Shapley Value Based Pricing for Auctions and Exchanges

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Abstract

This paper explores how the Shapley value can be used as the basis of a payment rule for auctions and exchanges. The standard Shapley value is modified so that losing bidders do not make or receive any payments. The new rule, called the balanced winner contribution (BWC) rule, satisfies a variation of Myerson's balanced contribution property. The payment rule is fair in the sense that, with respect to reported values, the members of every pair of traders make equal contributions to each other's share of the gains from trade. BWC payments can be used in single-item auctions and more complex auctions and exchanges with multiple items and package bidding. A series of examples is presented to illustrate how the BWC rule works and how the payments compare to those based on competitive prices, the core, and the Vickrey-Clarke-Groves mechanism.

Keywords: Shapley value, auctions, exchanges, package bidding, balanced budget
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1. Introduction

This paper explores how the Shapley value (Shapley, 1953) can be used as the basis of a payment rule for auctions and exchanges. Deciding what payment rule to use is a fundamental market design question in allocation problems with money. First price (pay-as-bid) rules are simple to implement. Participants, however, must bid strategically (rather than bidding their true values), to receive a share of the gains from trade. In an exchange setting, pay-as-bid typically generates a revenue surplus. Second price, Vickrey-Clarke-Groves (VCG), rules have attractive incentive properties, but in auction settings can generate low revenues and in exchange settings typically result in a revenue deficit. Payment rules based on competitive prices or surplus divisions in the core result in balanced budgets in exchanges. However, for some allocation problems, payments satisfying the rules do not exist, while for other problems they exist but are not unique. Shapley value divisions of surplus always exist, are unique, and are budget balanced. However, the Shapley value in its standard form, unlike the other mecha-

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nisms considered above, can allocate surplus to losing bidders, making it, at least at first sight, appear unsuitable for use as a payment rule for auctions and exchanges.

The paper considers how the Shapley value can be modified to only allocate surplus to bidders with winning bids while still allowing the competition from losing bidders to influence the division of surplus. The surplus shares are average marginal contributions over permutations of the bidders where the losing bidders appear before the winning bidders. This results in losing bidders receiving zero surplus but influencing the division of surplus by defining outside options for winning bidders. The prices satisfy a *balanced winner contribution* (BWC) property. When a single item is being sold, the prices produced by the mechanism can be thought of as intermediate between those that would be produced by a first price and a second price auction. When multiple units or multiple types of good are being traded, the pricing rule can be used in combination with package bidding.

Several examples are discussed, some involve selling a single item and others multiple items. In multiple item cases, when traders have preferences over combinations of items and items are traded individually, the exposure problem can occur. It arises when getting to the desired allocation requires a sequence of trades that if started and not completed, leaves the trader with a loss. A good example to illustrate this is flight tickets. Flights can be substitutes. A traveler may want to fly between two cities but be flexible about the departure time or where layovers occur. Flights can also be complements. A traveler may want an outward flight only if a return flight is also available. A family may want four seats on a particular flight or none at all. Similarly, an airline may want to sell tickets only if the quantity sold generates sufficient revenue to cover the costs that could be avoided by canceling the flight. Using all-or-nothing package bids and XOR constraints which allow at most one of a set of bids to be winning can protect traders from the exposure problem. This is because traders can submit bids that guarantee they will not end up holding unintended combinations of goods. In these settings, BWC payments always exist, even in cases where there are no competitive prices and the core is empty.

The rest of this paper is organized as follows. Section 2 discusses related literature. Section 3 describes the environment studied and the notation used in the paper. The mechanism uses a procedure that has two stages. First, winning bids are determined (and hence the allocation of commodities), then the division of surplus is calculated (and hence the payments for winning bids). Section 4 describes how the winner determination problem is solved. Section 5 describes how solutions of the winner determination problem for different sets of bidders can be used to construct a coalition game. Section 6 describes how, given the winning bids, prices are chosen to determine the division of the surplus. The pricing rule is a variation of Myerson's (1977; 1980) balanced contribution property. Section 9 discusses the computational complexity of calculating BWC payments and outlines a strategy for calculating approximations within a fixed time constraint. Section 10 concludes. Proofs are in an appendix.

2. Related Literature

A number of studies have suggested using the Shapley value as a cost allocation rule. For instance, Littlechild and Owen (1973) consider how to share the cost of providing an airport runway when catering for larger aircraft costs more than catering for smaller ones. They suggest the Shapley value as a solution. Roth and Verrecchia (1979) consider how to allocate costs between departments within an organization. They show how, under certain assumptions, using the Shapley value cost allocations will be equivalent to the expected outcome when managers bargain amongst themselves. The BWC approach taken in this paper differs from approaches using the standard Shapley value in that traders are partitioned into two sets, those with winning bids and those without, and the value is calculated in such a way that those without winning bids make no payments. This partitioning of the bidders into two sets and then using these to calculate a modified Shapley value is similar to the approach used to calculate the Owen value (Owen, 1977). A key difference is that Owen considers 'a priori unions' where the unions might have been formed by signing some agreement or represent membership of a certain clan or family. Such unions may be independent of members' contributions to payoffs. In contrast, partitioning based on winning bids is based on contributions to payoffs, since only those with winning bids contribute to the realized gains from trade.

The BWC rule can be used in auctions and exchanges that allow package bidding, which protects traders from the exposure problem. Previous studies have identified a number of settings where simple market mechanisms without package bidding may perform poorly. In auctions, bidders face an exposure problem if the items beings sold are complements but are sold individually. Bidders risk winning some but not all of the items they bid on and ending up paying more than the items are worth to them. Notable examples include auctions for airport landing slots (Rassenti et al., 1982) and radio spectrum licenses (Bykowsky et al., 2000). Goeree and Lindsay (2012) find that a related exposure problem can occur in exchanges when traders view items as substitutes and traders are already endowed with an item they value. Another setting where exposure can occur is when sellers have avoidable fixed costs. In such settings, competitive prices may not exist and the core may also be empty (see Telser (1996) for an analysis of the conditions under which the core can be empty).

One consequence of the exposure problem is that institutions that mitigate the problem may emerge. For instance, Sjostrom (1989) argues that in ocean shipping, a potential empty core problem is mitigated by shipping conferences. The conferences, which set prices and quantity quotas for members, first appeared in the late ninetieth century and remained prevalent throughout the twentieth century.

Another consequence is that researchers and practitioners have actively designed market mechanisms to alleviate the problem (see Milgrom (2007) for an overview of package bidding, Bichler and Goeree (2017) for many of the important papers on spectrum auction design, and Milgrom (2017) for a broad analysis of designing auctions with complex constraints). Package bidding solves the exposure problem by letting bidders submit all or nothing package bids. Most research has focused on the auction case with one seller and several buyers. Here, competitive prices may not exist but the core is always non-empty (since there is a single seller who is required for a coalition to get a positive payoff). One approach is to select payments in the core. Day and Milgrom (2007) analyze the properties of core selecting auctions. Since core payments are typically not unique, some rule is needed to choose between them. Day and Cramton (2012) present a procedure that can, for instance, select core payments that are closest to the Vickrey-Clarke-Groves payments.

There has been less research on package exchanges. Parkes et al. (2001) introduce and test a number of pricing rules that select payments as close as possible to the Vickrey-Clarke-Groves payments subject to the constraint that the exchange is budget balanced and participating is individually rational. They find that a threshold rule works best. Agents with a winning bid pay $\min(b, \pi_{vcg} + x)$ where b is the amount bid, π_{vcg} is the VCG payment, and x is the threshold. Hoffman and Menon (2010) suggest using the *nucleolus* as a payment rule for a package exchange. Schmeidler (1969) introduced the nucleolus as a way to divide the surplus in a coalition game which is unique and always exists. Maschler et al. (1979) explored its relation to other concepts. When the core exists, it contains the nucleolus and when the core is empty, the nucleolus minimizes the amount coalitions can gain by leaving the grand coalition. Fine et al. (2017) describe a mechanism called ACE that was used since 1995 to trade emissions permits. It is a package exchange that uses approximate competitive equilibrium prices. That is, the per unit price for winning bids is as close as possible to uniform subject to budget balance and participating being individually rational. These three approaches, like the BWC rule, achieve budget balance but at the expense of strategy proofness.[1] The three approaches above approximate VCG payments, core payments, and competitive prices respectively. The approximations are necessary to achieve budget balance and make participation individually rational. In contrast, BWC payments do not require an approximation to satisfy these properties.

3. Environment and Notation

Let $N = \{1, \ldots, n\}$ represent the set of $n \geq 2$ traders. In an auction, the seller is counted as a trader. Subsets of N will sometimes be denoted S and T with s and t denoting their respective sizes. Traders have quasi-linear utility. Let $M = \{1, \ldots, m\}$ represent the set of $m \geq 1$ commodities being traded. There may be multiple units of each commodity but each unit is indivisible.

Bids are submitted simultaneously. The mechanism is detail-free in the sense that it does not require any information about the distribution of traders' types. The mechanism treats bids as sincere reports of value.[2] A bid is a pair $\langle b; \boldsymbol{q} \rangle$ where $b \in \mathbb{R}$ is the amount bid and $\boldsymbol{q} \in \mathbb{Z}^m$ is a vector specifying the quantity of each commodity supplied or demanded. Positive quantities indicate giving money or commodities and negative ones indicate taking. Traders may submit multiple bids but at most one bid per trader can be winning (XOR bidding). Trader i's bids are indexed $B_i = \{1, 2, \ldots, |B_i|\}$. The amount trader $i \in N$ offers to pay in the bid with index $k \in B_i$ is denoted b_{ik} and the quantity of commodity $j \in M$ for the same bid is denoted q_{ikj}.

An example illustrates the notation. Consider a market with two goods. An offer to sell a package consisting of one unit of each good for 15 would be $\langle -15; 1, 1 \rangle$. To

[1] A recent survey of the trade-offs between strategy proofness and other desirable properties can be found in Lubin and Parkes (2012) and some recent developments of rules involving such trade-offs can be found in Erdil and Klemperer (2010) and Lubin (2015).

[2] A series of examples are included in sections 7 and 8. Nash equilibria where traders do not bid sincerely are considered.

buy either a unit of good 1 or good 2 for 10 but not both, two bids would be submitted: $\langle 10; -1, 0\rangle$ and $\langle 10; 0, -1\rangle$.

4. Winner Determination

Agents may submit multiple bids. Let w_{ik} indicate whether agent i's kth bid is winning. The winner determination problem, for any subset of bidders $S \subseteq N$, is as follows.

$$\begin{aligned} wd(S, M, B) = \arg\max_{\boldsymbol{w}} \quad & \sum_{i \in S} \sum_{k \in B_i} b_{ik} w_{ik} \\ \text{subject to} \quad & \sum_{i \in S} \sum_{k \in B_i} q_{ikj} w_{ik} \geq 0 \ \forall j \in M \\ & \sum_{k \in B_i} w_{ik} \leq 1 \ \forall i \in S \\ & w_{ik} \in \{0, 1\} \ \forall k \in B_i,\ \forall i \in S \end{aligned} \tag{1}$$

The objective is to maximize the value of winning bids. The first constraint is that for each commodity, the quantity demanded cannot exceed the quantity supplied. The second constraint is that bids cannot be partially winning. This ensures that if an agent submits a bid to buy two units and no other bids, if the bid is winning, the agent receives exactly two units. The third constraint prevents more than one bid from the same trader being winning. This allows bidders to express a preference to have at most one of a number of items.

It is worth noting that the solution to the winner determination problem, the set of winning bids that gives the maximum possible surplus, will not always be unique. For instance, if a seller submits the bid $\langle 0; 1\rangle$ and two buyers each submit $\langle 10; -1\rangle$, then a surplus of 10 can be obtained with the bid of either one of the buyers. The mechanism described in this paper only requires that a single solution is found; in the case of multiple solutions, one will be chosen arbitrarily.[3]

5. Auction Coalition Game

An auction or exchange and the submitted bids can be described as a coalition game.

Definition 1. A *coalition game* (N, v) consists of a finite set of players N and a characteristic function $v : 2^N \mapsto \mathbb{R}$ that satisfies $v(\emptyset) = 0$.

Definition 2. An *auction coalition game* is a coalition game whose characteristic function gives the surplus generated by the solution of the winner determination problem for different sets of traders.

[3]Software for solving integer linear programming problems (such as the winner determination problem) is commonly designed to find a single solution, not all solutions. Hence, only requiring a single solution to be found makes the mechanism easier to implement.

Suppose M and B are fixed and let $\boldsymbol{w^{S*}}$ denote a solution to equation 1 for bidders in S. We can now write a characteristic function for the auction as follows.

$$v(S) = \sum_{i \in S} \sum_{k \in B_i} b_{ik} w_{ik}^{S*} \tag{2}$$

The auction coalition game is monotone and superadditive.

Definition 3. A game (N, v) is *monotone* if for all $S, T \subseteq N$, if $S \subseteq T$, then $v(S) \leq v(T)$.

Adding a bidder never reduces the surplus because the surplus obtained without the new bidder is still feasible (it can be obtained by holding the set of winning bids constant when the bidder is added).

Definition 4. A game (N, v) is *superadditive* if for all $S, T \subseteq N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$.

When auctions with two distinct sets of bidders are combined, the union of the sets of winning bids in the un-combined auctions is feasible. This means that the surplus in the combined auction will be at least the sum of the surpluses in the un-combined auctions.

6. BWC Surplus Division

The auction coalition game describes the gains from trade that different coalitions can realize but does not specify how the gains are divided between players. The division of surplus can be described using a *value* function.

Definition 5. A *value* is a function $\phi(v, T)$ that for all coalition games (N, v) and $T \subseteq N$ gives a unique payoff vector in $\mathbb{R}^n$ such that $\sum_{i \in T} \phi_i(v, T) = v(T)$.

The value specifies the surplus share for each bidder but does not specify how this surplus is derived in terms of an allocation of commodities and money. Bidders trade the quantities specified in their winning bids.

$$\boldsymbol{q_i} = \sum_{k \in B_i} w_{ik}^{N*} \boldsymbol{q_{ik}} \tag{3}$$

The amount the bidder pays depends on the rule used to divide the surplus. Suppose surplus shares are given by value ψ. Bidders will pay the amount they bid for any winning bids less their share of the surplus.

$$\pi_i = \sum_{k \in B_i} w_{ik}^{N*} b_{ik} - \psi_i(N, v) \tag{4}$$

The standard Shapley value using our notation is:

$$\psi_i^{\dagger}(T, v) = \sum_{S \subseteq T \setminus \{i\}} \frac{s!\,(t-s-1)!}{t!} \left[v(S \cup \{i\}) - v(S) \right] \tag{5}$$

The standard Shapley value does not use any information about which bidders have winning bids. As a consequence, it can assign a share of the surplus to bidders who have no winning bids. I propose the following modification that takes into account the status of bids. For a given solution to $wd(N, M, B)$, let N_W denote the set of bidders with a winning bid and $N_L = N \setminus N_W$ denote those with no winning bids. A modified value is then defined using $T \subseteq N_W$ instead of $T \subseteq N$.

Definition 6. The *balanced winner contribution* (BWC) surplus shares are

$$\psi_i(T, v) = \sum_{S \subseteq T \setminus \{i\}} \frac{s!\,(t-s-1)!}{t!} \Big[v\,(N_L \cup S \cup \{i\}) - v\,(N_L \cup S\,) \Big] \tag{6}$$

One way to view BWC surplus shares is as a coalition game among all bidders, but where the Shapley value is modified by only considering orderings of the bidders where losing bidders appear before winning bidders.

The BWC surplus shares can also be described using a coalition game among the winners.

Definition 7. The *winner coalition game* for auction coalition game (N, v) and solution to the winner determination problem is (N_W, v') where N_W is the set of traders with winning bids and $v'(S) = v(S \cup N_L)$ with $S \subseteq N_W$.

Using this definition, the BWC surplus shares are simply the standard Shapley value of the winner coalition game.

The key components of the BWC mechanism have now been presented. For a given set of bidders N, range of commodities M and bids B, the allocation and payments are calculated as follows. First, the winner determination problem (Equation 1) is solved, which gives the set of winning bids. Then the surplus shares are calculated using Equation 6. Finally, payments are calculated as specified by Equation 4.

Table 1 shows the outcome of these three steps for a variation of the simple market game example used by Roth (1988, p. 3). A seller values an item at 10; buyer L values it at 22; and buyer H who values it at 26.[4] The efficient outcome is for the seller to sell the item to buyer H realizing the gain of 16. Applying the Shapley value to the game (N, v) results in the following division of surplus: the seller gets 10, buyer L gets 2, and buyer H gets 4. The problem with this division of surplus is that buyer L gets a share of the surplus despite not being involved in any trades. An alternative would be to completely exclude buyer L from the calculation by using the game $(\{1, 3\}, v)$, which results in an equal division of surplus between the seller and buyer H. The problem with this division is that the competition between the buyers is not taken into account. At the implied price of 18, both buyer L and H would be willing to buy the item. Using the BWC rule, in contrast, gives a price of 24. At this price, contributions are balanced in the sense that buyer H gets a surplus of 2 relative to not trading and the seller gets a

[4]The problem can be described as a coalition game (N, v) with players $N = \{1, 2, 3\}$. The gains from trade that different coalitions can realize is given by v with $v(1) = v(2) = v(3) = v(2, 3) = 0$, $v(1, 2) = 12$, and $v(1, 3) = v(1, 2, 3) = 16$. An alternative formulation is to define v as the value of outcomes rather than realized gains. This does not change the allocation of surplus.

Table 1: Example of winner determination, surplus division and payments

Agent	Bids	Surplus	Payment
(1) Seller (v=10)	$\underline{\boldsymbol{\langle -10; 1\rangle}}$	14	-24
(2) Buyer L(v=22)	$\langle 22; -1\rangle$		
(3) Buyer H(v=26)	$\underline{\boldsymbol{\langle 26; -1\rangle}}$	2	24

Note: Winning bids are underlined.

surplus of 2 relative to trading with the other buyer at 22. Notice that in this example 24 is the midpoint of the range of competitive prices $[22, 26]$.

The remainder of this section establishes and discusses some of the properties of the BWC mechanism. Proofs of the propositions are in an appendix.

Proposition 1 (BB)**.** *The BWC mechanism is budget balanced.*

If a mechanism runs a deficit, some party needs to be found to provide a subsidy. Conversely, if it runs a surplus that is extracted by a non-participant, participants have an incentive not to take part and instead divide the gains from trade amongst themselves. A budget balanced mechanism avoids these problems.

Proposition 2 (IR)**.** *The BWC mechanism is ex post individually rational.*

This means that if participants do not bid more for items than they value the items (or the converse in selling), then participants will not end up worse off than if they had not taken part in the mechanism. Hence, participants do not face an *exposure problem*. The problem can occur when getting to a desired allocation requires a number of trades and if some but not all are executed, traders end up worse off than had they not traded at all.

The two properties above are common to the BWC rule and the standard Shapley value.

Proposition 3 (No loser payments)**.** *The BWC mechanism satisfies no payments for losing bidders.*

This property could be seen as advantageous based on an appeal to fairness. One could argue that traders who do not contribute to the gains from trade should not receive a share of the surplus from trade. The standard Shapley value does not satisfy this property. In order to satisfy it, the BWC rule treats the winners and losers differently when calculating surplus shares. This violates Shapley's Symmetry axiom which implies that a player's surplus share depends only on the player's effect through the characteristic function.

While the BWC payment rule does not satisfy Shapley's axioms, it does satisfy a variation of Myerson's (1977; 1980) balanced contribution property. Myerson showed that the standard Shapley value satisfies a *balanced contribution property* and it is the only value to do so. This relationship was further developed by Hart and Mas-Colell (1989) who showed that for every coalition game, there is a unique *potential* function and the marginal contributions of the players to the potential coincide with the Shapley value. This, in turn, allows the Shapley value to be derived from a single axiom. In this paper, the contribution property is modified by restricting it to bidders with winning bids instead of including all players in the game.

Definition 8. A value ψ satisfies the *balanced winner contribution property* if for every auction coalition game (N, v) with winners N_W,

$$\psi_i(N_W, v) - \psi_i(N_W \setminus \{j\}, v) = \psi_j(N_W, v) - \psi_j(N_W \setminus \{i\}, v) \tag{7}$$

for all $i \in N_W$ and $j \in N_W$.

Proposition 4 (BWC)**.** *The BWC rule satisfies the balanced winner contribution property.*

Proposition 5 (Uniqueness)**.** *The BWC rule is the only payment rule that satisfies the balanced winner contribution property.*

Satisfying the balanced winner contribution property could be seen as a reason to use the BWC payment rule over possible alternatives. It could be justified using an argument similar to the one Myerson (1980) makes in favor of the balanced contribution property. Namely, it is an *equal-gains* principle with respect to reported values. For any two traders A and B with winning bids, A's gain from trade due to B is equal to B's gain due to A. This is arguably a fair way to divide the surplus.

7. Comparing BWC with the Core and Competitive Equilibrium

BWC payments, competitive prices, and core payments all satisfy no payments for losing bidders and budget balance. There are some settings where the core and competitive prices are guaranteed to exist. For example, in the assignment game analyzed by Shapley and Shubik (1971). The example shown in Table 1 is a simple instance of such an assignment game. In this particular example, the BWC payments coincide with the mid-point of the core.

Shapley (1971) established several results for convex games, notably that every convex game has a non-empty core and for such games, the Shapley value is in the core. These results can be used to identify settings where BWC payments coincide with core payments.

Definition 9. A game (N, v) is *convex* if for all $S, T \subseteq N$, $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.

Proposition 6 (Convex games)**.** *If the winner coalition game is convex, then the core of the auction coalition game is non-empty and contains the BWC payments.*

This provides sufficient conditions for the core and BWC payments to coincide.

It is also interesting to consider cases where BWC payments do not coincide with competitive prices or core payments. BWC payments always exist and are unique for a given profile of winning bids. In contrast, competitive prices and core payments do not always exist, and when they do exist are not necessarily unique. Hence, as a market design tool, an advantage of BCW payments is that they can be used in settings where

Table 2: Competitive prices and the core

Agent	Bids	Surplus	Payment
Two demand types and no competitive equilibrium			
Airline	$\underline{\boldsymbol{\langle 0; 1, 1\rangle}}$, $\langle 0; 1, 0\rangle$, $\langle 0; 0, 1\rangle$	75	-75
Tourist	$\underline{\boldsymbol{\langle 90; -1, -1\rangle}}$	15	75
Co-authors	$\langle 60; -1, 0\rangle$, $\langle 60; 0, -1\rangle$		
Larger market with two demand types			
Airline 1	$\underline{\boldsymbol{\langle 0; 1, 1\rangle}}$, $\langle 0; 1, 0\rangle$, $\langle 0; 0, 1\rangle$	86	−86
Airline 2	$\underline{\boldsymbol{\langle 0; 1, 1\rangle}}$, $\langle 0; 1, 0\rangle$, $\langle 0; 0, 1\rangle$	86	−86
Tourist 1	$\underline{\boldsymbol{\langle 90; -1, -1\rangle}}$	6	84
Tourist 2	$\langle 90; -1, -1\rangle$		
Co-authors 1	$\underline{\boldsymbol{\langle 60; -1, 0\rangle}}$, $\langle 60; 0, -1\rangle$	16	44
Co-authors 2	$\langle 60; -1, 0\rangle$, $\underline{\boldsymbol{\langle 60; 0, -1\rangle}}$	16	44
Avoidable fixed costs and empty core			
Small Airline	$\langle -85; 1\rangle$, $\underline{\boldsymbol{\langle -85; 2\rangle}}$	15	-100
Large Airline	$\langle -150; 1\rangle$, $\langle -150; 2\rangle$, $\langle -150; 3\rangle$		
Traveler 1	$\langle 55; -1\rangle$		
Traveler 2	$\underline{\boldsymbol{\langle 60; -1\rangle}}$	12.5	47.5
Traveler 3	$\underline{\boldsymbol{\langle 70; -1\rangle}}$	17.5	52.5

Note: Winning bids are underlined.

competitive prices and core payments cannot. Three examples are presented in Table 2.

The top panel of the table shows a variation of an example used by Baldwin and Klemperer (2016) to illustrate demand types. An airline is offering two flights for a particular weekend: one from London to Paris and one from Paris to London. A tourist would like to visit Paris for the weekend and so views the flights as complements and is offering 90 for both as a package. A pair of co-authors, one based in London and one in Paris would like to work together in the same city the following week, and so views the flights as substitutes and are offering to pay 60 for either flight (but not both). It is readily verified that it is efficient for the tourist to buy both flights but there are no competitive prices that support this outcome. Solving the winner determination problem selects this outcome and the BWC rule gives a price of 75 for the two-flight bundle. It is easy to verify that despite the absence of competitive prices, this is a core outcome since the airline cannot achieve a surplus of 75 or more by forming a coalition with the co-authors. The profile of bids shown is not a Nash equilibrium. Both the airline and the tourist can increase their payoffs by adjusting their bids. There is, however, a Nash equilibrium that produces the payoffs shown in the table. Suppose the tourist bids 75, the airline bids -75 for each bundle, and the co-authors' bids are unchanged. This gives the payoffs shown in the table and no player can gain by changing their bids.[5]

The middle panel of Table 2 illustrates what can happen as the market gets larger. The example has agents with the same demand types as the top panel but there are two copies of each type. The efficient allocation is for a return-flight bundle to be allocated to one of the tourists and one flight to be allocated to each of the pairs of co-authors. Interestingly, unlike the first example, this allocation can be supported by competitive prices. When both types of flight are priced at 45, the tourists are indifferent between buying and not buying the bundle, and each of the pairs of co-authors wants to buy one flight but is indifferent about which type. The competitive prices would give zero surplus to tourist 1 and a positive surplus to each of the pairs of co-authors and the airlines. The BWC surplus shares differ from this, with tourist 1 receiving a positive surplus. The reason for this is that there are some orderings of the set of winners where tourist 1 makes a positive marginal contribution (e.g. tourist 2, airline 1, airline 2, tourist 1). The profile of bids is not a Nash equilibrium. The efficient allocation of flights can be supported by a Nash equilibrium where all agents place bids at the competitive prices, but this gives a different profile of payoffs.

The example in the bottom panel of Table 2 differs from the two above in that it concerns a market with an empty core. The example is used by Telser (1994) to illustrate how avoidable fixed costs can lead to an empty core. A small airline has a plane with capacity 2 and the cost of a flight is 85. A larger airline has a plane with capacity 3 and the cost of a flight is 150. For both airlines, the cost of a flight does not depend on the number of passengers. There are three travelers who are willing to pay 55, 60, and 70 respectively for a seat on a flight. The efficient outcome is for the small

[5] In this equilibrium, each bidder can be thought of as having a profit target. Such equilibria in first price menu auctions are studied by Bernheim and Whinston (1986), who called them 'Truthful Nash Equilibria'.

airline to fly with travelers 2 and 3. However, not only are the no competitive prices that support this outcome, there is also no profile of payments that cannot be blocked by some coalition, that is, the core is empty. Consequently, a payment rule for this market cannot be based on competitive prices or core payments. The BWC surplus shares and payments are shown in the table. The two travelers with winning bids pay different amounts with traveler 3 paying more but also enjoying a higher surplus. Again, the profile of bids is not a Nash equilibrium but there is a Nash equilibrium that supports the efficient allocation.[6]

8. Comparing BWC with VCG

Both VCG payments and BWC payments always exist and are unique for a given profile of winning bids.[7] Both mechanisms feature no loser payments. VCG is incentive compatible but is not budget balanced. Conversely, BWC is not incentive compatible but is budget balanced.

To aid comparison with the BWC mechanism, surplus shares in the VCG mechanism can be expressed in terms of the auction coalition game as follows.

Proposition 7 (VCG surplus)**.** *In the VCG mechanism, bidder i's surplus is* $\phi_i^{VCG}(N, v) = v(N) - v(N \setminus \{i\})$.

Both the BWC and VCG surplus shares can be described in terms of marginal contributions. The difference between them is due to the weights given to different orderings. The BWC surplus shares take the average marginal contributions over all orderings where the winners, N_W, appear after the losers, N_L. The same set of orderings is used to calculate the surplus shares of all players, which results in a balanced budget. The VCG surplus shares are calculated using different orderings for each player. As can be seen above, rather than the average marginal contribution across a set of orderings, just the ordering where the player in question appears last is used.

An exchange implementing VCG payments will typically run a budget deficit and so require a subsidy from the auctioneer. In auction settings, the seller often also plays the role of the auctioneer, in which case the revenue is not the surplus calculated using Proposition 7. Instead, it is the sum of the buyer payments and the seller effectively absorbs any deficit. Consequently, auctions implementing VCG payments for buyers with the seller acting as auctioneer risk low seller revenues.

Ausubel and Milgrom (2006) suggest one of the reasons the VCG mechanism is rarely used in practice is the potential low seller revenue. The top panel of Table 3 shows a variation of an example they use to illustrate this point, and compares the

[6]Suppose the small airline bids -120 for both bundles, the large airline bids -300 for each bundle, traveler 1 bids 1, and travelers 2 and 3 bid 60. No player can individually increase their payoff by changing their bids. However, a coalition of traveler 1 and the large airline can jointly adjust their bids and increase their payoffs.

[7]When there are multiple solutions to the winner determination problem, there are multiple ways to allocate items that maximize reported surplus. Each of these allocations will be associated with a different profile of winning bids and payments. This is the case for the BWC and VCG mechanisms as well as the pay-as-bid mechanism.

Table 3: Low VCG revenue

Agent	Bids	VCG payment	BWC payment
Low VCG revenue with 3 buyers			
Airline	$\langle 0;1\rangle$, $\boldsymbol{\langle 0;2\rangle}$	-4	$-2\,{}^{2}\!/_{3}$
Couple	$\langle 2;-2\rangle$		
Single traveler 1	$\boldsymbol{\langle 2;-1\rangle}$	0	$1\,{}^{1}\!/_{3}$
Single traveler 2	$\boldsymbol{\langle 2;-1\rangle}$	0	$1\,{}^{1}\!/_{3}$
Higher VCG revenue with 2 buyers			
Airline	$\boldsymbol{\langle 0,1\rangle}$, $\langle 0,2\rangle$	-2	-2
Couple	$\langle 2,-2\rangle$		
Single traveler	$\boldsymbol{\langle 2,-1\rangle}$	2	2

Note: Winning bids are underlined.

VCG and BWC payments. An airline is offering two seats on a flight. A couple desires a package of two seats. Two single travelers each desire a single seat. All are offering to pay 2. Despite the competition for seats, the VCG revenue is zero. In contrast, the BWC revenue with sincere bidding is $2\,{}^{2}\!/_{3}$. Notice that even if the two single travelers reduce their bids to 1 (the lowest they can bid and remain winning), the revenue is still 2 compared to zero under the VCG mechanism.

Ausubel and Milgrom show how in the VCG mechanism the seller revenue can decrease when a bidder is added. They argue that this non-monotonicity creates loopholes and vulnerabilities that bidders can sometimes exploit by colluding or using shills (multiple identities). The bottom panel of Table 3 shows the effect of removing a bidder, illustrating the non-monotonicity. The VCG revenue increases from zero to 2. In contrast, the BWC revenue, with sincere bidding, falls from $2\,{}^{2}\!/_{3}$ to 2.[8]

9. Computational Complexity

If a market mechanism is going to be used in practice, it is necessary to consider not just its economic properties but also whether the computations involved in determining allocations and payments are tractable. Solving the winner determination problem for a combinatorial market is NP-hard.[9] This suggests that for markets above a certain

[8] Without sincere bidding, there are Nash equilibria such that the revenue remains at 2 when the number of bidders is reduced. When there are 3 buyers, the two single travelers need to bid such that the sum of their bids just exceeds the bid of the couple. When there are 2 buyers, the remaining single traveler and the couple both bid 2. One of the bids is selected at random to be the winner. There is no incentive to adjust bids.

[9] In computational complexity theory, NP-hard problems are those that are at least as hard as the hardest problems in the class NP. The hardest problems in NP are classed as NP-complete. A number of important combinatorial problems are NP-complete and there are no known algorithms that can solve them in polynomial time. It is readily verified that the combinatorial market winner determination is NP-hard since it can be used to solve some well-known NP-complete problems. For instance, the *knapsack problem* can be modeled as an auction with a seller offering to sell any quantity of a good up to some capacity and buyers bidding different amounts for different sized packages. The *set packing problem* can be modeled as an auction with

size, it may not be feasible to solve the winner determination problem. The problem applies whatever pricing rule is used, but may be more acute for BWC pricing. If a pay-as-bid rule is used, the winner determination problem needs to be solved once. For VCG payments, it needs to be solved $n_w + 1$ times where n_w is the number of bidders with winning bids. For BWC payments, it needs to be solved $2^{n_w} - 1$ times.

In computationally challenging cases, one option would be to calculate payments that approximate the BWC payments.[10] This could be done by employing two strategies. First, instead of solving the winner determination problem for the global optimal, use a procedure that returns the best solution found so far after a certain amount of time has expired. Solving the winner determination problem is an integer linear programming problem. Currently available software for solving such problems typically uses a *branch and bound* algorithm, which allows the best solution found so far to be retrieved. Second, instead of basing payments on all $n_w!$ orderings of the winners, base payments on a sample of the orderings. This could reduce the number of times the winner determination problem needs to be solved. Note that when global optimal solutions to the winner determination problem are used to calculate marginal contributions, marginal contributions are always non-negative. But when *best-so-far* solutions are used, this is not necessarily true. Accordingly, some adjustments may be required to ensure surplus shares are always non-negative and their sum is the total available surplus.

Algorithm 1 illustrates how the two strategies could be used. On line 1, the subroutine $wd(N, \bar{t})$ is called which begins solving the winner determination problem with the set of bidders N and time limit $\bar{t}$. If the time limit is reached before a global optimal solution is found, the best-so-far solution is returned.[11] On line 2, the subroutine $permutations(N_W, \bar{n})$ is called. It returns a set of permutations of the set of winners N_W. If $\bar{n}$ is less than the number of permutations of N_W, a random sample of $\bar{n}$ unique permutations is returned; otherwise, all permutations are returned. Line 3 initializes surplus shares to zero. Lines 4 to 15 iterate over permutations and bidders within each permutation. On line 9, the winner determination subroutine is called with a subset of bidders and time limit $\bar{t}$. The returned surplus is used to calculate marginal contributions. In cases where a best-so-far solution is returned, the solution may not be globally optimal. As a result, it is possible adding a bidder will decrease the surplus. Line 10 ensures that, in such cases, marginal contributions are non-negative, which is needed for ex post individual rationality. Line 11 ensures the sum of marginal contributions for an ordering does not exceed v^{N_W}, which is needed for budget balance. Assuming only solving the winner determination problem takes a non-negligible amount of time, the algorithm provides a way to determine the winning bids and calculate surplus shares

a seller offering one unit of each of a number of different goods and buyers bidding to buy different combinations. The *subset sum problem* can be modeled as a market for a single good with traders submitting package bids to buy or sell different quantities (solving the winner determination problem gives a subset of orders whose quantities sum to zero).

[10]Another option is to avoid computationally challenging cases by placing restrictions on the packages agents can bid on. Rothkopf et al. (1998) analyze how such restrictions can be used to make determining winning bids in combinatorial auctions manageable and Goeree and Holt (2010) test this approach experimentally.

[11]Assume that results are stored so that if the routine has previously been called for a given set of bidders, the previous result is returned immediately.

Algorithm 1: Find winning bids and approximate BWC surplus shares

Input: Bidders with bids N, time limit $\bar{t}$, permutations limit $\bar{n}$

Output: Winners N_W, winning bids $\boldsymbol{w}^{\boldsymbol{N*}}$, BWC surplus shares $\boldsymbol{\psi}$

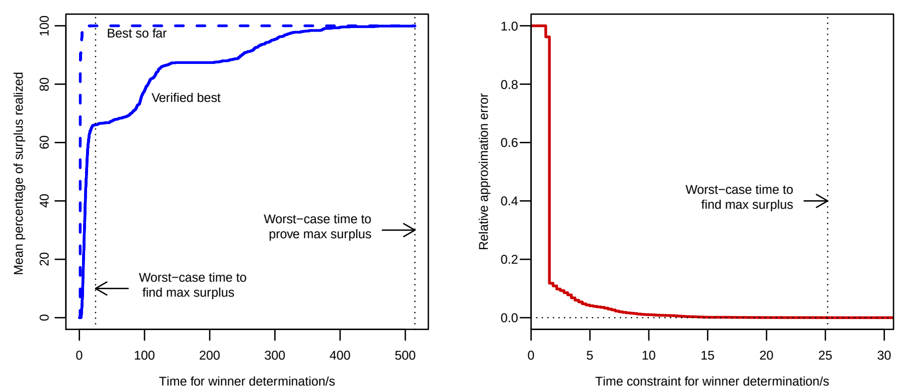
$\langle N_W, \boldsymbol{w}^{\boldsymbol{N*}}, v^{N_W} \rangle \leftarrow wd(N, \bar{t})$;
$P \leftarrow permutations(N_W, \bar{n})$;
foreach *bidder i in N* **do** $\psi_i \leftarrow 0$;
foreach *permutation $\boldsymbol{p}$ in P* **do**
    $v_{last} \leftarrow 0$;
    $S \leftarrow N \setminus N_W$;
    foreach *bidder i in $\boldsymbol{p}$* **do**
        $S \leftarrow S \cup \{i\}$;
        $v \leftarrow wd(S, \bar{t})$;
        if $v < v_{last}$ **then** $v \leftarrow v_{last}$;
        if $v > v^{N_W}$ **then** $v \leftarrow v^{N_W}$;
        $\psi_i \leftarrow \psi_i + (v - v_{last})/|P|$;
        $v_{last} \leftarrow v$;
    end
end
return $\langle N_W, \boldsymbol{w}^{\boldsymbol{N*}}, \boldsymbol{\psi} \rangle$

within a specified time-limit even for computationally challenging cases.

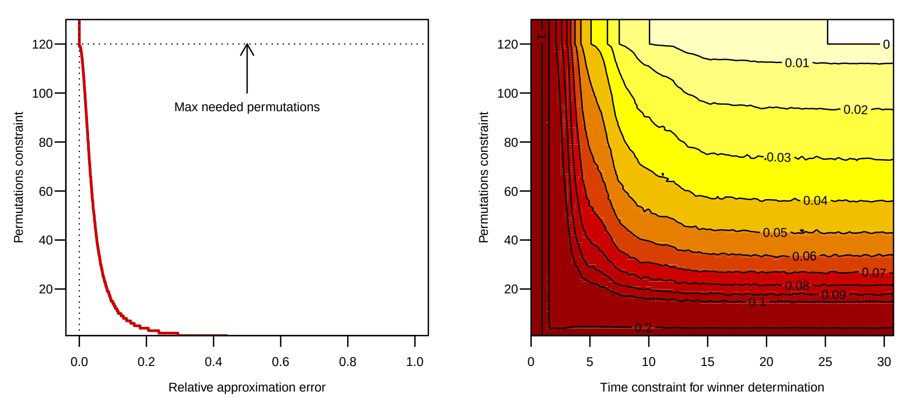
A numeric simulation was run to measure how closely the outcomes of the algorithm approximate the true BWC outcomes. An environment was chosen where calculating the true BWC outcomes took a moderate amount of time. This meant there were gains to be had from using the algorithm but calculating the true BWC outcomes was not prohibitively time consuming, so the approximate outcomes generated by the algorithm could be compared to the true ones. The environment had a single seller with 24 items and 5 buyers. Each buyer bid for every possible package of 5 items (hence each buyer placed 42,504 bids). The amounts buyers bid were drawn from the exponential distribution with parameter $\lambda = 1$. The seller bid zero. Running the auction always results in the seller and four buyers having winning bids and one buyer having no winning bids. This is because each of the 5 buyers wants exactly 5 items but only 24 items are offered. Accordingly, the true BWC surplus shares are calculated based on the 120 permutations of the 5 winners. A total of 1,000 auctions were simulated. For each auction, outcomes were calculated with different constraints on the time allowed for the winner determination problem and on the number of permutations evaluated.[12]

Fig. 1 shows results of the simulation. Panel (a) shows how long it takes to solve the winner determination problem. Determining the winning bids involves solving a linear integer programming problem. The branch and bound procedure used to solve the problem produces a sequence of best-so-far candidate solutions and terminates when

[12]The simulations were run on a desktop PC with an Intel i5-6500 3.20GHz CPU and 16GB RAM. The winner determination problem was solved using Gurobi Optimizer version 7.5.1.



(a) Realized surplus against time determining winners (b) The effect of time constraints on surplus shares



(c) The effect of permutation constraints on surplus shares

(d) The combined effect of time and permutation constraints on surplus shares

Fig. 1: The results shown are based on 1,000 auctions with 1 seller, 5 buyers and 24 item, with each buyer bidding for every package of 5 items. Panel (a) shows how time constraints on the winner determination problem affect the percentage of the available surplus realized when either only verified global optimal solutions are counted (the solid line) or best-so-far ones are used (the dashed line). Panels (b), (c) and (d) show how surplus shares are affected by constraints on the time allowed for winner determination and on the number of permutations of the winners evaluated. 'Relative approximation error' is the normalized Euclidean distance between the approximate and true BWC surplus shares.

it has been proved that there is no solution better than the current best candidate. The solid line labeled 'verified best' shows the percentage of the 1,000 auctions where the winner determination procedure had terminated at different times. All the auctions had the same number of bids, bidders and items but the time taken for the procedure to terminate varied considerably. Suppose the procedure is run with a time constraint and trades are only implemented if it has terminated before the time limit. The solid line can be interpreted as the average percentage of the available surplus that would be realized in this case. Now suppose that on reaching the time limit before the procedure has terminated, the best-so-far solution is implemented instead of no trade. The dashed line shows the average percentage of the available surplus that would be realized in this case. When *best-so-far* solutions are used, considerably less time is needed to realize surplus. Half the surplus is realized after 1.3 seconds and 100 percent after 24.9 seconds compared to half after 11.1 seconds and 100 percent after 514.8 seconds when only *verified-best* solutions are used. These results show there is indeed scope to limit the worst-case time spent on the winner determination problem while still preserving most of the surplus.

Panel (b) shows how imposing a time constraint on the winner determination problem affects individual surplus shares as opposed to overall surplus. Approximate BWC surplus shares were calculated with constraints on the time for the winner determination problem. Then the *relative approximation error* was calculated as the Euclidean distance between the approximate and true BWC surplus shares normalized by the Euclidean distance between the true shares and zero. Notice that the approximation error falls to zero after 24.9 seconds which is the worst-case time to find the maximum surplus labeled on Panel (a).

Panel (c) shows how imposing a constraint on the number of permutations evaluated affects the surplus shares. Note that the true BWC surplus shares are the average marginal contribution over all permutations of the winners. Hence, if a simple random sample of permutations is drawn and used to calculate approximate BWC surplus shares, the expected value of the approximate shares is the true shares. Furthermore, if the sample is drawn without replacement, the approximate shares equal the true shares when the number of draws is equal to the number of permutations.

Panel (d) is a heat map showing how different combinations of time constraints and permutation constraints affect the approximate BWC surplus shares. The lighter shades indicate smaller approximation errors. The contour lines show different combinations of constraints that on average result in the same approximation error (they are analogous to iso-quants for a production function). The area in the top right above the '0' contour represents combinations where the error is zero. Notice the relation to the points the approximation error reaches zero in panels (b) and (c). At the '0' contour, the constraints become binding and tightening them increases the approximation error. If, for example, a 0.05 error level is tolerable, there is considerable scope to tighten the constraints beyond the points where they are binding. This, in turn, allows computationally challenging cases to be made more tractable.

10. Discussion

A remarkable feature of the Shapley value is that for every coalition game it gives a unique profile of payoffs. Payoffs satisfying other solution concepts such as the core and competitive equilibrium do not always exist, and when they do exist, they are not necessarily unique. This paper has shown how the standard Shapley value can be modified so that it can be used to determine payments in auctions and exchanges such that losing bidders do not make or receive any payments. The resultant BWC payment rule is budget balanced and ex post individually rational. It is also the unique payment rule that satisfies the balanced-winner-contribution property. This is arguably a fair way to divide the surplus.

A number of examples were used to illustrate how BWC payments work in practice and how the payments compare to those based on other rules. For some simple cases, BWC payments coincide with core payments and payments based on competitive prices. In other cases, the payments do not coincide. The BWC payment rule can produce payments in cases where competitive prices do not exist and where the core is empty. The BWC mechanism shares some properties with the VCG mechanism. Both always produce unique payments for a given profile of winning bids. The mechanisms differ in that VCG is incentive compatible but not budget balanced whereas BWC is not incentive compatible but is budget balanced.

There are several directions future research could explore. One would be exploring the incentive properties of BWC payments. This could be done analytically for simple cases or by using computer simulations for more complex ones. Another would be developing refinements to the BWC payments rule. These could involve realizing additional market design objectives or protecting against potential vulnerabilities to gaming. Finally, the BWC mechanism could be tested in the laboratory.

Appendix A. Proofs

Proof of Proposition 1 (BB). Budget balance implies that the sum of the surplus shares equals the available surplus, $v(N)$. The BWC surplus shares are based on the $|N_W|!$ orderings of the winners N_W. The marginal contributions are calculated after the losing bids have been added, $v(N_L \cup S \cup \{i\}) - v(N_L \cup S)$. Note that $v(N_L) = 0$. Hence the marginal contributions of any ordering of the winners also sum to $v(N)$. The BWC surplus shares are the mean marginal contributions across all orderings. It follows that $\sum_{i \in N_W} \psi_i(N_W, v) = v(N)$. This implies budget balance. □

Proof of Proposition 2 (IR). Suppose that there is some bidder who pays more than he or she bid. The payment rule defined by equation 4 shows that it must be because $\psi_i(N_W, v) < 0$. Since ψ_i is a weighted average of marginal contributions, at least one of the marginal contributions must be negative, that is $v(N_L \cup S \cup \{i\}) < v(N_L \cup S)$. But this is a contradiction. Adding a bidder cannot decrease the surplus from the winner determination problem since the solution without the new bidder is still feasible. □

Proof of Proposition 3 (No loser payments). The proposition follows immediately from the definition of BWC payments (see equation 4). □

The next two propositions (4 and 5) are variations of theorems from Myerson (1977) and Myerson (1980). Using Definition 7, the auction coalition game (N, v) and the solution to the winner determination problem can be used to write a *winner coalition game* (N_W, v') where N_W is the set of traders with winning bids and $v'(S) = v(S \cup N_L)$ with $S \subseteq N_W$. The standard Shapley value of the winner coalition game is equivalent to the BWC surplus shares of the auction coalition game.

Proof of Proposition 4 (BWC). The proposition follows from Lemmas 1 and 2 from Myerson (1980) applied to the winner coalition game. $\square$

Proof of Proposition 5 (Uniqueness). The proposition follows from Lemmas 3 and 4 from Myerson (1980) applied to the winner coalition game. $\square$

Proof of Proposition 6 (Convex games). From Shapley (1971) Theorem 4 and Theorem 7 it follows that if the winner coalition game is convex, then the game has a non-empty core which contains the BWC surplus shares. This is because the standard Shapley value of the winner coalition game is equivalent to the BWC surplus shares. If a coalition C can block a profile of payoffs in the auction coalition game, the coalition $C \cap N_W$ can block the associated profile in the winner coalition game. Hence, if a profile of payoffs is in the core of the winner coalition game, then there is an associated profile of payoffs in the core of the auction coalition game. It follows that if the winner coalition game is convex, then the BWC surplus shares are in the core of the auction coalition game. $\square$

Proof of Proposition 7 (VCG surplus). Let $\boldsymbol{w}^{N*}$ represent the solution to the winner determination problem when i's bids are included and $\boldsymbol{w}^{N\setminus\{i\}*}$ when he or she is excluded. Bidder i's VCG payment $\pi_i = \sum_{j\in N\setminus\{i\}} \sum_{k\in B_j} b_{jk} \left[w_{jk}^{N\setminus\{i\}*} - w_{jk}^{N*}\right]$. Bidder i's surplus gain is $\phi_i^{VCG}(N, v) = \sum_{k\in B_i} b_{ik} w_{ik}^{N*} - \pi_i$. Substituting the expression for π_i and using (2) gives $\phi_i^{VCG}(N, v) = v(N) - v(N \setminus \{i\})$. $\square$

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