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Abstract

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Approximation, Euclidean median, Haar measure, Group actions, Multivariate median, Projection, Stability

Disciplines

Business | Statistics and Probability

The Projection Median of a Set of Points in $\mathbb{R}^d$ *

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Abstract. The projection median of a finite set of points in $\mathbb{R}^2$ was introduced by Durocher and Kirkpatrick [*Computational Geometry: Theory and Applications*, Vol. 42 (5), 364–375, 2009]. They proved that the projection median in $\mathbb{R}^2$ provides a better approximation of the 2-dimensional Euclidean median, than the center of mass or the rectilinear median, while maintaining a fixed degree of stability. In this paper we study the projection median of a set of points in $\mathbb{R}^d$ for $d \geq 2$. Using results from the theory of integration over topological groups, we show that the d-dimensional projection median provides a $(d/\pi)B(d/2, 1/2)$-approximation to the d-dimensional Euclidean median, where $B(\alpha, \beta)$ denotes the Beta function. We also show that the stability of the d-dimensional projection median is at least $\frac{1}{(d/\pi)B(d/2,1/2)}$, and its breakdown point is 1/2. Based on the stability bound and the breakdown point, we compare the d-dimensional projection median with the rectilinear median and the center of mass, as a candidate for approximating the d-dimensional Euclidean median. For the special case of $d = 3$, our results imply that the 3-dimensional projection median is a (3/2)-approximation of the 3-dimensional Euclidean median, which settles a conjecture posed by Durocher.

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1 Introduction

A *median function* on $\mathbb{R}^d$ is a function from the set of all finite non-empty sets contained in $\mathbb{R}^d$ to $\mathbb{R}^d$. The median of a set S of n real numbers is a point $\mathcal{M}(S)$ which partitions the points in S such that there are at most $n/2$ points of S greater than $\mathcal{M}(S)$ and at most $n/2$ points of S that are less than $\mathcal{M}(S)$. Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of n distinct real numbers arranged in increasing order. If $n = 2m + 1$ is odd, the median of S is the point p_{m+1}, and when $n = 2m$ is even any point on the line segment joining p_m and p_{m+1} is a median of S. In such cases, the midpoint of the line segment joining p_m and p_{m+1} is often selected to represent $\mathcal{M}(S)$.

Several attempts have been made to generalize the notion of median to higher dimensions. Hayford [13] suggested the *vector-of-medians* of orthogonal coordinates. This involves selecting an orthogonal coordinate system and then computing the coordinate-wise univariate median along these axes. However, this definition of multivariate median depends on the choice of the orthogonal coordinate system. The vector-of-medians of a finite set of points S in $\mathbb{R}^d$ is to be denoted by $\mathcal{M}_R(S)$. It is easy to see that for a set S in $\mathbb{R}^d$ $\sum_{s \in S} |\boldsymbol{s} - \boldsymbol{x}|$ is minimized when $\boldsymbol{x} = \mathcal{M}_R(S)$, where $|.|$ denotes the ℓ_1 norm. For this reason, the vector-of-medians is also referred to as the *rectilinear median*. The rectilinear median is invariant under translation and uniform scaling, but not under rotation or reflection. If there are an even number of points in S, then $\mathcal{M}_R(S)$ may not be unique, and we select $\mathcal{M}_R(S)$ to be the midpoint of the d-dimensional rectangular region of points that define rectilinear medians of S. Since the one-dimensional median defined above can be computed in $O(n)$

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time, the d-dimensional rectilinear $\mathcal{M}_R(S)$ can be computed in $O(dn)$ time by computing d independent one dimensional medians.

Analogous to Hayford's definition, the Euclidean median of a set S in $\mathbb{R}^d$ (to be denoted by $\mathcal{M}_E(S)$) is defined as the point in $\mathbb{R}^d$ which minimizes $\sum_{s \in S} ||\boldsymbol{s} - \boldsymbol{x}||$, when $\boldsymbol{x} = \mathcal{M}_E(S)$ and where $||.||$ denotes the ℓ_2 norm. The Euclidean median problem on three points in the plane was first posed by Fermat and solved geometrically by Torricelli early in the 17-th century [14]. The problem was later revived by Weber [22] in 1909 in the context of optimal facility location. The Euclidean median is invariant under uniform scaling, reflection, translation, and rotation. This makes it much more suitable candidate for a multivariate median compared to the rectilinear median. However, solving for the exact location of the Euclidean median in two or more dimensions is, in general, difficult. Bajaj [3] showed that even for 5 points, the coordinates of the Euclidean median may not be representable even if we allow radicals, and that it is impossible to construct an optimal solution by means of ruler and compass. The most famous of all existing algorithms is the iterative algorithm due to Weiszfeld [24]. For a comprehensive discussion about the Euclidean median and its various properties refer to Kupitz and Martini [15].

A median function $\mathcal{M}$ is said to be a λ-*approximation* of the Euclidean median, if $\sum_{\boldsymbol{p} \in S} ||\boldsymbol{p} - \mathcal{M}(S)|| \leq \lambda \sum_{\boldsymbol{p} \in S} ||\boldsymbol{p} - \mathcal{M}_E(S)||$ for all nonempty finite sets S in $\mathbb{R}^d$. Recently, motivated from several problems in mobile facility location, Durocher and Kirkpatrick [9] introduced the notion of stability of a median function, which measures the behavior of the median function to slight perturbations of the data. Given $\varepsilon > 0$ and a finite set S of $\mathbb{R}^d$, a function $f : S \to \mathbb{R}^d$ is an ε-*perturbation* on S if for all $\boldsymbol{p} \in S$, $||\boldsymbol{p} - f(\boldsymbol{p})|| \leq \varepsilon$. Let $\mathcal{F}_\varepsilon(S)$ denote the set of all ε-perturbations on S. A median function $\mathcal{M}$ is said to be κ-*stable* if for all $\varepsilon > 0$ and for all $f \in \mathcal{F}_\varepsilon(S)$, $\kappa||\mathcal{M}(S) - \mathcal{M}(f(S))|| \leq \varepsilon$, for all nonempty finite sets S in $\mathbb{R}^d$.

Using a 4-point example, Durocher and Kirkpatrick [9] showed that the Euclidean median is not continuous even for small point sets, thus proving that the Euclidean median is not κ-stable for any $\kappa > 0$. They also showed that no median function can ensure any fixed degree of stability while also guaranteeing an arbitrarily-close approximation of the Euclidean median sum.

It is well known that the center of mass of a set S of n points in $\mathbb{R}^d$ is the point in $\mathbb{R}^d$ given by $\frac{1}{n} \sum_{\boldsymbol{p} \in S} \boldsymbol{p}$. The center of mass is invariant under affine transformations and it is the unique point that minimizes the sum of the squares of the distances to the points of S [23]. It follows from results of Bereg et al. [4] that the center of mass of a set of n points in $\mathbb{R}^d$ is 1-stable and provides a $(2 - 2/n)$-approximation of the d-dimensional Euclidean median, and both the bounds are tight.

Bereg et al. [4] also proved that the rectilinear median in $\mathbb{R}^2$ provides a $\sqrt{2}$-approximation of the Euclidean median. Later, Durocher [8] showed that the d-dimensional rectilinear median provides a $\sqrt{d}$-approximation of the Euclidean median, and proved a $\frac{1+\sqrt{d-1}}{\sqrt{d}}$ lower bound on the approximation factor, for $d \geq 1$. Generalizing the results of Bereg et al. [4], Durocher [8] also proved a tight stability bound of $(1/\sqrt{d})$ on the d-dimensional rectilinear median for any $d \geq 1$.

Another classical measure for comparing different median functions is the notion of *breakdown point* [7], which is defined as the proportion of points which must be moved to infinity so that the median function will do the same. It easy to see that in $\mathbb{R}^1$ for a set S of n points, the standard 1-dimensional median has a breakdown point of $1/2$ and the 1-dimensional mean has a breakdown point of $1/n$. This immediately implies that for any $d \geq 2$, the d-dimensional rectilinear median and the d-dimensional center of mass also has a breakdown point of $1/2$ and $1/n$, respectively. The breakdown point of the Euclidean median is known to be $1/2$ [17].

The main result of Durocher and Kirkpatrick [9] is the introduction of the notion of the projection median. Given a fixed positive integer $d \geq 2$ and a finite set of S points in in $\mathbb{R}^d$, the d-dimensional projection median of S is defined as

$$\mathcal{M}_P(S) = d\frac{\int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{u}})\mathrm{d}\boldsymbol{u}}{\int_{S^{d-1}} \mathrm{d}\boldsymbol{u}} = d\int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{u}})\mathrm{d}\mu(\boldsymbol{u}) \tag{1}$$

where $S^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : ||\boldsymbol{x}|| = 1\}$ is the unit d-dimensional hypersphere, $\mathsf{med}(S_{\boldsymbol{u}})$ is the median of the projection of S onto the line through the origin parallel to vector $\boldsymbol{u}$, and μ is the normalized uniform measure over S^{d-1}.

