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# MULTICOMPONENT MIXTURE MODEL: THE ISSUE OF EXISTENCE VIA TIME DISCRETIZATION*

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**Abstract.** We prove the existence of global-in-time weak solutions to a model of chemically reacting mixture. We consider a coupling between the compressible Navier–Stokes system and the reaction diffusion equations for chemical species when the thermal effects are neglected. We first prove the existence of weak solutions to the semi-discretization in time. Based on this, the existence of solutions to the evolutionary system is proven.

**Key words.** Weak solutions to compressible flows, mixture models, time discretization, chemically reacting gases, barotropic flows.

**AMS subject classifications.** 35B45, 35D30, 76N10.

## 1. Introduction

We consider the model of motion for the $n$-component gaseous mixture undergoing an isothermal chemical reaction. We focus on the Fick approximation of diffusion fluxes which is often used to model the lean one-reaction flow [13],

$$\nu_F F + \nu_O O \to \nu_P P,$$

where $F$ denotes the fuel, $O$ denotes the oxidant, $P$ denotes the products, and $\nu_F$, $\nu_O$, and $\nu_P$ denote stoichiometric coefficients. When the reaction takes place in the presence of dilutant denoted by $N$, and when the oxidant and dilutant are in excess, one may ignore the cross-effects in the diffusion fluxes and simply assume that they are proportional to the gradients of species concentrations. Such a model was investigated e.g. by Feireisl, Petzeltová, and Trivisa in [12]. They proved the existence of weak variational entropy solutions to a system with an arbitrary large number of reversible reactions and diffusion flux $\mathbf{F}_k$ determined by the Fick law

$$\mathbf{F}_k = -D\nabla Y_k, \quad k \in \{1,\dots,n\},$$

where $Y_k$ denotes the species $k$ mass fraction, termed also *the concentration of species* $k$.

The analysis performed in this paper was motivated by previous studies of Klein et al. [16] in which the authors assumed that the pressure does not depend on the chemical composition of the mixture. Another application of such a result is to model the mixtures of isotopes. Then the molar masses of species $m_k$ are almost the same, and so the mean molar mass $\overline{m}$ is close to a constant

$$\frac{1}{\overline{m}} = \sum_{k=1}^{n} \frac{Y_k}{m_k} \approx c.$$

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In the present work, we aim to extend this result to a more general equation of state like, for example, the Boyl law describing the pressure $p$ of a mixture of ideal gases

$$p=\sum_{k=1}^{n}\frac{R\varrho Y_k}{m_k}, \tag{1.1}$$

where $R$ is the ideal gas constant and $\varrho$ denotes the density of the mixture. This leads to a stronger coupling between the fluid equations and the mass balances of the species. However, as mentioned in [36], the consistency of this model with the second law of thermodynamics requires a more general form of diffusion, the so called *multicomponent diffusion*.

The properties of the reaction-diffusion systems with this form of diffusion were investigated first by Giovangigli (see [13] and the references therein) in the case of data sufficiently close to an equilibrium. The extension to the framework of global-in-time weak solutions under some regularity assumptions on total density and the velocity vector field is due to Mucha, Pokorný, and Zatorska [24]. For the incompressible, isobaric, isothermal case, the so called *Maxwell–Stefan* system was investigated by Bothe [2], and by Jüngel and Stelzer [15] for the molar-based approach (when the sum of the molar concentrations of species is assumed to be constant). For the mass-based case approach, we refer to the work of Herberg, Meyries, Prüss, and Wilke [14]. The coupling of such systems with compressible Navier–Stokes-type systems was studied by Zatorska and by Mucha, Pokorný, and Zatorska in [36, 23] and in the incompressible case by Marion and Temam [19] and Chen and Jüngel [3]. The one-dimensional model for a model of combustion was studied, for example, in [17, 37].

Our goal is to investigate a system with fluxes of a simplified form in comparison to [36, 23] by assuming the Fick approximation. At the same time, we want to extend the result from [12] to a more general form of the pressure including the dependence of the species concentrations, as in (1.1). We prove the global-in-time existence of weak solutions by semi-discretization in time using similar methods as in [35, 25] devoted to single-component flow. Our approach relies on an existence result for the stationary Navier–Stokes-like model of the 4-component reactive mixture, due to Zatorska [34].

As far as the weak solutions with large data are concerned, the first existence result for the steady as well as the non-steady barotropic Navier–Stokes system is due to Lions [18]. He essentially used the properties of the so-called *effective viscous flux*. A compactness of this quantity has already been studied by Novotný [27] using the method of decomposition from [28]. Later on, this approach was extended by Feireisl [7]. He established a tool for studying density oscillations which allowed him to treat the case when density is not a-priori bounded in $L^2$. This technique was later on adopted by Novo and Novotný [26] to treat the steady case. The comparison of these methods together with complete approximation scheme can be found in the book of Novotný and Straškraba [31], mostly for the Dirichlet boundary conditions. For the steady problem with slip boundary conditions, we refer to the papers of Mucha and Pokorný [20, 32], where a new idea of construction of approximate solutions has also been introduced. For completeness, let us also mention the recent generalization of these results to the full Navier–Stokes–Fourier system in the evolutionary [8, 10, 11], the stationary [21, 22, 29, 30], and the time periodic case [9].

The paper is organized as follows. In Section 2, we formulate the model. We specify the constitutive relations and the assumptions on the transport coefficients. Further, we introduce the notion of a weak solution and we state the first main result of the paper, given in Theorem 2.2. In Section 3, we introduce the discretized system, and at the end, in Theorem 3.3, we give the second main result of this paper—the existence result for the fixed time step $\triangle t$. Then, in Section 4, we present the proof of this result by several regularizations and subsequent limit passages. Finally, in Section 5, we show a convergence to the continuous system when $\triangle t \to 0$.

**2. Presentation of the continuous model** The continuous model is given by

$$\left.\begin{aligned}\partial_t\varrho+\operatorname{div}(\varrho\mathbf{u})=0\\ \partial_t(\varrho\mathbf{u})+\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})-\operatorname{div}\mathbf{S}+\nabla\pi=\varrho\mathbf{f}\\ \partial_t\varrho_k+\operatorname{div}(\varrho_k\mathbf{u})+\operatorname{div}\mathbf{F}_k=\varrho\omega_k,\quad k\in\{1,\dots,n\}\end{aligned}\right\}\quad\text{in } (0,T)\times\Omega. \tag{2.1}$$

This model is characterized by the state variables: the density of the mixture $\varrho=\varrho(t,x)$, the velocity vector field $\mathbf{u}=\mathbf{u}(t,x)$, and the species mass fractions $Y_k$ for $k\in\{1,\dots,n\}$.

In system (2.1), the quantity $\mathbf{S}$ stands for the viscous stress tensor, $\pi$ denotes the internal pressure of the fluid, $\mathbf{f}$ denotes the external force, $\omega_k$ stands for the production rate of the $k$-th species, and by $\mathbf{F}_k$, we denote the diffusion flux.

The mass fractions $Y_k$ $k\in\{1,\dots,n\}$ are defined by

$$Y_k=\frac{\varrho_k}{\varrho},$$

where $m_k$ is the molar mass of species $k$.

The diffusion fluxes and the species production rates satisfy

$$\sum_{k=1}^n\mathbf{F}_k=0,\quad \sum_{k=1}^n\omega_k=0. \tag{2.2}$$

The system is supplemented by the following initial conditions:

$$\varrho(x)|_{t=0}=\varrho^0(x),\quad Y_k(x)|_{t=0}=Y_k^0(x),\quad \mathbf{u}(x)|_{t=0}=\mathbf{u}^0(x). \tag{2.3}$$

We assume that

$$0\le Y_k^0\le 1,\quad \sum_{k=1}^n Y_k^0=1,\quad 0<\underline{\varrho^0}\le\varrho^0\le\overline{\varrho^0}<\infty,$$

and that the total mass is given by

$$\int_\Omega\varrho^0\ \mathrm{d}x=M>0.$$

We consider $\Omega$, a bounded sufficiently smooth subset of $\mathbb{R}^3$, and we impose the following boundary conditions:

$$\mathbf{u}|_{\partial\Omega}=\mathbf{0}\quad\text{and}\quad \mathbf{F}_k\cdot\mathbf{n}|_{\partial\Omega}=\mathbf{0}. \tag{2.4}$$

The internal pressure is of the form

$$\pi(\varrho,Y)=\varrho^\gamma+\varrho\sum_{k=1}^n\frac{Y_k}{m_k},\quad \gamma>1. \tag{2.5}$$

The first term describes the barotropic pressure, and the latter summand represents the thermodynamic pressure for the mixture of $n$ species given by the Boyle law (1.1) (with $R$=1).

The fluxes $\mathbf{F}_i$ are given by

$$\mathbf{F}_i = -D\nabla Y_i, \quad D > 0. \tag{2.6}$$

The form of the viscous stress tensor $\mathbf{S}$ is determined by the Newton rheological law

$$\mathbf{S} = 2\mu\mathbf{D}(\mathbf{u}) + \nu \operatorname{div}\mathbf{u}\mathbf{I}, \tag{2.7}$$

where $\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$ and $\mu$ and $\nu$ are constant viscosity coefficients satisfying

$$\mu > 0, \quad 2\mu + 3\nu \geq 0. \tag{2.8}$$

The production rates $\omega_k$ are defined as

$$\omega_k = \omega_k(Y_1,\ldots,Y_n) = \omega_k^p(Y_1,\ldots,Y_n) - Y_k\omega_k^r(Y_1,\ldots,Y_n), \tag{2.9}$$

where $\omega_k^p$ and $Y_k\omega_k^r$ denote the rate of production and reduction of species $k$, respectively. We assume that $\omega_k^p$ and $\omega_k^r$ are bounded on $[0,1]^n$ and that

$$\omega_k^p(Y_1,\ldots,Y_n) \geq 0, \quad \omega_k^r(Y_1,\ldots,Y_n) \geq 0 \quad \text{for all } 0 \leq Y_i \leq 1. \tag{2.10}$$

Thus, in particular

$$\omega_k(Y_1,\ldots,Y_n) \geq 0 \quad \text{whenever} \quad Y_k = 0.$$

For the above system, we will look for a global in time weak solution in the following sense.

