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Joining Longer Queues: Information Externalities in Queue Choice

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Joining Longer Queues: Information Externalities in Queue Choice

Abstract

A classic example that illustrates how observed customer behavior impacts other customers' decisions is the selection of a restaurant whose quality is uncertain. Customers often choose the busier restaurant, inferring that other customers in that restaurant know something that they do not. In an environment with random arrival and service times, customer behavior is reflected in the lengths of the queues that form at the individual servers. Therefore, queue lengths could signal two factors—potentially higher arrivals to the server or potentially slower service at the server. In this paper, we focus on both factors when customers' waiting costs are negligible. This allows us to understand how information externalities due to congestion impact customers' service choice behavior.

In our model, based on private information about both the service-quality and queue-length information, customers decide which queue to join. When the service rates are the same and known, we confirm that it may be rational to ignore private information and purchase from the service provider with the longer queue when only one additional customer is present in the longer queue. We find that, due to the information externalities contained in queue lengths, there exist cycles during which one service firm is thriving whereas the other is not. Which service provider is thriving depends on luck; i.e., it is determined by the private signal of the customer arriving when both service providers are idle. These phenomena continue to hold when each service facility has multiple servers, or when a facility may go out of business when it cannot attract customers for a certain amount of time. Finally, we find that when the service rates are unknown but are negatively correlated with service values, our results are strengthened; long queues are now doubly informative. The market share of the high-quality firm is higher when there is service rate uncertainty, and it increases as the service rate decreases. When the service rates are positively correlated with unknown service values, long queues become less informative and customers might even join shorter queues.

Keywords

customer herding, behavioral operations, information externalities, market dynamics, private signals, service capacity

Disciplines

Organizational Behavior and Theory | Other Business | Other Economics

Online Technical Appendix

Joining Longer Queues: Information Externalities in Queue Choice

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PROOF of **Proposition 3**: We show the detailed proof that \mathcal{A}^b is not an equilibrium when fixed queue difference thresholds are greater than one, i.e. $b > 1$. Consider a strategy whose fixed threshold is $b > 1$. We write out the steady state probability transition equations for all boundary states under the \mathcal{A}^b strategy.

$$\pi_{k,0}(\lambda + \mu) = \pi_{k-1,0}\lambda + \pi_{k+1,0}\mu + \pi_{k,1}\mu \quad \forall k \geq b + 2.$$

$$\pi_{k,0}(\lambda + \mu) = \pi_{k-1,0}\lambda g + \pi_{k+1,0}\mu + \pi_{k,1}\mu \quad \forall 1 \leq k \leq b + 1.$$

Adding the above equations for all values of k , and writing a similar equation for all states $(0, n)$ for all n , we have the following equations below:

$$\rho(1 - g) \sum_{k=1}^b \pi_{k,0} + \pi_{1,0} = \rho g \pi_{0,0} + \sum_{k=1}^{\infty} \pi_{k,1} \quad (11)$$

$$\rho g \sum_{k=1}^b \pi_{0,k} + \pi_{0,1} = \rho(1 - g)\pi_{0,0} + \sum_{k=1}^{\infty} \pi_{1,k} \quad (12)$$

Since each customer follows his/her signal at the state $(1, 0)$, we have $\pi_{1,0} \leq \frac{g}{1-g}\pi_{0,1}$. Now consider equation 11:

$$\begin{aligned} \rho(1 - g) \sum_{k=1}^b \pi_{k,0} &= \rho g \pi_{0,0} - \pi_{1,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1} \\ &= -\rho(1 - g)\pi_{0,0} + \pi_{0,1} + \sum_{k=1}^{\infty} \pi_{k,1} \text{ by using } \pi_{1,0} + \pi_{0,1} = \rho\pi_{0,0} \\ &> \rho(1 - g)\pi_{0,0} - \pi_{0,1} + \sum_{k=1}^{\infty} \pi_{1,k} \\ &= \rho g \sum_{k=1}^b \pi_{0,k}. \end{aligned}$$