They show that in $\mathbb{R}^2$, the projection median is $(\pi/4)$-stable and it provides a $4/\pi$-approximation to the Euclidean median. This implies that the projection median in $\mathbb{R}^2$ maintains a fixed degree of stability while providing a better approximation of the 2-dimensional Euclidean median than the center of mass or the rectilinear median. They also showed that the stability bound is tight and a lower bound on the approximation factor is $\sqrt{4/\pi^2 + 1}$.

In this paper, we study the projection median of a set S of n points in $\mathbb{R}^d$. Using results from the theory of integration over topological groups, we show that the d-dimensional projection median provides a $J(d)$-approximation to the d-dimensional Euclidean median, where $J(d) = (d/\pi)B(d/2, 1/2)$ and $B(\alpha, \beta)$ denotes the Beta function. We also show that the d-dimensional projection median has a stability bound of $1/J(d)$, and that its breakdown point is $1/2$. Using these results, we compare the projection median, with the rectilinear median and the center of mass, as a candidate for approximating of the Euclidean median in $\mathbb{R}^d$. For the special case $d = 3$, our results imply that the 3-dimensional projection median is a $(3/2)$-approximation of the 3-dimensional Euclidean median, which settles a conjecture posed by Durocher [8]. We also characterize the locus of $\mathsf{med}(S_{\boldsymbol{u}})$ as $\boldsymbol{u}$ varies over S^{d-1} and show that its combinatorial complexity is same as the number of vertices in the median level of the arrangement of $(d-1)$-dimensional hyperplanes in the dual, corresponding to the points in S.

2 Topological Preliminaries

In this section we present the relevant results from the theory of integration over topological groups, which will give us the necessary mathematical machinery to deal with the projection median of a finite set of points in $\mathbb{R}^d$. A rotation ϑ is a isometry of $\mathbb{R}^d$, which keeps the origin and the orientation fixed. A rotation ϑ can be represented as a linear transformation $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A}$ is a $d \times d$ orthogonal matrix with determinant 1. The group of all rotations in $\mathbb{R}^d$ with the operation of composition is denoted by $\mathsf{SO}(d)$, which stands for the *special orthogonal group*. Algebraically, the group $\mathsf{SO}(d)$ is the set of all orthogonal matrices of order d with determinant 1, under matrix multiplication. With natural topology, obtained by regarding the matrices in $\mathsf{SO}(d)$ as points in $\mathbb{R}^{d^2}$, it is a compact group.

A Borel measure λ on a topological group $\mathcal{H}$, that is, a measure defined on Borel sets of $\mathcal{H}$, is said to be *left-invariant* if $\lambda(hH) = \lambda(H)$ for all $h \in \mathcal{H}$ and for all Borel subsets H of $\mathcal{H}$. Similarly, one can define a *right invariant* Borel measure on a toplogical group $\mathcal{H}$. A Borel measure on $\mathcal{H}$ is said to be *invariant* if it is both left and right invariant. Existence of invariant Borel measures and its uniqueness upto scalar multiplication can be proved, in general, for locally compact groups [18]. Now, since $\mathsf{SO}(d)$ is a compact group, it follows from Royden [Theorem 14.6.20, [18]] that there exists an unique Borel measure ν on $\mathsf{SO}(d)$, which is invariant under the action of the elements of $\mathsf{SO}(d)$, with $\nu(\mathsf{SO}(d)) = 1$. This measure ν is called the *normalized Haar measure* of $\mathsf{SO}(d)$ [18].

As mentioned before, S^{d-1} denotes the unit hypersphere in $\mathbb{R}^d$, that is, $S^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : ||\boldsymbol{x}|| = 1\}$. It is well-known that S^{d-1} is a compact and separable metric space, being a subspace of the separable metric space $\mathbb{R}^d$. For a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in S^{d-1}$, the d-dimensional spherical coordinates are given by

$$\begin{aligned} x_1 &= \cos(\phi_1) \\ x_2 &= \sin(\phi_1)\cos(\phi_2) \\ x_3 &= \sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\ &\vdots \\ x_{d-1} &= \sin(\phi_1)\ldots\sin(\phi_{d-2})\cos(\phi_{d-1}) \\ x_d &= \sin(\phi_1)\ldots\sin(\phi_{d-2})\sin(\phi_{d-1}) \end{aligned}$$

where each angle $\phi_1, \phi_2, \ldots \phi_{d-2}$ has a range of π and the angle ϕ_{d-1} has a range of 2π. Moreover, the volume element of the $(d-1)$-sphere is

$$\mathrm{d}_{S^{d-1}}V = \sin^{d-2}(\phi_1)\sin^{d-3}(\phi_2)\ldots\sin(\phi_{d-2})\mathrm{d}\phi_1\mathrm{d}\phi_2\ldots\mathrm{d}\phi_{d-1}.$$

Induced by the volume element of S^{d-1}, the *normalized uniform measure* μ, over S^{d-1} is given by

$$\begin{aligned} \mathrm{d}\mu &= \frac{\mathrm{d}_{S^{d-1}}V}{\int_0^\pi\int_0^\pi\ldots\int_0^{2\pi}\mathrm{d}_{S^{d-1}}V} \\ &= \frac{\sin^{d-2}(\phi_1)\sin^{d-3}(\phi_2)\ldots\sin(\phi_{d-2})\mathrm{d}\phi_1\mathrm{d}\phi_2\ldots\mathrm{d}\phi_{d-1}}{\int_0^\pi\sin^{d-2}(\phi_1)\mathrm{d}\phi_1\int_0^\pi\sin^{d-3}(\phi_2)\mathrm{d}\phi_2\ldots\int_0^\pi\sin(\phi_{d-2})\mathrm{d}\phi_{d-2}\int_0^{2\pi}\mathrm{d}\phi_{d-1}}. \end{aligned} \tag{2}$$

The group $\mathsf{SO}(d)$ has a natural action on S^{d-1} given by the map $\varphi : \mathsf{SO}(d) \times S^{d-1} \to S^{d-1}$, such that $\varphi(\boldsymbol{A}, \boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$, for $\boldsymbol{A} \in \mathsf{SO}(d)$ and $\boldsymbol{u} \in S^{d-1}$. For a fixed $\boldsymbol{u} \in S^{d-1}$, let us also define the map $\varphi_{\boldsymbol{u}}$ from $\mathsf{SO}(d)$ to S^{d-1} as follows: $\varphi_{\boldsymbol{u}}(\boldsymbol{A}) = \boldsymbol{A}\boldsymbol{u}$. The action of a fixed $\boldsymbol{A} \in \mathsf{SO}(d)$ is just a rotation of S^{d-1}, with the rotation matrix $\boldsymbol{A}$. It is clear that μ is invariant under this action of $\mathsf{SO}(d)$, that is, for any $\boldsymbol{A} \in \mathsf{SO}(d)$ and for any Borel subset B of S^{d-1} we have $\mu(B) = \mu(\boldsymbol{A}B)$.

A group action is *transitive* if it possesses only a single orbit. A group action of a topological group $\mathcal{H}$ on a topological space $\mathcal{X}$, $(h, x) \to hx$ is said to be a *proper* group action if the map $\psi_x : h \to hx$, where $h \in \mathcal{H}$, is proper for every fixed $x \in \mathcal{X}$, that is, $\psi_x^{-1}[K]$ is a compact subset of $\mathcal{H}$ for all compact subsets K of $\mathcal{X}$.

In the following two observations we prove that the action φ, defined above, is both transitive and proper.

Observation 1 *For $d \geq 2$, given any $\boldsymbol{x}, \boldsymbol{y} \in S^{d-1}$, there exists $\boldsymbol{A} \in \mathsf{SO}(d)$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$, that is, the action φ is transitive.*

Proof. Construct a set of orthonormal vectors $\boldsymbol{a}_1, ..., \boldsymbol{a}_d$ with $\boldsymbol{a}_1 = \boldsymbol{y}$. As $d \geq 2$, it is possible to change signs of $\boldsymbol{a}_2, \ldots, \boldsymbol{a}_d$, if necessary such that the matrix $\boldsymbol{A}_{\boldsymbol{y}} = [\boldsymbol{a}_1 : \ldots : \boldsymbol{a}_d]$ has determinant 1. Then $\boldsymbol{A}_{\boldsymbol{y}} \in \mathsf{SO}(d)$ and $\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{e}_1 = \boldsymbol{y}$. Similarly, construct a set of orthonormal vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d$ with $\boldsymbol{b}_1 = \boldsymbol{x}$, and assume that the matrix $\boldsymbol{A}_{\boldsymbol{x}} = [\boldsymbol{b}_1 : \ldots : \boldsymbol{b}_d]$ has determinant 1. Then $\boldsymbol{A}_{\boldsymbol{x}} \in \mathsf{SO}(d)$ and $\boldsymbol{A}_{\boldsymbol{x}}\boldsymbol{e}_1 = \boldsymbol{x}$. This implies that $\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{A}_{\boldsymbol{x}}^{-1} \in \mathsf{SO}(d)$ and $\boldsymbol{A}_{\boldsymbol{y}}\boldsymbol{A}_{\boldsymbol{x}}^{-1}\boldsymbol{x} = \boldsymbol{y}$, thus proving that the action φ is transitive. □

