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The Valuation of Callable Bonds

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Abstract
Callable bond indentures contain provisions that allow the issuing entity to retire the bond at a predetermined price before the maturity of the bond.[1] As such a callable bond is often viewed as a combination of an otherwise identical but non-callable bond and an option to call that bond. The writer of the call option is the holder of the bond, and the buyer of the call is the stockholder of the issuing corporation. Thus, the price of a callable bond is the value of the straight bond less the value of the call provision.

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THE VALUATION OF CALLABLE BONDS

Callable bond indentures contain provisions that allow the issuing entity to retire the bond at a predetermined price before the maturity of the bond.[1] As such a callable bond is often viewed as a combination of an otherwise identical but non-callable bond and an option to call that bond. The writer of the call option is the holder of the bond, and the buyer of the call is the stockholder of the issuing corporation. Thus, the price of a callable bond is the value of the straight bond less the value of the call provision.

But note that, unlike a detachable warrant on a bond, the call provision is not a separable option in the sense that it could trade in the market. Additionally, the comparison between the call on a bond and the call on a common stock can be misleading. To see this, recall that the holder of an American call on a common stock that pays no dividends will exercise it only at maturity. Prior to maturity, arbitrage arguments show that the call is more valuable unexercised than exercised. It is thus better to sell the call than exercise it.

The value of the call feature on a bond derives from the issuer's option to call the bond and refinance at a lower cost. Extant literature discusses the value of the call loosely in terms of the present value of the expected gain from refinancing. However, the call option feature on a bond is quite different from a stock option. For example, since the bond price converges to par at maturity, the value of the right to call a bond must converge to zero at maturity. Moreover, delay in exercising a call after rates have declined carries with it the risk that rates may again increase and extinguish the refinancing opportunity, as well as the carrying cost of paying higher coupons. Indeed, Brennan and Schwartz (1977A) conclude that after the call

protection period, an issuer should exercise the call as soon as there is any gain at all from exercising.

Immediately upon the expiration of the call protection period, there would be a positive gain from exercising the call if interest rates have declined sufficiently. After the expiration period, the story is quite different if trading is continuous and if bond prices follow a continuous path. Under these conditions, the following section shows that the gain from exercising a call after the call protection period is zero, or more precisely, approaches zero as the trading interval approaches zero. This result follows from the assumed continuity of the path of bond prices. Intuitively, with continuous trading and continuity in price movements, the issuer will exercise the call whenever there is any profit at all, and in the limit, this profit becomes arbitrarily small.

Yet, in a strategy of continually calling and reissuing new callable bonds, the right to call a bond has a significant effect upon the time pattern of coupon payments.[2] Each time an issuer calls such a bond, the issuer is able to issue an otherwise identical bond, but with an infinitesimally lower coupon. If the issuer can play this game sufficiently often, the series of infinitesimal reductions in coupons can "add up" to a finite reduction in the coupon.[3] Thus, if there are two bonds identical in all respects except that one is callable, the coupon on the callable bond must be greater than the coupon on the non-callable bond to induce an investor to hold the callable bond.

In effect, the strategy of repeatedly calling and reissuing new callable bonds is like "marking to market" changes in interest rates, but only in favorable directions. After the call protection period, the value of the call provision arises not from the exercise of a single call, but rather from a

strategy that allows the issuer to capture over time successively lower coupons.

In section II, we contrast the dynamic strategy with the continuous time model of Brennan and Schwartz (1977A, 1977B). The penultimate section presents some numerical examples to illustrate the dynamic nature of call provisions and the simultaneous effects of interest rate uncertainty and call provisions on the price and duration of coupon bearing bonds. The details of the numerical approximation used to value the bond are contained in the Appendix, along with a discussion of the errors in such numerical approximations. The paper concludes with some brief remarks.

I. VALUING THE CALL PROVISION

To begin, consider a bond with one period to maturity. There are N equally spaced decision points during the period. Time is measured by the remaining time to maturity or alternatively by the number of decision points that remain to maturity. Thus time 1 or decision point N refers to the beginning of the period and time 0 or decision point 0 the end. Since the concern of this paper is with the theoretical pricing of a callable bond, we assume perfectly functioning capital markets with no taxes or trading costs. Moreover, we assume away any risk of default.

The state of the system at any instant in time is characterized by the variable r, defined over the interval $[0,\infty)$. Subsequently, it will be assumed that r is the instantaneous risk free rate and follows a Weiner process. However, it is not necessary at this point to make this assumption.

If the current state is r at decision point n, define $p_n(r, m, k/N)$ as the contingent state price of one dollar to be received at decision point (n - k) if the state at that time is m. The following assumes that the state prices are invariant to n, so that the subscript can be dropped. Finally, we

assume that p is twice differentiable with respect to r, once differentiable with respect to k/N, $\partial p/\partial r$ is negative, and $0 \leq p \leq 1$.

Now consider an agent who wishes to borrow \$100 at decision point N. One alternative is to issue a non-callable bond paying a continuous coupon at the rate c. If the current state variable is k, the agent would then set c so that the current value of the bond is 100.

Another alternative is to issue a bond that is immediately callable at 100. This represents a somewhat unrealistic case, but helps to highlight the role of the call feature. Most bonds issued at their call price are only callable after some period of time, or, if immediately callable, are issued at a discount from their call price. Yet, there is certainly some coupon rate that would cause an investor to be indifferent between buying the non-callable or the callable bond. Let this rate be $\hat{c}$. In general, we use a hat to indicate that a variable corresponds to a callable bond.

The following discussion uses dynamic programming techniques to determine c and $\hat{c}$. After developing the general formulas, we then take the limit of a sequence as N approaches infinity.

Determining the coupon c for a non-callable bond with a current value of \$100 is straightforward. Let $B(n/N, r|N)$ be the price of the bond with n decision points or time $\tau = n/N$ to maturity, given that the state at time n is r. With one step to maturity, we have

$$B(1/N, r|N) = \int \left[p(r, m, 1/N)\ 100 + \int p(r, m, \tau)\ c\ d\tau\right]dm, \qquad (1)$$

where the region of integration for m is $[0, \infty)$ and for τ is $[0, 1/N]$. At step n, we have

$$B(n/N, r|N) = \int \left[p(r, m, 1/N)\ B[(n-1)/N, m|N] + \int p(r, m, \tau)\ c\ d\tau\right]dm \qquad (2)$$

If the current state at N is k, the agent then sets c to equate $B(1, k|N)$ to 100.

Let $\hat{B}(n/N, r|N)$ be the value of the callable bond at decision point n, given that the state at point n is r. Using dynamic programming, we begin at point 1, or equivalently one decision point prior to maturity. Were the bond not called at point 1, the value of the callable bond would be

$$f(1/N, r|N) = \int [p(r, m, 1/N)\ 100 + \int p(r, m, \tau)\ \hat{c}\ d\tau]dm \qquad (3)$$

At time 1, the optimal decision is to call the bond if $f(1/N, r|N) > 100$ for a "gain" to the issuer is $f(1/N, r|N) - 100$. To raise the 100 dollars necessary to call the bond, the issuer could issue a new bond at time 1/N maturing at time 0, but with a lower coupon than $\hat{c}$. Thus, the value of the callable bond at time 1/N is

$$\hat{B}(1/N, r|N) = \min [f(1/N, r|N), 100] . \qquad (4)$$

The term "gain" is enclosed in quotes since there really is no gain in that the lender would have already taken this gain into account in setting $\hat{c}$. This "gain" could equivalently be viewed as the "loss" from not following an optimal strategy.

