quad = \sum_{j=1}^{i-1}\big(\nu^{\otimes n}(A + [0,r]^j \times \{0\}^{n-j}) - \nu^{\otimes n}(A + [0,r]^{j-1} \times \{0\}^{n-j+1})\big),\end{aligned}$$

and thus it is sufficient to find $\delta > 0$ such that for all $0 < r < \delta$ and for all $1 \le j \le i-1$,

$$\nu^{\otimes n}(A + [0,r]^j \times \{0\}^{n-j}) - \nu^{\otimes n}(A + [0,r]^{j-1} \times \{0\}^{n-j+1}) \le \frac{\varepsilon}{4nC_\varepsilon}.$$

Since for any decreasing $D \subset \mathbb{R}^n$,

$$|\nu^{\otimes n}(D + \{0\}^{j-1} \times [0,r] \times \{0\}^{n-j}) - \nu^{\otimes n}(D)| \le \|\lambda\|_\infty r,$$

we can choose $\delta = \min\{\frac{\varepsilon}{4nC_\varepsilon\|\lambda\|_\infty}, \frac{\varepsilon}{2\|\lambda'\|_\infty}\}$. This completes the proof. □

REMARK 2.2. We note that the same proof (with minor modifications) holds for any convex set A. The only nonobvious change is noting that the Minkowski content of a segment $[a,b]$ is $\nu^+([a,b]) = \lambda(a) + \lambda(b)$, where λ is the density of the measure ν. On the other hand, it is clear that the statement of Proposition 1.3 does not hold for general measurable sets. For example, if $A = \mathbb{Q}^n$ where $\mathbb{Q}$ is the set of rational numbers, then the size of the boundary of A with respect to a uniform enlargement is ∞, while the sum of geometric influences of A is zero. It seems an interesting question to determine to which classes of measurable sets Proposition 1.3 applies.

2.2. *Proof of Proposition* 1.6. Define a function $\Pi : \mathbb{R}^n \to [0,\infty)$ by

$$\Pi(\alpha_1,\ldots,\alpha_n) = \nu_{\alpha_1} \otimes \cdots \otimes \nu_{\alpha_n}(A).$$

The partial derivative of Π with respect to the ith coordinate can be written as

$$\frac{\partial\Pi(\alpha_1,\ldots,\alpha_n)}{\partial\alpha_i} = \lim_{r\downarrow 0}\frac{\mathbb{E}_x\nu_{\alpha_i+r}(A_i^x) - \mathbb{E}_x\nu_{\alpha_i}(A_i^x)}{r}. \tag{2.12}$$

For $x \in \mathbb{R}^n$, define

$$s_i(A; x) := \inf\{y : y \in A_i^x\} \in [-\infty, \infty].$$

Since A is monotone increasing, for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\frac{\nu_{\alpha_i+r}(A_i^x) - \nu_{\alpha_i}(A_i^x)}{r} &= \frac{\nu_{\alpha_i+r}([s_i(A;x),\infty)) - \nu_{\alpha_i}([s_i(A;x),\infty))}{r} \\ &= \frac{1}{r}\int_{s_i(A;x)-r}^{s_i(A;x)} \lambda_{\alpha_i}(z)\,dz,\end{aligned} \tag{2.13}$$

and by the fundamental theorem of calculus, this expression converges to $\lambda_{\alpha_i}(s_i(A;x))$ as $r \to 0$. Moreover, (2.13) is uniformly bounded by $\|\lambda_{\alpha_i}\|_\infty = \|\lambda\|_\infty$ (which is finite since λ is bounded by the hypothesis). Therefore, by the dominated convergence theorem, it follows that the first-order partial derivatives of Π exist and are given by

$$\frac{\partial \Pi(\alpha_1, \ldots, \alpha_n)}{\partial \alpha_i} = \mathbb{E}_{x \sim \nu_{\alpha_1} \otimes \cdots \otimes \nu_{\alpha_n}} \lambda_{\alpha_i}(s_i(A;x)) = I_i^{\mathcal{G}}(A),$$

where the influence is with respect to the measure $\nu_{\alpha_1} \otimes \cdots \otimes \nu_{\alpha_n}$. [For the last equality, see observation (2) in the proof of Proposition 1.3 above. Here we use the convention that $\lambda_{\alpha_i}(-\infty) = \lambda_{\alpha_i}(\infty) = 0$.]

Hence, by the chain rule, it is sufficient to check that all the partial derivatives of Π are continuous at $(\alpha, \ldots, \alpha)$. Without loss of generality, we assume that $\alpha = 0$. Note that

$$\mathbb{E}_{x \sim \nu_{\alpha_1} \otimes \cdots \otimes \nu_{\alpha_n}} \lambda_{\alpha_i}(s_i(A;x)) = \mathbb{E}_{x \sim \nu \otimes \cdots \otimes \nu} \left(\prod_{j=1}^n \frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} \right) \lambda_{\alpha_i}(s_i(A;x)). \tag{2.14}$$

For each $x \in \mathbb{R}^n$,

$$\prod_{j=1}^n \frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} \lambda_{\alpha_i}(s_i(A;x)) \to \prod_{j=1}^n \lambda(s_i(A;x)) \tag{2.15}$$

as $\max|\alpha_i| \to 0$. Hence, the continuity of the partial derivatives would follow from the dominated convergence theorem if (2.15) was uniformly bounded. In order to obtain such bound, we consider a compact subset.

There exist $\kappa_L < K_L < K_R < \kappa_R$ and $\delta > 0$ such that $\nu([K_L + \delta, K_R - \delta]) \geq 1 - \varepsilon$. Let $c := \min_{z \in [K_L, K_R]} \lambda(z)$. Note that by the hypothesis on λ, we have $c > 0$. If $|\alpha_j| \leq \delta$ for all j, then

$$\begin{aligned}&\left| (2.14) - \mathbb{E}_{x \sim \nu \otimes \cdots \otimes \nu} \left(\prod_{j=1}^n \frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} 1_{\{K_L \leq x_j \leq K_R\}} \right) \lambda_{\alpha_i}(s_i(A;x)) \right| \\ &\quad \leq \varepsilon \cdot n \cdot \|\lambda\|_\infty.\end{aligned} \tag{2.16}$$

Indeed, denoting $S = \{x \in \mathbb{R}^n : \exists j, x_j \notin [K_L, K_R]\}$ and using (2.14), we have

$$(2.16) = |\mathbb{E}_{x\sim\nu_{\alpha_1}\otimes\cdots\otimes\nu_{\alpha_n}} 1_S \lambda_{\alpha_i}(s_i(A;x))| \le \|\lambda\|_\infty \mathbb{E}_{x\sim\nu_{\alpha_1}\otimes\cdots\otimes\nu_{\alpha_n}} 1_S \le \varepsilon \cdot n \cdot \|\lambda\|_\infty,$$

where the last inequality is a union bound using the choice of K_L and K_R.

Similarly, by a union bound we have

$$\begin{aligned} &\left|\mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu}\lambda(s_i(A;x)) - \mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu}1_{\{K_L\le x_j\le K_R\ \forall j\}}\lambda(s_i(A;x))\right| \\ &\quad \le \varepsilon\cdot n\cdot\|\lambda\|_\infty. \end{aligned} \tag{2.17}$$

Combining (2.16) with (2.17), it is sufficient to prove that

$$\begin{aligned} &\mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu}\prod_{j=1}^n \frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} 1_{\{K_L\le x_j\le K_R\}}\lambda_{\alpha_i}(s_i(A;x)) \\ &\qquad \to \mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu}\prod_{j=1}^n 1_{\{K_L\le x_j\le K_R\}}\lambda(s_i(A;x)). \end{aligned}$$

This indeed follows from the dominated convergence theorem, since for each $x \in \mathbb{R}^n$,

$$\prod_{j=1}^n \frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} 1_{\{K_L\le x_j\le K_R\}}\lambda_{\alpha_i}(s_i(A;x)) \to \prod_{j=1}^n 1_{\{K_L\le x_j\le K_R\}}\lambda(s_i(A;x))$$

as $\max|\alpha_i| \to 0$ and is uniformly bounded by $c^{-n}\|\lambda\|_\infty^{n+1}$. This completes the proof.