Observation 2 *For all $\boldsymbol{u} \in S^{d-1}, \varphi_{\boldsymbol{u}}^{-1}[K]$ is a compact subset of $\mathsf{SO}(d)$ for all compact subsets K of S^{d-1}, that is, the action φ is proper.*

Proof. It is clear that, for all $\boldsymbol{u} \in S^{d-1}$, the map $\varphi_{\boldsymbol{u}}$ is continuous and hence inverse images of closed sets are closed. The result now follows from noting that both $\mathsf{SO}(d)$ and S^{d-1} are compact, and a subset of either space is closed if and only if it is compact. □

From the above two observations it follows that $\mathsf{SO}(d)$, which is a compact group, acts naturally on the separable metric space S^{d-1}, and the natural action is both proper and transitive. Equipped with these facts, we now recall the following theorem from Royden (Proposition 14.6.25 [18]):

Theorem 1 (Royden [18]). *Let $\mathcal{H}$ be a compact group acting transitively and properly on a compact separable metric space $\mathcal{X}$. Then there exists an unique Borel measure μ' on $\mathcal{X}$, with $\mu'(\mathcal{X}) = 1$, which is invariant under the action of $\mathcal{H}$.* □

The following corollary is now immediate from above theorem and from Observations 1 and 2.

Corollary 1. *The normalized uniform measure μ over S^{d-1} (defined in Equation (2)) is the unique Borel probability measure on S^{d-1} which is invariant under the action of the elements of $\mathsf{SO}(d)$.* □

Before we proceed to prove the main topological result which will be used in the proofs of our subsequent results, we shall a few definitions and results from measure theory. Recall that if $\mathcal{F}$ and $\mathcal{F}_0$ are two sigma fields on Ω and Ω_0, respectively then a map $T : (\Omega, \mathcal{F}) \to (\Omega_0, \mathcal{F}_0)$ is called *measurable* if $T^{-1}(B_0) \in \mathcal{F}$ for all $B_0 \in \mathcal{F}_0$. With this definition we now state the following change of variable theorem from Ash and Doléans-Dade [Theorem 1.6.12, [2]]:

Theorem 2 (Ash and Doléans-Dade [2]). *Let $T : (\Omega, \mathcal{F}) \to (\Omega_0, \mathcal{F}_0)$ be a measurable map, and let λ be a measure defined on $(\Omega, \mathcal{F})$. Define a measure λ_0 on $(\Omega_0, \mathcal{F}_0)$ by $\lambda_0(X) = \lambda(T^{-1}(X))$, where $X \in \mathcal{F}_0$. If $f : \Omega_0 \to \mathbb{R}$ and $X \in \mathcal{F}_0$, then*

$$\int_{T^{-1}(X)} f(T(\omega))\mathrm{d}\lambda(\omega) = \int_X f(\omega)\mathrm{d}\lambda_0(\omega),$$

in the sense that if one of the integral exists, so does the other, and the two integrals are equal. □

With the help of the results stated above, we now prove a crucial result which will be used to obtain the bounds on the approximation factor of the d-dimensional projection median.

Result 1 *Consider the function $\psi : \mathsf{SO}(d) \to S^{d-1}$ given by $\psi(\boldsymbol{A}) = \boldsymbol{A}\boldsymbol{u}_1$, for $\boldsymbol{A} \in \mathsf{SO}(d)$ and for a fixed $\boldsymbol{u}_1 \in S^{d-1}$. If $f : S^{d-1} \to \mathbb{R}$ is a continuous function, then $\int_{\mathsf{SO}(d)} f(\psi(\boldsymbol{A}))\mathrm{d}\nu(\boldsymbol{A}) = \int_{S^{d-1}} f(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x})$.*

Proof. For any Borel subset B of S^{d-1}, let us define a measure ν^* as follows: $\nu^*(B) = \nu(\psi^{-1}(B))$. Clearly, $\nu^*(S^{d-1}) = \nu(\mathsf{SO}(d)) = 1$, that is, ν^* is normalized. As the function f is continuous and S^{d-1} is compact, the function f is bounded and $\int_{S^{d-1}} f(\boldsymbol{x})\mathrm{d}\nu^*(\boldsymbol{x})$ exists. Moreover, the continuity of f guarantees the measurability of f, and from Theorem 2 it follows that,

$$\int_{\mathsf{SO}(d)} f(\psi(\boldsymbol{A}))\mathrm{d}\nu(\boldsymbol{A}) = \int_{S^{d-1}} f(\boldsymbol{x})\mathrm{d}\nu^*(\boldsymbol{x}) \tag{3}$$

It is easy to check that ν^* is also invariant under action of $\mathsf{SO}(d)$, as $\nu^*(\boldsymbol{A}B) = \nu(\psi^{-1}(\boldsymbol{A}B)) = \nu(\boldsymbol{A}(\psi^{-1}(B))) = \nu(\psi^{-1}(B)) = \nu^*(B)$. By Corollary 1, it follows that ν^* and μ must agree on all the Borel sets of S^{d-1}, and the result follows. □

3 The Projection Median in $\mathbb{R}^d$

In this section we study the properties of the projection median of a set of points in $\mathbb{R}^d$. For a set of S of n points in $\mathbb{R}^d$, the projection median is defined as $\mathcal{M}_P(S) = d\int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{u}})\mathrm{d}\mu(\boldsymbol{u})$ where $\mathsf{med}(S_{\boldsymbol{u}})$ is the median of the projection of S onto the line through the origin in the direction of the vector $\boldsymbol{u}$, and μ is the normalized uniform measure over S^{d-1}. It is easy to see that $\mathcal{M}_P(S)$ is invariant under both rotation and translation of the underlying coordinate system.

Durocher [8] showed that in $\mathbb{R}^2$ the projection median and the rectilinear median satisfy the following identity: $\mathcal{M}_P(S) = \frac{2}{\pi}\int_0^{\pi/2} \mathcal{M}_\alpha(S)\mathrm{d}\alpha$, where $\mathcal{M}_\alpha(S)$ denotes the rectilinear median of S relative to a rotation of the reference axis by an angle α. In this section, using Result 1, we obtain the generalization of this result to higher dimensions, which also provides a reinterpretation of the d-dimensional projection median in terms of the d-dimensional rectilinear median.

3.1 Reinterpretation in Terms of the Rectilinear Median

Given a vector $\boldsymbol{x}$ or a matrix $\boldsymbol{M}$, we denote its transpose as $\boldsymbol{x}'$ or $\boldsymbol{M}'$, respectively. For two vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^d$, denote by $\langle \boldsymbol{x}_1, \boldsymbol{x}_2\rangle = \boldsymbol{x}_1'\boldsymbol{x}_2 = \boldsymbol{x}_2'\boldsymbol{x}_1$ the standard Euclidean inner product between the vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d$ be the canonical basis of $\mathbb{R}^d$, which also corresponds to the direction vectors along the d orthogonal coordinate axes of C. Let $C_{\boldsymbol{A}}$ denote the coordinate system obtained by rotating C by an orthogonal matrix $\boldsymbol{A} \in \mathsf{SO}(d)$. The coordinate axes of $C_{\boldsymbol{A}}$ are then given by $\boldsymbol{A}\boldsymbol{e}_1, \boldsymbol{A}\boldsymbol{e}_2, \ldots, \boldsymbol{A}\boldsymbol{e}_d$. Let $S_{\boldsymbol{A}} = \{\boldsymbol{A}'\boldsymbol{p}_1, \boldsymbol{A}'\boldsymbol{p}_2, \ldots, \boldsymbol{A}'\boldsymbol{p}_d\}$ be the points of S in $C_{\boldsymbol{A}}$. Consider the rectilinear median of the set of points $S_{\boldsymbol{A}}$ in $C_{\boldsymbol{A}}$. The coordinates of this point in C is to be denoted by $\mathcal{M}_R(S_{\boldsymbol{A}})$.

Now, we have the following simple observation:

Observation 3 *In the coordinate system C, $\mathcal{M}_R(S_{\boldsymbol{A}}) = \sum_{i=1}^d \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i})$.*

Proof. Let $Z = \{\langle \boldsymbol{p}_j, \boldsymbol{A}\boldsymbol{e}_i\rangle | \boldsymbol{p}_j \in S\} = \{\boldsymbol{p}_j'\boldsymbol{A}\boldsymbol{e}_i | \boldsymbol{p}_j \in S\}$. As $||\boldsymbol{A}\boldsymbol{e}_i|| = 1$, $\mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i})$ denotes the projection of S onto the line through the origin parallel to $\mathcal{A}\boldsymbol{e}_i$. By definition, $\mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i}) = \mathcal{M}(Z)\boldsymbol{A}\boldsymbol{e}_i$. Now, observe that $\mathcal{M}_R(S_{\boldsymbol{A}}) = \sum_{i=1}^d \mathcal{M}(Z)\boldsymbol{A}\boldsymbol{e}_i = \sum_{i=1}^d \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i})$. □

In the following lemma, using Result 1, we show that the d-dimensional projection median is equal to the d-dimensional rectilinear median integrated over the set of all rotations ($\mathsf{SO}(d)$) with respect to the Haar measure over $\mathsf{SO}(d)$.

