This is a preprint of: "Transcritical and zero-Hopf bifurcations in the Genesio system", Pedro T. Cardin, Jaume Llibre, *Nonlinear Dynam.*, vol. 88, 547–553, 2017.
DOI: [10.1007/s11071-016-3259-2]

TRANSCRITICAL AND ZERO–HOPF BIFURCATIONS IN THE GENESIO SYSTEM

PEDRO TONIOL CARDIN AND JAUME LLIBRE

Abstract. In this paper we study the existence of transcritical and zero–Hopf bifurcations of the third–order ordinary differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx - x^2 = 0,$$

called the Genesio equation, which has a unique quadratic nonlinear term and three real parameters. More precisely, writing this differential equation as a first order differential system in $\mathbb{R}^3$ we prove: first that the system exhibits a transcritical bifurcation at the equilibrium point located at the origin of coordinates when $c = 0$ and the parameters (a, b) are in the set $\{(a, b) \in \mathbb{R}^2 : b \neq 0\} \setminus \{(0, b) \in \mathbb{R}^2 : b > 0\}$, and second that the system has a zero–Hopf bifurcation producing a periodic orbit near the equilibrium point located at the origin when $a = c = 0$ and $b > 0$.

1. Introduction

In [4] Genesio and Tesi, inspired by the problem of determining conditions under which a nonlinear dynamical system presents chaotic behaviour, introduced the following third–order ordinary differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx - x^2 = 0, \tag{1}$$

where a, b and c are parameters and the dot indicates derivative with respect to the time t. If we define $y = \dot{x}$ and $z = \dot{y}$ the differential equation (1) becomes the first–order differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -cx - by - az + x^2, \end{aligned} \tag{2}$$

which is commonly known as the *Genesio system*. Based on the harmonic balance principle the authors in [4] presented two practical methods for predicting the existence and the location of chaotic motions. For instance, system (2) exhibits chaotic dynamical behaviors when $a = 1.2$, $b = 2.92$ and $c = 6$.

1991 *Mathematics Subject Classification.* 34C23; 34C25; 37G10.

Key words and phrases. Genesio system, transcritical bifurcation, zero–Hopf bifurcation, averaging theory.

We can find in the literature several articles concerning system (2). For instance, issues on synchronization of Genesio chaotic system have been studied in the articles [9, 10, 3, 15]. Already in [16] the authors studied the Hopf bifurcation and the existence of Silnikov homoclinic orbit for this system. Stability analysis and Hopf bifurcation of the Genesio system with distributed delay feedback has been studied in [5].

In this paper we have two main objectives. The first one is to show that system (2) exhibits a transcritical bifurcation, i.e. there is an exchange of stability that takes place at the critical points of this system at a certain bifurcation value. The analysis of transcritical bifurcation occurring in the Genesio system will be carried out with respect to the parameter c.

The second objective is to study the existence of the zero–Hopf equilibria and of the zero–Hopf bifurcations in the Genesio system (2). We recall that a *zero–Hopf equilibrium* of a three–dimensional autonomous differential system is an isolated equilibrium point of system whose linear part at the equilibrium has a zero eigenvalue and a pair of purely imaginary eigenvalues.

Usually the main tool for studying a zero–Hopf bifurcation is to pass the system to the normal form of a zero–Hopf bifurcation. However, our analysis of the zero–Hopf bifurcation occurring in the Genesio system will use the averaging theory, a summary of the results of this theory that we need here is given in section 2. The averaging theory has already been used to study Hopf and zero–Hopf bifurcations in some others differential systems, see for instance [1, 2, 7, 8].

As far as we know nobody has studied the existence or non–existence of transcritical bifurcations, zero–Hopf equilibria, and zero–Hopf bifurcations in the Genesio system (2).

Our main results are the following ones.

Theorem 1. *Consider the Genesio system* (2) *and assume that the parameters a and b vary in the set K given by*

$$K = \{(a, b) \in \mathbb{R}^2 : b \neq 0\} \setminus \{(0, b) \in \mathbb{R}^2 : b > 0\}.$$

Then system (2) *exhibits a transcritical bifurcation at the equilibrium point located at the origin of coordinates when $c = 0$.*

Next proposition characterizes when the equilibrium points of system (2) are zero–Hopf equilibria.