At time 2 the value of the callable bond, if not called at time 2 but optimally called at time 1, is

$$f(2/N, r|N) = \int [p(r, m, 1/N)\ \hat{B}(1/N, m|N) + \int p(r, m, \tau)\ \hat{c}\ d\tau]dm \qquad (5)$$

As at time 1, the optimal decision at time 2 is to call the bond if $f(2/N, r|N) > 100$ for a "gain" of $[f(2/N, r|N) - 100]$. Thus, the value of the callable bond at time 2 is

$$\hat{B}(2/N, r|N) = \min [f(2/N, r|N), 100] . \qquad (6)$$

Just as at time 1, the issuer who calls a bond at time 2 can finance the call by reissuing another bond.

In general at step n, the value of a callable bond following an optimal call strategy is given by

$$\hat{B}(n/N, r|N) = \min \left[f(n/N, r|N), 100 \right] , \qquad (7)$$

where $f(n/N, r|N)$ is the value of the callable bond at time n if not called at time n but following an optimal call strategy at every step from n-1 to 1.[4] The "gain" from exercising the call is $f(n/N, r|N) - 100$. The $\hat{c}$ that equates $f(1, r|N)$ to 100 is the required coupon.

For later reference note that, if called, the bond ceases to exist, and a new bond could be issued. Assuming that securities are correctly priced, the exact provisions of the new bond are a matter of indifference to the issuer as long as the new bond has a value of \$100. For example, the borrower could issue a new callable bond with a slightly lower coupon, the lower coupon being a reflection of the "gain" to the issuer from calling the bond. Alternatively, the borrower could issue a non-callable bond with, in a sense to be discussed below, a more favorable coupon.

With discrete decision points, the "gain" from exercising a call on a bond is $f(n/N, r|N) - 100$, a positive number. However, if f moves continuously over time, this "gain" approaches zero as the time between each decision approaches zero or equivalently the number of equally spaced decision points per interval of time approaches infinity. Specifically,

Theorem: Assume that r follows a Weiner process, state prices p are defined as above, $\tau = n/N$, where n and N are predetermined integers such that $n \in [1, N]$ and r is such that the call will be exercised for a gain G of

$[f(\tau, r|kN) - 100] > 0$, so that the call will be exercised. Then, in the limit of the sequence as k approaches infinity, G approaches zero.

Proof: If bond is not called at τ but optimally called at decision points after τ, the bond price f at τ would follow a Weiner process. This statement follows from an application of Ito's Lemma and from noting that the assumption that p is twice differentiable with respect to r insures that f is twice differentiable.[5] If

$$dr = \mu(r)dt + \sigma(r)dz,$$

where dz is a Weiner Process and $dt = -d\tau$. The application of Ito's Lemma and substituting $(dr)^2$ yields

$$df = \left(-f_n + \mu(r)f_r + \frac{1}{2}\sigma^2(r)f_{rr}\right)dt + \sigma(r)f_r dz \ .$$

Since f is a Weiner process, the price f is almost surely continuous at τ. Thus, for any level of gain $\varepsilon > 0$, there will exist a $k' > 0$ such that for any integer $k > k'$, $f(\tau, r|kN) - 100 < \varepsilon$ except for price paths with measure zero.

Q.E.D.

This theorem is really not very surprising. The "gain" from not exercising the call is a mirror image of the "loss" from not following one possible optimal strategy of reissuing a new callable bond with a slightly lesser coupon. In a short enough period of time in which prices change continuously, this loss can be made arbitrarily small. This theorem shows that the "gain" from exercising the call is always positive but approaches zero as the number of trading intervals approaches infinity.

A corollary is that the reduction in the coupon on a newly issued callable bond is always positive but approaches zero as the number of trading

intervals approaches infinity. Even though in the limit the change in coupon at each exercise of a call is zero, the reduction in coupon over a finite period from repeatedly reissuing new callable bonds is finite.[6]

The total reduction in the coupon over time is the sum of the reductions associated with each exercise. If the issuer exercises the call sufficiently often, loosely speaking an infinite number of times, the sum of the infinitesimal reductions can "add up" to a finite reduction. Except for its probabilistic nature, this problem is conceptually similar to a certain game played N times. At each play, the player receives a gain of 1/N. As N approaches infinity, the gain per play approaches zero, but the overall gain is always one.

This discussion leads to the following:

Theorem: The coupon rate $\hat{c}$ on a callable bond with no call protection is greater than the coupon rate c on an otherwise identical non-callable bond.

Proof: Consider the following strategy: Issue a callable bond with no call protection, call it when there is any profit at all, and issue a new callable bond with an infinitesimally smaller coupon. Repeat this strategy over and over again. Thus will be captured any favorable changes in coupon rates.

Now, we show that no investor would participate in this strategy if $\hat{c} \leq c$. If $\hat{c} < c$, the non-callable bond dominates. If $\hat{c} = c$, over a finite period of time, there is a probability greater than zero that the state variable r will decrease by a sufficient amount so that $\hat{c}$ on the newly issued callable bonds will be less than the original $\hat{c}$ or c by a finite amount. Thus, $\hat{c}$ must be sufficiently greater than c to induce an investor to purchase the callable bond since the investor realizes that there is some probability

that over time the coupon on the repeatedly reissued callable bonds will be less than c by a finite amount.

Q.E.D.

Since the coupon on a callable bond would initially exceed the coupon on a non-callable bond with the same initial value, a borrower who continually calls and reissues callable bonds must believe that there is some possibility that the coupon associated with the series of callable bonds will ultimately be less than the coupon for the non-callable bond that could have originally been issued. If this were not the case, the borrower would prefer the non-callable bond. Section IIIA presents a numerical example that illustrates this anticipated decrease in the coupon.

It is instructive to examine the effect of issuing a non-callable bond instead of a callable bond to fund the call. What makes it optimal to call a callable bond is that the state variable has changed so as to increase the price of the callable bond above $100. The same change in the state variable will also cause the price of the non-callable bond to increase. Thus, the borrower can finance the call by issuing a non-callable bond with a coupon rate somewhat less than c, the original rate on a non-callable bond. In the limit, the difference between c and the new coupon for a non-callable bond will approach zero. But this is precisely the point. A single exercise of a call has no value in the limit. It is the strategy of continuing calling and reissuing a new callable bond that gives value to the right to call a bond.

II. RELATION TO PREVIOUS WORK

Previous research uses numerical analysis to estimate the value of a callable bond. In this section we demonstrate the equivalence of the analysis of the value of the call provision in the previous section to these numerical techniques. As an illustration, we consider the pioneering work of Brennan

and Schwartz (1977B). They begin with a continuous time model of bond price dynamics, approximate the model with finite differences, and then recursively solve the set of difference equations to estimate the value of the bond. The following briefly reviews the essential features of the model of Brennan and Schwartz for a non-callable bond. Subsequently we discuss incorporation of the call feature.

Define the following:

τ the time to maturity of the bond, where $\tau = 0$ at maturity;

c the instantaneous rate of coupon payments per par value of 100;

r the instantaneous risk free rate of interest.

Let the interest rate r follow the stochastic process

$$dr = \mu(r)dt + \sigma(r)dz, \tag{8}$$

where dz is a Gauss-Weiner process, with $E(dz) = 0$ and $E(dz^2) = dt$, and $dt = -d\tau$. Further, if money exists, the nominal interest rate cannot be negative. This latter condition is satisfied if $\lim_{r \to 0} \sigma(r) = 0$ and $\mu(0) \geq 0$.

The drift term $\mu(r)$ in (8) can take various forms. Cox, Ingersoll and Ross (1985) assume that $\mu(r)$ takes the form of a mean reverting process, $\kappa(r - \theta)$, where κ and θ are constants. Brennan and Schwartz (1977) assume that r follows a martingale in which $\mu(r) = 0$, a special case of the mean reverting process.