Hence we have $\frac{\sum_{k=1}^b \pi_{k,0}}{\sum_{k=1}^b \pi_{0,k}} > \frac{g}{1-g}$. There is some n such that at least for some $n, 1 \leq n \leq b$ we have $\frac{\pi_{n,0}}{\pi_{0,n}} > \frac{g}{1-g}$. There is some state whose queue difference is *within* fixed threshold where the best response of an arriving customer would be NOT to follow the signal if all other customers were to follow the signal within this fixed queue difference. ■

Before we prove Proposition 5, as a first step, we show that any mixing between threshold strategies is not in equilibrium. Without loss of generality let server 1 be better than server 2 (i.e. $v_1 > v_2$) again. Given a strategy (set of actions at every state by all customers) we write the steady state transition equations and then solve for stationary probabilities. When server 2 provides higher valuation than server 1 we can write the same steady state by suitably replacing g with $1-g$. Strategy \mathcal{A}^{0+p} is the strategy in which all customers follow their private signal when the queue lengths are equal. When the queue length difference is greater than one, the customers follow the longer queue. When $|n_1 - n_2| = 1$, i.e. when the queue lengths differ by one, the customers mix between two strategies: following the longer queue and following their private signal. They follow their signal with probability p or ignore their signal, and follow the longer queue with probability $1-p$. Consequently, they join the longer queue 1 with probability $p' = g + (1-g)(1-p)$ and join queue 2 with probability $1-p' = p(1-g)$. Let $\pi_{m,n}$ be the long-run stationary probabilities of the state (m, n) under some strategy when $v_1 > v_2$.

$$\pi_{k,0}(\lambda + \mu) = \pi_{k-1,0}\lambda + \pi_{k+1,0}\mu + \pi_{k,1}\mu \quad \forall k \geq 3$$

$$\pi_{2,0}(\lambda + \mu) = \pi_{1,0}\lambda p' + \pi_{3,0}\mu + \pi_{2,1}\mu$$

$$\pi_{1,0}(\lambda + \mu) = \pi_{0,0}\lambda g + \pi_{2,0}\mu + \pi_{1,1}\mu$$

Adding all the equations for $k = 1, \dots, \infty$

$$(\lambda + \mu) \sum_{k=1}^{\infty} \pi_{k,0} = \pi_{0,0}\lambda g + \pi_{1,0}\lambda p' + \lambda \sum_{k=2}^{\infty} \pi_{k,0} + \mu \sum_{k=2}^{\infty} \pi_{k,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1} \quad (13)$$

$$\lambda \pi_{1,0} + \mu \pi_{1,0} = \lambda g \pi_{0,0} + \pi_{1,0}\lambda p' + \mu \sum_{k=1}^{\infty} \pi_{k,1} \quad (14)$$

$$\lambda(1-p')\pi_{1,0} + \mu \pi_{1,0} = \lambda g \pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1} \quad (15)$$

The above equation becomes,

$$(\lambda(1-p') + \mu)\pi_{1,0} = \lambda g \pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1} \quad (16)$$

$$(\rho(1-p') + 1)\pi_{1,0} = \rho g \pi_{0,0} + \sum_{k=1}^{\infty} \pi_{k,1} \quad (17)$$

Writing a similar expression for $\pi_{0,1}$ we have,

$$(\rho p' + 1)\pi_{0,1} = \rho(1 - g)\pi_{0,0} + \sum_{k=1}^{\infty} \pi_{1,k} \quad (18)$$