Proposition 1. *Differential system* (2) *has a unique zero–Hopf equilibrium localized at the origin of coordinates when $a = c = 0$ and $b > 0$.*

In what follows we shall study when the Genesio system (2) having a zero–Hopf equilibrium point at the origin of coordinates has a zero–Hopf bifurcation producing some periodic orbit. For doing this we consider ε–perturbations of the values of the parameters for which system (2) has a

zero–Hopf equilibrium. The small parameter ε is necessary in order to apply the averaging theory, and the analysis of the zero–Hopf bifurcation will be carried out with respect to it.

Theorem 2. *Consider the Genesio system* (2) *with the parameters* $a = \varepsilon\alpha$, $b = \omega^2 + \varepsilon\beta$ *and* $c = \varepsilon\gamma$, *with* $\omega > 0$ *and* ε *a sufficiently small parameter. Then this system exhibits a zero–Hopf bifurcation at the equilibrium point located at the origin of coordinates when* $\varepsilon = 0$ *if* $\gamma^2 - \alpha^2\omega^4 > 0$. *Moreover, the periodic orbit* $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ *bifurcating from this equilibrium point satisfies that* $(x(0,\varepsilon), y(0,\varepsilon), z(0,\varepsilon))$ *is*

$$\varepsilon\left(\frac{\gamma - \alpha\omega^2}{2} - \frac{\sqrt{\gamma^2 - \alpha^2\omega^4}}{\sqrt{2}}, 0, \frac{\omega^2\sqrt{\gamma^2 - \alpha^2\omega^4}}{\sqrt{2}}\right) + \mathcal{O}(\varepsilon^2),$$

if $\varepsilon > 0$ *is sufficiently small. If* $\lambda_\pm = (-\alpha\omega^2 \pm \sqrt{3\alpha^2\omega^4 - 2\gamma^2})/(2\omega^3)$, *then this periodic orbit is stable when* $Re(\lambda_\pm) < 0$, *and unstable if* $Re(\lambda_+) > 0$ *or* $Re(\lambda_-) > 0$.

Theorem 1 is proved in section 3, and Proposition 1 and Theorem 2 are proved in section 4. The rest of the article is organized as follows. In section 2, we present the basic definitions and results necessary for proving Theorems 1 and 2.

2. Preliminaries

2.1. **Transcritical bifurcation.** Consider the following differential equation in $\mathbb{R}^n$

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu) \tag{3}$$

depending on a parameter $\mu \in \mathbb{R}$. We assume that f is enough differentiable. The following theorem (see [13]) states the necessary conditions in order that system (3) exhibits a transcritical bifurcation. See also [6] page 149, or [11] page 338. We will use Theorem 3 for proving Theorem 1.

In the theorem below we use the notation $D_{\mathbf{x}}f$ to denote the Jacobian matrix of the function f. We also use the notation $(\partial f/\partial\mu)$ to indicate the vector of partial derivatives of the components of f with respect to $\mu \in \mathbb{R}$. A^T will denote the transpose of the matrix A.

Theorem 3. *Consider the one–parameter family* (3) *and assume that there is* $\mathbf{x}_0 \in \mathbb{R}^n$ *such that* $f(\mathbf{x}_0, \mu) = 0$ *for all* μ, *i.e.* $\mathbf{x}_0$ *is an equilibrium point of system* (3) *for all parameter values. Furthermore, when* $\mu = \mu_0$ *suppose that the following hypotheses hold.*

(H1) *The Jacobian matrix* $M = D_{\mathbf{x}}f(\mathbf{x}_0, \mu_0)$ *has a simple eigenvalue* $\lambda = 0$ *with eigenvector* v, *and* M^T *has an eigenvector* w *corresponding to the eigenvalue* $\lambda = 0$.

(H2) *M has k eigenvalues with negative real parts, and $n-k-1$ eigenvalues with positive real parts.*

(H3) $w^T\big((\partial f/\partial\mu)(\mathbf{x}_0,\mu_0)\big)=0$.

(H4) $w^T\big(D_{\mathbf{x}}(\partial f/\partial\mu)(\mathbf{x}_0,\mu_0)v\big)\neq 0$.

