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AVERAGING APPROACH TO CYCLICITY OF HOPF BIFURCATION IN PLANAR LINEAR-QUADRATIC POLYNOMIAL DISCONTINUOUS DIFFERENTIAL SYSTEMS

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ABSTRACT. It is well known that the cyclicity of a Hopf bifurcation in continuous quadratic polynomial differential systems in $\mathbb{R}^2$ is 3. In contrast here we consider discontinuous differential systems in $\mathbb{R}^2$ defined in two half–planes separated by a straight line. In one half plane we have a general linear center at the origin of $\mathbb{R}^2$, and in the other a general quadratic polynomial differential system having a focus or a center at the origin of $\mathbb{R}^2$. Using averaging theory, we prove that the cyclicity of a Hopf bifurcation for such discontinuous differential systems is at least 5. Our computations show that only one of the averaged functions of fifth order can produce 5 limit cycles and there are no more limit cycles up to sixth order averaged function.

1. Introduction and statement of the main result

A *polynomial differential system* in $\mathbb{R}^2$ is a differential system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are polynomials in the real variables x and y, and the dot denotes derivative with respect to an independent variable t. The *degree* of the polynomial differential system (1) is the maximum of the degrees of the polynomials P and Q.

A *linear differential system* or here simply a *linear system* is a polynomial differential system in $\mathbb{R}^2$ of degree one. Similarly a *quadratic differential system* or simply a *quadratic system* is a polynomial differential system in $\mathbb{R}^2$ of degree two.

A *limit cycle* of a differential system is a periodic orbit of that system which is isolated in the set of all periodic orbits of the system. The study of the limit cycles of the planar differential systems is one of the main topics of the qualitative theory of the differential systems in $\mathbb{R}^2$, see for instance [9, 18, 22].

A *Hopf bifurcation* takes place at a singular point p of a differential system when p changes its stability and one or several limit cycles arise from p. Here the *cyclicity of a Hopf bifurcation* at the singular point p inside a family of differential systems is the maximum number of limit cycles which can bifurcate from p inside the family considered.

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Bautin in [2] proved that the cyclicity of a Hopf bifurcation in the family of quadratic systems is three. Since linear systems have no limit cycles, the singular points of the linear systems cannot exhibit Hopf bifurcations. The maximal cyclicity of a Hopf bifurcation for the family of polynomial differential systems with a given degree larger than two is an open problem.

Here we shall consider *discontinuous differential systems* in $\mathbb{R}^2$ formed by two differential systems one in the half-plane $y > 0$ and the other in the half-plane $y < 0$. We can say that the study of the discontinuous differential systems started with Andronov, Vitt and Khaikin [1] and nowadays continues receiving the attention of many researchers, because these kind of differential systems are used to model many phenomena appearing in mechanics, electronics, economy, ..., see for instance the books of di Bernardo et al. [3], Filippov [14], Kunze [21] and Simpson [27], the survey of Makarenkov and Lamb [25], and the references cited in these last five works.

For discontinuous planar differential systems with a single straight line of discontinuity the limit cycles arising from a Hopf bifurcation were investigated in some papers, see for instance [7, 10, 17]. Our objective is to study the cyclicity of a Hopf bifurcation of a center-focus singular point of a discontinuous differential system in $\mathbb{R}^2$ formed by two differential systems, each of which is defined in a half-plane separated by a straight line. More precisely, we consider a discontinuous differential system of form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \mathcal{X}_+(x,y) & \text{if } y > 0, \\ \mathcal{X}_-(x,y) & \text{if } y < 0, \end{cases} \tag{2}$$

where the vector fields $\mathcal{X}_+(x,y), X_-(x,y)$ have a quadratic center-focus singular point at the origin of coordinates and a linear center at the origin, respectively, and have the same rotating direction.

A subclass of discontinuous differential systems (2) was considered by Coll, Gasull and Prohens [10] in 2001 with the particular vector fields $X_-(x,y) = iz$ and

$$\mathcal{X}_+(x,y) = (\lambda + i)z + p_{20}z^2 + p_{11}z\bar{z} + p_{02}\bar{z}^2,$$

written in complex notation where $z = x + y\,i$,

$$p_{20} = -\frac{11}{6} + 2b\,\pi - \frac{15}{32}c + \frac{6-3a}{8}i,$$

$$p_{11} = \frac{11}{12} + \frac{5}{8}c - 4b\,\pi + i,$$

$$p_{02} = \frac{37}{48} + 2b\,\pi - \frac{5}{32}c - \frac{6-3a}{8}i,$$

and $\lambda, a, b, c \in \mathbb{R}$. These authors proved for this subclass of systems that 4 limit cycles can bifurcate from the center-focus localized at the origin of coordinates. In 2003 Gasull and Torregrosa [15] investigated a particular family of discontinuous differential systems (2) with $\mathcal{X}_+(x,y) = iz + h.o.t$ and $X_-(x,y) = iz$, and constructed a bifurcation producing 5 limit cycles from the center-focus O. This result was obtained by computing Liapunov constants. In 2010 Chen and Du [6] and in 2015 Chen, Romanovski and Zhang [7] studied the cyclicity of a Hopf bifurcation of a center-focus singular point of a discontinuous differential systems in $\mathbb{R}^2$ formed by two quadratic systems, showing that the cyclicity of that Hopf bifurcation is at least 5 in the case of weak focus and at least 9 in the case of center.

In this paper we extend the results of [15] to an arbitrary quadratic polynomial differential system for $\mathcal{X}_+(x,y)$. Thus the main result of this paper is the following.

Theorem 1. *For the discontinuous differential systems* (2) *the cyclicity of the Hopf bifurcation at the origin is at least* 5 *using the averaging theory up to fifth order. Moreover there is no more limit cycles up to sixth order averaged function.*

In the proof of Theorem 1 we obtain several averaged functions of fifth order coming from the different sets of conditions for doing identically zero the previous averaged function of less order. Only one of these averaged functions of fifth order can produce 5 limit cycles, all the others produce less number of limit cycles. The discontinuous differential systems associated to this averaged function of fifth order producing 5 limit cycles contains the particular family of discontinuous differential systems studied by Gasull and Torregrosa [15]. We remark that our result of at least 5 limit cycles given in Theorem 1 is for the full family of discontinuous differential systems (2), and that it was obtained by the averaging theory, different from the Liapunov method used in [15]. Moreover, we find that the averaged functions of sixth order do not provide more than 4 limit cycles.