Lemma 1. *For a finite set S of points in $\mathbb{R}^d$, $\mathcal{M}_P(S) = \int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}}) \mathrm{d}\nu(\boldsymbol{A})$.*

Proof. Observation 3 implies that $\int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}}) \mathrm{d}\nu(\boldsymbol{A}) = \sum_{i=1}^d \int_{\mathsf{SO}(d)} \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i}) \mathrm{d}\nu(\boldsymbol{A})$. Consider the map $\psi_i : \mathsf{SO}(d) \to S^{d-1}$ given by $\psi(\boldsymbol{A}) = \boldsymbol{A}\boldsymbol{e}_i$ and the function $f : S^{d-1} \to \mathbb{R}^d$ given by $f(\boldsymbol{x}) = \mathsf{med}(S_{\boldsymbol{x}})$, where $||\boldsymbol{x}|| = 1$. Therefore, by Result 1, we have $\int_{\mathsf{SO}(d)} \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i}) \mathrm{d}\nu(\boldsymbol{A}) = \int_{\mathsf{SO}(d)} f(\psi_i(\boldsymbol{A})) \mathrm{d}\nu(\boldsymbol{A}) = \int_{S^{d-1}} f(x) \mathrm{d}\mu(\boldsymbol{x}) = \int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{x}}) \mathrm{d}\mu(\boldsymbol{x})$, where μ is the normalized uniform measure over S^{d-1}. Therefore,

$$\int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}}) \mathrm{d}\nu(\boldsymbol{A}) = \sum_{i=1}^d \int_{\mathsf{SO}(d)} \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i}) \mathrm{d}\nu(\boldsymbol{A}) = d \int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{x}}) \mathrm{d}\mu(\boldsymbol{x}),$$

and the result now follows from Equation (1). □

The above lemma establishes that the projection median of a point set S is same integral of the rectilinear median of S over the set of all rotations with respect to the normalized Haar measure over $\mathsf{SO}(d)$. This representation helps us to extend the results of Durocher and Kirkpatrick [9] to higher dimensions.

However, before we proceed to extend these results to higher dimensions, the following remarks are in order:

Remark **1:** For the special case $d = 2$, the unit sphere $S^{d-1} = S^1$ is itself a topological group with the group operation being complex multiplication. As a result, S^1 can be identified with the rotation group $\mathsf{SO}(2)$ via the following bijective correspondence: $(\cos\theta, \sin\theta) \leftrightarrow \boldsymbol{A}_\theta$, where $\boldsymbol{A}_\theta$ denotes the rotation matrix corresponding to a counterclockwise rotation of angle θ. It is easy to see that this correspondence is a group isomorphism as well as a homeomorphism of topological spaces. Also note that the normalized Haar measure on S^1 is given by $\mathrm{d}\mu(\theta) = \frac{1}{2\pi}\mathrm{d}\theta$. Hence, it follows from Lemma 1 that

$$\mathcal{M}_P(S) = \int_{\mathsf{SO}(2)} \mathcal{M}_R(S_{\boldsymbol{A}_\theta}) \mathrm{d}\nu(A_\theta) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}_\theta(S) \mathrm{d}\theta = \frac{2}{\pi} \int_0^{\pi/2} \mathcal{M}_\theta(S) \mathrm{d}\theta. \tag{4}$$

Note that the last equality follows from the observation that for any angle θ, $\mathcal{M}_\theta(S) = \mathsf{med}(S_\theta) + \mathsf{med}(S_{\theta+\pi/2})$, where $\mathsf{med}(S_\theta)$ is the median of the projection of S on the line $y = \tan\theta x$. This implies that for any angle θ, $\mathcal{M}_\theta(S) = \mathcal{M}_{-\theta}(S)$, and $\mathcal{M}_{\theta-\pi/2}(S) = \mathcal{M}_\theta(S)$ and the last step of Equation (4) follows. Equation (4) was used by Durocher and Kirkpatrick [9] to obtain their results on approximation factor of projection median of a set of points in $\mathbb{R}^2$. Lemma 1 is a natural generalization of that result.

Remark **2:** Observe that for any fixed $\boldsymbol{u} \in S^{d-1}$, $\mathsf{med}(S_{\boldsymbol{u}})$ has breakdown point greater than or equal to $1/2$. This means that if less than $1/2$ fraction of the points of a set S are moved to infinity, $\mathsf{med}(S_{\boldsymbol{u}})$ is finite, for every $\boldsymbol{u} \in S^{d-1}$. Moreover, it can be shown that when α ($< 1/2$) fraction of the points are moved to infinity, $\mathsf{med}(S_{\boldsymbol{u}})$ is continuous in $\boldsymbol{u}$ almost everywhere, and it is uniformly bounded. This implies that $\mathcal{M}_P(S) = \int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{u}}) \mathrm{d}\boldsymbol{u}$ is finite, when α ($< 1/2$) fraction of the points are moved to infinity. Hence, the breakdown point of the projection median of a set of points n points in $\mathbb{R}^d$ is at least $1/2$. To show that it is indeed $1/2$, we consider a set S of n points in $\mathbb{R}^d$ along the vector $\boldsymbol{e}_1$. If more than half of the points of S are moved to infinity along $\boldsymbol{e}_1$, then $\mathsf{med}(S_{\boldsymbol{u}}) = \infty$ on the set $\boldsymbol{u} \in Z = S^{d-1} \backslash \{\boldsymbol{e}_1^\perp\}$, where $\boldsymbol{e}_1^\perp$ is the $(d-1)$-dimensional hyperplane passing through the origin orthogonal to the unit vector $\boldsymbol{e}_1$. As the set Z has positive measure, $\mathcal{M}_P(S) = \int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{u}}) \mathrm{d}\boldsymbol{u}$ is infinite.

4 Approximation Factor

Equipped with the results of the previous section, we now proceed to determine an upper bound on the approximation factor of the projection median with respect to the Euclidean median in $\mathbb{R}^d$. The approximation factor is given by the ratio:

$$\lambda(d) = \frac{\sum_{i=1}^{n} ||\mathcal{M}_P(S) - \boldsymbol{p}_i||}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \tag{5}$$

Using Lemma 1, we now write the above ratio as:

$$\begin{aligned} \lambda(d) &= \frac{\sum_{i=1}^{n} || \int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}}) \mathrm{d}\nu(\boldsymbol{A}) - \boldsymbol{p}_i||}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &= \frac{\sum_{i=1}^{n} || \int_{\mathsf{SO}(d)} (\mathcal{M}_R(S_{\boldsymbol{A}}) - \boldsymbol{p}_i) \mathrm{d}\nu(\boldsymbol{A})||}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &\leq \frac{\sum_{i=1}^{n} \int_{\mathsf{SO}(d)} ||\mathcal{M}_R(S_{\boldsymbol{A}}) - \boldsymbol{p}_i|| \mathrm{d}\nu(\boldsymbol{A})}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \end{aligned} \tag{6}$$

where the last step follows from Minkowski's Integral Inequality [11]. Now, let $\boldsymbol{u}_i = \frac{\mathcal{M}_E(S) - \boldsymbol{p}_i}{||\mathcal{M}_E(S) - \boldsymbol{p}_i||}$ and observe that for all $\boldsymbol{A}$ and $\boldsymbol{x}$, $||\boldsymbol{x}|| \leq |\boldsymbol{x}|_{\boldsymbol{A}}$, where $|\boldsymbol{x}|_{\boldsymbol{A}}$ is the ℓ_1 norm of $\boldsymbol{x}$ in the coordinate system $C_{\boldsymbol{A}}$. This implies that

$$\begin{aligned} \lambda(d) &\leq \frac{\sum_{i=1}^{n} \int_{\mathsf{SO}(d)} |\mathcal{M}_R(S_{\boldsymbol{A}}) - \boldsymbol{p}_i|_{\boldsymbol{A}} \mathrm{d}\nu(\boldsymbol{A})}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &\leq \frac{\sum_{i=1}^{n} \int_{\mathsf{SO}(d)} |\mathcal{M}_E(S) - \boldsymbol{p}_i|_{\boldsymbol{A}} \mathrm{d}\nu(\boldsymbol{A})}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &= \frac{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i|| \int_{\mathsf{SO}(d)} |\boldsymbol{u}_i|_{\boldsymbol{A}} \mathrm{d}\nu(\boldsymbol{A})}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &= \frac{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i|| \int_{\mathsf{SO}(d)} \sum_{j=1}^{d} |\langle \boldsymbol{u}_i, \boldsymbol{A}\boldsymbol{e}_j \rangle| \mathrm{d}\nu(\boldsymbol{A})}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &= \frac{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i|| \sum_{j=1}^{\mathrm{d}} \int_{\mathsf{SO}(d)} |(\boldsymbol{A}'\boldsymbol{u}_i)'\boldsymbol{e}_j| \mathrm{d}\nu(\boldsymbol{A})}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||}. \end{aligned} \tag{7}$$

We can simplify Equation (7) using Result 1 as follows:

Observation 4 $\lambda(d) \leq d \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_1| \mathrm{d}\mu(\boldsymbol{x})$.