(H5) $w^T\big(D^2_{\mathbf{x}}f(\mathbf{x}_0,\mu_0)(v,v)\big)\neq 0$.

Then system (3) *exhibits a transcritical bifurcation at the equilibrium point* $\mathbf{x}_0$ *at the bifurcation value* $\mu=\mu_0$.

2.2. **Averaging theory.** In this subsection we present some basic results on the averaging theory, which will be used in the proof of Theorem 2. For a general introduction to the averaging theory see for instance the book of Sanders, Verhulst and Murdock [12].

Consider the following initial value problem

$$\dot{\mathbf{x}}=\varepsilon F(t,\mathbf{x})+\varepsilon^2 G(t,\mathbf{x},\varepsilon),\quad \mathbf{x}(0)=\mathbf{x}_0, \tag{4}$$

and the averaged differential equation

$$\dot{\mathbf{y}}=\varepsilon f(\mathbf{y}),\quad \mathbf{y}(0)=\mathbf{x}_0. \tag{5}$$

In equations (4) and (5), $\mathbf{x},\mathbf{y}\in D$, where $D\subset\mathbb{R}^n$ is an open set, $t\in[0,\infty)$ and ε is a small positive parameter. The functions $F:[0,\infty)\times D\to\mathbb{R}^n$ and $G:[0,\infty)\times D\times(0,\varepsilon_0]\to\mathbb{R}^n$ are assumed be periodic of period T in the variable t, and $f:D\to\mathbb{R}^n$ is given by

$$f(\mathbf{y})=\frac{1}{T}\int_0^T F(t,\mathbf{y})dt. \tag{6}$$

The next theorem establishes that, under certain conditions, the equilibrium points of the averaged equation (5) correspond to T–periodic solutions of system (4). See [14] for a proof.

Theorem 4. *Consider the initial value problems* (4) *and* (5) *and suppose that F, its Jacobian $D_{\mathbf{x}}F$, its Hessian $D_{\mathbf{xx}}F$, G and its Jacobian $D_{\mathbf{x}}G$ are continuous and bounded by a constant independent of ε in $[0,\infty)\times D$ and $\varepsilon\in(0,\varepsilon_0]$. Further we assume that F and G are T–periodic in t, with T independent of ε. Then the following statements hold.*

(a) *For $t\in[0,1/\varepsilon]$ we have $\mathbf{x}(t)-\mathbf{y}(t)=\mathcal{O}(\varepsilon)$ as $\varepsilon\to 0$.*

(b) *If p is an equilibrium point of system* (5) *such that*

$$\det D_{\mathbf{y}}f(p)\neq 0, \tag{7}$$

then there exists a periodic solution $\mathbf{x}(t,\varepsilon)$ of period T of system (4) *such that $\mathbf{x}(0,\varepsilon)-p=\mathcal{O}(\varepsilon)$ as $\varepsilon\to 0$.*

(c) *If all the real parts of the eigenvalues of the matrix $D_{\mathbf{y}}f(p)$ are negative, then the periodic solution $\mathbf{x}(t,\varepsilon)$ is stable. If some real part of the eigenvalues is positive, then the periodic solution $\mathbf{x}(t,\varepsilon)$ is unstable.*

3. Proof of Theorem 1

We recall that the analysis of transcritical bifurcation occurring in the Genesio system will be carried out with respect to the parameter c. So using the notation of subsection 2.1, we have $\mu = c$ and the vector field f associated with the Genesio system (2) is given by

$$f(\mathbf{x}, c) = (y, z, -cx - by - az + x^2),$$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Note that, in order to simplify the notation, we are using (x, y, z) instead of (x_1, x_2, x_3).

The vector field f has two equilibrium points $\mathbf{x}_0 = (0, 0, 0)$ and $\mathbf{x}_c = (c, 0, 0)$ which collide at the origin when $c = 0$. Moreover, when $c = 0$ we have that the matrix

$$M = D_{\mathbf{x}}f(\mathbf{x}_0, 0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -b & -a \end{pmatrix}$$

has a simple eigenvalue $\lambda = 0$. In fact, the characteristic polynomial of M is given by

$$p(\lambda) = -\lambda^3 - a\lambda^2 - b\lambda,$$

whose roots are

$$\lambda = 0, \quad \lambda_{\pm} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

Since by hypothesis the parameters a and b belong to the set $K = \{(a, b) \in \mathbb{R}^2 : b \neq 0\} \setminus \{(0, b) \in \mathbb{R}^2 : b > 0\}$, then both eigenvalues $\lambda_{\pm}$ have nonzero real part.