3. Relation to h-influences and a general lower bound on geometric influences.

In this section we analyze the geometric influences by reduction to problems concerning h-influences introduced in a recent paper by the first author [14]. First we describe and extend the results on h-influences, and then we show their relation to geometric influences.

3.1. *h-influences.*

DEFINITION 3.1. Let $h : [0, 1] \to [0, \infty)$ be a measurable function. For a measurable subset A of X^n equipped with a product measure $\nu^{\otimes n}$, the h-influence of the ith coordinate on A is

$$I_i^h(A) := \mathbb{E}_x[h(\nu(A_i^x))].$$

The two main results concerning h-influences are a monotonization lemma and an analog of the KKL theorem.

LEMMA 3.2 ([14]). *Consider the space* $[0, 1]^n$, *endowed with the product Lebesgue measure* $u^{\otimes n}$. *Let* $h : [0, 1] \to [0, 1]$ *be a concave continuous function. For every Borel measurable set* $A \subseteq [0, 1]^n$, *there exists a monotone increasing set* $B \subseteq [0, 1]^n$ *such that*:

(1) $u^{\otimes n}(A) = u^{\otimes n}(B)$;
(2) *for all* $1 \le i \le n$, *we have* $I_i^h(A) \ge I_i^h(B)$.

THEOREM 3.3 ([14]). *Denote the entropy function as* $\mathrm{Ent}(x) := -x \log x - (1 - x)\log(1 - x)$ *for all* $0 < x < 1$, *and* $\mathrm{Ent}(0) = \mathrm{Ent}(1) = 0$. *Consider the space* $[0, 1]^n$, *endowed with the product Lebesgue measure* $u^{\otimes n}$. *Let* $h : [0, 1] \to [0, 1]$ *such that* $h(x) \ge \mathrm{Ent}(x)$ *for all* $0 \le x \le 1$. *Then for every measurable set* $A \subseteq [0, 1]^n$ *with* $u^{\otimes n}(A) = t$, *there exists* $1 \le i \le n$ *such that the* h*-influence of the* i*th coordinate on* A *satisfies*

$$I_i^h(A) \ge ct(1 - t)\log n/n,$$

where $c > 0$ *is a universal constant.*

Other results on h-influences which we shall use later include analogs of several theorems concerning influences on the discrete cube: Talagrand's lower bound on the vector of influences [21], a variant of the KKL theorem for functions with low influences [9] and Friedgut's theorem asserting that a function with a low influence sum essentially depends on a few coordinates [8].

In the application to geometric influences we would like to use h-influences for certain functions h that do not dominate the entropy function. In order to overcome this problem, we use the following lemma that allows to relate general h-influences to the entropy-influence [i.e., h-influence for $h(x) = \mathrm{Ent}(x)$].

LEMMA 3.4. *Consider the product space* $(\mathbb{R}^n, \nu^{\otimes n})$, *where* ν *has a continuous cumulative distribution function* Λ. *Let* $h : [0, 1] \to [0, \infty)$, *and let* $A \subseteq \mathbb{R}^n$ *be a Borel-measurable set. For all* $1 \le i \le n$,

$$I_i^h(A) \ge \tfrac{1}{2}\delta \cdot I_i^{\mathrm{Ent}}(A), \tag{3.1}$$

where

$$\delta = \delta(A, i) = \inf_{x \in [\vartheta(I_i^{\mathrm{Ent}}(A)/2), 1-\vartheta(I_i^{\mathrm{Ent}}(A)/2)]} \frac{h(x)}{\mathrm{Ent}(x)} \tag{3.2}$$

and $\vartheta(y) = y/(-2\log y)$.

PROOF. Set $f = 1_A$. Let u be the Lebesgue measure on $[0, 1]$. Define $g(x_1, \dots, x_n) := f(\Lambda^{-1}(x_1), \dots, \Lambda^{-1}(x_n))$ and write B for the set $\{x \in \mathbb{R}^n : g(x) = 1\}$. Since $\Lambda^{-1}(u) \stackrel{d}{=} \nu$, the set B satisfies $u^{\otimes n}(B) = \nu^{\otimes n}(A) = t$ and

$$I_i^h(B)|_{u^{\otimes n}} = I_i^h(A)|_{\nu^{\otimes n}} \qquad \text{for each } 1 \le i \le n.$$

Denote by α the unique value in the segment $[0, 1/2]$ which satisfies the equation $\alpha = \mathrm{Ent}^{-1}(I_i^{\mathrm{Ent}}(A)/2)$. It is clear that for any $x \notin [\alpha, 1-\alpha]$,

$$\mathrm{Ent}(x) \leq \mathrm{Ent}\bigl(\mathrm{Ent}^{-1}\bigl(I_i^{\mathrm{Ent}}(A)/2\bigr)\bigr) = I_i^{\mathrm{Ent}}(A)/2,$$

and thus,

$$\begin{aligned}\mathbb{E}_x\bigl[\mathrm{Ent}(u(B_i^x))1_{\{u(B_i^x)\in[\alpha,1-\alpha]\}}\bigr] &= I_i^{\mathrm{Ent}}(B)|_{u^{\otimes n}} - \mathbb{E}_x\bigl[\mathrm{Ent}(u(B_i^x))1_{\{u(B_i^x)\notin[\alpha,1-\alpha]\}}\bigr] \\ &\geq I_i^{\mathrm{Ent}}(A)/2.\end{aligned}$$

Therefore, by (3.2),

$$\begin{aligned} I_i^h(A)|_{\nu^{\otimes n}} &= I_i^h(B)|_{u^{\otimes n}} \geq \mathbb{E}_x\bigl[h(u(B_i^x))1_{\{u(B_i^x)\in[\alpha,1-\alpha]\}}\bigr] \\ &\geq \Bigl(\inf_{x\in[\mathrm{Ent}^{-1}(I_i^{\mathrm{Ent}}(A)/2),1-\mathrm{Ent}^{-1}(I_i^{\mathrm{Ent}}(A)/2)]}\frac{h(x)}{\mathrm{Ent}(x)}\Bigr) I_i^{\mathrm{Ent}}(A)/2 \\ &\geq \delta \cdot I_i^{\mathrm{Ent}}(A)/2, \end{aligned}$$

where the last step follows from the fact that $\vartheta(x) \leq \mathrm{Ent}^{-1}(x)$ for $x \leq 1/2$ which is easy to verify. $\square$

3.2. *Relation between geometric influences and h-influences for log-concave measures*. It is straightforward to check the following relation between the geometric influences and the h-influences for monotone sets. The proof follows immediately from observation (2) in the proof of Proposition 1.3.

LEMMA 3.5. *Consider the product space $(\mathbb{R}^n, \nu^{\otimes n})$ where ν has a continuous density λ. Let Λ denote the cumulative distribution function of ν. Then for any monotone set $A \subseteq \mathbb{R}^n$,*

$$I_i^{\mathcal{G}}(A) = I_i^h(A) \qquad \forall 1 \leq i \leq n,$$

where $h(t) = \lambda(\Lambda^{-1}(t))$ when A is decreasing and $h(t) = \lambda(\Lambda^{-1}(1-t))$ when A is increasing. Here Λ^{-1} denotes the unique inverse of the function Λ.

Using Lemmas 3.2 and 3.5, we can obtain a monotonization lemma for geometric influences that holds if the underlying measure has a log-concave density. In order to show this, we use the following isoperimetric inequality satisfied by log-concave distributions (see, e.g., [3]).

THEOREM 3.6 ([3]). *Let ν have a log-concave density λ, and let Λ be the corresponding cumulative distribution function. Denote the* (*unique*) *inverse of the function Λ by Λ^{-1}. Fix any $t \in (0, 1)$; that is, for $t \in (0, 1)$ and for every Borel-measurable set $A \subseteq R$ with $\nu(A) = t$,*

$$\nu(A + [-r, r]) \geq \min\bigl\{\Lambda\bigl(\Lambda^{-1}(t) + r\bigr), 1 - \Lambda\bigl(\Lambda^{-1}(1-t) - r\bigr)\bigr\} \qquad \forall r > 0. \tag{3.3}$$

Moreover, in the class of all Borel-measurable sets of ν*-measure* t, *the extremal sets, that is, the sets for which* (3.3) *holds as an equality, are intervals of the form* $(-\infty, a]$ *or* $[a, \infty)$ *for some* $a \in \mathbb{R}$.