Proof. Consider the function $\psi_i : \mathsf{SO}(d) \to S^{d-1}$ given by $\psi_i(\boldsymbol{A}) = \boldsymbol{A}'\boldsymbol{u}_i$, for $i \in \{1, 2, \ldots, n\}$. Let for $j \in \{1, 2, \ldots, d\}$, $f_j : S^{d-1} \to \mathbb{R}$ be defined as $f_j(\boldsymbol{x}) = |\boldsymbol{x}'\boldsymbol{e}_j|$, where $||\boldsymbol{x}|| = 1$. Then from Result 1, $\int_{\mathsf{SO}(d)} |(\boldsymbol{A}'\boldsymbol{u}_i)'\boldsymbol{e}_j| \mathrm{d}\nu(\boldsymbol{A}) = \int_{\mathsf{SO}(d)} f_j(\psi_i(\boldsymbol{A})) \mathrm{d}\nu(\boldsymbol{A}) = \int_{S^{d-1}} f_j(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}) = \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_j| \mathrm{d}\mu(\boldsymbol{x})$. Since the integral are taken over all the units vectors in S^{d-1}, it is easy to see that for all $j, k \in \{1, 2, \ldots, d\}$ we have, $\int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_j| \mathrm{d}\mu(\boldsymbol{x}) = \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_k| \mathrm{d}\mu(\boldsymbol{x})$. The proof now follows from Equation (7). □

We shall now prove the main result of this paper, where we determine a upper bound on $\lambda(d)$.

Theorem 3. *For any $d \geq 2$, the d-dimensional projection median provides a $J(d)$-approximation of the d-dimensional Euclidean median, where $J(d) = (d/\pi)B(d/2, 1/2)$.*

Proof. It follows from Observation 4 that $\lambda(d) \leq d \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_1| \mathrm{d}\mu(\boldsymbol{x})$. For a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)' \in S^{d-1}$, let $\phi_1, \phi_2, \ldots, \phi_{d-1}$ denote the angles, corresponding to the d-dimensional spherical coordinates of $\boldsymbol{x}$. Then from Equation (2) it follows that $\boldsymbol{x}'\boldsymbol{e}_1 = x_1 = \cos\phi_1$. Using the definition of the normalized uniform measure μ over S^{d-1}, as given in Equation (2), we obtain

$$
\begin{aligned}
\lambda(d) &\leq d \frac{\int_0^{\pi}\int_0^{\pi}\cdots\int_0^{2\pi} |\cos(\phi_1)| \mathrm{d}_{S^{d-1}}V}{\int_0^{\pi}\int_0^{\pi}\cdots\int_0^{2\pi} \mathrm{d}_{S^{d-1}}V} \\
&= d \frac{\int_0^{\pi} |\cos(\phi_1)| \sin^{d-2}(\phi_1)\mathrm{d}\phi_1 \int_0^{\pi} \sin^{d-3}(\phi_2)\mathrm{d}\phi_2 \ldots \int_0^{\pi} \sin(\phi_{d-2})\mathrm{d}\phi_{d-2} \int_0^{2\pi} \mathrm{d}\phi_{d-1}}{\int_0^{\pi} \sin^{d-2}(\phi_1)\mathrm{d}\phi_1 \int_0^{\pi} \sin^{d-3}(\phi_2)\mathrm{d}\phi_2 \ldots \int_0^{\pi} \sin(\phi_{d-2})\mathrm{d}\phi_{d-2} \int_0^{2\pi} \mathrm{d}\phi_{d-1}} \\
&= d \frac{\int_0^{\pi} |\cos(\phi_1)| \sin^{d-2}(\phi_1)\mathrm{d}\phi_1}{\int_0^{\pi} \sin^{d-2}(\phi_1)\mathrm{d}\phi_1}. \\
&= d \frac{\int_0^{\pi/2} \cos(\phi_1) \sin^{d-2}(\phi_1)\mathrm{d}\phi_1 - \int_{\pi/2}^{\pi} \cos(\phi_1) \sin^{d-2}(\phi_1)\mathrm{d}\phi_1}{2\int_0^{\pi/2} \sin^{d-2}(\phi_1)\mathrm{d}\phi_1}. \qquad (8)
\end{aligned}
$$

Substituting $\phi_1 = \varphi_1 - \pi/2$ in the second integral in the numerator of Equation (8), and using the fact that for any two reals $a, b > -1$, $\int_0^{\pi/2} \sin^a z \cos^b z dz = \frac{1}{2} \cdot B(\frac{a+1}{2}, \frac{b+1}{2})$ [12], we get

$$
\begin{aligned}
\lambda(d) &\leq d \frac{\int_0^{\pi/2} \cos(\phi_1) \sin^{d-2}(\phi_1)\mathrm{d}\phi_1 + \int_0^{\pi/2} \sin(\varphi_1) \cos^{d-2}(\varphi_1) d\varphi_1}{2\int_0^{\pi/2} \sin^{d-2}(\phi_1)\mathrm{d}\phi_1} \\
&= d \frac{B(1, \frac{d-1}{2})}{B(\frac{1}{2}, \frac{d-1}{2})} \\
&= \frac{d}{\pi} B(d/2, 1/2) = J(d). \qquad (9)
\end{aligned}
$$

The last equality follows from the fact that $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Gamma(1/2) = \sqrt{\pi}$, where $\Gamma(\alpha)$ denotes the Gamma function. □

Now, we make a series of remarks where some of the immediate consequences of the above theorem are discussed.

Remark **3:** Since d is an integer, using standard formulae of Beta functions [12], it is possible to express $J(d)$ explicitly as follows

$$
J(d) = \begin{cases} \frac{d(d-2)(d-4)\ldots 4.2}{(d-1)(d-3)\ldots 5.3} \cdot \frac{2}{\pi}, & \text{if } d \text{ is even;} \\ \frac{d(d-2)(d-4)\ldots 3.1}{(d-1)(d-3)\ldots 4.2}, & \text{if } d \text{ is odd.} \end{cases} \qquad (10)
$$

Remark **4:** For the special case $d = 3$, $J(d) = 3/2$, which implies that in $\mathbb{R}^3$ the projection median gives a $(3/2)$-approximation of the Euclidean median. This proves a conjecture posed by Durocher [8].

Remark **5:** It is known that the rectilinear median is a $\sqrt{d}$-approximation of the Euclidean median. In Remark 6 it will shown that $\lambda(d) \leq J(d) < \sqrt{d}$. Now, if $\sqrt{d}$ was a tight bound on the approximation factor of the d-dimensional rectilinear median, then this would imply that the worst-case approximation factor of the projection median is better than the worst-case approximation factor of the rectilinear median. Unfortunately, since the tight bound on the approximation factor of the d-dimensional rectilinear median is not known for $d \geq 3$, such a conclusion cannot be drawn. However, it follows from Equation (6) that

$$\begin{aligned}
\lambda(d) &\leq \frac{\sum_{i=1}^{n} \int_{\mathsf{SO}(d)} ||\mathcal{M}_R(S_{\boldsymbol{A}}) - \boldsymbol{p}_i|| \mathrm{d}\nu(\boldsymbol{A})}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\
&= \int_{\mathsf{SO}(d)} \frac{\sum_{i=1}^{n} ||\mathcal{M}_R(S_{\boldsymbol{A}}) - \boldsymbol{p}_i|| \mathrm{d}\nu(\boldsymbol{A})}{\sum_{i=1}^{n} ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\
&= \int_{\mathsf{SO}(d)} \lambda_R(S_{\boldsymbol{A}}) \mathrm{d}\nu(\boldsymbol{A}) \qquad (11)
\end{aligned}$$

where $\lambda_R(S_{\boldsymbol{A}})$ is the approximation factor of the rectilinear median of S in the coordinate system $C_{\boldsymbol{A}}$. Note that $\int_{\mathsf{SO}(d)} \lambda_R(S_{\boldsymbol{A}}) \mathrm{d}\nu(\boldsymbol{A})$ can be interpreted as the average approximation factor of the rectilinear median of a set S in $\mathbb{R}^d$, with the average being taken over all rotations of the coordinate system C. Since the approximation factor $\lambda(d)$ is invariant under rotation, Equation (11) implies that for every point set S, the average approximation factor of the projection median is not worse than the average approximation factor of the rectilinear median.