From the balance equations, we note that the probability transition matrix when $v_1 > v_2$ is a transpose of the transition matrix when $v_2 > v_1$. Hence, we have the likelihood ratio at $(1, 0)$ as

$$\begin{aligned} l(1, 0) &= \frac{\pi_{1,0}(v_1 > v_2)}{\pi_{1,0}(v_2 > v_1)} = \frac{\pi_{1,0}(v_1 > v_2)}{\pi_{0,1}(v_1 > v_2)} \\ &= \frac{\lambda g \pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1}}{\lambda(1 - g)\pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{1,k}} \frac{\lambda p' + \mu}{\lambda(1 - p') + \mu} \\ &> \frac{\lambda g \pi_{0,0} + \mu A}{\lambda(1 - g)\pi_{0,0} + \mu B} \frac{\lambda g + \mu}{\lambda(1 - g) + \mu} \end{aligned}$$

where $A = \sum_{k=1}^{\infty} \pi_{k,1}$ and $B = \sum_{k=1}^{\infty} \pi_{1,k}$. Then under the \mathcal{A}^{0+p} we have,

$$\begin{aligned} \pi_{k,1} &\geq \frac{g}{1 - g} \pi_{1,k} \quad \forall k \geq 3 \text{ and } \pi_{2,1} = \frac{g}{1 - g} \pi_{1,2} \\ \text{which gives } \sum_{k \geq 2} \pi_{k,1} &\geq \frac{g}{1 - g} \sum_{k \geq 2} \pi_{1,k} > \sum_{k \geq 2} \pi_{1,k} \\ \frac{A}{B} &\geq \frac{g}{1 - g} \\ \text{i.e. } A &> B \end{aligned}$$

We provide a proof by contradiction. If \mathcal{A}^{0+p} is in equilibrium, we have one necessary condition at $(1, 0)$ as $l(1, 0) = \frac{g}{1 - g}$

$$\begin{aligned} l(1, 0) &= \frac{g}{1 - g} \\ \frac{\lambda g \pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1}}{\lambda(1 - g)\pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{1,k}} \left(\frac{\lambda p' + \mu}{\lambda(1 - p') + \mu} \right) &= \frac{g}{1 - g} \\ \frac{\rho g \pi_{0,0} + A}{\rho(1 - g)\pi_{0,0} + B} \frac{\rho p' + 1}{\rho(1 - p') + 1} &= \frac{g}{1 - g} \\ (\rho g \pi_{0,0} + A)(\rho p' + 1)(1 - g) &= g(\rho(1 - g)\pi_{0,0} + B)(\rho(1 - p') + 1) \end{aligned}$$

$$\text{we need } A(\rho p' + 1)(1 - g) + \rho^2 g(1 - g)p' \pi_{0,0} = Bg(\rho(1 - p') + 1) + \rho^2 g(1 - g)(1 - p') \pi_{0,0}$$

But we have, $\frac{A}{B} \geq \frac{g}{1 - g} \Rightarrow A(1 - g) \geq Bg$. Further, $p' = g + (1 - g)(1 - p) > g > 1 - g > 1 - p'$. Therefore, $\rho p' + 1 > \rho(1 - p') + 1$ and consequently, $A(1 - g)(\rho p' + 1) > Bg(\rho(1 - p') + 1)$.

Since $p' > 1 - p'$ we also have $\rho^2 g(1 - g)p' \pi_{0,0} > \rho^2 g(1 - g)(1 - p') \pi_{0,0}$. Therefore $LHS = A(\rho p' + 1)(1 - g) + \rho^2 g(1 - g)p' \pi_{0,0} > Bg(\rho(1 - p') + 1) + \rho^2 g(1 - g)(1 - p') \pi_{0,0} = RHS$ which is a contradiction of the necessary condition. The necessary indifference condition for applying a

mixed strategy at $(1, 0)$ is violated at $(1, 0)$ if all other customers follow A^{0+p} . So A^{0+p} is not an equilibrium strategy. Using similar arguments, it can be shown that a similarly defined modified strategy A^{b+p} is also not in equilibrium. ■

PROOF of Proposition 5:

Without loss of generality let server 1 be better than server 2 (i.e. $v_1 > v_2$) again. Given a strategy (set of actions at every state by all customers) we write the steady state transition equations and then solve for stationary probabilities. When server 2 provides higher valuation than server 1 we can write the same steady state by suitably replacing g with $1 - g$.