2. Prelimilaries

By [2, 19, 20] the quadratic systems having a center-focus singular point must be non-degenerate, i.e. the linear part of the vector field $\mathcal{X}_+(x,y)$ at such singular point is non-degenerate. This means that the Jacobian matrix of $\mathcal{X}_+(x,y)$ at that singular point has eigenvalues $\alpha^+ \pm i\beta^+$, where $\beta^+ \neq 0$. In the following proposition we simplify the linear part of the discontinuous differential systems (2).

Proposition 2. *There is a time-rescaling and a homeomorphism on* $\mathbb{R}^2$ *which restricted to* $\mathbb{R}^2 \setminus \{y=0\}$ *is a diffeomorphism such that the linear parts of* $\mathcal{X}_+(x,y)$ *and* $\mathcal{X}_-(x,y)$ *of the discontinuous differential system* (2) *are normalized as*

$$\begin{pmatrix} \lambda_1 x - y \\ x + \lambda_1 y \end{pmatrix} \text{ and } \begin{pmatrix} -y \\ x \end{pmatrix}, \tag{3}$$

respectively.

Proof. We rewrite the vector fields $\mathcal{X}_\pm(x,y)$ of the discontinuous differential system (2) as

$$\mathcal{X}_+(x,y) = A^+ \begin{pmatrix} x \\ y \end{pmatrix} + h.o.t, \qquad \mathcal{X}_-(x,y) = A^- \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4}$$

respectively, and assume that A^+ (resp. A^-) has eigenvalues $\alpha^+ \pm i\beta^+$ (resp. $\pm i\beta^-$), where $\beta^\pm \neq 0$. Clearly, $4\det A^\pm - (\mathrm{tr}A^\pm)^2 > 0$ and $A^\pm_{21} \neq 0$, where $A^\pm_{ij}$s are the elements of $A^\pm$ and $\det A^\pm, \mathrm{tr}A^\pm$ are the determinants, traces of $A^\pm$ respectively.

By the transformation

$$(5)\quad \begin{pmatrix} u\\ v\end{pmatrix}=P(x,y)=\begin{cases}\begin{pmatrix}2 & (A^+_{22}-A^+_{11})/A^+_{21}\\ 0 & \left\{4\det A^+-(\mathrm{tr}A^+)^2\right\}^{\frac{1}{2}}/|A^+_{21}|\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} & \text{if } y\geq 0,\\ \\ \begin{pmatrix}2 & (A^-_{22}-A^-_{11})/A^-_{21}\\ 0 & \left\{4\det A^--(\mathrm{tr}A^-)^2\right\}^{\frac{1}{2}}/|A^-_{21}|\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} & \text{if } y\leq 0,\end{cases}$$

we change the vector fields $\mathcal{X}_\pm(x,y)$ given in (4) into

$$(6)\quad \mathcal{Y}_+(u,v)=\begin{pmatrix}\alpha^+ & -\beta^+\\ \beta^+ & \alpha^+\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}+h.o.t,\qquad \mathcal{Y}_-(u,v)=\begin{pmatrix}0 & -\beta^-\\ \beta^- & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}.$$

Further, by $(u,v)\to(x,y)$ and time-rescalings $t\to t/\beta^+$, $t\to t/\beta^-$ the linear parts of the vector fields $\mathcal{Y}_\pm(u,v)$ given in (6) are changed into the linear parts given in (3) respectively, where $\lambda_1=\alpha^+/\beta^+$. On the other hand, it is easy to check that the transformation P defined on $\mathbb{R}^2$ in (5) is a diffeomorphism on $\mathbb{R}^2\setminus\{y=0\}$ and a homeomorphism to $\{y=0\}$, i.e. a homeomorphism on $\mathbb{R}^2$. This completes the proof of this proposition. □

By Proposition 2 we only need to consider the discontinuous differential system (2) with

$$(7)\qquad \mathcal{X}_+(x,y)=\begin{pmatrix}\lambda_1x-y-\lambda_3x^2+(2\lambda_2+\lambda_5)xy+\lambda_6y^2\\ x+\lambda_1y+\lambda_2x^2+(2\lambda_3+\lambda_4)xy+(\lambda_7-\lambda_2)y^2\end{pmatrix},$$

and

$$(8)\qquad \mathcal{X}_-(x,y)=\begin{pmatrix}-y\\ x\end{pmatrix}.$$

As we shall see in section 2 the quadratic part of the vector field $\mathcal{X}_+(x,y)$ consists of general quadratic polynomials.

It is well known that a periodic orbit of a quadratic system must surround a focus or a center, see Theorem 6 of Coppel [12], see also Proposition 8.13 of [13]. So, with a translation of this focus or center to the origin, any quadratic system having having a periodic orbit can be reduced to the form

$$(9)\qquad \begin{aligned}\dot{x}&=\lambda_1x-y+a_{20}x^2+a_{11}xy+a_{02}y^2,\\ \dot{y}&=x+\lambda_1y+b_{20}x^2+b_{11}xy+b_{02}y^2.\end{aligned}$$

Applying the invertible transformation $(a_{20},a_{11},a_{02},b_{20},b_{11},b_{02})^\top=A(\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7)^\top$ where

$$A=\begin{pmatrix}0 & -1 & 0 & 0 & 0 & 0\\ 2 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 1 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

we rewrite the quadratic system (9) can be written as

$$\begin{aligned} \dot{x} &= \lambda_1 x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy + (\lambda_7 - \lambda_2)y^2. \end{aligned} \tag{10}$$

where $\lambda_1, ..., \lambda_7$ are real constants. It is obvious that the 6 coefficients of the quadratic terms of system (10) remain independent of each other. Note that systems (10) with $\lambda_7 = 0$ are the quadratic systems in the classical Bautin form. But here we shall work with system (10).

We are using for the discontinuous differential systems (2) the expression of $\mathcal{X}_+(x, y)$ given in (10) instead of (9) because later on we shall see that there is a relationship between some coefficients of the averaged functions and the Liapunov constants of the quadratic system (10), see section 4.