If λ *is symmetric* (*around the median*), *then inequality* (3.3) *is simplified to*

$$\nu(A + [-r, r]) \geq \Lambda\big(\Lambda^{-1}(t) + r\big) \qquad \forall r > 0. \tag{3.4}$$

Now we are ready to present the monotonization lemma.

LEMMA 3.7. *Consider the product measure* $\nu^{\otimes n}$ *on* $\mathbb{R}^n$ *where* ν *is a probability distribution with a continuous symmetric log-concave density* λ *satisfying* $\lim_{|z|\to\infty} \lambda(z) = 0$. *Then for any Borel set* $A \subset \mathbb{R}^n$:

(i) $I_i^{\mathcal{G}}(A) \geq I_i^h(A)$ *for all* $1 \leq i \leq n$, *where* $h(t) = \lambda(\Lambda^{-1}(t))$;
(ii) *there exists an increasing set* B *such that* $\nu^{\otimes n}(B) = \nu^{\otimes n}(A)$ *and*

$$I_i^{\mathcal{G}}(B) \leq I_i^{\mathcal{G}}(A) \qquad \textit{for all } 1 \leq i \leq n.$$

PROOF. Let Λ be the cumulative distribution of ν. Fix $x \in \mathbb{R}^n$. By Theorem 3.6, we have, for all $r > 0$,

$$\frac{\nu(A_i^x + [-r, r]) - \nu(A_i^x)}{r} \geq \frac{\Lambda(\Lambda^{-1}(\nu(A_i^x)) + r) - \Lambda(\Lambda^{-1}(\nu(A_i^x)))}{r}.$$

Taking limit of the both sides as $r \downarrow 0$, we obtain

$$\nu^+(A_i^x) \geq \lambda(\Lambda^{-1}(\nu(A_i^x))) = h(\nu(A_i^x)),$$

which implies the first part of the lemma.

For a proof of the second part, we start by noting that the assumptions on ν imply that h is concave and continuous. Thus we can invoke Lemma 3.2 to find an increasing set B such that $\nu^{\otimes n}(B) = \nu^{\otimes n}(A)$ and $I_i^h(B) \leq I_i^h(A)$ for all $1 \leq i \leq n$. By the first part of the lemma, $I_i^h(A) \leq I_i^{\mathcal{G}}(A)$ for all $1 \leq i \leq n$. On the other hand, it follows from Lemma 3.5 that $I_i^{\mathcal{G}}(B) = I_i^h(B)$ for all $1 \leq i \leq n$. Hence,

$$I_i^{\mathcal{G}}(B) = I_i^h(B) \leq I_i^h(A) \leq I_i^{\mathcal{G}}(A)$$

as asserted. $\square$

To keep our exposition simple, we will restrict our attention to an important family of log-concave distributions known as Boltzmann measures for the rest of the section. We mention in passing that some of the techniques that we are going to develop can be applied to other log-concave measures with suitable isoperimetric properties.

3.3. *Lower bounds on geometric influences for Boltzmann measures.*

DEFINITION 3.8 (Boltzmann measure). The density of the Boltzmann measure μ_ρ with parameter $\rho \geq 1$ is given by

$$\phi_\rho(x) := \frac{1}{2\Gamma(1+1/\rho)} e^{-|x|^\rho}\,dx, \qquad x \in \mathbb{R}.$$

Note that $\rho = 2$ corresponds to the Gaussian measure with variance $1/2$ while $\rho = 1$ gives the two-sided exponential measure.

We have the following estimates on the tail probability of Boltzmann measures.

LEMMA 3.9. *Let Φ_ρ denote the cumulative distribution function of the Boltzmann distribution with parameter ρ. Then for $z > 0$, we have*

$$\frac{1}{2\rho\Gamma(1+1/\rho)}\left(\frac{z}{(\rho-1)/\rho + z^\rho}\right)e^{-z^\rho} \leq 1 - \Phi_\rho(z) \leq \frac{1}{2\rho\Gamma(1+1/\rho)}\frac{1}{z^{\rho-1}}e^{-z^\rho}.$$

In particular,

$$\phi_\rho(\Phi_\rho^{-1}(x)) \asymp x(1-x)\bigl(-\log\bigl(x(1-x)\bigr)\bigr)^{(\rho-1)/\rho} \tag{3.5}$$

for x close to zero or one.

PROOF. Set $Z_\rho = 2\Gamma(1+1/\rho)$. For the upper bound, note that

$$Z_\rho\bigl(1-\Phi_\rho(z)\bigr) = \int_z^\infty e^{-t^\rho}\,dt \leq \frac{1}{\rho z^{\rho-1}}\int_z^\infty \rho t^{\rho-1}e^{-t^\rho}\,dt \leq \frac{1}{\rho z^{\rho-1}}e^{-z^\rho}.$$

On the other hand, the lower bound is derived as follows:

$$\begin{aligned} Z_\rho\left(1+\frac{\rho-1}{\rho z^\rho}\right)\bigl(1-\Phi_\rho(z)\bigr) &\geq \int_z^\infty \left(1+\frac{\rho-1}{\rho t^\rho}\right)e^{-t^\rho}\,dt \\ &= -\frac{e^{-t^\rho}}{\rho t^{\rho-1}}\bigg|_z^\infty = \frac{e^{-z^\rho}}{\rho z^{\rho-1}}. \end{aligned}$$ □

It follows from Lemma 3.7(i) and Lemma 3.9 that for Boltzmann measures, the geometric influences lie between previously studied h-influences. On the one hand, they are greater than variance-influences [i.e., h-influences with $h(t) = t(1-t)$], that were studied in, for example, [11, 16]. On the other hand, for monotone sets they are smaller than the entropy-influences.

It is well known that there is no analog of the KKL influence bound for the variance-influence, and a tight lower bound on the maximal variance-influence is the trivial bound

$$\max_{1\leq i\leq n} I_i^{\mathrm{Var}}(A) \geq ct(1-t)/n,$$

where t is the measure of the set A. This inequality is an immediate corollary of the Efron–Stein inequality (see, e.g., [19]), and the tightness is shown by the standard example of one-sided boxes (considered in Section 4.1 below). On the other hand, the analog of the KKL bound proved in [14] holds only for h-influences with $h(t) \geq \mathrm{Ent}(t)$. In order to show KKL-type lower bounds for geometric influences, we use the following two results.

The first result is a dimension-free isoperimetric inequality for the Boltzmann measures.

LEMMA 3.10 ([2]). *Fix $\rho > 1$, and let μ_ρ denote the Boltzmann measure with parameter ρ. Then there exists a constant $k = k(\rho) > 0$ such that for any $n \geq 1$ and any measurable $A \in \mathbb{R}^n$, we have*

$$\mu_\rho^{\otimes n}(A + [-r, r]^n) \geq \mu_\rho\{(-\infty, \Phi_\rho^{-1}(t) + kr]\}, \qquad t = \mu_\rho^{\otimes n}(A).$$

The second key ingredient is a simple corollary of Lemma 3.4.

LEMMA 3.11. *Consider the product spaces $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, where μ_ρ denotes the Boltzmann measure with parameter $\rho > 1$. For any $A \subset \mathbb{R}^n$ and for all $1 \leq i \leq n$,*

$$I_i^{\mathcal{G}}(A) \geq c I_i^{\mathrm{Ent}}(A)(-\log(I_i^{\mathrm{Ent}}(A)))^{-1/\rho},$$

where $c = c(\rho) > 0$ is a universal constant.

PROOF. In view of Lemma 3.7, it is sufficient to prove that

$$I_i^h(A) \geq c I_i^{\mathrm{Ent}}(A)(-\log(I_i^{\mathrm{Ent}}(A)))^{-1/\rho}$$

for $h(x) := \phi_\rho(\Phi_\rho^{-1}(x))$. This indeed follows immediately from Lemma 3.4 using the estimate on $h(x)$ given in (3.5). □

Now we are ready to prove the KKL-type lower bounds. We start with an analog of the KKL theorem [12].