5 Stability

In 2D, Durocher and Kirkpatrick [9] showed the tight bound on the stability of the projection median is the reciprocal of the upper bound on the approximating factor. In this section we generalize this result to $\mathbb{R}^d$, $d \geq 3$. The proof follows by generalizing the techniques of Durocher [8] to higher dimensions, which finally involves integration of functions over the volume element of the $(d-1)$-dimensional sphere.

Theorem 4. *For any $d \geq 2$, the d-dimensional projection median is $1/J(d)$-stable.* □

Proof. Choose any nonempty and finite set U in $\mathbb{R}^d$. Let $f : U \to \mathbb{R}^d$ be any ε-perturbation of U, and let the set $V = f(U)$. Since the projection median is invariant under rotation and translation, without loss of generality assume that $\mathcal{M}_P(U)$ and $\mathcal{M}_P(V)$ have identical first $(d-1)$ coordinates. Let $U_{\boldsymbol{x}}$ and $V_{\boldsymbol{x}}$ be the projection of U and V onto the line through the origin parallel to the unit vector $\boldsymbol{x}$, respectively. Now, for any function f which is an ε-perturbation of S, and for any $\boldsymbol{x}$ such that $||\boldsymbol{x}|| = 1$, we have,

$$\begin{aligned}
|\mathsf{med}(U_{\boldsymbol{x}})' \boldsymbol{e}_d - \mathsf{med}(V_{\boldsymbol{x}})' \boldsymbol{e}_d| &= \left| \frac{\mathsf{med}(U_{\boldsymbol{x}})' - \mathsf{med}(V_{\boldsymbol{x}})'}{||\mathsf{med}(U_{\boldsymbol{x}}) - \mathsf{med}(V_{\boldsymbol{x}})||} \boldsymbol{e}_d \right| ||\mathsf{med}(U_{\boldsymbol{x}}) - \mathsf{med}(V_{\boldsymbol{x}})|| \\
&= |\boldsymbol{s}' \boldsymbol{e}_d| \cdot ||\mathsf{med}(U_{\boldsymbol{x}}) - \mathsf{med}(V_{\boldsymbol{x}})|| \qquad (12)
\end{aligned}$$

where $\boldsymbol{s} = \frac{\mathsf{med}(U_{\boldsymbol{x}}) - \mathsf{med}(V_{\boldsymbol{x}})}{||\mathsf{med}(U_{\boldsymbol{x}}) - \mathsf{med}(V_{\boldsymbol{x}})||}$. Observe that both $\mathsf{med}(U_{\boldsymbol{x}})$ and $\mathsf{med}(V_{\boldsymbol{x}})$ are in the direction of the unit vector $\boldsymbol{x}$. This implies that the normalized vector $\boldsymbol{s}$ is either the vector $\boldsymbol{x}$, or the vector $-\boldsymbol{x}$. Therefore, $|\boldsymbol{s}'\boldsymbol{e}_d| = |\boldsymbol{x}'\boldsymbol{e}_d|$. Now, since the one-dimensional median is 1-stable, Equation (12) can be written as follows:

$$|\mathsf{med}(U_{\boldsymbol{x}})'\boldsymbol{e}_d - \mathsf{med}(V_{\boldsymbol{x}})'\boldsymbol{e}_d| \leq |\boldsymbol{x}'\boldsymbol{e}_d| \left(\max_{\boldsymbol{p}\in S} ||\boldsymbol{p} - f(\boldsymbol{p})||\right) \leq \varepsilon|\boldsymbol{x}'\boldsymbol{e}_d|. \tag{13}$$

Therefore, we have

$$\begin{aligned} ||\mathcal{M}_P(U) - \mathcal{M}_P(V)|| &= |\mathcal{M}_P(U)'\boldsymbol{e}_d - \mathcal{M}_P(V)'\boldsymbol{e}_d| \\ &= \left| d \int_{S^{d-1}} \left(\mathsf{med}(U_{\boldsymbol{x}})' - \mathsf{med}(V_{\boldsymbol{x}})'\right) \boldsymbol{e}_d \mathrm{d}\mu(\boldsymbol{x}) \right| \\ &\leq d \int_{S^{d-1}} |\mathsf{med}(U_{\boldsymbol{x}})'\boldsymbol{e}_d - \mathsf{med}(V_{\boldsymbol{x}})'\boldsymbol{e}_d| \mathrm{d}\mu(\boldsymbol{x}) \\ &\leq \varepsilon \cdot \left(d \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_d| \mathrm{d}\mu(\boldsymbol{x}) \right) \\ &= \varepsilon \cdot \frac{d}{\pi} B(d/2, 1/2) = \varepsilon \cdot J(d). \end{aligned} \tag{14}$$

Therefore, for all non-empty finite subsets S of $\mathbb{R}^d$ and for all $f \in \mathcal{F}_\varepsilon(S)$, with $\varepsilon > 0$, we have $\frac{1}{J(d)} \cdot ||\mathcal{M}_P(S) - \mathcal{M}_P(f(S))|| \leq \varepsilon$. This proves that the d-dimensional projection median is $1/J(d)$-stable. □

Remark **6:** Observe that for every fixed $d \geq 2$,

$$J(d) = d \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_1| \mathrm{d}\mu(\boldsymbol{x}) = \sum_{i=1}^{d} \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_i| \mathrm{d}\mu(\boldsymbol{x}) = \int_{S^{d-1}} \sum_{i=1}^{d} |\boldsymbol{x}'\boldsymbol{e}_i| \mathrm{d}\mu(\boldsymbol{x}).$$

Now, by the Cauchy-Schwartz inequality and the fact that $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)' \in S^{d-1}$ it follows that

$$\sum_{i=1}^{d} |\boldsymbol{x}'\boldsymbol{e}_i| \leq \sqrt{d \sum_{i=1}^{d} |\boldsymbol{x}'\boldsymbol{e}_i|^2} = \sqrt{d \sum_{i=1}^{d} |x_i|^2} = \sqrt{d}.$$

Equality holds in the above equation if and only if $\boldsymbol{x}'\boldsymbol{e}_i = \boldsymbol{x}'\boldsymbol{e}_j$ for all i, j, that is, if and only if $\boldsymbol{x} = \frac{1}{\sqrt{d}} \cdot \mathbf{1} = \frac{1}{\sqrt{d}} \cdot (1, 1, \ldots, 1)'$. Therefore, for all $\boldsymbol{x} \in S^{d-1} \backslash \{\frac{1}{\sqrt{d}} \cdot \mathbf{1}\}$, $J(d) < \sqrt{d}$. This implies that for all $d \geq 2$, we have $J(d) = \int_{S^{d-1}} \sum_{i=1}^{d} |\boldsymbol{x}'\boldsymbol{e}_i| \mathrm{d}\mu(\boldsymbol{x}) < \sqrt{d}$. Since the stability bounds on the rectilinear median is tight in all the dimensions, the worst-case stability of the projection median is better than the worst-case stability of the rectilinear median.

6 Comparing the Different Median Functions

In this section, we compare the approximation factor, stability, and breakdown point of the d-dimensional Euclidean median ($\mathcal{M}_E$), with those of the d-dimensional rectilinear median ($\mathcal{M}_R$) and center of mass ($\mathcal{M}_C$). The bounds on these three quantities for the different median functions are summarized in Table 1.

As mentioned earlier, the Euclidean median $\mathcal{M}_E$, is arbitrarily unstable, that is, it has 0 stability. Ensuring any degree of stability in a median function implies an increase in the Euclidean median sum and thus necessitates approximation by some other median function.

This requirement leads to the concept of the projection median, which was introduced by Durocher and Kirkpatrick [9]. In $\mathbb{R}^2$, it provides a better approximation of the 2-dimensional Euclidean median than the center of mass or the rectilinear median, while maintaining a fixed degree of stability.

In this paper we obtain new bounds on the approximation factor and the stability of the d-dimensional projection median and generalize earlier results to higher dimensions. We have shown that for every $d \geq 2$, the projection median $\mathcal{M}_P(S)$ provides an approximation of the Euclidean median $\mathcal{M}_E(S)$, which is on the average no worse than the approximation of $\mathcal{M}_E(S)$ by the rectilinear median $\mathcal{M}_R(S)$, where the average is taken over the set of all the rotations of $\mathbb{R}^d$ (Remark 5). We also show that the worst-case stability of the projection median is better than the worst-case stability of the rectilinear median (Remark 6). Moreover, the projection median, unlike the rectilinear median, is invariant under rotation of the coordinate system. Therefore, for approximating the Euclidean median, $\mathcal{M}_P(S)$ is clearly a better candidate than $\mathcal{M}_R(S)$ not only in 2D, but in higher dimensions as well.

The center of mass is a 2-approximation of the Euclidean median and it has stability 1 for all dimensions. It is easy to see that for a dimension less than 6, the approximation factor of the projection median $J(d) = \frac{d}{\pi}B(d/2, 1/2)$ is less than 2. This implies that the worst-case approximation factor of the projection median is better than that of the center of mass for dimensions less than 6. Moreover, the projection median has a constant breakdown point of 1/2. Thus, the projection median is a more suitable candidate for approximating the Euclidean median, than the center of mass, for dimensions $d \leq 5$.