3. Proof of Theorem 1 via the averaging theory

We shall write the discontinuous differential system (2) with (7) and (8) in polar coordinates $(r, \theta) \in \mathbb{R} \times \mathbb{S}^1$, where $x = r\cos\theta$ and $y = r\sin\theta$. Then system (2) with (7) and (8) becomes

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{cases} \mathcal{Y}_+(r, \theta, \lambda) & \text{if } \theta \in [0, \pi], \\ \mathcal{Y}_-(r, \theta, \lambda) & \text{if } \theta \in [\pi, 2\pi], \end{cases} \tag{11}$$

where $\lambda = (\lambda_1, ..., \lambda_7) \in \mathbb{R}^7$, $\mathcal{Y}_+(r, \theta, \lambda) = (\mathcal{H}(r, \theta, \lambda), \mathcal{G}(r, \theta, \lambda))^T$, $\mathcal{Y}_-(r, \theta, \lambda) = (0, 1)^T$ and

$$\begin{aligned} \mathcal{H}(r, \theta, \lambda) = \;& \lambda_1 r + r^2(-\lambda_3 \cos^3\theta + (3\lambda_2 + \lambda_5)\cos^2\theta \sin\theta \\ & + (2\lambda_3 + \lambda_4 + \lambda_6)\cos\theta \sin^2\theta + (-\lambda_2 + \lambda_7)\sin^3\theta), \\ \mathcal{G}(r, \theta, \lambda) = \;& 1 + r(\lambda_2 \cos^3\theta + (3\lambda_3 + \lambda_4)\cos^2\theta \sin\theta \\ & - (3\lambda_2 + \lambda_5 - \lambda_7)\cos\theta \sin^2\theta - \lambda_6 \sin^3\theta). \end{aligned}$$

Since we want to study the Hopf bifurcation at the origin of the discontinuous differential system (11), we do the rescaling $r \to r\varepsilon$ and take

$$\lambda_i := \sum_{j=0}^{7} \lambda_{ij} \varepsilon^j \tag{12}$$

for each $i = 1, ..., 7$, with $\lambda_{10} = 0$. Then system (11) is reduced to the differential equation

$$\frac{dr}{d\theta} = \begin{cases} \dfrac{\mathcal{H}(r, \theta, \lambda, \varepsilon)}{\mathcal{G}(r, \theta, \lambda, \varepsilon)} = \displaystyle\sum_{i=1}^{\infty} \varepsilon^i F_i(\theta, r, \lambda) & \text{if } \theta \in [0, \pi], \\ 0 & \text{if } \theta \in [\pi, 2\pi], \end{cases} \tag{13}$$

when we take as new independent variable the angle θ. Here every F_i is a polynomial in the variables r, $\sin\theta$, $\cos\theta$ and λ.

Now the problem of estimating the cyclicity of the Hopf bifurcation at the origin of system (11) becomes the problem of estimating the cyclicity of the Hopf bifurcation at the origin of system (13), which is written in the normal form for applying the averaging theory of arbitrary order for studying its periodic orbits.

The classical averaging theory (see for instance [4, 5, 26]) recently has been extended for computing periodic orbits of analytical differential equations of one variable with arbitrary order in the small parameter ε by Giné, Grau and Llibre [16]. Later on this theory was extended by Llibre, Novaes and Teixeira [23] to arbitrary order in ε for continuous differential systems in n variables. Using the arguments of the paper of Llibre, Novaes and Teixeira [24] the formulas obtained for the averaged functions of arbitrary order in [16] or [23] also work for the discontinuous differential systems.

Let $\varphi(\cdot, z) : [0, t_z] \to \mathbb{R}^n$ be the solution of the unperturbed system, $r'(\theta) = 0$ such that $\varphi(0, z) = z$, i.e. $\varphi(\theta, z) = z$. Using the results and notations of [23, 24] we have that the *averaged function* $f_i : (0, \infty) \to \mathbb{R}$ of order $i = 1, 2, \ldots, k$ is

$$f_i(z) = \frac{w_i(\pi, z)}{i!}, \tag{14}$$

where $w_i : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ for $i = 1, 2, \ldots, k-1$ are defined recurrently by

$$\begin{aligned} w_i(t, z) = \quad & i! \int_0^t \Bigg(F_i\left(s, \varphi(s, z)\right) + \\ & \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L}{\partial r^L} F_{i-l}\left(s, \varphi(s, z)\right) \prod_{j=1}^{l} w_j(s, z)^{b_j} \Bigg) ds, \end{aligned}$$

where S_l is the set of all l-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$. Note that in (14) we have $w_i(\pi, z)$ instead of $w_i(2\pi, z)$ because the contribution to the averaged function of the linear vector field on the half-plane $y < 0$ is zero, see for more details [24]. In the subsection 4.1 of [23] are given explicitly all the terms of the function $f_i(r)$ for $i = 1, \ldots, 5$.

Theorem A of [23] and [24] say us that if the first averaged function $f_1(r) \not\equiv 0$, then each one of its positive simple zeros provides a limit cycle of the discontinuous differential equation (13) on the cylinder $(0, \infty) \times \mathbb{S}^1$, and consequently also provides a limit cycle of the discontinuous differential system (2) with (7) and (8) in $\mathbb{R}^2$. Moreover if $f_1(r) \equiv 0$ and $f_2(r) \not\equiv 0$, then each one of the positive simple zeros of $f_2(r)$ provides a limit cycle of the discontinuous differential equation (13) on the cylinder $(0, \infty) \times \mathbb{S}^1$, and consequently also provides a limit cycle of the discontinuous differential system (2) with (7) and (8) in $\mathbb{R}^2$. If $f_1(r) \equiv f_2(r) \equiv 0$ and $f_3(r) \not\equiv 0$ the same for the function $f_3(r)$, and so on.

In summary, for proving Theorem 1 we must compute the averaged functions $f_i(r)$, and the positive simple zeros of $f_i(r)$ when $f_k(r) \equiv 0$ for $k = 1, \ldots, i-1$.

Now using the software Mathematica we have computed

$$\begin{aligned} f_1(r) = \quad & \lambda_{11}\pi r + 2(\lambda_{20} + \lambda_{50} + 2\lambda_{70}) r^2, \\ f_2(r) = \quad & \lambda_{12}\pi r + \frac{2}{3}(\lambda_{21} + \lambda_{51} + 2\lambda_{71}) r^2 - \frac{\pi}{8}(\lambda_{30}\lambda_{50} - \lambda_{50}\lambda_{60} \\ & + 2\lambda_{30}\lambda_{70} + \lambda_{40}\lambda_{70} - 2\lambda_{60}\lambda_{70}) r^3, . \end{aligned} \tag{15}$$

Of course the above expression for the function $f_2(r)$ has been computed when $f_1(r) \equiv 0$.