THEOREM 3.12. *Consider the product spaces $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, where μ_ρ denotes the Boltzmann measure with parameter $\rho > 1$. There exists a constant $c = c(\rho) > 0$ such that for all $n \geq 1$ and for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\nu^{\otimes n}(A) = t$, we have*

$$\max_{1 \leq i \leq n} I_i^{\mathcal{G}}(A) \geq ct(1-t)\frac{(\log n)^{1-1/\rho}}{n}.$$

PROOF. The proof is divided into two cases, according to $\nu^{\otimes n}(A) = t$. If $t(1-t)$ is not very small, the proof uses Lemmas 3.7 and 3.11. If $t(1-t)$ is very small, the proof relies on Lemmas 3.7 and 3.10.

Case A: $t(1-t) > n^{-1}$. By Theorem 3.3, there exists $1 \le i \le n$, such that

$$I_i^{\mathrm{Ent}}(A) \ge ct(1-t)\frac{\log n}{n}.$$

Since $t(1-t) > 1/n$, it follows from Lemma 3.11 that

$$I_i^{\mathcal{G}}(A) \ge c I_i^{\mathrm{Ent}}(A)(-\log(I_i^{\mathrm{Ent}}(A)))^{-1/\rho} \ge c' t(1-t)\frac{\log n}{n} \cdot (\log n)^{-1/\rho},$$

where c' is a universal constant, as asserted.

Case B: $t(1-t) \le n^{-1}$. In view of Lemma 3.7, we can assume without loss of generality that the set A is increasing. In that case, by Proposition 1.3, we have

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(A) = \liminf_{r\downarrow 0} \frac{\mu_\rho^{\otimes n}(A + [-r, r]^n) - \mu_\rho^{\otimes n}(A)}{r}.$$

By Lemma 3.10,

$$\liminf_{r\downarrow 0} \frac{\mu_\rho^{\otimes n}(A + [-r, r]^n) - \mu_\rho^{\otimes n}(A)}{r} \ge k\phi_\rho(\Phi_\rho^{-1}(t)). \tag{3.6}$$

Since in this case $t(1-t) \le n^{-1}$, it follows from Lemma 3.9 that

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(A) \ge k\phi_\rho(\Phi_\rho^{-1}(t)) \ge k' t(1-t)(\log n)^{(\rho-1)/\rho}$$

for some constant $k'(\rho) > 0$. This completes the proof. □

Theorem 1.4 is an immediate consequence of Theorem 3.12. The derivation of Corollary 1.7 from Theorem 1.4 and Proposition 1.6 is exactly the same as the proof of Theorem 2.1 in [9] (which is the analogous result for Bernoulli measures on the discrete cube), and thus is omitted here.

We conclude this section with several analogs of results for influences on the discrete cube. In the theorem below, part (1) corresponds to Talagrand's lower bound on the vector of influences [21], part (2) corresponds to a variant of the KKL theorem for functions with low influences established in [9], part (3) corresponds to Friedgut's characterization of functions with a low influence sum [8] and part (4) corresponds to Hatami's characterization of functions with a low influence sum in the continuous case [11]. Statements (1), (3) and (4) of the theorem follow immediately using Lemma 3.11 from the corresponding statements for the Entropy-influence proved in [14], and statement (2) is an immediate corollary of statement (1).

THEOREM 3.13. *Consider the product spaces* $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, *where* μ_ρ *denotes the Boltzmann measure with parameter* $\rho > 1$. *For all* $n \ge 1$, *for any Borel-measurable set* $A \subset \mathbb{R}^n$, *and for all* $\alpha > 0$, *we have*:

(1) *if* $\mu_\rho^{\otimes n}(A) = t$, *then*

$$\sum_{i=1}^{n} \frac{I_i^{\mathcal{G}}(A)}{(-\log I_i^{\mathcal{G}}(A))^{1-1/\rho}} \geq c_1 t(1-t);$$

(2) *if* $\mu_\rho^{\otimes n}(A) = t$ *and* $\max_{1\leq i\leq n} I_i^{\mathcal{G}}(A) \leq \alpha$, *then*

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(A) \geq c_1 t(1-t)(-\log\alpha)^{1-1/\rho};$$

(3) *if* A *is monotone and* $\sum_{i=1}^{n} I_i^{\mathcal{G}}(A)(-\log I_i^{\mathcal{G}}(A))^{1/\rho} = s$, *then there exists a set* $B \subset \mathbb{R}^n$ *such that* 1_B *is determined by at most* $\exp(c_2 s/\epsilon)$ *coordinates and* $\mu_\rho^{\otimes n}(A \triangle B) \leq \epsilon$;

(4) *if* $\sum_{i=1}^{n} I_i^{\mathcal{G}}(A)(-\log I_i^{\mathcal{G}}(A))^{1/\rho} = s$, *then there exists a set* $B \subset \mathbb{R}^n$ *such that* 1_B *can be represented by a decision tree of depth at most* $\exp(c_3 s/\epsilon^2)$ *and* $\mu_\rho^{\otimes n}(A \triangle B) \leq \epsilon$,[4]

where c_1, c_2, *and* c_3 *are positive constants which depend only on* ρ.

Theorem 1.5 is a special case of statements (1) and (3) of Theorem 3.13 obtained for $\rho = 2$.

3.4. *A remark on geometric influences for more general product measures*. It is worth mentioning that variants of Theorems 3.12 and 3.13 hold for any measure ν on $\mathbb{R}$ which satisfies the following two conditions:

- ν is absolutely continuous with respect to the Lebesgue measure;
- there exist constants $\rho \geq 1$, $a > 0$, such that for the isoperimetric function $\mathcal{I}_\nu$ of ν we have

$$\mathcal{I}_\nu(t) \geq a \min(t, 1-t)\big(-\log \min(t, 1-t)\big)^{1-1/\rho}, \qquad t \in [0,1].$$

The proofs are similar to those given for Boltzmann measures, except for the following changes:

- Lemma 3.7(i) now holds with $h(t) = \mathcal{I}_\nu(t)$;
- Lemma 3.7(ii) does not hold in general, but this is not a problem since for the proof of Theorem 3.12 we only need the first part of the lemma;
- instead of Lemma 3.10, we use the following dimension-free isoperimetric inequality which holds for the product measure $\nu^{\otimes n}$ (see [2]):

[4] See, for example, [11] for the definition of a decision tree.

For all $n \geq 1$ and for any measurable set $A \subseteq \mathbb{R}^n$,

$$\liminf_{r \downarrow 0} \frac{\nu^{\otimes n}(A + [-r, r]^n) - \nu^{\otimes n}(A)}{r} \geq \frac{a}{K} \min(t, 1-t)\bigl(-\log \min(t, 1-t)\bigr)^{1-1/\rho},$$

where $t = \nu^{\otimes n}(A)$ and $K > 0$ is a universal constant.

4. Boundaries of transitive sets under uniform enlargement. In the Gaussian space, the isoperimetric inequality for uniform enlargement follows from the classical Gaussian isoperimetric inequality by Sudakov and Tsirelson [20], Borell [5] (see also [1, 3, 7]) and the fact that the boundary of a set under uniform enlargement always dominates its usual boundary (i.e., the boundary under L^2 enlargement). To be specific, the boundary under uniform enlargement of any measurable set $A \subset \mathbb{R}^n$ with $\mu^{\otimes n}(A) = t$, where μ is the Gaussian measure on $\mathbb{R}$, obeys the following lower bound:

$$\liminf_{r \downarrow 0} \frac{\mu^{\otimes n}(A + [-r, r]^n) - \mu^{\otimes n}(A)}{r} \geq \phi(\Phi^{-1}(t)) \tag{4.1}$$

(where Φ and ϕ are the cumulative distribution function and the density of the Gaussian distribution in $\mathbb{R}$), and it is easy to check that the bound is achieved when A is an "axis-parallel" halfspace (i.e., sets of the form $\{x \in \mathbb{R}^n : x_i \leq a\}$ or its complement) with $\mu^{\otimes n}(A) = t$.

In this section we consider the same isoperimetric problem under an additional symmetry condition:

Find a lower bound on the boundary measure (*under uniform enlargement*) *of sets in* $\mathbb{R}^n$ *that are transitive.*

The invariance under permutation condition rules out candidates like the axis-parallel halfspaces and one might expect that under this assumption, a set should have "large" boundary. This intuition is confirmed by Theorem 1.8. In this section we prove a stronger version of this theorem that holds for all Boltzmann measures.

