A refined version of Grothendieck's anabelian conjecture for hyperbolic curves over finite fields

MOHAMED SAÏDI and AKIO TAMAGAWA

Abstract. In this paper we prove a refined version of a theorem by Tamagawa and Mochizuki on isomorphisms between (tame) arithmetic fundamental groups of hyperbolic curves over finite fields, where one "ignores" the information provided by a "small" set of primes.

§0. Introduction. Let k be a finite field of characteristic $p > 0$ and U a hyperbolic curve over k. Namely, $U = X \setminus S$, where X is a proper, smooth, geometrically connected curve of genus g over k and $S \subset X$ is a divisor which is finite étale of degree r over k, such that $2 - 2g - r < 0$. We have the following commutative diagram of profinite groups:

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(U \times_k \bar{k}, *) & \to & \pi_1(U, *) & \to & G_k & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \pi_1^t(U \times_k \bar{k}, *) & \to & \pi_1^t(U, *) & \to & G_k & \to & 1 \end{array}$$

in which both rows are exact and all vertical arrows are surjective (and bijective for $r = 0$). Here, G_k is the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$, $*$ means a suitable geometric point, and π_1 (resp. π_1^t) stands for the étale (resp. tame) fundamental group. The following result is fundamental in the anabelian geometry of hyperbolic curves over finite fields.

Theorem A (Tamagawa, Mochizuki). *Let U, V be hyperbolic curves over finite fields k_U, k_V, respectively. Let*

$$\alpha : \pi_1(U, *) \xrightarrow{\sim} \pi_1(V, *)$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\sim} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale universal coverings determined by the profinite groups $\pi_1(U, *)$, $\pi_1(V, *)$, respectively.

Theorem A was proved by Tamagawa (cf. [Tamagawa], Theorem (4.3)) in the affine case (together with the variant where π_1 is replaced by π_1^t), and by Mochizuki (cf. [Mochizuki1], Theorem 3.2) in the proper case. It implies, in particular, that one can embed a suitable category of hyperbolic curves over finite fields into the category of profinite groups. It is essential in the anabelian philosophy of Grothendieck, as was formulated in [Grothendieck], to be able to determine the image of this functor. Recall that the full structure of the profinite group $\pi_1(U \times_k \bar{k}, *)$ is unknown (for any single example of U which is hyperbolic). Hence, a fortiori, the structure of $\pi_1(U, *)$ is unknown. (Even if we replace the fundamental groups $\pi_1(U \times_k \bar{k}, *)$, $\pi_1(U, *)$ by the tame fundamental groups $\pi_1^t(U \times_k \bar{k}, *)$, $\pi_1^t(U, *)$, respectively, the situation is just the same.) Thus, the problem of determining the image of the above functor seems to be quite difficult, at least for the moment. In this paper we investigate the following question:

Question 0.1. Is it possible to prove any result analogous to the above Theorem A where $\pi_1(U, *)$ is replaced by some (continuous) quotient of $\pi_1(U, *)$ whose structure is better understood?

Let $\mathfrak{Primes}$ be the set of all prime numbers. Let $\Sigma = \Sigma_X \subset \mathfrak{Primes}$ be a set of prime numbers containing at least one prime number different from the characteristic p. Let $\mathcal{C}$ be the full class of finite groups whose cardinality is divisible only by primes in Σ. Let $\Delta_U \stackrel{\text{def}}{=} \pi_1^t(U \times_k \bar{k}, *)^\Sigma$ be the maximal pro-$\mathcal{C}$ quotient of $\pi_1^t(U \times_k \bar{k}, *)$. Here, if Σ does not contain p, the structure of Δ_U is well understood: Δ_U is isomorphic to the pro-Σ completion of a certain well-known finitely generated discrete group (i.e., either a free group or a surface group). Let $\Pi_U \stackrel{\text{def}}{=} \pi_1^t(U, *)/\operatorname{Ker}(\pi_1^t(U \times_k \bar{k}, *) \twoheadrightarrow \pi_1^t(U \times_k \bar{k}, *)^\Sigma)$ be the corresponding quotient of $\pi_1^t(U, *)$. We shall refer to Π_U as the maximal geometrically pro-Σ quotient of the tame fundamental group $\pi_1^t(U, *)$ or, in short, the geometrically pro-Σ tame fundamental group of U. (When Σ does not contain p, we may and shall refer to it as the maximal geometrically pro-Σ quotient of the fundamental group $\pi_1(U, *)$ or, in short, the geometrically pro-Σ fundamental group of U.)

Question 0.2. Is it possible to prove any result analogous to the above Theorem A where $\pi_1(U, *)$ is replaced by Π_U, for some non-empty set of prime numbers Σ containing at least one prime number different from the characteristic p?

The first set Σ to consider is the set $\Sigma \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \{\text{characteristic} = p\}$. In this case we shall refer to Π_U as the maximal geometrically prime-to-characteristic quotient of the fundamental group $\pi_1(U, *)$. We have the following result:

Theorem B (A Prime-to-p Version of Grothendieck's Anabelian Conjecture for Hyperbolic Curves over Finite Fields). *Let U, V be hyperbolic curves over finite fields k_U, k_V, respectively. Let $\Sigma_U \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \{\operatorname{char}(k_U)\}$, $\Sigma_V \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \{\operatorname{char}(k_V)\}$, and write Π_U, Π_V for the geometrically pro-Σ_U étale fundamental group of U, and the geometrically pro-Σ_V étale fundamental group of V, respectively. Let*

$$\alpha : \Pi_U \xrightarrow{\sim} \Pi_V$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\sim} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups Π_U, Π_V, respectively.

Theorem B was proved by Saïdi and Tamagawa (cf. [Saïdi-Tamagawa1], Corollary 3.10). Our main result in this paper is the following refined version of the above Theorems A and B (cf. Theorem 4.22).

Theorem C (A Refined Version of the Grothendieck Anabelian Conjecture for Proper Hyperbolic Curves over Finite Fields). *Let X, Y be proper hyperbolic curves over finite fields k_X, k_Y of characteristic p_X, p_Y, respectively. Let $\Sigma_X, \Sigma_Y \subset \mathfrak{Primes}$ be sets of prime numbers and set $\Sigma'_X \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma_X$, $\Sigma'_Y \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma_Y$. Assume that neither the Σ'_X-adic representation $\rho_{\Sigma'_X} : G_{k_X} \to \prod_{l \in \Sigma'_X} \mathrm{GL}(T_l(J_X))$ nor the Σ'_Y-adic representation $\rho_{\Sigma'_Y} : G_{k_Y} \to \prod_{l \in \Sigma'_Y} \mathrm{GL}(T_l(J_Y))$, arising from the Jacobian varieties J_X, J_Y of X, Y, respectively, is injective. Write Π_X, Π_Y for the geometrically pro-Σ_X étale fundamental group of X and the geometrically pro-Σ_Y étale fundamental group of Y, respectively. Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sim} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups Π_X, Π_Y, respectively.

Note that the extra assumptions on Σ_X and Σ_Y in Theorem C are satisfied if Σ'_X, Σ'_Y are finite. We show that sets of primes Σ_X and Σ_Y satisfying the conditions in Theorem C must be of (natural) density $\neq 0$, while given any $\epsilon > 0$ there exist sets of primes Σ_X and Σ_Y of (natural) density $< \epsilon$ satisfying the conditions in Theorem C (cf. Remark 2.8).

Theorem C above implies a "similar" version for affine hyperbolic curves (cf. Theorem 4.23).

Theorem D (A Refined Version of the Grothendieck Anabelian Conjecture for (Not Necessarily Proper) Hyperbolic Curves over Finite Fields). *Let U, V be (not necessarily proper) hyperbolic curves over finite fields k_U, k_V of characteristic p_U, p_V, respectively. Let $\Sigma_U, \Sigma_V \subset \mathfrak{Primes}$, be sets of* *prime numbers, and set* $\Sigma'_U \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma_U$, $\Sigma'_V \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma_V$. *Write* Π_U, Π_V *for the geometrically pro-*Σ_U *tame fundamental group of* U *and the geometrically pro-*Σ_V *tame fundamental group of* V*, respectively. Let*

$$\alpha : \Pi_U \xrightarrow{\sim} \Pi_V$$

be an isomorphism of profinite groups. Assume that there exist open subgroups $\Pi_{U'} \subset \Pi_U$, $\Pi_{V'} \subset \Pi_V$*, which correspond to each other via* α*, i.e.,* $\Pi_{V'} = \alpha(\Pi_{U'})$*, corresponding to étale coverings* $U' \to U$, $V' \to V$*, such that the smooth compactifications* X' *of* U' *and* Y' *of* V' *are hyperbolic, and that neither the* Σ'_U*-adic representation* $\rho_{\Sigma'_U} : G_{k_{U'}} \to \prod_{l \in \Sigma'_U} \mathrm{GL}(T_l(J_{X'}))$ *nor the* Σ'_V*-adic representation* $\rho_{\Sigma'_V} : G_{k_{V'}} \to \prod_{l \in \Sigma'_V} \mathrm{GL}(T_l(J_{Y'}))$*, arising from the Jacobian varieties* $J_{X'}$, $J_{Y'}$ *of* X', Y'*, respectively, is injective. (Here,* $k_{U'}$, $k_{V'}$ *denote the fields of constants of* U', V'*, respectively.) Then* α *arises from a uniquely determined commutative diagram of schemes:*

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\sim} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups Π_U, Π_V*, respectively.*

In what follows we explain the steps/ideas of the proof of Theorem C. Starting from an isomorphism

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

between profinite groups, one can first, using well-known results on the group-theoretic characterization of decomposition groups in arithmetic fundamental groups as in [Tamagawa] (the so-called local theory), establish a set-theoretic bijection

$$\phi : X^{\mathrm{cl}} \setminus E_X \xrightarrow{\sim} Y^{\mathrm{cl}} \setminus E_Y$$

between the set of closed points of X, Y, outside some "exceptional sets" $E_X \subsetneq X^{\mathrm{cl}}$ and $E_Y \subsetneq Y^{\mathrm{cl}}$, respectively, such that $\alpha(D_x) = D_{\phi(x)}$ for $x \in X^{\mathrm{cl}} \setminus E_X$ where D_x, $D_{\phi(x)}$ denote the decomposition group of x, $\phi(x)$ in Π_X, Π_Y, respectively (which are only defined up to conjugation). It is not difficult to prove $p \stackrel{\mathrm{def}}{=} p_X = p_Y$ and $\Sigma \stackrel{\mathrm{def}}{=} \Sigma_X = \Sigma_Y$. As a technical step in the proof we resort to a specific auxiliary prime number l and consider the $\mathbb{Z}_l$-extensions k^l_X, k^l_Y, of k_X, k_Y, respectively. Let $X^l \stackrel{\mathrm{def}}{=} X \times_{k_X} k^l_X$, and $Y^l \stackrel{\mathrm{def}}{=} Y \times_{k_Y} k^l_Y$. Write $E_{X^l} \stackrel{\mathrm{def}}{=} E_X \times_{k_X} k^l_X$ (resp. $E_{Y^l} \stackrel{\mathrm{def}}{=} E_Y \times_{k_Y} k^l_Y$), $\mathcal{O}_{E_{X^l}}$, $\mathcal{O}_{E_{Y^l}}$ for the rings of rational functions on X^l, Y^l whose poles are disjoint from E_{X^l}, E_{Y^l}, respectively, and $\mathcal{O}^\times_{E_{X^l}}$, $\mathcal{O}^\times_{E_{Y^l}}$ the multiplicative groups of $\mathcal{O}_{E_{X^l}}$, $\mathcal{O}_{E_{Y^l}}$, respectively. We have a natural set-theoretic bijection $\phi^l : (X^l)^{\mathrm{cl}} \setminus E_{X^l} \xrightarrow{\sim} (Y^l)^{\mathrm{cl}} \setminus E_{Y^l}$. Next, certain finite index subgroups $H^\times_{X^l}$, $H^\times_{Y^l}$ of $\mathcal{O}^\times_{E_{X^l}}$, $\mathcal{O}^\times_{E_{Y^l}}$ are naturally associated with α via Kummer theory, such that $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ induces a commutative diagram:

$$\begin{array}{ccc} H^\times_{X^l}/((k^l_X)^\times\{\Sigma'\}) & \xleftarrow{\rho'} & H^\times_{Y^l}/((k^l_Y)^\times\{\Sigma'\}) \\ \downarrow & & \downarrow \\ H^\times_{X^l}/(k^l_X)^\times & \xleftarrow{\bar{\rho}} & H^\times_{Y^l}/(k^l_Y)^\times \end{array}$$

in which the vertical arrows are the natural surjective homomorphisms and the horizontal arrows are natural isomorphisms induced by α, where $\Sigma' \overset{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma$, and $(k_X^l)^\times\{\Sigma'\}$ (resp. $(k_Y^l)^\times\{\Sigma'\}$) is the Σ'-primary part of the multiplicative group $(k_X^l)^\times$ (resp. $(k_Y^l)^\times$).

The isomorphism $\bar{\rho} : H_{Y^l}^\times/(k_Y^l)^\times \xrightarrow{\sim} H_{X^l}^\times/(k_X^l)^\times$ between subgroups of groups of principal divisors supported outside exceptional sets has the property that it preserves the valuations of functions, with respect to the set-theoretic bijection $\phi^l : (X^l)^{\text{cl}} \setminus E_{X^l} \xrightarrow{\sim} (Y^l)^{\text{cl}} \setminus E_{Y^l}$. We think of elements of $\mathcal{O}_{E_{X^l}}^\times/((k_X^l)^\times\{\Sigma'\})$ and $\mathcal{O}_{E_{Y^l}}^\times/((k_Y^l)^\times\{\Sigma'\})$ as "pseudo-functions", i.e., classes of rational functions with divisor supported outside exceptional sets, modulo Σ'-primary constants. In particular, given a pseudo-function $f' \in \mathcal{O}_{E_{X^l}}^\times/((k_X^l)^\times\{\Sigma'\})$ (resp. $g' \in \mathcal{O}_{E_{Y^l}}^\times/((k_Y^l)^\times\{\Sigma'\})$), and a closed point $x \in X^{\text{cl}} \setminus E_X$ (resp. $y \in Y^{\text{cl}} \setminus E_Y$) it makes sense to consider the Σ-value $f'(x) \in (k(x)^\times)^\Sigma$ (resp. $g'(y) \in (k(y)^\times)^\Sigma$) of f' (resp. g') (cf. Lemma 4.5 and the discussion before it). Here, $(k(x)^\times)^\Sigma$, $(k(y)^\times)^\Sigma$ denote the maximal Σ-primary quotient of the multiplicative group of the residue fields $k(x)$, $k(y)$, respectively. Then the isomorphism $\rho' : H_{Y^l}^\times/((k_Y^l)^\times\{\Sigma'\}) \to H_{X^l}^\times/((k_X^l)^\times\{\Sigma'\})$ has the property that it preserves the Σ-value of the pseudo-functions with respect to the set-theoretic bijection $\phi^l : (X^l)^{\text{cl}} \setminus E_{X^l} \xrightarrow{\sim} (Y^l)^{\text{cl}} \setminus E_{Y^l}$. Let $R_{X^l} \overset{\text{def}}{=} \langle H_{X^l}^\times \rangle$, $R_{Y^l} \overset{\text{def}}{=} \langle H_{Y^l}^\times \rangle$ denote the abelian subgroups of $\mathcal{O}_{E_{X^l}}$, $\mathcal{O}_{E_{Y^l}}$ generated by $H_{X^l}^\times$, $H_{Y^l}^\times$, respectively. In fact, R_{X^l}, R_{Y^l} are subalgebras of $\mathcal{O}_{E_{X^l}}$, $\mathcal{O}_{E_{Y^l}}$ over k_X^l, k_Y^l, respectively, having the same fields of fractions, and $\mathcal{O}_{E_{X^l}}$, $\mathcal{O}_{E_{Y^l}}$ are the normalizations of R_{X^l}, R_{Y^l}, respectively. We think of the multiplicative groups $H_{X^l}^\times/(k_X^l)^\times$, $H_{Y^l}^\times/(k_Y^l)^\times$ as subsets of the projective spaces $\mathbb{P}(R_{X^l})$, $\mathbb{P}(R_{Y^l})$ associated to the infinite-dimensional k_X^l-vector space R_{X^l}, k_Y^l-vector space R_{Y^l}, respectively. Using again, in an essential way, the fact that the set Σ satisfies the assumptions in Theorem C we show that the isomorphism $\bar{\rho} : H_{Y^l}^\times/(k_Y^l)^\times \to H_{X^l}^\times/(k_X^l)^\times$ viewed as a bijection between subsets of the projective spaces $\mathbb{P}(R_{Y^l})$ and $\mathbb{P}(R_{X^l})$ preserves "partial" collineations in the following sense: given a line $\ell \subset \mathbb{P}(R_{Y^l})$ such that $\ell \cap (H_{Y^l}^\times/(k_Y^l)^\times) \neq \varnothing$ then there exists a unique line $\ell' \subset \mathbb{P}(R_{X^l})$ such that $\ell' \cap (H_{X^l}^\times/(k_X^l)^\times) \neq \varnothing$ and $\bar{\rho}(\ell \cap (H_{Y^l}^\times/(k_Y^l)^\times)) = \ell' \cap (H_{X^l}^\times/(k_X^l)^\times)$. If $H_{X^l}^\times/(k_X^l)^\times = \mathcal{O}_{E_{X^l}}^\times/(k_X^l)^\times$, $H_{Y^l}^\times/(k_Y^l)^\times = \mathcal{O}_{E_{Y^l}}^\times/(k_Y^l)^\times$, and $E_X = E_Y = \varnothing$, then $\bar{\rho} : K_{Y^l}^\times/(k_Y^l)^\times \to K_{X^l}^\times/(k_X^l)^\times$ is a bijection between points of the projective spaces $\mathbb{P}(K_{Y^l})$ and $\mathbb{P}(K_{X^l})$, which preserves collineations, where K_{X^l} (resp. K_{Y^l}) is the function field of X^l (resp. Y^l). Thus, by the fundamental theorem of projective geometry, it arises from a unique semi-linear isomorphism $(K_{X^l}, +) \xrightarrow{\sim} (K_{Y^l}, +)$. Unfortunately, at this stage we are even not able to prove that the exceptional sets E_X and E_Y are finite. This causes a very serious difficulty. To overcome this difficulty, we prove, in §5, a refined version of the fundamental theorem of projective geometry, which may be of interest independently of the topic of this paper (cf. Theorem 5.7), and which applies well in our situation in order to recover the ring structures of $\mathcal{O}_{E_{X^l}}$, $\mathcal{O}_{E_{Y^l}}$, respectively. More precisely, given a commutative field k we define the notion of an admissible set $\mathcal{S}$ of subsets of $\mathbb{P}^1(k)$ (cf. Definition 5.4) (roughly speaking these are sets consisting of "small" subsets of $\mathbb{P}^1(k)$). For a subset $\mathcal{U} \subset \mathbb{P}(V)$ of a projective space $\mathbb{P}(V)$ associated to a k-vector space V, we define the notion of being $\mathcal{S}$-ample where $\mathcal{S}$ as above is admissible (cf. Definition 5.6). Roughly speaking, being $\mathcal{S}$-ample means that $\mathcal{U}$ is "sufficiently large" in some sense (cf. loc. cit.). Let $\mathbb{L}(V)$ be the set of lines in $\mathbb{P}(V)$, and $\mathbb{L}(V)_{\mathcal{U}} \stackrel{\text{def}}{=} \{\ell \in \mathbb{L}(V) \mid \ell \cap \mathcal{U} \neq \varnothing\}$. Our main result is the following (cf. Theorem 5.7).

Theorem E (A Refined Version of the Fundamental Theorem of Projective Geometry). *Let V_i be a k_i-vector space for $i = 1, 2$. Assume that $\dim_{k_i}(V_i) \geq 3$ for $i = 1, 2$. Let U_i be a subset of $\mathbb{P}(V_i)$ for $i = 1, 2$, and assume that U_i is $\mathcal{S}_i$-ample for some admissible set $\mathcal{S}_i$ of subsets of $\mathbb{P}^1(k_i)$ for $i = 1, 2$. Let $\sigma : U_1 \xrightarrow{\sim} U_2$ and $\tau : \mathbb{L}(V_1)_{U_1} \xrightarrow{\sim} \mathbb{L}(V_2)_{U_2}$ be bijections such that for each $\ell \in \mathbb{L}(V_1)_{U_1}$, one has $\tau(\ell)_{U_2} = \sigma(\ell_{U_1})$. Then, each such $(\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \xrightarrow{\sim} (U_2, \mathbb{L}(V_2)_{U_2})$ uniquely extends to a collineation $(\tilde{\sigma}, \tilde{\tau}) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))$. Thus, $\tilde{\sigma}$ is a bijection between projective spaces which preserves collineation. In particular, there exists an isomorphism $\mu : k_1 \xrightarrow{\sim} k_2$, and a μ-semi-linear isomorphism of abelian groups $\lambda : (V_1, +) \xrightarrow{\sim} (V_2, +)$ that induces $(\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \xrightarrow{\sim} (U_2, \mathbb{L}(V_2)_{U_2})$. Moreover, such an isomorphism (μ, λ) is unique up to scalar multiplication.*

Theorem E applies well in our case. More precisely, applying Theorem E to the above situation we deduce that there exists a unique isomorphism $\tilde{\rho} : \mathbb{P}(R_{Y^l}) \xrightarrow{\sim} \mathbb{P}(R_{X^l})$ which extends the bijection $\bar{\rho} : H^{\times}_{Y^l}/(k^l_Y)^{\times} \to H^{\times}_{X^l}/(k^l_X)^{\times}$ and $\tilde{\rho}$ preserves collineation. In particular, the bijection $\tilde{\rho}$ arises from a ψ_0-isomorphism

$$\psi : (R_{Y^l}, +) \xrightarrow{\sim} (R_{X^l}, +),$$

where $\psi_0 : k^l_Y \xrightarrow{\sim} k^l_X$ is a field isomorphism. Namely, ψ is an isomorphism of abelian groups which is semilinear with respect to ψ_0 in the sense that $\psi(ax) = \psi_0(a)\psi(x)$ for $a \in k^l_Y$ and $x \in R_{Y^l}$. Further, ψ_0 is uniquely determined and ψ is uniquely determined up to scalar multiplication. Moreover, if we normalize the isomorphism $\psi : (R_{Y^l}, +) \xrightarrow{\sim} (R_{X^l}, +)$, by the condition $\psi(1) = 1$, it becomes a ring isomorphism such that the diagram

$$\begin{array}{ccc} R_{X^l} & \xleftarrow{\psi} & R_{Y^l} \\ \uparrow & & \uparrow \\ k^l_X & \xleftarrow{\psi_0} & k^l_Y \end{array}$$

commutes. Further, ψ induces a natural commutative diagram

$$\begin{array}{ccc} X^l & \xrightarrow{\psi} & Y^l \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi} & Y \end{array}$$

where the horizontal maps are scheme isomorphisms and the vertical maps are natural morphisms. By passing to open subgroups of Π_X and Π_Y which correspond to each other via α, one constructs the desired scheme isomorphism $\tilde{X} \xrightarrow{\sim} \tilde{Y}$ which is compatible with the isomorphism $\psi : X \xrightarrow{\sim} Y$. Here, one has to overcome the difficulty that the assumptions on the set Σ in Theorem C are not preserved by passing to open subgroups: even if the representation $\rho_{\Sigma', X} : G_{k_X} \to \prod_{l \in \Sigma'} \mathrm{GL}(T_l(J_X))$ is not injective and $\Pi_{X'} \subset \Pi_X$ is an open subgroup, the representation $\rho_{\Sigma', X'}$:

$G_{k_{X'}} \to \prod_{l\in\Sigma'} \mathrm{GL}(T_l(J_{X'}))$ might be injective. We overcome this problem by introducing certain (weaker but more technical) conditions which are preserved by passing to open subgroups.

In §1, we review the main results of the local theory mainly from [Saïdi-Tamagawa1], and how various invariants of the curve X can be recovered group-theoretically from Π_X. In §2, we define and discuss the notion of large set of primes relative to a hyperbolic curve over a finite field. In §3, we review the main results of Mochizuki's theory of cuspidalization of étale fundamental groups of proper hyperbolic curves, which plays an essential role in this paper. In §4, we prove our main results: Theorem C and Theorem D. In §5, we prove the refined version of the fundamental theorem of projective geometry: Theorem E.

Remark 0.3. (i) A function field version of the main results of the present paper is given in [Saïdi-Tamagawa3]. (See also [Saïdi-Tamagawa1] and [Saïdi-Tamagawa2] for the special case $\Sigma = \mathfrak{Primes} \setminus \{p\}$.)
(ii) At the moment of writing this paper, we do not know (even in the function field case) if pro-l versions of the above results hold, namely if the above Theorems C and D hold (under a certain Frobenius-preserving assumption) in the case where $\Sigma = \{l\}$ consists of a single prime l which is different form p.

§1. Review of the local theory.

In this section we briefly review the main results in [Saïdi-Tamagawa1], §1 concerning the local theory in arithmetic fundamental groups of hyperbolic curves over finite fields. Let X be a proper, smooth, geometrically connected curve over a finite field $k = k_X$ of characteristic $p = p_X > 0$. Write $K = K_X$ for the function field of X.

Let S be a (possibly empty) finite set of closed points of X, and set $U = U_S \stackrel{\text{def}}{=} X \setminus S$. We assume that U is hyperbolic.

Fix a separable closure $K^{\text{sep}} = K_X^{\text{sep}}$ of K, and write $\overline{k} = \overline{k_X}$ for the algebraic closure of k in K^{sep}. Write

$$G_K \stackrel{\text{def}}{=} \mathrm{Gal}(K^{\text{sep}}/K),$$

$$G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$$

for the absolute Galois groups of K and k, respectively.

The tame fundamental group $\pi_1^t(U)$ with respect to the base point defined by K^{sep} (where "tame" is with respect to the complement of U in X) can be naturally identified with a quotient of G_K. Write $\mathrm{Gal}(K_U^t/K)$ for this quotient. (In case $S = \varnothing$, we also write K_U^{ur} for K_U^t.) It is easy to see that K_U^t contains $K\overline{k}$.

Let $\Sigma = \Sigma_X$ be a set of prime numbers that contains at least one prime number different from p. Write

$$\Sigma^\dagger \stackrel{\text{def}}{=} \Sigma \setminus \{p\}.$$

Thus, $\Sigma^\dagger \neq \varnothing$ by our assumption. Denote by $\hat{\mathbb{Z}}^{\Sigma^\dagger}$ the maximal pro-$\Sigma^\dagger$ quotient of $\hat{\mathbb{Z}}$. Set $\Sigma' = \Sigma'_X = \mathfrak{Primes} \setminus \Sigma_X$. We say that Σ is cofinite if $\sharp(\Sigma') < \infty$.

We define $\tilde{K}_U$ to be the maximal pro-Σ subextension of $K\overline{k}$ in K_U^t. Now, set

$$\Pi_U = \mathrm{Gal}(\tilde{K}_U/K),$$

which is a quotient of $\pi_1^t(U) = \mathrm{Gal}(K_U^t/K)$. This fits into the exact sequence

$$1 \to \Delta_U \to \Pi_U \stackrel{\mathrm{pr}_U}{\to} G_k \to 1.$$

Here, Δ_U is the maximal pro-Σ quotient of $\pi_1^t(\overline{U})$, where, for a k-scheme Z, we set $\overline{Z} \stackrel{\text{def}}{=} Z \times_k \overline{k}$.

Define $\tilde{X}_U$ to be the integral closure of X in $\tilde{K}_U$. Define $\tilde{U}$ to be the integral closure of U in $\tilde{K}_U$, which can be naturally identified with the inverse image (as an open subscheme) of U in $\tilde{X}_U$. Define $\tilde{S}_U$ to be the inverse image (as a set) of S in $\tilde{X}_U$.

For a scheme Z, write Z^{cl} for the set of closed points of Z. Then we have

$$X^{\text{cl}} = U^{\text{cl}} \coprod S,$$

$$(\tilde{X}_U)^{\text{cl}} = \tilde{U}^{\text{cl}} \coprod \tilde{S}_U.$$

Moreover, $(\tilde{X}_U)^{\text{cl}}$ admits a natural action of Π_U, and the corresponding quotient can be naturally identified with X^{cl}.

For each $\tilde{x} \in (\tilde{X}_U)^{\text{cl}}$, we define the decomposition group $D_{\tilde{x}} \subset \Pi_U$ (respectively, the inertia group $I_{\tilde{x}} \subset D_{\tilde{x}}$) to be the stabilizer at $\tilde{x}$ of the natural action of Π_U on $(\tilde{X}_U)^{\text{cl}}$ (respectively, the kernel of the natural action of $D_{\tilde{x}}$ on $k(\tilde{x}) = \overline{k(x)} = \overline{k}$, where x is the image of $\tilde{x}$ in X^{cl}). These groups fit into the following commutative diagram in which both rows are exact:

$$\begin{array}{ccccccccc}
1 & \to & I_{\tilde{x}} & \to & D_{\tilde{x}} & \to & G_{k(x)} & \to & 1 \\
 & & \cap & & \cap & & \cap & & \\
1 & \to & \Delta_U & \to & \Pi_U & \to & G_k & \to & 1
\end{array}$$

Moreover, $I_{\tilde{x}} = \{1\}$ (respectively, $I_{\tilde{x}}$ is (non-canonically) isomorphic to $\hat{\mathbb{Z}}^{\Sigma^\dagger}$), if $\tilde{x} \in \tilde{U}^{\text{cl}}$ (respectively, $\tilde{x} \in \tilde{S}_U$). Since $I_{\tilde{x}}$ is normal in $D_{\tilde{x}}$, $D_{\tilde{x}}$ acts on $I_{\tilde{x}}$ by conjugation. Since $I_{\tilde{x}}$ is abelian, this action factors through $D_{\tilde{x}} \to G_{k(x)}$ and induces a natural action of $G_{k(x)}$ on $I_{\tilde{x}}$.

Let G be a profinite group. Then, define $\text{Sub}(G)$ (respectively, $\text{OSub}(G)$) to be the set of closed (respectively, open) subgroups of G.

By conjugation, G acts on $\text{Sub}(G)$. More generally, let H and K be closed subgroups of G such that K normalizes H. Then, by conjugation, K acts on $\text{Sub}(H)$. We denote by $\text{Sub}(H)_K$ the quotient $\text{Sub}(H)/K$ by this action. In particular, $\text{Sub}(G)_G$ is the set of conjugacy classes of closed subgroups of G.

For any closed subgroups H, K of G with $K \subset H$, we have a natural inclusion $\text{Sub}(K) \subset \text{Sub}(H)$, as well as a natural map $\text{Sub}(H) \to \text{Sub}(K)$, $J \mapsto J \cap K$. By using this latter natural map, we define

$$\overline{\text{Sub}}(G) \stackrel{\text{def}}{=} \varinjlim_{H \in \text{OSub}(G)} \text{Sub}(H).$$

Observe that $\overline{\text{Sub}}(G)$ can be identified with the set of commensurate classes of closed subgroups of G. (Closed subgroups J_1 and J_2 of G are called commensurate (to each other), if $J_1 \cap J_2$ is open both in J_1 and in J_2.)

With these notations, we obtain natural maps

$$D = D[U] : (\tilde{X}_U)^{\text{cl}} \to \text{Sub}(\Pi_U), \tilde{x} \mapsto D_{\tilde{x}},$$

$$I = I[U] : (\tilde{X}_U)^{\mathrm{cl}} \to \mathrm{Sub}(\Delta_U) \subset \mathrm{Sub}(\Pi_U), \tilde{x} \mapsto I_{\tilde{x}},$$

which fit into the commutative diagram

$$\begin{array}{ccc} (\tilde{X}_U)^{\mathrm{cl}} & \xrightarrow{D} & \mathrm{Sub}(\Pi_U) \\ \| & & \downarrow \\ (\tilde{X}_U)^{\mathrm{cl}} & \xrightarrow{I} & \mathrm{Sub}(\Delta_U) \end{array}$$

where the vertical arrow stands for the natural map $\mathrm{Sub}(\Pi_U) \to \mathrm{Sub}(\Delta_U)$, $J \mapsto J \cap \Delta_U$. By composition with the natural map $\mathrm{Sub}(\Pi_U) \to \overline{\mathrm{Sub}}(\Pi_U)$, D, I yield

$$\overline{D} = \overline{D}[U] : (\tilde{X}_U)^{\mathrm{cl}} \to \overline{\mathrm{Sub}}(\Pi_U),$$

$$\overline{I} = \overline{I}[U] : (\tilde{X}_U)^{\mathrm{cl}} \to \overline{\mathrm{Sub}}(\Delta_U) \subset \overline{\mathrm{Sub}}(\Pi_U).$$

Note that, unlike the case of D, I, the maps $\overline{D}, \overline{I}$ are essentially unchanged if we replace U by any covering corresponding to an open subgroup of Π_U.

Since the maps D, I are Π_U-equivariant, they induce natural maps

$$D_{\Pi_U} = D[U]_{\Pi_U} : X^{\mathrm{cl}} \to \mathrm{Sub}(\Pi_U)_{\Pi_U},$$

$$I_{\Pi_U} = I[U]_{\Pi_U} : X^{\mathrm{cl}} \to \mathrm{Sub}(\Delta_U)_{\Pi_U} \subset \mathrm{Sub}(\Pi_U)_{\Pi_U},$$

respectively.

Definition 1.1. Let $f : A \to B$ be a map of sets.
(i) We define $\mu_f : B \to \mathbb{Z} \cup \{\infty\}$ by $\mu_f(b) = \sharp(f^{-1}(b))$. (Thus, f is injective (respectively, surjective) if $\mu_f(b) \leq 1$ (respectively, $\mu_f(b) \geq 1$) for any $b \in B$. We also have $f(A) = \{b \in B \mid \mu_f(b) \geq 1\}$.)
(ii) We say that f is quasi-finite, if $\mu_f(b) < \infty$ for any $b \in B$.
(iii) We say that an element a of A is an exceptional element of f (in A), if $\mu_f(f(a)) > 1$. We refer to the set of exceptional elements as the exceptional set.
(iv) We say that a pair (a_1, a_2) of elements of A is an exceptional pair of f (in A), if $a_1 \neq a_2$ and $f(a_1) = f(a_2)$ hold.
(v) We say that f is almost injective (in the strong sense), if the exceptional set of f is finite. (Observe that almost injectivity implies quasi-finiteness.)

Definition 1.2. Denote by $E_{\tilde{U}}$ the exceptional set of $\overline{D}$ in $(\tilde{X}_U)^{\mathrm{cl}}$.

Proposition 1.3. *Let $\overline{\rho}$ denote the natural morphism $\tilde{X}_U \to \overline{X}$. Then, for each $\overline{x} \in \overline{X}^{\mathrm{cl}}$, $\overline{D}|_{\overline{\rho}^{-1}(\overline{x})}$ is injective.*

Proof. Cf. [Saïdi-Tamagawa1], Proposition 1.8(iii). □

Definition 1.4. We define E_U to be the image of $E_{\tilde{U}}$ in X^{cl}. (This can be identified with $E_{\tilde{U}}/\Pi_U$.)

Next, we shall explain how various invariants and structures of U can be recovered group-theoretically (or φ-group-theoretically) from Π_U, in the following sense.

Definition 1.5. (i) We say that $\Pi = (\Pi, \Delta, \varphi_\Pi)$ is a φ-(profinite) group, if Π is a profinite group, Δ is a closed normal subgroup of Π and φ_Π is an element of Π/Δ.
(ii) An isomorphism from a φ-group $\Pi = (\Pi, \Delta, \varphi_\Pi)$ to another φ-group $\Pi' = (\Pi', \Delta', \varphi_{\Pi'})$ is an isomorphism $\Pi \xrightarrow{\sim} \Pi'$ as profinite groups that induces $\Delta \xrightarrow{\sim} \Delta'$, hence also $\Pi/\Delta \xrightarrow{\sim} \Pi'/\Delta'$, such that the last isomorphism sends φ_Π to $\varphi_{\Pi'}$.

From now on, we regard Π_U as a φ-group by $\Pi_U = (\Pi_U, \Delta_U, \varphi_k)$, where φ_k stands for the $\sharp(k)$-th power Frobenius element in $G_k = \Pi_U/\Delta_U$. We shall say that an isomorphism $\alpha : \Pi_U \xrightarrow{\sim} \Pi_{U'}$ as profinite groups is Frobenius-preserving, if α is an isomorphism as φ-groups.

Definition 1.6. (i) Given an invariant $F(U)$ (e.g., a number, a set of numbers, etc.) that depends on the isomorphism class (as a scheme) of a hyperbolic curve U over a finite field, we say that $F(U)$ can be recovered group-theoretically (respectively, φ-group-theoretically) from Π_U, if any isomorphism (respectively, any Frobenius-preserving isomorphism) $\Pi_U \xrightarrow{\sim} \Pi_V$ implies $F(U) = F(V)$ for two such curves U, V.
(ii) Given an additional structure $\mathcal{F}(U)$ (e.g., a family of subgroups, quotients, elements, etc.) on the profinite group Π_U that depends functorially on a hyperbolic curve U over a finite field (in the sense that, for any isomorphism (as schemes) between two such curves U, V, any isomorphism $\Pi_U \xrightarrow{\sim} \Pi_V$ induced by this isomorphism $U \xrightarrow{\sim} V$ (which is unique up to composition with inner automorphisms) preserves the structures $\mathcal{F}(U)$ and $\mathcal{F}(V)$, we say that $\mathcal{F}(U)$ can be recovered group-theoretically (respectively, φ-group-theoretically) from Π_U, if any isomorphism (respectively, any Frobenius-preserving isomorphism) $\Pi_U \xrightarrow{\sim} \Pi_V$ between two such curves U, V preserves the structures $\mathcal{F}(U)$ and $\mathcal{F}(V)$.

Proposition 1.7. *The following invariants and structures can be recovered group-theoretically from Π_U:*
(i) *The subgroup Δ_U of Π_U, hence the quotient $G_k = \Pi_U/\Delta_U$.*
(ii) *The subsets Σ and $\Sigma^\dagger$ of $\mathfrak{Primes}$.*

Proof. Cf. [Saïdi-Tamagawa1], Proposition 1.15(i)(ii). $\square$

Finally, we shall explain that the set of decomposition groups in Π_U can be recovered group-theoretically from Π_U. First, we shall treat decomposition groups at points of $\tilde{S}_U$.

Theorem 1.8. (i) *The set of inertia groups at points of $\tilde{S}_U$ (i.e., the image of the injective map $I|_{\tilde{S}_U} : \tilde{S}_U \to \mathrm{Sub}(\Delta_U) \subset \mathrm{Sub}(\Pi_U)$) can be recovered φ-group-theoretically from Π_U.*
(ii) *The set of decomposition groups at points of $\tilde{S}_U$ (i.e., the image of the injective map $D|_{\tilde{S}_U} : \tilde{S}_U \to \mathrm{Sub}(\Pi_U)$) can be recovered φ-group-theoretically from Π_U.*

Proof. Cf. [Saïdi-Tamagawa1], Theorem 1.18. $\square$

Next, we shall consider decomposition groups at points of $\tilde{U}^{\mathrm{cl}}$. This is done along the lines of [Tamagawa], §2, but slightly more subtle than the case of [Tamagawa], due to the existence of the exceptional set $E_{\tilde{U}}$.

Theorem 1.9. *The following hold.*
(i) *The set of decomposition groups at points of $\tilde{U}^{\mathrm{cl}}$ (respectively, $\tilde{U}^{\mathrm{cl}} \setminus E_{\tilde{U}}$, respectively, $E_{\tilde{U}}$) (i.e., the image of the map $D|_{\tilde{U}^{\mathrm{cl}}} : \tilde{U}^{\mathrm{cl}} \to \mathrm{Sub}(\Pi_U)$ (respectively,*

$D|_{\tilde{U}^{\mathrm{cl}}\setminus E_{\tilde{U}}} : \tilde{U}^{\mathrm{cl}} \setminus E_{\tilde{U}} \to \mathrm{Sub}(\Pi_U)$, *respectively,* $D|_{E_{\tilde{U}}} : E_{\tilde{U}} \to \mathrm{Sub}(\Pi_U)$)) *can be recovered* φ*-group-theoretically from* Π_U*.*
(ii) *The set of decomposition groups at points of* $(\tilde{X}_U)^{\mathrm{cl}}$ *(i.e., the image of the map* $D : (\tilde{X}_U)^{\mathrm{cl}} \to \mathrm{Sub}(\Pi_U)$*) can be recovered* φ*-group-theoretically from* Π_U*.*

Proof. Cf. [Saïdi-Tamagawa1], Theorem 1.24 and Corollary 1.25. $\square$

§2. Large and small sets of primes relative to a hyperbolic curve over a finite field. Throughout this section, let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers, and set $\Sigma' \overset{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma$. Let k be a finite field of characteristic $p > 0$ and set $\Sigma^\dagger = \Sigma \setminus \{p\}$. Write

$$\hat{\mathbb{Z}}^{\Sigma^\dagger} \overset{\mathrm{def}}{=} \prod_{l \in \Sigma^\dagger} \mathbb{Z}_l.$$

For a prime number $l \in \mathfrak{Primes} \setminus \{p\}$ let

$$\chi_l : G_k \to \mathbb{Z}_l^\times$$

be the l-adic cyclotomic character, and define the Σ-part of the cyclotomic character by:

$$\chi_\Sigma \overset{\mathrm{def}}{=} (\chi_l)_{l\in\Sigma^\dagger} : G_k \to (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times = \prod_{l\in\Sigma^\dagger} \mathbb{Z}_l^\times.$$

Thus, we have

$$\bar{k}^{\mathrm{Ker}(\chi_\Sigma)} = k_\Sigma \overset{\mathrm{def}}{=} k(\zeta_{l^j} \mid l \in \Sigma^\dagger, j \in \mathbb{Z}_{\geq 0}).$$

For a prime number $l \in \mathfrak{Primes}$, let $G_{k,l} \subset G_k$ be the pro-l-Sylow subgroup of G_k. (Recall that $G_k \simeq \hat{\mathbb{Z}}$ and $G_{k,l} \simeq \mathbb{Z}_l$.) Next, we recall the notion of k-small and k-large set of primes. (Cf. [Saïdi-Tamagawa3], §3 for more details).

Definition 2.1. (k-Small/k-Large Set of Primes) (i) We say that the set Σ is k-small if the Σ-part χ_Σ of the cyclotomic character is not injective.
(ii) We say that the set Σ is k-large if the set Σ' is k-small.

Note that a k-large set of primes is not k-small, by [Saïdi-Tamagawa3], Proposition 3.3.

The following results are slight generalizations of results in [Saïdi-Tamagawa3], §3. Let X be a proper, smooth, geometrically connected curve over k, $f, g : X \to \mathbb{P}^1_k$ nonconstant k-morphisms, and F a proper subfield of $\bar{k}$ containing k. Write $X(F)^{\mathrm{cl}} \subset X^{\mathrm{cl}}$ for the image of $X(F)$ in X^{cl}.

Definition 2.2. We say that the pair (f, g) has property P_Σ (respectively, $Q_{F,\Sigma}$, $\overline{P}_\Sigma$ and $\overline{Q}_{F,\Sigma}$) if the following holds:

$P_\Sigma(f, g)$: $\exists a, b \in k^\times\{\Sigma'\}$, such that $f = a + bg$.

$Q_{F,\Sigma}(f, g)$: $\forall' x \in X^{\mathrm{cl}} \setminus X(F)^{\mathrm{cl}}$, $\exists a_x, b_x \in k(x)^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.

$\overline{P}_\Sigma(f, g)$: $\exists a, b \in \bar{k}^\times\{\Sigma'\}$, such that $f = a + bg$.

$\overline{Q}_{F,\Sigma}(f, g)$: $\forall' x \in X^{\mathrm{cl}} \setminus X(F)^{\mathrm{cl}}$, $\exists a_x, b_x \in \bar{k}^\times\{\Sigma'\}$, such that $f(x) = a_x + b_x g(x)$.

Here the symbol $\forall'$ means "for all but finitely many".

Proposition 2.3. (i) *We have the following implications:*

$$\begin{array}{ccc} P_\Sigma(f,g) & \Longleftrightarrow & \overline{P}_\Sigma(f,g) \\ \Downarrow & & \Downarrow \\ Q_{F,\Sigma}(f,g) & \Longrightarrow & \overline{Q}_{F,\Sigma}(f,g) \end{array}$$

(ii) *Assume that Σ is k-large. Then we have the following implication:*

$$\overline{Q}_{F,\Sigma}(f,g) \Longrightarrow \overline{P}_\Sigma(f,g).$$

Proof. (i) The equivalence in the first row is given in [Saïdi-Tamagawa3], Definition/Proposition 3.5. The remaining implications are immediate.
(ii) Similar to the proof of [Saïdi-Tamagawa3], Proposition 3.11. Indeed, as in the proof of loc. cit., if property $\overline{P}_\Sigma(f,g)$ does not hold one deduces that there exists a non-empty open subscheme $V \subset X$ such that for every $x \in V^{\mathrm{cl}} \setminus X(F)^{\mathrm{cl}}$, one has $k(x) \subset K$ where K/k is a subextension of $\bar{k}/k$ such that $\bar{k}/K$ is infinite. In particular, for every $x \in V^{\mathrm{cl}}$, one has $k(x) \subset K$ or $k(x) \subset F$. This is not possible: let $\phi : V \to \mathbb{A}^1_k$ be a finite k-morphism, $a \in \bar{k} \setminus K \cup F$, and $x \in \phi^{-1}(a) \subset V^{\mathrm{cl}}$, then $k(a) \subset k(x) \subset K \cup F$, which is absurd. $\square$

Definition/Proposition 2.4. *For a pair (f,g) as in the above discussion, a positive integer m, and a set of prime numbers $\Sigma \subset \mathfrak{Primes}$. We define the following properties:*
$P_\Sigma^{(m)}(f,g) : \exists a, c \in k^\times\{\Sigma'\}$, *such that* $f = a(1+cg)^m$.
$\overline{P}_\Sigma^{(m)}(f,g) : \exists a, c \in \bar{k}^\times\{\Sigma'\}$, *such that* $f = a(1+cg)^m$.
$\overline{Q}_{F,\Sigma}^{(m)}(f,g) : \forall' x \in X^{\mathrm{cl}} \setminus X(F)^{\mathrm{cl}}$, $\exists a_x, c_x \in \bar{k}^\times\{\Sigma'\}$, *such that* $f(x) = a_x(1+c_x g(x))^m$.
Then:
(i) *The implications*

$$P_\Sigma^{(m)}(f,g) \Longleftrightarrow \overline{P}_\Sigma^{(m)}(f,g) \Longrightarrow \overline{Q}_{F,\Sigma}^{(m)}(f,g)$$

hold.
(ii) *If Σ is k-large, then the implication*

$$\overline{Q}_{F,\Sigma}^{(m)}(f,g) \Longrightarrow \overline{P}_\Sigma^{(m)}(f,g)$$

holds.

Proof. Similar to the proof of Proposition 2.3. $\square$

The following is the first application of the k-largeness property to the (geometrically pro-Σ, tame) fundamental groups of hyperbolic curves over k.

Proposition 2.5. *Let U be a hyperbolic curve over k, X the smooth compactification of U, g the genus of X, r the cardinality of $X_{\bar{k}} \setminus U_{\bar{k}}$, and Π_U the geometrically pro-Σ tame fundamental group of U (cf. §1). Assume that Σ is k-large. Then the following invariants and structures can be recovered group-theoretically from Π_U (cf. Definition 1.6 for the meaning of being recovered group-theoretically).*
(i) *The prime number p.*
(ii) *The $\sharp(k)$-th power Frobenius element $\varphi_k \in G_k$.*

(iii) *The cardinality* $q \stackrel{\text{def}}{=} \sharp(k)$ *(or, equivalently, the isomorphism class of the finite field* k*).*

Proof. First, consider the natural character

$$\rho^{\det} : G_k \to \mathrm{Aut}(\bigwedge_{\hat{\mathbb{Z}}^{\Sigma^\dagger}}^{\max} (\Delta_X^{\mathrm{ab}})^{\Sigma^\dagger}) = (\hat{\mathbb{Z}}^{\Sigma^\dagger})^\times,$$

which can be group-theoretically recovered, by Proposition 1.7(i)(ii). As in [Tamagawa], Proposition 3.4 and its proof, $\rho^{\det}$ coincides with $\lambda \cdot (\chi_\Sigma)^a$, where $a = g$ (resp. $a = g + r - 1$) for $r = 0$ (resp. $r > 0$), and λ is a certain character with values in $\{\pm 1\}$. (Note that $\lambda = 1$ when $r = 0$.) It follows from the hyperbolicity assumption $2 - 2g - r < 0$ that $a > 0$. In particular, we have $(\rho^{\det})^2 = (\chi_\Sigma)^{2a}$.
(i) For each $N \in \mathbb{Z}_{>0}$, let k_N/k denote the unique finite subextension of $\bar{k}/k$ of degree N. Then $G_{k_N} = (G_k)^N \subset G_k$ can be recovered group-theoretically. Consider the coinvariant quotient $\hat{\mathbb{Z}}^{\Sigma^\dagger}_{(\rho^{\det})^2(G_{k_N})}$ and define $w_{X,N}$ to be its cardinality, which is a group-theoretic invariant. As $(\rho^{\det})^2 = (\chi_\Sigma)^{2a}$, this invariant is computed as:

$$w_{X,N} = (q^{2aN} - 1)_\Sigma = \frac{q^{2aN} - 1}{(q^{2aN} - 1)_{\Sigma'}},$$

where, for a positive integer n, $n = n_\Sigma n_{\Sigma'}$ stands for the unique decomposition where every prime divisor of n_Σ (resp. $n_{\Sigma'}$) belongs to Σ (resp. Σ'). Set

$$M_0 \stackrel{\text{def}}{=} \inf \mathcal{M},\ \mathcal{M} \stackrel{\text{def}}{=} \{M \in \mathbb{R}_{>0} \mid \exists C > 0, \forall N \in \mathbb{Z}_{>0}, w_{X,N} \leq CM^N\},$$

which is also group-theoretic.

We claim that $M_0 = q^{2a}$. Indeed, since $w_{X,N} \leq q^{2aN} - 1 \leq q^{2aN}$, we have $q^{2a} \in \mathcal{M}$. On the other hand, set $F_0 \stackrel{\text{def}}{=} \bar{k}^{\mathrm{Ker}((\rho^{\det})^2)}$. As Σ is k-large, F_0 is a proper subfield of $\bar{k}$. Take a prime l so that $k(l) \stackrel{\text{def}}{=} F_0 \cap k^l$ is finite, where k^l denotes the unique $\mathbb{Z}_l$-extension of k (cf. Proposition 2.13 below). Write $[k(l) : k] = l^{n_0}$. Then for each $n \geq 0$, we have

$$w_{X,l^n} = (q^{2al^n} - 1)_\Sigma = \frac{q^{2al^n} - 1}{(q^{2al^n} - 1)_{\Sigma'}} \geq \frac{q^{2al^n} - 1}{(q^{2al^{n_0}} - 1)_{\Sigma'}} \geq \frac{q^{2al^n} - 1}{q^{2al^{n_0}} - 1} \geq \frac{q^{2al^n}}{q^{2al^{n_0}}}.$$

Thus, if $(0 <)M < q^{2a}$, we have $M \notin \mathcal{M}$. The claim now follows.

Now, p can be recovered as the unique prime divisor of $M_0 = q^{2a}$.
(ii) Similar to [Tamagawa], Proposition 3.4(i)(ii).
(iii) Similar to [Tamagawa], Proposition 3.4(iii). □

The notion of k-small/k-large set of primes can be naturally generalized as follows, by replacing the multiplicative group $\mathbb{G}_{m,k}$ by an abelian variety. Let A be an abelian variety over k, and $T(A) = \prod_{l \in \mathfrak{Primes}} T_l(A)$ the (full) Tate module of $A \times_k \bar{k}$. Let $T_\Sigma(A) = \prod_{l \in \Sigma} T_l(A)$ be the maximal pro-Σ quotient of $T(A)$. Recall that one has a natural Galois representation $\rho_{A,\Sigma} : G_k \to \mathrm{Aut}(T_\Sigma(A))$.

Definition 2.6. (A-Small/A-Large Set of Primes) Let A be an abelian variety over k.
(i) We say that the set Σ is A-small if the Galois representation $\rho_{A,\Sigma} : G_k \to \mathrm{Aut}(T_\Sigma(A))$ is not injective.
(ii) We say that the set Σ is A-large if the set Σ' is A-small.

Lemma 2.7. *Let A be an abelian variety over k of dimension $g > 0$ and let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers. If Σ is A-large, then Σ is k-large.*

Proof. It is well-known that the $2g$-th exterior power of the representation $\rho_{A,\Sigma}$ coincides with the g-th power of χ_Σ. Hence, an open subgroup of $\mathrm{Ker}(\rho_{A,\Sigma})$ (of index $\mid g$) is contained in $\mathrm{Ker}(\chi_\Sigma)$, from which the assertion follows. $\square$

Remark 2.8. Let A be an abelian variety over k of dimension $g > 0$ and let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers.
(i) By Lemma 2.7 and [Saïdi-Tamagawa3], Remark 3.4.1, if a set of primes $\Sigma \subset \mathfrak{Primes}$ is A-large, then Σ is of (natural) density $\neq 1$.
(ii) On the other hand, for any given $\epsilon > 0$, there exists a set of primes $\Sigma \subset \mathfrak{Primes}$ such that Σ is A-large and that the (natural) density of Σ is $< \epsilon$. Indeed, take a prime number $r \neq p$ satisfying $\frac{2g(2g+1)}{2\epsilon} < r-1$. Let $\Sigma \stackrel{\mathrm{def}}{=} \cup_{k=1}^{2g}\{l \in \mathfrak{Primes} \mid l^k \equiv 1 \pmod{r}\} \cup \{r, p\}$. Observe that the condition $l^k \equiv 1 \pmod{r}$ is equivalent to saying that $l \bmod r \in \mu_k(\mathbb{F}_r)$, hence the (natural) density of Σ is $\leq \frac{\sum_{k=1}^{2g} k}{r-1} \leq \frac{2g(2g+1)}{2(r-1)} < \epsilon$. We claim that $\Sigma' \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma$ is A-small. Indeed, $l \in \Sigma'$ implies that $l \neq r, p$ and r does not divide $\sharp\, \mathrm{GL}_{2g}(\mathbb{F}_l) = (l^{2g}-1)(l^{2g-1}-1)\cdots(l^2-1)(l-1)l^{\frac{2g(2g-1)}{2}}$. Consider the following homomorphism $G_k \to \mathrm{Aut}(T_{\Sigma'}(A)) \to \mathrm{Aut}(T_l(A)) \to \mathrm{Aut}(A[l])$ where the far left map is the natural Σ'-adic representation and the right maps are the natural ones. Note that the kernel of the natural surjective map $\mathrm{Aut}(T_l(A)) \to \mathrm{Aut}(A[l])$ is pro-l. In particular, the image of the r-Sylow subgroup of G_k in $\mathrm{Aut}(A[l])$ is trivial, hence the image of the r-Sylow subgroup of G_k in $\mathrm{Aut}(T_{\Sigma'}(A))$ is trivial and Σ' is A-small.

Next, let X be a proper, smooth, and geometrically connected hyperbolic curve over the finite field k. We apply the notations in §1 to $U = X$. In particular, Δ_X and Π_X denote the maximal pro-Σ quotient of the geometric fundamental group $\pi_1(X \times_k \bar{k})$ and the maximal geometrically pro-Σ quotient of the arithmetic fundamental group $\pi_1(X)$, respectively. For the definition of the exceptional set $E_X \subset X^{\mathrm{cl}}$, see Definition 1.4. Further, let J_X denote the Jacobian variety of X.

Definition 2.9. (i) We denote by F_X the compositum of $k(x)$ in $\bar{k}$ for all $x \in E_X$. (Note that F_X depends on Σ, as so does E_X.)
(ii) Let F be a proper subfield of $\bar{k}$ containing k: $k \subset F \subsetneq \bar{k}$. We say that X is almost Σ-separated with respect to F if $E_X \subset X(F)^{\mathrm{cl}}$ or, equivalently, if $F_X \subset F$.
(iii) We say that X is almost Σ-separated if it is almost Σ-separated with respect to some proper subfield of $\bar{k}$ containing k or, equivalently, if $F_X \subsetneq \bar{k}$.

Let k, k' be finite fields and X, X' proper, smooth, geometrically connected curves over k, k', respectively. Let $f : X' \to X$ be a finite, generically étale morphism (as schemes), which induces a finite separable extension $k'(X')/k(X)$ of function fields and a finite extension k'/k of constant fields. (In particular, we may identify $\bar{k}' = \bar{k}$.) Let $L'/k(X)$ denote the Galois closure of $k'(X')/k(X)$.

Definition 2.10. (i) We say that f is a Σ-covering if the cardinality of the finite group $\mathrm{Gal}(L'\bar{k}/k(X)\bar{k})$ is divisible only by primes in Σ.
(ii) We say that f is tame-Galois if $k'(X')/k(X)$ is a Galois extension (i.e., $L' = k'(X')$) and is at most tamely ramified everywhere on X.

Proposition 2.11. *Assume that $f : X' \to X$ is a Σ-covering.*
(i) *Assume that f is étale. Then we have $E_{X'} = f^{-1}(E_X)$. Further, X is almost Σ-separated if and only if so is X'. More precisely, if X' is almost Σ-separated with respect to F', then X is almost Σ-separated with respect to F'; and, if X is almost Σ-separated with respect to F, then X' is almost Σ-separated with respect to some finite extension of Fk'.*
(ii) *Assume that f is tame-Galois. Then we have $E_{X'} \subset f^{-1}(E_X) \cup S'$, where $S' \subset (X')^{\mathrm{cl}}$ is the non-étale locus of f. Further, if X is almost Σ-separated, then so is X'. More precisely, if X is almost Σ-separated with respect to F, then X' is almost Σ-separated with respect to some finite extension of Fk'.*

Proof. (i) When f is an étale Σ-covering, we have the following commutative diagram:

$$\begin{array}{ccc} (\tilde{X}')^{\mathrm{cl}} & \stackrel{\overline{D}}{\to} & \overline{\mathrm{Sub}}(\Pi_{X'}) \\ \| & & \| \\ (\tilde{X})^{\mathrm{cl}} & \stackrel{\overline{D}}{\to} & \overline{\mathrm{Sub}}(\Pi_X) \end{array}$$

from which we get $E_{X'} = f^{-1}(E_X)$. It is clear that, if X' is almost Σ-separated with respect to F', then X is almost Σ-separated with respect to F'. Next, assume that X is almost Σ-separated with respect to F. Define F' to be the finite extension of F corresponding to the open subgroup $(G_F)^{d!} \subset G_F$ (of index $\mid d!$), where d is the degree of f. Then $F' \supset Fk'$ (as $[Fk' : F] \mid [k' : k] \mid [k'(X') : k(X)] = d$), and X' is almost Σ-separated with respect to F'.
(ii) Set $G \stackrel{\mathrm{def}}{=} \mathrm{Gal}(k'(X')/k(X))$ and $\Delta_G \stackrel{\mathrm{def}}{=} \mathrm{Gal}(k'(X')\bar{k}/k(X)\bar{k})$. Let $\tilde{X} \to X$ (resp. $\tilde{X}' \to X'$) be the profinite covering corresponding to Π_X (resp. $\Pi_{X'}$). Note that $\tilde{X}' \to X$ is a profinite Galois covering with group $\Pi_{X',G}$ which sits naturally in the following exact sequences $1 \to \Pi_{X'} \to \Pi_{X',G} \to G \to 1$, $1 \to \Delta_{X',G} \to \Pi_{X',G} \to G_k \to 1$, where $\Delta_{X',G}$ is defined so that the latter sequence is exact and sits naturally in the following exact sequence $1 \to \Delta_{X'} \to \Delta_{X',G} \to \Delta_G \to 1$. Note that if we view X as an orbicurve, being the stack-theoretic quotient of X' by the action of the finite group G, then $\Pi_{X',G}$ is nothing but the geometrically pro-Σ étale fundamental group of the orbicurve X.

Now, let $x'_1 \in E_{X'} \subset (X')^{\mathrm{cl}}$. Then there exists $\tilde{x}'_1, \tilde{x}'_2 \in (\tilde{X}')^{\mathrm{cl}}$, $\tilde{x}'_1 \neq \tilde{x}'_2$, such that $\tilde{x}'_1$ is above x'_1 and that $D_{\tilde{x}'_1}, D_{\tilde{x}'_2}$ are commensurate in $\Pi_{X'}$. Let $\tilde{x}_1$, $\tilde{x}_2 \in \tilde{X}^{\mathrm{cl}}$, $x'_1, x'_2 \in (X')^{\mathrm{cl}}$ and x_1, $x_2 \in \tilde{X}^{\mathrm{cl}}$, the images of $\tilde{x}'_1, \tilde{x}'_2$, respectively. Then $D_{\tilde{x}_1}, D_{\tilde{x}_2} \subset \Pi_X$ are commensurate to each other, hence either $\tilde{x}_1 = \tilde{x}_2$ or $\tilde{x}_1, \tilde{x}_2 \in E_{\tilde{X}}$. In the latter case, we have $x'_1 \in f^{-1}(E_X)$, as desired. In the former case, in particular, the images of $\tilde{x}'_1$, $\tilde{x}'_2$ in $\overline{X}^{\mathrm{cl}}$ are equal, hence there exists $\sigma \in \Delta_{X',G}$ such that $\sigma \cdot \tilde{x}'_1 = \tilde{x}'_2$. Write $Z \stackrel{\mathrm{def}}{=} D_{\tilde{x}'_1} \cap D_{\tilde{x}'_2} \subset \Pi_{X'} \subset \Pi_{X',G}$ for simplicity. First, we follow the proof of [Saïdi-Tamagawa1], Lemma 1.7 in order to deduce that σ is torsion. More precisely, let $Z_0 \stackrel{\mathrm{def}}{=} Z \cap \sigma Z \sigma^{-1}$. Then as in loc. cit. we deduce that σ commutes with any element of Z_0, i.e., $Z_0 \subset Z_{\Pi_{X',G}}(\langle \sigma \rangle)$. (Here, given a profinite group G and a closed subgroup H, we write $Z_G(H)$ for the centralizer of H in G.) Moreover, arguing by contradiction, suppose $\sigma \neq 1$ and let $\overline{N}$ be a sufficiently small open characteristic subgroup of $\Delta_{X',G}$ such that $\sigma \notin \overline{N}$ and set $\overline{H} \stackrel{\mathrm{def}}{=} \langle \overline{N}, \sigma \rangle \subset \Delta_{X',G}$. Then, as in loc. cit. Z_0 normalizes $\overline{H}$ and the image of σ in $\overline{H}^{\mathrm{ab}}$ is nontrivial and fixed by the action of Z_0. By observing the Frobenius weights in the action of Z_0 we deduces that $\langle\sigma\rangle \cap \Delta_{X'} = \{1\}$ and σ has finite order.

If $\sigma = 1$, then $\tilde{x}'_1 = \tilde{x}'_2$ and we are done. Suppose that $\sigma \neq 1$. Then it follows from [Mochizuki2], Lemma 4.1(iii) that there exists a unique closed point $\tilde{y}' \in \tilde{X}'$ such that $\langle\sigma\rangle \subset I_{\tilde{y}'}$, where $I_{\tilde{y}'}$ is the inertia subgroup at $\tilde{y}'$ (necessarily finite). We claim that $Z_{\Pi_{X',G}}(\langle\sigma\rangle)$ is commensurate to the decomposition group $D_{\tilde{y}'}$ at $\tilde{y}'$. Indeed, first there exists an open subgroup $D^o_{\tilde{y}'} \subset D_{\tilde{y}'}$ of $D_{\tilde{y}'}$ such that $D^o_{\tilde{y}'} \subset Z_{\Pi_{X',G}}(\langle\sigma\rangle)$, as follows easily from the fact that $D_{\tilde{y}'}$ acts by inner conjugation on its normal subgroup $I_{\tilde{y}'}$ and the group of automorphisms of $I_{\tilde{y}'}$ is finite. Second, we have a natural exact sequence $1 \to Z_{\Delta_{X',G}}(\langle\sigma\rangle) \to Z_{\Pi_{X',G}}(\langle\sigma\rangle) \to G_k$. Now, our second claim is that $Z_{\Delta_{X',G}}(\langle\sigma\rangle)$ is finite. Indeed, after possibly replacing the orbicurve X by a suitable étale cover, corresponding to an open subgroup of $\Pi_{X',G}$, we can assume that $\langle\sigma\rangle = I_{\tilde{y}'}$. The assertion then follows from the fact that $I_{\tilde{y}'}$ is normally terminal in $\Delta_{X',G}$ (cf. [Mochizuki2], Lemma 4.1(i)). This implies that the subgroups Z_0 and $D^o_{\tilde{y}'}$ of $Z_{\Pi_{X',G}}(\langle\sigma\rangle)$ are commensurate (cf. above exact sequence), hence $D_{\tilde{x}'_1}$ and $D_{\tilde{y}'}$ are commensurate in $\Pi_{X',G}$, since Z_0 is open in $D_{\tilde{x}'_1}$, and $D^o_{\tilde{y}'}$ is open in $D_{\tilde{y}'}$. In particular, $D_{\tilde{x}_1}$ and $D_{\tilde{y}}$ are commensurate in Π_X, where $\tilde{y}$ is the image of $\tilde{y}'$ in $\tilde{X}^{\mathrm{cl}}$. Hence, either $\tilde{x}_1 = \tilde{y}$ or $\tilde{x}_1 \in E_{\tilde{X}}$. In the former case, we have $x_1 = y$, hence $x'_1 \in S'$, as desired, In the latter case, we have $x_1 \in E_X$, hence $x'_1 \in f^{-1}(E_{X'})$, as desired. □

The following application of the J_X-largeness property is crucial in later sections.

Proposition 2.12. *If Σ is J_X-large, then X is almost Σ-separated. More precisely, then X is almost Σ-separated with respect to some finite extension ($\neq \bar{k}$) of $\bar{k}^{\mathrm{Ker}(\rho_{J_X,\Sigma'})}$.*

Proof. Let $F_0 \overset{\mathrm{def}}{=} \bar{k}^{\mathrm{Ker}(\rho_{J_X,\Sigma'})}$. Then $k \subset F_0 \subsetneq \bar{k}$. Let F denote the finite extension of F_0 corresponding to the open subgroup $(G_{F_0})^e \subset G_{F_0}$ (of index $\leq e$), where $e = 2$ (resp. $e = 1$) if X is hyperelliptic (resp. otherwise). Now, we claim that the field F satisfies the above property.

Consider the morphism $\delta : X \times X \to J_X$, $(P, Q) \mapsto \mathrm{cl}(P - Q)$. Then $\delta|_{X\times X\setminus\iota(X)}$ is quasi-finite with geometric fibers of cardinality $\leq e$, where $\iota : X \to X \times X$ is the diagonal map (cf. [Saïdi-Tamagawa1], Claim 1.9(i) and its proof). Thus, we have a quasi-finite map $X(\bar{k}) \times X(\bar{k}) \setminus \iota(X)(\bar{k}) \to J_X(\bar{k})$ with fibers of cardinality $\leq e$.

Let $\tilde{x}, \tilde{x}' \in \tilde{X}^{\mathrm{cl}}$ be an exceptional pair of the map $\overline{D}$ and $\bar{x}, \bar{x}' \in \overline{X}^{\mathrm{cl}} = X(\bar{k})$ the images of $\tilde{x}, \tilde{x}'$, respectively. By Proposition 1.3, we have $\bar{x} \neq \bar{x}'$. Then the image of $(\bar{x}, \bar{x}') \in X(\bar{k}) \times X(\bar{k}) \setminus \iota(X)(\bar{k})$ via the composed map $X(\bar{k}) \times X(\bar{k}) \setminus \iota(X)(\bar{k}) \xrightarrow{\delta} J_X(\bar{k}) \twoheadrightarrow J_X(\bar{k})/(J_X(\bar{k})\{\Sigma'\})$ is trivial (cf. [Saïdi-Tamagawa1], proof of Proposition 1.8(vi)). Thus, the image of $(\bar{x}, \bar{x}')$ in $J_X(\bar{k})$ lies in the subgroup $J_X(\bar{k})\{\Sigma'\}$ which is contained in $J_X(F_0)$ by the choice of F_0. It follows from this that $(\bar{x}, \bar{x}') \in X(F) \times X(F)$ since the above map $\delta|_{X\times X\setminus\iota(X)}$ is quasi-finite with geometric fibers of cardinality $\leq e$. □

Proposition 2.13. (i) *If X is almost Σ-separated, then there exists a prime number l, such that, for every finite extension k'/k^l and every finite extension F/F_X, the field $k' \cap F$ is finite, where k^l denotes the unique $\mathbb{Z}_l$-extension of k. (We shall refer to such a prime number l as being (X, Σ)-admissible.) In particular, the set $E_X \cap X(k')^{\mathrm{cl}}$ is finite.*
(ii) *If, moreover, Σ is J_X-large (resp. Σ is J_X-large and J_X has positive p-rank), then the prime number l in (i) can be chosen to be in $\Sigma \cup \{p\}$ (resp. Σ).*

Proof. (i) As $1 \neq G_{F_X} \subset G_k = \hat{\mathbb{Z}} = \prod_{l \in \mathfrak{Primes}} \mathbb{Z}_l$, there exists an $l \in \mathfrak{Primes}$ such that the image of G_{F_X} in $\mathbb{Z}_l$ is nontrivial, or, equivalently, open in $\mathbb{Z}_l$. Such an l satisfies the desired property.
(ii) By Proposition 2.12, there exists an open subgroup $1 \neq H \subset \mathrm{Ker}(\rho_{J_X,\Sigma'})$ that is contained in G_{F_X}. So, as in the proof of (i), take any $l \in \mathfrak{Primes}$ such that the image of H under the projection $G_k = \hat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}_l$ is nontrivial. Then the assertions in (i) hold for this l.

Now, assume that Σ is J_X-large (resp. Σ is J_X-large and J_X has positive p-rank), and suppose $l \in \Sigma' \setminus \{p\}$ (resp. $l \in \Sigma'$). Then the image of the l-adic representation $\rho_{J_X,\{l\}} : G_k \to \mathrm{GL}(T_l(J_X))$ is infinite and almost pro-l, hence the image of $\mathrm{Ker}(\rho_{J_X,\Sigma'}) \subset \mathrm{Ker}(\rho_{J_X,\{l\}})$ in $\mathbb{Z}_l$ is trivial. Thus, we must have $l \in \Sigma \cup \{p\}$ (resp. $l \in \Sigma$) automatically. $\square$

§3. Review of Mochizuki's cuspidalization theory of proper hyperbolic curves over finite fields. In this §, we review the main results of Mochizuki's theory of cuspidalizations of arithmetic fundamental groups of proper hyperbolic curves over finite fields, developed in [Mochizuki1], which plays an important role in this paper. We maintain the notations of §1 and further assume $X = U$. (Thus, the finite set S in §1 is empty, and, in this §, we save the symbol S for another finite set of closed points of X.) Accordingly, X is a proper hyperbolic curve over a finite field $k = k_X$.

Recall that Δ_X stands for the maximal pro-Σ quotient of $\pi_1(\overline{X})$, that Π_X stands for $\pi_1(X)/\mathrm{Ker}(\pi_1(\overline{X}) \twoheadrightarrow \Delta_X)$, and that they fit into the following exact sequence:

$$1 \to \Delta_X \to \Pi_X \xrightarrow{\mathrm{pr}_X} G_k \to 1.$$

Similarly, if we write $X \times X \stackrel{\mathrm{def}}{=} X \times_k X$, then we obtain (by considering the maximal pro-Σ quotient $\Delta_{X\times X}$ of $\pi_1(\overline{X \times X})$) an exact sequence:

$$1 \to \Delta_{X\times X} \to \Pi_{X\times X} \to G_k \to 1,$$

where $\Pi_{X\times X}$ (respectively, $\Delta_{X\times X}$) may be identified with $\Pi_X \times_{G_k} \Pi_X$ (respectively, $\Delta_X \times \Delta_X$).

Definition 3.1. (cf. [Mochizuki1], Definition 1.1(i).) Let H be a profinite group equipped with a homomorphism $H \to \Pi_X$. Then we shall refer to the kernel I_H of $H \to \Pi_X$ as the cuspidal subgroup of H (relative to $H \to \Pi_X$). We shall refer to an inner automorphism of H by an element of I_H as a cuspidally inner automorphism. We shall say that H is cuspidally abelian (respectively, cuspidally pro-Σ^*, where Σ^* is a set of prime numbers) (relative to $H \to \Pi_X$) if I_H is abelian (respectively, if I_H is a pro-Σ^* group). If H is cuspidally abelian, then observe that H/I_H acts naturally (by conjugation) on I_H. We shall say that H is cuspidally central (relative to $H \to \Pi_X$) if this action of H/I_H on I_H is trivial. Also, we shall use the same terminology for $H \to \Pi_X$ when Π_X is replaced by Δ_X, $\Pi_{X\times X}$, or $\Delta_{X\times X}$.

For a finite subset $S \subset X^{\mathrm{cl}}$ write $U_S \stackrel{\mathrm{def}}{=} X \setminus S$. Let Δ_{U_S} be the maximal cuspidally (relative to the natural map to Δ_X) pro-$\Sigma^\dagger$ quotient of the maximal pro-Σ quotient of the tame fundamental group of $\overline{U_S}$ (where "tame" is with respect to the complement of U_S in X), and let Π_{U_S} be the corresponding quotient $\pi_1(U_S)/\mathrm{Ker}(\pi_1(\overline{U_S}) \twoheadrightarrow \Delta_{U_S})$ of $\pi_1(U_S)$. Thus, we have an exact sequence:

$$1 \to \Delta_{U_S} \to \Pi_{U_S} \to G_k \to 1,$$

which fits into the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & \Delta_{U_S} & \to & \Pi_{U_S} & \to & G_k & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \Delta_X & \to & \Pi_X & \to & G_k & \to & 1 \end{array}$$

Further, let $\iota : X \to X \times X$ be the diagonal morphism, and write

$$U_{X\times X} \stackrel{\text{def}}{=} X \times X \setminus \iota(X).$$

We shall denote by $\Delta_{U_{X\times X}}$ the maximal cuspidally (relative to the natural map to $\Delta_{X\times X}$) pro-$\Sigma^\dagger$ quotient of the maximal pro-Σ quotient of the tame fundamental group of $(U_{X\times X})_{\bar{k}}$ (where "tame" is with respect to the divisor $\iota(X) \subset X \times X$), and by $\Pi_{U_{X\times X}}$ the corresponding quotient $\pi_1(U_{X\times X})/\operatorname{Ker}(\pi_1(\overline{U_{X\times X}}) \twoheadrightarrow \Delta_{U_{X\times X}})$ of $\pi_1(U_{X\times X})$. Thus, we have an exact sequence:

$$1 \to \Delta_{U_{X\times X}} \to \Pi_{U_{X\times X}} \to G_k \to 1,$$

which fits into the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & \Delta_{U_{X\times X}} & \to & \Pi_{U_{X\times X}} & \to & G_k & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \Delta_{X\times X} & \to & \Pi_{X\times X} & \to & G_k & \to & 1 \end{array}$$

Finally, set

$$M_X \stackrel{\text{def}}{=} \operatorname{Hom}_{\hat{\mathbb{Z}}^{\Sigma^\dagger}}(H^2(\Delta_X, \hat{\mathbb{Z}}^{\Sigma^\dagger}), \hat{\mathbb{Z}}^{\Sigma^\dagger}).$$

Thus, M_X is a free $\hat{\mathbb{Z}}^{\Sigma^\dagger}$-module of rank 1, and M_X is isomorphic to $\hat{\mathbb{Z}}^{\Sigma^\dagger}(1)$ as a G_k-module (where the "(1)" denotes a "Tate twist", i.e., G_k acts on $\hat{\mathbb{Z}}^{\Sigma^\dagger}(1)$ via the cyclotomic character) (cf. [Mochizuki1], the discussion following Proposition 1.1).

For the rest of this §, let X, Y be proper, hyperbolic curves over finite fields k_X, k_Y of characteristic p_X, p_Y, respectively. Let Σ_X (respectively, Σ_Y) be a set of prime numbers that contains at least one prime number different from p_X (respectively, p_Y). Write Δ_X (respectively, Δ_Y) for the maximal pro-Σ_X quotient of $\pi_1(\overline{X})$ (respectively, the maximal pro-Σ_Y quotient of $\pi_1(\overline{Y})$), and Π_X (respectively, Π_Y) for the quotient $\pi_1(X)/\operatorname{Ker}(\pi_1(\overline{X}) \twoheadrightarrow \Delta_X)$ of $\pi_1(X)$ (respectively, the quotient $\pi_1(Y)/\operatorname{Ker}(\pi_1(\overline{Y}) \twoheadrightarrow \Delta_Y)$ of $\pi_1(Y)$).

Let

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an isomorphism of profinite groups.

The following is one of the main results of Mochizuki's theory (cf. [Mochizuki1], Theorem 1.1(iii)).

Theorem 3.2. *(Reconstruction of Maximal Cuspidally Abelian Extensions) Let $\iota_X : X \to X \times X$ (respectively, $\iota_Y : Y \to Y \times Y$) be the diagonal morphism, and write $U_{X\times X} \stackrel{\text{def}}{=} X \times X \setminus \iota(X)$ (respectively, $U_{Y\times Y} \stackrel{\text{def}}{=} Y \times Y \setminus \iota(Y)$). Denote by $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$, $\Pi_{U_{Y\times Y}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{Y\times Y}}$ the maximal cuspidally (relative to the natural surjections $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_{X\times X}$, $\Pi_{U_{Y\times Y}} \twoheadrightarrow \Pi_{Y\times Y}$, respectively) abelian quotients. Then there is a commutative diagram:*

$$\begin{array}{ccc} \Pi^{\text{c-ab}}_{U_{X\times X}} & \xrightarrow{\alpha^{\text{c-ab}}} & \Pi^{\text{c-ab}}_{U_{Y\times Y}} \\ \downarrow & & \downarrow \\ \Pi_{X\times X} & \xrightarrow{\alpha\times\alpha} & \Pi_{Y\times Y} \end{array}$$

where $\alpha^{\text{c-ab}}$ is an isomorphism which is well-defined up to cuspidally inner automorphism (i.e., an inner automorphism of $\Pi^{\text{c-ab}}_{U_{Y\times Y}}$ by an element of the cuspidal subgroup $\text{Ker}(\Pi^{\text{c-ab}}_{U_{Y\times Y}} \twoheadrightarrow \Pi_{Y\times Y})$). Moreover, the correspondence

$$\alpha \mapsto \alpha^{\text{c-ab}}$$

is functorial (up to cuspidally inner automorphism) with respect to α.

Proof. See [Mochizuki1], Theorem 1.1(iii). (See also [Saïdi-Tamagawa1], Theorem 2.2.) $\square$

Let $\tilde{x} \in \tilde{X}^{\text{cl}}$ and x the image of $\tilde{x}$ in X. In this and next §§, we sometimes refer to the decomposition group $D_{\tilde{x}}$ as the decomposition group of Π_X at x, and denote it simply by D_x. Thus, D_x is well-defined only up to conjugation by an element of Π_X.

For the rest of this §, we shall assume that α is Frobenius-preserving (cf. Definition 1.5). (Note that this assumption is automatically satisfied in the case where Σ_X is k_X-large and Σ_Y is k_Y-large, cf. Proposition 2.5(ii).) Thus, by Theorem 1.9, one deduces naturally from α a bijection

$$\phi : X^{\text{cl}} \setminus E_X \xrightarrow{\sim} Y^{\text{cl}} \setminus E_Y$$

such that

$$\alpha(D_x) = D_{\phi(x)}$$

holds (up to conjugation) for any $x \in X^{\text{cl}} \setminus E_X$. Here, considering the images of D_x and $D_{\phi(x)}$ in $G_{k_X} = \Pi_X/\Delta_X \xrightarrow{\sim} \Pi_Y/\Delta_Y = G_{k_Y}$ (cf. Proposition 1.7(i)), we obtain $[k(x) : k_X] = [k(\phi(x)) : k_Y]$, hence $\sharp(k(x)) = \sharp(k(\phi(x)))$ by Proposition 2.5(iii).

As an important consequence of Theorem 3.2 we deduce the following:

Corollary 3.3. *With the above assumptions, let $S \subset X^{\text{cl}} \setminus E_X$, $T \subset Y^{\text{cl}} \setminus E_Y$ be finite subsets that correspond to each other via ϕ. Then α, $\alpha^{\text{c-ab}}$ induce isomorphisms (well-defined up to cuspidally inner automorphisms, i.e., inner automorphisms by elements of $\text{Ker}(\Pi^{\text{c-ab}}_{V_T} \to \Pi_Y)$)*

$$\alpha^{\text{c-ab}}_{S,T} : \Pi^{\text{c-ab}}_{U_S} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{V_T}$$

lying over α, where $U_S \stackrel{\text{def}}{=} X \setminus S$, $V_T \stackrel{\text{def}}{=} Y \setminus T$, and $\Pi_{U_S} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_S}$, $\Pi_{V_T} \twoheadrightarrow \Pi^{\text{c-ab}}_{V_T}$, are the maximal cuspidally abelian quotients (relative to the maps $\Pi_{U_S} \twoheadrightarrow \Pi_X$,

$\Pi_{V_T} \twoheadrightarrow \Pi_Y$, *respectively). These isomorphisms are functorial with respect to* α, S, T, *as well as with respect to passing to connected finite étale coverings of* X, Y, *which arise from open subgroups of* Π_X, Π_Y, *in the following sense: Let* $\xi : X' \to X$ *(respectively,* $\eta : Y' \to Y$*) be a finite étale covering which arises from the open subgroup* $\Pi_{X'} \subset \Pi_X$ *(respectively,* $\Pi_{Y'} \subset \Pi_Y$*), such that* $\alpha(\Pi_{X'}) = \Pi_{Y'}$*; set* $U'_{S'} \stackrel{\text{def}}{=} X' \setminus S'$, $V'_{T'} \stackrel{\text{def}}{=} Y' \setminus T'$, $S' \stackrel{\text{def}}{=} \xi^{-1}(S)$, $T' \stackrel{\text{def}}{=} \eta^{-1}(T)$*; and denote by* α' *the isomorphism* $\Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$ *induced by* α*. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \Pi^{\text{c-ab}}_{U'_{S'}} & \xrightarrow{(\alpha')^{\text{c-ab}}_{S',T'}} & \Pi^{\text{c-ab}}_{V'_{T'}} \\ \downarrow & & \downarrow \\ \Pi^{\text{c-ab}}_{U_S} & \xrightarrow{\alpha^{\text{c-ab}}_{S,T}} & \Pi^{\text{c-ab}}_{V_T} \end{array}$$

where the vertical arrows are the natural maps.

Proof. The proof of [Mochizuki1], Theorem 2.1(i) (where $E_X = E_Y = \varnothing$ is assumed) works as it is. See also [Saïdi-Tamagawa1], Corollary 2.3. □

Next, let

$$1 \to M_X \to \mathcal{D} \to \Pi_{X \times X} \to 1$$

be a fundamental extension, i.e., an extension whose corresponding extension class in $H^2_{\text{et}}(X \times X, M_X)$ (via the natural identification $H^2(\Pi_{X\times X}, M_X) \xrightarrow{\sim} H^2_{\text{et}}(X \times X, M_X)$ (cf. [Mochizuki1], Proposition 1.1)) coincides with the first (étale) Chern class of the diagonal $\iota(X)$ (cf. [Mochizuki1], Proposition 1.5). Let $x, y \in X(k)$ and write $D_x, D_y \subset \Pi_X$ for the associated decomposition groups (which are well-defined up to conjugation). Set

$$\mathcal{D}_x \stackrel{\text{def}}{=} \mathcal{D}|D_x \times_{G_k} \Pi_X, \quad \mathcal{D}_{x,y} \stackrel{\text{def}}{=} \mathcal{D}|D_x \times_{G_k} D_y.$$

Thus, $\mathcal{D}_x$ (respectively, $\mathcal{D}_{x,y}$) is an extension of Π_X (respectively, G_k) by M_X. Similarly, if $D = \sum_i m_i.x_i$, $E = \sum_j n_j.y_j$ are divisors on X supported on k-rational points, then set

$$\mathcal{D}_D \stackrel{\text{def}}{=} \sum_i m_i.\mathcal{D}_{x_i}, \ \mathcal{D}_{D,E} \stackrel{\text{def}}{=} \sum_{i,j} m_i.n_j.\mathcal{D}_{x_i,y_j}$$

where the sums are to be understood as sums of extensions of Π_X, G_k, respectively, by M_X, i.e., the sums are induced by the additive structure of M_X.

For a finite subset $S \subset X(k)$, we shall write

$$\mathcal{D}_S \stackrel{\text{def}}{=} \prod_{x\in S} \mathcal{D}_x$$

where the product is to be understood as a fiber product over Π_X. Thus, $\mathcal{D}_S$ is an extension of Π_X by a product of copies of M_X indexed by the points of S. We shall refer to $\mathcal{D}_S$ as the S-cuspidalization of Π_X. Observe that if $T \subset X(k)$ is a finite subset containing S, then we obtain a natural projection morphism $\mathcal{D}_T \to \mathcal{D}_S$. More generally, for a finite subset $S \subset X^{\text{cl}}$ which does not necessarily consist of k-rational points, one can still construct the object $\mathcal{D}_S$ by passing to a finite extension k_S of k over which the points of S are rational, performing the above construction over k_S, and then descending to k. (See [Mochizuki1], Remark 5 for more details.)

Proposition 3.4. *(Maximal Geometrically Cuspidally Central Quotients)*
(i) *For $S \subset X^{\mathrm{cl}}$ a finite subset, the S-cuspidalization $\mathcal{D}_S$ of Π_X may be identified with the quotient $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\text{c-cn}} \stackrel{\mathrm{def}}{=} \Pi_{U_S} / \operatorname{Ker}(\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{\text{c-cn}})$ of Π_{U_S}, where $\Delta_{U_S}^{\text{c-cn}}$ is the maximal cuspidally central quotient of Δ_{U_S} relative to the natural map $\Delta_{U_S} \twoheadrightarrow \Delta_X$.*
(ii) *The fundamental extension $\mathcal{D}$ may be identified with the quotient $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_{U_{X\times X}}^{\text{c-cn}} \stackrel{\mathrm{def}}{=} \Pi_{U_{X\times X}} / \operatorname{Ker}(\Delta_{U_{X\times X}} \twoheadrightarrow \Delta_{U_{X\times X}}^{\text{c-cn}})$ of $\Pi_{U_{X\times X}}$, where $\Delta_{U_{X\times X}}^{\text{c-cn}}$ is the maximal cuspidally central quotient of $\Delta_{U_{X\times X}}$ relative to the natural map $\Delta_{U_{X\times X}} \twoheadrightarrow \Delta_{X\times X}$.*

Proof. See [Mochizuki1], Proposition 1.6(iii)(iv). (Precisely speaking, Proposition 1.6(iii) loc. cit. only treats the special case where $S \subset X(k)$ holds. However, the proof for the general case is easily reduced to this special case by passing to a finite extension of k. cf. Remark 5, loc. cit.) See also [Saïdi-Tamagawa1], Proposition 2.4. □

Remark 3.5. Let $\mathcal{D}$ (respectively, $\mathcal{E}$) be a fundamental extension of X (respectively, Y). The isomorphism $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ induces an isomorphism:

$$\mathcal{D} \xrightarrow{\sim} \mathcal{E}$$

up to cyclotomically inner automorphisms (i.e., inner automorphisms by elements of M_X, M_Y) and the actions of $(k_X^\times)^{\Sigma_X^\dagger}, (k_Y^\times)^{\Sigma_Y^\dagger}$, where $(k_X^\times)^{\Sigma_X^\dagger}$ (respectively, $(k_Y^\times)^{\Sigma_Y^\dagger}$) is the maximal $\Sigma_X^\dagger$- (respectively, $\Sigma_Y^\dagger$-) quotient of $k_X^\times$ (respectively, $k_Y^\times$) (cf. [Mochizuki1], Proposition 1.4(ii)). Moreover, let $S \subset X^{\mathrm{cl}} \setminus E_X$ and $T \subset Y^{\mathrm{cl}} \setminus E_Y$ be as in Corollary 3.3 and write $\mathcal{D}_S$ (respectively, $\mathcal{E}_T$) for the S-cuspidalization of Π_X (respectively, the T-cuspidalization of Π_Y). Then the isomorphism $\mathcal{D} \xrightarrow{\sim} \mathcal{E}$ induces an isomorphism

$$\mathcal{D}_S \xrightarrow{\sim} \mathcal{E}_T$$

lying over α.

On the other hand, let $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\text{c-cn}}$ and $\Pi_{V_T} \twoheadrightarrow \Pi_{V_T}^{\text{c-cn}}$ be the maximal geometrically cuspidally central quotients (here, $U_S \stackrel{\mathrm{def}}{=} X \setminus S$, $V_T \stackrel{\mathrm{def}}{=} Y \setminus T$) (cf. Proposition 3.4). Note that the isomorphism $\alpha_{S,T}^{\text{c-ab}} : \Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$ in Corollary 3.3 naturally induces an isomorphism

$$\Pi_{U_S}^{\text{c-cn}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-cn}}$$

lying over α, which is well-defined up to cuspidally inner automorphism. Now, by Proposition 3.4(i), $\Pi_{U_S}^{\text{c-cn}}$ (respectively, $\Pi_{V_T}^{\text{c-cn}}$) may be identified with $\mathcal{D}_S$ (respectively, $\mathcal{E}_T$). Thus, we deduce another isomorphism

$$\mathcal{D}_S \xrightarrow{\sim} \mathcal{E}_T$$

lying over α.

Now, the above two isomorphisms between $\mathcal{D}_S$ and $\mathcal{E}_T$ coincide with each other up to cyclotomically inner automorphisms and the actions of $(k_X^\times)^{\Sigma_X^\dagger}, (k_Y^\times)^{\Sigma_Y^\dagger}$.

Another main result of Mochizuki's theory is the following, which allows us to recover φ-group-theoretically the maximal cuspidally pro-l extension of Π_X, in the case where the set of cusps consists of a single rational point.

Theorem 3.6. *(Reconstruction of Maximal Cuspidally Pro-l Extensions)*
Let $x_* \in X(k_X)$, $y_* \in Y(k_Y)$, *and set* $S \stackrel{\text{def}}{=} \{x_*\}$, $T \stackrel{\text{def}}{=} \{y_*\}$, $U_S \stackrel{\text{def}}{=} X \setminus S$, $V_T \stackrel{\text{def}}{=} Y \setminus T$. *Assume that the Frobenius-preserving isomorphism* $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ *maps the decomposition group of* x_* *in* Π_X *(which is well-defined up to conjugation) to the decomposition group of* y_* *in* Π_Y *(which is well-defined up to conjugation). Then, for each prime* $l \in \Sigma^\dagger$ *(thus,* $l \neq p$*), there exists a commutative diagram:*

$$\begin{array}{ccc} \Pi_{U_S}^{\text{c-}l} & \xrightarrow{\alpha^{\text{c-}l}} & \Pi_{V_T}^{\text{c-}l} \\ \downarrow & & \downarrow \\ \Pi_X & \xrightarrow{\alpha} & \Pi_Y \end{array}$$

in which $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\text{c-}l}$, $\Pi_{V_T} \twoheadrightarrow \Pi_{V_T}^{\text{c-}l}$ *are the maximal cuspidally pro-l quotients (relative to the maps* $\Pi_{U_S} \twoheadrightarrow \Pi_X$, $\Pi_{V_T} \twoheadrightarrow \Pi_Y$*, respectively), the vertical arrows are the natural surjections, and* $\alpha^{\text{c-}l}$ *is an isomorphism well-defined up to composition with a cuspidally inner automorphism (i.e., an inner automorphism by an element of* $\text{Ker}(\Pi_{V_T}^{\text{c-}l} \to \Pi_Y)$*), which is compatible relative to the natural surjections*

$$\Pi_{U_S}^{\text{c-}l} \twoheadrightarrow \Pi_{U_S}^{\text{c-ab},l}, \qquad \Pi_{V_T}^{\text{c-}l} \twoheadrightarrow \Pi_{V_T}^{\text{c-ab},l}$$

*where the subscript "*c-ab, l*" denotes the maximal cuspidally pro-l abelian quotient, with the isomorphism*

$$\alpha_{S,T}^{\text{c-ab}} : \Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$$

in Corollary 3.3. Moreover, $\alpha^{\text{c-}l}$ *is compatible, up to cuspidally inner automorphisms, with the decomposition groups of* x_*, y_* *in* $\Pi_{U_S}^{\text{c-}l}$, $\Pi_{V_T}^{\text{c-}l}$. □

Proof. See [Mochizuki1], Theorem 3.1. (See also [Saïdi-Tamagawa1], Theorem 2.6.) □

If n is an integer all of whose prime factors belong to $\Sigma^\dagger$, then we have the Kummer exact sequence

$$1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1,$$

where $\mathbb{G}_m \to \mathbb{G}_m$ is the n-th power map. We shall identify μ_n with M_X/nM_X according to the identification in [Mochizuki1], the discussion at the beginning of §2.

Consider a subset

$$E \subset X^{\text{cl}}.$$

(We will set $E = E_X$ eventually, but E is arbitrary for the present.) Let $S \subset X^{\text{cl}} \setminus E$ be a finite set. If we consider the above Kummer exact sequence on the étale site of $U_S \stackrel{\text{def}}{=} X \setminus S$ and pass to the inverse limit with respect to n, then we obtain a natural homomorphism

$$\Gamma(U_S, \mathcal{O}_{U_S}^\times) \to H^1(\Pi_{U_S}, M_X)$$

(cf. loc. cit.). (Note that here it suffices to consider the group cohomology of Π_{U_S} (i.e., as opposed to the étale cohomology of U_S), since the extraction of n-th roots of an element of $\Gamma(U_S, \mathcal{O}_{U_S}^\times)$ yields finite étale coverings of U_S that correspond to open subgroups of Π_{U_S}.) The above homomorphism induces a natural injective homomorphism

$$\Gamma(U_S, \mathcal{O}_{U_S}^\times)/(k^\times\{\Sigma'\}) \to H^1(\Pi_{U_S}, M_X)$$

where $k^\times\{\Sigma'\}$ stands for the Σ'-primary part of the multiplicative group $k^\times$ (since the abelian group $\Gamma(U_S, \mathcal{O}_{U_S}^\times)/(k^\times\{\Sigma'\})$ is finitely generated and free of Σ'-primary torsion, hence injects into its pro-Σ completion). In particular, by allowing S to vary among all finite subsets of $X^{\text{cl}} \backslash E$, we obtain a natural injective homomorphism

$$\mathcal{O}_E^\times/(k^\times\{\Sigma'\}) \to \varinjlim_S H^1(\Pi_{U_S}, M_X),$$

where

$$\mathcal{O}_E^\times \stackrel{\text{def}}{=} \{f \in K_X^\times \mid \sup(\text{div}(f)) \cap E = \varnothing\}$$

is the multiplicative group of the units in the ring $\mathcal{O}_E$ of functions on X which are regular at all points of E. (Here, K_X denotes the function field of X.)

Proposition 3.7. *(Kummer Classes of Functions) Suppose that $S \subset X^{\text{cl}} \setminus E$ is a finite subset. Write*

$$\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{\text{c-ab}} \twoheadrightarrow \Delta_{U_S}^{\text{c-cn}}$$

for the maximal cuspidally abelian and the maximal cuspidally central quotients, respectively, relative to the map $\Delta_{U_S} \twoheadrightarrow \Delta_X$, and

$$\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\text{c-ab}} \twoheadrightarrow \Pi_{U_S}^{\text{c-cn}}$$

for the corresponding quotients of Π_{U_S} (i.e., $\Pi_{U_S}^{\text{c-ab}} \stackrel{\text{def}}{=} \Pi_{U_S}/\operatorname{Ker}(\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{\text{c-ab}})$, $\Pi_{U_S}^{\text{c-cn}} \stackrel{\text{def}}{=} \Pi_{U_S}/\operatorname{Ker}(\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{\text{c-cn}})$). If $x \in X^{\text{cl}}$, then we shall write

$$D_x[U_S] \subset \Pi_{U_S}$$

for the decomposition group at x in Π_{U_S} (which is well-defined up to conjugation), and $I_x[U_S] \stackrel{\text{def}}{=} D_x[U_S] \cap \Delta_{U_S}$ for the inertia subgroup of $D_x[U_S]$. Thus, when $x \in S$ we have a natural isomorphism of M_X with $I_x[U_S]$ (cf. [Mochizuki1], Proposition 1.6(ii)(iii)). Then:

(i) *The natural surjections above induce the following isomorphisms:*

$$H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}, M_X).$$

In particular, we obtain the following natural injective homomorphisms:

$$\Gamma(U_S, \mathcal{O}_{U_S}^\times)/(k^\times\{\Sigma'\}) \hookrightarrow H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}, M_X),$$

$$\mathcal{O}_E^\times/(k^\times\{\Sigma'\}) \hookrightarrow \varinjlim_S H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} \varinjlim_S H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} \varinjlim_S H^1(\Pi_{U_S}, M_X),$$

where S varies among all finite subsets of $X \setminus E$.

(ii) *Restricting cohomology classes of Π_{U_S} to the various $I_x[U_S]$ for $x \in S$ yields a natural exact sequence:*

$$1 \to (k^\times)^{\Sigma^\dagger} \to H^1(\Pi_{U_S}, M_X) \to (\bigoplus_{s \in S} \hat{\mathbb{Z}}^{\Sigma^\dagger})$$

(where we identify $\mathrm{Hom}_{\hat{\mathbb{Z}}^{\Sigma^\dagger}}(I_x[U_S], M_X)$ *with* $\hat{\mathbb{Z}}^{\Sigma^\dagger}$*, and* $(k^\times)^{\Sigma^\dagger}$ *is the maximal pro-*$\Sigma^\dagger$ *quotient of the multiplicative group* $k^\times$*). Moreover, the image (via the natural homomorphism given in* (i)*) of* $\Gamma(U_S, \mathcal{O}_{U_S}^\times)/(k^\times\{\Sigma'\})$ *in* $H^1(\Pi_{U_S}, M_X)$ *is equal to the inverse image in* $H^1(\Pi_{U_S}, M_X)$ *of the submodule of*

$$(\bigoplus_{s\in S} \mathbb{Z}) \subset (\bigoplus_{s\in S} \hat{\mathbb{Z}}^{\Sigma^\dagger})$$

determined by the principal divisors (with support in S*). A similar statement holds when* Π_{U_S} *is replaced by* $\Pi_{U_S}^{\text{c-cn}}$ *or* $\Pi_{U_S}^{\text{c-ab}}$*.*
(iii) *If* $f \in \Gamma(U_S, \mathcal{O}_{U_S}^\times)$*, write* f' *for its image in* $\Gamma(U_S, \mathcal{O}_{U_S}^\times)/(k^\times\{\Sigma'\})$*. Write*

$$\kappa_{f'}^{\text{c-cn}} \in H^1(\Pi_{U_S}^{\text{c-cn}}, M_X), \quad \kappa_{f'}^{\text{c-ab}} \in H^1(\Pi_{U_S}^{\text{c-ab}}, M_X), \quad \kappa_{f'} \in H^1(\Pi_{U_S}, M_X)$$

for the associated Kummer classes. If $x \in (X^{\text{cl}} \setminus E) \setminus S$*, then* $D_x[U_S]$ *maps, via the natural surjection* $\Pi_{U_S} \twoheadrightarrow G_k$*, isomorphically onto the open subgroup* $G_{k(x)} \subset G_k$ *(where* $k(x)$ *is the residue field of* X *at* x*). Moreover, the images of the pulled back classes*

$$\begin{aligned}\kappa_{f'}^{\text{c-cn}}|_{D_x[U_S]} = \kappa_{f'}^{\text{c-ab}}|_{D_x[U_S]} = \kappa_{f'}|_{D_x[U_S]} \in H^1(D_x[U_S], M_X) &\simeq H^1(G_{k(x)}, M_X) \\ &\simeq (k(x)^\times)^{\Sigma^\dagger}\end{aligned}$$

in $(k(x)^\times)^{\Sigma^\dagger}$ *are equal to the image in* $(k(x)^\times)^{\Sigma^\dagger}$ *of the value* $f(x) \in k(x)^\times$ *of* f *at* x*.*

Proof. See [Saïdi-Tamagawa1], Proposition 3.1. (See also [Mochizuki1], Proposition 2.1.) □

Remark 3.8. (cf. [Mochizuki1], Remark 12.) In the situation of Proposition 3.7(iii), assume $x \in X(k)$ and $S \subset X(k)$ for simplicity. If we think of the extension $\Pi_{U_S}^{\text{c-cn}}$ of Π_X as being given by the extension $\mathcal{D}_S$, where $\mathcal{D}$ is a fundamental extension of $\Pi_{X\times X}$ (cf. Proposition 3.4(i)), then it follows that the image of $D_x[U_S]$ in $\Pi_{U_S}^{\text{c-cn}}$ may be thought of as the image of $D_x[U_S]$ in $\mathcal{D}_S$. This image of $D_x[U_S]$ in $\mathcal{D}_S$ amounts to a section of $\mathcal{D}_S \twoheadrightarrow \Pi_X \twoheadrightarrow G_k$ lying over the section $s_x : G_k \to \Pi_X$ corresponding to the rational point x (which is well-defined up to conjugation). Since $\mathcal{D}_S$ is defined as a certain fiber product, this section is equivalent to a collection of sections (regarded as "cyclotomically outer homomorphisms", i.e., well-defined up to composition with an inner automorphism of $\mathcal{D}_{y,x}$ by an element of $\mathrm{Ker}(\mathcal{D}_{y,x} \twoheadrightarrow G_k)$)

$$\gamma_{y,x} : G_k \to \mathcal{D}_{y,x},$$

where y ranges over all points of S. Namely, from this point of view, Proposition 3.7(iii) may be regarded as saying that the image in $(k(x)^\times)^{\Sigma^\dagger} = (k^\times)^{\Sigma^\dagger}$ of the value $f(x)$ of the function $f \in \Gamma(U_S, \mathcal{O}_{U_S}^\times)$ at $x \in X(k)$ may be computed from its Kummer class, as soon as one knows the sections $\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$ for $y \in S$. Observe that $\gamma_{y,x}$ depends only on x, y, and not on the choice of S.

Definition 3.9. (cf. [Mochizuki1], Definition 2.1.) For $x, y \in X(k)$ with $x \neq y$, we shall refer to the above section (regarded as a cyclotomically outer homomorphism)

$$\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$$

as the Green's trivialization of $\mathcal{D}$ at (y, x). If D is a divisor on X supported on k-rational points $\neq x$, then multiplication of the various Green's trivializations for the points in the support of D yields a section (regarded as a cyclotomically outer homomorphism)

$$\gamma_{D,x} : G_k \to \mathcal{D}_{D,x}$$

which we shall refer to as the Green's trivialization of $\mathcal{D}$ at (D, x).

Definition 3.10. (cf. [Mochizuki1], Definition 2.2.) Let the notations and the assumptions as in Corollary 3.3.
(i) Write $\mathcal{D}$ (respectively, $\mathcal{E}$) for the fundamental extension of $\Pi_{X\times X}$ (respectively, $\Pi_{Y\times Y}$) that arises as the quotient of $\Pi^{\text{c-ab}}_{U_{X\times X}}$ (respectively, $\Pi^{\text{c-ab}}_{U_{Y\times Y}}$) by the kernel of the maximal cuspidally central quotient $\Delta^{\text{c-ab}}_{U_{X\times X}} \twoheadrightarrow \Delta^{\text{c-cn}}_{U_{X\times X}}$ (respectively, $\Delta^{\text{c-ab}}_{U_{Y\times Y}} \twoheadrightarrow \Delta^{\text{c-cn}}_{U_{Y\times Y}}$) (cf. Proposition 3.4(ii)). The isomorphism $\alpha^{\text{c-ab}}$ induces naturally an isomorphism:

$$\alpha^{\text{c-cn}} : \mathcal{D} \xrightarrow{\sim} \mathcal{E}$$

We shall say that α is (S, T)-locally Green-compatible outside exceptional sets if, for every pair of points $(x_1, x_2) \in X(k_X) \times X(k_X)$ corresponding via ϕ to a pair of points $(y_1, y_2) \in Y(k_Y) \times Y(k_Y)$, such that $x_1 \in (X^{\text{cl}} \setminus E_X) \setminus S$, $y_1 \in (Y^{\text{cl}} \setminus E_Y) \setminus T$, $x_2 \in S$, $y_2 \in T$, the isomorphism

$$\mathcal{D}_{x_1,x_2} \xrightarrow{\sim} \mathcal{E}_{y_1,y_2}$$

(obtained by restricting $\alpha^{\text{c-cn}}$ to the various decomposition groups) is compatible with the Green's trivializations. We shall say that α is (S, T)-locally bi-principally Green-compatible outside exceptional sets if, for every point $x \in X(k_X) \cap S$ and every principal divisor P supported on k_X-rational points $\neq x$ contained in $X^{\text{cl}} \setminus E_X$ corresponding via ϕ to a pair (y, Q) (so $y \in Y(k_Y) \cap T$) with Q principal, the isomorphism

$$\mathcal{D}_{P,x} \xrightarrow{\sim} \mathcal{E}_{Q,y}$$

obtained from $\alpha^{\text{c-cn}}$ is compatible with the Green's trivializations.
(ii) We shall say that α is totally globally Green-compatible (respectively, totally globally bi-principally Green-compatible) outside exceptional sets if, for all pair of connected finite étale coverings $\xi : X' \to X$, $\eta : Y' \to Y$ that arise from open subgroups $\Pi_{X'} \subset \Pi_X$, $\Pi_{Y'} \subset \Pi_Y$, corresponding to each other via α, then for any subset $S \subset X^{\text{cl}} \setminus E_X$ that corresponds, via ϕ, to $T \subset Y \setminus E_Y$ the isomorphism

$$\Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$$

induced by α is (S', T')-locally Green-compatible (respectively, (S', T')-locally bi-principally Green-compatible) outside exceptional sets, where $S' \overset{\text{def}}{=} \xi^{-1}(S) \subset X'^{\text{cl}}$, $T' \overset{\text{def}}{=} \eta^{-1}(T) \subset Y'^{\text{cl}}$ are the inverse images of S, T, respectively.

Remark 3.11. In [Saïdi-Tamagawa1], Definition 3.4, we adopted a slightly different notion of being "(S, T)-locally (or totally globally) principally Green-compatible outside exceptional sets", where the divisor Q of Y appearing in (i) above is not assumed to be principal. Here we adopt the above notion of being "(S, T)-locally (or totally globally) bi-principally Green-compatible outside exceptional sets", since it is more natural in our settings (although both notions work).

Proposition 3.12. *(Total Global Green-Compatibility Outside Exceptional Sets) In the situation of Theorem 3.2, assume further that α is Frobenius-preserving. Then the isomorphism α is totally globally, and totally globally bi-principally, Green-compatible outside exceptional sets.*

Proof. Similar to the proof of Proposition 3.8 in [Saïdi-Tamagawa1]. (See also [Mochizuki1], Corollary 3.1.) $\square$

§4. Isomorphisms between geometrically pro-Σ arithmetic fundamental groups.

We maintain the notations of §3.

Definition/Remark 4.1. Let $J = J_X$ be the Jacobian variety of X. Let $\mathrm{Div}^0_{X\setminus E}$ be the group of degree zero divisors on X which are supported on points in $X \setminus E$. Write $D_{X\setminus E}$ for the kernel of the natural homomorphism $\mathrm{Div}^0_{X\setminus E} \to J(k)^\Sigma$. Here, $J(k)^\Sigma$ stands for the maximal pro-Σ quotient $J(k)/(J(k)\{\Sigma'\})$ of $J(k)$, where, for an abelian group M, $M\{\Sigma'\}$ stands for the subgroup of torsion elements a of M such that every prime divisor of the order of a belongs to Σ'. Then $D_{X\setminus E}$ sits naturally in the following exact sequence:

$$0 \to \mathrm{Pri}_{X\setminus E} \to D_{X\setminus E} \to J(k)\{\Sigma'\} \to 0,$$

where $\mathrm{Pri}_{X\setminus E} \stackrel{\mathrm{def}}{=} \mathcal{O}_E^\times / k_X^\times$ stands for the group of principal divisors supported in $X \setminus E$. Further, let $\mathcal{D}_{X\setminus E}$ be the inverse image of $D_{X\setminus E}$ in $\varinjlim H^1(\Pi^{\text{c-ab}}_{U_S}, M_X)$ (cf. Proposition 3.7(ii)). Then $\mathcal{D}_{X\setminus E}$ sits naturally in the following exact sequence

$$0 \to \mathcal{O}_E^\times / (k_X^\times\{\Sigma'\}) \to \mathcal{D}_{X\setminus E} \to J(k)\{\Sigma'\} \to 0.$$

Now, let X, Y be proper hyperbolic curves over finite fields k_X, k_Y of characteristic p_X, p_Y, respectively, and define K_X, K_Y to be the function fields of X, Y, respectively. Let Σ_X (respectively, Σ_Y) be a set of prime numbers that contains at least one prime number different from p_X (respectively, p_Y). Write Δ_X (respectively, Δ_Y) for the maximal pro-Σ_X quotient of $\pi_1(\overline{X})$ (respectively, the maximal pro-Σ_Y quotient of $\pi_1(\overline{Y})$), and Π_X (respectively, Π_Y) for the quotient $\pi_1(X)/\operatorname{Ker}(\pi_1(\overline{X}) \twoheadrightarrow \Delta_X)$ of $\pi_1(X)$ (respectively, the quotient $\pi_1(Y)/\operatorname{Ker}(\pi_1(\overline{Y}) \twoheadrightarrow \Delta_Y)$ of $\pi_1(Y)$).

Let

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an isomorphism of profinite groups and write $\Sigma \stackrel{\mathrm{def}}{=} \Sigma_X = \Sigma_Y$ (cf. Proposition 1.7(ii)).

Theorem 4.2. *(Reconstruction of Pseudo-Functions) Assume that α is Frobenius-preserving. Then:*
(i) *The bijection $\phi : X^{\mathrm{cl}} \setminus E_X \xrightarrow{\sim} Y^{\mathrm{cl}} \setminus E_Y$ induced by α (where E_X and E_Y are the exceptional sets), together with the isomorphisms in Corollary 3.3, induce natural bijections $\bar{\rho} : D_{Y\setminus E_Y} \xrightarrow{\sim} D_{X\setminus E_X}$, $\rho' : \mathcal{D}_{Y\setminus E_Y} \xrightarrow{\sim} \mathcal{D}_{X\setminus E_X}$, which fit into the following commutative diagrams*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pri}_{X\setminus E_X} = \mathcal{O}_{E_X}^{\times}/k_X^{\times} & \longrightarrow & D_{X\setminus E_X} & \longrightarrow & J_X(k_X)\{\Sigma'\} & \longrightarrow & 0 \\ & & & & \bar{\rho}\big\uparrow & & & & \\ 0 & \longrightarrow & \mathrm{Pri}_{Y\setminus E_Y} = \mathcal{O}_{E_Y}^{\times}/k_Y^{\times} & \longrightarrow & D_{Y\setminus E_Y} & \longrightarrow & J_Y(k_Y)\{\Sigma'\} & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{E_X}^{\times}/(k_X^{\times}\{\Sigma'\}) & \longrightarrow & \mathcal{D}_{X\setminus E_X} & \longrightarrow & J_X(k_X)\{\Sigma'\} & \longrightarrow & 0 \\ & & & & \rho'\big\uparrow & & & & \\ 0 & \longrightarrow & \mathcal{O}_{E_Y}^{\times}/(k_Y^{\times}\{\Sigma'\}) & \longrightarrow & \mathcal{D}_{Y\setminus E_Y} & \longrightarrow & J_Y(k_Y)\{\Sigma'\} & \longrightarrow & 0 \end{array}$$

Moreover, the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}_{X\setminus E_X} & \xleftarrow{\rho'} & \mathcal{D}_{Y\setminus E_Y} \\ \big\downarrow & & \big\downarrow \\ D_{X\setminus E_X} & \xleftarrow{\bar{\rho}} & D_{Y\setminus E_Y} \end{array}$$

where the vertical maps are the natural ones.
(ii) *The bijection $\bar{\rho}$ in (i) induces a natural isomorphism*

$$\bar{\rho} : \overline{H}_Y \xrightarrow{\sim} \overline{H}_X.$$

Here, $\overline{H}_X \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathrm{Pri}_{X\setminus E_X} \xrightarrow{\bar{\rho}^{-1}} D_{Y\setminus E_Y} \to J_Y(k_Y)\{\Sigma'\})$ and $\overline{H}_Y \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathrm{Pri}_{Y\setminus E_Y} \xrightarrow{\bar{\rho}} D_{X\setminus E_X} \to J_X(k_X)\{\Sigma'\})$ are finite index subgroups of $\mathcal{O}_{E_X}^{\times}/k_X^{\times}$ and $\mathcal{O}_{E_Y}^{\times}/k_Y^{\times}$, respectively. (More precisely, the quotients $(\mathcal{O}_{E_X}^{\times}/k_X^{\times})/\overline{H}_X$ and $(\mathcal{O}_{E_Y}^{\times}/k_Y^{\times})/\overline{H}_Y$ are Σ'-primary finite abelian groups that are embeddable into $J_Y(k_Y)\{\Sigma'\}$ and $J_X(k_X)\{\Sigma'\}$, respectively.) Moreover, let H'_X (resp. H'_Y) be the inverse image of $\overline{H}_X$ (resp. $\overline{H}_Y$) in $\mathcal{O}_{E_X}^{\times}/(k_X^{\times})\{\Sigma'\}$ (resp. $\mathcal{O}_{E_Y}^{\times}/(k_Y^{\times}\{\Sigma'\})$). Then the isomorphism ρ' in (i) induces a natural isomorphism $\rho' : H'_Y \xrightarrow{\sim} H'_X$ which fits into the following commutative diagram

$$\begin{array}{ccc} H'_X & \xleftarrow{\rho'} & H'_Y \\ \big\downarrow & & \big\downarrow \\ \overline{H}_X & \xleftarrow{\bar{\rho}} & \overline{H}_Y \end{array} \tag{4.1}$$

where the vertical maps are the natural surjections.

In particular, ρ' induces a natural isomorphism

$$\tau : (k_Y^\times)^\Sigma = k_Y^\times/(k_Y^\times\{\Sigma'\}) \xrightarrow{\sim} k_X^\times/(k_X^\times\{\Sigma'\}) = (k_X^\times)^\Sigma.$$

(iii) *The diagram in (ii) is functorial in X, Y, in the following sense: if $\xi : X' \to X$ is a finite étale covering, arising from an open subgroup $\Pi_{X'} \subset \Pi_X$, which corresponds to a finite étale covering $Y' \to Y$ via α (thus, $\Pi_{Y'} \stackrel{\text{def}}{=} \alpha(\Pi_{X'})$), then α induces natural isomorphisms $\gamma' : H'_{Y'} \xrightarrow{\sim} H'_{X'}$ and $\overline{\gamma} : \overline{H}_{Y'} \xrightarrow{\sim} \overline{H}_{X'}$, which fit into the following commutative diagrams*

$$\begin{array}{ccc} H'_{X'} & \xleftarrow{\gamma'} & H'_{Y'} \\ \uparrow & & \uparrow \\ H'_X & \xleftarrow{\rho'} & H'_Y \end{array}$$

and

$$\begin{array}{ccc} \overline{H}_{X'} & \xleftarrow{\overline{\gamma}} & \overline{H}_{Y'} \\ \uparrow & & \uparrow \\ \overline{H}_X & \xleftarrow{\overline{\rho}} & \overline{H}_Y \end{array}$$

where the vertical maps are natural homomorphisms induced by the natural injective homomorphisms $\mathcal{O}_{E_X}^\times/k_X^\times\{\Sigma'\} \hookrightarrow \mathcal{O}_{E_{X'}}^\times/k_{X'}^\times\{\Sigma'\}$, $\mathcal{O}_{E_Y}^\times/k_Y^\times\{\Sigma'\} \hookrightarrow \mathcal{O}_{E_{Y'}}^\times/k_{Y'}^\times\{\Sigma'\}$. $\mathcal{O}_{E_X}^\times/k_X^\times \hookrightarrow \mathcal{O}_{E_{X'}}^\times/k_{X'}^\times$, and $\mathcal{O}_{E_Y}^\times/k_Y^\times \hookrightarrow \mathcal{O}_{E_{Y'}}^\times/k_{Y'}^\times$.

In particular, α induces a natural isomorphism

$$\tau : (\bar{k}_Y^\times)^\Sigma = \bar{k}_Y^\times/(\bar{k}_Y^\times\{\Sigma'\}) \xrightarrow{\sim} \bar{k}_X^\times/(\bar{k}_X^\times\{\Sigma'\}) = (\bar{k}_X^\times)^\Sigma$$

which extends $\tau : (k_Y^\times)^\Sigma \xrightarrow{\sim} (k_X^\times)^\Sigma$ in (ii).

Proof. Similar to the proof of [Saïdi-Tamagawa1], Theorem 3.6. (See also [Saïdi-Tamagawa3], Lemma 4.5 for a similar statement in the birational case.) Here, the commutativity of the diagrams in (iii) follows basically from the functoriality of Kummer theory, together with the commutativity of the diagram

$$\begin{array}{ccc} M_{X'} & \longleftarrow & M_{Y'} \\ \uparrow & & \uparrow \\ M_X & \longleftarrow & M_Y \end{array}$$

where the horizontal arrows are isomorphisms induced by α (or, more precisely, by α^{-1}, via the functoriality of H^2) and the vertical arrows are isomorphisms defined geometrically via the identification of M_X and $M_{X'}$ (resp. M_Y and $M_{Y'}$) with the Tate module of $\mathbb{G}_m$ over $\bar{k}_X = \bar{k}_{X'}$ (resp. $\bar{k}_Y = \bar{k}_{Y'}$). The commutativity of this last diagram follows from the fact that the above (geometrically defined) isomorphism $M_X \xrightarrow{\sim} M_{X'}$ (resp. $M_Y \xrightarrow{\sim} M_{Y'}$) is identified with the composite of the $(X'_{\bar{k}_X} : X_{\bar{k}_X})$-multiplication map $M_X \to M_X$ (resp. the $(Y'_{\bar{k}_Y} : Y_{\bar{k}_Y})$-multiplication map $M_Y \to M_Y$) and the inverse of the natural isomorphism $M_{X'} \xrightarrow{\sim} (X'_{\bar{k}_X} : X_{\bar{k}_X})M_X \subset M_X$ (resp. $M_{Y'} \xrightarrow{\sim} (Y'_{\bar{k}_Y} : Y_{\bar{k}_Y})M_Y \subset M_Y$) induced by the inclusion $M_{X'} \to M_X$ (resp. $M_{Y'} \to M_Y$) arising from the functoriality of H^2, and the fact that $(X'_{\bar{k}_X} : X_{\bar{k}_X}) = (\Delta_X : \Delta_{X'}) = (\Delta_Y : \Delta_{Y'}) = (Y'_{\bar{k}_Y} : Y_{\bar{k}_Y})$. $\square$

Definition 4.3. We say that α is pseudo-constants-additive if $\tau : (\bar{k}_Y^\times)^\Sigma \xrightarrow{\sim} (\bar{k}_X^\times)^\Sigma$ in Theorem 4.2(iii) satisfies the following: For $\eta \in \overline{k}_Y^\times$ and $\zeta \in \overline{k}_X^\times$, if

$$1 + \eta \neq 0 \text{ and } \tau(\eta') = \zeta',$$

then there exist $\alpha, \beta \in \overline{k}_X^\times\{\Sigma'\}$, such that

$$\alpha + \beta\zeta \neq 0 \text{ and } \tau((1+\eta)') = (\alpha + \beta\zeta)'.$$

Here, for an element ξ of $\bar{k}_X^\times$ (resp. $\bar{k}_Y^\times$), ξ' denotes its image in $(\bar{k}_X^\times)^\Sigma$ (resp. $(\bar{k}_Y^\times)^\Sigma$).

Theorem 4.4. *(Pseudo-Constants-Additive Isomorphisms) Assume that Σ is k_X-large and k_Y-large, that at least one of X and Y is almost Σ-separated, and that α is pseudo-constants-additive. Then α arises from a uniquely determined commutative diagram of schemes:*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sim} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings determined by the groups Π_X, Π_Y.

The rest of this section is devoted to the proof of Theorem 4.4 and deducing corollaries.

First, since Σ is k_X-large and k_Y-large, α is Frobenius-preserving by Proposition 2.5(ii). Thus, we may apply Theorem 1.9, Theorem 3.12 and Theorem 4.2 to α. In particular, α is totally globally bi-principally Green-compatible outside exceptional sets by Theorem 3.12. Also, we see that both X and Y are almost Σ-separated. Indeed, let F_X (resp. F_Y) denote the compositum of $k(x)$ (resp. $k(y)$) in $\bar{k}_X$ (resp. $\bar{k}_Y$) for all $x \in E_X$ (resp. $y \in E_Y$). Then it follows from Theorem 1.9(i) that $G_{F_X} \subset G_{k_X}$ corresponds to $G_{F_Y} \subset G_{k_Y}$ via the natural isomorphism $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ induced by α (cf. Proposition 1.7(i)). Thus, $F_X \subsetneq \bar{k}_X$ (i.e., $G_{F_X} \neq \{1\}$) is equivalent to $F_Y \subsetneq \bar{k}_Y$ (i.e., $G_{F_Y} \neq \{1\}$).

Let $x \in X^{\mathrm{cl}} \setminus E_X$ and $y \stackrel{\mathrm{def}}{=} \phi(x) \in Y^{\mathrm{cl}} \setminus E_Y$. Then, as α preserves the decomposition groups at x, y, respectively, α induces naturally an isomorphism

$$\tau_{x,y} : (k(y)^\times)^\Sigma \xrightarrow{\sim} (k(x)^\times)^\Sigma$$

(cf. Proposition 3.7(iii))), which fits into a commutative diagram

$$\begin{array}{ccc} (k(x)^\times)^\Sigma & \xleftarrow{\tau_{x,y}} & (k(y)^\times)^\Sigma \\ \uparrow & & \uparrow \\ (k_X^\times)^\Sigma & \xleftarrow{\tau} & (k_Y^\times)^\Sigma \end{array}$$

where the vertical maps are the natural homomorphisms.

Next, we shall think of elements of $\mathcal{O}_{E_X}^\times/(k_X)^\times$, $\mathcal{O}_{E_Y}^\times/(k_Y)^\times$ as principal divisors of rational functions on X, Y, respectively, and denote them $\bar{f}, \bar{g}, \ldots$, where f, g are rational functions on X, Y, whose supports of divisors are disjoint from E_X, E_Y, respectively. We will denote the elements of $\mathcal{O}_{E_X}^\times/(k_X^\times\{\Sigma'\})$, $\mathcal{O}_{E_Y}^\times/(k_Y^\times\{\Sigma'\})$, by $f', g', \ldots$, where f, g, are rational functions on X, Y, whose supports of divisors are disjoint from E_X, E_Y, respectively, and refer to them as "pseudo-functions" $\stackrel{\text{def}}{=}$ classes of elements of $\mathcal{O}_{E_X}^\times$, $\mathcal{O}_{E_Y}^\times$, modulo $k_X^\times\{\Sigma'\}$, $k_Y^\times\{\Sigma'\}$, respectively. For each $x \in X^{\text{cl}} \setminus E_X$ (resp. $y \stackrel{\text{def}}{=} \phi(x) \in Y^{\text{cl}} \setminus E_Y$), we denote by $v_x : \mathcal{O}_{E_X}^\times/(k_X)^\times \to \mathbb{Z}$ (resp. $v_y : \mathcal{O}_{E_Y}^\times/(k_Y)^\times \to \mathbb{Z}$) the function induced by the (normalized, additive) valuation $v_x : K_X^\times \to \mathbb{Z}$ at x (resp. $v_y : K_Y^\times \to \mathbb{Z}$ at y). Further, we denote by $\deg : \mathcal{O}_{E_X}^\times/(k_X)^\times \to \mathbb{Z}_{\geq 0}$ (resp. $\deg : \mathcal{O}_{E_Y}^\times/(k_Y)^\times \to \mathbb{Z}_{\geq 0}$) the function induced by the degree function $\deg : K_X^\times \to \mathbb{Z}_{\geq 0}$ (resp. $\deg : K_Y^\times \to \mathbb{Z}_{\geq 0}$) that sends $f \in K_X^\times$ (resp. $f \in K_Y^\times$) to the degree of the pole divisor of f.

Lemma 4.5. *(Recovering the Valuations and the Σ-Values of Pseudo-Functions) Consider the commutative diagram (4.1). Let $x \in X^{\text{cl}} \setminus E_X$ and $y \stackrel{\text{def}}{=} \phi(x) \in Y^{\text{cl}} \setminus E_Y$. Then the following implications hold:*
(i) *For $\bar{f} \in \overline{H}_Y$ and $\bar{g} \in \overline{H}_X$:*

$$\bar{\rho}(\bar{f}) = \bar{g} \Longrightarrow v_y(\bar{f}) = v_x(\bar{g}).$$

In particular, in terms of divisors, if:

$$\bar{f} = y_1 + y_2 + \cdots + y_n - y_1' - \cdots - y_{n'}',$$

then:

$$\bar{g} = x_1 + x_2 + \cdots + x_n - x_1' - \cdots - x_{n'}',$$

where $y_i \stackrel{\text{def}}{=} \phi(x_i)$ (resp. $y_{i'}' \stackrel{\text{def}}{=} \phi(x_{i'}')$) for $i \in \{1, \ldots, n\}$ (resp. $i' \in \{1, \ldots, n'\}$). In other words, the isomorphism $\bar{\rho} : \overline{H}_Y \xrightarrow{\sim} \overline{H}_X$ preserves the valuations of the classes of functions in $\overline{H}_X, \overline{H}_Y$ with respect to the bijection $\phi : X^{\text{cl}} \setminus E_X \xrightarrow{\sim} Y^{\text{cl}} \setminus E_Y$ between points. Further, the isomorphism $\bar{\rho}$ preserves the degrees of the classes of functions in $\overline{H}_X, \overline{H}_Y$.
(ii) *For $f' \in H_Y'$ and $g' \in H_X'$:*

$$v_y(\bar{f}) = 0 \text{ and } \rho(f') = g' \Longrightarrow v_x(\bar{g}) = 0 \text{ and } \tau_{x,y}(f'(y)) = g'(x)$$

where

$$y = \phi(x) \text{ and } \tau_{x,y} : (k(y)^\times)^\Sigma \xrightarrow{\sim} (k(x)^\times)^\Sigma$$

is the natural identification above. In other words, the isomorphism $\rho' : H_Y' \xrightarrow{\sim} H_X'$ preserves the Σ-values of the pseudo-functions in H_X', H_Y' with respect to the bijection $\phi : X^{\text{cl}} \setminus E_X \xrightarrow{\sim} Y^{\text{cl}} \setminus E_Y$ between points.

Proof. Assertion (i) follows from Proposition 3.7(i)(ii), together with the fact that $[k(x) : k_X] = [k(\phi(x)) : k_Y]$ for each $x \in X^{\text{cl}} \setminus E_X$ (cf. discussion before Corollary 3.3), and assertion (ii) follows from Proposition 3.7(iii), together with the fact that α is totally globally bi-principally Green-compatible outside exceptional sets. □

Lemma 4.6. *Let $x \in X^{\mathrm{cl}} \backslash E_X$ and $y \stackrel{\mathrm{def}}{=} \phi(x) \in Y^{\mathrm{cl}} \backslash E_Y$. The natural identification*

$$\tau_{x,y} : (k(y)^\times)^\Sigma \xrightarrow{\sim} (k(x)^\times)^\Sigma,$$

induced by α, satisfies the following property: For $\eta \in k(y)^\times$ and $\zeta \in k(x)^\times$, if

$$1 + \eta \neq 0 \text{ and } \tau_{x,y}(\eta') = \zeta',$$

then there exist $\alpha, \beta \in \overline{k}_X^\times\{\Sigma'\}$, such that

$$\alpha + \beta\zeta \neq 0 \text{ and } \tau_{x,y}((1+\eta)') = (\alpha + \beta\zeta)'.$$

Proof. Similar to the proof of [Saïdi-Tamagawa3], Lemma 4.10. □

Let $l \in \mathfrak{Primes}$ be a prime number which is both (X, Σ)-admissible and (Y, Σ)-admissible, i.e., for every finite extension k' of k_X (resp. k_Y), $(k')^l \cap F_X$ (resp. $(k')^l \cap F_Y$) is finite, where $(k')^l$ is the maximal pro-l extension of k', and, in particular, $E_X \cap X((k')^l)^{\mathrm{cl}}$ (resp. $E_Y \cap Y((k')^l)^{\mathrm{cl}}$) is finite. Let X^l, Y^l be the normalization of X, Y in $K_X k_X^l$, $K_Y k_Y^l$, respectively. Set $E_{X^l} \stackrel{\mathrm{def}}{=} E_X \times_{k_X} k_X^l$, $E_{Y^l} \stackrel{\mathrm{def}}{=} E_Y \times_{k_Y} k_Y^l$, and write $\mathcal{O}_{E_{X^l}}^\times$, $\mathcal{O}_{E_{Y^l}}^\times$ for the group of multiplicative functions on X^l, Y^l whose supports of divisors are disjoint from E_{X^l}, E_{Y^l}, respectively. Define $D_{X^l \backslash E_{X^l}}$, $\mathcal{D}_{X^l \backslash E_{X^l}}$ (resp. $D_{Y^l \backslash E_{Y^l}}$, $\mathcal{D}_{Y^l \backslash E_{Y^l}}$) in a similar way as in Definition 4.1. Thus, we have natural exact sequences

$$0 \to \mathcal{O}_{E_{X^l}}^\times / (k_X^l)^\times \to D_{X^l \backslash E_{X^l}} \to J_X(k_X^l)\{\Sigma'\} \to 0$$

and

$$0 \to \mathcal{O}_{E_{X^l}}^\times / (k_X^l)^\times\{\Sigma'\}) \to \mathcal{D}_{X^l \backslash E_{X^l}} \to J_X(k_X^l)\{\Sigma'\} \to 0$$

(resp. similar sequences for $D_{Y^l \backslash E_{Y^l}}$ and $\mathcal{D}_{Y^l \backslash E_{Y^l}}$).

The isomorphism $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ induces natural isomorphisms $\bar{\rho} : D_{Y^l \backslash E_{Y^l}} \xrightarrow{\sim} D_{X^l \backslash E_{X^l}}$, and $\rho' : \mathcal{D}_{Y^l \backslash E_{Y^l}} \xrightarrow{\sim} \mathcal{D}_{X^l \backslash E_{X^l}}$ (by passing to finite subextensions of k_X^l / k_X and k_Y^l / k_Y corresponding to each other by α, cf. Theorem 4.2(i)), which fit into the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{X^l \backslash E_{X^l}} & \xleftarrow{\rho'} & \mathcal{D}_{Y^l \backslash E_{Y^l}} \\ \downarrow & & \downarrow \\ D_{X^l \backslash E_{X^l}} & \xleftarrow{\bar{\rho}} & D_{Y^l \backslash E_{Y^l}} \end{array}$$

where the vertical maps are the natural ones. The above isomorphism $\bar{\rho}$ implies the existence of subgroups $\overline{H}_{X^l}$ (resp. $\overline{H}_{Y^l}$) of $\mathcal{O}_{E_{X^l}}^\times / (k_X^l)^\times$ (resp. $\mathcal{O}_{E_{Y^l}}^\times / (k_Y^l)^\times$), which are functorial in X and Y, and a natural isomorphism

$$\bar{\rho} : \overline{H}_{Y^l} \xrightarrow{\sim} \overline{H}_{X^l},$$

which lies above the isomorphism $\bar{\rho} : \overline{H}_Y \xrightarrow{\sim} \overline{H}_X$ in Theorem 4.2(ii). (Use similar arguments as those in the proof of Theorem 4.2(ii).) Moreover, let H'_{X^l}

(resp. H'_{Y^l}) be the inverse image of $\overline{H}_{X^l}$ (resp. $\overline{H}_{Y^l}$) in $\mathcal{O}^\times_{E_{X^l}}/((k^l_X)^\times\{\Sigma'\})$ (resp. $\mathcal{O}^\times_{E_{Y^l}}/((k^l_Y)^\times\{\Sigma'\})$). Then there exists a natural isomorphism $\rho' : H'_{Y^l} \xrightarrow{\sim} H'_{X^l}$ (cf. Theorem 4.2(ii)), and H'_{X^l} (resp. H'_{Y^l}) is a subgroup of $\mathcal{O}^\times_{E_{X^l}}/((k^l_X)^\times\{\Sigma'\})$ (resp. $\mathcal{O}^\times_{E_{Y^l}}/((k^l_Y)^\times\{\Sigma'\})$). Further, let

$$H^\times_{X^l} \stackrel{\text{def}}{=} \{f \in \mathcal{O}^\times_{E_{X^l}} \mid \bar{f} \in \overline{H}_{X^l}\} \text{ (resp. } H^\times_{Y^l} \stackrel{\text{def}}{=} \{f \in \mathcal{O}^\times_{E_{Y^l}} \mid \bar{f} \in \overline{H}_{Y^l}\} \text{).}$$

Then $H^\times_{X^l}$ (resp. $H^\times_{Y^l}$) is a subgroup of $\mathcal{O}^\times_{E_{X^l}}$ (resp. $\mathcal{O}^\times_{E_{Y^l}}$), and the quotient $\mathcal{O}^\times_{E_{X^l}}/H^\times_{X^l}$ (resp. $\mathcal{O}^\times_{E_{Y^l}}/H^\times_{Y^l}$) is embeddable into $J_Y(k^l_Y)\{\Sigma'\}$ (resp. $J_X(k^l_X)\{\Sigma'\}$), hence is a Σ'-primary torsion abelian group. (More precisely, let r be a prime number. Then the r-primary part of $\mathcal{O}^\times_{E_{X^l}}/H^\times_{X^l}$ (resp. $\mathcal{O}^\times_{E_{Y^l}}/H^\times_{Y^l}$) is trivial (resp. finite, resp. embeddable into a finite direct sum of $\mathbb{Q}_r/\mathbb{Z}_r$), if $r \in \Sigma$ (resp. $r \in \Sigma' \setminus \{l\}$, resp. r is any prime.) We have a commutative diagram

$$\begin{array}{ccc} H'_{X^l} & \xleftarrow{\rho'} & H'_{Y^l} \\ \downarrow & & \downarrow \\ \overline{H}_{X^l} & \xleftarrow{\bar{\rho}} & \overline{H}_{Y^l} \end{array}$$

where the vertical maps are the natural surjections (cf. Theorem 4.2(ii)), and lies above the commutative diagram (4.1) in Theorem 4.2(ii).

We state the following results (Remark 4.7, Lemma 4.8, Corollary 4.10 and Lemmas 4.11-4.13) only for X, but similar statments also hold for Y.

Remark 4.7. If Σ is J_Y-large, then F_Y is contained in a finite extension ($\neq \bar{k}_Y$) of $\bar{k}_Y^{\mathrm{Ker}(\rho_{J_Y,\Sigma'})}$ by Proposition 2.12. Thus, any l such that the image of the pro-l Sylow subgroup $G_{k_Y,l}$ of G_{k_Y} under $\rho_{J_Y,\Sigma'}$ is finite is (Y,Σ)-admissible. (Further, by Proposition 2.13(ii), such an l is automatically in $\Sigma \cup \{p\}$.) Let us take such an l. Then it follows that $(\rho_{J_Y,\Sigma'}(G_{k_Y}) : \rho_{J_Y,\Sigma'}(G_{k^l_Y})) < \infty$, hence $\sharp(J_Y(k^l_Y)\{\Sigma'\}) < \infty$. Thus, in this case, the quotient $\mathcal{O}^\times_{E_{X^l}}/H^\times_{X^l}$ is finite.

Lemma 4.8. *Let $f_1, \ldots, f_n \in \mathcal{O}_{E_{X^l}}$. Then, for all but finitely many $c \in k^l_X$, $f_1 - c, \ldots, f_n - c \in \mathcal{O}^\times_{E_{X^l}}$.*

Proof. We may and shall assume $n = 1$ and set $f \stackrel{\text{def}}{=} f_1$. (The assertion for general n can be reduced to this special case immediately.) The function $f \in \mathcal{O}_{E_{X^l}}$ descends to a finite extension of k_X. More precisely, there exists a finite subextension k_0/k_X of k^l_X/k_X, such that $f \in \mathcal{O}_{E_{X_{k_0}}}$, where $E_{X_{k_0}} \stackrel{\text{def}}{=} E_X \times_{k_X} k_0$ and $\mathcal{O}_{E_{X_{k_0}}} = \mathcal{O}_{E_X} \otimes_{k_X} k_0$. Then f defines a k_0-morphism $f : X_{k_0} \to \mathbb{P}^1_{k_0}$ of degree, say, d. One has $f^{-1}(\mathbb{P}^1_{k_0}(k^l)^{\mathrm{cl}}) \subset X(k')^{\mathrm{cl}}$, where k' is the compositum of finite extensions of k^l of degree $\leq d$, which is finite over k^l. On the other hand, $E_{X_{k_0}} \subset X_{k_0}(F)^{\mathrm{cl}}$, where $F \stackrel{\text{def}}{=} F_X k_0$. By Proposition 2.13, we see that $S := f^{-1}(\mathbb{P}^1_{k_0}(k^l)^{\mathrm{cl}}) \cap E_{X_{k_0}}$ is finite. Now, for each $c \in k^l \setminus f(S)(k^l)$ ($\subset \mathbb{P}^1_{k_0}(k^l)$), one has $f - c \in \mathcal{O}^\times_{E_{X^l}}$, as desired. $\square$

Definition 4.9. (i) For a subset S of a field of characteristic $p > 0$, we denote by $\langle S \rangle$ the abelian subgroup (or, equivalently, $\mathbb{F}_p$-vector subspace) generated by S.
(ii) $R_{X^l} \stackrel{\text{def}}{=} \langle H_{X^l}^\times \rangle$, $R_{Y^l} \stackrel{\text{def}}{=} \langle H_{Y^l}^\times \rangle$.

Corollary 4.10. (i) $(k_X^l)^\times + \mathcal{O}_{E_{X^l}}^\times = \mathcal{O}_{E_{X^l}}$.
(ii) $\langle \mathcal{O}_{E_{X^l}}^\times \rangle = \mathcal{O}_{E_{X^l}}$. *In particular,* $\langle \mathcal{O}_{E_{X^l}}^\times \rangle$ *coincides with the* k_X^l*-subalgebra generated by* $\mathcal{O}_{E_{X^l}}^\times$.

Proof. (i) Clearly $k_X^l + \mathcal{O}_{E_{X^l}}^\times \subset \mathcal{O}_{E_{X^l}}$. To show the opposite, take any $f \in \mathcal{O}_{E_{X^l}}$. Then, by Lemma 4.8, there exists $c \in (k_X^l)^\times$ such that $f - c \in \mathcal{O}_{E_{X^l}}^\times$. Thus, $f = c + (f - c) \in (k_X^l)^\times + \mathcal{O}_{E_{X^l}}^\times$, as desired.
(ii) The first assertion follows directly from (i). The second assertion follows from the first assertion, together with the fact that $\mathcal{O}_{E_{X^l}}$ is a k_X^l-algebra. $\square$

So far, we have only resorted to the assumptions that Σ is k_X-large and k_Y-large and that at least one of X and Y is almost Σ-separated. From now, we will resort to the other assumption that α is pseudo-constants-additive.

Lemma 4.11. *Let* $f \in H_{X^l}^\times$*, and assume that* $1 + f \in \mathcal{O}_{E_{X^l}}^\times$*. Then* $1 + f \in H_{X^l}^\times$.

Proof. Replacing k_X, k_Y by suitable finite subextensions of k_X^l/k_X and k_Y^l/k_Y corresponding to each other by α, we may assume that $f \in H_X^\times$. Then the proof is similar to [Saïdi-Tamagawa3], Lemma 3.16, using Proposition 2.4. More precisely, write $(\rho')^{-1}(f') = g'$. We have $((1 + f)')^m \in H_X'$ and we may write $(\rho')^{-1}(((1 + f)')^m) = h'$ with $h \in H_Y^\times$ for some positive integer m divisible only by primes in Σ'. By evaluating the pseudo-function h' at all points in $Y^{\text{cl}} \backslash Y(F_Y)^{\text{cl}}$, and using Lemma 4.6 and Proposition 2.4, we deduce that $h' = (\rho')^{-1}(((1 + f)')^m) = ((1 + cg)')^m \in H_Y' \subset \mathcal{D}_{Y \backslash E_Y}$ for some $c \in (k_Y)^\times\{\Sigma'\}$ (cf. loc. cit. for more details). Hence, $(\rho')^{-1}((1 + f)') = (1 + cg)' \in \mathcal{D}_{Y \backslash E_Y}$ since $\mathcal{D}_{Y \backslash E_Y}$ admits no nontrivial Σ'-primary torsion ($D_{Y \backslash E_Y}$ has no nontrivial torsion, and the kernel of the natural surjective map $\mathcal{D}_{Y \backslash E_Y} \to D_{Y \backslash E_Y}$ is Σ-primary torsion). As $(1 + cg)' \in \mathcal{O}_{E_Y}^\times/(k_Y^\times\{\Sigma'\})$, we have $(1 + f)' \in H_X'$ by definition, as desired. $\square$

Lemma 4.12. (i) R_{X^l} *coincides with the* k_X^l*-subalgebra generated by* $H_{X^l}^\times$.
(ii) *Let* $f_1, \ldots, f_n \in R_{X^l}$*. Then, for all but finitely many* $c \in k_X^l$*,* $f_1 - c, \ldots, f_n - c \in H_{X^l}^\times$.
(iii) $(k_X^l)^\times + H_{X^l}^\times = R_{X^l}$.
(iv) $R_{X^l} \cap \mathcal{O}_{E_{X^l}}^\times = R_{X^l}^\times = H_{X^l}^\times$.

Proof. (i) First, note that $k_X^l = (k_X^l)^\times \cup \{0\}$ is contained in $R_{X^l} = \langle H_{X^l}^\times \rangle$, as $(k_X^l)^\times \subset H_{X^l}^\times$. Thus, it suffices to prove that R_{X^l} is stable under multiplication. But this just follows from the fact that $H_{X^l}^\times$ (which is a multiplicative subgroup) is stable under multiplication.
(ii) We may and shall assume $n = 1$ and set $f \stackrel{\text{def}}{=} f_1$. (The assertion for general n can be reduced to this special case immediately.) As $f \in R_{X^l} = \langle H_{X^l}^\times \rangle$, f can be written as $f = g_1 + \cdots + g_m$ with $g_1, \ldots, g_m \in H_{X^l}^\times$. We shall prove the assertion for f by induction on m. The case $m = 0$ (i.e., $f = 0$) is easy: any $c \in k_X^l \setminus \{0\}$ satisfies the desired property. The case $m = 1$ (i.e., $f = g_1 \in H_{X^l}^\times$) follows immediately from Lemma 4.8 and Lemma 4.11. More precisely, by Lemma 4.8 (and the case

$m = 0$), for all but finitely many $c \in k_X^l$, one has $f - c \in \mathcal{O}_{E_{X^l}}^\times$ and $-c \in H_{X^l}^\times$, hence

$$f - c = \frac{f-c}{f} \cdot f = \left(\frac{-c}{f} + 1\right) f \in H_{X^l}^\times$$

by Lemma 4.11. Now, assume $m > 1$ and suppose that the assertion holds for $m - 1$. Then it follows from Lemma 4.8 and the induction hypothesis that, for all but finitely many $c \in k_X^l$, one has $f - c \in \mathcal{O}_{E_{X^l}}^\times$ and $(g_1 + \cdots + g_{m-1}) - c \in H_{X^l}^\times$. Now, as

$$f - c = \frac{f-c}{g_m} \cdot g_m = \left(\frac{(g_1 + \cdots + g_{m-1}) - c}{g_m} + 1\right) g_m,$$

one has $f - c \in H_{X^l}^\times$ by Lemma 4.11.
(iii) Clearly $(k_X^l)^\times + H_{X^l}^\times \subset R_{X^l}$. To show the opposite, take any $f \in R_{X^l}$. Then, by (ii), there exists $c \in (k_X^l)^\times$ such that $f - c \in H_{X^l}^\times$. Thus, $f = c + (f - c) \in (k_X^l)^\times + H_{X^l}^\times$, as desired.
(iv) Clearly $R_{X^l} \cap \mathcal{O}_{E_{X^l}}^\times \supset R_{X^l}^\times \supset H_{X^l}^\times$, hence it suffices to prove $R_{X^l} \cap \mathcal{O}_{E_{X^l}}^\times \subset H_{X^l}^\times$. So, take any $f \in R_{X^l} \cap \mathcal{O}_{E_{X^l}}^\times$. By (iii), there exist $c \in (k_X^l)^\times$ and $g \in H_{X^l}^\times$ such that $f = c + g$. As

$$f = \frac{f}{c} \cdot c = \left(1 + \frac{g}{c}\right) c,$$

one has $f \in H_{X^l}^\times$ by Lemma 4.11. □

Lemma 4.13. *$\mathrm{Fr}(\mathcal{O}_{E_{X^l}}) = \mathrm{Fr}(R_{X^l})$ and $\mathcal{O}_{E_{X^l}}$ is the normalization of R_{X^l}.*

Proof. Write $K_{X^l} = \mathrm{Fr}(\mathcal{O}_{E_{X^l}})$ $(= K_X k_X^l)$ and $N_{X^l} = \mathrm{Fr}(R_{X^l})$.

Step 1. $\mathcal{O}_{E_{X^l}}$ is the integral closure of R_{X^l} in K_{X^l}.

Indeed, as $\mathcal{O}_{E_{X^l}}$ is the intersection of discrete valuation rings $\mathcal{O}_{X^l,x}$ ($x \in E_{X^l}$), $\mathcal{O}_{E_{X^l}}$ is integrally closed. On the other hand, as $\mathcal{O}_{E_{X^l}}^\times / R_{X^l}^\times$ is torsion, each element of $\mathcal{O}_{E_{X^l}}^\times$ is integral over R_{X^l}. As $\mathcal{O}_{E_{X^l}} = \langle \mathcal{O}_{E_{X^l}}^\times \rangle$, $\mathcal{O}_{E_{X^l}}$ is integral over R_{X^l}, as desired.

Step 2. $\mathcal{O}_{E_{X^l}}^\times / (k_X^l)^\times$ and $R_{X^l}^\times / (k_X^l)^\times$ are free $\mathbb{Z}$-modules of countably infinite rank.

Indeed, each of these groups is injectively mapped into $\mathrm{Div}^0_{X^l \setminus E_{X^l}}$ with torsion cokernel. Now, the assertion follows from the fact that $\mathrm{Div}^0_{X^l \setminus E_{X^l}}$ itself is a free $\mathbb{Z}$-module of countably infinite rank. (Note that $(X^l)^{\mathrm{cl}} \setminus E_{X^l}$ is a countably infinite set.)

Step 3. K_{X^l}/N_{X^l} is finite.

Indeed, since $k_X^l \subset N_{X^l} \subset K_{X^l}$ and K_{X^l} is a (regular) function field of one variable over k_X^l, it suffices to prove that $N_{X^l} \supsetneq k_X^l$. But this follows, for example, from Step 1 or Step 2.

Step 4. K_{X^l}/N_{X^l} is separable.

Indeed, otherwise, $N_{X^l} \subset (K_{X^l})^p$, since K_{X^l}/N_{X^l} is a finite extension of (regular) function fields of one variable over a perfect field k_X^l. Then $R_{X^l} \subset \mathcal{O}_{E_{X^l}} \cap (K_{X^l})^p = (\mathcal{O}_{E_{X^l}})^p$ (where the last equality follows from the fact that $\mathcal{O}_{E_{X^l}}$ is normal), and $R_{X^l}^\times \subset ((\mathcal{O}_{E_{X^l}})^p)^\times = (\mathcal{O}_{E_{X^l}}^\times)^p$. Thus, one has $\mathcal{O}_{E_{X^l}}^\times / R_{X^l}^\times \twoheadrightarrow \mathcal{O}_{E_{X^l}}^\times / (\mathcal{O}_{E_{X^l}}^\times)^p$. But this is impossible, since the p-primary part of the torsion abelian group $\mathcal{O}_{E_{X^l}}^\times / R_{X^l}^\times$

is embeddable into a finite direct sum of $\mathbb{Q}_p/\mathbb{Z}_p$ (hence is isomorphic to a direct sum of a finite number of copies of $\mathbb{Q}_p/\mathbb{Z}_p$ and a p-primary finite abelian group), while $\mathcal{O}^\times_{E_{X^l}}/(\mathcal{O}^\times_{E_{X^l}})^p$ is an $\mathbb{F}_p$-vector space of countably infinite dimension by Step 2.

Step 5. $K^\times_{X^l}/N^\times_{X^l}$ has finite torsion.

Indeed, the homomorphism $N_{X^l} \to K_{X^l}$ of fields comes from a finite, generically étale k^l_X-morphism $X^l \to Z$ (where Z is a proper, smooth, geometrically connected curve over k^l_X with function field N_{X^l}) of degree $d \stackrel{\text{def}}{=} [K_{X^l} : N_{X^l}]$. Then

$$K^\times_{X^l}/N^\times_{X^l} = (K^\times_{X^l}/(k^l_X)^\times)/(N^\times_{X^l}/(k^l_X)^\times) = \mathrm{Pri}_{X^l} / \mathrm{Pri}_Z .$$

Considering the commutative diagram with two rows exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pri}_Z & \longrightarrow & \mathrm{Div}_Z & \longrightarrow & \mathrm{Pic}_Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pri}_{X^l} & \longrightarrow & \mathrm{Div}_{X^l} & \longrightarrow & \mathrm{Pic}_{X^l} & \longrightarrow & 0 \end{array}$$

in which the vertical arrows are induced by the pull-back of divisors by the morphism $X^l \to Z$, one obtains an exact sequence

$$0 \to \mathrm{Ker}(\mathrm{Pic}_Z \to \mathrm{Pic}_{X^l}) \to \mathrm{Pri}_{X^l} / \mathrm{Pri}_Z \to \mathrm{Div}_{X^l} / \mathrm{Div}_Z .$$

Now, on the one hand, by considering the norm map $\mathrm{Pic}_{X^l} \to \mathrm{Pic}_Z$, one sees that

$$\mathrm{Ker}(\mathrm{Pic}_Z \to \mathrm{Pic}_{X^l}) \subset \mathrm{Pic}_Z[d] = \mathrm{Pic}^0_Z[d] = J_Z(k^l_X)[d],$$

hence that $\mathrm{Ker}(\mathrm{Pic}_Z \to \mathrm{Pic}_{X^l})$ is finite. On the other hand, by considering the definition of $\mathrm{Div}_Z \to \mathrm{Div}_{X^l}$, one sees that the torsion of $\mathrm{Div}_{X^l} / \mathrm{Div}_Z$ (all of which arises from the finitely many ramified points of the generically étale morphism $X^l \to Z$) is finite. Thus, the assertion follows.

Step 6. Let $\tilde{R}_{X^l}$ denote the normalization of R_{X^l} in N_{X^l}. Then $(\mathcal{O}^\times_{E_{X^l}} : \tilde{R}^\times_{X^l}) < \infty$.

Indeed, as $\mathcal{O}^\times_{E_{X^l}}/\tilde{R}^\times_{X^l} \twoheadleftarrow \mathcal{O}^\times_{E_{X^l}}/R^\times_{X^l}$, $\mathcal{O}^\times_{E_{X^l}}/\tilde{R}^\times_{X^l}$ is torsion. On the other hand, as $\mathcal{O}_{E_{X^l}}$ is integral over $\tilde{R}_{X^l}$ and $\tilde{R}_{X^l}$ is integrally closed, one has $\mathcal{O}^\times_{E_{X^l}}/\tilde{R}^\times_{X^l} \hookrightarrow K^\times_{X^l}/N^\times_{X^l}$. Now, the assertion follows from Step 5.

Step 7. End of proof: $\mathcal{O}_{E_{X^l}} = \tilde{R}_{X^l}$ and $K_{X^l} = N_{X^l}$.

Indeed, by Step 6, we may write $\mathcal{O}^\times_{E_{X^l}} = \tilde{R}^\times_{X^l} f_1 \cup \cdots \cup \tilde{R}^\times_{X^l} f_r$ for some $f_1, \ldots, f_r \in \mathcal{O}^\times_{E_{X^l}}$. This, together with Corollary 4.10(i), implies that $\mathcal{O}_{E_{X^l}} = (\tilde{R}_{X^l} f_1 \cup \cdots \cup \tilde{R}_{X^l} f_r) + k^l_X$, and that $\mathcal{O}_{E_{X^l}}/k^l_X = \overline{\tilde{R}_{X^l} f_1} \cup \cdots \cup \overline{\tilde{R}_{X^l} f_r}$ as a union of k^l_X-vector subspaces (where $\overline{\tilde{R}_{X^l} f_i}$ denotes the image of $\tilde{R}_{X^l} f_i$ in $\mathcal{O}_{E_{X^l}}/k^l_X$). As k^l_X is an infinite field, we must have $\mathcal{O}_{E_{X^l}}/k^l_X = \overline{\tilde{R}_{X^l} f_i}$ for some i, hence $\mathcal{O}_{E_{X^l}} = k^l_X + \tilde{R}_{X^l} f_i$.

We claim that $f \stackrel{\text{def}}{=} f_i \in N_{X^l}$. Indeed, otherwise, $1, f \in K_{X^l}$ are linearly independent over N_{X^l}. Namely, $N_{X^l} \oplus N_{X^l} f \hookrightarrow K_{X^l}$, hence, in particular, $\tilde{R}_{X^l} \oplus \tilde{R}_{X^l} f \hookrightarrow \mathcal{O}_{E_{X^l}}$. As $k^l_X \subsetneq R_{X^l} \subset \tilde{R}_{X^l}$, one has $k^l_X \oplus \tilde{R}_{X^l} f \subsetneq \tilde{R}_{X^l} \oplus \tilde{R}_{X^l} f$. This implies that $k^l_X + \tilde{R}_{X^l} f \subsetneq R_{X^l} + \tilde{R}_{X^l} f \subset \mathcal{O}_{E_{X^l}}$, which is absurd.

Now, $f \in N_{X^l} \cap \mathcal{O}_{X^l}^\times = \tilde{R}_{X^l}^\times$, where the last equality follows from Step 1, together with the definition of $\tilde{R}_{X^l}$. Thus, $\mathcal{O}_{E_{X^l}} = k_X^l + \tilde{R}_{X^l} f = k_X^l + \tilde{R}_{X^l} = \tilde{R}_{X^l}$, as desired. In particular, $K_{X^l} = \mathrm{Fr}(\mathcal{O}_{E_{X^l}}) = \mathrm{Fr}(\tilde{R}_{X^l}) = N_{X^l}$. $\square$

Next, we will denote by $\mathbb{P}(R_{X^l}) \overset{\text{def}}{=} (R_{X^l} \setminus \{0\})/(k_X^l)^\times$ (resp. $\mathbb{P}(R_{Y^l}) \overset{\text{def}}{=} (R_{Y^l} \setminus \{0\})/(k_Y^l)^\times$) the projective space associated to the infinite-dimensional k_X^l-vector space R_{X^l} (resp. the k_Y^l-vector space R_{Y^l}) and by $\mathbb{L}(R_{X^l})$ ($\subset 2^{\mathbb{P}(R_{X^l})}$) (resp. $\mathbb{L}(R_{Y^l})$ ($\subset 2^{\mathbb{P}(R_{Y^l})}$)) the set of lines on $\mathbb{P}(R_{X^l})$ (resp. $\mathbb{P}(R_{Y^l})$). We view $\mathcal{U}_{X^l} \overset{\text{def}}{=} \overline{H}_{X^l} = R_{X^l}^\times/(k_X^l)^\times$ and $\mathcal{U}_{Y^l} \overset{\text{def}}{=} \overline{H}_{Y^l} = R_{Y^l}^\times/(k_Y^l)^\times$ (cf. Lemma 4.12(iv)) as subsets of the projective spaces $\mathbb{P}(R_{X^l})$ and $\mathbb{P}(R_{Y^l})$, respectively. Let F_X/k_X and F_Y/k_Y be the extensions in Definition 2.9. We define the following sets of subsets of $\mathbb{P}^1(k_X^l)$ and $\mathbb{P}^1(k_Y^l)$:

$$\mathcal{S}_{X^l} \overset{\text{def}}{=} \{S \subset \mathbb{P}^1(k_X^l) \mid \sharp(S) < \infty\},\ \mathcal{S}_{Y^l} \overset{\text{def}}{=} \{S \subset \mathbb{P}^1(k_Y^l) \mid \sharp(S) < \infty\}.$$

Lemma 4.14. *The sets $\mathcal{S}_{X^l}$ and $\mathcal{S}_{Y^l}$ are admissible (cf. Definition 5.4(ii) for the meaning of being admissible).*

Proof. This follows from the fact that k_X^l and k_Y^l are infinite. See Remark 5.5. $\square$

Lemma 4.15. *The subset $\mathcal{U}_{X^l} \subset \mathbb{P}(R_{X^l})$ is $\mathcal{S}_{X^l}$-ample, and the subset $\mathcal{U}_{Y^l} \subset \mathbb{P}(R_{Y^l})$ is $\mathcal{S}_{Y^l}$-ample (cf. Definition 5.6(iii) for the meaning of being $\mathcal{S}_{X^l}$-ample and $\mathcal{S}_{Y^l}$-ample).*

Proof. We prove that $\mathcal{U}_{X^l}$ is $\mathcal{S}_{X^l}$-ample. The proof that $\mathcal{U}_{Y^l}$ is $\mathcal{S}_{Y^l}$-ample is similar. We have to prove the following equality $\mathbb{L}(R_{X^l})_{\mathcal{U}_{X^l}} = \mathbb{L}(R_{X^l})_{\mathcal{U}_{X^l}, \mathcal{S}_{X^l}}$ (cf. Definition 5.6(iii)(2)). More precisely, let $\ell \in \mathbb{L}(R_{X^l})$ be a line in the projective space $\mathbb{P}(R_{X^l})$ such that $\ell_{\mathcal{U}_{X^l}} \overset{\text{def}}{=} \ell \cap \mathcal{U}_{X^l} \neq \varnothing$, then we have to prove that $\ell \setminus \ell_{\mathcal{U}_{X^l}} \in \mathcal{S}_{X^l}$, i.e., $\sharp(\ell \setminus \ell_{\mathcal{U}_{X^l}}) < \infty$. We have $\ell = \mathbb{P}(V)$, where V is a 2-dimensional k_X^l-vector subspace of R_{X^l}. As $\ell_{\mathcal{U}_{X^l}} \neq \varnothing$ we can take $f \in V \cap R_{X^l}^\times$. Further, taking any $g \in V \setminus k_X^l f$, we may write $V = \{af + bg \mid a, b \in k_X^l\}$. Then $\ell = \{\overline{f}\} \cup \{\overline{cf+g},\ c \in k_X^l\} = \{\overline{f}\} \cup \{\overline{(h-c)f},\ c \in k_X^l\}$, where $h \overset{\text{def}}{=} \frac{g}{f} \in R_{X^l}$. Now, by Lemma 4.12(ii), $h - c \in R_{X^l}^\times$ (hence $\overline{(h-c)f} \in \mathcal{U}_{X^l}$) for all but finitely many $c \in k_X^l$, as desired. $\square$

Lemma 4.16. *The natural isomorphism $\bar{\rho} : \mathcal{U}_{Y^l} \xrightarrow{\sim} \mathcal{U}_{X^l}$ induces a natural bijection $\bar{\tau} : \mathbb{L}(R_{Y^l})_{\mathcal{U}_{Y^l}} \xrightarrow{\sim} \mathbb{L}(R_{X^l})_{\mathcal{U}_{X^l}}$ with the following property: If $\ell \in \mathbb{L}(R_{Y^l})_{\mathcal{U}_{Y^l}}$ then $\bar{\tau}(\ell)_{\mathcal{U}_{X^l}} = \bar{\rho}(\ell_{\mathcal{U}_{Y^l}})$, where $\bar{\tau}(\ell)_{\mathcal{U}_{X^l}} \overset{\text{def}}{=} \bar{\tau}(\ell) \cap \mathcal{U}_{X^l}$ and $\ell_{\mathcal{U}_{Y^l}} \overset{\text{def}}{=} \ell \cap \mathcal{U}_{Y^l}$.*

Proof. We will define the map $\bar{\tau}$. Let $\ell \in \mathbb{L}(R_{Y^l})_{\mathcal{U}_{Y^l}}$. Then by Lemma 4.15, $\ell \setminus \ell_{\mathcal{U}_{Y^l}}$ is finite, hence $\ell_{\mathcal{U}_{Y^l}}$ is infinite. Take two distinct points $\overline{f}_1, \overline{g}_1 \in \ell_{\mathcal{U}_{Y^l}}$, and take any liftings $f_1, g_1 \in R_{Y^l}^\times$ of $\overline{f}_1, \overline{g}_1$, respectively. Set $V_1 = \{af_1 + bg_1 \mid a, b \in k_Y^l\}$, so that V_1 is a 2-dimensional k_Y^l-vector subspace of R_{Y^l} and that $\ell = \mathbb{P}(V_1) \subset \mathbb{P}(R_{Y^l})$. Then we have $\ell = \{\overline{g}_1\} \cup \{\overline{(1+ch_1)f_1} \mid c \in k_Y^l\}$, where $h_1 \overset{\text{def}}{=} \frac{g_1}{f_1} \in R_{Y^l}^\times$, and $\ell_{\mathcal{U}_{Y^l}} = \{\overline{g}_1\} \cup \{\overline{(1+ch_1)f_1} \mid c \in k_Y^l, (1+ch_1) \in R_{Y^l}^\times\}$. Take any $c \in (k_Y^l)^\times$ such that $(1 + ch_1) \in R_{Y^l}^\times$. Set $f_2' \overset{\text{def}}{=} \rho'(f_1'), g_2' \overset{\text{def}}{=} \rho'(g_1')$, where $\rho' : H_{Y^l}' = R_{Y^l}^\times/((k_Y^l)^\times\{\Sigma'\}) \xrightarrow{\sim} H_{X^l}' = R_{X^l}^\times/((k_X^l)^\times\{\Sigma'\})$ is the natural isomorphism (cf. discussion before Lemma 4.11), and take any liftings $f_2, g_2 \in R_{X^l}^\times$ of f_2', g_2', respectively. Set $h_2 \overset{\text{def}}{=} \frac{g_2}{f_2} \in R_{X^l}^\times$, so that $h_2' = \frac{g_2'}{f_2'} = \rho'(h_1')$. Further set $V_2 = \{af_2 + bg_2 \mid a, b \in k_X^l\}$. Since $\bar{\rho}$ is bijective, one has $\overline{f}_2 \neq \overline{g}_2$, hence V_2 is a 2-dimensional k_X^l-vector subspace of R_{X^l}. Set $\bar{\tau}(\ell) \overset{\text{def}}{=} \ell' \overset{\text{def}}{=} \mathbb{P}(V_2) \subset \mathbb{P}(R_{X^l})$. As $\overline{f}_2 \in \ell'$, one has $\ell' \in \mathbb{L}(R_{X^l})_{\mathcal{U}_{X^l}}$.

By evaluating the pseudo-function $\rho'((1+ch_1)')$ at all points $x \in (X^l)^{\text{cl}}$ above $X^{\text{cl}} \setminus X(F_X)^{\text{cl}}$, and applying Lemma 4.5, Lemma 4.6, and Proposition 2.3 to suitable finite subextensions of k_X^l/k_X and k_Y^l/k_Y, we deduce that $\rho'((1+ch_1)') = (\alpha + \beta h_2)' \in R_{X^l}^\times/(k_{X^l}^\times)\{\Sigma'\}$ for some $\alpha \in (k_X^l)^\times\{\Sigma'\}$ and $\beta \in (k_X^l)^\times$, hence $\bar{\rho}(\overline{1+ch_1}) = \overline{1+\gamma h_2}$ for $\gamma = \frac{\beta}{\alpha} \in (k_X^l)^\times$. This, together with $\bar{\rho}(\overline{f}_1) = \overline{f}_2, \bar{\rho}(\overline{g}_1) = \overline{g}_2$, implies that $\bar{\rho}(\ell_{\mathcal{U}_{Y^l}}) \subset \bar{\tau}(\ell)_{\mathcal{U}_{X^l}}$. Further, $\bar{\tau}$ is bijective since it has an inverse $\bar{\tau}' : \mathbb{L}(R_{X^l})_{\mathcal{U}_{X^l}} \xrightarrow{\sim} \mathbb{L}(R_{Y^l})_{\mathcal{U}_{Y^l}}$ which is naturally deduced from $\alpha^{-1} : \Pi_Y \xrightarrow{\sim} \Pi_X$. By using $\bar{\tau}'$, we also conclude that $\bar{\rho}(\ell_{\mathcal{U}_{Y^l}}) = \ell'_{\mathcal{U}_{X^l}}$, as desired. $\square$

Lemma 4.17. *(Recovering the Additive Structure of R_{X^l} and R_{Y^l}) The following hold.*
(i) *There exist natural isomorphisms $\tilde{\rho} : \mathbb{P}(R_{Y^l}) \xrightarrow{\sim} \mathbb{P}(R_{X^l})$ and $\tilde{\tau} : \mathbb{L}(R_{Y^l}) \xrightarrow{\sim} \mathbb{L}(R_{X^l})$ which extend the isomorphisms $\bar{\rho} : \mathcal{U}_{Y^l} \xrightarrow{\sim} \mathcal{U}_{X^l}$ and $\bar{\tau} : \mathbb{L}(R_{Y^l})_{\mathcal{U}_{Y^l}} \xrightarrow{\sim} \mathbb{L}(R_{X^l})_{\mathcal{U}_{X^l}}$, respectively, such that for every $\ell \in \mathbb{L}(R_{Y^l})$, we have $\tilde{\tau}(\ell) = \tilde{\rho}(\ell)$ set-theoretically.*
(ii) *The bijection $\tilde{\rho}$ arises from a ψ_0-isomorphism*

$$\psi : (R_{Y^l}, +) \xrightarrow{\sim} (R_{X^l}, +),$$

where $\psi_0 : k_Y^l \xrightarrow{\sim} k_X^l$ is a field isomorphism. Namely, ψ is an isomorphism of abelian groups which is compatible with ψ_0 in the sense that $\psi(ax) = \psi_0(a)\psi(x)$ for $a \in k_Y^l$ and $x \in R_{Y^l}$. Further, ψ_0 is uniquely determined and ψ is uniquely determined up to scalar multiplication.

Proof. Assertion (i) follows formally from Lemma 4.16 and Theorem 5.7. Assertion (ii) follows from Theorem 5.7. $\square$

Lemma 4.18. *(Recovering the Ring Structure of R_{X^l} and R_{Y^l}) The following hold.*
(i) *If we normalize the isomorphism*

$$\psi : (R_{Y^l}, +) \xrightarrow{\sim} (R_{X^l}, +),$$

in Lemma 4.17 by the condition $\psi(1) = 1$ (which is possible as $\bar{\rho}(\bar{1}) = \bar{1}$), it becomes a ring isomorphism such that the diagram

$$\begin{array}{ccc} R_{X^l} & \xleftarrow{\psi} & R_{Y^l} \\ \uparrow & & \uparrow \\ k_X^l & \xleftarrow{\psi_0} & k_Y^l \end{array}$$

commutes.
(ii) *ψ induces a natural commutative diagram*

$$\begin{array}{ccc} K_{X^l} & \xleftarrow{\psi} & K_{Y^l} \\ \uparrow & & \uparrow \\ K_X & \xleftarrow{\psi} & K_Y \end{array}$$

where the horizontal maps are field isomorphisms and the vertical maps are natural inclusions. Further, $\psi : K_{Y^l} \xrightarrow{\sim} K_{X^l}$ *is Galois-equivariant with respect to the isomorphism* $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ *induced by* $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ *(cf. Proposition 1.7(i)).*
(iii) *ψ induces a natural commutative diagram*

$$\begin{array}{ccc} X^l & \xrightarrow{\psi} & Y^l \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi} & Y \end{array}$$

where the horizontal maps are scheme isomorphisms and the vertical maps are natural projections. Further, $\psi : X^l \xrightarrow{\sim} Y^l$ *is Galois-equivariant with respect to the isomorphism* $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ *induced by* $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ *(cf. Proposition 1.7(i)).*

Proof. (i) The proof is similar to the proof of [Saïdi-Tamagawa3], Lemma 4.14. More precisely, let $f \in R_{Y^l}^\times$, then $\psi \circ \mu_f$ and $\mu_{\psi(f)} \circ \psi$ are ψ_0-isomorphisms $(R_{Y^l}, +) \xrightarrow{\sim} (R_{X^l}, +)$, where μ_g denotes the g-multiplication map. Note that $\psi(f) \in R_{X^l}^\times$, since $\overline{\psi(f)} = \bar{\rho}(\overline{f}) \in R_{X^l}^\times/(k_X^l)^\times \subset K_{X^l}^\times/(k_X^l)^\times$. The isomorphisms $R_{Y^l}^\times/(k_Y^l)^\times \xrightarrow{\sim} R_{X^l}^\times/(k_X^l)^\times$ they induce coincide with each other:

$$\overline{\psi \circ \mu_f} = \bar{\rho} \circ \mu_{\bar{f}} = \mu_{\bar{\rho}(\overline{f})} \circ \bar{\rho} = \overline{\mu_{\psi(f)} \circ \psi},$$

where the second equality follows from the multiplicativity of $\bar{\rho}$. Further, we have

$$\psi \circ \mu_f(1) = \psi(f) = \mu_{\psi(f)}(1) = \mu_{\psi(f)} \circ \psi(1).$$

Thus, the equality $\psi \circ \mu_f = \mu_{\psi(f)} \circ \psi$ follows from the uniqueness in Theorem 5.7. This equality means that $\psi(fg) = \psi(f)\psi(g)$ holds for $f \in R_{Y^l}^\times$ and $g \in R_{Y^l}$, which, together with the fact that $R_{Y^l} = \langle R_{Y^l}^\times \rangle$, implies the multiplicativity of ψ.
(ii) By considering fields of fractions, we see that $\psi : R_{Y^l} \xrightarrow{\sim} R_{X^l}$ naturally induces $\psi : K_{Y^l} \xrightarrow{\sim} K_{X^l}$. Since the isomorphism $\bar{\rho} : \mathcal{U}_{Y^l} \xrightarrow{\sim} \mathcal{U}_{X^l}$ is Galois-equivariant under the compatible actions of G_{k_X} and G_{k_Y} respectively, and since the isomorphism $\psi : R_{Y^l} \xrightarrow{\sim} R_{X^l}$, $1 \mapsto 1$ is uniquely determined by $\bar{\rho}$, it follows that $\psi : R_{Y^l} \xrightarrow{\sim} R_{X^l}$ is Galois-equivariant under the compatible actions of G_{k_X} and G_{k_Y}, respectively, hence so is $\psi : K_{Y^l} \xrightarrow{\sim} K_{X^l}$. Now, the fact that the field isomorphism $\psi : K_{Y^l} \xrightarrow{\sim} K_{X^l}$ maps K_Y isomorphically to K_X follows from this.
(iii) This follows immediately from (ii). □

Next, let $X' \to X$ be a finite étale covering corresponding to an open subgroup $\Pi_{X'} \subset \Pi_X$ and $Y' \to Y$ the corresponding étale covering via α (i.e., $\Pi_{Y'} \stackrel{\text{def}}{=} \alpha(\Pi_{X'})$). (We refer to the case where $X' \to X$ is Galois (or, equivalently, $Y' \to Y$ is Galois) as a Galois case.) Then α induces, by restriction to $\Pi_{X'}$, an isomorphism

$$\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}.$$

To apply the preceding arguments to this isomorphism, we need to show that the assumptions of Theorem 4.4 hold.

Lemma 4.19. (i) Σ *is* $k_{X'}$*-large and* $k_{Y'}$*-large.*
(ii) X' *and* Y' *are* Σ*-separated. More precisely,* $F_{X'}$ *and* $F_{Y'}$ *are finite extensions of* F_X *and* F_Y*, respectively. In particular, a prime number is* (X, Σ)*-admissible (resp.* (Y, Σ)*-admissible) if and only if it is* (X', Σ)*-admissible (resp.* (Y', Σ)*-admissible).*
(iii) $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$ *is pseudo-constants-additive.*

Proof. (i) This follows from the fact that $k_{X'}$ and $k_{Y'}$ are finite extensions of k_X and k_Y, respectively.
(ii) This follows from Proposition 2.11(i).
(iii) By Theorem 4.2(iii), the isomorphism $\tau : (\bar{k}_{Y'}^{\times})^{\Sigma} \xrightarrow{\sim} (\bar{k}_{X'}^{\times})^{\Sigma}$ induced by $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$ just coincides with the isomorphism $\tau : (\bar{k}_{Y}^{\times})^{\Sigma} \xrightarrow{\sim} (\bar{k}_{X}^{\times})^{\Sigma}$ induced by $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$. Thus, the pseudo-constants-additivity of $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$ follows from that of $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$. $\square$

Now, we may apply Lemma 4.18 to $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$ to obtain: (i) a ring isomorphism $\psi' : R_{(Y')^l} \xrightarrow{\sim} R_{(X')^l}$ compatible with a field isomorphism $\psi'_0 : k_{Y'}^l \xrightarrow{\sim} k_{X'}^l$; (ii) a field isomorphism $\psi' : K_{(Y')^l} \xrightarrow{\sim} K_{(X')^l}$ compatible with a field isomorphism $\psi' : K_{Y'} \xrightarrow{\sim} K_{X'}$ and Galois-equivariant with respect to the isomorphism $G_{k_{X'}} \xrightarrow{\sim} G_{k_{Y'}}$ induced by $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$; and (iii) a scheme isomorphism $\psi' : (X')^l \xrightarrow{\sim} (Y')^l$ compatible with a scheme isomorphism $\psi' : X' \xrightarrow{\sim} Y'$ and Galois-equivariant with respect to the isomorphism $G_{k_{X'}} \xrightarrow{\sim} G_{k_{Y'}}$ induced by $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$. Further, in the Galois case, these isomorphisms are Galois-equivariant with respect to the isomorphism $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$. Indeed, this Galois-equivariance can be proved just similarly as in the proof of Lemma 4.18(ii).

Lemma 4.20. (i) *The following diagram of rings is commutative:*

$$\begin{array}{ccc} R_{(X')^l} & \xleftarrow{\psi'} & R_{(Y')^l} \\ \uparrow & & \uparrow \\ R_{X^l} & \xleftarrow{\psi} & R_{Y^l} \end{array}$$

In particular, the following diagram of fields is commutative:

$$\begin{array}{ccc} k_{X'}^l & \xleftarrow{\psi'_0} & k_{Y'}^l \\ \uparrow & & \uparrow \\ k_X^l & \xleftarrow{\psi_0} & k_Y^l \end{array}$$

(ii) *The following diagram of fields is commutative:*

$$\begin{array}{ccc} K_{(X')^l} & \xleftarrow{\psi'} & K_{(Y')^l} \\ \uparrow & & \uparrow \\ K_{X^l} & \xleftarrow{\psi} & K_{Y^l} \end{array}$$

In particular, the following diagram of fields is commutative:

$$\begin{array}{ccc} K_{X'} & \xleftarrow{\psi'} & K_{Y'} \\ \uparrow & & \uparrow \\ K_X & \xleftarrow{\psi} & K_Y \end{array}$$

(iii) *The following diagram of schemes is commutative:*

$$\begin{array}{ccc} (X')^l & \xrightarrow{\psi'} & (Y')^l \\ \downarrow & & \downarrow \\ X^l & \xrightarrow{\psi} & Y^l \end{array}$$

In particular, the following diagram of schemes is commutative:

$$\begin{array}{ccc} X' & \xrightarrow{\psi'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi} & Y \end{array}$$

Proof. (i) To prove the commutativity of the first diagram, we may and shall assume that we are in the Galois case. Let ι_X and ι_Y denote the natural inclusions $R_{X^l} \to R_{(X')^l}$ and $R_{Y^l} \to R_{(Y')^l}$, respectively. Applying Theorem 4.2(iii) to various finite covers of X and Y, we see that the diagram in question induces a commutative diagram

$$\begin{array}{ccc} R^{\times}_{(X')^l}/(k^l_{X'})^{\times} & \longleftarrow & R^{\times}_{(Y')^l}/(k^l_{Y'})^{\times} \\ \uparrow & & \uparrow \\ R^{\times}_{X^l}/(k^l_X)^{\times} & \longleftarrow & R^{\times}_{Y^l}/(k^l_Y)^{\times} \end{array}$$

In particular, we have

$$\psi' \circ \iota_Y(R^{\times}_{Y^l}) \cdot (k^l_{X'})^{\times} = \iota_X \circ \psi(R^{\times}_{Y^l}) \cdot (k^l_{X'})^{\times}.$$

Consider the Π_{X^l}-fixed parts of both sides of this equality. Then, since $\psi' : R_{(Y')^l} \xrightarrow{\sim} R_{(X')^l}$ is Galois-equivariant with respect to $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ and since $\alpha(\Pi_{X^l}) = \Pi_{Y^l}$, we obtain

$$\psi' \circ \iota_Y(R^{\times}_{Y^l}) = \iota_X \circ \psi(R^{\times}_{Y^l}),$$

hence

$$R \stackrel{\mathrm{def}}{=} \psi' \circ \iota_Y(R_{Y^l}) = \langle \psi' \circ \iota_Y(R^{\times}_{Y^l}) \rangle = \langle \iota_X \circ \psi(R^{\times}_{Y^l}) \rangle = \iota_X \circ \psi(R_{Y^l}).$$

Accordingly, each of $\psi' \circ \iota_Y$ and $\iota_X \circ \psi$ induces an isomorphism $R_{Y^l} \xrightarrow{\sim} R$ that maps 1 to 1. Now, the desired equality $\psi' \circ \iota_X = \iota_Y \circ \psi$ follows from the uniqueness assertion in Theorem 5.7.

The commutativity of the second diagram follows from that of the first diagram.
(ii) The commutativity of the first diagram follows from that of the first diagram in (i). The commutativity of the second diagram follows from that of the first diagram.
(iii) This follows immediate from (ii). □

Corollary 4.21. *The isomorphism α induces a natural isomorphism $\tilde{X} \xrightarrow{\sim} \tilde{Y}$ which fits into a commutative diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sim} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

where the vertical maps are the pro-étale coverings corresponding to Π_X and Π_Y, respectively.

Proof. This follows from Lemma 4.20 by passing to open subgroups of Π_X and Π_Y which correspond to each other via α, together with the Galois-equivariance of various isomorphisms in the Galois case. □

This finishes the proof of Theorem 4.4. □

Theorem 4.22. *(A Refined Version of the Grothendieck Conjecture for Proper Hyperbolic Curves over Finite Fields) Let X, Y be proper hyperbolic curves over finite fields k_X, k_Y of characteristic p_X, p_Y, respectively. Let $\Sigma_X, \Sigma_Y \subset \mathfrak{Primes}$ be sets of prime numbers containing at least one prime number different from p_X, p_Y, respectively, and set $\Sigma'_X \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma_X$, $\Sigma'_Y \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma_Y$. Assume that Σ_X is J_X-large and that Σ_Y is J_Y-large. Write Π_X, Π_Y for the geometrically pro-Σ_X étale fundamental group of X and the geometrically pro-Σ_Y étale fundamental group of Y, respectively. Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sim} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups Π_X, Π_Y, respectively.

Proof. By Proposition 1.7, we have $\Sigma \stackrel{\text{def}}{=} \Sigma_X = \Sigma_Y$. By Lemma 2.7, Σ is k_X-large and k_Y-large. By Proposition 2.5(i), we have $p \stackrel{\text{def}}{=} p_X = p_Y$. By Proposition 2.12, X and Y are almost Σ-separated. Now, by Theorem 4.4, it suffices to prove that α is pseudo-constants-additive.

Let the notations be as in the proof of Theorem 4.4. To prove that α is pseudo-constants-additive, set

$$\tilde{T}_{X^l} \stackrel{\text{def}}{=} \{f \in \mathcal{O}_{E_{X^l}} \mid \text{At least one pole of } f \text{ is } k_X^l\text{-rational.}\},$$

$$\tilde{T}^\times_{X^l} \stackrel{\text{def}}{=} \{f \in \mathcal{O}^\times_{E_{X^l}} \mid f, f^{-1} \in \tilde{T}_{X^l}\},$$

and define $T^\times_{X^l}$ to be the intersection of $\tilde{T}_{X^l}$ with $H^\times_{X^l}$. Further, set $\overline{T^\times_{X^l}} \stackrel{\text{def}}{=} T^\times_{X^l}/(k^l_X)^\times$. We define $\tilde{T}_{Y^l}$, $\tilde{T}^\times_{Y^l}$, $T^\times_{Y^l}$ and $\overline{T^\times_{Y^l}}$ similarly. Since the divisors of functions are preserved under $\bar{\rho}$ (cf. Lemma 4.5(i)) and since the degrees of extensions of residue fields are preserved under $\phi : X^{\mathrm{cl}} \setminus E_X \xrightarrow{\sim} Y^{\mathrm{cl}} \setminus E_Y$ (cf. discussion before Corollary 3.3), we have $\bar{\rho}(\overline{T_{Y^l}^\times}) = \overline{T_{X^l}^\times}$.

Observe that $\mathrm{PGL}_2(k_X^l)$ acts on $K_{X^l} \backslash k_X^l = \mathbb{P}^1(K_{X^l}) \backslash \mathbb{P}^1(k_X^l)$ via linear fractional transformation:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod (k_X^l)^\times \in \mathrm{PGL}_2(k_X^l),\ h \in K_{X^l} \setminus k_X^l \implies A \cdot h \stackrel{\mathrm{def}}{=} \frac{ah+b}{ch+d}.$$

Claim: For any $h \in K_{X^l} \setminus k_X^l$, there exists $A \in \mathrm{PGL}_2(k_X^l)$ such that $f \stackrel{\mathrm{def}}{=} A \cdot h$ satisfies $f, f+1 \in H_{X^l}^\times$. (In particular, $T_{X^l}^\times \neq \varnothing$.) A similar statement holds for Y.

Indeed, for simplicity, set $P_X \stackrel{\mathrm{def}}{=} \mathrm{PGL}_2(k_X^l)$ and write $P_X h$ for the P_X-orbit of h. Take a finite subextension k_0/k_X of k_X^l/k_X such that $h \in K_X k_0$. Then h can be regarded as a finite k_0-morphism $X_{k_0} \to \mathbb{P}^1_{k_0}$. On the one hand, by the Weil estimate, we have $\sharp(X_{k_0}(k_0^l)^{\mathrm{cl}}) = \infty$. On the other hand, by Proposition 2.13, $\sharp(h(X_{k_0}(k_0^l)^{\mathrm{cl}}) \cap h(E_X \times_{k_X} k_0)) < \infty$. Thus, there exist infinitely many points $x \in X(k_X^l)$ whose image under h is not contained in $h(E_X \times_{k_X} k_0) \cup \{\infty\}$. Take such an x and set $d \stackrel{\mathrm{def}}{=} h(x) \in k_X^l = \mathbb{P}^1(k_X^l) \setminus \{\infty\}$. Then the set of zeros of $h - d \in K_X^l$ intersects trivially with E_{X^l} and includes at least one k_X^l-rational point. Thus, $h_1 \stackrel{\mathrm{def}}{=} \frac{1}{h-d} \in \tilde{T}_{X^l} \cap P_X h$. Next, it follows from the above argument again (applied to h_1 instead of h) that for all but finitely many $c \in k_X^l$, the set of zeros of $h_1 - c \in K_X^l$ intersects trivially with E_{X^l} and includes at least one k_X^l-rational point, hence $h_1 - c \in \tilde{T}_{X^l}^\times \cap P_X h$. Further, by Remark 4.7, $(\mathcal{O}_{E_{X^l}}^\times : H_{X^l}^\times) < \infty$, hence there exists an infinite subset C of k_X^l such that for all $c \in C$, $h_1 - c \in \tilde{T}_{X^l}^\times \cap P_X h$ and $h_1 - c$ belongs to the same coset of $\mathcal{O}_{E_{X^l}}^\times / H_{X^l}^\times$. Now, take mutually distinct three elements $a, b, c \in C$. Then

$$f \stackrel{\mathrm{def}}{=} \frac{b-c}{a-b} \cdot \frac{h_1 - a}{h_1 - c} \in \tilde{T}_{X^l}^\times \cap H_{X^l}^\times \cap P_X h \subset T_{X^l}^\times \cap P_X h$$

and

$$f + 1 \stackrel{\mathrm{def}}{=} \frac{a-c}{a-b} \cdot \frac{h_1 - b}{h_1 - c} \in \tilde{T}_{X^l}^\times \cap H_{X^l}^\times \cap P_X h \subset T_{X^l}^\times \cap P_X h,$$

as desired.

As the degree of a function depends only on its P_X-orbit, the above claim particularly implies that the set of degrees of functions in $K_{X^l} \setminus k_X^l$ coincides with that of $H_{X^l}^\times$: $\deg(K_{X^l} \setminus k_X^l) = \deg(H_{X^l}^\times)$, and, similarly, that the set of degrees of functions in $K_{Y^l} \setminus k_Y^l$ coincides with that of $H_{Y^l}^\times$: $\deg(K_{Y^l} \setminus k_Y^l) = \deg(H_{Y^l}^\times)$. In particular, it follows from Lemma 4.5(i) that the gonality γ_{X^l} of X^l coincides with the gonality γ_{Y^l} of Y^l.

Now, take any $h \in K_{Y^l} \setminus k_Y^l$ attaining the gonality: $\deg(h) = \gamma_{Y^l}$. Then, by the above claim, there exists $f \in P_Y h$ (where $P_Y \stackrel{\mathrm{def}}{=} \mathrm{PGL}_2(k_Y^l)$) such that $f, f+1 \in H_{Y^l}^\times$. As $f \in P_Y h$, we have $\deg(f) = \deg(h) = \gamma_{Y^l}$.

Set $g' \stackrel{\mathrm{def}}{=} \rho'(f') \in H'_{X^l}$, $g'_1 \stackrel{\mathrm{def}}{=} \rho'((f+1)') \in H'_{X^l}$ and take any lifts $g, g_1 \in H_{X^l}^\times$ of g', g'_1, respectively. Then, by Lemma 4.5(i), $\deg(g) = \deg(f) = \gamma_{Y^l} = \gamma_{X^l}$.

As the pole divisors of f and $f+1$ coincide, the pole divisors of g and g_1 coincide by Lemma 4.5(i). Also, as f admits at least one k_Y^l-rational pole, g admits at least one k_X^l-rational pole, say, x. Now, by considering the leading terms of Laurent expansions of g and g_1 at the k_X^l-rational pole x, we see that there exists $\beta \in (k_X^l)^\times$, such that the pole divisor D_β of $g_1 - \beta g$ is strictly smaller than the pole divisor D of g. (That is to say, the divisor $D - D_\beta$ is effective and non-zero.) As $\deg(g) = \gamma_{X^l}$, this implies that $g_1 - \beta g$ is constant, hence we may write $g_1 = \alpha + \beta g$ with $\alpha \in k_X^l$. By evaluating this equation at $\phi^{-1}(y_1)$, where $y_1 \in (Y^l)^{\mathrm{cl}}$ is a zero of f, we see $\alpha' = 1'$, i.e., $\alpha \in (k_X^l)^\times\{\Sigma'\}$. Similarly, by evaluating the equation $\frac{g_1}{g} = \frac{\alpha}{g} + \beta$ at $\phi^{-1}(y_2)$, where $y_2 \in (Y^l)^{\mathrm{cl}}$ is a pole of g, we see $\beta' = 1'$, i.e., $\beta \in (k_X^l)^\times\{\Sigma'\}$. Now, the proof of the assertion follows just similarly to the last paragraph of the proof of [Saïdi-Tamagawa3], Lemma 4.9. $\square$

As a consequence of Theorem 4.22 we can deduce the following refined version of the Grothendieck conjecture for (not necessarily proper) hyperbolic curves over finite fields.

Theorem 4.23. *(A Refined Version of the Grothendieck Conjecture for (Not Necessarily Proper) Hyperbolic Curves over Finite Fields) Let U, V be (not necessarily proper) hyperbolic curves over finite fields k_U, k_V of characteristic p_U, p_V, respectively. Let $\Sigma_U, \Sigma_V \subset \mathfrak{Primes}$ be sets of prime numbers and set $\Sigma'_U \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma_U$, $\Sigma'_V \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma_V$. Write Π_U, Π_V, for the geometrically pro-Σ_U tame fundamental group of U and the geometrically pro-Σ_V tame fundamental group of V, respectively. Let*

$$\alpha : \Pi_U \xrightarrow{\sim} \Pi_V$$

be an isomorphism of profinite groups. Assume that there exist open subgroups $\Pi_{U'} \subset \Pi_U$, $\Pi_{V'} \subset \Pi_V$, which correspond to each other via α, i.e., $\Pi_{V'} \stackrel{\mathrm{def}}{=} \alpha(\Pi_{U'})$, corresponding to étale coverings $U' \to U$, $V' \to V$, such that the smooth compactifications X' of U' and Y' of V' are hyperbolic, that Σ_U is $J_{X'}$-large and that Σ_V is $J_{Y'}$-large. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\sim} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups Π_U, Π_V, respectively.

The rest of this section is devoted to the proof of Theorem 4.23. Let X, Y, X', Y' be the smooth compactifications of U, V, U', V', respectively. We consider any open subgroup $\Pi_{U''} \subset \Pi_U$ corresponding to an étale covering $U'' \to U$ such that $\Pi_{U''}$ is normal in Π_U, and that $\Pi_{U''}$ is contained in $\Pi_{U'}$. We refer to such a subgroup $\Pi_{U''} \subset \Pi_U$ (resp. a covering $U'' \to U$) as a nice subgroup of Π_U (resp. a nice covering of U). Set $\Pi_{V''} \stackrel{\mathrm{def}}{=} \alpha(\Pi_{U''})$, which corresponds to an étale covering $V'' \to V$. Let X'', Y'' be the smooth compactification of U'', V'', respectively. (Note that X'', Y'' are hyperbolic, as they dominate X', Y', respectively.) By Theorem 1.8(i), $\alpha : \Pi_{U''} \xrightarrow{\sim} \Pi_{V''}$ induces $\alpha : \Pi_{X''} \xrightarrow{\sim} \Pi_{Y''}$.

Note that by (the proof of) Theorem 4.22, applied to $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$ induced by $\alpha : \Pi_{U'} \xrightarrow{\sim} \Pi_{V'}$, we have $\Sigma \stackrel{\text{def}}{=} \Sigma_U = \Sigma_V$; $p \stackrel{\text{def}}{=} p_U = p_V$; Σ is $k_{X'}$-large and $k_{Y'}$-large; X' and Y' are almost Σ-separated; and $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$ is pseudo-constants-additive.

Lemma 4.24. (i) *Σ is $k_{X''}$-large and $k_{Y''}$-large.*
(ii) *X'' and Y'' are almost Σ-separated.*
(iii) *$\alpha : \Pi_{X''} \xrightarrow{\sim} \Pi_{Y''}$ is pseudo-constants-additive.*

Proof. (i) This follows from the fact that $k_{X''}$ (resp. $k_{Y''}$) is a finite extension of $k_{X'}$ (resp. $k_{Y'}$).
(ii) This follows from Proposition 2.11, together with the fact that $X'' \to X'$, $Y'' \to Y'$ are tame-Galois.
(iii) This follows from the fact that the diagram

$$\begin{array}{ccc} (\bar{k}^{\times}_{X''})^{\Sigma} & \stackrel{\tau}{\longleftarrow} & (\bar{k}^{\times}_{Y''})^{\Sigma} \\ \uparrow & & \uparrow \\ (\bar{k}^{\times}_{X'})^{\Sigma} & \stackrel{\tau}{\longleftarrow} & (\bar{k}^{\times}_{Y'})^{\Sigma} \end{array}$$

is commutative, where the horizontal arrows are isomorphisms induced by $\alpha : \Pi_{X''} \xrightarrow{\sim} \Pi_{Y''}$ and $\alpha : \Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$, and the vertical arrows are isomorphisms induced (via Kummer theory) by the natural homomorphisms $\Pi_{X''} \to \Pi_{X'}$ and $\Pi_{Y''} \to \Pi_{Y'}$ (induced by $\Pi_{U''} \hookrightarrow \Pi_{U'}$ and $\Pi_{V''} \hookrightarrow \Pi_{V'}$, respectively). Here, as in the proof of Theorem 4.2, the commutativity of this diagram follows basically from the functoriality of Kummer theory, together with the commutativity of the diagram

$$\begin{array}{ccc} M_{X''} & \longleftarrow & M_{Y''} \\ \uparrow & & \uparrow \\ M_{X'} & \longleftarrow & M_{Y'} \end{array}$$

where the horizontal arrows are isomorphisms induced by α (or, more precisely, by α^{-1}, via the functoriality of H^2) and the vertical arrows are isomorphisms defined geometrically via the identification of $M_{X'}$ and $M_{X''}$ (resp. $M_{Y'}$ and $M_{Y''}$) with the Tate module of $\mathbb{G}_m$ over $\bar{k}_{X'} = \bar{k}_{X''}$ (resp. $\bar{k}_{Y'} = \bar{k}_{Y''}$). The commutativity of this last diagram follows from the fact that the above (geometrically defined) isomorphism $M_{X'} \xrightarrow{\sim} M_{X''}$ (resp. $M_{Y'} \xrightarrow{\sim} M_{Y''}$) is identified with the composite of the $(X''_{\bar{k}_{X''}} : X'_{\bar{k}_{X'}})$-multiplication map $M_{X'} \to M_{X'}$ (resp. the $(Y''_{\bar{k}_{Y''}} : Y'_{\bar{k}_{Y'}})$-multiplication map $M_{Y'} \to M_{Y'}$) and the inverse of the natural isomorphism $M_{X''} \xrightarrow{\sim} (X''_{\bar{k}_{X''}} : X'_{\bar{k}_{X'}})M_{X'} \subset M_{X'}$ (resp. $M_{Y''} \xrightarrow{\sim} (Y''_{\bar{k}_{Y''}} : Y'_{\bar{k}_{Y'}})M_{Y'} \subset M_{Y'}$) induced by the inclusion $M_{X''} \to M_{X'}$ (resp. $M_{Y''} \to M_{Y'}$) arising from the functoriality of H^2, and the fact that $(X''_{\bar{k}_{X''}} : X'_{\bar{k}_{X'}}) = (\Delta_{U'} : \Delta_{U''}) = (\Delta_{V'} : \Delta_{V''}) = (Y''_{\bar{k}_{Y''}} : Y'_{\bar{k}_{Y'}})$. $\square$

Thus, by Theorem 4.4, we obtain an isomorphism $X'' \xrightarrow{\sim} Y''$. Further, this isomorphism is Galois-equivariant with respect to $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$. Indeed, this Galois-equivariance can be proved just similarly as in the proof of Lemma 4.18(ii).

Further, let $U''_1 \to U$, $U''_2 \to U$ be nice coverings of U such that $\Pi_{U''_2} \subset \Pi_{U''_1}$, $V''_2 \to V$, $V''_1 \to V$ corresponding nice coverings of V (via $\alpha : \Pi_U \xrightarrow{\sim} \Pi_V$), and

$X_1'', X_2'', Y_1'', Y_2''$ the smooth compactifications of $U_1'', U_2'', V_1'', V_2''$, respectively. Then the following diagram

$$\begin{array}{ccc} X_2'' & \longrightarrow & Y_2'' \\ \big\downarrow & & \big\downarrow \\ X_1'' & \longrightarrow & Y_1'' \end{array}$$

is commutative, where the horizontal arrows are (Galois-equivariant) isomorphisms induced by $\alpha : \Pi_{X_2''} \xrightarrow{\sim} \Pi_{Y_2''}$ and $\alpha : \Pi_{X_1''} \xrightarrow{\sim} \Pi_{Y_1''}$, and the vertical arrows are natural (finite) morphisms induced by finite étale coverings $U_2'' \to U_1''$ and $V_2'' \to V_1''$. Indeed, the proof of this commutativity is similar to the proof of Lemma 4.20. More precisely, recall the proof of Theorem 4.4. The isomorphism $X_i'' \xrightarrow{\sim} Y_i''$ ($i \in \{1,2\}$) is induced by the isomorphism $R_{(Y_i'')^l} \xrightarrow{\sim} R_{(X_i'')^l}$ obtained by applying Theorem 5.7 to

$$\begin{array}{ccc} \mathbb{P}(R_{(X_i'')^l}) & & \mathbb{P}(R_{(Y_i'')^l}) \\ \cup & & \cup \\ R^\times_{(X_i'')^l}/(k^l_{X_i''})^\times & \xleftarrow{\sim} & R^\times_{(Y_i'')^l}/(k^l_{Y_i''})^\times \end{array}$$

where

$$\begin{array}{ccccc} \langle H^\times_{(X_i'')^l}\rangle & = & R_{(X_i'')^l} & \subset & \mathcal{O}_{E_{(X_i'')^l}} \\ \cup & & \cup & & \cup \\ H^\times_{(X_i'')^l} & = & R^\times_{(X_i'')^l} & \subset & \mathcal{O}^\times_{E_{(X_i'')^l}} \end{array}$$

and

$$\begin{array}{ccccc} \langle H^\times_{(Y_i'')^l}\rangle & = & R_{(Y_i'')^l} & \subset & \mathcal{O}_{E_{(Y_i'')^l}} \\ \cup & & \cup & & \cup \\ H^\times_{(Y_i'')^l} & = & R^\times_{(Y_i'')^l} & \subset & \mathcal{O}^\times_{E_{(Y_i'')^l}} \end{array}$$

But the problem here (which does not occur in the proof of Lemma 4.20) is that in general the natural inclusions $K_{(X_1'')^l} \hookrightarrow K_{(X_2'')^l}$, $K_{(Y_1'')^l} \hookrightarrow K_{(Y_2'')^l}$ induced by the (ramified) coverings $f : X_2'' \to X_1''$, $g : Y_2'' \to Y_1''$, respectively, may not induce inclusions $\mathcal{O}_{E_{(X_1'')^l}} \hookrightarrow \mathcal{O}_{E_{(X_2'')^l}}$, $\mathcal{O}_{E_{(Y_1'')^l}} \hookrightarrow \mathcal{O}_{E_{(Y_2'')^l}}$, respectively. This is because it is unclear if $f^{-1}(E''_{X_1''}) = E''_{X_2''}$, $g^{-1}(E''_{Y_1''}) = E''_{Y_2''}$ hold.

Here, the remedy is to resort to Proposition 2.11(ii) instead of Proposition 2.11(i). So, for each nice covering $U'' \to U$ (resp. $V'' \to V$), define $\mathcal{E}_{X''}$ (resp. $\mathcal{E}_{Y''}$) to be the inverse image of $E_{X'} \cup (X' \setminus U')$ (resp. $E_{Y'} \cup (Y' \setminus V')$) in X'' (resp. Y''). Then we have $E_{X''} \subset \mathcal{E}_{X''}$, $E_{Y''} \subset \mathcal{E}_{Y''}$. In particular, let l be a (X', Σ)-admissible prime number (then l is automatically (Y', Σ)-admissible). Then l is (X'', Σ)-admissible and (Y'', Σ)-admissible for all nice coverings $U'' \to U$ and $V'' \to V$. Now, replacing $E_{X''}$, $E_{Y''}$ by $\mathcal{E}_{X''}$, $\mathcal{E}_{Y''}$ in the various definitions, we obtain

$$\begin{array}{ccccc} \langle \mathcal{H}^\times_{(X'')^l}\rangle & = & \mathcal{R}_{(X'')^l} & \subset & \mathcal{O}_{\mathcal{E}_{(X'')^l}} \\ \cup & & \cup & & \cup \\ \mathcal{H}^\times_{(X'')^l} & = & \mathcal{R}^\times_{(X'')^l} & \subset & \mathcal{O}^\times_{\mathcal{E}_{(X'')^l}} \end{array}$$

and

$$\begin{array}{ccccc} \langle \mathcal{H}^{\times}_{(Y'')^l} \rangle & = & \mathcal{R}_{(Y'')^l} & \subset & \mathcal{O}_{\mathcal{E}_{(Y'')^l}} \\ \cup & & \cup & & \cup \\ \mathcal{H}^{\times}_{(Y'')^l} & = & \mathcal{R}^{\times}_{(Y'')^l} & \subset & \mathcal{O}^{\times}_{\mathcal{E}_{(Y'')^l}} \end{array}$$

As in Lemma 4.13, we have $\mathrm{Fr}(\mathcal{R}_{(X'')^l}) = K_{(X'')^l}$ and $\mathrm{Fr}(\mathcal{R}_{(Y'')^l}) = K_{(Y'')^l}$. Then we may apply Theorem 5.7 to

$$\begin{array}{ccc} \mathbb{P}(\mathcal{R}_{(X'')^l}) & & \mathbb{P}(\mathcal{R}_{(Y'')^l}) \\ \cup & & \cup \\ \mathcal{R}^{\times}_{(X'')^l}/(k^l_{X''})^{\times} & \stackrel{\sim}{\leftarrow} & \mathcal{R}^{\times}_{(Y'')^l}/(k^l_{Y''_i})^{\times} \end{array}$$

to obtain $\mathcal{R}_{(Y'')^l} \xrightarrow{\sim} \mathcal{R}_{(X'')^l}$. By the uniqueness assertion of Theorem 5.7, the diagram

$$\begin{array}{ccc} R_{(X'')^l} & \longleftarrow & R_{(Y'')^l} \\ \uparrow & & \uparrow \\ \mathcal{R}_{(X'')^l} & \longleftarrow & \mathcal{R}_{(Y'')^l} \end{array}$$

commutes, where the vertical arrows are natural inclusions. In particular, $R_{(Y'')^l} \xrightarrow{\sim} R_{(X'')^l}$ and $\mathcal{R}_{(Y'')^l} \xrightarrow{\sim} \mathcal{R}_{(X'')^l}$ induce the same isomorphisms $K_{Y''} \xrightarrow{\sim} K_{X''}$ and $X'' \xrightarrow{\sim} Y''$.

Now, as in the proof of Lemma 4.20, we can prove that the diagrams

$$\begin{array}{ccc} \mathcal{R}_{(X''_2)^l} & \longleftarrow & \mathcal{R}_{(Y''_2)^l} \\ \uparrow & & \uparrow \\ \mathcal{R}_{(X''_1)^l} & \longleftarrow & \mathcal{R}_{(Y''_1)^l} \end{array}$$

$$\begin{array}{ccc} K_{X''_2} & \longleftarrow & K_{Y''_2} \\ \uparrow & & \uparrow \\ K_{X''_1} & \longleftarrow & K_{Y''_1} \end{array}$$

and

$$\begin{array}{ccc} X''_2 & \longrightarrow & Y''_2 \\ \downarrow & & \downarrow \\ X''_1 & \longrightarrow & Y''_1 \end{array}$$

commute, as desired.

Now, passing to nice open subgroups of Π_U and Π_V which correspond to each other via α, we obtain an isomorphism

$$\tilde{X}_U \xrightarrow{\sim} \tilde{Y}_V$$

which is Galois-equivariant with respect to $\alpha : \Pi_U \xrightarrow{\sim} \Pi_V$, where $\tilde{X}_U$, $\tilde{Y}_V$ are the integral closures of X, Y in (the function fields of) $\tilde{U}$, $\tilde{V}$, respectively. (Note that in general $\tilde{X}_U$, $\tilde{Y}_V$ do not coincide with $\tilde{X}$, $\tilde{Y}$.) Further, dividing both sides of this isomorphism by the actions of Π_U and Π_V, it follows that the isomorphism $\tilde{X}_U \xrightarrow{\sim} \tilde{Y}_V$ fits into a commutative diagram

$$\begin{array}{ccc} \tilde{X}_U & \xrightarrow{\sim} & \tilde{Y}_V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

Finally, removing the ramification loci from this last diagram, we obtain the desired commutative diagram

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\sim} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups Π_U, Π_V, respectively.

This finishes the proof of Theorem 4.23. $\square$

§**5. On the fundamental theorem of projective geometry.** Throughout this section all fields are assumed to be commutative. For a field k and a vector space V over k, define $\mathbb{P}(V)$ to be the projective space associated to V:

$$\mathbb{P}(V) \stackrel{\text{def}}{=} (V \setminus \{0\})/k^\times,$$

and define $\mathbb{L}(V)$ $(\subset 2^{\mathbb{P}(V)})$ to be the set of lines on $\mathbb{P}(V)$. Thus, $\mathbb{P}(V)$ (resp. $\mathbb{L}(V)$) is also identified with the set of 1-dimensional (resp. 2-dimensional) k-vector subspaces of V. For each $x \in V \setminus \{0\}$, denote the point of $\mathbb{P}(V)$ corresponding to x by $\bar{x}$. We say that a set of points in $\mathbb{P}(V)$ is collinear, if there exists a line in $\mathbb{L}(V)$ which contains all of them. We say that a set of lines in $\mathbb{L}(V)$ is concurrent, if there exists a point in $\mathbb{P}(V)$ which is contained in all of them. We set $\mathbb{P}^n(k) \stackrel{\text{def}}{=} \mathbb{P}(k^n)$ for each $n \geq 0$.

It is well-known and easily proved that the projective space $(\mathbb{P}(V), \mathbb{L}(V))$ satisfies the following proposition.

Proposition 5.1 (cf. [EDM], 343). *Let k be a field and V a vector space over k.*
(i) *(Axioms of projective geometry) The following (I)-(III) hold.*
(I) *If $p, q \in \mathbb{P}(V)$, $p \neq q$, then there exists a unique $\ell \in \mathbb{L}(V)$, such that $p, q \in \ell$. We denote this line ℓ by $p \vee q$.*
(II) *If $p_0, p_1, p_2, q_1, q_2 \in \mathbb{P}(V)$, p_0, p_1, p_2 are not collinear, $q_1 \neq q_2$, p_0, p_1, q_1 are collinear and p_0, p_2, q_2 are collinear, then $p_1 \vee p_2, q_1 \vee q_2$ are concurrent.*
(III) *If $\ell \in \mathbb{L}(V)$, then $\sharp(\ell) \geq 3$.*
(ii) *(Desargues' theorem) If $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{P}(V)$, p_1, p_2, p_3 are not collinear, q_1, q_2, q_3 are not collinear, $p_1 \neq q_1$, $p_2 \neq q_2$ and $p_3 \neq q_3$, then: "$p_1 \vee q_1, p_2 \vee$*

$q_2, p_3 \vee q_3$ *are concurrent" if and only if "*$(p_2 \vee p_3) \cap (q_2 \vee q_3)$, $(p_3 \vee p_1) \cap (q_3 \vee q_1)$, $(p_1 \vee p_2) \cap (q_1 \vee q_2)$ *are collinear".* $\square$

Now, roughly speaking, the fundamental theorem of projective geometry asserts that the information carried by the pair (k, V) is equivalent to that carried by the pair $(\mathbb{P}(V), \mathbb{L}(V))$.

To be more precise, from now on, let k_i be a field and V_i a vector space over k_i, for $i = 1, 2$.

Definition 5.2. (i) An (A semilinear) isomorphism $(k_1, V_1) \xrightarrow{\sim} (k_2, V_2)$ is a pair (μ, λ), where μ is an isomorphism $k_1 \xrightarrow{\sim} k_2$ of fields and λ is an isomorphism $V_1 \xrightarrow{\sim} V_2$ of abelian groups, such that, for each $a \in k_1$ and $x \in V_1$, one has $\lambda(ax) = \mu(a)\lambda(x)$. (In fact, when $V_i \neq 0$, μ is determined uniquely by λ, hence we may say that λ is an (a semilinear) isomorphism.)
(ii) A collineation $(\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))$ is a pair (σ, τ), where σ is a bijection $\mathbb{P}(V_1) \xrightarrow{\sim} \mathbb{P}(V_2)$ and τ is a bijection $\mathbb{L}(V_1) \xrightarrow{\sim} \mathbb{L}(V_2)$, such that, for each $\ell \in \mathbb{L}(V_1)$, one has $\tau(\ell) = \sigma(\ell)(\stackrel{\text{def}}{=} \{\sigma(p) \mid p \in \ell\})$. (In fact, τ is determined uniquely by σ, hence we may say that σ is a collineation.)

Theorem 5.3 (Fundamental Theorem of Projective Geometry, cf. [Artin]). (i) *Each isomorphism* $(\mu, \lambda) : (k_1, V_1) \xrightarrow{\sim} (k_2, V_2)$ *naturally induces a collineation* $(\sigma, \tau) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))$ *by setting* $\sigma(\bar{x}) \stackrel{\text{def}}{=} \overline{\lambda(x)}$ *for* $x \in V_1 \setminus \{0\}$.
(ii) *Assume that* $\dim_{k_i}(V_i) \geq 3$ *for* $i = 1, 2$. *Then, for each collineation* $(\sigma, \tau) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))$, *there exists an isomorphism* $(\mu, \lambda) : (k_1, V_1) \xrightarrow{\sim} (k_2, V_2)$ *that induces* $(\sigma, \tau) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))$ *(in the sense of (i)). Further, such an isomorphism* (μ, λ) *is unique up to scalar multiplication. More precisely, if* $(\mu, \lambda), (\mu', \lambda')$ *are such isomorphisms (that induce the same collineation* (σ, τ)*), then there exists an (in fact, a unique) element* $a \in k_1^\times$ *such that* $\mu' = \mu$ *and* $\lambda'(-) = \lambda(a \cdot -)$. $\square$

The aim of this section is to give a refined version of the fundamental theorem of projective geometry, where certain "partial" collineations defined over "sufficiently large" subsets of projective spaces are considered. To formulate it precisely, let us first define what are "sufficiently large" subsets of projective spaces.

Definition 5.4. Let k be a field. Let $\mathcal{S}$ be a set of subsets of $\mathbb{P}^1(k)$.
(i) We say that $\mathcal{S}$ is PGL_2-stable, if, for any $S \in \mathcal{S}$ and any $\sigma \in \mathrm{PGL}_2(k)$, one has $\sigma(S) \in \mathcal{S}$.
(ii) Let m, n be integers ≥ 0. Then we say that $\mathcal{S}$ is (m, n)-admissible, if $\mathcal{S}$ is PGL_2-stable and, for any $0 \leq m' \leq m$, $0 \leq n' \leq n$, any $S_1, \dots, S_{m'} \in \mathcal{S}$, and any $p_1, \dots, p_{n'} \in \mathbb{P}^1(k)$, one has

$$S_1 \cup \cdots \cup S_{m'} \cup \{p_1, \dots, p_{n'}\} \subsetneq \mathbb{P}^1(k).$$

(Thus, if $m_1 \geq m_2 \geq 0$, $n_1 \geq n_2 \geq 0$, then $\mathcal{S}$: (m_1, n_1)-admissible $\Longrightarrow$ $\mathcal{S}$: (m_2, n_2)-admissible.))
(iii) We say that $\mathcal{S}$ is admissible, if $\mathcal{S}$ is (m, n)-admissible for all integers $m, n \geq 0$.
(iv) Assume that $\mathcal{S}$ is PGL_2-stable. Then, for each 1-dimensional projective space ℓ over k, we set

$$\mathcal{S}_\ell \stackrel{\text{def}}{=} \{S \subset \ell \mid \alpha(S) \in \mathcal{S} \text{ for some } \alpha : \ell \underset{k}{\xrightarrow{\sim}} \mathbb{P}^1(k)\}.$$

(By assumption, we see that "for some α" in this definition may be replaced by "for all α" and that $\mathcal{S} = \mathcal{S}_{\mathbb{P}^1(k)}$.)

Remark 5.5. We have:

$\mathcal{S}$: (m,n)-admissible $\Longleftarrow \forall S \in \mathcal{S}, \sharp(\mathbb{P}^1(k)) > m\sharp(S) + n \Longleftrightarrow \sharp(k) > m\sharp(S) + n - 1.$

In particular, if $\mathcal{S} = \{\varnothing\}$, we have:

$$\mathcal{S}\text{: } (m,n)\text{-admissible} \Longleftrightarrow \sharp(P^1(k)) > n \Longleftrightarrow \sharp(k) \geq n.$$

Definition 5.6. Let k be a field, V a vector space over k, and U a subset of $\mathbb{P}(V)$.
(o) For each line $\ell \in \mathbb{L}(V)$, we write $\ell_U \stackrel{\text{def}}{=} \ell \cap U$ and $\ell_{U^c} \stackrel{\text{def}}{=} \ell \cap (\mathbb{P}(V) \setminus U) = \ell \setminus \ell_U$.
(i) We define $\mathbb{L}(V)_U \subset \mathbb{L}(V)$ by:

$$\mathbb{L}(V)_U \stackrel{\text{def}}{=} \{\ell \in \mathbb{L}(V) \mid \ell_U \neq \varnothing\}.$$

(ii) Let $\mathcal{S}$ be a PGL_2-stable set of subsets of $\mathbb{P}^1(k)$. Then we define $\mathbb{L}(V)_{U,\mathcal{S}} \subset \mathbb{L}(V)$ by:

$$\mathbb{L}(V)_{U,\mathcal{S}} \stackrel{\text{def}}{=} \{\ell \in \mathbb{L}(V) \mid \ell_{U^c} \in \mathcal{S}_\ell\}.$$

(iii) Let $\mathcal{S}$ be a PGL_2-stable set of subsets of $\mathbb{P}^1(k)$. We say that U is $\mathcal{S}$-ample, if the following conditions (1)(2) hold.
(1) $U \neq \varnothing$.
(2) $\mathbb{L}(V)_U \subset \mathbb{L}(V)_{U,\mathcal{S}}$. Equivalently, for each $\ell \in \mathbb{L}(V)$, either $\ell_U = \varnothing$ or $\ell_{U^c} \in \mathcal{S}_\ell$.
(When $\mathcal{S}$ is $(1,0)$-admissible, one automatically has $\mathbb{L}(V)_{U,\mathcal{S}} \subset \mathbb{L}(V)_U$, and the above condition (2) is then equivalent to: $\mathbb{L}(V)_U = \mathbb{L}(V)_{U,\mathcal{S}}$.)

Now, return to the situation of the fundamental theorem of projective geometry. Namely, let k_i be a field and V_i a vector space over k_i, for $i = 1, 2$. The main result in this section is the following refinement of the fundamental theorem of projective geometry.

Theorem 5.7. *Assume that* $\dim_{k_i}(V_i) \geq 3$ *for* $i = 1, 2$. *Let* U_i *be a subset of* $\mathbb{P}(V_i)$ *for* $i = 1, 2$, *and assume that* U_i *is* $\mathcal{S}_i$*-ample for some* $(3,2)$*-admissible set* $\mathcal{S}_i$ *of subsets of* $\mathbb{P}^1(k_i)$ *for* $i = 1, 2$. *Let* $\sigma : U_1 \xrightarrow{\sim} U_2$ *and* $\tau : \mathbb{L}(V_1)_{U_1} \xrightarrow{\sim} \mathbb{L}(V_2)_{U_2}$ *be bijections such that for each* $\ell \in \mathbb{L}(V_1)_{U_1}$, *one has* $\tau(\ell)_{U_2} = \sigma(\ell_{U_1})$. *Then, each such* $(\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \xrightarrow{\sim} (U_2, \mathbb{L}(V_2)_{U_2})$ *uniquely extends to a collineation* $(\tilde{\sigma}, \tilde{\tau}) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))$. *In particular, there exists an isomorphism* $(\mu, \lambda) : (k_1, V_1) \xrightarrow{\sim} (k_2, V_2)$ *that induces* $(\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \xrightarrow{\sim} (U_2, \mathbb{L}(V_2)_{U_2})$, *and such an isomorphism* (μ, λ) *is unique up to scalar multiplication.*

Proof. Step 0. The second assertion follows from the first assertion and Theorem 5.3. So, let us concentrate on the proof of the first assertion.

Step 1. Claim: If $p \in U_1$, then $\tau : \mathbb{L}(V_1)_{U_1} \xrightarrow{\sim} \mathbb{L}(V_2)_{U_2}$ induces a bijection of subsets $\mathbb{L}(V_1)_{\{p\}} \xrightarrow{\sim} \mathbb{L}(V_2)_{\{\sigma(p)\}}$.

Indeed, for each $\ell \in \mathbb{L}(V_1)_{U_1}$, one has

$$\begin{aligned}
\ell \in \mathbb{L}(V_1)_{\{p\}} &\Longleftrightarrow p \in \ell \\
&\Longleftrightarrow p \in \ell_{U_1} \\
&\Longleftrightarrow \sigma(p) \in \sigma(\ell_{U_1}) = \tau(\ell)_{U_2} \\
&\Longleftrightarrow \sigma(p) \in \tau(\ell) \\
&\Longleftrightarrow \tau(\ell) \in \mathbb{L}(V_2)_{\{\sigma(p)\}},
\end{aligned}$$

as desired.

Step 2. Claim: If $p \in \mathbb{P}(V_1)$, then there exists a unique point $p' \in \mathbb{P}(V_2)$, such that $\tau : \mathbb{L}(V_1)_{U_1} \xrightarrow{\sim} \mathbb{L}(V_2)_{U_2}$ induces a bijection of subsets $\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1} \xrightarrow{\sim} \mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2}$. If, moreover, $p \in U_1$, then $p' = \sigma(p)$.

Step 2-1. Claim: Two lines $\ell_1, \ell_2 \in \mathbb{L}(V_1)_{U_1}$ are concurrent, if and only if so are $\tau(\ell_1), \tau(\ell_2) \in \mathbb{L}(V_2)_{U_2}$.

Indeed, it suffices to prove the 'only if' part, since the 'if' part is obtained by applying the 'only if' part to $\sigma^{-1} : U_2 \xrightarrow{\sim} U_1$. Now, the 'only if' part is clear if $\ell_1 = \ell_2$. So, assume $\ell_1 \neq \ell_2$. Then, as ℓ_1, ℓ_2 are concurrent, there is a (unique) point $p \in \mathbb{P}(V)$ such that $\ell_1 \cap \ell_2 = \{p\}$. If $p \in U_1$, then $\sigma(p) \in \tau(\ell_1), \tau(\ell_2)$ by Step 1, hence $\tau(\ell_1), \tau(\ell_2)$ are concurrent, as desired. So, we may and shall assume that $p \notin U_1$. For each $i = 1, 2$, choose $p_i \in (\ell_i)_{U_1} = \ell_i \setminus ((\ell_i)_{U_1^c}) \neq \varnothing$ ((1,0)-admissibility). As $p_1 \in \ell_1 \setminus \ell_2$ and $p_2 \in \ell_2 \setminus \ell_1$, one has $p_1 \neq p_2$. So, set $m \stackrel{\text{def}}{=} p_1 \vee p_2$. As $p_1 \in U_1$, one has $m \in \mathbb{L}(V_1)_{U_1}$. Next, take $q \in m_{U_1} \setminus \{p_1, p_2\} = m \setminus (m_{U_1^c} \cup \{p_1, p_2\}) \neq \varnothing$ ((1,2)-admissibility). Consider the projection (or perspective mapping) $\alpha : \ell_1 \xrightarrow{\sim} \ell_2$ with respect to the center q. More precisely, α is defined by $\{\alpha(x)\} = (x \vee q) \cap \ell_2$ for each $x \in \ell_1$. (In particular, $\alpha(p_1) = p_2$ and $\alpha(p) = p$.) Take $q_1 \in ((\ell_1)_{U_1} \cap \alpha^{-1}((\ell_2)_{U_1})) \setminus \{p_1\} = \ell_1 \setminus ((\ell_1)_{U_1^c} \cup \alpha^{-1}((\ell_2)_{U_1^c}) \cup \{p_1\}) \neq \varnothing$ ((2,1)-admissibility), and set $q_2 \stackrel{\text{def}}{=} \alpha(q_1)$ and $n = q_1 \vee q$ $(= q_2 \vee q = q_1 \vee q_2)$. (Thus, $q_2 \in ((\ell_2)_{U_1} \cap \alpha((\ell_1)_{U_1})) \setminus \{p_2\}$, and, as $q_1 \in U_1$, one has $n \in \mathbb{L}(V_1)_{U_1}$.) Now, one has $q, p_1, q_1, p_2, q_2 \in U_1$, q, p_1, q_1 are not collinear, $p_2 \neq q_2$, $q, p_1, p_2 \in m$ and $q, q_1, q_2 \in n$. Accordingly, one has $\sigma(q), \sigma(p_1), \sigma(q_1), \sigma(p_2), \sigma(q_2) \in U_2$, $\sigma(q), \sigma(p_1), \sigma(q_1)$ not collinear, $\sigma(p_2) \neq \sigma(q_2)$, and, by Step 1, $\sigma(p_1), \sigma(p_2), \sigma(q) \in \tau(m)$ and $\sigma(q_1), \sigma(q_2), \sigma(q) \in \tau(n)$. Further, as $p_1, q_1 \in \ell_1$ $(p_1 \neq q_1)$ and $p_2, q_2 \in \ell_2$ $(p_2 \neq q_2)$, one has $\sigma(p_1), \sigma(q_1) \in \tau(\ell_1)$ (with $\sigma(p_1) \neq \sigma(q_1)$) and $\sigma(p_2), \sigma(q_2) \in \tau(\ell_2)$ (with $\sigma(p_2) \neq \sigma(q_2)$) by Step 1. Now, by Proposition 5.1(i)(II), this implies that $\tau(\ell_1) = \sigma(p_1) \vee \sigma(q_1)$ and $\tau(\ell_2) = \sigma(p_2) \vee \sigma(q_1)$ are concurrent, as desired.

Step 2-2. Claim: Three lines $\ell_1, \ell_2, \ell_3 \in \mathbb{L}(V_1)_{U_1}$ are concurrent, if and only if so are $\tau(\ell_1), \tau(\ell_2), \tau(\ell_3) \in \mathbb{L}(V_2)_{U_2}$.

Indeed, it suffices to prove the 'only if' part, since the 'if' part is obtained by applying the 'only if' part to $\sigma^{-1} : U_2 \xrightarrow{\sim} U_1$. Now, the 'only if' part follows from (the 'only if' part of) Step 2-1 if $\ell_i = \ell_j$ for some $i \neq j$. So, assume that ℓ_1, ℓ_2, ℓ_3 are mutually distinct and set $\ell_1 \cap \ell_2 \cap \ell_3 = \{p\}$. If $p \in U_1$, then $\sigma(p) \in \tau(\ell_1), \tau(\ell_2), \tau(\ell_3)$ by Step 1, hence $\tau(\ell_1), \tau(\ell_2), \tau(\ell_3)$ are concurrent, as desired. So, we may and shall assume that $p \notin U_1$. Take $p_1 \in (\ell_1)_{U_1} \neq \varnothing$ and $p_2 \in (\ell_2)_{U_1} \neq \varnothing$. Take $p_3 \in (\ell_3)_{U_1} \setminus ((p_1 \vee p_2) \cap \ell_3) = \ell_3 \setminus ((\ell_3)_{U_1^c} \cup ((p_1 \vee p_2) \cap \ell_3)) \neq \varnothing$ ((1,1)-admissibility). Take $q_1 \in (\ell_1)_{U_1} \setminus \{p_1\} = \ell_1 \setminus ((\ell_1)_{U_1^c} \cup \{p_1\}) \neq \varnothing$ ((1,1)-admissibility). Let $\alpha : \ell_2 \xrightarrow{\sim} (p_1 \vee p_2)$ be the projection with respect to the center q_1: $\{\alpha(x)\} = (x \vee q_1) \cap (p_1 \vee p_2)$. Take $q_2 \in (\ell_2)_{U_1} \cap \alpha^{-1}((p_1 \vee p_2)_{U_1}) \setminus \{p_2\} = \ell_1 \setminus ((\ell_1)_{U_1^c} \cup \alpha^{-1}((p_1 \vee p_2)_{U_1^c}) \cup \{p_2\}) \neq \varnothing$ ((2,1)-admissibility), and set $r_{12} \stackrel{\text{def}}{=} \alpha(q_2) \in (p_1 \vee p_2)_{U_1} \setminus \{p_1, p_2\}$. Let $\beta : \ell_3 \xrightarrow{\sim} (p_2 \vee p_3)$ be the projection with respect to the center q_2: $\{\beta(x)\} = (x \vee q_2) \cap (p_2 \vee p_3)$ and $\gamma : \ell_3 \xrightarrow{\sim} (p_3 \vee p_1)$ the projection with respect to the center q_1: $\{\gamma(x)\} = (x \vee q_1) \cap (p_3 \vee p_1)$. Take $q_3 \in ((\ell_3)_{U_1} \cap \beta^{-1}((p_2 \vee p_3)_{U_1}) \cap \gamma^{-1}((p_3 \vee p_1)_{U_1})) \setminus (\{p_3\} \cup ((q_1 \vee q_2) \cap \ell_3)) = \ell_3 \setminus ((\ell_3)_{U_1^c} \cup \beta^{-1}((p_2 \vee p_3)_{U_1^c}) \cup \gamma^{-1}((p_3 \vee p_1)_{U_1^c}) \cup \{p_3\} \cup ((q_1 \vee q_2) \cap \ell_3)) \neq \varnothing$ (here, we use the (3,2)-admissibility assumption fully), and set $r_{23} \stackrel{\text{def}}{=} \beta(q_3) \in$

$(p_2 \vee p_3)_{U_1} \setminus \{p_2, p_3\}$ and $r_{31} \stackrel{\text{def}}{=} \gamma(q_3) \in (p_3 \vee p_1)_{U_1} \setminus \{p_3, p_1\}$. Now, one has $p_1, p_2, p_3, q_1, q_2, q_3, r_{12}, r_{23}, r_{31} \in U_1$ with p_1, p_2, p_3 not collinear and q_1, q_2, q_3 not collinear, $p_1, q_1 \in \ell_1$ with $p_1 \neq q_1$, $p_2, q_2 \in \ell_2$ with $p_2 \neq q_2$, $p_3, q_3 \in \ell_3$ with $p_3 \neq q_3$, $\{r_{12}\} = (p_1 \vee p_2) \cap (q_1 \vee q_2)$, $\{r_{23}\} = (p_2 \vee p_3) \cap (q_2 \vee q_3)$, $\{r_{31}\} = (p_3 \vee p_1) \cap (q_3 \vee q_1)$, and $\ell_1 = p_1 \vee q_1$, $\ell_2 = p_2 \vee q_2$, $\ell_3 = p_3 \vee q_3$ are concurrent. Then, by (the '$\Longrightarrow$' part of) Proposition 5.1(ii), r_{12}, r_{23}, r_{31} are collinear. Accordingly (by Step 1), one has $\sigma(p_1), \sigma(p_2), \sigma(p_3), \sigma(q_1), \sigma(q_2), \sigma(q_3), \sigma(r_{12}), \sigma(r_{23}), \sigma(r_{31}) \in U_2$ with $\sigma(p_1), \sigma(p_2), \sigma(p_3)$ not collinear and $\sigma(q_1), \sigma(q_2), \sigma(q_3)$ not collinear, $\sigma(p_1), \sigma(q_1) \in \tau(\ell_1)$ with $\sigma(p_1) \neq \sigma(q_1)$, $\sigma(p_2), \sigma(q_2) \in \tau(\ell_2)$ with $\sigma(p_2) \neq \sigma(q_2)$, $\sigma(p_3), \sigma(q_3) \in \tau(\ell_3)$ with $\sigma(p_3) \neq \sigma(q_3)$, $\{\sigma(r_{12})\} = (\sigma(p_1) \vee \sigma(p_2)) \cap (\sigma(q_1) \vee \sigma(q_2))$, $\{\sigma(r_{23})\} = (\sigma(p_2) \vee \sigma(p_3)) \cap (\sigma(q_2) \vee \sigma(q_3))$, $\{\sigma(r_{31})\} = (\sigma(p_3) \vee \sigma(p_1)) \cap (\sigma(q_3) \vee \sigma(q_1))$, and $\sigma(r_{12}), \sigma(r_{23}), \sigma(r_{31})$ are collinear. Now, by (the '$\Longleftarrow$' part of) Proposition 5.1(ii), $\tau(\ell_1) = \sigma(p_1) \vee \sigma(q_1)$, $\tau(\ell_2) = \sigma(p_2) \vee \sigma(q_2)$, $\tau(\ell_3) = \sigma(p_3) \vee \sigma(q_3)$ are concurrent, as desired.

Step 2-3. Claim: If $p \in \mathbb{P}(V_i)$ for $i = 1, 2$, one has

$$\bigcap_{\ell \in \mathbb{L}(V_i)_{\{p\}} \cap \mathbb{L}(V_i)_{U_i}} \ell = \{p\}.$$

Indeed, we may assume that $i = 1$. First, "$\supset$" is clear. So, to prove "$\subset$", it suffices to show that the left-hand side is of cardinality at most one. Then, since two distinct lines intersect at at most one point, it suffices to prove that there are at least two elements $\ell \in \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}$. Consider the two cases separately: (i) $p \in U_1$; and (ii) $p \notin U_1$. In case (i), as $\dim(\mathbb{P}(V_1)) \geq 2$, hence there exist $q, r \in \mathbb{P}(V_1)$ such that p, q, r are not collinear. Then $p \vee q$ and $p \vee r$ are two distinct lines that belong to $\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}$. In case (ii), as $U_1 \neq \varnothing$, take $q \in U_1$. (Thus, $q \neq p$.) As $\dim(\mathbb{P}(V_1)) \geq 2$, hence there exists $r \in \mathbb{P}(V_1)$ such that p, q, r are not collinear. Observe $q \vee r \in \mathbb{L}(V_1)_{U_1}$, and take $s \in (q \vee r)_{U_1} \setminus \{q, r\} = (q \vee r) \setminus ((q \vee r)_{U_1^c} \cup \{q, r\}) \neq \varnothing$ ((1, 2)-admissibility). Now, $p \vee q$ and $p \vee s$ are two distinct lines that belong to $\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}$.

Step 2-4. Claim: If $p \in \mathbb{P}(V_1)$,

$$\bigcap_{\ell \in \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}} \tau(\ell)$$

is a subset of $\mathbb{P}(V_2)$ of cardinality one. (Denote it by $\{p'\}$.)

Indeed, for each pair $\ell, m \in \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}$ with $\ell \neq m$, $\tau(\ell) \cap \tau(m)$ is of cardinality one by Step 2-1. So, set $\tau(\ell) \cap \tau(m) = \{p'_{\ell,m}\}$. In fact, the point $p'_{\ell,m}$ does not depend on the pair ℓ, m. Indeed, let $\ell', m' \in \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}$ be any pair with $\ell' \neq m'$. If $\sharp\{\ell, m, \ell', m'\} = 2$, then it is clear that $p'_{\ell,m} = p'_{\ell',m'}$. If $\sharp\{\ell, m, \ell', m'\} = 3$, then it follows from Step 2-2 that $\tau(\ell), \tau(m), \tau(\ell'), \tau(m')$ are concurrent, hence $p'_{\ell,m} = p'_{\ell',m'}$. If $\sharp\{\ell, m, \ell', m'\} = 4$, then, again by Step 2-2, one has

$$p'_{\ell,m} = p'_{\ell,m'} = p'_{\ell',m'}.$$

Now, write $p' = p'_{\ell,m}$ for some (or, equivalently, all) pair $\ell, m \in \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}$ with $\ell \neq m$. Here, note that, as shown in the proof of Step 2-3, $\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}$

is of cardinality at least two, hence at least one such pair ℓ, m exists. Then, by definition, one has

$$\bigcap_{\ell \in \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}} \tau(\ell) = \{p'\},$$

as desired.

Step 2-5. End of Step 2.

Let $p \in \mathbb{L}(V_1)$ and define $p' \in \mathbb{L}(V_2)$ as in Step 2-4. Then one has

$$\tau(\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}) \subset \mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2}.$$

Applying this to $\sigma^{-1} : U_2 \xrightarrow{\sim} U_1$ and $p' \in \mathbb{L}(V_2)$, we obtain

$$\tau^{-1}(\mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2}) \subset \mathbb{L}(V_1)_{\{p''\}} \cap \mathbb{L}(V_1)_{U_1}$$

for some unique $p'' \in \mathbb{L}(V_1)$. Combining these containment relations, we conclude

$$\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1} \subset \mathbb{L}(V_1)_{\{p''\}} \cap \mathbb{L}(V_1)_{U_1},$$

which, together with Step 2-3, implies that $\{p\} \supset \{p''\}$, hence $p = p''$, and that

$$\tau(\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}) = \mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2},$$

as desired.

The uniqueness of p' is clear by Step 2-3. This uniqueness, together with Step 1, implies $p' = \sigma(p)$ for $p \in U_1$.

Step 3. We define $\tilde{\sigma} : \mathbb{P}(V_1) \to \mathbb{P}(V_2)$ to be the map that sends $p \in \mathbb{P}(V_1)$ to $p' \in \mathbb{P}(V_2)$ defined in Step 2. (Thus, in particular, $\tilde{\sigma}(p) = \sigma(p)$ if $p \in U_1$.)

Claim: $\tilde{\sigma}$ is a collineation. (More precisely, there exists a (unique) bijection $\tilde{\tau} : \mathbb{L}(V_1) \xrightarrow{\sim} \mathbb{L}(V_2)$, such that $(\tilde{\sigma}, \tilde{\tau})$ is a collineation.)

Step 3-1. Claim: $\tilde{\sigma} : \mathbb{P}(V_1) \to \mathbb{P}(V_2)$ is a bijection.

Indeed, starting with $\sigma^{-1} : U_2 \xrightarrow{\sim} U_1$ instead of σ, we obtain $\widetilde{\sigma^{-1}} : \mathbb{P}(V_2) \to \mathbb{P}(V_1)$, which turns out to be the inverse of $\tilde{\sigma}$ from the uniqueness assertion of Step 2.

Step 3-2. Claim: If $\ell \in \mathbb{L}(V_1)_{U_1}$, then $\tilde{\sigma}(\ell) = \tau(\ell)$ $(\in \mathbb{L}(V_2)_{U_2})$.

Indeed, let $\ell \in \mathbb{L}(V_1)_{U_1}$ and $p \in \ell$. Then, just by the definition of $\tilde{\sigma}$, we have $\tilde{\sigma}(p) \in \tau(\ell)$. Namely, we obtain $\tilde{\sigma}(\ell) \subset \tau(\ell)$. Applying this to $\sigma^{-1} : U_2 \xrightarrow{\sim} U_1$ and $\tau(\ell) \in \mathbb{L}(V_2)_{U_2}$ (and noting that $\widetilde{\sigma^{-1}} = \tilde{\sigma}^{-1}$ as shown in Step 3-1), we obtain $\tilde{\sigma}^{-1}(\tau(\ell)) \subset \tau^{-1}(\tau(\ell)) = \ell$, or, equivalently, $\tau(\ell) \subset \tilde{\sigma}(\ell)$. Combining these, we obtain $\tilde{\sigma}(\ell) = \tau(\ell)$, as desired.

Step 3-3. Claim: Three points $p_1, p_2, p_3 \in \mathbb{P}(V_1)$ are collinear, if and only if so are $\tilde{\sigma}(p_1), \tilde{\sigma}(p_2), \tilde{\sigma}(p_3)$.

Indeed, it suffices to prove the 'only if' part, since the 'if' part is obtained by applying the 'only if' part to $\sigma^{-1} : U_2 \xrightarrow{\sim} U_1$. If $\sharp\{p_1, p_2, p_3\} \leq 2$, the assertion is clear. So, we may assume that p_1, p_2, p_3 are mutually distinct. In particular, the line ℓ containing p_1, p_2, p_3 (whose existence is ensured by the collinearity of

p_1, p_2, p_3) is unique. Next, if $\ell \in \mathbb{L}(V_1)_{U_1}$, then the assertion follows immediately from Step 3-2. So, we may assume that $\ell \notin \mathbb{L}(V_1)_{U_1}$, i.e., $\ell \cap U_1 = \varnothing$.

Take $q_1 \in U_1 \neq \varnothing$. As $\ell \cap U_1 = \varnothing$, we have $q_1 \notin \ell$, hence there exists a unique plane $P \subset \mathbb{P}(V_1)$ containing both ℓ and q_1. We shall construct various points in $P \cap U_1$. Let $\alpha : (q_1 \vee p_3) \xrightarrow{\sim} (q_1 \vee p_2)$ be the projection with respect to the center p_1: $\{\alpha(x)\} = (x \vee p_1) \cap (q_1 \vee p_2)$. Take $q_2 \in (q_1 \vee p_3)_{U_1} \cap \alpha^{-1}((q_1 \vee p_2)_{U_1}) \setminus \{q_1\} = (q_1 \vee p_3) \setminus ((q_1 \vee p_3)_{U_1^c} \cup \alpha^{-1}((q_1 \vee p_2)_{U_1^c}) \cup \{q_1\}) \neq \varnothing$ ((2, 1)-admissibility), and set $q_3 \stackrel{\text{def}}{=} \alpha(q_2) \in (q_1 \vee p_2)_{U_1} \cap \alpha((q_1 \vee p_3)_{U_1}) \setminus \{q_1\}$. Next, take $r_1 \in P \cap U_1 \setminus ((q_1 \vee p_3) \cup (q_1 \vee p_2)) \supset (q_1 \vee p_1)_{U_1} \setminus \{q_1\} = (q_1 \vee p_1) \setminus ((q_1 \vee p_1)_{U_1^c} \cup \{q_1\}) \neq \varnothing$ ((1, 1)-admissibility). Let $\beta : (r_1 \vee p_3) \xrightarrow{\sim} (r_1 \vee p_2)$ be the projection with respect to the center p_1: $\{\beta(x)\} = (x \vee p_1) \cap (r_1 \vee p_2)$. Let $\gamma : (r_1 \vee p_3) \xrightarrow{\sim} (r_1 \vee q_1)$ be the projection with respect to the center q_2: $\{\gamma(x)\} = (x \vee q_2) \cap (r_1 \vee q_1)$. Take $r_2 \in (r_1 \vee p_3)_{U_1} \cap \beta^{-1}((r_1 \vee p_2)_{U_1}) \cap \gamma^{-1}((r_1 \vee q_1)_{U_1}) \setminus \{r_1\} = (r_1 \vee p_3) \setminus ((r_1 \vee p_3)_{U_1^c} \cup \beta^{-1}((r_1 \vee p_2)_{U_1^c}) \cup \gamma^{-1}((r_1 \vee q_1)_{U_1^c}) \cup \{r_1\}) \neq \varnothing$ ((3, 1)-admissibility), and set $r_3 \stackrel{\text{def}}{=} \beta(r_2) \in (r_1 \vee p_2)_{U_1} \setminus \{r_1\}$.

First, q_1, q_2, q_3 are not collinear. Indeed, otherwise, q_1, p_2, p_3 must also be collinear, which contradicts the choice of q_1. Second, r_1, r_2, r_3 are not collinear. Indeed, otherwise, r_1, p_3, p_2 must also be collinear, which contradicts the choice of r_1. ($r_1 \in U_1$ and $p_3 \vee p_2 = \ell \subset U_1^c$.) Third, $q_1 \neq r_1$. Indeed, this follows from the definition of r_1. Fourth, $q_2 \neq r_2$. Indeed, otherwise, $q_2 = r_2 \in r_1 \vee p_3$, hence $r_1 \in q_2 \vee p_3 = q_1 \vee p_3$, which contradicts the choice of r_1. Fifth, $q_3 \neq r_3$. Indeed, otherwise, $q_3 = r_3$, hence $r_1 \in r_1 \vee p_2 = r_3 \vee p_2 = q_3 \vee p_2 = q_1 \vee p_2$, which contradicts the choice of r_1.

Thus, one may apply (the '$\Longleftarrow$' part of) Proposition 5.1(ii) to $q_1, q_2, q_3, r_1, r_2, r_3$. Then there exists $s \in \mathbb{P}(V)$, such that s, q_1, r_1 are collinear, s, q_2, r_2 are collinear, and s, q_3, r_3 are collinear. By definition, $s = \gamma(r_2) \in U_1$.

By Step 3-2, $\sigma(q_1), \sigma(q_2), \sigma(q_3)$ are not collinear, $\sigma(r_1), \sigma(r_2), \sigma(r_3)$ are not collinear, $\sigma(s), \sigma(q_1), \sigma(r_1)$ are collinear, $\sigma(s), \sigma(q_2), \sigma(r_2)$ are collinear, and $\sigma(s), \sigma(q_3), \sigma(r_3)$ are collinear. Also, as σ is a bijection, one has $\sigma(q_1) \neq \sigma(r_1)$, $\sigma(q_2) \neq \sigma(r_2)$ and $\sigma(q_3) \neq \sigma(r_3)$. Thus, applying (the '$\Longrightarrow$' part of) Proposition 5.1(ii) to $\sigma(q_1)$,$\sigma(q_2)$,$\sigma(q_3)$,$\sigma(r_1)$,$\sigma(r_2)$,$\sigma(r_3)$, we conclude that $\tilde{\sigma}(p_1) = (\sigma(q_2) \vee \sigma(q_3)) \cap (\sigma(r_2) \vee \sigma(r_3))$, $\tilde{\sigma}(p_2) = (\sigma(q_3) \vee \sigma(q_1)) \cap (\sigma(r_3) \vee \sigma(r_1))$, $\tilde{\sigma}(p_3) = (\sigma(q_1) \vee \sigma(q_2)) \cap (\sigma(r_1) \vee \sigma(r_2))$ are collinear, as desired.

Step 3-4. Claim: If $\ell \in \mathbb{L}(V_1)$, then $\tilde{\sigma}(\ell) \in \mathbb{L}(V_2)$.

Indeed, for each pair $p, q \in \ell$ with $p \neq q$, set $\ell'_{p,q} \stackrel{\text{def}}{=} \tilde{\sigma}(p) \vee \tilde{\sigma}(q) \in \mathbb{L}(V_2)$. In fact, the line $\ell'_{p,q}$ does not depend on the pair p, q. Indeed, let $p', q' \in \ell$ be any pair with $p' \neq q'$. If $\sharp\{p, q, p', q'\} = 2$, then it is clear that $\ell'_{p,q} = \ell'_{p',q'}$. If $\sharp\{p, q, p', q'\} = 3$, then it follows from Step 3-3 that $\tilde{\sigma}(p), \tilde{\sigma}(q), \tilde{\sigma}(p'), \tilde{\sigma}(q')$ are collinear, hence $\ell'_{p,q} = \ell'_{p',q'}$. If $\sharp\{p, q, p', q'\} = 4$, then, again by Step 3-3, one has

$$\ell'_{p,q} = \ell'_{p,q'} = \ell'_{p',q'}.$$

Now, write $\ell' = \ell'_{p,q}$ for some (or, equivalently, all) pair $p, q \in \ell$ with $p \neq q$. Here, note that, by Proposition 5.1(i)(III), at least one such pair p, q exists. Then, by definition, $\tilde{\sigma}(\ell) \subset \ell'$ or, equivalently, $\sigma(\ell) \subset \tilde{\sigma}^{-1}(\ell')$.

Applying this to $\sigma^{-1} : U_2 \xrightarrow{\sim} U_1$ and $\ell' \in \mathbb{L}(V_2)$, we obtain $\tilde{\sigma}^{-1}(\ell') = \widetilde{\sigma^{-1}}(\ell') \subset \ell''$ for some $\ell' \in \mathbb{L}(V_1)$. Combining these containment relations, we conclude $\ell \subset \ell''$, which implies $\ell = \ell''$ and $\tilde{\sigma}(\ell) = \ell'$, as desired.

Step 3-5. End of Step 3.

By Step 3-4, we may define $\tilde{\tau} : \mathbb{L}(V_1) \to \mathbb{L}(V_2)$ to be the map that sends $\ell \in \mathbb{L}(V_1)$ to $\tilde{\sigma}(\ell) \in \mathbb{L}(V_2)$. (Note that $\tilde{\tau} : \mathbb{L}(V_1) \to \mathbb{L}(V_2)$ is an extension of $\tau : \mathbb{L}(V_1)_{U_1} \to \mathbb{L}(V_2)_{U_2}$, by Step 3-2.) Applying this to $\sigma^{-1} : U_2 \xrightarrow{\sim} U_1$, we may also define $\tilde{\tau}' : \mathbb{L}(V_2) \to \mathbb{L}(V_1)$ to be the map that sends $\ell' \in \mathbb{L}(V_2)$ to $\tilde{\sigma}^{-1}(\ell') = \widetilde{\sigma^{-1}}(\ell') \in \mathbb{L}(V_1)$. By definition, it is immediate to prove that $\tilde{\tau}'$ is the inverse map of $\tilde{\tau}$, and, in particular, that $\tilde{\tau}$ is a bijection. Now, by the very definition of $\tilde{\tau}$, $(\tilde{\sigma}, \tilde{\tau}) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))$ is a collineation, as desired.

Step 4. Uniqueness. If $(\tilde{\sigma}, \tilde{\tau}) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))$ is a collineation, then, for each $p \in \mathbb{P}(V_1)$, the bijection $\tilde{\tau} : \mathbb{L}(V_1) \xrightarrow{\sim} \mathbb{L}(V_2)$ induces a bijection of subsets $\mathbb{L}(V_1)_{\{p\}} \xrightarrow{\sim} \mathbb{L}(V_2)_{\{\tilde{\sigma}(p)\}}$. So, if $(\tilde{\sigma}, \tilde{\tau})$ extends $(\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \xrightarrow{\sim} (U_2, \mathbb{L}(V_2)_{U_2})$, then $\tilde{\tau}$ (or, equivalently, τ) induces a bijection of subsets $\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1} \xrightarrow{\sim} \mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2}$. Thus, the uniqueness assertion in Theorem 5.7 follows from the uniqueness assertion in Step 2. (Recall the fact that $\tilde{\tau}$ is determined by $\tilde{\sigma}$ uniquely.) □

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Mohamed Saïdi
College of Engineering, Mathematics, and Physical Sciences
University of Exeter
Harrison Building
North Park Road
EXETER EX4 4QF
United Kingdom
M.Saidi@exeter.ac.uk

Akio Tamagawa
Research Institute for Mathematical Sciences
Kyoto University
KYOTO 606-8502
Japan
tamagawa@kurims.kyoto-u.ac.jp