Let $\#Z_+(f_i)$ denote the cardinal of $Z_+(f_i)$, the set of all positive simple zeros of the averaged function $f_i(r)$.

$\#Z_+(f_1)$	condition for $f_1 \equiv 0$	$\#Z_+(f_2)$	condition for $f_2 \equiv 0$	$\#Z_+(f_3)$
1	C_1	2	C_{11}	3
			C_{12}	2
			C_{13}	3

Table 1. The numbers of positive simple zeros of the averaged function $f_i(r)$ for $i = 1, 2, 3$.

Lemma 3. *The numbers $\#Z_+(f_i)$ for $i = 1, 2, 3$ are given in Table* 1 *under the conditions C_1, C_{11}, C_{12} and C_{13}, which are given in* (16), (17), (18) *and* (19), *respectively.*

Proof. Clearly from the expression of the polynomial $f_1(r)$ it follows that it can have at most one positive simple zero, and that there are polynomials $f_1(r)$ having 1 positive simple zero.

Note that $f_1(r) \equiv 0$ if and only if

$$\lambda_{11} = 0, \text{ and } \lambda_{20} = -\lambda_{50} - 2\lambda_{70}. \tag{16}$$

These two conditions are denoted by C_1 in Table 1.

When (16) holds from the expression of the polynomial $f_2(r)$ given in (15), we obtain that $f_2(r)$ has at most 2 positive simple zeros, and that there are polynomials $f_2(r)$ with 2 positive simple zeros.

Note that $f_2(r) \equiv 0$ if and only if either

$$\begin{aligned} &\lambda_{12} = 0,\ \lambda_{21} = -\lambda_{51} - 2\lambda_{71},\ \lambda_{70} \neq 0, \\ &\lambda_{40} = \frac{\lambda_{50}\lambda_{60} - \lambda_{30}\lambda_{50} - 2\lambda_{30}\lambda_{70} + 2\lambda_{60}\lambda_{70}}{\lambda_{70}}; \end{aligned} \tag{17}$$

or

$$\lambda_{12} = 0,\ \lambda_{21} = -\lambda_{51} - 2\lambda_{71},\ \lambda_{50} = 0,\ \lambda_{70} = 0; \tag{18}$$

or

$$\lambda_{12} = 0,\ \lambda_{21} = -\lambda_{51} - 2\lambda_{71},\ \lambda_{30} = \lambda_{60},\ \lambda_{70} = 0, \tag{19}$$

denoted by C_{11}, C_{12}, C_{13} in Table 1, respectively.

When (17) holds, we get

$$\begin{aligned} f_3(r) \;=\; & \lambda_{13}\pi r + \frac{2}{3}(\lambda_{22} + \lambda_{52} + 2\lambda_{72})r^2 - \frac{\pi}{8\lambda_{70}}(\lambda_{31}\lambda_{50}\lambda_{70} \\ & + \lambda_{30}\lambda_{51}\lambda_{70} - \lambda_{51}\lambda_{60}\lambda_{70} - \lambda_{50}\lambda_{61}\lambda_{70} + 2\lambda_{31}\lambda_{70}^2 \\ & + \lambda_{41}\lambda_{70}^2 - 2\lambda_{61}\lambda_{70}^2 - \lambda_{30}\lambda_{50}\lambda_{71} + \lambda_{50}\lambda_{60}\lambda_{71})r^3 \\ & + \frac{2}{15}(\lambda_{50} + 2\lambda_{70})(6\lambda_{30}\lambda_{60} - 3\lambda_{30}^2 + 4\lambda_{50}\lambda_{70} + 12\lambda_{70}^2)r^4. \end{aligned}$$

We obtain that $f_3(r)$ has at most 3 positive simple zeros, and that there are polynomials $f_3(r)$ with 3 positive simple zeros.

Similarly, when (18) (resp. (19)) holds, we get

$$\begin{aligned} f_3(r) &= \lambda_{13}\pi r + \frac{2}{3}(\lambda_{22} + \lambda_{52} + 2\lambda_{72})r^2 - \frac{\pi}{8}(\lambda_{30}\lambda_{51} \\ &\quad -\lambda_{60}\lambda_{51} + 2\lambda_{71}\lambda_{30} + \lambda_{40}\lambda_{71} - 2\lambda_{60}\lambda_{71})r^3 \end{aligned}$$

(resp.

$$\begin{aligned} f_3(r) &= \lambda_{13}\pi r + \frac{2}{3}(\lambda_{22} + \lambda_{52} + 2\lambda_{72})r^2 - \frac{\pi}{8}(\lambda_{31}\lambda_{50} \\ &\quad -\lambda_{50}\lambda_{61} + \lambda_{40}\lambda_{71})r^3 + \frac{2}{5}\lambda_{50}\lambda_{60}^2 r^4.) \end{aligned}$$

Further, we obtain that $f_3(r)$ has at most 2 (resp. 3) positive simple zeros, and that there are polynomials $f_3(r)$ with 2 (resp. 3) positive simple zeros. □

condition for $f_3 \equiv 0$	$\#Z_+(f_4)$	condition for $f_4 \equiv 0$	$\#Z_+(f_5)$
C_{111}	3	C_{1111}	3
		C_{1112}	4
C_{112}	3	C_{1121}	4
		C_{1122}	4
C_{113}	2	C_{1131}	4
C_{114}	4	C_{1141}	4
		C_{1142}	5
C_{121}	3	C_{1211}	3
		C_{1212}	4
		C_{1213}	3
C_{122}	2	C_{1221}	2
		C_{1222}	3
		C_{1223}	3
C_{123}	3	C_{1231}	3
		C_{1232}	3
		C_{1233}	2
C_{131}	3	C_{1311}	3
		C_{1312}	2
		C_{1313}	3
C_{132}	3	C_{1321}	3
		C_{1322}	3
		C_{1323}	2
C_{133}	3	C_{1331}	3

TABLE 2. The numbers of positive simple zeros of the averaged functions $f_4(r)$ and $f_5(r)$.

By the number of $\#Z_+(f_i)$ for $i = 1, 2, 3$ given in Lemma 3 we get the some lower bounds of the cyclicity of the Hopf bifurcation at the origin of the discontinuous differential system (2) with (7) and (8). In order to find a greater bound, we continues to consider averaged functions of higher orders.