THEOREM 4.1. *Consider the product spaces* $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, *where* μ_ρ *denotes the Boltzmann measure with parameter* $\rho > 1$. *There exists a constant* $c = c(\rho) > 0$ *such that the following holds for all* $n \geq 1$:

For any transitive Borel-measurable set $A \subset \mathbb{R}^n$, *we have*

$$\liminf_{r \downarrow 0} \frac{\mu_\rho^{\otimes n}(A + [-r, r]^n) - \mu_\rho^{\otimes n}(A)}{r} \geq c t(1-t)(\log n)^{1-1/\rho},$$

where $t = \mu_\rho^{\otimes n}(A)$.

The transitivity assumption on A implies that Theorem 4.1 is an immediate consequence of Theorem 3.12, once we establish the following lemma.

LEMMA 4.2. *Let λ be a continuous symmetric log-concave density on $\mathbb{R}$. Let A be any Borel-measurable subset of $\mathbb{R}^n$. Then*

$$\liminf_{r\downarrow 0} \frac{\nu^{\otimes n}(A+[-r,r]^n) - \nu^{\otimes n}(A)}{r} \geq \sum_{i=1}^{n} I_i^h(A),$$

where $h(x) = \lambda(\Lambda^{-1}(x))$ for all $x \in [0,1]$.

PROOF. The proof is similar to the proof of Proposition 1.3. For all $1 \leq i \leq n$, define

$$B_r^i = A + [-r,r]^{i-1} \times \{0\}^{n-i+1}.$$

Like in the proof of Proposition 1.3, it is sufficient to show that for each i,

$$\liminf_{r\downarrow 0} \frac{\nu^{\otimes n}(B_r^i + \{0\}^{i-1} \times [-r,r] \times \{0\}^{n-i}) - \nu^{\otimes n}(B_r^i)}{r} \geq I_i^h(A). \tag{4.2}$$

Note that for all $x \in \mathbb{R}^n$, both $\nu^{\otimes n}(B_r^i)$ and $\nu((B_r^i)_i^x)$ are increasing as functions of r, and thus they tend to some limit as $r \searrow 0$. Furthermore, we can assume that $\nu^{\otimes n}(\bar{A} \setminus A) = 0$, since otherwise,

$$\liminf_{r\downarrow 0} \frac{\nu^{\otimes n}(A+[-r,r]^n) - \nu^{\otimes n}(A)}{r} \geq \liminf_{r\downarrow 0} \frac{\nu^{\otimes n}(\bar{A} \setminus A)}{r} \to \infty.$$

Therefore,

$$\nu^{\otimes n}(B_r^i) \searrow \nu^{\otimes n}(\bar{A}) = \nu^{\otimes n}(A)$$

and

$$\nu((B_r^i)_i^x) \searrow \nu(A_i^x) \tag{4.3}$$

for almost every $x \in \mathbb{R}^n$ (with respect to the measure $\nu^{\otimes n}$).

Now observe that by the one-dimensional isoperimetric inequality for symmetric log-concave distributions (Theorem 3.6),

$$\begin{aligned} \nu^{\otimes n}(B_r^i + \{0\}^{i-1} \times [-r,r] \times \{0\}^{n-i}) &= \mathbb{E}_x \nu\big((B_r^i)_i^x + [-r,r]\big) \\ &\geq \mathbb{E}_x \Lambda\big(\Lambda^{-1}(\nu((B_r^i)_i^x)) + r\big). \end{aligned}$$

Therefore, using the mean value theorem like in the proof of Proposition 1.3, we get

$$\begin{aligned} &\liminf_{r\downarrow 0} \frac{\nu^{\otimes n}(B_r + \{0\}^{i-1} \times [-r,r] \times \{0\}^{n-i}) - \nu^{\otimes n}(B_r)}{r} \\ &\quad \geq \liminf_{r\downarrow 0} \mathbb{E}_x \inf_{z \in [\Lambda^{-1}(\nu((B_r^i)_i^x)), \Lambda^{-1}(\nu((B_r^i)_i^x))+r]} \lambda(z). \end{aligned} \tag{4.4}$$

Finally, by (4.3), for almost every $x \in \mathbb{R}^n$,

$$\lim_{r \downarrow 0} \inf_{z \in [\Lambda^{-1}(\nu((B_r^i)_i^x)), \Lambda^{-1}(\nu((B_r^i)_i^x))+r]} \lambda(z) = \lambda(\Lambda^{-1}(\nu(A_i^x))),$$

and thus, by the dominated convergence theorem,

$$\liminf_{r \downarrow 0} \mathbb{E}_x \inf_{z \in [\Lambda^{-1}(\nu((B_r^i)_i^x)), \Lambda^{-1}(\nu((B_r^i)_i^x))+r]} \lambda(z) = \mathbb{E}_x \lambda(\Lambda^{-1}(\nu(A_i^x))) = I_i^h(A).$$

This completes the proof of the lemma, and thus also the proof of Theorem 4.1. $\square$

4.1. *Tightness of Theorems* 3.12, 3.13 *and* 4.1. We conclude this section with showing that Theorems 3.12, 3.13 and 4.1 are tight (up to constant factors) among sets with constant measure, which we set for convenience to be $1/2$. We demonstrate this by choosing an appropriate sequence of "*one-sided boxes*."

PROPOSITION 4.3. *Consider the product spaces* $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, *where* μ_ρ *denotes the Boltzmann measure with parameter* $\rho \geq 1$. *Let* $B_n := (-\infty, a_n]^n$ *where* a_n *is chosen such that* $\Phi_\rho(a_n)^n = 1/2$. *Then there exists a constant* $c = c(\rho)$ *such that*

$$I_i^{\mathcal{G}}(B_n) \leq c \cdot \frac{(\log n)^{1-1/\rho}}{n}$$

for all $1 \leq i \leq n$.

PROOF. Fix an i. By elementary calculation,

$$I_i^{\mathcal{G}}(B_n) = \Phi_\rho(a_n)^{n-1}\phi_\rho(a_n) = (1/2)^{(n-1)/n}\phi_\rho(a_n).$$

Note that $1 - \Phi_\rho(a_n) \asymp n^{-1}$, and thus, by Lemma 3.9, $a_n \asymp (\log n)^{1/\rho}$. Furthermore, since by Lemma 3.9, $\phi_\rho(z) \asymp z^{\rho-1}(1 - \Phi_\rho(z))$ for large z, we have $I_i^{\mathcal{G}}(B_n) \asymp n^{-1}(\log n)^{1-1/\rho}$, as asserted. $\square$

The tightness of Theorem 3.12 and Theorem 3.13(1) follows immediately from Proposition 4.3. The tightness of Theorem 4.1 follows using Proposition 1.3 since B is monotone. The tightness of Theorem 3.13(2) and the tightness in s in Theorem 3.13(3) and Theorem 3.13(4) follows by considering the subset $B_k \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$.

5. Geometric influences under rotation. Consider the product Gaussian measure $\mu^{\otimes n}$ on $\mathbb{R}^n$. In Section 3 we obtained lower bounds on the sum of geometric influences, and, in particular, we showed that for a transitive set $A \subset \mathbb{R}^n$, the sum is at least $\Omega(t(1-t)\sqrt{\log n})$, where $t = \mu^{\otimes n}(A)$.

In this section we consider a different symmetry group, the group of rotations of $\mathbb{R}^n$. The interest in this group comes from the fact that the Gaussian measure is invariant under rotations while the influence sum is not.

Indeed, a halfspace of measure $1/2$ may have influence sum as small as of order 1 when it is aligned with one of the axis and as large as of order $\sqrt{n}$ when it is aligned with the diagonal direction $(1, 1, \dots, 1)$.

In this section we show that under some mild conditions (that do not contain any invariance assumption), rotation allows us to increase the sum of geometric influences up to $\Omega(t(1-t)\sqrt{-\log(t(1-t))}\sqrt{n})$. The dependence on n in this lower bound is tight for several examples, including halfspaces and L^2-balls. We note that on the other extreme, rotation cannot decrease the sum of geometric influences below $\Omega(t(1-t)\sqrt{-\log(t(1-t))})$, as follows from a combination of Proposition 1.3, Lemma 3.7(ii) and the isoperimetric inequality (4.1).