DEFINITION 2.1. *We say that $(\varrho,\mathbf{u},Y_1,\ldots,n)$ is a weak solution to the problem (2.1)–(2.5), (2.6)–(2.10) provided $\varrho \in L^\infty(0,T;L^\gamma(\Omega))$, $\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))$, $Y \in L^\infty((0,T)\times\Omega) \cap L^2(0,T;W^{1,2}(\Omega))$, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$, $Y_k, \varrho \geq 0$, $\int_\Omega \varrho(x,t)dx = M$, and $\sum_{k=1}^n Y_k = 1$ a.e. in $\Omega$, and the system (2.1) is fulfilled in the distributional sense in $(0,T)\times\Omega$.*

The main theorem of this work reads as follows.

THEOREM 2.2. *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mu > 0$, $\nu + \frac{2}{3}\mu > 0$, $\gamma \geq 2$, $M > 0$, $\rho_0 \geq 0$, $\mathbf{u}_0 \in L^2(\Omega)$, $\rho_0 \in L^\gamma(\Omega)$, and $0 \leq Y_k \leq 1$. Then there exists a weak solution to (2.1)–(2.5), (2.6)–(2.10) in the sense of Definition 2.1.*

The main idea of the proof of Theorem 2.2 is to use time-discretization, which will be introduced in Section 3. In this section, we also introduce further approximation involving elliptic regularization in the continuity equation $-\varepsilon\Delta\varrho$ and artificial pressure $\delta\varrho^\Gamma$. Existence of regular solutions for all parameters being fixed is proven by the Galerkin approximation for the momentum equation combined with the fixed point argument. The continuous system is recovered in Section 5, still with $\delta$ fixed and $\Gamma$ large enough, which plays an important role in the derivation of the *effective flux equality.* At the end of this section, we perform the last limit passage $\delta \to 0$. Since this is done for the continuous system, the proof is an easy combination of reasoning from previous sections and the techniques from [12]. The only substantial difference is the energy estimate, which arises due to a more general form of the pressure (2.5) and asks for $\gamma \geq 2$. For the reader's convenience, we also recall some classical facts about the Riesz transform that are used in the course of the proof. They are collected in the appendix in Section A.

## 3. The time-discretized model

We will prove the existence of solutions to system (2.1) by a relevant time discretization and by letting the length of the time step go to 0. We also introduce the first approximation parameter $\delta>0$ in front of the artificial pressure $\varrho^\Gamma$.

Below, we define uniform partitions of the time interval $[0,T]$, $0=t_0<t_1<\cdots<t_{\tilde N}=T$, such that $(\triangle t)=t_j-t_{j-1}=const$. Then the discretized system reads

$$
\begin{aligned}
&(\triangle t)^{-1}\left(\varrho^j-\varrho^{j-1}\right)+\operatorname{div}(\varrho^j\mathbf{u}^j)=0\\
&(\triangle t)^{-1}\left(\varrho^j\mathbf{u}^j-\varrho^{j-1}\mathbf{u}^{j-1}\right)+\operatorname{div}(\varrho^j\mathbf{u}^j\otimes\mathbf{u}^j)-\operatorname{div}\mathbf{S}(\mathbf{u}^j)+\nabla\pi(\varrho^j,Y^j)+\nabla\delta(\varrho^j)^\Gamma\\
=&\varrho^j\mathbf{f}^j(\triangle t)^{-1}\left(\varrho^jY_k^j-\varrho^{j-1}Y_k^{j-1}\right)+\operatorname{div}(\varrho^jY_k^j\mathbf{u}^j)+\operatorname{div}\mathbf{F}_k(Y_k^j)\\
=&\varrho^j\omega_k(Y^j),\quad k\in\{1,\dots,n\}
\end{aligned}
\tag{3.1}
$$

with boundary conditions

$$\mathbf{u}^j|_{\partial\Omega}=\mathbf{F}_k(Y^j)\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{3.2}$$

where, for brevity, we denote $Y^j=\{Y_1^j,\dots,Y_n^j\}$.

For the purposes of this part of the paper, we introduce the following definition of a weak solution.

Definition 3.1. *We say $(\varrho^j,\mathbf{u}^j,Y^j)$ is a weak solution to the problem (3.1)–(3.2) provided $\varrho^j\in L^\gamma(\Omega)$, $\mathbf{u}^j\in W_0^{1,2}(\Omega)$, $Y^j\in W^{1,2}(\Omega)$, $\mathbf{F}_k(\varrho^j,Y^j)\cdot\mathbf{n}|_{\partial\Omega}=0$, $Y_k^j,\varrho^j\geq 0$, and $\sum_{k=1}^n Y_k^j=1$ a.e. in $\Omega$, and the following integral equalities hold:*

$$(\triangle t)^{-1}\int_\Omega(\varrho^j-\varrho^{j-1})\xi\,\mathrm{d}x-\int_\Omega\varrho^j\mathbf{u}^j\cdot\nabla\xi\,\mathrm{d}x=0, \tag{3.3}$$

*for all $\xi\in C^\infty(\overline{\Omega})$,*

$$
\begin{aligned}
(\triangle t)^{-1}\int_\Omega(\varrho^j\mathbf{u}^j-\varrho^{j-1}\mathbf{u}^{j-1})\boldsymbol{\varphi}\,\mathrm{d}x-\int_\Omega\left(\varrho^j\left(\mathbf{u}^j\otimes\mathbf{u}^j\right):\nabla\boldsymbol{\varphi}\right)\,\mathrm{d}x-\int_\Omega\mathbf{S}(\mathbf{u}^j):\nabla\boldsymbol{\varphi}\,\mathrm{d}x\\
-\int_\Omega\left(\pi(\varrho^j,Y^j)+\delta(\varrho^j)^\Gamma\right)\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x=\int_\Omega\varrho^j\mathbf{f}^j\cdot\boldsymbol{\varphi}\,\mathrm{d}x,
\end{aligned}
$$

*for $\boldsymbol{\varphi}\in C_0^\infty(\Omega)$, and*

$$
\begin{aligned}
(\triangle t)^{-1}\int_\Omega(\varrho^jY_k^j-\varrho^{j-1}Y_k^{j-1})\phi\,\mathrm{d}x-\int_\Omega\varrho^jY_k^j\mathbf{u}^j\cdot\nabla\phi\,\mathrm{d}x\\
-\int_\Omega\mathbf{F}_k(Y_k^j)\cdot\nabla\phi\,\mathrm{d}x=\int_\Omega\varrho^j\omega_k(Y^j)\phi\,\mathrm{d}x,
\end{aligned}
$$

*for all $\phi\in C^\infty(\overline{\Omega})$ and for $k\in\{1,\dots,n\}$.*

We will also use the notion of a renormalized solution to the continuity equation.

Definition 3.2. *Let $\mathbf{u}^j\in W_{loc}^{1,2}(\mathbb{R}^3)$ and $\varrho^{j-1}$, $\varrho^j\in L_{loc}^{6/5}(\mathbb{R}^3)$ solve*

$$(\triangle t)^{-1}\left(\varrho^j-\varrho^{j-1}\right)+\operatorname{div}(\varrho^j\mathbf{u}^j)=0$$

*in the sense of distributions on $\mathbb{R}^3$. Then the pair $(\varrho^j,\mathbf{u}^j)$ is called a renormalized solution to the continuity equation if*

$$(\triangle t)^{-1}\left(\varrho^j-\varrho^{j-1}\right)b'(\varrho^j)+\operatorname{div}\left(b(\varrho^j)\mathbf{u}^j\right)+(\varrho^jb'(\varrho^j)-b(\varrho)j)\operatorname{div}\mathbf{u}^j\,\mathrm{d}x=0, \tag{3.4}$$

*in the sense of distributions on* $\mathbb{R}^3$*, for all* $b\in W^{1,\infty}(0,\infty)\cap C^1([0,\infty))$*, such that* $sb'(s)\in L^\infty(0,\infty)$*.*

Our main result for this system is as follows.

THEOREM 3.3. *Let* $\Omega\in C^2$ *be a bounded domain in* $\mathbb{R}^3$*,* $\mu>0$*,* $\nu+\frac{2}{3}\mu>0$*,* $\gamma\geq 2$*,* $M>0$*, and let* $(\triangle t)^{-1}$ *be constant, and let* $\delta>0$ *and* $\Gamma\geq 3$ *be fixed. Let* $(\varrho^{j-1},\mathbf{u}^{j-1},Y_k^{j-1})\in L^\gamma(\Omega)\times W^{1,2}(\Omega)\times W^{1,2}(\Omega)$ *be given functions such that*

$$Y_k^{j-1},\ \varrho^{j-1}\geq 0,\quad \sum_{k=1}^n Y_k^{j-1}=1 \text{ a.e. in } \Omega,\quad \varrho^{j-1}\left(Y_k^{j-1}\right)^2,\ \varrho^{j-1}|\mathbf{u}^{j-1}|^2\in L^1(\Omega),$$

$$\mathbf{F}_k(Y^{j-1})\cdot\mathbf{n}|_{\partial\Omega}=\mathbf{u}^{j-1}|_{\partial\Omega}=0.$$

*Then there exists a weak solution to (3.1)–(3.2) in the sense of Definition 3.1. Additionally, the pair* $(\varrho^j,\mathbf{u}^j)$*, extended by* $0$ *outside* $\Omega$ *is a renormalized solution to the continuity equation in the sense of Definition 3.2.*

## 4. Approximation

The purpose of this section is to prove Theorem 3.3. For this purpose we introduce a suitable regularization of system (3.1)–(3.2), indicated by the presence of three parameters $\varepsilon>0$ responsible for smoothing the solution to the continuity equation and the parameter of the Galerkin approximation $N\in\mathbb{N}$. The artificial pressure parameter $\delta>0$ was introduced in the previous section.