The transcritical bifurcation is characterized by the exchange of stability of the equilibrium point $\mathbf{x}_c = (c, 0, 0)$ when the parameter c passes through the bifurcation value $c = 0$. Note that it is a difficult task to study the stability of the equilibrium point $\mathbf{x}_c$, for $c \neq 0$, by analyzing the roots of the characteristic polynomial of the matrix $D_{\mathbf{x}}f(\mathbf{x}_c, c)$, that is the polynomial $q(\lambda) = -\lambda^3 - a\lambda^2 - b\lambda + c$. Thus, we will use Theorem 3 to show that the system (2) exhibits a transcritical bifurcation.

Note that the vectors $v = (1, 0, 0)$ and $w = (b, a, 1)$ are eigenvectors of the matrices M and M^T, respectively, corresponding to the eigenvalue $\lambda = 0$. Furthermore, we have that

$$w^T\big((\partial f/\partial c)(\mathbf{x}_0, 0)\big) = \begin{pmatrix} b & a & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0,$$

$$w^T\big(D_{\mathbf{x}}(\partial f/\partial c)(\mathbf{x}_0,0)v\big) = \begin{pmatrix} b & a & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -1 \neq 0,$$

$$w^T\big(D^2_{\mathbf{x}} f(\mathbf{x}_0,0)(v,v)\big) = \begin{pmatrix} b & a & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = 2 \neq 0.$$

Thus, all the hypotheses of Theorem 3 are satisfied. Therefore, the system (2) exhibits a transcritical bifurcation at the equilibrium point at the origin at the bifurcation value $c = 0$. This completes the proof of Theorem 1.

4. Proof of Proposition 1 and Theorem 2

Proof of Proposition 1. We saw that the characteristic polynomial of the linear part of system (2) at the equilibrium point $\mathbf{x}_c = (c, 0, 0)$ is $q(\lambda) = -\lambda^3 - a\lambda^2 - b\lambda + c$. We want to find the parameter values for which the polynomial q has a zero eigenvalue and a pair of purely imaginary eigenvalues, that is the parameter values for which q is of the form $-\lambda(\lambda^2 + B)$ with $B > 0$. In order to simplify the expressions, we will put $B = \omega^2$, with $\omega > 0$. Thus, imposing the condition $q(\lambda) = -\lambda(\lambda^2 + \omega^2)$, we obtain that $a = c = 0$ and $b = \omega^2$. Hence, when $a = c = 0$ and $b > 0$ there is a unique zero–Hopf equilibrium point at the origin of coordinates. Moreover, if we put $b = \omega^2$, with $\omega > 0$, then the eigenvalues are 0 and $\pm i\omega$. This completes the proof of Proposition 1. □

Proof of Theorem 2. We shall use the averaging theory of first order described in subsection 2.2 (see Theorem 4) in order to study if from the zero–Hopf equilibrium point located at the origin of coordinates, it bifurcates some periodic orbit by moving the parameters a, b and c of system (2). Thus, let the parameters a, b and c of system (2) be given by $a = \varepsilon\alpha$, $b = \omega^2 + \varepsilon\beta$ and $c = \varepsilon\gamma$, with $\varepsilon > 0$ a sufficiently small parameter. Then, the Genesio system (2) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\varepsilon\gamma x - (\omega^2 + \varepsilon\beta)y - \varepsilon\alpha z + x^2. \end{aligned} \tag{8}$$

The first step in order to write our differential system (8) in the normal form for applying the averaging theory is to write the linear part at the origin of system (8) when $\varepsilon = 0$ into its real Jordan normal form, that is into the form