DEFINITION 5.1. Let $B(x,r) := \{y \in \mathbb{R}^n : \|y-x\|_2 < r\}$ be the open ball in $\mathbb{R}^n$ with center at x and radius r, and let $\bar{B}(x,r)$ be the corresponding closed ball. For $\varepsilon > 0$ and $A \subseteq R^n$, define

$$A_\varepsilon := \{x \in A : \bar{B}(x,\varepsilon) \cap A^c = \varnothing\} \quad \text{and} \quad A^\varepsilon := \{x \in \mathbb{R}^n : B(x,\varepsilon) \cap A \neq \varnothing\}.$$

Finally, denote by $\mathcal{J}_n$ the collection of all measurable sets $B \subseteq \mathbb{R}^n$ for which there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$, we have

$$(B_\varepsilon)^{2\varepsilon} \supseteq B. \tag{5.1}$$

An example of a subset of $\mathbb{R}^2$ that does not belong to the class $\mathcal{J}_2$ is $\{(x,y) : 1 \le x < \infty, 0 \le y < 1/x\}$.

The crucial ingredient in the proof Theorem 1.9 is a lemma asserting that under the conditions of the theorem, an enlargement of A by a *random* rotation of the cube $[-r,r]^n$ increases $\mu^{\otimes n}(A)$ significantly.

NOTATION 5.2. Let $O = O(n, \mathbb{R})$ be the set of all orthogonal transformations on $\mathbb{R}^n$, and let π be the (unique) Haar measure on O. Denote by M a random element of O distributed according to the measure π.

LEMMA 5.3. *There exists a constant $K > 0$ such that for any $A \in \mathcal{J}_n$, we have*

$$\mathbb{E}_{M\sim\pi}\big[\mu^{\otimes n}\big(A + M^{-1}(Kn^{-1/2}[-r,r]^n)\big)\big] \ge \mu^{\otimes n}(A) + \tfrac{1}{2}\mu^{\otimes n}(A^{r/3} \setminus A)$$

for all sufficiently small $r > 0$ (depending on A).

First we show that Lemma 5.3 implies Theorem 1.9.

PROOF. Note that for any $g \in O$, $g(A)$ is convex, and that $\mu^{\otimes n}$ is invariant under g. Thus by Proposition 1.3,[5] we have

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(g(A)) = \lim_{r\downarrow 0} \frac{\mu^{\otimes n}(g(A)+[-r,r]^n) - \mu^{\otimes n}(g(A))}{r} = \lim_{r\downarrow 0} \frac{\mu^{\otimes n}(A+g^{-1}([-r,r]^n)) - \mu^{\otimes n}(A)}{r}.$$

Furthermore, note that for any $g \in O$,

$$\lim_{r\downarrow 0} \frac{\mu^{\otimes n}(A+g^{-1}([-r,r]^n)) - \mu^{\otimes n}(A)}{r} \leq \lim_{r\downarrow 0} \frac{\mu^{\otimes n}(A+\sqrt{n}[-r,r]^n) - \mu^{\otimes n}(A)}{r} = \sqrt{n} \times \sum_{i=1}^{n} I_i^{\mathcal{G}}(A).$$

Therefore, by the dominated convergence theorem,

$$\mathbb{E}_{M\sim\pi}\left[\sum_{i=1}^{n} I_i^{\mathcal{G}}(M(A))\right] = \lim_{r\downarrow 0} \frac{\mathbb{E}_{M\sim\pi}[\mu^{\otimes n}(A+M^{-1}([-r,r]^n))] - \mu^{\otimes n}(A)}{r}. \tag{5.2}$$

By Lemma 5.3, we have (for a sufficiently small r)

$$\mathbb{E}_{M\sim\pi}\big[\mu^{\otimes n}\big(A+M^{-1}(Kn^{-1/2}[-r,r]^n)\big)\big] - \mu^{\otimes n}(A) \geq \tfrac{1}{2}\mu^{\otimes n}(A^{r/3}\setminus A).$$

By the standard Gaussian isoperimetric inequality,

$$\mu^{\otimes n}(A^{r/3}\setminus A) \geq \mu\big((-\infty, \Phi^{-1}(t)+r/3]\big).$$

Substituting into (5.2), we get

$$\begin{aligned}\mathbb{E}_{M\sim\pi}\left[\sum_{i=1}^{n} I_i^{\mathcal{G}}(M(A))\right] &\geq \limsup_{r\downarrow 0} \frac{\mu((-\infty, \Phi^{-1}(t)+K^{-1}n^{1/2}r/3])}{2r} \\ &\geq \frac{\sqrt{n}}{6K}\phi(\Phi^{-1}(t)) \\ &\geq ct(1-t)\sqrt{-\log\big(t(1-t)\big)} \times \sqrt{n}\end{aligned}$$

for some constant $c > 0$ (where the last inequality follows from the estimation given in Lemma 3.9, with $\rho = 2$). Thus there exists at least one orthogonal transformation $g \in O$ such that

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(g(A)) \geq ct(1-t)\sqrt{-\log\big(t(1-t)\big)} \times \sqrt{n}$$

[5] Note that by Remark 2.2, Proposition 1.3 holds for convex sets.

as asserted. $\square$

Now we present the proof of Lemma 5.3.

PROOF OF LEMMA 5.3. By Fubini's theorem, we have

$$\begin{aligned}
&\mathbb{E}_{M\sim\pi}\big[\mu^{\otimes n}\big(A+M^{-1}(Kn^{-1/2}[-r,r]^n)\big)\big]\\
&\quad=\mathbb{E}_{M\sim\pi}\big[\mu^{\otimes n}\{x\in\mathbb{R}^n : x=y+z,\, y\in A,\, z\in M^{-1}(Kn^{-1/2}[-r,r]^n)\}\big]\\
&\quad=\mathbb{E}_{x\sim\mu^{\otimes n}}\big[\pi\{g\in O : x=y+z,\, y\in A,\, z\in g^{-1}(Kn^{-1/2}[-r,r]^n)\}\big].
\end{aligned}\tag{5.3}$$

Since each $x\in A$ can be trivially represented as $y+z$ with $y=x\in A$, $z=0\in g^{-1}(Kn^{-1/2}[-r,r]^n)$ for any $g\in O$, the assertion of the lemma would follow immediately from (5.3) once we show that for all $x\in A^{r/3}\setminus A$,

$$\pi\{g\in O : x=y+z,\, y\in A,\, z\in g^{-1}(Kn^{-1/2}[-r,r]^n)\}\geq 1/2.\tag{5.4}$$

Since $A\in\mathcal{J}_n$, we can choose r sufficiently small such that $A\subset(A_{r/3})^{2r/3}$, and thus $A^{r/3}\subset(A_{r/3})^r$. Therefore, for any $x\in A^{r/3}\setminus A$, there exists $y\in A_{r/3}$, such that $\|x-y\|_2<r$. If there exists $y'\in B(y,r/3)$ such that $x-y'\in g^{-1}(Kn^{-1/2}[-r,r]^n)$, then x can be represented as $y'+(x-y')$, as required in the left-hand side of (5.4). Therefore, it is sufficient to prove the following claim:

CLAIM 5.4. *For any $x,y\in\mathbb{R}^n$ such that $\|x-y\|_2<r$,*

$$\pi\{g\in O : \exists y'\in B(y,r/3) \text{ such that } x-y'\in g^{-1}(Kn^{-1/2}[-r,r]^n)\}\geq 1/2.$$

PROOF. Fix $x,y\in\mathbb{R}^n$ such that $\|x-y\|_2<r$. We have

$$\begin{aligned}
&\{g\in O : \exists y'\in B(y,r/3) \text{ such that } x-y'\in g^{-1}(Kn^{-1/2}[-r,r]^n)\}\\
&\quad=\{g\in O : \exists y'\in B(y,r/3) \text{ such that } g(x-y')\in Kn^{-1/2}[-r,r]^n\}\\
&\quad=\{g\in O : \exists y''\in B(0,r/3) \text{ such that } g(x-y)-y''\in Kn^{-1/2}[-r,r]^n\}\\
&\quad=\Big\{g\in O : \inf_{y''\in B(0,r/3)}\|g(x-y)-y''\|_\infty\leq Kn^{-1/2}r\Big\}.
\end{aligned}$$

Note that

$$\pi\Big\{g\in O : \inf_{y''\in B(0,r/3)}\|g(x-y)-y''\|_\infty\leq Kn^{-1/2}r\Big\}\tag{5.5}$$

is invariant under rotation of the vector $(x-y)$, and in particular,

$$(5.5)=\pi\Big\{g\in O : \inf_{y''\in B(0,r/3)}\big\|g(\|x-y\|_2\times e_1)-y''\big\|_\infty\leq Kn^{-1/2}r\Big\},$$

where $e_1=(1,0,\dots,0)\in\mathbb{R}^n$ is the unit vector along the first coordinate axis.