For fixed $\varepsilon$, $N$, and $\delta$, we will look for $(\varrho^j_{\delta,\varepsilon,N},\mathbf{u}^j_{\delta,\varepsilon,N},Y^j_{\delta,\varepsilon,N})$ (we will skip the subindexes when no confusion can arise) satisfying:

- the approximate continuity equation

$$\begin{aligned}(\triangle t)^{-1}\left(\varrho^j-\varrho^{j-1}\right)+\operatorname{div}(\varrho^j\mathbf{u}^j)-\varepsilon\Delta\varrho^j=0,\\ \nabla\varrho^j\cdot\mathbf{n}|_{\partial\Omega}=0,\end{aligned}\tag{4.1}$$

- the approximate momentum equation

$$\begin{aligned}&(\triangle t)^{-1}\int_\Omega(\varrho^j\mathbf{u}^j-\varrho^{j-1}\mathbf{u}^{j-1})\cdot\boldsymbol{\varphi}\,\mathrm{d}x-\int_\Omega(\varrho^j\mathbf{u}^j\otimes\mathbf{u}^j):\nabla\boldsymbol{\varphi}\,\mathrm{d}x+\int_\Omega\mathbf{S}(\mathbf{u}^j):\boldsymbol{\varphi}\,\mathrm{d}x\\ &-\int_\Omega\left(\pi(\varrho^j,\hat{Y}^j)+\delta(\varrho^j)^\Gamma\right)\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x+\varepsilon\int_\Omega\nabla\varrho^j\cdot\nabla\mathbf{u}^j\cdot\boldsymbol{\varphi}\,\mathrm{d}x=\int_\Omega\varrho^j\mathbf{f}^j\boldsymbol{\varphi}\,\mathrm{d}x,\end{aligned}\tag{4.2}$$

is satisfied for each

$$\boldsymbol{\varphi}\in W_N=\operatorname{span}\{\boldsymbol{\varphi}^1,\dots,\boldsymbol{\varphi}^N\}\subset W_0^{1,2}(\Omega);$$

i.e. the first $N$ eigenfunctions of the Laplace operator with Dirichlet boundary conditions,

- the approximate species balance equations

$$\begin{aligned}(\triangle t)^{-1}(\varrho^jY_k^j-\varrho^{j-1}Y_k^{j-1})+\operatorname{div}(\varrho^jY_k^j\mathbf{u}^j)+\operatorname{div}\mathbf{F}_k(Y_k^j)-\varepsilon\Delta\left(Y_k^j\varrho^j\right)=\varrho^j\omega_k(Y^j),\\ \mathbf{F}_k^\varepsilon\cdot\mathbf{n}|_{\partial\Omega}=0,\quad k\in\{1,\dots,n\}.\end{aligned}\tag{4.3}$$

The aim of this section is to prove the following theorem.

THEOREM 4.1. *Let $\varepsilon$, $\delta$, $\triangle t>0$, $N\in\mathbb{N}$, and $\Gamma>\frac{9}{2}$ be fixed. Let $(\varrho^{j-1},\mathbf{u}^{j-1},Y_k^{j-1})$ satisfy the assumptions of Theorem 3.3. Then, there exists $(\varrho^j,\mathbf{u}^j,Y^j)$, a regular solution to (4.1)–(4.3), such that $\varrho^j\in W^{2,p}(\Omega)$, $\mathbf{u}^j\in W_N$, $Y^j\in W^{2,p}(\Omega)$, and $k\in\{1,\dots,n\}$, for all $p<\infty$. Moreover, $\varrho^j\geq 0$ in $\Omega$, $\int_\Omega \varrho^j\,\mathrm{d}x=\int_\Omega \varrho^{j-1}\,\mathrm{d}x=M$, $Y_k^j\geq 0$, and $\sum_{k=1}^n Y_k^j=1$.*

The proof of this theorem is based on several auxiliary lemmas, and it is presented in the next subsection.

**4.1. Existence for fixed parameters.**
**Step 1:** We define the operator

$$\mathcal{S}\,:\,W_N\to W^{2,p}(\Omega),$$

$1\leq p<\infty$, $\mathcal{S}(\mathbf{u}^j)=\varrho^j$, where $\varrho^j$ solves the approximate continuity equation (4.1) with the Neumann boundary condition. We then claim that the following result holds true.

LEMMA 4.2. *Let the assumptions of Theorem 4.1 be satisfied. Then the operator $\mathcal{S}$ is well defined for all $p<\infty$. Moreover, if $\mathcal{S}(\mathbf{u}^j)=\varrho^j$, then $\varrho^j\geq 0$ in $\Omega$ and $\int_\Omega\varrho^j\,\mathrm{d}x=\int_\Omega\varrho^{j-1}\,\mathrm{d}x$. Additionally, if $\|\mathbf{u}^j\|_{W_N}\leq L$ and $L>0$, then*

$$\|\varrho^j\|_{2,p}\leq C(\varepsilon,p,\Omega)(1+L)\|\varrho^{j-1}\|_p,\quad 1<p<\infty. \tag{4.4}$$

The above lemma is an analogue of Proposition 4.29 from [31], so we omit the proof.

**Step 2:** Our next aim is to show the non-negativity of the species concentrations under the assumption that the solution to (4.1)–(4.3) is sufficiently smooth; i.e. $\varrho^j$, $\mathbf{u}^j$, and $Y_k^j\in W^{2,p}(\Omega)$, for any $p<\infty$, $k\in\{1,\dots,n\}$, and $\varrho^j\geq 0$.

For the first $n-1$ species, it will follow directly from the features of the species production terms. We test Equation (4.3) by $Y^j_{k\,-}=\min\{Y_k^j,0\}$. Note that this is a continuous function and that $Y_k^jY^j_{k\,-}=(Y^j_{k\,-})^2$ and $Y_k^j\nabla Y_k^j=Y^j_{k\,-}\nabla Y^j_{k\,-}$. Thus integrating by parts, we obtain

$$\begin{aligned}(\triangle t)^{-1}\int_\Omega\varrho^j(Y^j_{k\,-})^2\,\mathrm{d}x-\int_\Omega\varrho^jY_k^j\mathbf{u}^j\nabla Y^j_{k\,-}\,\mathrm{d}x+\int_\Omega(D+\varepsilon\varrho^j)|\nabla Y^j_{k\,-}|^2\,\mathrm{d}x\\ +\varepsilon\int_\Omega Y^j_{k\,-}\nabla\varrho^j\cdot\nabla Y^j_{k\,-}\,\mathrm{d}x=\int_\Omega\varrho^j\omega_k(Y^j)\,\mathrm{d}x+(\triangle t)^{-1}\int_\Omega\varrho^{j-1}Y_k^{j-1}Y^j_{k\,-}\,\mathrm{d}x.\end{aligned}$$

Note, that the first integral on the r.h.s. is non-positive due to assumptions imposed on $\omega_k$ (2.10). The second integral is non-positive due to assumptions on $Y_k^{j-1}$.

Next, we multiply (4.1) by $\frac{1}{2}(Y^j_{k\,-})^2$, and we add the resulting expression to the above equality to get

$$\frac{1}{2}(\triangle t)^{-1}\int_\Omega\varrho^j(Y^j_{k\,-})^2+\varrho^{j-1}(Y^j_{k\,-})^2\,\mathrm{d}x+\int_\Omega(D+\varepsilon\varrho^j)|\nabla Y^j_{k\,-}|^2\,\mathrm{d}x\leq 0.$$

By the fact that $\int_\Omega\varrho^j\,\mathrm{d}x=M>0$, we can hence conclude that $Y^j_{k\,-}=0$, and thus $Y_k^j\geq 0$.

Thus $Y_k^j\geq 0$ for $k\in\{1,\dots,n-1\}$, however, so far we do not know if $Y_k^j\leq 1$.

To show this, we define $Y_n^j=1-\sum_{k=1}^{n-1}Y_k^j$, derive the equation for $Y_n^j$ from the approximate continuity equation (4.1) and the first $n-1$ species Equation (4.3), and repeat the above procedure to deduce that $Y_n^j\geq 0$ in $\Omega$. Note, however, that this is possible only under the assumption that all the diffusion coefficients in (2.6) are equal to the same constant $D$.

REMARK 4.3. Note that the lower and upper bounds for $Y_i$, $i\in\{1,\dots,n\}$, do not depend on the approximation parameters. Thus in the course of subsequent limit passages, we will get that

$$0\le Y_k\le 1 \quad \text{a.e. in } \Omega. \tag{4.5}$$

**Step 3:** We now prove the existence of solutions to the momentum and the species mass balance equations for a given $\mathbf{u}$ and $\varrho$. The main idea consists of applying the Leray–Schauder fixed point theorem to the mapping

$$\mathcal{T}: W_N\times[W^{2,p}]^n\to W_N\times[W^{2,p}]^n, \quad \mathcal{T}(\mathbf{u}^j,Y_k^j)\to(\mathbf{w},X_k),$$

where $(\mathbf{w},X_k)$ is a solution to the boundary-value problem

$$\begin{aligned}&\int_\Omega \mathbf{S}(\mathbf{w}):\nabla\boldsymbol{\varphi}\,\mathrm{d}x\\ &=(\triangle t)^{-1}\int_\Omega(\varrho^{j-1}\mathbf{u}^{j-1}-\varrho^j\mathbf{u}^j)\boldsymbol{\varphi}\,\mathrm{d}x+\int_\Omega(\varrho^j\mathbf{u}^j\otimes\mathbf{u}^j)\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}x\\ &\quad+\int_\Omega\left(\pi(\varrho^j,\hat{Y}^j)+\delta(\varrho^j)^\Gamma\right)\mathrm{div}\boldsymbol{\varphi}\,\mathrm{d}x-\varepsilon\int_\Omega\nabla\varrho^j\cdot\nabla\mathbf{u}^j\cdot\boldsymbol{\varphi}\,\mathrm{d}x+\int_\Omega\varrho^j\mathbf{f}^j\boldsymbol{\varphi}\,\mathrm{d}x,\end{aligned} \tag{4.6}$$

$$\begin{aligned}&-\mathrm{div}((D+\varepsilon\varrho)\nabla X_k)\\ &=\varrho^j\omega_k(Y^j)+(\triangle t)^{-1}\varrho^{j-1}Y_k^{j-1}-(\triangle t)^{-1}\varrho^jY_k^j-\mathrm{div}\left(\varrho^jY_k^j\mathbf{u}^j\right)+\varepsilon\,\mathrm{div}(Y_k^j\nabla\varrho^j),\\ &\quad\nabla Y_k^j\cdot\mathbf{n}|_{\partial\Omega}=0,\end{aligned}$$

satisfied for $\boldsymbol{\varphi}\in W_N$ and $k\in\{1,\dots,n\}$ and with $\hat{Y}_k^j=Y_k^j$ for $Y_k^j\le 1$, or 1, elsewhere.