Consider the class of strategies where customers mix between following the signal and following the longer queue, along all the recurrent states in \mathcal{A}^0 strategy. In particular, at state $(k, 0)$ (and $(0, k)$) they follow their signal with probability p_k ($0 \leq p_k \leq 1$) or ignore their signal, and follow the longer queue with probability $1 - p_k$. Consequently, they join the longer queue 1 with probability $p'_k = g + (1 - g)(1 - p_k)$ and join queue 2 with probability $1 - p'_k = p_k(1 - g)$. Let $\pi_{m,n}$ be the long-run stationary probabilities of the state (m, n) under some strategy when $v_1 > v_2$.

$$\begin{aligned}\pi_{k,0}(\lambda + \mu) &= \pi_{k-1,0}\lambda p'_k + \pi_{k+1,0}\mu + \pi_{k,1}\mu \quad \forall k \geq 2 \\ \pi_{1,0}(\lambda + \mu) &= \pi_{0,0}\lambda g + \pi_{2,0}\mu + \pi_{1,1}\mu\end{aligned}$$

Adding all the equations for $k = 1, \dots, \infty$ and assigning $p'_0 = g$.

$$(\lambda + \mu) \sum_{k=1}^{\infty} \pi_{k,0} = \lambda \sum_{k=1}^{\infty} p'_k \pi_{k,0} + \mu \sum_{k=2}^{\infty} \pi_{k,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1} \quad (19)$$

$$\lambda \sum_{k=1}^{\infty} (1 - p'_k) \pi_{k,0} + \mu \pi_{1,0} = \lambda g \pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1} \quad (20)$$

The above equation becomes,

$$\rho \sum_{k=1}^{\infty} (1 - p') \pi_{k,0} + \pi_{1,0} = \rho g \pi_{0,0} + \sum_{k=1}^{\infty} \pi_{k,1} \quad (21)$$

Similarly, at states $(0, k) \forall k$, they join the longer queue 1 with probability $p''_k = g + (1 - g)p_k$ and join queue 2 with probability $1 - p'' = (1 - p_k)(1 - g)$.

$$\rho \sum_{k=1}^{\infty} (1 - p'') \pi_{0,k} + \pi_{0,1} = \rho(1 - g) \pi_{0,0} + \sum_{k=1}^{\infty} \pi_{1,k} \quad (22)$$

From the steady state balance equations, we note that the probability transition matrix when

$v_1 > v_2$ is a transpose of the transition matrix when $v_2 > v_1$.

$$\begin{aligned} \frac{\lambda \sum_{k=1}^{\infty} (1-p')\pi_{k,0} + \mu\pi_{1,0}}{\lambda \sum_{k=1}^{\infty} (1-p'')\pi_{0,k} + \mu\pi_{0,1}} &= \frac{\lambda g\pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{k,1}}{\lambda(1-g)\pi_{0,0} + \mu \sum_{k=1}^{\infty} \pi_{1,k}} \\ &= \frac{\lambda g\pi_{0,0} + \mu A}{\lambda(1-g)\pi_{0,0} + \mu B}. \end{aligned}$$

where $A = \sum_{k=1}^{\infty} \pi_{k,1}$ and $B = \sum_{k=1}^{\infty} \pi_{1,k}$.

For mixing to be in equilibrium we require,

$$\frac{\rho \sum_{k=1}^{\infty} (1-p')\pi_{k,0} + \pi_{1,0}}{\rho \sum_{k=1}^{\infty} (1-p'')\pi_{0,k} + \pi_{0,1}} = \frac{\rho g\pi_{0,0} + \sum_{k=1}^{\infty} \pi_{k,1}}{\rho(1-g)\pi_{0,0} + \sum_{k=1}^{\infty} \pi_{1,k}}. \quad (23)$$

If customers mix at every indifferent state, we have $l(k, 0) = \frac{g}{1-g} \forall k$.