A well-known property of the Haar measure says that if $M \in O$ is distributed according to π, then any column of M is distributed like a normalized vector of independent standard Gaussians. That is,

$$M_{\text{column}} \sim \frac{Z}{\|Z\|_2},$$

where $Z = (Z_1, \dots, Z_n)$ is a random n-vector with i.i.d. standard Gaussian entries. Thus, $M(\|x-y\|_2 \times e_1)$ is distributed like $\|x-y\|_2 \times Z/\|Z\|_2$. Therefore, we have

$$\begin{aligned}(5.5) &= \mathbb{P}_{Z\sim\mu^{\otimes n}}\left(\inf_{y''\in B(0,r/3)}\left\|\|x-y\|_2 \times \frac{Z}{\|Z\|_2} - y''\right\|_\infty \le Kn^{-1/2}r\right)\\ &\ge \mathbb{P}_{Z\sim\mu^{\otimes n}}\left(\inf_{y'''\in B(0,1/3)}\left\|\frac{Z}{\|Z\|_2} - y'''\right\|_\infty \le Kn^{-1/2}\right).\end{aligned}$$

Note that if $Z \in \mathbb{R}^n$ satisfies

$$\frac{\sum_i Z_i^2 1_{|Z_i|/\|Z\|_2 > Kn^{-1/2}}}{\|Z\|_2^2} < 1/9,$$

then the vector y''' defined by $y_i''' = (Z_i \cdot 1_{|Z_i|/\|Z\|_2 > Kn^{-1/2}})/\|Z\|_2$ satisfies

$$y''' \in B(0,1/3) \quad \text{and} \quad \left\|\frac{Z}{\|Z\|_2} - y'''\right\|_\infty \le Kn^{-1/2}.$$

Hence,

$$\begin{aligned}(5.5) &\ge \mathbb{P}_{Z\sim\mu^{\otimes n}}\left(\inf_{y'''\in B(0,1/3)}\left\|\frac{Z}{\|Z\|_2} - y'''\right\|_\infty \le Kn^{-1/2}\right)\\ &\ge \mathbb{P}_{Z\sim\mu^{\otimes n}}\left(\frac{\sum_i Z_i^2 1_{|Z_i|/\|Z\|_2 > Kn^{-1/2}}}{\|Z\|_2^2} < 1/9\right).\end{aligned}$$

For $\gamma < 1$ and $t > 0$, by the Markov inequality,

$$\begin{aligned}\mathbb{P}(\|Z\|^2 < \gamma n) &\le \mathbb{P}(e^{-t\|Z\|^2} < e^{-t\gamma n}) \le e^{t\gamma n}(\mathbb{E}e^{-tZ_1^2})^n\\ &= e^{t\gamma n}(1+2t)^{-n/2}.\end{aligned}$$

Optimizing over $t > 0$, we have

$$\mathbb{P}(\|Z\|^2 < \gamma n) \le (\gamma e^{1-\gamma})^{n/2}. \tag{5.6}$$

Finally, again by the Markov inequality,

$$\mathbb{P}_{Z\sim\mu^{\otimes n}}\left[\sum_{i:|Z_i|>K/2} Z_i^2 \ge \frac{n}{450}\right] \le \frac{n \times [\mathbb{E}Z_1^2 1_{\{|Z_1|>K/2\}}]}{n/450} \le 1/4$$

for sufficiently large $K > 0$, and by (5.6), $\mathbb{P}[\|Z\|_2^2 > n/50] \geq 3/4$. Therefore,

$$
\begin{aligned}
(5.5) &\geq \mathbb{P}_{Z\sim\mu^{\otimes n}}\left(\frac{\sum_i Z_i^2 1_{|Z_i|/\|Z\|_2 > Kn^{-1/2}}}{\|Z\|_2^2} < 1/9\right) \\
&\geq \mathbb{P}_{Z\sim\mu^{\otimes n}}\left[\left(\sum_{i\,:\,|Z_i|>K/2} Z_i^2 \leq \frac{n}{450}\right) \wedge \left(\|Z\|_2 > \sqrt{n}/50\right)\right] \\
&\geq 3/4 + 3/4 - 1 = 1/2.
\end{aligned}
$$

This completes the proof of the claim and of Lemma 5.3. □

Intuitively, the condition $A \in \mathcal{J}_n$ means that the boundary of A is "sufficiently smooth." One can easily check that if $A \in \mathcal{J}_n$, then the boundary of A is a porous set and thus has Hausdorff dimension strictly less than n (see [22] and references therein to know more about porous sets). However, this condition is far from being sufficient. Here we give a sufficient condition for a set to belong to $\mathcal{J}_n$ in terms of smoothness of its boundary.

DEFINITION 5.5. Let $A \subset \mathbb{R}^n$ be a measurable set. We write $\partial A \in C^1$ and say that the boundary of A is of class C^1 if for any point $z \in \partial A$, there exists $r = r(z) > 0$ and a one-to-one mapping ψ of $B(z,r)$ onto an open set $D = D \subseteq \mathbb{R}^n$ such that:

- $\psi \in C^1(\bar{B}(z,r))$ and $\psi^{-1} \in C^1(\bar{D})$;
- $\psi(B(z,r) \cap \partial A) = D \cap \{x \in \mathbb{R}^n : x_1 = 0\}$;
- $\psi(B(z,r) \cap \operatorname{int}(A)) \subseteq (0,\infty) \times \mathbb{R}^{n-1}$.

PROPOSITION 5.6. *Let $A \subset \mathbb{R}^n$ be a bounded set with $\partial A \in C^1$. Then $A \in \mathcal{J}_n$.*

PROOF. Suppose on the contrary that $A \notin \mathcal{J}_n$. Then there exists a sequence $\{x^m\}_{m=1}^{\infty}$ such that $x^m \in A$ but $x^m \notin (A_{1/m})^{2/m}$. Since A is bounded, the sequence contains a subsequence $\{x^{m_k}\}$ converging to a point x^0. Clearly, $x^0 \in \partial A$.

Since $\partial A \in C^1$, we can define a new set of local coordinates $(y_1, y_2, \ldots, y_n)$ [also denoted by (y_1, y'), where $y' \in \mathbb{R}^{n-1}$], such that:

(1) the point x^0 is the origin with respect to the y-coordinates;

(2) there exists an open neighborhood $(-\delta_0, \delta_0) \times U \subseteq \mathbb{R} \times \mathbb{R}^{n-1}$ containing the origin and a continuously differentiable function $f : U \to \mathbb{R}_+$, such that in the y-coordinates,

$$\partial A \cap [(-\delta_0, \delta_0) \times U] = \{(f(y'), y') : y' \in U\}$$

and

$$\operatorname{int} A \cap [(-\delta_0, \delta_0) \times U] = \{(y_1, y') : y' \in U,\ f(y') < y_1 < \delta_0\}. \tag{5.7}$$

By the construction of the new coordinates, $f(y') \geq 0$ for all $y' \in U$ and $f(0) := f(0, 0, \ldots, 0) = 0$. Since $f \in C^1(U)$, it follows that $\nabla f(0) = 0$. Hence, by the continuity of the partial derivatives of f, there exists $r_0 > 0$ such that $\|\nabla f(y')\|_\infty \leq 1/(3\sqrt{n})$ for all $y' \in B_{n-1}(0, r_0) \subseteq U$.