We prove the following lemma.

LEMMA 4.4. *Let the assumptions of Theorem 4.1 be fulfilled, and let $\varrho^j$ be given by Lemma 4.2. Then, the operator $\mathcal{T}$ is continuous and compact from $W_N\times[W^{2,p}(\Omega)]^n$ into itself.*

*Proof.* The existence and uniqueness of a solution to the system (4.6) is a consequence of the Lax–Milgram Theorem. Evidently, the mapping $\mathcal{T}$ is compact. Since the r.h.s. of (4.6) is sufficiently smooth and of lower order, it is also continuous. □

To conclude, we should show boundedness of possible fixed points to

$$\lambda\mathcal{T}(\mathbf{u}^j,Y_k^j)=(\mathbf{u}^j,Y_k^j), \quad \lambda\in[0,1]. \tag{4.7}$$

We first prove the following lemma.

LEMMA 4.5. *Let the assumptions of Theorem 4.1 be satisfied. Then there exists $c>0$ such that the solutions of (4.7) in the class $W_N\times[W^{2,p}(\Omega)]^n$ fulfill*

$$\|\mathbf{u}^j\|_{W_N}+\|Y_k^j\|_{W^{2,p}(\Omega)}\le c,$$

*independently of $t$.*

The second equality in (4.6) rewrites as

$$\begin{aligned}&-\mathrm{div}((D+\varepsilon\varrho^j)\nabla Y_k^j)\\ &=\lambda\left(\varrho^j\omega_k+(\triangle t)^{-1}\varrho^{j-1}Y_k^{j-1}-(\triangle t)^{-1}\varrho^jY_k^j-\mathrm{div}\left(\varrho^jY_k^j\mathbf{u}^j\right)+\varepsilon\,\mathrm{div}(Y_k^j\nabla\varrho^j)\right).\end{aligned} \tag{4.8}$$

Multiplying it by $Y_k^j$, integrating by parts, and using the boundary conditions, we get

$$\begin{aligned}&\int_\Omega (D+\varepsilon\varrho^j)|\nabla Y_k^j|^2\,\mathrm{d}x\\ =&\lambda(\triangle t)^{-1}\int_\Omega\left(\varrho^{j-1}Y_k^{j-1}Y_k^j-\varrho^j\left(Y_k^j\right)^2\right)\,\mathrm{d}x-\frac{\lambda\varepsilon}{2}\int_\Omega\nabla\varrho^j\cdot\nabla\left(Y_k^j\right)^2\,\mathrm{d}x\\ &-\frac{\lambda}{2}\int_\Omega\mathrm{div}(\varrho^j\mathbf{u}^j)\left(Y_k^j\right)^2\,\mathrm{d}x+\lambda\int_\Omega\varrho^j\omega_kY_k^j\,\mathrm{d}x.\end{aligned}$$

From the approximate continuity equation, we obtain

$$\begin{aligned}&\int_\Omega (D+\varepsilon\varrho^j)|\nabla Y_k^j|^2\,\mathrm{d}x+\lambda(\triangle t)^{-1}\int_\Omega\left(\varrho^{j-1}\frac{(Y_k^j)^2}{2}+\varrho^j\frac{(Y_k^j)^2}{2}\right)\,\mathrm{d}x\\ =&\lambda\int_\Omega\varrho^j\omega_kY_k^j\,\mathrm{d}x+\lambda(\triangle t)^{-1}\int_\Omega\varrho^{j-1}Y_k^{j-1}Y_k^j\,\mathrm{d}x.\end{aligned}\tag{4.9}$$

Using $\varphi=u^j$ in the first equality of (4.6), we obtain

$$\begin{aligned}&\int_\Omega\mathbf{S}(\mathbf{u}^j):\nabla\mathbf{u}^j\,\mathrm{d}x\\ =&\lambda\int_\Omega\left(\pi(\varrho^j,\hat{Y}^j)+\delta(\varrho^j)^\Gamma\right)\mathrm{div}\,\mathbf{u}^j\,\mathrm{d}x+\lambda(\triangle t)^{-1}\int_\Omega\left(\varrho^{j-1}\mathbf{u}^{j-1}-\varrho^j\mathbf{u}^j\right)\cdot\mathbf{u}^j\,\mathrm{d}x\\ &-\lambda\int_\Omega\mathrm{div}(\varrho^j\mathbf{u}^j\otimes\mathbf{u}^j)\cdot\mathbf{u}^j\,\mathrm{d}x-\varepsilon\lambda\int_\Omega\nabla\varrho^j\cdot\nabla\mathbf{u}^j\cdot\mathbf{u}^j\,\mathrm{d}x+\lambda\int_\Omega\varrho^j\mathbf{f}^j\cdot\mathbf{u}^j\,\mathrm{d}x.\end{aligned}\tag{4.10}$$

The first term on the l.h.s. can be used to control the norm of $\mathbf{u}$ in $W_0^{1,2}(\Omega)$. This is due to a simple generalization of the Korn inequality. We will prove that there exists a constant $c$ depending on $\Omega$ and $\mu$ such that

$$c\|\mathbf{u}^j\|^2_{W^{1,2}(\Omega)}\le\int_\Omega\mathbf{S}(\mathbf{u}^j):\nabla\mathbf{u}^j\,\mathrm{d}x.$$

Rewriting the viscous part of the stress tensor in the form

$$\mathbf{S}(\mathbf{u}^j)=2\mu\left(\mathbf{D}(\mathbf{u}^j)-\frac{1}{3}\mathrm{div}\,\mathbf{u}^j\mathbf{I}\right)+\xi\,\mathrm{div}\,\mathbf{u}^j\mathbf{I},$$

we can estimate

$$\begin{aligned}\int_\Omega\mathbf{S}(\mathbf{u}^j):\nabla\mathbf{u}^j\,\mathrm{d}x&\ge\mu\int_\Omega\left(|\nabla\mathbf{u}^j|^2+(\nabla\mathbf{u}^j)^T:\nabla\mathbf{u}-\frac{2}{3}(\mathrm{div}\,\mathbf{u}^j)^2\right)\,\mathrm{d}x\\ &=\mu\int_\Omega\left(|\nabla\mathbf{u}^j|^2+\frac{1}{3}(\mathrm{div}\,\mathbf{u}^j)^2\right)\,\mathrm{d}x,\end{aligned}$$

and we conclude by application of the Poincaré inequality.

Next, we use (4.1) with $\varrho=\mathcal{S}(\mathbf{u})$ to express

$$\begin{aligned}&\int_\Omega\nabla(\varrho^j)^\gamma\cdot\mathbf{u}^j\,\mathrm{d}x\\ =&\varepsilon\gamma\int_\Omega(\varrho^j)^{\gamma-2}|\nabla\varrho^j|^2\,\mathrm{d}x+(\triangle t)^{-1}\frac{s}{s-1}\int_\Omega(\varrho^j)^\gamma\,\mathrm{d}x-(\triangle t)^{-1}\frac{\gamma}{\gamma-1}\int_\Omega\varrho^{j-1}(\varrho^j)^{\gamma-1}\,\mathrm{d}x,\end{aligned}$$

and the same for the artificial pressure $\delta(\varrho^j)^\Gamma$. Then, integrating (4.10) by parts, we derive

$$\begin{aligned}&\int_\Omega \mathbf{S}(\mathbf{u}^j):\nabla\mathbf{u}^j\;\mathrm{d}x+\frac{t(\triangle t)^{-1}}{2}\int_\Omega\left(\varrho^{j-1}+\varrho^j\right)|\mathbf{u}^j|^2\;\mathrm{d}x+t\varepsilon\gamma\int_\Omega(\varrho^j)^{\gamma-2}|\nabla\varrho^j|^2\;\mathrm{d}x\\&+t\varepsilon\delta\Gamma\int_\Omega(\varrho^j)^{\Gamma-2}|\nabla\varrho^j|^2\;\mathrm{d}x+t(\triangle t)^{-1}\frac{\gamma}{\gamma-1}\int_\Omega(\varrho^j)^\gamma\;\mathrm{d}x+t(\triangle t)^{-1}\delta\frac{\Gamma}{\Gamma-1}\int_\Omega(\varrho^j)^\Gamma\;\mathrm{d}x\\=&t(\triangle t)^{-1}\frac{\gamma}{\gamma-1}\int_\Omega\varrho^{j-1}(\varrho^j)^{\gamma-1}\;\mathrm{d}x+t(\triangle t)^{-1}\frac{\delta\Gamma}{\Gamma-1}\int_\Omega\varrho^{j-1}(\varrho^j)^{\Gamma-1}\;\mathrm{d}x\\&+t\int_\Omega\sum_{k=1}^n\frac{\hat{Y}_k^j}{m_k}\varrho^j\operatorname{div}\mathbf{u}^j\;\mathrm{d}x+t(\triangle t)^{-1}\int_\Omega\varrho^{j-1}\mathbf{u}^{j-1}\cdot\mathbf{u}^j\;\mathrm{d}x+t\int_\Omega\varrho^j\mathbf{f}^j\cdot\mathbf{u}^j\;\mathrm{d}x.\end{aligned}\tag{4.11}$$

Summing up equations (4.9) and (4.11), using the Cauchy inequality, boundedness of $\omega_k$, and the equivalency of norms on $W_N$, we show

$$\|\mathbf{u}^j\|_{W_N}+\|Y_k^j\|_{W^{1,2}(\Omega)}\le c,\tag{4.12}$$

with $c$ a constant independent of $t$. Finally, we may estimate the norm of the second gradient of $Y_k$ directly from (4.8):

$$\begin{aligned}&-(D+\varepsilon\varrho^j)\Delta Y_k^j\\&=-t\left(\varrho^j\omega_k+(\triangle t)^{-1}\varrho^{j-1}Y_k^{j-1}-(\triangle t)^{-1}\varrho^jY_k^j-\operatorname{div}\left(\varrho^jY_k^j\mathbf{u}^j\right)+\varepsilon\operatorname{div}\left(Y_k^j\nabla\varrho^j\right)\right)\\&\quad-\varepsilon\nabla\varrho^j\cdot\nabla Y_k^j.\end{aligned}$$

Due to the regularity of $\varrho^j$ and $\mathbf{u}^j$ and the estimate (4.12), we first justify that $Y_k^j\in W^{2,2}(\Omega)\hookrightarrow W^{1,6}(\Omega)\hookrightarrow L^\infty(\Omega)$. Then, by the bootstrap procedure, we arrive at $\|Y_k\|_{W^{2,p}(\Omega)}\le c$ which finishes the proof of Lemma 4.5.