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To do this, we apply the linear change of variables $(x, y, z) \to (X, Y, Z)$, where

$$x = \frac{Z - \omega X}{\omega^2}, \quad y = Y, \quad z = \omega X. \tag{9}$$

In the new variables (X, Y, Z), system (8) becomes

$$\begin{aligned} \dot{X} &= \frac{Z^2 - 2\omega XZ + \omega^2 X^2 - \omega^6 Y}{\omega^5} + \varepsilon \frac{(\gamma - \alpha\omega^2)\omega X - \beta\omega^2 Y - \gamma Z}{\omega^3}, \\ \dot{Y} &= \omega X, \\ \dot{Z} &= \frac{(\omega X - Z)^2}{\omega^4} + \varepsilon \frac{(\gamma - \alpha\omega^2)\omega X - \beta\omega^2 Y - \gamma Z}{\omega^2}. \end{aligned} \tag{10}$$

Now we re–scale the variables (X, Y, Z) as follows $(X, Y, Z) \to (\varepsilon u, \varepsilon v, \varepsilon w)$. Then system (10) becomes

$$\begin{aligned} \dot{u} &= -\omega v + \varepsilon \frac{(\gamma - \alpha\omega^2)\omega^3 u - \beta\omega^4 v - \gamma\omega^2 w + \omega^2 u^2 - 2\omega uw + w^2}{\omega^5}, \\ \dot{v} &= \omega u, \\ \dot{w} &= \varepsilon \frac{(\gamma - \alpha\omega^2)\omega^3 u - \beta\omega^4 v - \gamma\omega^2 w + \omega^2 u^2 - 2\omega uw + w^2}{\omega^4}. \end{aligned} \tag{11}$$

Now we pass the differential system (11) to cylindrical coordinates (r, θ, w) defined by $u = r\cos\theta$ and $v = r\sin\theta$, and we obtain

$$\begin{aligned} \dot{r} &= \frac{\varepsilon}{\omega^5}\Big[(w - \gamma\omega^2)w - \omega r(2w - \gamma\omega^2 + \alpha\omega^4)\cos\theta \\ &\quad + \omega^2 r^2\cos^2\theta - \beta\omega^4 r\sin\theta\Big]\cos\theta, \\ \dot{\theta} &= \omega + \frac{\varepsilon}{\omega^5 r}\Big[(\gamma\omega^2 - w)w + \omega r(2w - \gamma\omega^2 + \alpha\omega^4)\cos\theta \\ &\quad - \omega^2 r^2\cos^2\theta + \beta\omega^4 r\sin\theta\Big]\sin\theta, \\ \dot{w} &= \frac{\varepsilon}{\omega^4}\Big[(w - \gamma\omega^2)w - \omega r(2w - \gamma\omega^2 + \alpha\omega^4)\cos\theta \\ &\quad + \omega^2 r^2\cos^2\theta - \beta\omega^4 r\sin\theta\Big]. \end{aligned} \tag{12}$$

In system (12) we take θ as the new independent variable, and we get

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon F_1(\theta, r, w) + O(\varepsilon^2), \\ \frac{dw}{d\theta} &= \varepsilon F_2(\theta, r, w) + O(\varepsilon^2), \end{aligned} \tag{13}$$

where

$$F_1(\theta,r,w) = \frac{1}{\omega^6}\Big[(w-\gamma\omega^2)w - \omega r(2w-\gamma\omega^2+\alpha\omega^4)\cos\theta + \omega^2 r^2\cos^2\theta - \beta\omega^4 r\sin\theta\Big]\cos\theta,$$

$$F_2(\theta,r,w) = \frac{1}{\omega^5}\Big[(w-\gamma\omega^2)w - \omega r(2w-\gamma\omega^2+\alpha\omega^4)\cos\theta + \omega^2 r^2\cos^2\theta - \beta\omega^4 r\sin\theta\Big].$$

Using the notation of subsection 2.2, we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,w)^T$ and

$$F(\theta,r,w) = \begin{pmatrix} F_1(\theta,r,w) \\ F_2(\theta,r,w) \end{pmatrix} \quad \text{and} \quad f(r,w) = \begin{pmatrix} f_1(r,w) \\ f_2(r,w) \end{pmatrix}.$$

It is immediate to check that system (13) satisfies all the assumptions of Theorem 4.