For dimensions greater than 5, we have $J(d) > 2$. Moreover, both the approximation factor and the stability of the projection median worsen with increase in the dimension. On the other hand, the center of mass admits a fixed approximation factor and stability of 2 and 1 across all dimensions, respectively. However, the center of mass is highly non-robust as its breakdown point approaches 0, when the size of the given point set increases. Therefore, there is a clear trade-off between approximation factor and the value of the breakdown point for these two median functions. Note that stability of an estimator determines the change in the location of a median function subject to slight perturbation of the data, and has relevance in problems of mobile facility location. On the other hand, breakdown point determines whether the location of the median function changes dramatically by a significant change of some of the data points, that is, the breakdown point determines the sensitivity of a median function towards outliers. Therefore, for a point set with few outliers, the center of mass may be preferred over the projection median, for approximating the Euclidean median and maintaining a fixed degree of stability. On the other hand, for a point set with many outliers, the projection median is the preferred choice of the median function.

7 Geometric Interpretation of the Projection Median

So far we have studied the various properties of the projection median in $\mathbb{R}^d$ and compared them with those of other standard median functions. In this section, we try to provide a geometric interpretation of the projection median in $\mathbb{R}^d$ by characterizing the locus of the median of the projections on a unit vector, as it rotates over the unit sphere. More formally, given a set $S = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$ of n points in $\mathbb{R}^d$ in a d-dimensional orthogonal coordinate system C, we characterize the set $\lambda(S) = \{\mathsf{med}(S_{\boldsymbol{u}}) : \boldsymbol{u} \in S^{d-1}\}$.

Let $\boldsymbol{o}$ be the origin of the coordinate system C. For the sake of simplicity, assume that $n = 2m + 1$ is odd and the points in S are in general position, that is, no $(d + 1)$ points of S lie on a d-dimensional hyperplane. For any point $\boldsymbol{q} \in \mathbb{R}^d$, let $\boldsymbol{u}(\boldsymbol{q})$ be projection of the

Table 1. Comparing median functions in $\mathbb{R}^d$

Median Function	Notation	Approximation	Stability	Breakdown Point
Euclidean Median	$\mathcal{M}_E$	1	0	1/2
Rectilinear Median	$\mathcal{M}_R$	$\left[\frac{1+\sqrt{d-1}}{\sqrt{d}}, \sqrt{d}\right]$	$1/\sqrt{d}$	1/2
Center of Mass	$\mathcal{M}_C$	$2-\frac{2}{n}$	1	$1/n$
Projection Median	$\mathcal{M}_P$	$\left[\sqrt{4/\pi^2+1}, \frac{d}{\pi}B(d/2,1/2)\right]$	$\left[\frac{1}{(d/\pi)B(d/2,1/2)}, 1\right]$	1/2

vector $\boldsymbol{q}$ on the unit vector $\boldsymbol{u} \in S^{d-1}$. This implies that $\boldsymbol{u}(\boldsymbol{q}) = \langle \boldsymbol{q}, \boldsymbol{u} \rangle \boldsymbol{u} = (\boldsymbol{q}' \boldsymbol{u})\boldsymbol{u}$. It is easy to see that the locus of $\boldsymbol{u}(\boldsymbol{q})$, as $\boldsymbol{u}$ varies over S^{d-1}, is the $(d-1)$-sphere $S(\boldsymbol{q})$ with the line segment $[\boldsymbol{o}, \boldsymbol{q}]$ as the diameter. We call the $(d-1)$-sphere $S(\boldsymbol{q})$ the *projection sphere* of point $\boldsymbol{q}$. The projection circle (1-sphere) for a point $\boldsymbol{q} \in \mathbb{R}^2$ is shown in Figure 1(a).

Corresponding to the n points in S, consider the arrangement $\mathcal{A}(S)$ of the projection spheres $\{S(\boldsymbol{p}_1), S(\boldsymbol{p}_2) \ldots, S(\boldsymbol{p}_n)\}$ in $\mathbb{R}^d$. Clearly, for every $\boldsymbol{u} \in S^{d-1}$, $\mathsf{med}(S_{\boldsymbol{u}}) = \mathcal{M}_R(\{\boldsymbol{u}(\boldsymbol{p}_i) : \boldsymbol{p}_i \in S\})$. For a fixed $\boldsymbol{u} \in S^{d-1}$, suppose the extended half-line emanating from the origin $\boldsymbol{o}$ in the direction of $\boldsymbol{u}$ intersects the projection spheres in $\mathcal{A}(S)$ in the following order: $S(\boldsymbol{p}_{\pi_{\boldsymbol{u}}(1)}), S(\boldsymbol{p}_{\pi_{\boldsymbol{u}}(2)}), \ldots, S(\boldsymbol{p}_{\pi_{\boldsymbol{u}}(n)})$, where $\pi_{\boldsymbol{u}}$ is some permutation of $\{1, 2, \ldots, n\}$ which depends on the vector $\boldsymbol{u}$. This implies that $\mathsf{med}(S_{\boldsymbol{u}}) = \mathcal{M}_R(\{\boldsymbol{u}(\boldsymbol{p}_i) : \boldsymbol{p}_i \in S\}) = \boldsymbol{u}(\boldsymbol{p}_{\pi_{\boldsymbol{u}}(m+1)})$. Then for the vector $\boldsymbol{u} \in S^{d-1}$, the point $\boldsymbol{p}_{\pi_{\boldsymbol{u}}(m+1)}$ will be called the *parent* of $\mathsf{med}(S_{\boldsymbol{u}})$, and the projection sphere $S(\boldsymbol{p}_{\pi_{\boldsymbol{u}}(m+1)})$ will be called the *parent sphere* of $\mathsf{med}(S_{\boldsymbol{u}})$. This means that for every fixed $\boldsymbol{u} \in S^{d-1}$, the point $\mathsf{med}(S_{\boldsymbol{u}})$ lies on the boundary of some projection sphere, and is in the interior m projection spheres and the exterior of the remaining m projection spheres. Therefore, for all $\boldsymbol{u} \in S^{d-1}$, the point $\mathsf{med}(S_{\boldsymbol{u}})$ lies on the median level in the arrangement $\mathcal{A}(S) = \{S(\boldsymbol{p}_1), S(\boldsymbol{p}_2), \ldots, S(\boldsymbol{p}_n)\}$ of the n projection spheres. Moreover, any point $\boldsymbol{q}_0$ on the median level of $\mathcal{A}(S)$, $\mathsf{med}(S_{\boldsymbol{u}_0}) = \boldsymbol{q}_0$ for the unit vector $\boldsymbol{u}_0$ in the direction of $\boldsymbol{q}_0$. Therefore, the locus of $\mathsf{med}(S_{\boldsymbol{u}})$ as $\boldsymbol{u}$ varies over S^{d-1} is the median level of the arrangement $\mathcal{A}(S)$ of the n projection spheres. The locus of $\mathsf{med}(S_{\boldsymbol{u}})$ for a set of 7 points in the plane, as the median level in the arrangement of the 7 projection circles, is shown in Figure 1(b). The locus is indicated in Figure 1(b) using thick black lines.

For a fixed $\boldsymbol{u} \in S^{d-1}$, denote by $\boldsymbol{u}^{\perp}$ the $(d-1)$-dimensional hyperplane orthogonal to $\boldsymbol{u}$ and passing through the parent of $\mathrm{med}(S_{\boldsymbol{u}})$ at $\boldsymbol{u}$. A vertex in $\lambda(S)$ is a point in $\lambda(S)$ formed by the intersection of the d projection spheres. At a vertex of $\lambda(S)$, the point $\mathsf{med}(S_{\boldsymbol{u}})$ lies on d projection spheres, which implies that the hyperplane $\boldsymbol{u}^{\perp}$ passes through d points of S. The point in the dual corresponding to the hyperplane $\boldsymbol{u}^{\perp}$ is a vertex of the median level in the arrangement of the n hyperplanes in the dual corresponding to the set of points $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$. Therefore, the combinatorial complexity $\lambda(S)$ is the number of vertices in the median level of the arrangement of these n hyperplanes in the dual.