To conclude, we observe that (4.5) is valid. In particular, $\hat{Y}_k{}^j=Y_k^j$ in (4.6), and the proof of Theorem 4.1 is complete.

**4.2. Limit passage in the Galerkin approximation.** First, we need estimates which are uniform with respect to $N$. They can be deduced easily from (4.9) and (4.11) taking $t=1$. We have

$$\|\mathbf{u}_N^j\|_{W^{1,2}(\Omega)}+\sum_{k=1}^n\|Y_{k,N}^j\|_{W^{1,2}(\Omega)}+\|\varrho_N^j\|_{L^\Gamma(\Omega)}+\|\nabla\left(\varrho_N^j\right)^{\Gamma/2}\|_{L^2(\Omega)}\le c.\tag{4.13}$$

Moreover, using standard elliptic theory, we can deduce that $\varrho_N^j$ satisfies

$$\|\varrho_N^j\|_{W^{2,2}(\Omega)}\le c.\tag{4.14}$$

Using (4.13) and (4.14) and the imbedding theorems, we may justify the existence of a subsequence (denoted by $N$) such that

$$\begin{array}{lll}\varrho_N^j\to\varrho^j, & \text{weakly in } W^{2,2}(\Omega) & \text{and strongly in } W^{1,q}(\Omega),\ q<6,\\ \varrho_N^j\to\varrho^j, & \text{weakly}^* \text{ in } L^\infty(\Omega) & \\ \mathbf{u}_N^j\to\mathbf{u}^j, & \text{weakly in } W^{1,2}(\Omega) & \text{and strongly in } L^q(\Omega),\ q<6,\\ Y_{k,N}^j\to Y_k^j, & \text{weakly in } W^{1,2}(\Omega) & \text{and strongly in } L^q(\Omega),\ q<6,\\ Y_{k,N}^j\to Y_k^j, & \text{weakly}^* \text{ in } L^\infty(\Omega). & \end{array}$$

Having this, justification of the limit in (4.1), (4.2), and (4.3) is an easy exercise, so we skip the details.

**4.3. Uniform estimate of the pressure.** We again start by deriving some uniform estimates. The uniform estimates resulting from (4.9) and (4.11) are the following:

$$\|\mathbf{u}_\varepsilon^j\|^2_{W^{1,2}(\Omega)} + \sum_{k=1}^n \|Y^j_{k,\varepsilon}\|^2_{W^{1,2}(\Omega)} + \delta(\triangle t)^{-1}\|\varrho^j_\varepsilon\|^\Gamma_{L^\Gamma(\Omega)} + (\triangle t)^{-1}\|\varrho^j_\varepsilon\|^\gamma_{L^\gamma(\Omega)} + \varepsilon\delta\|\nabla\left(\varrho^j_\varepsilon\right)^{\Gamma/2}\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\left(\varrho^j_\varepsilon\right)^{\gamma/2}\|^2_{L^2(\Omega)} \le c. \quad (4.15)$$

However, we still need a better estimate of the pressure which can be obtained by application of the Bogovskii operator. We will show that when $\Gamma \ge 3$, the barotropic component of the pressure is bounded in a space slightly better than $L^1(\Omega)$ which becomes important in the course of all subsequent limit passages.

We test the approximate momentum equation (4.2) by a function

$$\Phi = \mathcal{B}\left(\left(\varrho^j\right)^\beta - \frac{1}{|\Omega|}\int_\Omega \left(\varrho^j\right)^\beta \,\mathrm{d}x\right),$$

where $\beta \in (0,1]$ and $\mathcal{B}$ is the Bogovskii operator defined by the following lemma.

Lemma 4.6 ([31], Lemma 3.17). *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exits a linear operator $\mathcal{B}_\Omega = \left(\mathcal{B}^1_\Omega, \mathcal{B}^2_\Omega, \mathcal{B}^3_\Omega\right)$ with the following properties:*

$$\mathcal{B}_\Omega : \overline{L^p}(\Omega) \to (W^{1,p}_0(\Omega))^3, \quad 1 < p < \infty,$$

$$\mathrm{div}\,(\mathcal{B}_\Omega(f)) = f \quad \textit{a.e. in } \Omega, f \in \overline{L^p}(\Omega),$$

$$\|\nabla \mathcal{B}_\Omega(f)\|_{L^p(\Omega)} \le c(p,\Omega)\|f\|_{L^p(\Omega)}, \quad 1 < p < \infty.$$

*If*

$$f = \mathrm{div}\,(g), \ \textit{with } g \in L^p(\Omega), \mathrm{div}\,(g) \in L^q(\Omega), \quad 1 < q < \infty,$$

*then*

$$\|\mathcal{B}_\Omega(f)\|_{L^q(\Omega)} \le c(q,\Omega)\|g\|_{L^q(\Omega)},$$

*where* $\overline{L^p(\Omega)} = \{f \in L^p(\Omega) : \int_\Omega f(y)\mathrm{d}y = 0\}$.
More recent results on this issue can be found, for example, in [5].

We know, in particular, that

$$\|\nabla\Phi\|_p \le c(p,\Omega)\|\left(\varrho^j\right)^\beta\|_p,$$

and due to the Sobolev imbedding,

$$\|\Phi\|_{\bar p} \le c(p,\Omega)\|\left(\varrho^j\right)^\beta\|_p, \quad 1 < p < \infty, \qquad \bar p = \begin{cases} \frac{3p}{3-p} & \text{if } p < 3, \\ \in [1,\infty) & \text{if } p = 3, \\ \infty & \text{if } p > 3. \end{cases}$$

For more details concerning the Bogovskii operato,r we refer to [31]. This testing results in the following identity:

$$
\begin{aligned}
&\int_\Omega \left( \left(\varrho^j\right)^{\gamma+\beta} + \delta \left(\varrho^j\right)^{\Gamma+\beta} \right) \, \mathrm{d}x \\
=&(\triangle t)^{-1} \int_\Omega \left(\varrho^j \mathbf{u}^j - \varrho^{j-1}\mathbf{u}^{j-1}\right) \cdot \Phi \, \mathrm{d}x - \int_\Omega \varrho^j \left(\mathbf{u}^j \otimes \mathbf{u}^j\right) : \nabla \Phi \, \mathrm{d}x \\
&+ \int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla \Phi \, \mathrm{d}x + \varepsilon \int_\Omega \nabla \varrho^j \cdot \nabla \mathbf{u}^j \cdot \Phi \, \mathrm{d}x + \int_\Omega \sum_{k=1}^n \frac{Y^j}{m_k} \left(\varrho^j\right)^{\beta+1} \, \mathrm{d}x \\
&+ \frac{1}{|\Omega|} \int_\Omega \left( \pi(\varrho^j, Y^j) + \delta \left(\varrho^j\right)^\Gamma \right) \, \mathrm{d}x \int_\Omega \left(\varrho^j\right)^\beta \, \mathrm{d}x - \int_\Omega \varrho^j \mathbf{f}^j \cdot \Phi \, \mathrm{d}x \\
=&\sum_{i=1}^7 I_i. \qquad (4.16)
\end{aligned}
$$

We will only estimate the most restrictive terms. The first is the convective term. We have

$$
I_2 \le \|\varrho^j\|_p \| |\mathbf{u}^j|^2 \|_3 \| \left(\varrho^j\right)^\beta \|_q \le t \|\varrho^j\|_p \|\mathbf{u}^j\|_{1,2}^2 \|\varrho^j\|_{q\beta}^\beta \le c \|\varrho^j\|_p \|\varrho\|_{q\beta}^\beta \qquad (4.17)
$$

where $\frac{1}{p} + \frac{1}{q} = \frac{2}{3}$. In the last inequality, we used (4.15) to estimate the norm of $\mathbf{u}^j$. Now we choose $p = q\beta = \frac{3(\beta+1)}{2}$, and we apply the interpolation inequality of the type $\|\varrho^j\|_p \le \|\varrho^j\|_1^\vartheta \|\varrho^j\|_{\gamma+\beta}^{1-\vartheta}$ which leads to the restriction

$$
\beta \le 2\gamma - 3.
$$

Next, we handle $I_4$. Employing the Hölder inequality, we get for $\frac{1}{\overline{p}'} + \frac{1}{\overline{p}} = \frac{1}{2}$,

$$
I_4 = \varepsilon \int_\Omega \nabla \varrho^j \cdot \nabla \mathbf{u}^j \cdot \Phi \, \mathrm{d}x \le c\varepsilon \|\nabla \varrho^j\|_{\overline{p}'} \|\mathbf{u}\|_{1,2} \|\Phi\|_{\overline{p}} \le c\varepsilon \|\nabla \varrho^j\|_2 \|\varrho^j\|_{\beta p}^\beta,
$$

for some $p > 3$. We choose $p$ such that $\beta p = \beta + \Gamma$, so $\Gamma > 2\beta$.