Consider the left-hand side of the equation $\frac{\lambda \sum_{k=1}^{\infty} (1-p')\pi_{k,0} + \mu\pi_{1,0}}{\lambda \sum_{k=1}^{\infty} (1-p'')\pi_{0,k} + \mu\pi_{0,1}}$. Suppose $\frac{\pi_{1,0}}{\pi_{0,1}} > \frac{g}{1-g}$. Then the customers join the longer queue, and we are done. Suppose $\frac{\pi_{1,0}}{\pi_{0,1}} = \frac{g}{1-g}$. We intend to show that there is at least one state where customers would strictly prefer to deviate and always join the longer queue. We have $\pi_{1,0} = g\rho\pi_{0,0}$ and $\pi_{0,1} = (1-g)\rho\pi_{0,0}$. Hence, the equation (23) reduces to

$$\frac{\rho \sum_{k=1}^{\infty} (1-p')\pi_{k,0}}{\rho \sum_{k=1}^{\infty} (1-p'')\pi_{0,k}} = \frac{\sum_{k=1}^{\infty} \pi_{k,1}}{\sum_{k=1}^{\infty} \pi_{1,k}}. \quad (24)$$

If $\frac{\pi(k,0)}{\pi(0,k)} = \frac{g}{1-g} \forall k$, then we have $\frac{\sum p_k \pi(k,0)}{\sum p_k \pi(k,0)} = \frac{g}{1-g}$ for some constants p_k . Using the result, $1 - p'_k = p_k(1-g)$ and $1 - p''_k = p_k g$, we then have

$$\begin{aligned} \frac{\rho \sum_{k=1}^{\infty} (1-p')\pi_{k,0}}{\rho \sum_{k=1}^{\infty} (1-p'')\pi_{0,k}} &= \frac{\rho \sum_{k=1}^{\infty} p_k(1-g)\pi_{k,0}}{\rho \sum_{k=1}^{\infty} p_k g\pi_{0,k}} \\ &= \frac{1-g}{g} \frac{\sum_{k=1}^{\infty} p_k \pi_{k,0}}{\sum_{k=1}^{\infty} p_k \pi_{0,k}} \\ &= \frac{1-g}{g} \left(\frac{g}{1-g} \right). \end{aligned}$$

From RHS of equation (24) we have, for $g > 1/2$, $\frac{\sum_{k=1}^{\infty} \pi_{k,1}}{\sum_{k=1}^{\infty} \pi_{1,k}} > \frac{g}{1-g} > 1$, which would imply that $\frac{\rho \sum_{k=1}^{\infty} (1-p')\pi_{k,0}}{\rho \sum_{k=1}^{\infty} (1-p'')\pi_{0,k}} > 1$. Hence we would require that there exist at least some k such that $\frac{\pi_{k,0}}{\pi_{0,k}} > \frac{g}{1-g}$. Therefore, there is at least one state where customers deviate and join the longer queue. ■

PROOF of Proposition 6:

The proof to the first part of the proposition is similar to the Proof of Proposition 3 in the paper where higher fixed queue difference threshold strategies are ruled out. Consider any threshold strategy such that $T_0 \geq 1$. Note that when $T_0 \geq 1$ some interior states are recurrent. Writing the steady balance equations for states on the outer arm, we have,

$$\pi_{k,0}(\lambda + \mu) = \pi_{k-1,0}\lambda + \pi_{k+1,0}\mu + \pi_{k,1}\mu \quad \forall k \geq T_0 + 2.$$

$$\pi_{k,0}(\lambda + \mu) = \pi_{k-1,0}\lambda g + \pi_{k+1,0}\mu + \pi_{k,1}\mu \quad \forall 1 \leq k \leq T_0 + 1.$$