If the expectation hypothesis holds, the Markovian property of r implies that the price of the bond at time τ will be a function of only τ and r, $B(\tau, r)$. Using Ito's lemma and the pure expectations hypothesis,[7] Brennan and Schwartz show that the price of a default-free bond must satisfy the partial differential equation

$$\tfrac{1}{2}\,\sigma^2(r)B_{rr} + \mu(r)B_r - rB - B_\tau + c = 0 \qquad (9a)$$

with the following boundary conditions:

Maturity Date: At maturity, the value of the bond will be the par value, so that

$$B(0, r) = 100 \qquad (9b)$$

Zero Interest Rate: A natural boundary occurs when the interest rate is zero. Setting $r = 0$ in (9a) yields

$$B_\tau(\tau, 0) = c - \mu(0)B_r(\tau, 0) \qquad (9c)$$

Infinite Interest: As the interest rate approaches infinity, the bond (except at maturity) will have a value of zero. This condition is captured by

$$\lim_{r\to\infty} \quad B(\tau, r) = 0, \ \tau > 0 \qquad (9d)$$

The solution to (9) can be approximated by numerical techniques. The appendix contains a detailed discussion of the approach used by Brennan and Schwartz. Briefly, first define $s = 1/(1 + r)$, $0 \le s \le 1$, $b(\tau, s) = B(\tau, r)$ and $\sigma(s) = \sigma(r)$ and transform (9) in terms of s. Approximate each unit of time by N equal subintervals and the state variable s by M + 1 points equally spaced on the closed interval 0 to 1. After substituting the finite difference approximations for the partial derivatives, the difference equations developed by Brennan and Schwartz can be rewritten in a form similar to equation (2) of the prior section (see the Appendix).[8]

To incorporate the call provision of a bond, Brennan and Schwartz propose the boundary condition

$$B(\tau, r) \le 100, \ 0 < \tau \le \tau_c \qquad (10)$$

where τ_c is the time to maturity at which a bond can first be called and the call price is 100. They implement this constraint in their numerical approximation in exactly the same way as the call provision is incorporated in the previous section. Namely, they begin at n = 1 and if $B(1/N, r|N) > 100$, they set the value of the bond to 100, and so on. Thus, the constraint acts as an absorbing barrier since the implementation of their numerical approximation properly allows for the retirement of a callable bond upon exercise of the call. At that point, the issuer can finance the call by raising an additional \$100 in the financial markets.[9]

The discussion above highlights an important facet of the process giving rise to the call's value that is masked, however, by the bond price dynamics as posited in the differential equation (9a). Note that equation (9a) is defined over all r and all $\tau > 0$. In contrast, as shown in the last section, when an issuer exercises a call and finances the call with a new issue, the new issue will in some sense have more favorable terms than the issue called. Thus, whenever the bond is called, the differential equation (9a) changes. The possibility that an issuer can change the basic stochastic process governing the borrowings is what gives value to the right to call a bond. This possibility of changing the stochastic process is of much broader application. For example, Majd and Pindyck (1987) model the decision to delay an incremental investment on an ongoing project as the ability to alter the stochastic process underlying the value of the investment.

III. SOME NUMERICAL EXAMPLES

This section presents some numerical examples to illustrate the effect of call features and interest rate uncertainty on the prices, coupons and duration of callable bonds. The current price of a bond is a function of the coupon, the time to maturity, the call provisions, the current state, and the

uncertainty about future states. Thus, the pricing relationship implicitly involves six variables. To provide some comparative statics, we generally hold four of the six variables constant, vary one of the other two, and record the value of the remaining variable. To simplify the call provisions, we assume that the bond is callable at 100, so that the time to first call fully describes the call feature of a bond. Assume also that the risk free rate follows a diffusion process with $\mu(r) = 0$ and $\sigma(r) = \sqrt{r}\sigma$.[10]

The first subsection explores the effect of call features on current and future coupons. The second subsection shows that bond prices of callable bonds do not always increase with increases in the uncertainty about future interest rates in contrast to the results of Cox, Ingersoll, and Ross (1985) for non-callable bonds. The third subsection examines the effect of interest rate uncertainty on the duration of both callable and non-callable bonds.

A. The Call Feature and Bond Coupons

In our discussion of the origin of the value for a call provision in section I, we measured the value of the call option as the difference between the coupon rate for a callable bond and the coupon rate for an otherwise-equivalent non-callable bond. We now demonstrate this numerically. Assume a borrower requires \$100 for 20 years. The coupon that the borrower must pay today depends on the current riskfree rate r, the uncertainty in this rate, and the call provisions.

Table 1 contains the coupons required for various combination of σ and r. In all of these comparisons, we adjust the price of the bond to maintain the coupon on a 20-year non-callable bond at 10 percent. For example, if σ is 0.10 and r is 0.165, a 20-year non-callable bond with a coupon of 10 percent would have a current price of \$80. To borrow \$100, the agent would need to issue 1.25 bonds.

The required coupon increases as the call protection period decreases (Table 1). For example, if σ is 10% and the current value of r is 13.2%, the borrower could finance $100 at a coupon rate of 10 percent using a non-callable bond. Alternatively, holding σ and r constant, the borrower could issue $100 at a coupon rate of 18.4% using an immediately callable bond. With a call protection period of 5 years, the coupon is 14.1%, and with a call protection period of 10 years, the coupon is 12.3%. As would be expected, the increase in the coupon for a bond with call provisions is less when the bond is selling at a discount.

In section I, we stated that a borrower who issues a callable bond anticipates that the coupon on successively reissued callable bonds may gradually decrease over time and may ultimately be less than the coupon from originally issuing a non-callable bond. To illustrate this effect, we perform a simulation to compare the coupon from issuing a non-callable bond with the distribution of possible coupons from issuing and reissuing immediately callable bonds. The specific scenario illustrated in Figure 1 compares the coupon on a five-year non-callable bond with the distribution of future coupons associated with a strategy of calling and reissuing new callable bonds with continually lower coupons.[11]

The initial value of r is 7%, and σ in $\sqrt{r}\sigma$ is 15. Under these assumptions, the coupon on a five-year non-callable bond issued at par is 6.5%, and the coupon on a five-year immediately callable bond is 12.22%. Over time, there is an increasing probability that the coupon on the reissued callable bonds will be less than the coupon on the non-callable bond. Some readers may feel that the probability of coupons near zero is unrealistically large. The reason for this possibly large probability is that under the martingale process employed in this simulation, zero is an absorbing barrier

for the state variable r. A mean-reverting process with the same σ would lead to lesser probabilities of coupons near zero.

B. Prices of Coupon Bonds and Interest Rate Risk

Cox, Ingersoll, and Ross show that the derivative of the price of a non-callable zero-coupon bond with respect to σ is positive. Since a non-callable coupon-paying bond can be viewed as a portfolio of zero-coupon bonds, the same result applies to such bonds. This section shows numerically that this positive derivative of price with respect to σ does not generalize to callable bonds.

Numerically, we generate prices for a 20-year bond with a coupon of 10 percent as a function of the current interest rate and σ ranging from 0.0 to 0.35. As expected, the prices of non-callable bonds increase with interest rate risk for any level of r (Figure 2A).

This result, however, does not always hold for callable bonds. At the extreme, the price of a bond that is immediately callable at 100 is negatively related to σ for interest rates less than about 11.75 percent, and positively related to σ at rates greater than 11.75 percent (Figure 2B). This pattern persists for bonds with a call protection period of five years, although the relationship is greatly dampened (Figure 2C).

The intuition behind these results is the following: Consider a non-callable bond in which r is constant over time and equal to the coupon. In this case of certainty, the price of the bond will be a convex function of r Now, introduce some uncertainty about the future value of r. Before prices adjust, Jensen's inequality shows that the expected price of the bond at the next decision point is now greater than the current price of the bond. Since the coupon is still the same, the expected holding period return has increased, violating the pure expectation hypothesis assumed in the Brennan-

Schwartz model. To conform to the pure expectation hypothesis, the price of the existing bond must increase.[12] Since in a certain world, the price of a callable bond is not a convex function of r, Jensen's inequality is not applicable, and the price of a callable bond does not necessarily increase with increases in interest rate uncertainty.