To get the estimate for $\|\nabla \varrho\|_2$, we need to interpret the approximate continuity equation as a Neumann-boundary problem

$$
\begin{aligned}
-\varepsilon \Delta \varrho &= \operatorname{div} \mathbf{b} \quad \textit{in } \Omega \\
\frac{\partial \varrho}{\partial \mathbf{n}} &= b \cdot \mathbf{n} \quad \textit{at } \partial\Omega,
\end{aligned} \qquad (4.18)
$$

with the right hand side

$$
\mathbf{b} = (\triangle t)^{-1} \mathcal{B}(h - \varrho) - \varrho \mathbf{u}.
$$

From the classical theory, we know that if $\partial\Omega$ is smooth enough and if $b \in L^q(\Omega)$, then there exists a unique $\varrho \in W^{1,q}(\Omega)$ satisfying (4.18) in the weak sense, such that $\int_\Omega \varrho \, \mathrm{d}x =$ *const*. Moreover,

$$
\|\nabla \varrho\|_q \le \frac{c(q,\Omega)}{\varepsilon} \|\mathbf{b}\|_q. \qquad (4.19)
$$

In our case, it is enough to see that the q-norm of $b$ may be estimated as

$$
\|\mathbf{b}\|_2 \le c(\triangle t)^{-1} (\|\varrho\|_\gamma + \|h\|_\gamma) + \|\varrho\|_3 \|\mathbf{u}\|_6. \qquad (4.20)
$$

Thus, observation (4.19) together with (4.15) and assumption $\Gamma \ge 3$ yields the following estimate of $I_4$:

$$
I_4 \le c(\delta) \|\varrho\|_{\beta+\Gamma}^\beta.
$$

Therefore, from (4.16), we deduce that independently of $\varepsilon$ we have

$$\|\varrho\|_{(1+\beta)\gamma}+\delta\|\varrho\|_{(1+\beta)\Gamma}\leq c(\delta), \tag{4.21}$$

for $\gamma\geq 2$, $\Gamma\geq 3$.

**4.4. Limit passage $\varepsilon\to 0$.** The estimates from the previous sections, (4.12), (4.15), and (4.21), can be used to deduce that, at least for suitable subsequences, we have

$$\mathbf{u}^j_\varepsilon\to\mathbf{u} \quad \text{weakly in } W^{1,2}(\Omega), \tag{4.22}$$

$$\varrho^j_\varepsilon\to\varrho \quad \text{weakly in } L^{(1+\beta)\gamma}(\Omega)\cap L^{(1+\beta)\Gamma}(\Omega), \tag{4.23}$$

$$\varepsilon\nabla\varrho^j_\varepsilon\to 0 \quad \text{strongly in } L^2(\Omega), \tag{4.24}$$

$$Y^j_{k,\varepsilon}\to Y_k \quad \text{weakly in } W^{1,2}(\Omega), \text{ strongly in } L^p(\Omega),\ p<6, \tag{4.25}$$

$$Y^j_{k,\varepsilon}\to Y_k \quad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{4.26}$$

We are, hence, in a position to conclude that there exists $(\varrho^j,\mathbf{u}^j,Y^j)$ which satisfies the integral equalities:

$$(\triangle t)^{-1}\int_\Omega\left(\varrho^j-\varrho^{j-1}\right)\xi\ \mathrm{d}x-\int_\Omega\varrho^j\mathbf{u}^j\cdot\nabla\xi\ \mathrm{d}x=0, \quad \forall\xi\in C^\infty(\overline{\Omega}),$$

$$\begin{aligned}&(\triangle t)^{-1}\int_\Omega\left(\varrho^j\mathbf{u}^j-\varrho^{j-1}\mathbf{u}^{j-1}\right)\boldsymbol{\varphi}\ \mathrm{d}x-\int_\Omega\varrho^j\left(\mathbf{u}^j\otimes\mathbf{u}^j\right):\nabla\boldsymbol{\varphi}\ \mathrm{d}x\\&\quad+\int_\Omega\mathbf{S}(\mathbf{u}^j):\nabla\boldsymbol{\varphi}\ \mathrm{d}x-\int_\Omega\overline{\pi(\varrho,Y)+\delta\varrho^\Gamma}\,\mathrm{div}\boldsymbol{\varphi}\ \mathrm{d}x\\&=\int_\Omega\varrho^j\mathbf{f}^j\cdot\boldsymbol{\varphi}\ \mathrm{d}x, \quad \forall\boldsymbol{\varphi}\in C_0^\infty(\Omega),\end{aligned}$$

$$\begin{aligned}&(\triangle t)^{-1}\int_\Omega\left(\varrho^jY^j_k-\varrho^{j-1}Y^{j-1}_k\right)\phi\ \mathrm{d}x-\int_\Omega\varrho^jY^j_k\mathbf{u}^j\cdot\nabla\phi\ \mathrm{d}x-\int_\Omega D\nabla Y^j_k\cdot\nabla\phi\ \mathrm{d}x\\&=\int_\Omega\varrho^j\omega_k(Y^j)\phi\ \mathrm{d}x \quad \forall\phi\in C^\infty(\overline{\Omega}),\end{aligned}$$

for $k\in\{1,\dots,n\}$. Here and in the sequel, $\overline{g(\varrho^j,\mathbf{u}^j,Y^j)}$ denotes the weak limit of a sequence $g(\varrho^j_\varepsilon,\mathbf{u}^j_\varepsilon,Y^j_\varepsilon)$.

To conclude, one needs to verify if $\overline{\pi(\varrho,Y)+\delta\varrho^\Gamma}=\pi(\varrho,Y)+\delta\varrho^\Gamma$. In view of the strong convergence of $Y_k$ for $k\in K$, the positive answer to this question is in fact equivalent to the strong convergence of the density.

Since the strong convergence of the sequence approximating the density cannot be deduced from the system directly, we will apply the technique introduced by Lions [18]. It is based on an observation that the missing compactness can be replaced by the compactness of a quantity called the effective viscous flux or the effective pressure.

In order to proceed, we first observe that, since $\varrho_\varepsilon\mathbf{u}_\varepsilon$ and $\nabla\varrho_\varepsilon$ possess zero normal traces, it is possible to extend the approximate continuity equation to all of $\mathbb{R}^3$:

$$(\triangle t)^{-1}1_\Omega\varrho^j_\varepsilon+\mathrm{div}(1_\Omega\varrho^j_\varepsilon\mathbf{u}^j_\varepsilon)=\varepsilon\,\mathrm{div}(1_\Omega\nabla\varrho^j_\varepsilon)+(\triangle t)^{-1}1_\Omega\varrho^{j-1}_\varepsilon. \tag{4.27}$$

To derive a key equality for this reasoning, we introduce the inverse divergence operator $\mathcal{A}=\nabla\Delta^{-1}$ and the double Riesz transform $\mathcal{R}=\nabla\otimes\nabla\Delta^{-1}$, whose definition and main

properties are recalled in the appendix.
We test the approximate momentum equation by the function

$$\varphi(x) = \zeta(x)\phi, \quad \phi = (\nabla\Delta^{-1})[1_\Omega \varrho_\varepsilon^j],\ \zeta \in C_0^\infty(\Omega).$$

Note that this operation "gains" one derivative. Thus using only the $L^\Gamma(\Omega)$ integrability of $\varrho_\varepsilon$, we justify that this is an admissible test function. Evidently, $\sum_{i=1}^3 \mathcal{R}_{i,i}[v] = v$. Thus integrating by parts, we obtain the following equivalence:

$$\begin{aligned}
&\int_\Omega \zeta\left((\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma)\varrho_\varepsilon^j - \mathbf{S}(\mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j]\right)\, dx \\
&= -(\triangle t)^{-1} \int_\Omega \zeta\left(\varrho_\varepsilon^j u_{\varepsilon,i}^j - \varrho_\varepsilon^{j-1} u_{\varepsilon,i}^{j-1}\right)\mathcal{A}_i[1_\Omega \varrho_\varepsilon^j]\, dx \\
&\quad - \int_\Omega \zeta \varrho_\varepsilon^j u_{\varepsilon,i}^j u_{\varepsilon,k}^j \mathcal{R}_{i,k}[1_\Omega \varrho_\varepsilon^j]\, dx - \int_\Omega \varrho_\varepsilon^j u_{\varepsilon,i}^j u_{\varepsilon,k}^j \partial_k \zeta \mathcal{A}_i[1_\Omega \varrho_\varepsilon]\, dx \\
&\quad + \int_\Omega S_{i,k}\partial_k \zeta \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j]\, dx - \int_\Omega \left(\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma\right)\partial_i \zeta \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j]\, dx \\
&\quad + \varepsilon \int_\Omega \zeta \nabla \varrho_\varepsilon^j \cdot \nabla u_{\varepsilon,i}^j \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j]\, dx - \int_\Omega \zeta \varrho^j f_i^j \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j]\, dx,
\end{aligned} \tag{4.28}$$

where we used the Einstein summation convention. Adding and subtracting $\int_\Omega \zeta \varrho_\varepsilon^j u_{\varepsilon,k}^j \mathcal{R}_{i,k}[1_\Omega \varrho_\varepsilon^j u_{\varepsilon,i}^j]\, dx$, we may rewrite the r.h.s. in a form which lets the commutator appear. Then, using the fact that $R_{i,j}[v] = \partial_i \mathcal{A}_j[v]$, the basic properties of the Riesz operator,

$$\mathcal{R}_{i,j}[v] = \mathcal{R}_{j,i}[v], \quad \int_{\mathbb{R}^3} \mathcal{R}_{i,j}[v]u\, dx = \int_{\mathbb{R}^3} \mathcal{R}_{j,i}[v]u\, dx, \quad v \in L^p(\mathbb{R}^3),\ u \in L^{p'}(\mathbb{R}^3),$$

and the approximate continuity equation, we transform (4.28) into

$$\begin{aligned}
&\int_\Omega \zeta\left((\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma)\varrho_\varepsilon^j - \mathbf{S}(\mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j]\right)\, dx \\
&= \int_\Omega \zeta\left(\varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \mathcal{R}[1_\Omega \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j] - \varrho_\varepsilon^j(\mathbf{u}_\varepsilon^j \otimes \mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j]\right)\, dx \\
&\quad - \int_\Omega \zeta \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \nabla\Delta^{-1}[\operatorname{div} 1_\Omega \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j]\, dx + (\triangle t)^{-1}\int_\Omega \zeta\left(\varrho_\varepsilon^j \mathbf{u}_\varepsilon^j - \varrho_\varepsilon^{j-1}\mathbf{u}_\varepsilon^{j-1}\right)\cdot \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\, dx \\
&\quad - \int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon^j \otimes \mathbf{u}_\varepsilon^j) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\, dx + \int_\Omega \mathbf{S}(\mathbf{u}_\varepsilon^j) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\, dx \\
&\quad - \int_\Omega \left(\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma\right)\nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\, dx \\
&\quad + \varepsilon \int_\Omega \zeta \nabla \varrho_\varepsilon^j \cdot \nabla \mathbf{u}_\varepsilon^j \cdot \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\, dx - \int_\Omega \zeta \varrho_\varepsilon^j \mathbf{f}^j \cdot \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\, dx \\
&= \sum_{i=1}^8 I_i.
\end{aligned} \tag{4.29}$$