Let $y^m = (y_1^m, (y^m)')$ be the representation of the point x^m in the y-coordinates. Find m large enough such that $1/m < \min\{\delta_0/10, r_0/10\}$, and y^m lies within $A \cap [0, \delta_0/2] \times B_{n-1}(0, r_0/2)$. Define

$$z = (z_1, z_2, \ldots, z_n) = y^m + (1.5m^{-1}, 0, \ldots, 0, 0).$$

We claim that $B(z, 1/m) \subseteq A$. This would be a contradiction to the hypothesis $y^m \notin (A_{1/m})^{2/m}$.

Note that by the choice of m, we have $z \in A$, and moreover,

$$\begin{aligned} \operatorname{dist}(z, \partial A) &\geq \operatorname{dist}\bigl(z, \partial A \cap [(-\delta_0, \delta_0) \times B_{n-1}(0, r_0)]\bigr) \\ &= \inf_{y' \in B_{n-1}(0, r_0)} \bigl\|\bigl(y_1^m + 1.5m^{-1}, (y^m)'\bigr) - (f(y'), y')\bigr\|_2. \end{aligned} \tag{5.8}$$

We would like to show that if $\|(y^m)' - y'\|_2$ is "small," then $|y_1^m + 1.5m^{-1} - f(y')|$ is "big," and thus in total, the right-hand side of (5.8) cannot be "too small."

Define $w_1 := y_1^m + 1.5m^{-1} - f((y^m)')$. Note that since $y^m \in A$, it follows from (5.7) that $w_1 \geq 1.5m^{-1}$. By the mean value theorem, for each $y' \in B_{n-1}(0, r_0)$,

$$\begin{aligned} |f((y^m)') - f(y')| &\leq \Bigl(\sup_{y'' \in B_{n-1}(0, r_0)} \|\nabla f(y'')\|_\infty\Bigr) \|(y^m)' - y'\|_1 \\ &\leq \frac{\|(y^m)' - y'\|_1}{3\sqrt{n}} \leq \frac{\|(y^m)' - y'\|_2}{3}, \end{aligned}$$

and thus

$$|y_1^m + 1.5m^{-1} - f(y')| = \bigl|w_1 - \bigl(f(y') - f((y^m)')\bigr)\bigr| \geq 1.5m^{-1} - \frac{\|(y^m)' - y'\|_2}{3}.$$

Consequently, if $\|(y^m)' - y'\|_2 \geq 4.5m^{-1}$, then

$$\bigl\|\bigl(y_1^m + 1.5m^{-1}, (y^m)'\bigr) - (f(y'), y')\bigr\|_2 \geq \|(y^m)' - y'\|_2 \geq 4.5m^{-1},$$

and if $\|(y^m)' - y'\|_2 < 4.5m^{-1}$, then

$$\begin{aligned} &\bigl\|\bigl(y_1^m + 1.5m^{-1}, (y^m)'\bigr) - (f(y'), y')\bigr\|_2 \\ &\quad \geq \sqrt{\|(y^m)' - y'\|_2^2 + \left(1.5m^{-1} - \frac{\|(y^m)' - y'\|_2}{3}\right)^2} \\ &\quad = \min_{0 \leq s < 4.5m^{-1}} \sqrt{s^2 + (1.5m^{-1} - s/3)^2} = \sqrt{\frac{81}{40}} m^{-1}. \end{aligned}$$

Combining the two cases, we get

$$\operatorname{dist}(z, \partial A) \geq \inf_{y' \in B_{n-1}(0,r_0)} \|(y_1^m + 1.5m^{-1}, (y^m)') - (f(y'), y')\|_2$$
$$\geq \min\left(4.5m^{-1}, \sqrt{\frac{81}{40}}m^{-1}\right) > 1/m.$$

This completes the proof. □

If the condition $A \in \mathcal{J}_n$ is removed, we can prove only a weaker lower bound on the maximal sum of geometric influences that can be obtained by rotation.

PROPOSITION 5.7. *Consider the product Gaussian measure $\mu^{\otimes n}$ on $\mathbb{R}^n$. For any convex set A with $\mu^{\otimes n}(A) = t$, there exists an orthogonal transformation g on $\mathbb{R}^n$ such that*

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(g(A)) \geq ct(1-t)\sqrt{-\log(t(1-t))}\frac{\sqrt{n}}{\sqrt{\log n}},$$

where $c > 0$ is a universal constant.

The proof of Proposition 5.7 uses a weaker variant of Lemma 5.3:

LEMMA 5.8. *Let M be as defined in Notation 5.2. There exists a constant $K > 0$ such that for any $A \subset \mathbb{R}^n$ and for any $r > 0$, we have*

$$\mathbb{E}_{M\sim\pi}[\mu^{\otimes n}(A + M^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n))] \geq \mu^{\otimes n}(A) + \tfrac{1}{2}\mu^{\otimes n}(A^r \setminus A).$$

PROOF. By Fubini's theorem, we have

$$\mathbb{E}_{M\sim\pi}[\mu^{\otimes n}(A + M^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n))]$$
$$= \mathbb{E}_{x\sim\mu^{\otimes n}}[\pi\{g \in O : x \in A + g^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n)\}].$$

Thus it is sufficient to prove that for any $x \in A^r \setminus A$,

$$\pi\{g \in O : x \in A + g^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n)\} \geq 1/2.$$

Equivalently, it is sufficient to prove that for any $x \in B(0, r)$,

$$\pi\{g \in O : x \in g^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n)\} \geq 1/2.$$

We can assume without loss of generality that $x = r' \cdot e_1$ for some $r' < r$. By the argument used in the proof of Lemma 5.3, if $M \in O$ is distributed according to π, then $M(r' \cdot e_1)$ is distributed like $r' \cdot Z/\|Z\|_2$, where $Z = (Z_1, \dots, Z_n)$ is a random n-vector with i.i.d. standard Gaussian entries. Hence

$$\begin{aligned}
&\pi\{g \in O : x \in g^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n)\}\\
&\quad = \mathbb{P}_{Z\sim\mu^{\otimes n}}\left(\left\| r' \frac{Z}{\|Z\|_2}\right\|_\infty \le K\sqrt{\log n} \cdot n^{-1/2} r\right)\\
&\quad \ge \mathbb{P}_{Z\sim\mu^{\otimes n}}\left(\left\| \frac{Z}{\|Z\|_2}\right\|_\infty \le K\sqrt{\log n} \cdot n^{-1/2}\right)\\
&\quad \ge \mathbb{P}_{Z\sim\mu^{\otimes n}}[(\|Z\|_\infty \le K\sqrt{\log n}/\sqrt{50}) \wedge (\|Z\|_2 \ge \sqrt{n}/\sqrt{50})].
\end{aligned} \tag{5.9}$$

We have

$$\begin{aligned}
&\mathbb{P}_{Z\sim\mu^{\otimes n}}(\|Z\|_\infty \le K\sqrt{\log n}/\sqrt{50})\\
&\qquad \ge 1 - n\mathbb{P}(Z_i > K\sqrt{\log n}/\sqrt{50})\\
&\qquad \ge 1 - \frac{n}{\sqrt{2\pi}} \cdot n^{-K^2/100} \ge 3/4
\end{aligned}$$

for a sufficiently big K. Therefore,

$$(5.9) \ge 3/4 + 3/4 - 1 = 1/2,$$

and this completes the proof of the lemma. □

The derivation of Proposition 5.7 from Lemma 5.8 is the same as the derivation of Theorem 1.9 from Lemma 5.3.

Note that the convexity assumption on A is used only to apply Proposition 1.3 that relates the sum of influences to the size of the boundary with respect to uniform enlargement. Thus, our argument also shows that for *any* measurable set A with $\mu^{\otimes n}(A) = t$, there exists an orthogonal transformation g on $\mathbb{R}^n$ such that

$$\lim_{r\downarrow 0} \frac{\mu^{\otimes n}(g(A) + [-r, r]^n) - \mu^{\otimes n}(g(A))}{r} \ge ct(1-t)\sqrt{-\log(t(1-t))}\frac{\sqrt{n}}{\sqrt{\log n}},$$

where $c > 0$ is a universal constant.

Finally, we note that apparently the assertion of Proposition 5.7 is not optimal, and the lower bound asserted in Theorem 1.9 should hold for general convex sets.

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