Now we compute the integrals (6). We obtain that

$$f_1(r,w) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r,w)d\theta = \frac{r(\gamma\omega^2-\alpha\omega^4-2w)}{2\omega^5},$$

$$f_2(r,w) = \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r,w)d\theta = \frac{2w^2-2\gamma\omega^2 w+\omega^2 r^2}{2\omega^5}.$$

The system $f_1(r,w)=f_2(r,w)=0$ has a unique solution (r^*,w^*) with $r^*>0$, namely

$$r^* = \frac{\omega\sqrt{\gamma^2-\alpha^2\omega^4}}{\sqrt{2}}, \qquad w^* = \frac{\omega^2(\gamma-\alpha\omega^2)}{2}.$$

The Jacobian (7) at (r^*,w^*) takes the value

$$\det \frac{\partial(f_1,f_2)}{\partial(r,w)}\bigg|_{(r,w)=(r^*,w^*)} = \frac{\gamma^2-\alpha^2\omega^4}{2\omega^6},$$

which is nonzero by hypothesis. Moreover the eigenvalues of the Jacobian matrix

$$\frac{\partial(f_1,f_2)}{\partial(r,w)}\bigg|_{(r,w)=(r^*,w^*)}$$

are given by

$$\frac{-\alpha\omega^2 \pm \sqrt{3\alpha^2\omega^4-2\gamma^2}}{2\omega^3}.$$

The rest of the proof of Theorem 2 follows immediately from Theorem 4 if we show that the periodic solution corresponding to (r^*,w^*) provides a periodic orbit bifurcating from the origin of coordinates of the differential system (8) at $\varepsilon=0$.

Theorem 4 garantes for $\varepsilon > 0$ sufficiently small the existence of a periodic solution $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ of system (13) such that

$$(r(0,\varepsilon), w(0,\varepsilon)) \to \left(\frac{\omega\sqrt{\gamma^2-\alpha^2\omega^4}}{\sqrt{2}}, \frac{\omega^2(\gamma-\alpha\omega^2)}{2}\right),$$

when $\varepsilon \to 0$. From the second equation of system (12) we obtain that $\theta(t,\varepsilon) = \omega t + O(\varepsilon)$. Moreover, we have that $(r(t,\varepsilon), \theta(t,\varepsilon), w(t,\varepsilon))$ is a periodic solution of system (12) such that

$$(r(0,\varepsilon), \theta(0,\varepsilon), w(0,\varepsilon)) \to \left(\frac{\omega\sqrt{\gamma^2-\alpha^2\omega^4}}{\sqrt{2}}, 0, \frac{\omega^2(\gamma-\alpha\omega^2)}{2}\right),$$

when $\varepsilon \to 0$. So for $\varepsilon > 0$ sufficiently small system (11) has the periodic solution

$$(u(t,\varepsilon), v(t,\varepsilon), w(t,\varepsilon)) = (r(t,\varepsilon)\cos\theta(t,\varepsilon), r(t,\varepsilon)\sin\theta(t,\varepsilon), w(t,\varepsilon)),$$

such that

$$(u(0,\varepsilon), v(0,\varepsilon), w(0,\varepsilon)) \to \left(\frac{\omega\sqrt{\gamma^2-\alpha^2\omega^4}}{\sqrt{2}}, 0, \frac{\omega^2(\gamma-\alpha\omega^2)}{2}\right),$$

when $\varepsilon \to 0$. This periodic solution in the differential system (10) writes as $(X(t,\varepsilon),\ Y(t,\varepsilon), Z(t,\varepsilon)) = (\varepsilon u(t,\varepsilon), \varepsilon v(t,\varepsilon), \varepsilon w(t,\varepsilon))$, and it satisfies that

$$(X(0,\varepsilon), Y(0,\varepsilon), Z(0,\varepsilon)) \to \left(\frac{\varepsilon\omega\sqrt{\gamma^2-\alpha^2\omega^4}}{\sqrt{2}}, 0, \frac{\varepsilon\omega^2(\gamma-\alpha\omega^2)}{2}\right),$$

when $\varepsilon \to 0$. Finally, we have that system (8) has the periodic solution $(x(t,\varepsilon),\ y(t,\varepsilon), z(t,\varepsilon))$ obtained from solution $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon))$ through the change of variables (9). It satisfies that $(x(0,\varepsilon), y(0,\varepsilon), z(0,\varepsilon))$ is