Proposition 1. *For a set $S = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$ of n points in $\mathbb{R}^d$ the locus of $\mathsf{med}(S_{\boldsymbol{u}})$ as $\boldsymbol{u}$ varies over S^{d-1} is the median level in the arrangement $\mathcal{A}(S) = \{S(\boldsymbol{p}_1), S(\boldsymbol{p}_2), \ldots, S(\boldsymbol{p}_n)\}$ of the n projection spheres, and its combinatorial complexity is same as the number of vertices*

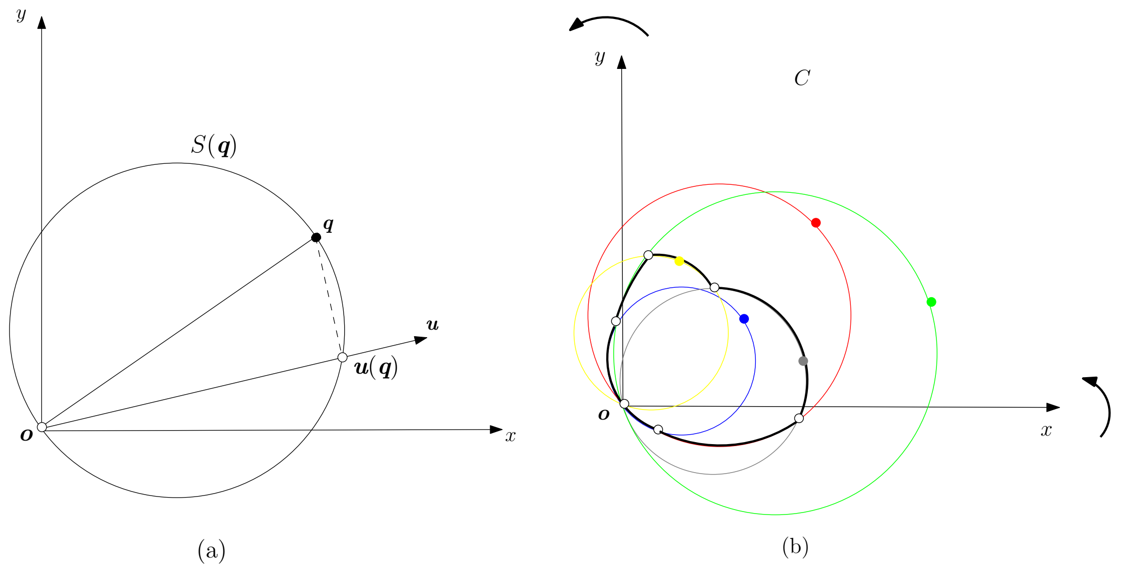

Fig. 1. (a) Projection circle for a point $\boldsymbol{q} \in \mathbb{R}^2$, and (b) Locus of $\mathsf{med}(S_{\boldsymbol{u}})$ as $\boldsymbol{u}$ varies over S^1 for a set of 7 points in the plane.

in the median level of the arrangement of the n hyperplanes in the dual, corresponding to the points in S. □

It follows from the above proposition that in any fixed dimension d, an upper bound on the number of vertices in the median level of a set of n hyperplanes in $\mathbb{R}^d$ is an upper bound on the combinatorial complexity of $\lambda(S)$. The best known upper bounds on the number of vertices in the median level of a set of n hyperplanes in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ are $O(n^{4/3})$ [6], $O(n^{5/2})$ [20], and $O(n^{4-1/18})$ [19], respectively. In $\mathbb{R}^d$, for $d \geq 5$, the upper bounds are of the form $O(n^{d-\delta_d})$, where $\delta_d = 1/(4d-3)^d$. The best known lower bound in $\mathbb{R}^d$ is $\Omega(n^{d-1} \cdot 2^{c\sqrt{\log n}})$ [16, 21], which implies that there exist a set S of n points in $\mathbb{R}^d$ such that the number of vertices in $\lambda(S)$ is $\Omega(n^{d-1} \cdot 2^{c\sqrt{\log n}})$.

The connection between the projection median and the median level of the hyperplane arrangement in the dual, can be used to obtain an algorithm for computing the projection median of a set of points in $\mathbb{R}^d$. We shall denote the median level of these n hyperplanes in the dual by $\mathcal{D}(S)$. Define $g : S^{d-1} \to \mathcal{D}(S)$, as follows: For $\boldsymbol{u} \in S^{d-1}$, let $g(\boldsymbol{u})$ be the point in $\mathcal{D}(S)$ corresponding to the hyperplane $\boldsymbol{u}^{\perp}$ in the primal. This is a bijective correspondence between S^{d-1} and $\mathcal{D}(S)$, and we denote the inverse by $h : \mathcal{D}(S) \to S^{d-1}$. Note that for any two points $\boldsymbol{x}, \boldsymbol{y}$ in some $(d-1)$-cell C of $\mathcal{D}(S)$, the parents of $\mathsf{med}(S_{h(\boldsymbol{x})})$ and $\mathsf{med}(S_{h(\boldsymbol{y})})$ are identical. Therefore, $\mathsf{med}(S_{\boldsymbol{u}})$ lies on the same projection sphere for all $\boldsymbol{u} \in h(C)$ and $\int_{h(C)} \mathsf{med}(S_{\boldsymbol{u}}) \mathrm{d}\boldsymbol{u}$ can be computed easily. Therefore, if the median level $\mathcal{D}(S)$ is known, then the problem of computing the projection median of S can be done by computing the integral $\int_{h(C)} \mathsf{med}(S_{\boldsymbol{u}}) \mathrm{d}\boldsymbol{u}$ for all $(d-1)$-cells C of $\mathcal{D}(S)$. As the number of $(d-1)$-cells in $\mathcal{D}(S)$ is at most the number vertices in the median level of $\mathcal{D}(S)$, the time required to compute the projection median in $\mathbb{R}^d$ is bounded by the time required to compute the median level of n hyperplanes in $\mathbb{R}^d$.

Efficient output-sensitive algorithms for level construction are known in $\mathbb{R}^2$ and $\mathbb{R}^3$. Edelsbrunner and Welzl [10] showed that in $\mathbb{R}^2$ a level of complexity b can be constructed in time $O(n \log n + b \log^2 n)$ time. This was later improved to $O(n \log n + b \log^{1+\epsilon} n)$ amortized time by Chan [5] using the dynamic convex hull data-structure. Durocher and Kirkpatrick

[9] used this to compute the projection median of a set of points in $\mathbb{R}^2$ in $O(n^{4/3}\log^{1+\epsilon} n)$ time. Agarwal and Matoušek [1] showed that a level of complexity b can be computed in time $O(n^{1+\epsilon} + bn^{\epsilon})$ in $\mathbb{R}^3$, where ϵ is an arbitrarily small positive constant. Therefore, the projection median of a set of points in $\mathbb{R}^3$ can be computed in $O(n^{5/2+\epsilon})$ time. In $\mathbb{R}^d$, with $d > 3$, the projection median can be obtained in $O(n^{d(1-\frac{\delta_d}{d+1})+\epsilon})$ time [1].

8 Conclusions

In this paper we have studied the projection median of a set of points in $\mathbb{R}^d$ for $d \geq 2$. Using results from the theory of integration over topological groups, we show that the d-dimensional projection median provides a $(d/\pi)B(d/2, 1/2)$-approximation to the d-dimensional Euclidean median. We also show that the d-dimensional projection median has a stability bound of $\frac{1}{(d/\pi)B(d/2,1/2)}$, and its breakdown point is 1/2. For the special case of $d = 3$, our results imply that the 3-dimensional projection median is a (3/2)-approximation of the 3-dimensional Euclidean median, which settles a conjecture posed by Durocher [8].

Based on these bounds on the approximation factor, stability and the breakdown point, we compare the d-dimensional projection median as a candidate for approximating the d-dimensional Euclidean median with the rectilinear median and the center of mass. It is shown that in all dimensions, the projection median is a better candidate for approximating the Euclidean median compared to the rectilinear median. For dimensions less than six, the projection median provides a better approximation of the Euclidean median than the center of mass, and maintains a fixed degree of stability, has a fixed breakdown point 1/2. As the dimension increases, the approximation factor and stability of the projection median worsen, while the center of mass maintains a fixed approximation factor of 2 and a fixed stability of 1. The projection median, however, has a constant breakdown point of 1/2, compared to the center of mass, which has a breakdown point of $1/n$.

As mentioned by Durocher and Kirkpatrick [9], the importance of the projection median also extends to several problems of mobile facility location [4, 8, 9]. Given a set of mobile clients moving continuously and with bounded velocity in $\mathbb{R}^d$, the suitability of a mobile facility is determined both by its approximation factor and also by its maximum velocity and continuity of its motion. Since, the stability of a median function is inversely related to the maximum velocity of a mobile facility [8], our results imply that the d-dimensional projection median defines the position of a mobile facility that approximates the mobile Euclidean median with a factor of $(d/\pi)B(d/2, 1/2)$ while maintaining a maximum velocity of at most $(d/\pi)B(d/2, 1/2)$ relative to the velocity of the clients.

Durocher and Kirkpatrick [9] proved the lower bound of $\sqrt{4/\pi^2 + 1}$ on the approximation factor of the projection median in $\mathbb{R}^2$. Note that the same quantity provides a lower bound on the approximation factor in $\mathbb{R}^d$ as well. However, the problem of obtaining the tight bound on the approximation factor in $\mathbb{R}^d$, for $d \geq 2$, remains open. Finding the tight bound on the approximation factor the d-dimensional rectilinear median, for $d \geq 3$, is another interesting problem.

In $\mathbb{R}^2$, Durocher and Kirkpatrick [9] proved that $1/J(2) = 4/\pi$ is a tight stability bound on the projection median. Proving a tight lower bound on the stability of the projection median for $d \geq 3$, remains to be solved.

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