C. Duration

Traditional measures of duration assume that future interest rates are known and often that the term structure is flat. As such, holding maturity and coupon constant, the price of a bond is a function only of the continuously compounded riskfree rate r, P(r). The total differential of P is

$$dP = P_r \, dr \ . \tag{11}$$

Rearranging (11) yields the usual duration measure

$$D = - \frac{\frac{dP}{P}}{dr} = - \frac{P_r}{P} \tag{12}$$

With uncertain interest rates, the price of a bond will be a function of r and σ (r), P(r, σ(r)). To define a comparable measure of duration in an uncertain world requires explicit assumptions as to what is held constant. For example, if $\sigma(r) = \sqrt{r}\sigma$, the total differential of P is

$$dP = P_r dr + P_{\sigma(r)} \frac{1}{2}\sigma r^{-\frac{1}{2}} dr + P_{\sigma(r)} r^{\frac{1}{2}} d\sigma \ . \tag{13}$$

Holding the overall level of risk constant, $(d\sigma(r) = 0)$ results in the standard duration measure (12). Alternatively, setting $d\sigma = 0$, results in the duration measure

$$D' = - \left| \frac{P_r}{P} + \frac{1}{2}\sigma r^{-\frac{1}{2}} \frac{P_{\sigma(r)}}{P} \right| \tag{14}$$

This duration measure accounts for the direct effect on bond prices of

interest rate changes, as well as the indirect effect of changes in r on prices through the level of uncertainty about interest rates. That is, in a world of uncertainty, changes in the level of the state variable r have both direct and indirect effects, and there is no longer an obvious single measure of duration as in a world of certainty.

To examine the effects of interest rate risk on duration, we use the numerical approximations for bond prices described in section II, and calculate the duration, as defined by (14), of a twenty-year bond with a continuously paid coupon of 10% per year for various call provisions, current interest rate r, and uncertainty σ. The results are reported in Figures 3A-C.

In the certainty case, the investor knows with certainty whether the bond will be called. Thus, in a certain world the duration of a callable bond is related to two distinct "maturity" dates: the actual maturity date, and the date of first call. If the price today is less than \$100, investors know that the bond will not be called regardless of the call protection period. As a consequence, the bond will behave as if it were a 20-year non-callable bond, and the duration of the bond will be unrelated to the length of the call protection period. If again $\sigma = 0$ but the price is greater than \$100, investors know that the bond will be called at the first call date. Thus, if the bond has call protection for five years, it will behave like a five-year bond and have a shorter duration than a bond with call protection for ten years, and so on. If price equals \$100 and $\sigma = 0$, duration is undefined.

Only when there is uncertainty regarding the future behavior of interest rates will there be uncertainty as to whether a bond will be called. In this case, duration will be a function of the probability of call. The probability of a call is related to the difference between the current price of the bond, the call price, and the uncertainty regarding the course of future interest

rates. In contrast to the certainty case, there is no longer a jump in duration as the price rises above the call price; and duration as a function of price becomes flatter as σ increases. Intuitively, as uncertainty about future interest rates increases, the current level of interest rates conveys less information about the future level and, therefore, the probability of an eventual call.

Finally, turning to the non-callable bond in Figure 3C, duration increases monotonically with increases in price (or decreases in interest rates)--a well known result. This relation is dampened in a world with interest rate uncertainty. As interest rate risk increases, the duration of a bond decreases.

III. Concluding Remarks

This paper explores the price behavior of callable bonds under the assumption of perfect capital markets. Using dynamic programming techniques, the paper shows that the call feature of a bond is quite different from the usual type of option, such as a call on a common stock. After the call protection period, the issuing firm will exercise a call whenever there is any gain from exercising. In the limit, the gain from the exercise of a single call is zero. Yet, by continually calling and reissuing a new callable bond, an issuer can continually capture any favorable changes in interest rates and thus change the time pattern of coupon payments. Some numerical examples were presented to illustrate this process.

In closing, it is important to note what this paper did not address. In a perfect capital market, any company would be indifferent between issuing a callable or a non-callable bond. If a call feature of a bond alters the time pattern of coupon payments, the coupon rate will adjust just enough to compensate the lender. If there are transaction costs associated with

exercising a call, however, the lender will always price a callable bond to be indifferent between the callable bond and a non-callable bond. The issuer thus will bear all of the costs associated with calling the bond. In this case, callable bonds would never be issued. Yet, virtually all long-term corporate bonds are issued with call provisions. This paper does not address the very interesting and important question of why corporations issue callable bonds if callable bonds entail greater transaction costs than non-callable bonds.

APPENDIX

Errors in the Numerical Approximation

Since analytical solutions to the differential equation (9) in the text are known only for special cases, the solution is approximated by first transforming it to a finite difference equation. In this appendix we demonstrate the approach and then examine the errors in the numerical approximation.

The approach requires that the state variable r be bounded. Since r is unbounded from above, Brennan and Schwartz suggest the transformation $s = 1/(1 + r)$, $0 \leq s \leq 1$, and define $b(\tau, s)$ as $B(\tau, r)$ and $\sigma(s)$ as $\sigma(r)$. This transformation yields

$$\frac{1}{2}s^4\sigma^2(r)b_{ss} + s^3\sigma^2(r)b_s - \frac{1 - s}{s}\, b - b_\tau + c = 0 \qquad \text{(A1)}$$

with the revised boundary conditions:

$$b(0, s) = 100 \qquad \text{(Maturity)} \qquad \text{(A1a)}$$

$$b_\tau(\tau, 1) = c \qquad \text{(Zero Interest)} \qquad \text{(A1b)}$$

$$b(\tau, 0) = 0 \qquad \text{(Infinite Interest)} \qquad \text{(A1c)}$$

Unlike r, the state variable s is bounded in the closed interval 0 to 1. They then approximate s by M + 1 points, equally spaced at increments of h, and each unit of time by $\tau N + 1$ points, equally spaced at increments of k. Setting τ to 1 makes this model identical notationally to the one in the text.

Incorporating the boundary conditions (A1a)-(A1c), the set of finite difference equations, at each point in time ($n = 1, \ldots, \tau N$), takes the form

$$\begin{pmatrix} \beta_1 & \gamma_1 & & & & \\ \alpha_2 & \beta_2 & \gamma_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \alpha_{M-1} & \beta_{M-1} & \gamma_{M-1} & \\ & & & & 1 & \end{pmatrix} \begin{pmatrix} b_n(1) \\ b_n(2) \\ \vdots \\ b_n(M-1) \\ b_n(M) \end{pmatrix} = \begin{pmatrix} b_{n-1}(1) + c \\ b_{n-1}(2) + c \\ \vdots \\ b_{n-1}(M-1) + c \\ b_{n-1}(M) + c \end{pmatrix} \tag{A2}$$

where

$$\alpha_i = - \frac{k s_i^4 \sigma^2(s_i)}{2h^2} \tag{A2a}$$

$$\beta_i = + \frac{k s_i^4 \sigma^2(s_i)}{h^2} + \frac{k}{h} s_i^3 \sigma^2(s_i) + k \left(\frac{1 - s_i}{s_i}\right) + 1 \tag{A2b}$$

$$\gamma_i = - \frac{k}{h} s_i^3 \sigma^2(s_i) - \frac{k}{2h^2} s_i^4 \sigma^2(s_i) \tag{A2c}$$

$$c = C/N \tag{A2d}$$

and C is the annual coupon.