$$\varepsilon\left(\frac{\gamma-\alpha\omega^2}{2} - \frac{\sqrt{\gamma^2-\alpha^2\omega^4}}{\sqrt{2}}, 0, \frac{\omega^2\sqrt{\gamma^2-\alpha^2\omega^4}}{\sqrt{2}}\right) + \mathcal{O}(\varepsilon^2),$$

if ε is sufficiently small. Thus $(x(0,\varepsilon), y(0,\varepsilon), z(0,\varepsilon)) \to (0,0,0)$ when $\varepsilon \to 0$. Therefore, it is a periodic solution starting at the zero–Hopf equilibrium point located at the origin of coordinates when $\varepsilon = 0$. This completes the proof of Theorem 2. □

5. Acknowledgments

The first author is supported by FAPESP grant number 2013/24541-0. The second author is partially supported by MINECO grant number MTM2013-40998-P, an AGAUR grant number 2014SGR-568 and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338. Both authors are supported by CAPES grant 88881.030454/2013–01 Program CSF–PVE.

References

[1] C. Buzzi, J. Llibre, and J. Medrado. Hopf and zero-Hopf bifurcations in the Hindmarsh-Rose system. *Nonlinear Dynam.*, 83(3):1549–1556, 2016.

[2] V. Castellanos, J. Llibre, and I. Quilantán. Simultaneous periodic orbits bifurcating from two zero–hopf equilibria in a tritrophic food chain model. *J. Appl. Math. Phys.*, 1(7):31–38, 2013.

[3] M. Chen and Z. Han. Controlling and synchronizing chaotic Genesio system via nonlinear feedback control. *Chaos Solitons Fractals*, 17(4):709–716, 2003.

[4] R. Genesio and A. Tesi. Harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems. *Automatica*, 28(3):531–548, 1992.

[5] J. Guan, F. Chen, and G. Wang. Chaos control and Hopf bifurcation analysis of the Genesio system with distributed delays feedback. *Adv. Difference Equ.*, pages 2012:166, 16, 2012.

[6] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[7] J. Llibre. Periodic orbits in the zero-hopf bifurcation of the Rossler system. *Romanian Astron. J.*, 24(1):49–60, 2014.

[8] J. Llibre, R. D. S. Oliveira, and C. Valls. On the integrability and the zero-Hopf bifurcation of a Chen-Wang differential system. *Nonlinear Dynam.*, 80(1-2):353–361, 2015.

[9] J. H. Park. Synchronization of Genesio chaotic system via backstepping approach. *Chaos Solitons Fractals*, 27(5):1369–1375, 2006.

[10] J. H. Park. Adaptive controller design for modified projective synchronization of Genesio-Tesi chaotic system with uncertain parameters. *Chaos Solitons Fractals*, 34(4):1154–1159, 2007.

[11] L. Perko. *Differential equations and dynamical systems*, volume 7 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1991.

[12] J. A. Sanders, F. Verhulst, and J. Murdock. *Averaging methods in nonlinear dynamical systems*, volume 59 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2007.

[13] J. Sotomayor. Generic bifurcations of dynamical systems. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 561–582. Academic Press, New York, 1973.

[14] F. Verhulst. *Nonlinear differential equations and dynamical systems*. Universitext. Springer-Verlag, Berlin, second edition, 1996. Translated from the 1985 Dutch original.

[15] X. Wu, Z.-H. Guan, Z. Wu, and T. Li. Chaos synchronization between Chen system and Genesio system. *Phys. Lett. A*, 364(6):484–487, 2007.

[16] L. Zhou and F. Chen. Hopf bifurcation and Si′lnikov chaos of Genesio system. *Chaos Solitons Fractals*, 40(3):1413–1422, 2009.

Departamento de Matemática, Faculdade de Engenharia de Ilha Solteira, UNESP – Univ Estadual Paulista, Rua Rio de Janeiro, 266, CEP 15385–000, Ilha Solteira, São Paulo, Brazil

E-mail address: pedrocardin@gmail.com; pedrocardin@mat.feis.unesp.br

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08913 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat