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GLOBAL BEHAVIOUR OF THE PERIOD FUNCTION FOR SOME DEGENERATE CENTERS

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ABSTRACT. We study the global behaviour of the period function on the period annulus of degenerate centers for two families of planar polynomial vector fields. These families are the quasi-homogeneous vector fields and the vector fields given by the sum of two quasi-homogeneous Hamiltonian ones. In the first case we prove that the period function is globally decreasing, extending previous results that deal either with the Hamiltonian quasi-homogeneous case or with the general homogeneous situation. In the second family, and after adding some more additional hypotheses, we show that the period function of the origin is either decreasing or has at most one critical period and that both possibilities may happen. This result also extends some previous results that deal with the situation where both vector fields are homogenous and the origin is a non-degenerate center.

1. INTRODUCTION AND MAIN RESULTS

A planar polynomial vector field $X(x,y) = (P(x,y), Q(x,y))$ is called (p,q) quasi-homogeneous of quasi-degree n if there exist $p,q,n \in \mathbb{N}$ such that

$$P(\lambda^p x, \lambda^q y) = \lambda^{n+p-1} P(x,y), \quad Q(\lambda^p x, \lambda^q y) = \lambda^{n+q-1} Q(x,y),$$

for all $\lambda \in \mathbb{R}$. It is not restrictive to take p and q coprime. The numbers p and q are usually called weights. It is well known that its associated differential equation

$$\begin{cases} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{cases}$$

can be integrated by writing it in the so called generalized polar coordinates, see for instance [21] or Section 2. Notice that homogeneous vector fields of degree n are quasi-homogeneous of quasi-degree n and weights $(1,1)$. Moreover, in this case the generalized polar coordinates are the usual polar ones.

In this paper we are concerned with vector fields having a degenerate critical point at the origin of center type, and being either quasi-homogeneous or given by the sum of two quasi-homogeneous ones sharing the same weights (p,q). In the latter case, additionally we will assume that the vector field is Hamiltonian. We write the vector field in both situations as $X(x,y) =$

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$X_n(x,y) + X_m(x,y)$, with associated differential equation

$$\begin{cases} \dot{x} &=& P_n(x,y) + P_m(x,y), \\ \dot{y} &=& Q_n(x,y) + Q_m(x,y), \qquad 1 < n < m, \end{cases} \tag{1.1}$$

where each $X_j = (P_j, Q_j)$, $j \in \{n, m\}$, is (p,q) quasi-homogeneous of quasi-degree j. We assume that $X_n(x,y) \not\equiv 0$ but we admit that $X_m(x,y) \equiv 0$.

We want to know the global behavior of the period function on the period annulus of the origin when we assume that the differential equation associated to X has a degenerate center at this point. Recall that a *center* is a critical point that has a punctured neighbourhood full of periodic orbits. The largest of such neighbourhoods is called the *period annulus* of the center. When the eigenvalues of DX at the center are not purely imaginary, then the center is called *degenerate*. This is our situation because $n > 1$. The function that associates to any closed curved of the period annulus its period is called the *period function* of the center. It is well known that the period function tends to infinity when the orbits in a period annulus approach either to a degenerate center or to a polycycle with some finite critical point, see for instance [9].

In general, given a system with a center, we will write $T(x,y)$ to denote the period of the orbit passing through the point (x,y). When the system is Hamiltonian, it is sometimes more convenient to parameterize the periodic orbits by their energy h and write $T(h)$ to denote their corresponding periods. The *critical periods* are the zeroes of the derivative of the period function once the continuum of periodic orbits is parameterized by a smooth one-parameter function. This parameter can be the energy in the Hamiltonian situation, or anyone describing a transversal section to the orbits. It is not difficult to prove that the number of critical periods does not depend neither on the transversal section, nor on its parametrization. When a zero of the derivative of the period function is simple we will say that the system has a *simple* critical period.

Some motivations to know properties of the period function come from its role in the study of several differential equations. For instance, it appears in mathematical models in physics or ecology, see [15, 17, 23, 24] and the references therein. From a more mathematical point of view, it is important in the study of the bifurcations from a continuum of periodic orbits giving rise to isolated ones, see [8, pp. 369-370], in the description of the dynamics of some discrete dynamical systems, see [5, 10, 11] or for counting the solutions of some boundary value problems, see [6, 7].

The period function for homogeneous vector fields (both Hamiltonian and non-Hamiltonian) was characterized in [14], while the quasi-homogeneous Hamiltonian were studied in [25]. Our main result for the quasi-homogeneous case, i. e. $X_m(x,y) \equiv 0$, completely characterizes the period function in the general case, extending their results.

Theorem A. *Consider a (p,q) quasi-homogeneous vector field of quasi-degree n, that is* (1.1) *with $X_m = 0$, with a degenerate center at the origin. Then its associated period function is monotonic decreasing. Moreover it can be written as*

$$T(\xi, 0) = T_1\, \xi^{\frac{1-n}{p}}, \qquad \textit{or} \qquad T(0,\eta) = T_2\, \eta^{\frac{1-n}{q}},$$

for $\xi, \eta \in \mathbb{R}^+$, *and some non-zero constants* T_1 *and* T_2.

Recall that a critical point is called *monodromic* if there are no orbits tending or leaving the point with a given direction. For analytic vector fields, monodromic points are either center or focus, and the problem of distinguishing between both options is called the *center-focus problem.* The solution of the center-focus problem for quasi-homogeneous vector fields is relatively easy. As we will see in the proof of the previous theorem, in order to have a center at the origin we only need to guarantee that the origin is monodromic and moreover that some definite integral, that can be obtained from the expression in quasi-homogeneous polar coordinates, is zero, see (3.4).

In the particular case that the system considered in Theorem A is also Hamiltonian the above result can be rewritten as follows, recovering the result in [25].

Corollary 1.1. *Under the hypotheses of Theorem A, if moreover the system is Hamiltonian, with* $H(0,0) = 0$ *and closed ovals* $H(x,y) = h \geq 0$, *then the period function parameterized by the energy level* h *is*

$$T(h) = T_3\, h^{\frac{1-n}{n+p+q-1}},$$

for some non-zero constant T_3.

As we will see, the constants $T_j, j = 1,2,3$ are closely related and all them can be expressed in terms of two iterated integrals, see (3.5). Moreover in some cases they can be explicitly computed. For instance, consider the $(1,2)$ quasi-homogeneous Hamiltonian system, of quasi-degree 2,

$$\begin{cases} \dot{x} = -y + bx^2, \\ \dot{y} = x^3 - 2bxy, \end{cases}$$

with Hamiltonian $H(x,y) = y^2/2 - bx^2y + x^4/4$. When $b^2 < 1/2$, it has a center at the origin and its period function, for $\xi > 0$, is (see Example 3.3)

$$T(\xi, 0) = \frac{T_1}{\xi} = \frac{2\pi^{3/2}}{\sqrt[4]{1-2b^2}\,\Gamma^2\,(3/4)}\,\frac{1}{\xi},$$

where Γ is the Gamma function. Equivalently, for $\eta > 0$,

$$T(0,\eta) = \frac{T_1}{\sqrt[4]{2}\,\sqrt{\eta}} \qquad \text{and} \qquad T(h) = \frac{T_1}{\sqrt{2}}\frac{1}{\sqrt[4]{h}}.$$

Other examples can be seen in Section 3.

For general systems (1.1) the center-focus problem is still an open question. Moreover, even for quadratic systems with a reversible center, the global behaviour of the period function is not fully understood, see for instance [22]. Therefore to ensure that the origin is a center and to start with the most tractable case, we will restrict our attention to the Hamiltonian subcase. Notice that for system (1.1) the condition of being a Hamiltonian vector field implies the existence of two (p,q) quasi-homogeneous functions $H_n(x,y)$ and $H_m(x,y)$, with respective quasi-degrees $n+p+q-1$ and

$m+p+q-1$, such that

$$H_k(\lambda^p x, \lambda^q y) = \lambda^{k+p+q-1} H_k(x,y),$$
$$\frac{\partial H_k(x,y)}{\partial x} = Q_k(x,y), \quad \frac{\partial H_k(x,y)}{\partial y} = -P_k(x,y), \quad k=n,m,$$

and the Hamiltonian is $H(x,y) = H_n(x,y) + H_m(x,y)$.

We obtain the following results, where recall that a center is called *global* if its associated basin of attraction is the whole plane.

Theorem B. *Consider a Hamiltonian system of the form* (1.1) *with a global center at the origin. Then its period function is monotone decreasing to zero.*

We remark that previous result strongly relies on two facts. The first one is that the associated vector field is given by the sum of two (p,q) quasi-homogenous ones, while the second fact is that $n>1$. As an example of the necessity of both hypotheses, consider for instance the globally linearizable isochronous system associated to the Hamiltonian $H(x,y) = x^2 + (y+x^2)^2$, for which all orbits have period π. The corresponding Hamiltonian vector field can be considered as the sum of three homogeneous ones, which violates the first required assumption. On the other hand, the same vector field can be also considered as the sum of two $(1,2)$ quasi-homogeneous ones of quasi-degrees 0 and 2, respectively. In this case the second assumption fails.

Theorem C. *Consider a Hamiltonian system of the form* (1.1) *with a center at the origin. For*

$$m \geq 2n-1 \quad \text{and} \quad p+q \leq \frac{(m-n)(3m^2+2mn-4n^2-8m+6n+1)}{(m-2n+1)(n-1)}$$

the period function of the origin has at most one critical period and, when it exists it is simple.

For the particular case when X_n and X_m are both homogeneous vector fields ($(p,q)=(1,1)$), we obtain the following result:

Corollary 1.2. *Consider a Hamiltonian system of the form* (1.1) *with $p=q=1$. If $m \geq 2n-1$ then the period function of the origin of system* (1.1) *has at most one critical period and, if it exists, it is simple. More specifically,*

(i) if m is even, it has exactly one critical period.
(ii) if m is odd, it can have none or one critical period. Moreover both possibilities may occur.

The above corollary extends the results obtained in [14, 16, 18] for the case of Hamiltonian vector fields of the form $X_n + X_m$, with $n=1<m$, where it is also proved that the period function on the period annulus of the origin has at most one critical period. When $n \geq 2$ (indeed n has to be odd to have a center at the origin) the same result holds when $m \geq 2n-1$. Our attempts to cover the remaining cases have not succeeded. For instance, by applying our result we know the behaviour of the period function on the period annulus of the origin for all Hamiltonian systems of the form $X_3 + X_m$, $m \geq 5$, and the only open case is $m=4$.

The paper is organized as follows: in Section 2 we give some preliminary results and we introduce the generalized polar coordinates; Section 3 deals with quasi-homogeneous vector fields, not necessarily Hamiltonian, and is devoted to prove Theorem A. Finally, the proofs of Theorems B and C and Corollary 1.2 about Hamiltonian vector fields of the form $X_n + X_m$ are given in Section 4.

2. Preliminary results

We start recalling the generalized polar coordinates and the generalized trigonometric functions. They were introduced by Lyapunov in his study of the stability of degenerate critical points, see [21]. These new functions are defined as the unique solution of the initial value problem

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y^{2p-1}, \\ \dot{y} & = & x^{2q-1}, \end{array} \right. \tag{2.1}$$

with the initial conditions $x(0) = \sqrt[2q]{1/p}$, $y(0) = 0$. We will denote them by $x(\theta) = \mathrm{Cs}(\theta)$, $y(\theta) = \mathrm{Sn}(\theta)$. When $p = q = 1$ we recover the usual trigonometric functions. The generalized trigonometric functions satisfy the equality $p\,\mathrm{Cs}^{2q}(\theta) + q\,\mathrm{Sn}^{2p}(\theta) = 1$ and they are periodic, with period

$$\Omega = \Omega_{p,q} = 2p^{\frac{-1}{2q}}\,q^{\frac{-1}{2p}}\,\frac{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{1}{2q}\right)}{\Gamma\left(\frac{1}{2p}+\frac{1}{2q}\right)}, \tag{2.2}$$

where Γ is the Gamma function. Associated to them we can introduce the quasi-homogenous polar coordinates by the change

$$x = r^p\,\mathrm{Cs}(\theta), \quad y = r^q\,\mathrm{Sn}(\theta). \tag{2.3}$$

With these coordinates, it holds that $px^{2q} + qy^{2p} = r^{2pq}$.

In general, a system

$$\left\{ \begin{array}{l} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{array} \right. \tag{2.4}$$

by doing the change to generalized polar coordinates, is transformed into

$$\left\{ \begin{array}{l} \dot{r} = r^{1-2pq}\big[x^{2q-1}P(x,y) + y^{2p-1}Q(x,y)\big], \\ \dot{\theta} = r^{-p-q}\big[pxQ(x,y) - qyP(x,y)\big], \end{array} \right. \tag{2.5}$$

where x and y have to be substituted using (2.3). In the particular case that the vector field $X = (P,Q)$ is (p,q) quasi-homogenous of quasi-degree n, we obtain

$$\left\{ \begin{array}{l} \dot{r} = r^n\big[\,\mathrm{Cs}^{2q-1}(\theta)P(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)) + \mathrm{Sn}^{2p-1}(\theta)Q(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta))\big], \\ \dot{\theta} = r^{n-1}\big[p\,\mathrm{Cs}(\theta)Q(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)) - q\,\mathrm{Sn}(\theta)P(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta))\big], \end{array} \right. \tag{2.6}$$

Moreover, system (2.1), which has quasi-degree $n = 2pq - p - q + 1$, is transformed into

$$\left\{ \begin{array}{l} \dot{r} = 0, \\ \dot{\theta} = r^{n-1} = r^{2pq-p-q}. \end{array} \right.$$

Notice that each polynomial vector field can be decomposed in different ways according to some chosen (p,q) weights. For instance, the vector field (y, x^2) decomposes as $(y,0) + (0,x^2)$ in its homogeneous components or as itself when one takes $(2,3)$ quasi-homogeneous ones. Therefore, given X, and a couple of weights (p,q), we have a unique decomposition

$$X(x,y)=\sum_{j=n}^{m} X_j(x,y),$$

where $n=n(p,q)\le m=m(p,q)$ and each $X_j(x,y)=X_j(x,y,p,q)$ is (p,q) quasi-homogeneous of quasi-degree j. Observe that in general many X_j are identically zero.

Associated to a given (p,q) decomposition and motivated by the expressions of $\dot\theta$ in (2.5) and (2.6) we define the (p,q) *characteristic quasi-directions at the origin* of the vector field $X=(P,Q)$, as the curves $\lambda\to(\lambda^p\bar x,\lambda^q\bar y)$, where $(\bar x,\bar y)\ne(0,0)$ is a real zero of the quasi-homogeneous polynomial

$$F^0_{p,q}(x,y):=pxQ_{n(p,q)}(x,y)-qyP_{n(p,q)}(x,y). \tag{2.7}$$

Similarly, we define the (p,q) *characteristic quasi-directions at infinity* of X as the curves $\lambda\to(\lambda^p\bar x,\lambda^q\bar y)$, where $(\bar x,\bar y)\ne(0,0)$ is a real zero of the quasi-homogeneous polynomial

$$F^\infty_{p,q}(x,y):=pxQ_{m(p,q)}(x,y)-qyP_{m(p,q)}(x,y). \tag{2.8}$$

Notice that, as a result of the quasi-homogeneity of $F^0_{p,q}$ and $F^\infty_{p,q}$, the control of the zeroes and signs of these functions is a one-variable problem. For instance, (u,v) gives a characteristic quasi-direction at the origin if either $F^0_{p,q}(0,1)=0$ or $F^0_{p,q}(1,w)=0$, where $w=v\,u^{-q/p}$. Using this fact it makes sense to talk about the multiplicity of the characteristic quasi-directions as the multiplicity of the one-variable associated functions.

Observe that if X is a (p,q) quasi-homogeneous vector field, then any (p,q) characteristic quasi-direction is invariant by X.

Based on the ideas of [1, 2] it is not difficult to prove the following result, wherein the definition for infinity to be monodromic is essentially the same as for the origin. This result states some folklore results that appear in many places only when $(p,q)=(1,1)$.

Proposition 2.1. *Consider a polynomial vector filed X. The following holds:*

(i) If the origin is a critical point and some orbit tends to it asymptotically to some curve $\lambda\to(\lambda^p\bar x,\lambda^q\bar y)$, then this direction has to be a (p,q) characteristic quasi-direction, that is a zero of $F^0_{p,q}$.

(ii) If the origin is a monodromic critical point then given any pair of weights p,q, either the point has not (p,q) characteristic quasi-directions at the origin or all its (p,q) characteristic quasi-directions at the origin have even multiplicity. Moreover when X is (p,q) quasi-homogeneous a necessary and sufficient condition to be monodromic is that the point has not (p,q) characteristic quasi-directions.

(iii) If the infinity is monodromic, given any pair of weights p,q, then either it has not (p,q) characteristic quasi-directions at infinity or all its (p,q) characteristic quasi-directions at infinity have even multiplicity.

Following with the vector field $X=(y,x^2)$ considered at the beginning of this section, we get

$$F^0_{1,1}(x,y)=y^2,\qquad F^0_{2,3}(x,y)=2x^3-3y^2.$$

On one hand, since $F^0_{1,1}$ has a double characteristic direction, the above proposition taking $(p,q)=(1,1)$ does not allow to conclude whether there is an orbit tending to the origin in positive or negative time. On the other hand taking $(p,q)=(2,3)$ we can conclude that the origin is a non-monodromic point because $F^0_{2,3}(1,w)$ has a simple root.

Next result gives a well known extension of Euler Theorem to smooth (p,q) quasi-homogeneous functions. We will need this extension to prove Lemma 2.3 on non-vanishing of quasi-characteristic polynomials of Hamiltonian systems.

Lemma 2.2. *Let $F:\mathbb{R}^2\to\mathbb{R}$ be a $\mathcal{C}^1$, (p,q) quasi-homogeneous function of quasi-degree k, i.e. such that for all $\lambda\in\mathbb{R}$,*

$$F(\lambda^p x,\lambda^q y)=\lambda^k F(x,y). \tag{2.9}$$

Then

$$px\frac{\partial F(x,y)}{\partial x}+qy\frac{\partial F(x,y)}{\partial y}=k\,F(x,y).$$

Proof. Derivating with respect to λ the equality (2.9) we get that

$$\frac{\partial F(\lambda^p x,\lambda^q y)}{\partial x}p\lambda^{p-1}x+\frac{\partial F(\lambda^p x,\lambda^q y)}{\partial y}q\lambda^{q-1}y=k\lambda^{k-1}F(x,y).$$

The result follows substituting $\lambda=1$ in the above expression. $\square$

Lemma 2.3. *The quasi-characteristic polynomial at the origin or at infinity of a Hamiltonian system can not be identically null.*

Proof. Consider a Hamiltonian function written in its (p,q) quasi-homogeneous components $H(x,y)=H_n(x,y)+\ldots+H_m(x,y)$, with $H_n,H_m\not\equiv 0$, and its associated system

$$\begin{cases}\dot{x} &= -\frac{\partial H(x,y)}{\partial y}=P(x,y)=P_n(x,y)+P_{n+1}(x,y)+\ldots+P_m(x,y),\\ \dot{y} &= \frac{\partial H(x,y)}{\partial x}=Q(x,y)=Q_n(x,y)+Q_{n+1}(x,y)+\ldots+Q_m(x,y).\end{cases}$$

Its quasi-characteristic polynomial at the origin is

$$\begin{aligned}F^0_{p,q}(x,y)=&-qyP_n(x,y)+pxQ_n(x,y)=qy\frac{\partial H_n(x,y)}{\partial y}+px\frac{\partial H_n(x,y)}{\partial x}=\\ =&(n+p+q-1)H_n(x,y),\end{aligned}$$

where in the last equality we have used Lemma 2.2. Then, since $H_n(x,y)\not\equiv 0$, $F^0_{p,q}$ can not be null.

The case of infinity is completely analogous but substituting H_n by H_m. $\square$

Next proposition can be proved as in [19, Lemma 3]. The results at infinity can be inferred from the ones at the origin by doing the change $r=1/\rho$. We use the following notation:

$$f(x)\sim g(x) \quad \text{at} \quad x=x_0\in\mathbb{R}\cup\{\infty\},$$

if $\lim_{x\to x_0} f(x)/g(x)=k\neq 0$.

Proposition 2.4. *Given $1\le n<m$, consider a vector field $X=X_n+X_{n+1}+\cdots+X_m$, with $X_n\not\equiv 0$, $X_m\not\equiv 0$ and each X_i being a (p,q) quasi-homogeneous polynomial of quasi-degree i.*

(i) If the origin is a center and has not (p,q) characteristic quasi-directions then for $\xi>0$, $T(\xi,0)\sim\xi^{\frac{1-n}{p}}$ at $\xi=0$.

(ii) If the infinity has a neighbourhood full of periodic orbits and has not (p,q) characteristic quasi-directions then $T(\xi,0)\sim\xi^{\frac{1-m}{p}}$ at $\xi=\infty$.

The following proposition extends the results of item (ii) of [12, Theorem C], that deals with polynomial Hamiltonian systems with Hamiltonian $H(x,y)=(x^2+y^2)/2+H_m(x,y)$, with H_m homogeneous, to Hamiltonian system of the form (1.1) with $p=q=1$. Actually, it extends the results to nonlinear vector fields. Its proof is similar to the one of that paper and we omit it. It will be one of the key points for proving Corollary 1.2.

Proposition 2.5. *Consider a Hamiltonian system of the form* (1.1) *with $p=q=1$ and a center at the origin. Then either it has a global center or its period annulus is bounded.*

In order to prove that the bound for the number of critical periods is one, a way is to compute the second derivative of the period function and verify that it does not change sign. Next result gives an alternative for this computation that, moreover, has the freedom of choosing a function φ.

Theorem 2.6 ([18])**.** *Let I be a real open interval. An analytic function $f:I\to\mathbb{R}$ has at most one simple critical point if and only if there exists an analytic function $\varphi:I\to\mathbb{R}$ such that for all $x\in I$*

$$f''(x)+\varphi(x)f'(x)\neq 0.$$

3. The quasi-homogeneous case

Consider a vector field $X=X_n+\cdots+X_m$, $n\geq 1$, decomposed as sum of homogeneous components X_j of respective degrees j. It is well known that if the origin is monodromic, then n must be odd. This can be seen, for instance by using item (ii) of Proposition 2.1, because either it has not characteristic directions or all its characteristic directions have to have even multiplicity. Hence, the polynomial that gives these directions must have even degree. Thus $n+1$ has to be even. Next result extends this property to (p,q) quasi-homogeneous vector fields.

Lemma 3.1. *Consider the (p,q) quasi-homogeneous system of quasi-degree n,*

$$\begin{cases}\dot{x} &= P_n(x,y),\\ \dot{y} &= Q_n(x,y).\end{cases}\tag{3.1}$$

If it has a monodromic point at the origin, then $n=2kpq-p-q+1$ for some $k\in\mathbb{N}$.

Proof. In order to be monodromic at the origin the function $P_n(x,y)$ must satisfy that $P_n(0,y)\neq 0$ and $Q_n(x,0)\neq 0$. Otherwise it would have an invariant line through it. Thus,

$$P_n(0,y)=a_1y^{k_1},\quad Q_n(x,0)=a_2x^{k_2},\quad \text{with}\quad a_1a_2\neq 0,$$

for some natural numbers k_1, $k_2\geq 1$.

Moreover, since the vector field is (p,q) quasi-homogeneous of quasi-degree n, it holds that:

$$P_n(0,\lambda^q y) = a_1\lambda^{k_1 q}y^{k_1} = \lambda^{n+p-1}P_n(0,y) = a_1\lambda^{n+p-1}y^{k_1},$$
$$Q_n(\lambda^p x,0) = a_2\lambda^{k_2 p}x^{k_2} = \lambda^{n+q-1}Q_n(x,0) = a_2\lambda^{n+q-1}x^{k_2}.$$

Consequently $n+p-1 = k_1 q$ and $n+q-1 = k_2 p$. From these equalities $(k_1+1)q = (k_2+1)p$. But p and q are coprime numbers, hence $k_1+1 = Kp$ and $k_2+1 = Kq$ for some $K \in \mathbb{N}$. By substituting one gets

$$n-1 = k_1 q - p = (Kp-1)q - p = Kpq - p - q.$$

It remains to be proved that K is even. By item (ii) of Proposition 2.1, since the origin is monodromic, it can not have (p,q) characteristic quasi-directions at the origin. Consequently, if we consider

$$F^0_{p,q}(x,y) = pxQ_n(x,y) - qyP_n(x,y),$$

it happens that $F^0_{p,q}(1,y) = pQ_n(1,y) - qyP_n(1,y)$ has no real roots. The term of higher degree of the previous expression is y^{k_1+1} and hence, $k_1+1 = Kp$ must be even. Doing the same reasoning but now with $F^0_{p,q}(x,1)$ one gets that $k_2+1 = Kq$ must also be even. But p,q can not be both even at the same time, as they are coprime. Consequently, $K = 2k$. □

Proof of Theorem A. By using the quasi-homogeneous polar coordinates we can write system (1.1) as

$$\begin{cases} \dot r = r^n A(\theta), \\ \dot\theta = r^{n-1}B(\theta), \end{cases} \tag{3.2}$$

where

$$A(\theta) = \mathrm{Cs}^{2q-1}(\theta)P(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)) + \mathrm{Sn}^{2p-1}(\theta)Q(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)),$$
$$B(\theta) = p\,\mathrm{Cs}(\theta)Q(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)) - q\,\mathrm{Sn}(\theta)P(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)),$$

see system (2.6). From the above expressions it is clear that the monodromy condition in this situation is: *the function $B(\theta)$ does not vanish.* Then, clearly the origin has not (p,q) characteristic quasi-directions. Under this monodromy assumption we can write the above system as

$$\frac{dr}{d\theta} = \frac{A(\theta)}{B(\theta)}\,r,$$

which can be easily integrated, giving

$$r(\theta;r_0) = r_0 \exp\Big(\int_0^\theta \frac{A(\psi)}{B(\psi)}d\psi\Big), \tag{3.3}$$

where $r_0 > 0$ denotes the initial condition at $\theta = 0$. Hence, the center condition $r(\Omega_{p,q};r_0) = r_0$ writes as

$$\int_0^{\Omega_{p,q}} \frac{A(\psi)}{B(\psi)}d\psi = 0. \tag{3.4}$$

Moreover, from the second equation of (3.2) and (3.3) it holds that

$$\begin{aligned}\widetilde{T}(r_0) =& \int_0^{\Omega_{p,q}} \frac{d\theta}{B(\theta)\, r^{n-1}(\theta; r_0)} \\ =& \Big(\int_0^{\Omega_{p,q}} \frac{1}{B(\theta)} \exp\Big[(1-n) \int_0^{\theta} \frac{A(\psi)}{B(\psi)} d\psi \Big] d\theta \Big) \frac{1}{r_0^{n-1}},\end{aligned} \tag{3.5}$$

where $\widetilde{T}(r_0)$ denotes the period of the orbit passing through the point with generalized polar coordinates $r = r_0$ and $\theta = 0$, that is the point $(x, y) = (r_0^p \sqrt[2q]{1/p}, 0)$. Hence

$$T(\xi, 0) = T_1\, \xi^{\frac{1-n}{p}},$$

for some constant $T_1 \neq 0$, as we wanted to prove. If the initial condition of the periodic orbit is taken to be $(0, \eta)$, $\eta > 0$, then similarly we get that $T(0, \eta) = T_2\, \eta^{\frac{1-n}{q}}$. □

Proof of Corollary 1.1. If the quasi-homogeneous vector field $X = (P_n, Q_n)$ is Hamiltonian, then its Hamiltonian function, satisfying $H(0, 0) = 0$, can be obtained as

$$H(x, y) = \int_0^x Q_n(u, y)\, du + R(y) = a_2 \frac{x^{k_2+1}}{k_2 + 1} + yS(x, y),$$

for some polynomial functions R and S, with $R(0) = 0$, where we keep the same notation as in the proof of Lemma 3.1. Then, using that $k_2 + 1 = 2kq$, see again Lemma 3.1, the energy level of the solution passing through the point $(\xi, 0)$, called h, satisfies $h = H(\xi, 0) = \frac{a_2}{2kq} \xi^{2kq}$. Applying now Theorem A we get that

$$T(h) = T_3\, h^{\frac{1-n}{2kpq}} = T_3\, h^{\frac{1-n}{n+p+q-1}},$$

because $2kpq = n + p + q - 1$. □

We end this section with a couple of examples.

Example 3.2. Consider the classical (p, q) quasi-homogeneous system:

$$\begin{cases} \dot{x} = -y^{2p-1}, \\ \dot{y} = x^{2q-1} \end{cases}$$

It has quasi-degree $n = 2pq - p - q + 1$ and it is Hamiltonian, with $H(x, y) = \frac{x^{2q}}{2q} + \frac{y^{2p}}{2p}$. Recall that in the generalized polar coordinates the previous system writes as $\dot{r} = 0$, $\dot{\theta} = r^{n-1}$. Since $H(r^p \operatorname{Cs}(\theta), r^q \operatorname{Sn}(\theta)) = \frac{r^{2pq}}{2pq}$ it holds that the orbit γ_h with energy $h > 0$ is $r = (2pqh)^{\frac{1}{2pq}}$.

By the proof of Theorem A, the period function of γ_h can be explicitly computed as

$$T(h) = \int_0^{\Omega_{p,q}} \frac{1}{r^{n-1}}\, d\theta = T_3\, h^{\frac{1-n}{2pq}}, \quad \text{with} \quad T_3 = (2pq)^{\frac{1-n}{2pq}} \Omega_{p,q}.$$

Example 3.3. Let us consider next $(1, 2)$ quasi-homogeneous systems of quasi-degree 2:

$$\begin{cases} \dot{x} = -y + bx^2, \\ \dot{y} = x^3 + axy, \end{cases} \tag{3.6}$$

with $(a-2b)^2<8$. As it is proved in [3] the previous system is the only cubic quasi-homogeneous (and non-homogeneous) system having a center at the origin (after a rescaling of the variables, if necessary). Notice that the condition $(a-2b)^2<8$ is, precisely, the condition of non-existence of characteristic quasi-directions, because this function

$$pxQ(x,y)-qyP(x,y)=x(x^3+axy)-2y(-y+bx^2)=x^4+(a-2b)x^2y+2y^2,$$

does not vanish at $(x,y)\neq(0,0)$ if and only if $(a-2b)^2-8<0$. Moreover, the origin is a center because it is invariant by the change of variables and time $(x,y,t)\to(-x,y,-t)$, and so it is reversible.

When $b=0$, system (3.6) is the one studied in [4], where an explicit expression for the period function is given. When $a=-2b$ the previous system is Hamiltonian with $H(x,y)=y^2/2-bx^2y+x^4/4$.

We will compute the period function in the general case, getting a closed expression when the system is Hamiltonian.

Following Theorem A and its proof, $T(\xi,0)=T_1\xi^{\frac{1-n}{p}}=T_1/\xi$, with

$$T_1=T_1(a,b)=\int_0^{\Omega_{1,2}}\frac{\exp\left(-\int_0^\theta\frac{\operatorname{Cs}(\varphi)(b\operatorname{Cs}^4(\varphi)+a\operatorname{Sn}^2(\varphi))}{1+(a-2b)\operatorname{Cs}^2(\varphi)\operatorname{Sn}(\varphi)}\,d\varphi\right)}{1+(a-2b)\operatorname{Cs}^2(\varphi)\operatorname{Sn}(\varphi)}\,d\theta.$$

When $b=0$ the formula given in [4] is recovered.

In the Hamiltonian case, $a=-2b$, the integral in the numerator of the expression of T_1 can be computed explicitly in the following way:

$$\int_0^\theta\frac{\operatorname{Cs}(\varphi)(b\operatorname{Cs}^4(\varphi)-2b\operatorname{Sn}^2(\varphi))}{1-4b\operatorname{Cs}^2(\varphi)\operatorname{Sn}(\varphi)}\,d\varphi=\frac{-1}{4}\ln\Big(1-4b\operatorname{Cs}^2(\theta)\operatorname{Sn}(\theta)\Big),$$

where we have used that $\dot{\operatorname{Cs}}(\theta)=-\operatorname{Sn}(\theta)$, $\dot{\operatorname{Sn}}(\theta)=\operatorname{Cs}^3(\theta)$.

Substituting now in the expression of T_1 one gets:

$$T_1=T_1(b)=\int_0^{\Omega_{1,2}}\frac{d\theta}{(1-4b\operatorname{Cs}^2(\theta)\operatorname{Sn}(\theta))^{\frac{3}{4}}}.$$

Now, by using the change of variables $x=\operatorname{Sn}(\theta)/\operatorname{Cs}^2(\theta)$, we can write

$$\begin{aligned}T_1(b)=&\int_{-\infty}^{\infty}\frac{2\,dx}{(1-4bx+2x^2)^{\frac{3}{4}}}=\frac{2\sqrt{2}}{\sqrt[4]{1-2b^2}}\int_0^\infty\frac{dx}{(1+x^2)^{\frac{3}{4}}}\\=&\frac{4}{\sqrt[4]{1-2b^2}}\,F\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right)=\frac{2\pi^{3/2}}{\sqrt[4]{1-2b^2}\,\Gamma^2(3/4)}=\frac{\Omega_{1,2}}{\sqrt[4]{1-2b^2}},\end{aligned}$$

where F is the elliptic integral of the first kind. See [20, §3.185.1, §8.11], for instance. We observe that $F(\frac{\pi}{2},k)=K(k)$, where K is a complete elliptic integral of the first kind.

4. Proofs of Theorems B and C

Proof of Theorem B. First of all, we transform system (1.1) into generalized polar coordinates. Using (2.5) we get

$$\left\{\begin{array}{rcl}\dot r&=&R(r,\theta)=a_n(\theta)r^n+a_m(\theta)r^m,\\ \dot\theta&=&\Phi(r,\theta)=(n+p+q-1)H_n(\theta)r^{n-1}+(m+p+q-1)H_m(\theta)r^{m-1},\end{array}\right.\tag{4.1}$$

where $H_k(\theta) = H_k(\operatorname{Cs}\theta, \operatorname{Sn}\theta)$, $k = n, m$, and $a_n(\theta)$ and $a_m(\theta)$ are Ω-periodic functions. Notice that we have used Euler Theorem for (p, q) quasi-homogeneous functions, see Lemma 2.2.

From the results of [13] we know that the periodic orbits of the system 4.1 that surround the origin never cut the curve $\Phi(r, \theta) = 0$. Moreover, the sign of $\Phi(r, \theta)$ in a neighbourhood of the origin is given by the sign of $H_n(\theta)$ that we will assume without loss of generality that is positive. Another important fact is that, as the period annulus is global and by item (iii) of Proposition 2.1, the function $F^\infty_{p,q}(x, y)$ does not change sign. Then, the same holds for

$$H_m(\theta) = \frac{1}{m+p+q-1} F^\infty_{p,q}(\operatorname{Cs}\theta, \operatorname{Sn}\theta),$$

where we have used Lemma 2.3. In fact $H_m(\theta)$ has to have the same sign as $H_n(\theta)$. Otherwise, the direction of rotation will be opposite at the origin and at infinity, what would imply that the orbits of the global center would cut $\Phi(r, \theta) = 0$.

Let us prove that the period function tends to zero as it approaches to infinity. If $H_m(\theta) > 0$ this is simply a consequence of item (ii) in Proposition 2.4. The proof in the case $H_m(\theta) \geq 0$ is more delicate. Observe that as $h = r^{n+p+q-1}H_n(\theta) + r^{m+p+q-1}H_m(\theta)$, then

$$\frac{dh}{dr} = r^{p+q-1}\Phi(r, \theta) = r^{p+q-1}\frac{d\theta}{dt}. \tag{4.2}$$

Consequently,

$$T(h) = \int_0^\Omega \frac{d\theta}{\Phi(r, \theta)} = \int_0^\Omega \frac{d\theta}{\Phi\big(r(\theta, h), \theta\big)}, \tag{4.3}$$

where $r = r(\theta, h)$ denotes the solution of the implicit closed curve given by $h = r^{n+p+q-1}H_n(\theta) + r^{m+p+q-1}H_m(\theta)$ and recall that

$$\Phi(r, \theta) = (n+p+q-1)H_n(\theta)r^{n-1} + (m+p+q-1)H_m(\theta)r^{m-1}.$$

Notice also that for each fixed $\theta = \theta^* \in \Omega$,

$$H_n(\theta^*) \geq 0, \quad H_m(\theta^*) \geq 0 \quad \text{and} \quad H_n^2(\theta^*) + H_m^2(\theta^*) > 0. \tag{4.4}$$

The last inequality holds because otherwise the ray $\theta = \theta^*$ would be invariant and this is not possible because the origin is a center.

Let us prove first that there exists $\widetilde{h} > 0$ such that for $h \geq \widetilde{h}$ and all $\theta \in [0, \Omega]$,

$$\frac{1}{\Phi\big(r(\theta, h), \theta\big)} \leq 1. \tag{4.5}$$

Recall that the origin is a global center. Additionally, the fact that system 1.1 is polynomial implies that for each given $\theta^* \in [0, \Omega]$,

$$\lim_{h\to\infty} r(\theta^*, h) = \infty. \tag{4.6}$$

Moreover, since (4.4) holds, we have that the function $h \to r(\theta^*, h)$ is increasing and

$$\lim_{h\to\infty} \Phi\big(r(\theta^*, h), \theta^*\big) = \infty. \tag{4.7}$$

Therefore, given $\theta = \theta^*$ there exists $h(\theta^*)$ such that $\Phi\big(r(\theta^*, h(\theta^*)), \theta^*\big) \geq 2$. By continuity, there exists an open neighbourhood of θ^*, say $\mathcal{U}_{\theta^*}$, such that

$$\Phi\big(r(\theta, h(\theta^*)), \theta\big) \geq 1, \quad \text{for all} \quad \theta \in \mathcal{U}_{\theta^*}.$$

By using the monotonicity of $h \to r(\theta, h)$ and of $h \to \Phi(r(\theta, h), \theta)$ it holds that

$$\Phi\big(r(\theta, h), \theta\big) \geq 1, \quad \text{for all} \quad \theta \in \mathcal{U}_{\theta^*} \quad \text{and all} \quad h \geq h(\theta^*).$$

By compactness of $[0, \Omega]$ we can cover it by finitely many $\mathcal{U}_{\theta_j}$, $j = 1, \ldots, k$, in such a way that for $h \geq \widetilde{h} := \max\big(h(\theta_1), h(\theta_2), \ldots, h(\theta_k)\big)$ it holds that

$$\Phi\big(r(\theta, h), \theta\big) \geq 1, \quad \text{for} \quad \theta \in [0, \Omega] \quad \text{and} \quad h \geq \widetilde{h}.$$

Then (4.5) follows. Moreover, by (4.7),

$$\lim_{h \to \infty} \frac{1}{\Phi\big(r(\theta^*, h), \theta^*\big)} = 0. \tag{4.8}$$

Since inequality (4.5) holds we can use the dominated convergence theorem to compute $\lim_{h \to \infty} T(h)$. Therefore

$$\lim_{h \to \infty} T(h) = \lim_{h \to \infty} \int_0^{\Omega} \frac{d\theta}{\Phi\big(r(\theta, h), \theta\big)} = \int_0^{\Omega} \lim_{h \to \infty} \frac{d\theta}{\Phi\big(r(\theta, h), \theta\big)} = 0,$$

as we wanted to prove.

Recall also that from the results of [9], as the origin is a degenerate center, its period function goes to infinity as it approaches to it.

We claim that the period function of the center at the origin of system (1.1) has at most one simple critical period. If the claim holds, as the behaviour of the function is the one proved above (begins at zero being infinity and tends to zero at infinity) the period function will have no simple critical periods and it will be monotone decreasing. Hence, Theorem B will be proved.

To prove the claim, our approach is based on Theorem 2.6 and uses similar ideas that the ones of [18]. We have to compute $T''(h) + \varphi(h)T'(h)$ for a suitable φ and prove that this expression does not change sign. By using (4.2) and (4.3) we get that

$$T(h) = \frac{d}{dh} \int_0^{\Omega} \frac{r^{p+q}}{p+q}\, d\theta, \quad \text{and} \quad T'(h) = \frac{d^2}{dh^2} \int_0^{\Omega} \frac{r^{p+q}}{p+q}\, d\theta.$$

Developing latter expression one gets:

$$\begin{aligned} T'(h) = \quad & -\int_0^{\Omega} \frac{1}{\Phi^3(r, \theta)} \Big((n+p+q-1)(n-1) r^{n-p-q-1} H_n(\theta) + \\ & \qquad + (m+p+q-1)(m-1) r^{m-p-q-1} H_m(\theta) \Big)\, d\theta, \end{aligned}$$

where $\Phi(r, \theta) > 0$ on all the period annulus. Recall again that in all the expressions $r = r(\theta, h)$ denotes the implicit closed curve given by $h = r^{n+p+q-1} H_n(\theta) + r^{m+p+q-1} H_m(\theta)$.

Similarly we compute the second derivative of the period function. In order to apply Theorem 2.6 we consider $\varphi(h) = k/h$, where k is a constant value that will be fixed according each one of the two cases in which we split the proof of this theorem. So, after several computations, we get that

$$T''(h)+\varphi(h)T'(h) = \int_0^{\Omega} \frac{1}{\Phi^5(r,\theta)hr^5}\Big(c_1H_n^2(\theta)H_m(\theta)r^{2n+m}+ \\ + c_2H_n(\theta)H_m^2(\theta)r^{n+2m} + c_3H_n^3(\theta)r^{3n} + c_4H_m^3(\theta)r^{3m}\Big)\,d\theta, \quad (4.9)$$

where $c_j = c_j(m,n,p,q,k)$, $j = 1,2,3,4$. Their expressions are large and for the sake of shortness we omit the explicit expressions of three of them. As an example,

$$c_3 = (1-n)(n+p+q-1)^2\big(k(n+p+q-1) - 2n - p - q + 2\big).$$

The proof of the theorem will be divided into two cases: the first one when $n < m < 2n-1$ and the second case the opposite, $m \geq 2n-1$. We begin with the first one: $n < m < 2n-1$.

In this case, in the expression (4.9) we choose a k such that $c_3 = 0$, that is

$$k = \frac{2n+p+q-2}{n+p+q-1}.$$

Hence, the parenthesis of the integrand of the previous expression (4.9) becomes:

$$P(h,\theta) = c_1H_n^2(\theta)H_m(\theta)r^{2n+m} + c_2H_n(\theta)H_m^2(\theta)r^{n+2m} + c_4H_m^3(\theta)r^{3m},$$

with

$$c_1 = (2n-m-1)(m-n)(m-n+p+q)(n+p+q-1) > 0,$$
$$c_2 = (m-n)(m+p+q-1)\Big((m-n)(2m-n-1) + 2(n-1)(p+q)\Big) > 0,$$
$$c_4 = \frac{(m-1)(m-n)(p+q)(m+p+q-1)^2}{n+p+q-1} > 0.$$

Consequently, $T''(h) + \varphi(h)T'(h) > 0$ and according to Theorem 2.6 the period function T has at most one critical period and, if it exists, it is simple.

Now we proceed with the second case $m \geq 2n-1$. In this situation we choose k in such a way that $c_1 = 0$ in the expression (4.9). It can be seen that the parenthesis of the integrand of (4.9) becomes:

$$P(h,\theta) \;=\; c_2H_n(\theta)H_m^2(\theta)r^{n+2m} + c_3H_n^3(\theta)r^{3n} + c_4H_m^3(\theta)r^{3m}, \quad (4.10)$$

where

$$c_2 = \frac{(m-n)(m+p+q-1)}{m+2n-3}\Big(4(m-n)^3 + 2(m-n)^2(4(n-1)+p+q) \\ + 3(m-n)(n-1)(p+q) + 3(n-1)^2(p+q)\Big) > 0, \quad (4.11)$$
$$c_3 = \frac{(n-1)(m-2n+1)(m-n)(n+p+q-1)^2(m-n+p+q)}{(m+2n-3)(m+p+q-1)} \geq 0,$$
$$c_4 = \frac{(m-1)(m-n)(m+p+q-1)^2}{(m+2n-3)(n+p+q-1)}\Big((m-2n+1)(m-n)+ \\ + 2(m-1)(p+q)\Big) > 0.$$

Again $T''(h) + \varphi(h)T'(h) > 0$ and according to Theorem 2.6 the period function has at most one critical period and, if it exists, it is simple. Then the claim is proved. □

Remark 4.1. By Proposition 2.1.(iii), global centers either have not (p, q) characteristic quasi-directions at infinity or have all them with even multiplicity. Moreover, by Proposition 2.4.(ii), in the former case and when $m > 1$ the period function tends to 0 when the orbits approach to infinity. On the other hand, in the latter case, even for Hamiltonian systems and $m > 1$, it is no more true in general that $\lim_{h\to\infty} T(h) = 0$. As an example, take again the Hamiltonian considered in the Introduction.

Proof of Theorem C. The proof starts with the same computations and notations that the one of the second case of previous theorem, $m \geq 2n - 1$. Hence we have to prove that the function $P(h, \theta)$ given in (4.10) and with the constants $c_j > 0$ given in (4.11) is positive, where recall that we are assuming without loss of generality that $H_n(\theta) > 0$. The main difference is that the period annulus is not necessarily global. Hence the function $H_m(\theta)$ can change sign along it and we do not still know if the sign of $P(h, \theta)$ is constant.

For the values of θ such that $H_m(\theta) \geq 0$ there is nothing to be proved because $P(h, \theta)$ is a sum of nonnegative quantities.

Consider a value of θ such that $H_m(\theta) < 0$. We rewrite the function $P(h, \theta)$ in the following way:

$$\begin{aligned} P(h,\theta) =& c_3 H_n^3(\theta) r^{3n} + \frac{c_2}{n+p+q-1} H_m^2(\theta) r^{2m+1} \times \\ & \times \Big((n+p+q-1) H_n(\theta) r^{n-1} + \frac{c_4(n+p+q-1)}{c_2} H_m(\theta) r^{m-1} \Big). \end{aligned}$$

We claim now that $\frac{c_4(n+p+q-1)}{c_2} \leq m + p + q - 1$. If that is true, it holds that

$$\frac{c_4(n+p+q-1)}{c_2} H_m(\theta) \geq (m+p+q-1) H_m(\theta).$$

Thus

$$\begin{aligned} &(n+p+q-1) H_n(\theta) r^{n-1} + \frac{c_4(n+p+q-1)}{c_2} H_m(\theta) r^{m-1} \geq \\ &(n+p+q-1) H_n(\theta) r^{n-1} + (m+p+q-1) H_m(\theta) r^{m-1} = \Phi(r,\theta) > 0. \end{aligned}$$

Then $P(h, \theta)$ will be also positive on the whole period annulus. Applying Theorem 2.6 to the period function with the φ given, we know that it will have at most one (simple) critical period

We prove now the claim. The previous inequality is equivalent to $c := (m+p+q-1)c_2 - (n+p+q-1)c_4 \geq 0$. This function c can be written in the following way:

$$\begin{aligned} c =& \frac{(m-n)(m+p+q-1)^2}{m+2n-3} \Big((m-n)\big(3m^2 + 2mn - 8m - \\ & - 4n^2 + 6n + 1\big) - (n-1)(m-2n+1)(p+q) \Big). \end{aligned}$$

It is a straightforward computation proving that $c \geq 0$ is equivalent to

$$p+q \leq \frac{(m-n)\left(3m^2+2mn-4n^2-8m+6n+1\right)}{(n-1)(m-2n+1)},$$

which is precisely one of the hypotheses of the theorem. Then the result follows. □

Proof of Corollary 1.2. The homogeneous case can be recovered from the quasi-homogeneous one by setting $p=q=1$ in Theorem C. Then it is enough with proving that

$$2 \leq \frac{(m-n)\left(3m^2+2mn-4n^2-8m+6n+1\right)}{(n-1)(m-2n+1)}.$$

The previous inequality is equivalent to the chain of inequalities,

$$(m-n)\left(3m^2+2mn-4n^2-8m+6n+1\right)-2(n-1)(m-2n+1) \geq 0,$$

$$3(m-n)^3+8(m-n)^2(n-1)+(m-n)(n-3)(n-1)+2(n-1)^2 \geq 0,$$

and this last inequality is obviously true for $n \geq 3$. It remains the case $n=2$, but it follows by a straightforward computation.

Let us now prove the second part of the corollary. We first study the case m even. As system (1.1) is Hamiltonian, then the quasi-characteristic polynomial at the infinity, $F_{1,1}^{\infty}$, can not be identically null, as it has been proved in Lemma 2.3. Then, as the degree of the characteristic polynomial at infinity is odd, it must have an orbit tending to infinity in positive or negative time. Consequently, the period annulus $\mathcal{P}$ of the origin can not be global. Then, by Proposition 2.5 the period annulus of the origin must be bounded. Therefore, there must exist another critical point in the exterior boundary $\partial\mathcal{P}$ of $\mathcal{P}$. As a consequence, since the period function tends to infinity when it approaches to the origin and also to $\partial\mathcal{P}$ (see [9]), we know that the period function must have, at least, one critical period. But we have just proved that it has at most one critical period. Hence, if m is even the period function has exactly one simple critical period.

Let us now study the case in which $m=2\ell-1$ is odd. We have to prove that there exist Hamiltonian systems with a center at the origin having one simple critical period, and systems with a center at the origin having zero simple critical periods. Consider the following Hamiltonian $H(x,y) = (x^2+y^2)^k + a(x^2+y^2)^{\ell}$, with $1<k<\ell$, $a \neq 0$, and the differential system associated to it:

$$\begin{cases} \dot{x} &= -2y(k(x^2+y^2)^{k-1}+a\,\ell(x^2+y^2)^{\ell-1}), \\ \dot{y} &= 2x(k(x^2+y^2)^{k-1}+a\,\ell(x^2+y^2)^{\ell-1}). \end{cases} \tag{4.12}$$

In polar coordinates it writes as

$$\begin{cases} \dot{r} &= 0, \\ \dot{\theta} &= 2(kr^{2k-2}+a\,\ell r^{2\ell-2}). \end{cases} \tag{4.13}$$

Observe that previous system has a continuum of critical points when $a<0$, and thus the period annulus is bounded, while the period annulus is global in the opposite case. Therefore when $a>0$ the period function is monotone decreasing and when $a<0$ it has exactly one (simple) critical period. Indeed, in this particular example, where the periodic orbits are circles, the period function parameterized by the radius, $\widetilde{T}(r)$, can be explicitly given, because

$$\widetilde{T}(r) = \int_0^{2\pi} \frac{d\theta}{2\big(k^{2k-2} + a\,\ell r^{2l-2}\big)} = \frac{\pi}{kr^{2k-2} + a\,\ell r^{2l-2}}.$$

Hence the decreasing behaviour of $\widetilde{T}$ when $a > 0$ and the existence of exactly one critical period when $a < 0$ is clear. Moreover, when $a < 0$, the critical period corresponds to $r = r_0$ with $\widetilde{T}'(r_0) = 0$. Then r_0 is the positive solution of $k(k-1) + a\,\ell(\ell-1)r^{2(l-k)} = 0$. $\square$

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References

[1] M.J. Álvarez, A. Ferragut, X. Jarque, *A survey on the blow up technique.* Internat. J. Bifur. Chaos Appl. Sci. Engrg., **21**, (2011) 3103–3118.

[2] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Meyer, *Qualitative theory of second–order dynamic systems.* John Wiley & Sons, New York-Toronto, Israel Program for Scientific Translations, Jerusalem-London, 1973.

[3] W. Aziz, J. Llibre, C. Pantazi, *Centers of quasi-homogeneous polynomial differential equations of degreee three*, Adv. Math. **254**, (2014) 233–250.

[4] J. Billeke, E. Saez, *Monotonicity of the period map of a non-Hamiltonian center*, Collect. Math. **45**, (1994) 177–182.

[5] F. Beukers, R. Cushman, *Zeemans monotonicity conjecture*, J. Differential Equations **143**, (1997) 191–200.

[6] C. Chicone, *The monotonicity of the period function for planar Hamiltonian vector fields*, J. Differential Equations **69**, (1987) 310–321.

[7] C. Chicone, *Geometric methods of two-point nonlinear boundary value problem*, J. Differential Equations **72**, (1988) 360–407.

[8] S.N. Chow, J.K. Hale, Methods of bifurcation theory. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 251. Springer-Verlag, New York-Berlin, 1982. xv+515 pp.

[9] C.J. Christopher, C.J. Devlin, *Isochronous centres in planar polynomial systems*, SIAM J. Math. Anal. **28**, (1997) 162–177.

[10] A. Cima, A. Gasull, V. Mañosa, *Studying discrete dynamical systems through differential equations*, J. Differential Equations **244**, (2008) 630–648.

[11] A. Cima, A. Gasull, V. Mañosa, *Non-integrability of measure preserving maps via Lie symmetries*, J. Differential Equations **259**, (2015) 5115–5136.

[12] A. Cima, A. Gasull, F. Mañosas, *On polynomial Hamiltonian planar vector fields*, J. Differential Equations **106**, (1993) 367–383.

[13] B. Coll, A. Gasull, R. Prohens, *Differential equations defined by the sum of two quasi-homogeneous vector fields,* Canadian Journal of Mathematics **49**, (1997) 212–231.

[14] C.B. Collins, *The period function for some polynomial systems of arbitrary degree,* Differential and Integral Equations **9**, 252–266 (1996).

[15] A. Constantin and G. Villari, *Particle trajectories in linear water waves*, J. Math. Fluid Mech. **10**, (2008) 1–18.

[16] W.A. Coppel, L. Gavrilov, *The period function of a Hamiltonian quadratic system*, Differential Integral Equations **6** (1993), 1357–1365.

[17] E. Freire, A. Gasull, A. Guillamon, *First derivative of the period function with applications*, J. Differential Equations, **204**, (2004) 139–162.
[18] A. Gasull, A. Guillamon, V. Mañosa, F. Mañosas, *The period function for Hamiltonian systems with homogeneous nonlinearities*, J. Differential Equations **139**, (1997) 237–260.
[19] A. Gasull, V. Mañosa, J. Villadelprat, *On the period of the limit cycles appearing in one-parameter bifurcations*, J. Differential Equations **213**, (2005) 255–288.
[20] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products.* **7**th ed. Academic Press, Amsterdam (2007).
[21] A.M. Lyapunov, *Stability of motion* Mathematics in Science and Engineering **30**, Academic Press, New York-London (1966).
[22] P. Mardešić, D. Marin and J. Villadelprat, *The period function of reversible quadratic center*, J. Differential Equations **224** (2006), 120–171.
[23] F. Rothe, *The periods of the Volterra-Lokta system*, J. Reine Angew. Math. **355**, (1985) 129–138.
[24] J. Waldvogel, *The period in the Lotka-Volterra system is monotonic*, J. Math. Anal. Appl. **114**, (1986) 178–184.
[25] Y. Zhao, Z. Zhang, *Abelian integrals and period function for quasihomogeneous Hamiltonian vector fields*, Applied Mathemtacis Letters **18**, 563–569 (2005).

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