Adding the above equations for all values of k and writing a similar equation for all states $(0, n)$ for all n , we have the following equations below:

$$\rho(1-g) \sum_{k=1}^{T_0} \pi_{k,0} + \pi_{1,0} = \rho g \pi_{0,0} + \sum_{k=1}^{\infty} \pi_{k,1} \quad (25)$$

$$\rho g \sum_{k=1}^{T_0} \pi_{0,k} + \pi_{0,1} = \rho(1-g) \pi_{0,0} + \sum_{k=1}^{\infty} \pi_{1,k} \quad (26)$$

Hence we have $\frac{\sum_{k=1}^{T_0} \pi_{k,0}}{\sum_{k=1}^{T_0} \pi_{0,k}} > \frac{g}{1-g}$ for any $T_0 \geq 1$. There is some n such that for some $1 \leq n \leq T_0$ we have $\frac{\pi_{n,0}}{\pi_{0,n}} > \frac{g}{1-g}$. There is some state *within* the threshold at the outer arm, where the best response of an arriving customer would be to NOT follow the signal if all other customers were to follow the queue within the threshold.

For the second part of the proof, note that when $T_0 = 0$, all the states (n_1, n_2) where $n_1 > 0$ and $n_2 > 0$ are transient for $\lambda < \mu$. Only recurrent states are $(n_1, 0)$ and $(0, n_2)$ for any $n_1 \geq 0$ or $n_2 \geq 0$. When $T_0 = 0$ the actions at all recurrent states are identical to the strategy \mathcal{A}^0 . Due to the various values that T_k could take for any $k > 0$, the strategies could be different only at zero-probability (transient) states. This completes the proof. ■

PROOF of Proposition 8:

We consider the likelihood of two servers being in the market. Let $F \in \{0, 1, 2\}$ be the number of servers in the market. Let us examine the strategy \mathcal{A}^0 conditioning on the presence of two servers in the market. We have to examine $\pi(n_1, 0 | v_1 > v_2, F = 2)$ and $\pi(n_1, 0 | v_1 < v_2, F = 2)$. Also let $P(F)$ be the probability arriving customers see F servers in the market. Since customers arrive according to a Poisson process PASTA property (Wolff 1982) applies. Given the equilibrium \mathcal{A}^0 and the customer arrival process, we can determine $P(F = i | v_1 > v_2) = P(F = i | v_1 < v_2)$ provided τ is generated from $\Phi(\cdot)$ for all the servers in the market.

$$\begin{aligned} l(1, 0 | F = 2) &= \frac{\pi(1, 0 | v_1 > v_2, F = 2)}{\pi(1, 0)(v_2 > v_1, F = 2)} \\ &= \frac{\pi_1(1, 0 | F = 2)}{\pi_2(1, 0 | F = 2)} \\ &= \frac{\pi_1(1, 0)P(F = 2 | v_1 > v_2)}{\pi_2(1, 0)P(F = 2 | v_1 < v_2)} \\ &= \frac{g}{1-g} \end{aligned}$$

Similarly, conditional likelihood properties for states $l(k, 0 | F = 2) = \frac{g}{1-g} \forall k$ and $l(0, k | F = 2) = \frac{1-g}{g} \forall k$. This completes the proof that \mathcal{A}^0 is in equilibrium conditional on customers observing two servers in the market. ■

PROOF of **Proposition 9**: We aim to prove that \mathcal{A}^0 is an equilibrium strategy for multi-server queues with each service provider having N servers. As before, we calculate the long-run equilibrium probabilities under the strategy of always following the longer queue. The customer follows the signal at $(n, n) \forall n$. At other states, she follows the longer queue. It is evident that states $(n_1, 0)$ and $(0, n_2)$ with $n_1, n_2 \geq 0$ will be recurrent states.

Calculating the steady state probabilities when $v_1 > v_2$, we get

$$\begin{aligned}\pi_1(n_1, 0) &= \frac{g}{n_1!} \rho^{n_1} K \forall 0 < n_1 \leq N. \\ &= \frac{g}{N! N^{n_1-N}} \rho^{n_1} K \forall N < n_1 \\ \pi_1(0, n_2) &= \frac{1-g}{n_2!} \rho^{n_2} K \forall 0 < n_2 \leq N. \\ &= \frac{1-g}{N! N^{n_2-N}} \rho^{n_2} K \forall N < n_2 \\ \pi_1(0, 0) &= K \text{ where } K = \left[\sum_{k=0}^{N-1} \frac{(N\rho)^k}{k!} + \frac{(N\rho)^N}{N!} (1-\rho)^{-1} \right]^{-1}.\end{aligned}$$

Similar expressions exist for π_2 when $(v_2 > v_1)$. Considering likelihood ratios we again obtain:

$$l(n_1, n_2) = \begin{cases} \frac{g}{1-g} & n_1 \geq 1, n_2 = 0 \\ 1 & n_1 = 0, n_2 = 0 \\ \frac{1-g}{g} & n_1 = 0, n_2 \geq 1 \end{cases}$$

Following the signal at $n_1 = 0, n_2 = 0$ is an equilibrium action if $\frac{1-g}{g} \leq l(n_1, n_2) \leq \frac{g}{1-g}$, which is satisfied since $\frac{1}{2} < g < 1$.

Consider all recurrent states along the outer arms, $(n_1, 0) \ n_1 > 0$ we have $l(n_1, 0) = \frac{g}{1-g}$, where following the longer queue is consistent with the strategy \mathcal{A}^0 . Similarly, the condition of following the longest queue when $n_2 \geq 1, n_1 = 0$ is also weakly satisfied. Therefore \mathcal{A}^0 continues to exist as an equilibrium strategy even when the providers have N multiple servers each. ■

PROOF of **Proposition 13** Part (i): It is straightforward to show \mathcal{A}^0 is not an equilibrium strategy since $l_{k0} = \frac{\pi_1(k,0)}{\pi_2(k,0)} = \frac{g\rho_F^k \pi_1(0,0)}{(1-g)\rho_S^k \pi_2(0,0)} = \left(\frac{g}{1-g}\right) \frac{\rho_F^k}{\rho_S^k} < \frac{g}{1-g}$. Suppose the customers follow a strategy \mathcal{A}^B . Writing the steady state equations for the states when $v_1 > v_2$, we have the following equations.

For $k > 1$,

$$\begin{aligned}
(\lambda + \mu_1 + \mu_2)\pi_{k+i,k} &= \lambda g \pi_{k+i-1,k} + \lambda(1-g)\pi_{k+i,k-1} + \mu_1 \pi_{k+i+1,k} + \mu_2 \pi_{k+i,k+1} \quad \forall 0 < i \leq B-1 \\
(\lambda + \mu_1 + \mu_2)\pi_{k+i,k} &= \lambda g \pi_{k+i-1,k} + \mu_1 \pi_{k+i+1,k} + \mu_2 \pi_{k+i,k+1} \quad \text{for } i = B \\
(\lambda + \mu_1 + \mu_2)\pi_{k+i,k} &= \lambda g \pi_{k+i-1,k} + \mu_1 \pi_{k+i+1,k} + \mu_2 \pi_{k+i,k+1} \quad \text{for } i = B+1 \\
(\lambda + \mu_1 + \mu_2)\pi_{k+i,k} &= \lambda g \pi_{k+i-1,k} + \mu_1 \pi_{k+i+1,k} + \mu_2 \pi_{k+i,k+1} \quad \text{for } i > B+1 \\
(\lambda + \mu_1)\pi_{i,0} &= \lambda g \pi_{i-1,0} + \mu_1 \pi_{i+1,0} + \mu_2 \pi_{i,1} \quad \forall 0 < i \leq B+1 \\
(\lambda + \mu_1)\pi_{i,0} &= \lambda g \pi_{i-1,0} + \mu_1 \pi_{i+1,0} + \mu_2 \pi_{i,1} \quad \text{for } i > B+1
\end{aligned}$$

In the next steps, we sum up the probabilities of all the states along the i^{th} diagonal. As $\rho_2 \approx 1$ we have $\pi_{i,0} \rightarrow 0 \forall i$. As the service rates are high, the process behaves asymptotically as a birth and death process on both sides of the diagonal with each state being one of ‘diagonals’ (where i^{th} state is defined as sum of the states $\cup_k(k+i, k)$). For example, under \mathcal{A}^1 , the process moves from the main diagonal to the lower diagonal with birth rate $(g\lambda + \mu_2)$ and death rate μ_1 . Similarly, the process moves from the main diagonal to the upper adjacent diagonal with birth rate $(1-g)\lambda + \mu_1$ and death rate μ_2 . Therefore let us consider the asymptotic limiting expression for first upper and first lower diagonals.

$$\begin{aligned}
\sum_{k=0}^{\infty} \pi_{k+1,k} &= \frac{(g\lambda + \mu_2)}{\mu_1} \sum_{k=0}^{\infty} \pi_{k,k} \\
\sum_{k=0}^{\infty} \pi_{k,k+1} &= \frac{((1-g)\lambda + \mu_1)}{\mu_2} \sum_{k=0}^{\infty} \pi_{k,k}
\end{aligned}$$

We have,

$$\begin{aligned}
\frac{\sum_{k=0}^{\infty} \pi_{k+1,k}^1}{\sum_{k=0}^{\infty} \pi_{k+1,k}^2} &= \frac{\sum_{k=0}^{\infty} \pi_{k+1,k}^1}{\sum_{k=0}^{\infty} \pi_{k,k+1}^1} = \frac{(g\lambda + \mu_2)}{\mu_1} \frac{\mu_2}{((1-g)\lambda + \mu_1)} = \frac{\mu_2^2(g\rho_2 + 1)}{\mu_1^2((1-g)\rho_1 + 1)} \\
&= \frac{\rho_1^2(g\rho_2 + 1)}{\rho_2^2((1-g)\rho_1 + 1)} < \frac{\rho_1^2 g \rho_2}{\rho_2^2(1-g)\rho_1} \quad \text{since } g\rho_2 > (1-g)\rho_1 \text{ always.} \\
&= \frac{\rho_1 g}{\rho_2(1-g)} \\
&< \frac{g}{(1-g)}
\end{aligned}$$

Hence there is at least one state along $(k+1, k)$ such that $\frac{\pi_{k+1,k}^1}{\pi_{k+1,k}^2} < \frac{g}{1-g}$ where the customers’ best response is to follow their signal. So \mathcal{A}^1 cannot be in equilibrium. A similar argument can show that any strategy \mathcal{A}^B for finite $B > 1$ is not in equilibrium by considering the asymptotic steady state probabilities at the corresponding diagonal.

Part (ii) When $\rho_1 \approx 0$, we have $\rho_1 \ll \rho_2$. Then at all states $\pi(m, n) = 0 \forall m > 1$. Also, consider the likelihood ratios at $(1, 0)$.

$$\begin{aligned}l_{00} &= \frac{\pi_1(0, 0)}{\pi_2(0, 0)} = 1 \quad \text{Follow Signal} \\l_{10} &\approx \varepsilon < \frac{(1-g)}{g} \quad \text{Follow the shorter queue (or queue 2).} \\l_{0j} &> \frac{g}{1-g} \quad \text{Follow queue 1.}\end{aligned}$$

■

References

Wolff, R.W. 1982. Poisson Arrivals See Time Averages. *Operations Research*. 30(2) pp. 223–231.