The coefficients α_i, β_i, and γ_i are invariant with respect to time. Thus, system (A2) can be rewritten in matrix form as

$$\mathbf{A}b_n = b_{n-1} + ce \tag{A3}$$

or

$$b_n = \mathbf{A}^{-1}(b_{n-1} + ce)$$

where b_j is the vector of b_{ij}'s, $\mathbf{A}$ is a tridiagonal matrix of coefficients, and e is a vector of ones. Beginning at j=1, solving (A3) recursively provides the vector of prices at any point j.[13]

In using numerical methods there is a complex trade-off among the specific algorithms used, rounding errors, and the fineness of the time and state variable grid as specified by k and h respectively. In the absence of rounding errors,[14] the error in the approximation declines as k and h decrease. However, as the grid becomes finer, solving (A3) recursively requires more and more calculations. Due to the finite and fixed number of significant figures in a number stored in a computer, there is always the possibility that a calculation will introduce rounding errors and that such errors will propagate themselves in an explosive fashion through successive iterations. This trade-off indicates the importance of the judicious choice of h and k in using any numerical approximation.

One way to examine the interaction of a finer grid with the possible introduction of rounding errors is to calculate the approximate values and compare them with the true values. In earlier studies, k and h were picked somewhat arbitrarily since the absence of a closed form solution precluded the calculation of the correct values. However, closed form solutions are now available for certain processes. As an example, Cox, Ingersoll and Ross (1985) derive a closed form solution for a default-free zero-coupon bond under the assumption that the standard deviation of the stochastic process describing interest rate behavior, $\sigma(r)$, is $\sqrt{r}\sigma$, where σ is a constant. Specializing their solution to a martingale gives the price of a zero coupon bond with τ to maturity as

$$P(\tau, r) = e^{-f(\tau)r} \qquad (A4)$$

where

$$f(\tau) = \frac{2(e^{\sqrt{2}\tau\sigma} - 1)}{\sqrt{2}\,\sigma(e^{\sqrt{2}\tau\sigma} - 1) + 2\sqrt{2}\sigma}$$

In the special case in which σ is zero, $f(\tau) = \tau$ and $P(\tau, r) = e^{-r\tau}$, so that r is the usual continuously compounded rate of discount.[15]

For the purposes of this paper, we do not need to have precise estimates of the stochastic return process, but we do need some rough range of plausible values. The empirical work of Marsh and Rosenfeld (1983) suggests that $\mu(r)$ in (A4) is not much different from zero. The value of σ in $\sqrt{r}\sigma$ varies greatly from one time period to another and from one set of data to another. Expressed in annual units, they find that σ ranges from .01 to .15. If r were to equal to 10 percent, $\sqrt{r}\sigma$ would thus range from .3 percent to 4.7 percent. Of note, their empirical work suggests that the log model in which $\sigma(r)$ is $r\sigma$ better describes the return process than the model used by Cox, Ingersoll, and Ross. Nonetheless, for exposition reasons, the numerical calculations in this paper assume somewhat arbitrarily that $\sigma(r)$ is $\sqrt{r}\sigma$.

Equation (A4) allows an analysis of the behavior of the numerical approximation to the bond value as a function of h and k. Table A1 contains the results of this experiment for $\sigma = 0$ and .20 and $r = .25$ for both a one-year and a twenty-year zero coupon bond.[16] The approximation errors in Panel A assume that $\sigma = 0$, and in this case we report only one set of values for each maturity. The reason is that, when $\sigma = 0$, all terms in (A2) involving h drop out of the system. Thus, the approximated bond value is unrelated to the fineness of the grid along the state variable dimension. The values in the table represent the approximation error as a percent of the correct value. For example, for a twenty-year bond with $\sigma = 0$, $r = .25$ and $1/k = 120$, the approximated bond value is 0.52% larger than the correct value of .006738.

One interesting feature of Panel A of Table A1 is that the approximated values uniformly overstate the correct values when $\sigma = 0$. This overstatement

is unambiguously related to the approximation of the compounding process in (A2). In this case of certainty, the tridiagonal matrix in (A3) reduces to a diagonal matrix and the estimated value of $P(\tau, r)$ is given by

$$\hat{P}(\tau, r) = 100(1 + kr)^{-(\tau/k)} \qquad (A5)$$

A comparison of (A5) with (A1) with $\sigma = 0$ shows that the estimated price $\hat{P}$ will always exceed P if $r > 0$. Economically, the finite difference approximation ignores the compounding of returns within each increment of time.

The calculations in Panel B of Table A1 assume that future interest rates are uncertain and demonstrate how the errors change as a function of the number of time increments per year (1/k), the size of the increments of the state variable (h), and the maturity of the zero coupon bond. Generally, the percentage errors for a one-year bond are less than those for a twenty-year bond.

For a one-year bond, most of the improvement in the approximation error occurs with increases in the number of time increments per year. Importantly for time increments of 60 or more per year, the approximation errors are very small--uniformly less than 0.06 percent. For a given number of time increments per year, there is little change in the percentage errors as the size of the increments in the state variable (h) decreases.

In contrast, for a twenty-year bond, most of the improvement in the approximation error occurs with decreases in the size of the increments of the state variables (h). For a given value of h, the approximation error shows some decrease as the number of time increments per year increases. Yet, the decreases in the errors are not as great in magnitude as they are with decreases in the increment for the state variable.

Although these calculation strictly apply only to the case in which $\sigma(r) = \sqrt{r}\sigma$, they do suggest some generalizations. When there is no uncertainty as

to the future course of interest rates, the approximation error will be only a function of the number of increments per year. For maturities of twenty years, the number of state variables becomes more important than the number of increments per year as interest rate uncertainty increases.

FOOTNOTES

[1]Some bonds are callable at 100 percent of par regardless of the length of time to maturity. Other bonds are initially callable at a premium above 100 (for example 108), and this premium decreases over time at discrete intervals until the call premium equals zero. The actual price at which the call takes place is the stated call price plus accrued interest. Furthermore, some bonds are callable before the normal call date provided the funds used to call the bonds are not obtained from new borrowings at an interest rate lower than a prespecified rate. The literature has generally ignored this possibility of early call, and this paper will also ignore this possibility. Additionally, even after the call protection period, the call cannot usually be exercised immediately but only after a notification period, frequently a month. Again, this paper will ignore this complication.

[2]As noted, most prior work on callable bonds has been directed towards valuing a single bond. A notable exception is Boyce and Kalotay (1979).

[3]An analogue to this type of game but in a certain world is a game that is played n times and each time pays 1/n. As n approaches infinity, the gain for each play approaches zero. Yet, the overall gain is 1.

[4]Brennan and Schwartz (1977A) argue that a firm will call its callable bonds "at the point at which their uncalled value is equal to the call price." Since they do not define the term "uncalled value," this statement of behavior is not well defined. However, if $f(n/N, r|N)$ is defined as the uncalled value, this statement becomes well defined.

[5]The value of the callable bond $\hat{B}$ is not differentiable at r^* given as the solution of $f(n/N, r^*|N) = 100$.

[6]Two individuals who have read this paper suggested that the "high contact" theorem of Merton (1973) could be used to derive these results directly. Merton's theorem holds time constant and varies the state variable. In contrast, this paper explicitly models the strategy of over time repeatedly issuing new callable bonds and shows that in the limit the reduction in the coupon approaches zero but is always positive, a result that could not be obtained from Merton's theorem.

[7]The pure expectations hypothesis states that the instantaneous expected rate of return on default-free securities of all maturities is equal to the instantaneously risk-free rate of interest, r. We also assume perfect capital markets with no transactions costs, taxes or restrictions on short sales.

[8]To make equation (2) identical, one would have to apply the same transformation to r as Brennan and Schwartz use and then approximate the new state variable by M + 1 points.

[9]In contrast, the call constraint as posed in equation (10) is a reflecting barrier and does not recognize that the bond ceases to exist when called. The numerical analysis implemented by Brennan and Schwartz, however, does not formally impose equation (10) as the boundary condition.

[10]In the examples that follow, unless stated otherwise, the numerical algorithm is implemented with 1001 state variable increments (h = .001) and 120 time increments per year (k = 1/120). Coupons are paid "continuously" at the end of each time increment with a value equal to C/120, where C is the annual coupon. The appendix contains a detailed description of the algorithm and its implementation. Although the discussion of numerical approximation errors in the Appendix suggests that a coarser grid (along both the state variable and time dimensions) than used here would suffice, we opt for greater accuracy in reporting the main results.

[11]In implementing this simulation, the algorithm to calculate prices assumed that there were 401 state variables and 120 increments per year. To conserve computer resources, the simulation assumed that the bond could be called only at the end of each month, i.e., every tenth time increment. The changes in the state variable were simulated 10 increments at a time by the formula

$$r_{\tau-10} = r_{\tau} + \sqrt{r_{\tau}}\, s\, x ,$$

where $s = .15/\sqrt{12}$, r_{τ} is the value of r at time τ, and x is a unit normal random variate. It will be recalled that the time at maturity is zero. The simulation was repeated 1000 times, and coupons varied in increments of .001. Because of the discreteness of the return generating mechanism, r_{τ} could become negative in which case it was set to 0.002.

[12]If there were liquidity premiums or if, in a general equilibrium model, r varied directly with σ, the total derivatives of the price of a non-callable bond with respect to σ could be of any sign.

[13]Press, Flannery, Teukolsky and Vetterling (1986) provide an algorithm that utilizes this structure to solve (A3) for $\mathbf{b}_j$.

[14]The term "rounding errors" is a very loose term and can cover a host of problems. For example, one of the most serious errors in computer operations involves the addition of two numbers of substantially different magnitudes. For example, if the computer stores four significant figures, the sum of 1000 and 0.001 will be stored as 1000. This type of "rounding errors" can introduce substantial errors in solving simultaneous sets of equations such as (A2).

[15]Using L'Hopital's rule, the limit of $f(\tau)$ as σ approaches zero is τ. Thus, in this special case of certainty, (A4) reduces to the usual formula where r is the continuously compounded rate of return.

[16]The values in Table A1 represent approximation errors for the value of zero coupon bonds. Determining the error for a non-callable bond that pays coupons every six months is straightforward. Since a non-callable coupon bond can be decomposed into a series of zero coupon bonds and since the matrix **A** in (A3) is the same for any zero coupon bond, one could obtain for any such coupon paying bond, tables similar to those presented here by summing the approximated prices of each of the appropriate zero-coupon bonds.

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TABLE 1
Coupon Required to Compensate the Lender for Call Feature
Numeraire is 20-Year Non-Callable Bond with 10% Coupon

20-Year Non-Callable Bond Price (10% Coupon)	Risk-Free Interest Rate	Coupon (% of Par)			
		No Call Protection	5 Year Call Protection	10 Year Call Protection	Non-Callable
A. σ = 10%					
80	16.5%	12.9%	12.5%	11.6%	10.0%
100	13.2	18.4	14.1	12.3	10.0
120	10.8	∞	16.1	13.1	10.0
B. σ = 20%					
80	24.5	19.1	16.8	13.8	10.0
100	19.9	29.7	18.8	14.7	10.0
120	16.4	∞	20.0	15.4	10.0

Table A1
Percentage Comparison of Approximated Values to Correct Values of Zero-Coupon Discount Bonds for Various Differencing Intervals

Time Increments Per Year (1/k)	Increments of the State Variable s (h)							
	0.04	0.02	0.01	.005	.0025	.00125	.000625	.0003125
	A. $\alpha = 0.0$; $r = .25$	B. $\alpha = 0.2$; $r = .25$						
	One Year Bond (Correct Value = .778801)	One Year Bond (Correct Value = .780090)						
15	0.206	0.220	0.225	0.228	0.229	0.230	0.230	0.230
30	0.104	0.105	0.110	0.113	0.114	0.115	0.115	0.116
60	0.052	0.047	0.052	0.055	0.057	0.057	0.058	0.058
120	0.026	0.018	0.023	0.026	0.028	0.028	0.029	0.029
240	0.013	0.003	0.009	0.012	0.013	0.014	0.014	0.014
480	0.007	-0.004	0.002	0.004	0.006	0.006	0.007	0.007
960	0.003	-0.008	-0.002	0.001	0.002	0.003	0.003	0.003
1920	0.002	-0.010	-0.004	-0.001	0.000	0.001	0.001	0.002
3840	0.001	-0.010	-0.005	-0.002	-0.001	0.000	0.000	0.001
	Twenty Year Bond (Correct Value = .006738)	Twenty Year Bond (Correct Value = .172828)						
15	4.207	3.325	1.646	0.844	0.452	0.258	0.162	0.114	0.091
30	2.093	3.294	1.614	0.811	0.419	0.225	0.129	0.081	0.057
60	1.044	3.279	1.598	0.794	0.402	0.208	0.112	0.064	0.040
120	0.521	3.271	1.590	0.786	0.394	0.200	0.104	0.056	0.032
240	0.261	3.267	1.586	0.782	0.390	0.196	0.100	0.052	0.028
480	0.130	3.265	1.584	0.780	0.388	0.194	0.098	0.050	0.026
960	0.065	3.264	1.583	0.779	0.387	0.193	0.097	0.049	0.025
1920	0.032	3.264	1.582	0.778	0.386	0.192	0.096	0.048	0.024
3840	0.016	3.264	1.582	0.778	0.386	0.192	0.096	0.048	0.024

Figure 1

Percentile Distribution of Future Coupons
Using Optimal Call Strategy
5-yr No Call Protection

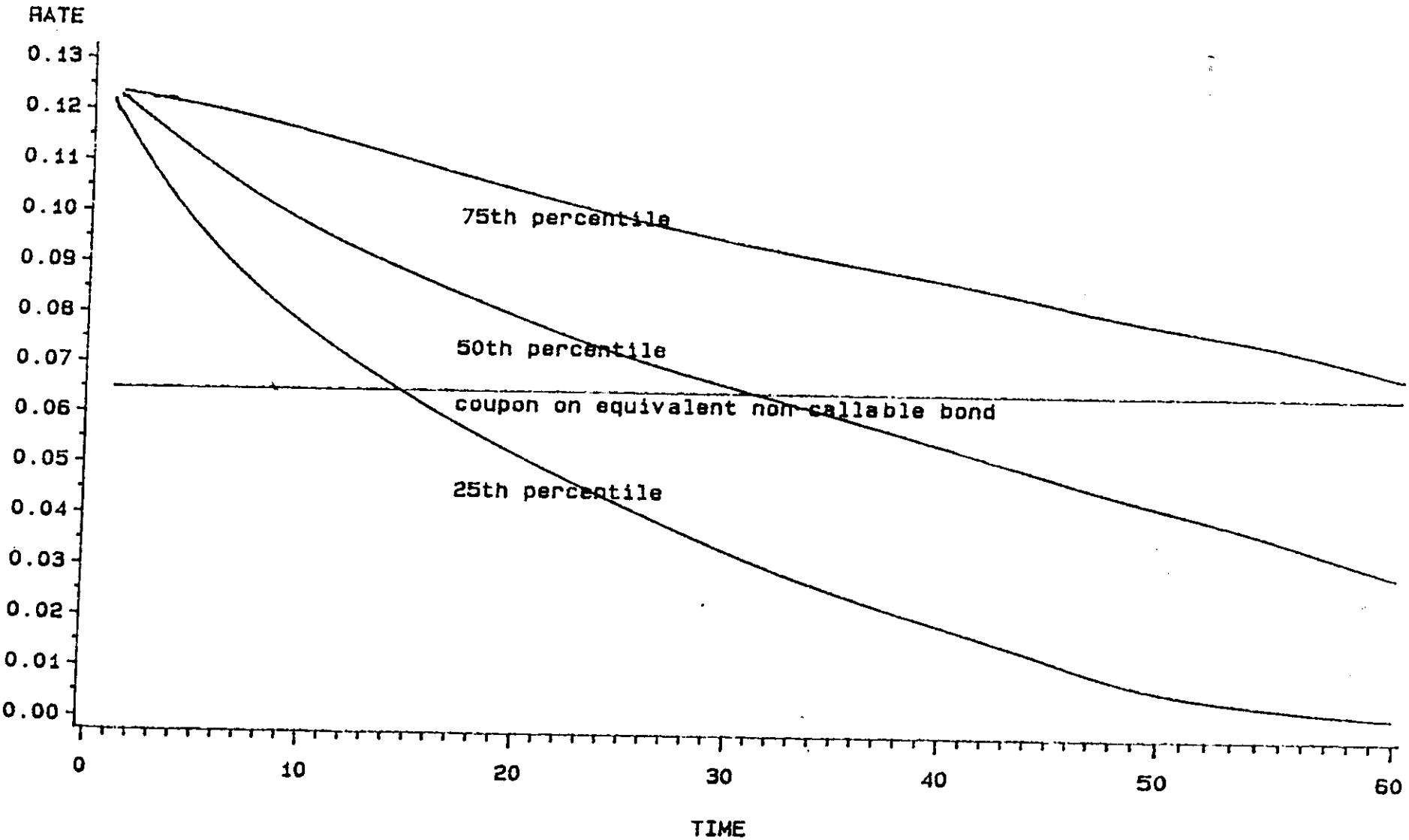

FIGURE 2A
Coupon Bond Prices and Interest Rate Risk
non-callable bond
(20-yr bond; 10% coupon)

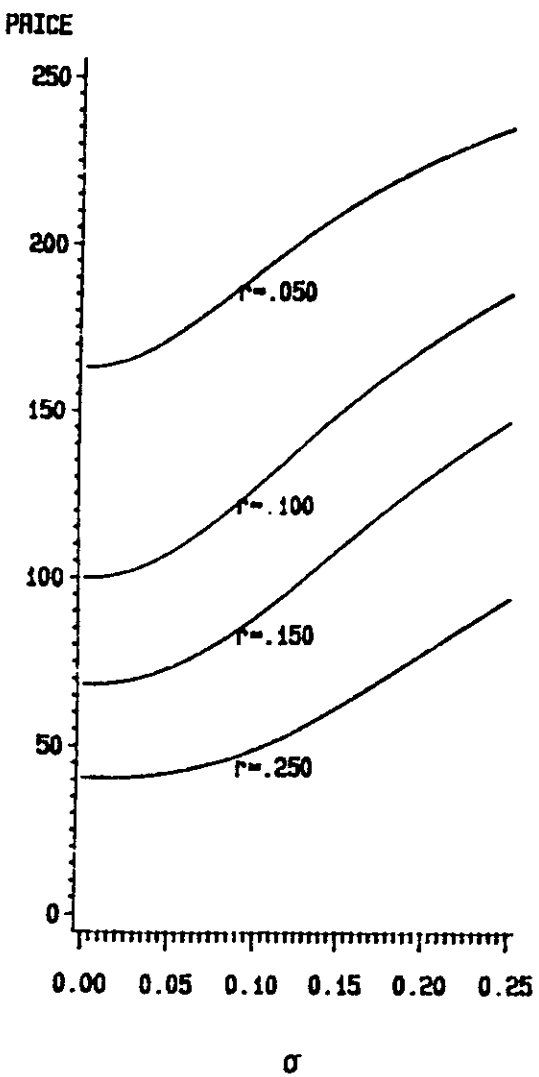

FIGURE 2B
Coupon Bond Prices and Interest Rate Risk
no call protection
(20-yr bond; 10% coupon)

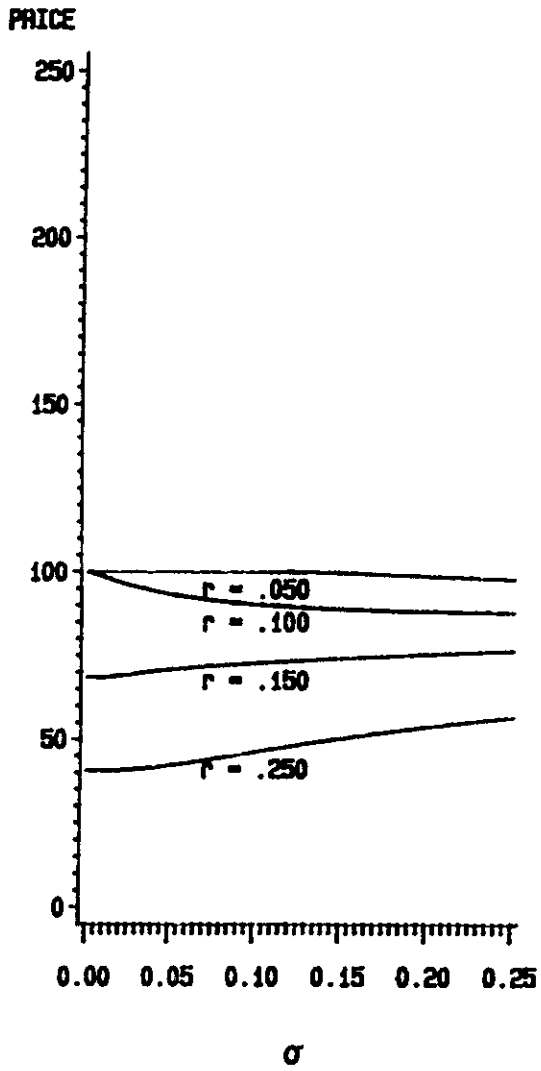

FIGURE 2C
Coupon Bond Prices and Interest Rate Risk
5-year call protection
(20-yr bond; 10% coupon)

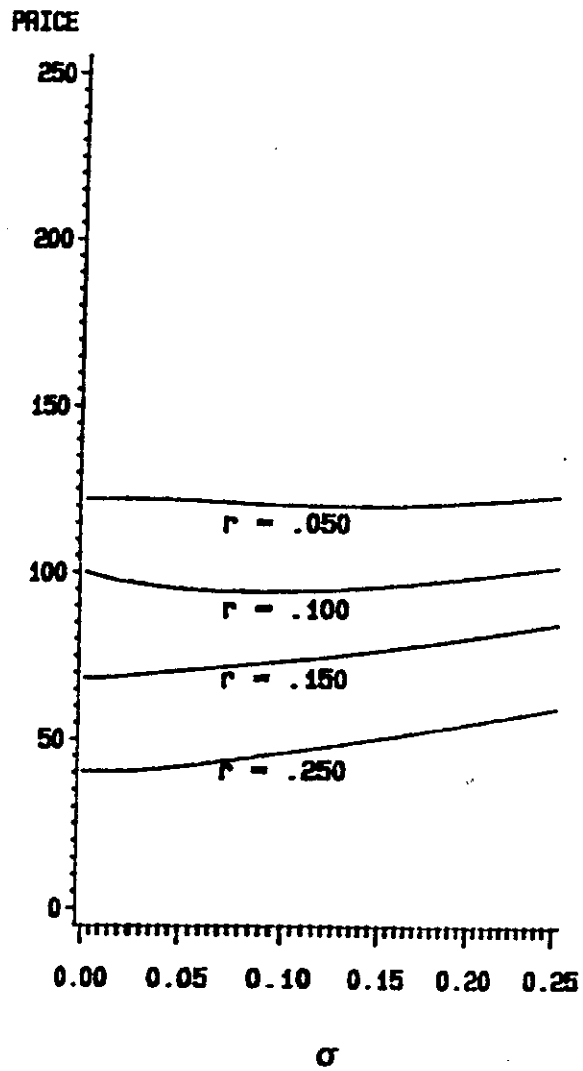

Figure 3A

Duration vs. Price
No Call Protection
(20-yr bond; coupon=10%; sigma constant)

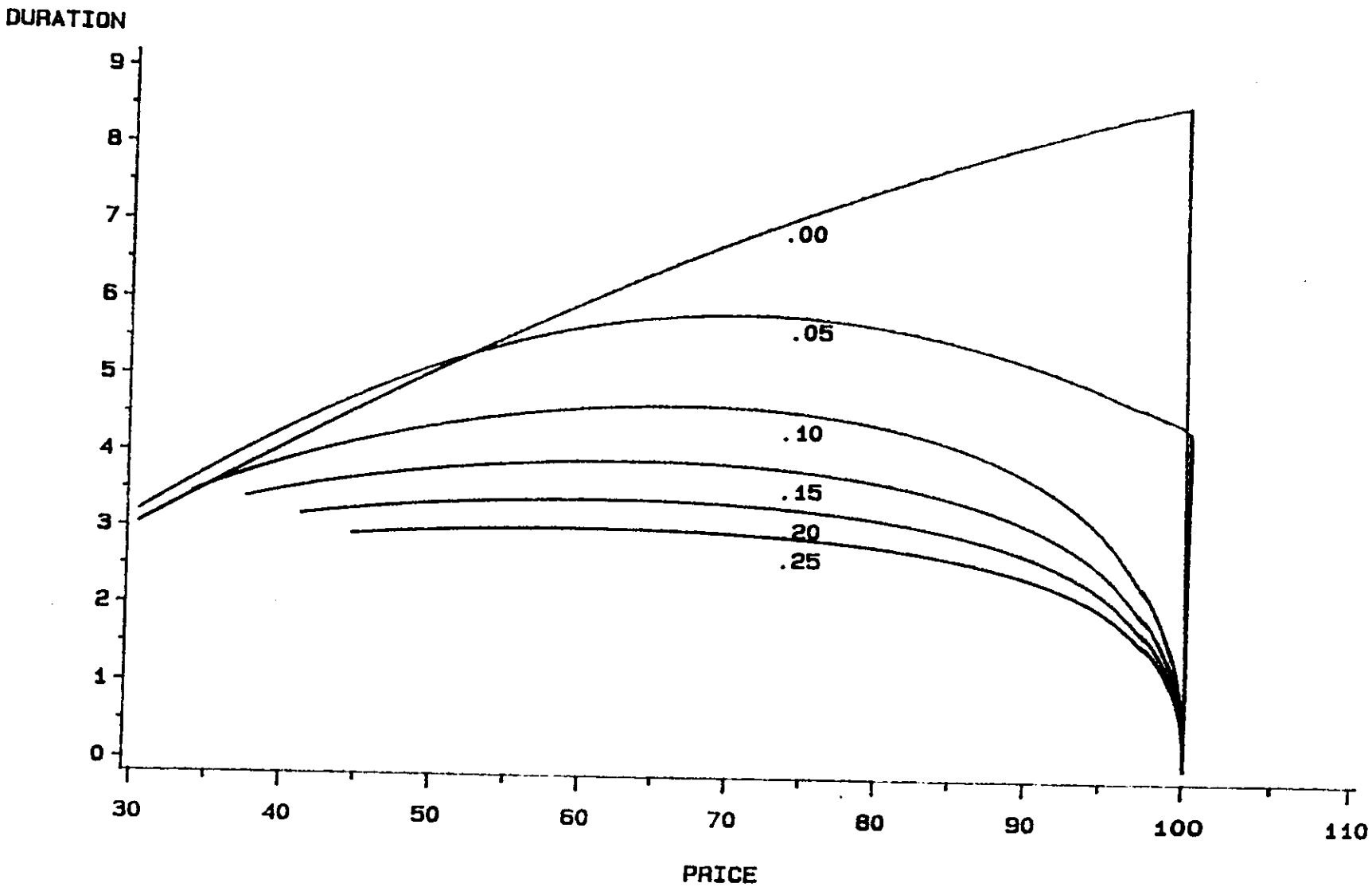

Figure 3B

Duration vs. Price
5-yr Call Protection
(20-yr bond; coupon=10%; sigma constant)

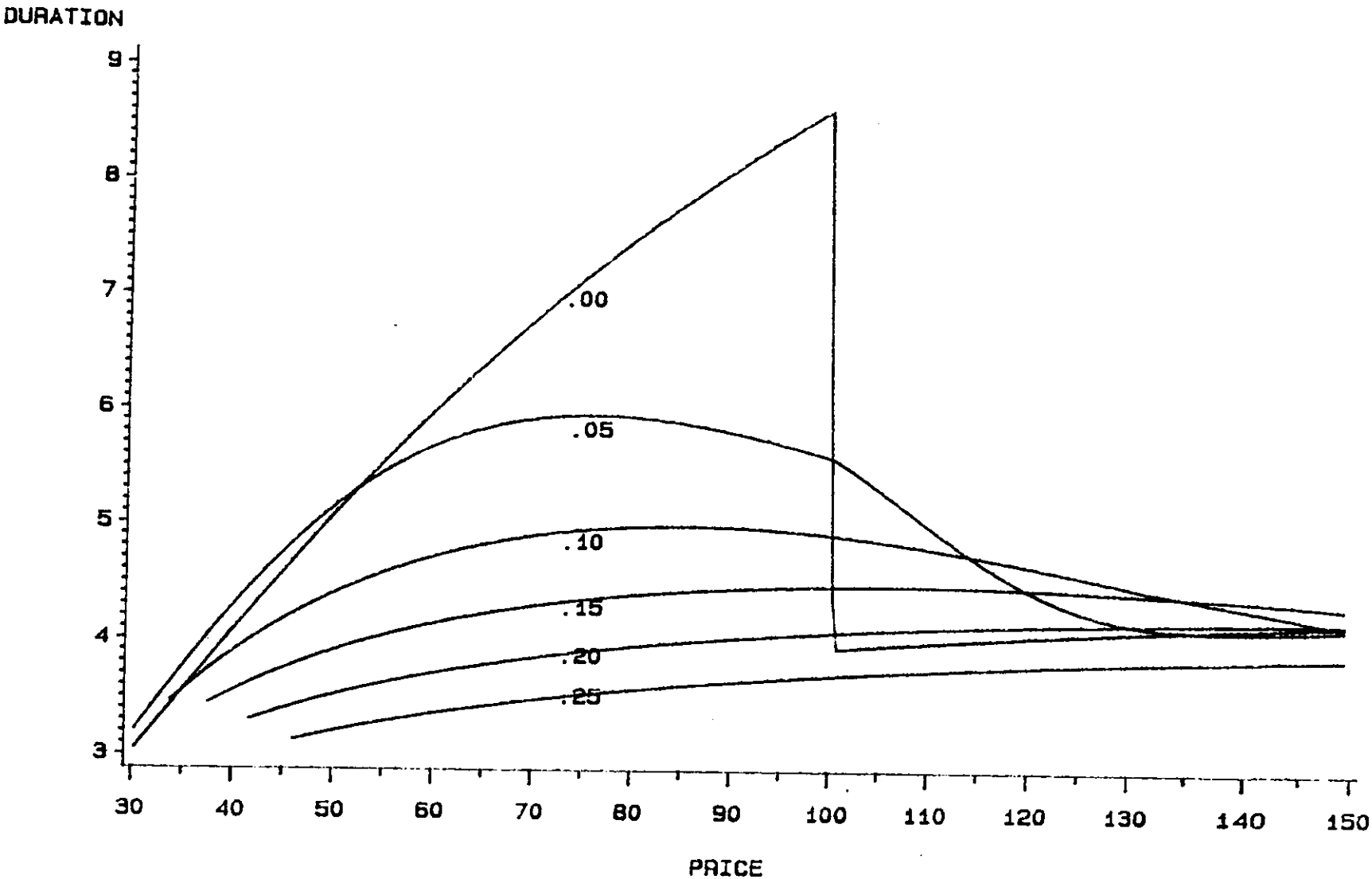

Figure 3C

Duration vs. Price
Non Callable
(20-yr bond; coupon=10%; sigma constant)