Lemma 4. *The numbers* $\#Z_+(f_i)$ *for* $i = 4, 5, 6$ *are given in Table* 2 *under the conditions* C_{111}, C_{112}, C_{113}, C_{114}, C_{1141}, C_{1142}, $\ldots$ *which are given in* (20), (21), (22), (23), (24), (25), $\ldots$, *respectively.*

Proof. In Table 2 we list the conditions C_{1ij}'s for $f_3(r) \equiv 0$ and conditions C_{1ijk}'s for $f_4(r) \equiv 0$. We explain these conditions firstly. C_{1ij} denotes the j-th condition obtained from $f_3(r) \equiv 0$ under the conditions C_1 and C_{1i}, where C_1 and C_{1i} cause $f_1(r)$ and $f_2(r)$ to be identically zero respectively as given in Table 1. For example, either C_{111}, or C_{112}, or C_{113}, or C_{114} holds while solving $f_3(r) \equiv 0$ under C_1 and C_{11}. Either C_{121}, or C_{122}, or C_{123} holds while solving $f_3(r) \equiv 0$ under C_1 and C_{12}, and so on. C_{1ijk} denotes the k-th obtained from $f_4(r) \equiv 0$ under the conditions C_1, C_{1i} and C_{1ij}, where C_1, C_{1i} and C_{1ij} cause $f_1(r)$, $f_2(r)$ and $f_3(r)$ to be identically zero respectively. For example, either C_{1141}, or C_{1142} holds while solving $f_4(r) \equiv 0$ under C_1, C_{11} and C_{114}. Either C_{1211} or C_{1212} or C_{1213} holds from solving $f_4(r) \equiv 0$ under the conditions C_1, C_{12} and C_{121}, and so on.

Under C_1 and C_{11}, $f_1(r) \equiv f_2(r) \equiv 0$ and the expression of $f_3(r)$ is as given in (15). It is easy to check that $f_3(r) \equiv 0$ if and only if either

$$\begin{aligned} &\lambda_{13} = 0,\ \lambda_{22} = -\lambda_{52} - 2\lambda_{72},\ \lambda_{50} = -2\lambda_{70},\ \lambda_{70} \neq 0, \\ &\lambda_{41} = \frac{\lambda_{51}\lambda_{60} - \lambda_{51}\lambda_{30} - 2\lambda_{30}\lambda_{71} + 2\lambda_{60}\lambda_{71}}{\lambda_{70}}; \end{aligned} \tag{20}$$

or

$$\begin{aligned} &\lambda_{13} = 0,\ \lambda_{22} = -\lambda_{52} - 2\lambda_{72},\ \lambda_{50} = \frac{3\lambda_{30}^2 - 6\lambda_{30}\lambda_{60}}{4\lambda_{70}} - 3\lambda_{70}, \\ &\lambda_{30} = 0,\ \lambda_{31} = \frac{\lambda_{41}\lambda_{70} - \lambda_{51}\lambda_{60} + \lambda_{61}\lambda_{70} - 3\lambda_{60}\lambda_{71}}{\lambda_{70}},\ \lambda_{70} \neq 0; \end{aligned} \tag{21}$$

or

$$\begin{aligned} &\lambda_{13} = 0,\ \lambda_{22} = -\lambda_{52} - 2\lambda_{72},\ \lambda_{50} = \frac{3\lambda_{30}^2 - 6\lambda_{30}\lambda_{60}}{4\lambda_{70}} - 3\lambda_{70}, \\ &\lambda_{30} \neq 0,\ \lambda_{60} = \frac{\lambda_{30}}{2} - \frac{2\lambda_{70}^2}{3\lambda_{30}},\ \lambda_{70} \neq 0, \\ &\lambda_{51} = -\frac{2(3\lambda_{30}\lambda_{41}\lambda_{70} + 3\lambda_{30}^2\lambda_{71} + 4\lambda_{70}^2\lambda_{71})}{3\lambda_{30}^2 + 4\lambda_{70}^2}; \end{aligned} \tag{22}$$

or

$$\begin{aligned} &\lambda_{13} = 0,\ \lambda_{22} = -\lambda_{52} - 2\lambda_{72},\ \lambda_{50} = \frac{3\lambda_{30}^2 - 6\lambda_{30}\lambda_{60}}{4\lambda_{70}} - 3\lambda_{70}, \\ &\lambda_{30} \neq 0,\ \lambda_{60} \neq \frac{\lambda_{30}}{2} - \frac{2\lambda_{70}^2}{3\lambda_{30}},\ \lambda_{70} \neq 0, \\ &\lambda_{31} = (6\lambda_{30}\lambda_{60}\lambda_{61}\lambda_{70} - 3\lambda_{30}^2\lambda_{61}\lambda_{70} + 4\lambda_{30}\lambda_{51}\lambda_{70}^2 - 4\lambda_{51}\lambda_{60}\lambda_{70}^2 \\ &\qquad +4\lambda_{41}\lambda_{70}^3 + 4\lambda_{61}\lambda_{70}^3 - 3\lambda_{30}^3\lambda_{71} + 9\lambda_{30}^2\lambda_{60}\lambda_{71} - 6\lambda_{30}\lambda_{60}^2\lambda_{71} \\ &\qquad +12\lambda_{30}\lambda_{70}^2\lambda_{71} - 12\lambda_{60}\lambda_{70}^2\lambda_{71})/(\lambda_{70}(6\lambda_{30}\lambda_{60} - 3\lambda_{30}^2 + 4\lambda_{70}^2)); \end{aligned} \tag{23}$$

denoted by $C_{111}, C_{112}, C_{113}, C_{114}$ in Table 2 , respectively.

Under C_{114}, we compute $f_4(r)$ and obtain

$$\begin{aligned} f_4(r) \quad = \quad & \lambda_{14}\pi r + 2(\lambda_{23} + \lambda_{53} + 2\lambda_{73})r^2/3 + (-9\lambda_{30}^4\lambda_{32}\lambda_{70} + 36\lambda_{30}^3\lambda_{32}\lambda_{60}\lambda_{70} \\ & -36\lambda_{30}^2\lambda_{32}\lambda_{60}^2\lambda_{70} + 9\lambda_{30}^4\lambda_{62}\lambda_{70} - 36\lambda_{30}^3\lambda_{60}\lambda_{62}\lambda_{70} + 36\lambda_{30}^2\lambda_{60}^2\lambda_{62}\lambda_{70} \end{aligned}$$

$$
\begin{aligned}
&-12\lambda_{30}^3\lambda_{52}\lambda_{70}^2+36\lambda_{30}^2\lambda_{52}\lambda_{60}\lambda_{70}^2-24\lambda_{30}\lambda_{52}\lambda_{60}^2\lambda_{70}^2+24\lambda_{30}^2\lambda_{32}\lambda_{70}^3\\
&-12\lambda_{30}^2\lambda_{42}\lambda_{70}^3+16\lambda_{30}\lambda_{51}^2\lambda_{70}^3-48\lambda_{30}\lambda_{32}\lambda_{60}\lambda_{70}^3+24\lambda_{30}\lambda_{42}\lambda_{60}\lambda_{70}^3\\
&-16\lambda_{51}^2\lambda_{60}\lambda_{70}^3-24\lambda_{30}^2\lambda_{62}\lambda_{70}^3+48\lambda_{30}\lambda_{60}\lambda_{62}\lambda_{70}^3+16\lambda_{41}\lambda_{51}\lambda_{70}^4\\
&+16\lambda_{30}\lambda_{52}\lambda_{70}^4-16\lambda_{52}\lambda_{60}\lambda_{70}^4-16\lambda_{32}\lambda_{70}^5+16\lambda_{42}\lambda_{70}^5+16\lambda_{62}\lambda_{70}^5\\
&-12\lambda_{30}^3\lambda_{51}\lambda_{70}\lambda_{71}+36\lambda_{30}^2\lambda_{51}\lambda_{60}\lambda_{70}\lambda_{71}-24\lambda_{30}\lambda_{51}\lambda_{60}^2\lambda_{70}\lambda_{71}\\
&-12\lambda_{30}^2\lambda_{41}\lambda_{70}^2\lambda_{71}+24\lambda_{30}\lambda_{41}\lambda_{60}\lambda_{70}^2\lambda_{71}+80\lambda_{30}\lambda_{51}\lambda_{70}^3\lambda_{71}+48\lambda_{41}\lambda_{70}^4\lambda_{71}\\
&-80\lambda_{51}\lambda_{60}\lambda_{70}^3\lambda_{71}-24\lambda_{30}^3\lambda_{70}\lambda_{71}^2+72\lambda_{30}^2\lambda_{60}\lambda_{70}\lambda_{71}^2-48\lambda_{30}\lambda_{60}^2\lambda_{70}\lambda_{71}^2\\
&+96\lambda_{30}\lambda_{70}^3\lambda_{71}^2-96\lambda_{60}\lambda_{70}^3\lambda_{71}^2+9\lambda_{30}^5\lambda_{72}-45\lambda_{30}^4\lambda_{60}\lambda_{72}+72\lambda_{30}^3\lambda_{60}^2\lambda_{72}\\
&-36\lambda_{30}^2\lambda_{60}^3\lambda_{72}-48\lambda_{30}^3\lambda_{70}^2\lambda_{72}+144\lambda_{30}^2\lambda_{60}\lambda_{70}^2\lambda_{72}-96\lambda_{30}\lambda_{60}^2\lambda_{70}^2\lambda_{72}\\
&+48\lambda_{30}\lambda_{70}^4\lambda_{72}-48\lambda_{60}\lambda_{70}^4\lambda_{72})\pi r^3)/(32\lambda_{70}^2(3\lambda_{30}^2-6\lambda_{30}\lambda_{60}-4\lambda_{70}^2))\\
&-(-18\lambda_{30}^2\lambda_{60}\lambda_{61}\lambda_{70}+36\lambda_{30}\lambda_{60}^2\lambda_{61}\lambda_{70}-36\lambda_{30}^2\lambda_{51}\lambda_{70}^2+72\lambda_{30}\lambda_{51}\lambda_{60}\lambda_{70}^2\\
&-24\lambda_{51}\lambda_{60}^2\lambda_{70}^2-24\lambda_{30}\lambda_{41}\lambda_{70}^3+24\lambda_{41}\lambda_{60}\lambda_{70}^3+24\lambda_{60}\lambda_{61}\lambda_{70}^3+16\lambda_{51}\lambda_{70}^4\\
&+9\lambda_{30}^4\lambda_{71}-36\lambda_{30}^3\lambda_{60}\lambda_{71}+54\lambda_{30}^2\lambda_{60}^2\lambda_{71}-36\lambda_{30}\lambda_{60}^3\lambda_{71}-96\lambda_{30}^2\lambda_{70}^2\lambda_{71}\\
&+192\lambda_{30}\lambda_{60}\lambda_{70}^2\lambda_{71}-72\lambda_{60}^2\lambda_{70}^2\lambda_{71}+48\lambda_{70}^4\lambda_{71})r^4/(30\lambda_{70}^2)\\
&-\lambda_{30}(\lambda_{30}^2-2\lambda_{30}\lambda_{60}-8\lambda_{70}^2)(3\lambda_{30}^2-6\lambda_{30}\lambda_{60}-4\lambda_{70}^2)(3\lambda_{30}^2-3\lambda_{30}\lambda_{60}\\
&-2\lambda_{70}^2)\pi r^5/(512\lambda_{70}^3).
\end{aligned}
$$

We obtain that $f_4(r)$ has at most 4 positive simple zeros, and that there are polynomials $f_4(r)$ with 4 positive simple zeros.

Straight computations show that $f_4(r)\equiv 0$ if and only if either

$$
\begin{aligned}
&\lambda_{14}=0,\ \lambda_{23}=-\lambda_{53}-2\lambda_{73},\ \lambda_{60}=\lambda_{30}-2\lambda_{70}^2/(3\lambda_{30}),\lambda_{30}\neq 0,\lambda_{70}\neq 0,\\
&\lambda_{32}=(27\lambda_{30}^5\lambda_{62}+36\lambda_{30}^3\lambda_{42}\lambda_{70}^2+48\lambda_{30}\lambda_{41}\lambda_{51}\lambda_{70}^3+24\lambda_{30}^2\lambda_{52}\lambda_{70}^3+32\lambda_{51}^2\lambda_{70}^4\\
&\quad+36\lambda_{30}^3\lambda_{41}\lambda_{70}\lambda_{71}+24\lambda_{30}^2\lambda_{51}\lambda_{70}^2\lambda_{71}+96\lambda_{30}\lambda_{41}\lambda_{70}^3\lambda_{71}+128\lambda_{51}\lambda_{70}^4\lambda_{71}\\
&\quad+48\lambda_{30}^2\lambda_{70}^2\lambda_{71}^2+128\lambda_{70}^4\lambda_{71}^2+18\lambda_{30}^4\lambda_{70}\lambda_{72}+48\lambda_{30}^2\lambda_{70}^3\lambda_{72})/(27\lambda_{30}^5),\\
&\lambda_{41}=(54\lambda_{30}^5\lambda_{61}\lambda_{70}+36\lambda_{30}^4\lambda_{51}\lambda_{70}^2-36\lambda_{30}^3\lambda_{61}\lambda_{70}^3-32\lambda_{51}\lambda_{70}^6-27\lambda_{30}^6\lambda_{71}\\
&\quad+144\lambda_{30}^4\lambda_{70}^2\lambda_{71}-24\lambda_{30}^2\lambda_{70}^4\lambda_{71}-64\lambda_{70}^6\lambda_{71})/(48\lambda_{30}\lambda_{70}^5);
\end{aligned}
\tag{24}
$$

or

$$\lambda_{14}=0,\ \lambda_{23}=-\lambda_{53}-2\lambda_{73},\ \lambda_{60}=(\lambda_{30}^2-8\lambda_{70}^2)/(2\lambda_{30}),$$

$$\lambda_{51}=3(-2\lambda_{30}^3\lambda_{41}\lambda_{70}-10\lambda_{30}^3\lambda_{61}\lambda_{70}-16\lambda_{30}\lambda_{41}\lambda_{70}^3+80\lambda_{30}\lambda_{61}\lambda_{70}^3 +3\lambda_{30}^4\lambda_{71}-72\lambda_{30}^2\lambda_{70}^2\lambda_{71}+192\lambda_{70}^4\lambda_{71})/(3\lambda_{30}^4+88\lambda_{30}^2\lambda_{70}^2+192\lambda_{70}^4),$$

$$\begin{aligned}\lambda_{52}=&(-90\lambda_{30}^9\lambda_{32}\lambda_{70}-18\lambda_{30}^9\lambda_{42}\lambda_{70}+36\lambda_{30}^8\lambda_{41}\lambda_{61}\lambda_{70}+180\lambda_{30}^8\lambda_{61}^2\lambda_{70}\\ &+90\lambda_{30}^9\lambda_{62}\lambda_{70}-5280\lambda_{30}^7\lambda_{32}\lambda_{70}^3-96\lambda_{30}^6\lambda_{41}^2\lambda_{70}^3-1056\lambda_{30}^7\lambda_{42}\lambda_{70}^3\\ &-192\lambda_{30}^6\lambda_{41}\lambda_{61}\lambda_{70}^3-1440\lambda_{30}^6\lambda_{61}^2\lambda_{70}^3+5280\lambda_{30}^7\lambda_{62}\lambda_{70}^3\\ &-88960\lambda_{30}^5\lambda_{32}\lambda_{70}^5-768\lambda_{30}^4\lambda_{41}^2\lambda_{70}^5-17792\lambda_{30}^5\lambda_{42}\lambda_{70}^5\\ &+1536\lambda_{30}^4\lambda_{41}\lambda_{61}\lambda_{70}^5-11520\lambda_{30}^4\lambda_{61}^2\lambda_{70}^5+88960\lambda_{30}^5\lambda_{62}\lambda_{70}^5\\ &-337920\lambda_{30}^3\lambda_{32}\lambda_{70}^7-67584\lambda_{30}^3\lambda_{42}\lambda_{70}^7-18432\lambda_{30}^2\lambda_{41}\lambda_{61}\lambda_{70}^7\\ &+92160\lambda_{30}^2\lambda_{61}^2\lambda_{70}^7+337920\lambda_{30}^3\lambda_{62}\lambda_{70}^7-368640\lambda_{30}\lambda_{32}\lambda_{70}^9\\ &-73728\lambda_{30}\lambda_{42}\lambda_{70}^9+368640\lambda_{30}\lambda_{62}\lambda_{70}^9-18\lambda_{30}^9\lambda_{41}\lambda_{71}\\ &-90\lambda_{30}^9\lambda_{61}\lambda_{71}-240\lambda_{30}^7\lambda_{41}\lambda_{70}^2\lambda_{71}+3120\lambda_{30}^7\lambda_{61}\lambda_{70}^2\lambda_{71}\\ &-9216\lambda_{30}^5\lambda_{41}\lambda_{70}^4\lambda_{71}-15360\lambda_{30}^3\lambda_{41}\lambda_{70}^6\lambda_{71}-199680\lambda_{30}^3\lambda_{61}\lambda_{70}^6\lambda_{71}\\ &-73728\lambda_{30}\lambda_{41}\lambda_{70}^8\lambda_{71}+368640\lambda_{30}\lambda_{61}\lambda_{70}^8\lambda_{71}-1440\lambda_{30}^8\lambda_{70}\lambda_{71}^2\\ &-7680\lambda_{30}^6\lambda_{70}^3\lambda_{71}^2-61440\lambda_{30}^4\lambda_{70}^5\lambda_{71}^2-737280\lambda_{30}^2\lambda_{70}^7\lambda_{71}^2\\ &+27\lambda_{30}^{10}\lambda_{72}+1800\lambda_{30}^8\lambda_{70}^2\lambda_{72}+39360\lambda_{30}^6\lambda_{70}^4\lambda_{72}+314880\lambda_{30}^4\lambda_{70}^6\lambda_{72}\\ &+921600\lambda_{30}^2\lambda_{70}^8\lambda_{72}+884736\lambda_{70}^10\lambda_{72})/((\lambda_{30}^2+8\lambda_{70}^2)(3\lambda_{30}^4\\ &+88\lambda_{30}^2\lambda_{70}^2+192\lambda_{70}^4)^2),\end{aligned} \tag{25}$$

$$\lambda_{30}\neq 0,\ \lambda_{70}\neq 0,$$

denoted by C_{1141} and C_{1142} in Table 2, respectively.

Under C_{1142}, we compute $f_5(r)$ and obtain

$$\begin{aligned}f_5(r)=\ &\lambda_{15}\pi r+2(\lambda_{24}+\lambda_{54}+2\lambda_{74})r^2/3+f_{5,3}r^3+f_{5,4}r^4\\ &+5\lambda_{30}^2\lambda_{70}(3\lambda_{30}^2+20\lambda_{70}^2)(\lambda_{30}^2\lambda_{41}+5\lambda_{30}^2\lambda_{61}+8\lambda_{41}\lambda_{70}^2\\ &-40\lambda_{61}\lambda_{70}^2+80\lambda_{30}\lambda_{70}\lambda_{71})\pi r^5/(32(3\lambda_{30}^4+88\lambda_{30}^2\lambda_{70}^2+192\lambda_{70}^4))\\ &-32\lambda_{70}^3(3\lambda_{30}^2+20\lambda_{70}^2)r^6/7,\end{aligned}$$

where $f_{5,3}$ and $f_{5,4}$ are rational functions in the λ's, and their numerators have 179 and 49 terms respectively. Clearly $f_5(r)$ has at most 5 positive simple zeros and cannot be identically zero because the coefficient of r^6 is nonzero. Moreover, by the Mathematica commander *MatrixRank* we find that the matrix

$$\begin{pmatrix}\frac{\partial f_{5,1}}{\partial\lambda_{15}} & \frac{\partial f_{5,1}}{\partial\lambda_{24}} & \cdots & \frac{\partial f_{5,1}}{\partial\lambda_{74}}\\ \frac{\partial f_{5,2}}{\partial\lambda_{15}} & \frac{\partial f_{5,2}}{\partial\lambda_{24}} & \cdots & \frac{\partial f_{5,2}}{\partial\lambda_{74}}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial f_{5,6}}{\partial\lambda_{15}} & \frac{\partial f_{5,6}}{\partial\lambda_{24}} & \cdots & \frac{\partial f_{5,6}}{\partial\lambda_{74}}\end{pmatrix}_{6\times 18}$$

has rank 6, where the $f_{5,i}$'s are the coefficients of r^i's of $f_5(r)$. Here all $f_{5,i}$'s have rational forms in $(\lambda_{15},\lambda_{24},...,\lambda_{74})\in\mathbb{R}^{18}$. Hence, from Lemma 4.5 of [11] we obtain that there are polynomials $f_5(r)$ having 5 positive simple zeros as given in Table 2.

Similarly, for other cases we obtain the upper bounds of the number of positive simple zeros for $f_4(r)$ and $f_5(r)$ respectively and list them in Table 2. □

$condition for f_4 \equiv 0$	$condition for f_5 \equiv 0$	$\#Z_+(f_6)$
C_{1111}	C_{11111}	3
	C_{11112}	4
C_{1112}	C_{11121}	4
C_{1121}	C_{11211}	4
C_{1122}	C_{11221}	4
C_{1131}	C_{11311}	4
C_{1141}	C_{11411}	4
C_{1211}	C_{12111}	3
	C_{12112}	3
	C_{12113}	3
C_{1212}	C_{12121}	3
	C_{12122}	3
	C_{12123}	4
C_{1213}	C_{12131}	3
	C_{12132}	4
	C_{12133}	3
C_{1221}	C_{12211}	2
	C_{12212}	3
	C_{12213}	3
C_{1222}	C_{12221}	3
	C_{12222}	3
	C_{12223}	2
C_{1223}	C_{12231}	3
	C_{12232}	4
	C_{12233}	3
C_{1231}	C_{12311}	2
	C_{12312}	3
	C_{12313}	3
C_{1232}	C_{12321}	3
	C_{12322}	3
	C_{12323}	2
C_{1233}	C_{12331}	3
C_{1311}	C_{13111}	3
	C_{13112}	3
	C_{13113}	2
C_{13121}	C_{13121}	3
C_{1313}	C_{13131}	3
	C_{13132}	3
	C_{13133}	2
C_{1321}	C_{13211}	3
	C_{13212}	3
	C_{13213}	2
C_{1322}	C_{13221}	3
	C_{13222}	3
	C_{13223}	2
C_{1323}	C_{13231}	3
C_{1331}	C_{13311}	4

TABLE 3. The number of positive simple zeros of the averaged function $f_6(r)$

Proof of Theorem 1. By Proposition 2, we only need to consider (2) with (7) and (8). The result of the Hopf cyclicity of at least 5 follows immediately from Lemma 4. In order to consider if the cyclicity of the Hopf bifurcation at the origin of (2) with (7) and (8) can be greater than 5, we continue to consider the number of positive simple zeros of the averaged function $f_6(r)$ when $f_1(r) \equiv ... \equiv f_5(r) \equiv 0$. Using the same method as in the proof of Lemma 4, we give the number $\#Z_+(f_6)$ in Table 3 according with the different conditions satisfied by the parameters. Since the maximum number of positive simple zeros of the averaged function $f_6(r)$ is at most 4 as it follows from Table 3, and this number is smaller than the maximum number (5) obtained for the number of positive simple zeros of the averaged function $f_5(r)$, we do not provide here the details for obtaining the results described in Table 3. □

From Theorem 1 we see that, in order to find a greater lower bound of the cyclicity of the Hopf bifurcation at the origin of the discontinuous differential system (2) than 5, it is necessary to consider some averaged functions $f_i(r)$'s of higher orders than 6, but when we compute all averaged functions of order 6 we do not find more limit cycles that the ones found with averaged functions of order 5. This is the reason for which we stop the computations at order 6.

4. Averaged functions and Liapunov constants

From [2] we know the Liapunov constants of the quadratic system (10) with $\lambda_7 = 0$ are

$$\begin{aligned} V_1 &= \lambda_1, \\ V_3 &= -\lambda_5(\lambda_3 - \lambda_6), \\ V_5 &= \lambda_2\lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5(\lambda_3 - \lambda_6)), \\ V_7 &= -\lambda_2\lambda_4(\lambda_3 - \lambda_6)^2(\lambda_6(\lambda_3 - 2\lambda_6) - \lambda_2^2). \end{aligned}$$

We remark that the coefficients of odd powers in r, of the averaged functions f_1 and f_2 given in (15) and of the averaged functions f_3 coming from the conditions (18) and (19) for $\lambda_7 = 0$, are the first and the second Liapunov constants of the quadratic systems. This relationship between the coefficients of odd powers in r of the averaged functions and the Liapunov constants also occurs with other averaged functions of higher orders, and does not happen when λ_7 is nonzero, for instance see the averaged function f_3 coming from condition (17).

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