Finally, $I_2$ may be expressed by means of an approximate (extended) continuity Equation (4.27) in the following way:

$$I_2 = -\varepsilon \int_\Omega \zeta \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \nabla \Delta^{-1}[\operatorname{div} 1_\Omega \nabla \varrho_\varepsilon^j]\, \mathrm{d}x + (\triangle t)^{-1} \int_\Omega \zeta \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \nabla \Delta^{-1}[1_\Omega(\varrho_\varepsilon^j - \varrho_\varepsilon^{j-1})]\, \mathrm{d}x. \tag{4.30}$$

We will compare (4.29) with a similar expression obtained by testing the limit momentum equation with the function

$$\varphi(x) = \zeta(x)\phi, \quad \phi = (\nabla \Delta^{-1})[1_\Omega \varrho^j],\ \zeta \in C_0^\infty(\Omega).$$

Note that this is still an admissible test function since $\Gamma \geq 3$. We have

$$\begin{aligned}
&\int_\Omega \zeta \left( \overline{\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma}\varrho^j - \mathbf{S}(\mathbf{u}^j) : \mathcal{R}[1_\Omega \varrho^j] \right) \mathrm{d}x \\
&= \int_\Omega \zeta \left( \varrho^j \mathbf{u}^j \cdot \mathcal{R}[1_\Omega \varrho^j \mathbf{u}^j] - \varrho^j (\mathbf{u}^j \otimes \mathbf{u}^j) : \mathcal{R}[1_\Omega \varrho^j] \right) \mathrm{d}x - \int_\Omega \zeta \varrho^j \mathbf{u}^j \cdot \nabla \Delta^{-1}[1_\Omega \varrho^{j-1}]\, \mathrm{d}x \\
&+ (\triangle t)^{-1} \int_\Omega \zeta \left( \varrho^j \mathbf{u}^j - \varrho^{j-1} \mathbf{u}^{j-1} \right) \cdot \nabla \Delta^{-1}[1_\Omega \varrho^j]\, \mathrm{d}x \\
&- \int_\Omega \varrho^j (\mathbf{u}^j \otimes \mathbf{u}^j) : \nabla \zeta \otimes \nabla \Delta^{-1}[1_\Omega \varrho^j]\, \mathrm{d}x + \int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla \zeta \nabla \Delta^{-1}[1_\Omega \varrho^j]\, \mathrm{d}x \\
&- \int_\Omega \overline{\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma} \nabla \zeta \otimes \nabla \Delta^{-1}[1_\Omega \varrho^j]\, \mathrm{d}x - \int_\Omega \zeta \varrho^j \mathbf{f}^j \cdot \nabla \Delta^{-1}[1_\Omega \varrho^j]\, \mathrm{d}x \\
&= \sum_{i=1}^7 I_i.
\end{aligned} \tag{4.31}$$

Then, we see that

$$(\nabla \Delta^{-1})[1_\Omega \varrho_\varepsilon^j] \to (\nabla \Delta^{-1})[1_\Omega \varrho^j] \quad \text{in } C(\overline{\Omega}), \tag{4.32}$$

which is the consequence of Lemma A.1. Recalling (4.22)–(4.25) we show that the $\varepsilon$-dependent integral on the r.h.s. of (4.29) vanishes, whence $I_2, \ldots, I_6$ converge to their counterparts in (4.31).

In what follows, we give some more details of these limit passages. Firstly, due to the compact imbedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p < 6$, we have

$$\mathbf{u}_\varepsilon^j \to \mathbf{u}^j \quad \text{strongly in } L^p(\Omega), \quad 1 \leq p < 6. \tag{4.33}$$

Also taking into account (4.23), we therefore get

$$\mathbf{u}_\varepsilon^j \varrho_\varepsilon^j \to \mathbf{u}^j \varrho^j \quad \text{weakly in } L^p(\Omega), \quad 1 \leq p < \frac{6(1+\beta)\gamma}{6 + (1+\beta)\gamma}. \tag{4.34}$$

Since $\frac{6(1+\beta)\gamma}{6+(1+\beta)\gamma} > 2$ for $\gamma \geq 2$, and by virtue of (4.24) and Lemma A.1, we check that the $\varepsilon$-dependent component in (4.30) tends to 0; i.e. $\varepsilon \int_\Omega \zeta \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \Delta^{-1}[\operatorname{div} 1_\Omega \nabla \varrho_\varepsilon]\, \mathrm{d}x \to 0$. Next, due to (4.22) and (4.24), $\varepsilon(\nabla \varrho_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \to 0$ weakly in $L^1(\Omega)$, which coupled with (4.32), implies the zero limit of $I_7$.

Therefore, by letting $\varepsilon$ go to 0 in (4.29) and comparing the limit with (4.31), we verify that

$$
\begin{aligned}
&\int_\Omega \zeta\left(\left(\pi(\varrho_\varepsilon^j,Y_\varepsilon^j)+\delta(\varrho_\varepsilon^j)^\Gamma\right)\varrho_\varepsilon^j-\mathbf{S}(\mathbf{u}_\varepsilon^j):\mathcal{R}[1_\Omega\varrho_\varepsilon^j]\right)\,\mathrm{d}x\\
&\quad-\int_\Omega \zeta\left(\varrho_\varepsilon^j\mathbf{u}_\varepsilon^j\cdot\mathcal{R}[1_\Omega\varrho_\varepsilon^j\mathbf{u}_\varepsilon^j]-\varrho_\varepsilon^j(\mathbf{u}_\varepsilon^j\otimes\mathbf{u}_\varepsilon^j):\mathcal{R}[1_\Omega\varrho_\varepsilon^j]\right)\,\mathrm{d}x\\
&\to\int_\Omega \zeta\left(\overline{\pi(\varrho^j,Y^j)+\delta(\varrho^j)^\Gamma}\varrho^j-\mathbf{S}(\mathbf{u}^j):\mathcal{R}[1_\Omega\varrho^j]\right)\,\mathrm{d}x\\
&\quad-\int_\Omega \zeta\left(\varrho^j\mathbf{u}^j\cdot\mathcal{R}[1_\Omega\varrho^j\mathbf{u}^j]-\varrho^j(\mathbf{u}^j\otimes\mathbf{u}^j):\mathcal{R}[1_\Omega\varrho^j]\right)\,\mathrm{d}x. \qquad (4.35)
\end{aligned}
$$

Our next aim is to show that the last terms on both sides cancel.

For this purpose, we take $\mathbf{V}_\varepsilon^j=\varrho_\varepsilon^j\mathbf{u}_\varepsilon^j$ and $r_\varepsilon^j=\varrho_\varepsilon^j$ and check that they fulfill the assumptions of Lemma A.2 with $p=\frac{6(1+\beta)\gamma}{(1+\beta)\gamma+6}$ and $q=(1+\beta)\gamma$, where by $\varrho_\varepsilon^j$ $\mathbf{u}_\varepsilon^j$, $\varrho^j$, and $\mathbf{u}^j$, we mean the functions extended by 0 outside $\Omega$. Thus, there is enough room to choose $s>2$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{s}$, and so Lemma A.2 yields

$$
\varrho_\varepsilon^j\mathbf{u}_\varepsilon^j\cdot\mathcal{R}[1_\Omega\varrho_\varepsilon^j\mathbf{u}_\varepsilon^j]-\varrho_\varepsilon^j(\mathbf{u}_\varepsilon^j\otimes\mathbf{u}_\varepsilon^j):\mathcal{R}[1_\Omega\varrho_\varepsilon^j]\to\varrho^j\mathbf{u}^j\cdot\mathcal{R}[1_\Omega\varrho^j\mathbf{u}^j]-\varrho^j(\mathbf{u}^j\otimes\mathbf{u}^j):\mathcal{R}[1_\Omega\varrho^j],
$$

weakly in $L^s(\Omega)$. Substituting this result into (4.35), we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\int_\Omega \zeta\left(\left(\pi(\varrho_\varepsilon,Y_\varepsilon)+\delta\varrho^\Gamma\right)\varrho_\varepsilon-\mathbf{S}(\mathbf{u}_\varepsilon):\mathcal{R}[1_\Omega\varrho_\varepsilon]\right)\,\mathrm{d}x\\
&=\int_\Omega \zeta\left(\overline{\pi(\varrho,Y)+\delta\varrho^\Gamma}\varrho-\mathbf{S}(\mathbf{u}):\mathcal{R}[1_\Omega\varrho]\right)\,\mathrm{d}x. \qquad (4.36)
\end{aligned}
$$

We express $\mathbf{S}(\mathbf{u}_\varepsilon^j):\mathcal{R}[1_\Omega\varrho_\varepsilon^j]$ and $\mathbf{S}(\mathbf{u}^j):\mathcal{R}[1_\Omega\varrho^j]$ in terms of the divergence of $\mathbf{u}_\varepsilon^j$ and $\mathbf{u}^j$, respectively. For the second part of (2.7), we have

$$
\nu\,\mathrm{div}\,\mathbf{u}_\varepsilon^j\mathbf{I}:\mathcal{R}[1_\Omega\varrho_\varepsilon^j]=\nu\sum_{i=1}^3\mathrm{div}\,\mathbf{u}_\varepsilon^j\mathcal{R}_{i,i}[1_\Omega\varrho_\varepsilon^j]=\nu 1_\Omega\,\mathrm{div}\,\mathbf{u}_\varepsilon^j\varrho_\varepsilon^j.
$$

To handle the first part, we integrate by parts and we check that

$$
\int_\Omega \zeta\mu\left(\nabla\mathbf{u}_\varepsilon^j+(\nabla\mathbf{u}_\varepsilon^j)^T\right):\mathcal{R}[1_\Omega\varrho_\varepsilon^j]\,\mathrm{d}x=\int_\Omega\mathcal{R}:[\zeta\mu\left(\nabla\mathbf{u}_\varepsilon^j+(\nabla\mathbf{u}_\varepsilon^j)^T\right)]\varrho_\varepsilon^j\,\mathrm{d}x. \qquad (4.37)
$$

Observe that $\mathcal{R}:\left[\nabla\mathbf{u}_\varepsilon+(\nabla\mathbf{u}_\varepsilon)^T\right]=2\sum_{i,j=1}^3\partial_i\mathcal{A}_j\partial_j u_{\varepsilon,i}=2\sum_i^3\partial_i\sum_{j=1}^3\mathcal{R}_{j,j}u_i=2\,\mathrm{div}\,\mathbf{u}_\varepsilon$. Thus, the r.h.s. of (4.37) can be rewritten as

$$
\begin{aligned}
&\int_\Omega\mathcal{R}:[\zeta2\mu\mathbf{D}(\mathbf{u}_\varepsilon)]\varrho_\varepsilon\,\mathrm{d}x\\
&=\int_\Omega\zeta2\mu\,\mathrm{div}\,\mathbf{u}_\varepsilon\varrho_\varepsilon\,\mathrm{d}x+\int_\Omega\left(\mathcal{R}:[\zeta2\mu\mathbf{D}(\mathbf{u}_\varepsilon)]-\zeta\mathcal{R}:[2\mu\mathbf{D}(\mathbf{u}_\varepsilon)]\right)\varrho_\varepsilon\,\mathrm{d}x.
\end{aligned}
$$

Repeating the same procedure for the limit stress tensor $\mathbf{S}(\mathbf{u})$, we obtain from (4.36) the following expression:

$$\begin{aligned}
&\int_\Omega \zeta\left(\overline{(\pi(\varrho^j,Y^j)+\delta(\varrho^j)^\Gamma)\,\varrho^j} - (2\mu+\nu)\overline{\operatorname{div}\mathbf{u}^j\varrho^j}\right)\,\mathrm{d}x\\
&=\int_\Omega \zeta\left(\overline{\pi(\varrho^j,Y^j)+\delta(\varrho^j)^\Gamma}\varrho^j - (2\mu+\nu)\operatorname{div}\mathbf{u}^j\varrho^j\right)\,\mathrm{d}x\\
&\quad+\lim_{\varepsilon\to 0}\int_\Omega \left(\mathcal{R}:[\zeta 2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)] - \zeta\mathcal{R}:[2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)]\right)\varrho^j_\varepsilon\,\mathrm{d}x\\
&\quad-\int_\Omega \left(\mathcal{R}:[\zeta 2\mu\mathbf{D}(\mathbf{u}^j)] - \zeta\mathcal{R}:[2\mu\mathbf{D}(\mathbf{u}^j)]\right)\varrho^j\,\mathrm{d}x. \qquad (4.38)
\end{aligned}$$

In order to show that the two last integrals cancel, we will apply Lemma A.3 to each row of the matrix $\mathbf{D}(\mathbf{u}^j_\varepsilon)$. I.e., we take

$$w=\zeta,\quad V^j_i=\partial_i u^j_{\varepsilon,k}+\partial_k u^j_{\varepsilon,i},\ j=1,2,3.$$

By virtue of (4.15), $\mathbf{V}^j\in L^2(\mathbb{R}^3)$. Since $\zeta$ extended by 0 outside $\Omega$ belongs, in particular, to $W^{1,\infty}(\mathbb{R}^3)$, we can take any $s\in(1,6)$ and $\alpha=\frac{6-s}{2s}$ for which

$$\|\mathcal{R}:[\zeta 2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)] - \zeta\mathcal{R}:[2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)]\|_{W^{\alpha,s}(\mathbb{R}^3)}\le c.$$

Next, we may use the fact that $W^{\alpha,s}(\mathbb{R}^3)$ is continuously embedded into $L^a(\mathbb{R}^3)$ for any $1\le a\le 6$ and that the embedding is compact for $a<6$. Moreover, since $\frac{1}{q}=\frac{1}{a}+\frac{1}{(1+\beta)\gamma}<1$, we have that

$$\left(\mathcal{R}:[\zeta 2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)] - \zeta\mathcal{R}:[2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)]\right)\varrho^j_\varepsilon\to\left(\mathcal{R}:[\zeta 2\mu\mathbf{D}(\mathbf{u}^j)] - \zeta\mathcal{R}:[2\mu\mathbf{D}(\mathbf{u}^j)]\right)\varrho^j$$

weakly in $L^q(\mathbb{R}^3)$. Therefore (4.38) reduces to the following remarkable identity:

$$\begin{aligned}
&\int_\Omega \zeta\left(\overline{\pi(\varrho^j,Y^j)\varrho^j+\delta(\varrho^j)^{\Gamma+1}} - (2\mu+\nu)\overline{\operatorname{div}\mathbf{u}^j\varrho^j}\right)\,\mathrm{d}x\\
&=\int_\Omega \zeta\left(\overline{\pi(\varrho^j,Y^j)+\delta(\varrho^j)^\Gamma}\varrho - (2\mu+\nu)\operatorname{div}\mathbf{u}^j\varrho^j\right)\,\mathrm{d}x. \qquad (4.39)
\end{aligned}$$

In what follows, we will exploit (4.39) by use of the renormalized continuity equation. The following result is a consequence of a technique introduced and developed by DiPerna and Lions [6]. Applying it to the continuity equation (extended by 0 outside $\Omega$), we obtain.

LEMMA 4.7. *Let $\varrho^{j-1}$, $\varrho^j\in L^p(\mathbb{R}^3)$, $p\ge 2$, $\varrho\ge 0$, a.e. in $\Omega$, and $\mathbf{u}\in W^{1,2}_0(\mathbb{R}^3)$ satisfy the continuity equation*

$$(\triangle t)^{-1}\varrho^j+\operatorname{div}(\varrho^j\mathbf{u}^j)=(\triangle t)^{-1}\varrho^{j-1}$$

*in the sense of distributions on $\mathbb{R}^3$.*

*Then the pair $(\varrho^j,\mathbf{u}^j)$ solves the renormalized continuity equation (3.4) in the sense of distributions on $\mathbb{R}^3$ where $b(\cdot)$ is specified as follows:*

$$\begin{gathered}
b\in C([0,\infty)\cap C^1((0,\infty)),\\
\lim_{s\to 0^+}(sb'(s)-b(s))\in\mathbb{R},\\
|b'(s)|\le Cs^\lambda,\quad s\in(1,\infty),\quad \lambda\le\tfrac{p}{2}-1.
\end{gathered}$$

The best general reference here is [10], Section 10.18; see also [31].

Applying Lemma 4.7 to the limit continuity equation, we can verify that the pair of functions $(\varrho,\mathbf{u})$ extended by zero outside of $\Omega$ is a solution to the renormalized continuity equation, as specified in Definition 3.2. Moreover, taking $b(\varrho^j)=\varrho^j\log\varrho^j$, it can be deduced from (3.4) and from (3.3) with $\xi=1$ that

$$(\triangle t)^{-1}\int_\Omega \left(\varrho^j-\varrho^{j-1}\right)\log\varrho^j \ \mathrm{d}x+\int_\Omega \varrho^j \operatorname{div}\mathbf{u}^j \ \mathrm{d}x=0.$$

We now test the approximate continuity equation (4.1) with $\log(\varrho^j_\varepsilon+\eta)$, $\eta>0$. Note that due to Lemma 4.2 this is an admissible test function:

$$(\triangle t)^{-1}\int_\Omega (\varrho^j_\varepsilon-\varrho^{j-1}_\varepsilon)\log(\varrho^j_\varepsilon+\eta)\ \mathrm{d}x-\int_\Omega \varrho^j_\varepsilon\mathbf{u}^j_\varepsilon\cdot\frac{\nabla\varrho^j_\varepsilon}{\varrho^j_\varepsilon+\eta}\ \mathrm{d}x-\varepsilon\int_\Omega \Delta\varrho^j_\varepsilon\log(\varrho^j_\varepsilon+\eta)\ \mathrm{d}x=0.$$

Integrating by parts in the last integral and using the boundary conditions, we obtain the following expression:

$$(\triangle t)^{-1}\int_\Omega (\varrho^j_\varepsilon-\varrho^{j-1}_\varepsilon)\log(\varrho^j_\varepsilon+\eta)\ \mathrm{d}x-\int_\Omega \varrho^j_\varepsilon\mathbf{u}^j_\varepsilon\cdot\frac{\nabla\varrho^j_\varepsilon}{\varrho^j_\varepsilon+\eta}\ \mathrm{d}x+\varepsilon\int_\Omega \frac{|\nabla\varrho^j_\varepsilon|^2}{\varrho^j_\varepsilon+\eta}\ \mathrm{d}x=0.$$

But the last term is nonnegative, so we have

$$(\triangle t)^{-1}\int_\Omega (\varrho^j_\varepsilon-\varrho^{j-1}_\varepsilon)\log(\varrho^j_\varepsilon+\eta)\ \mathrm{d}x-\int_\Omega \varrho^j_\varepsilon\mathbf{u}^j_\varepsilon\cdot\frac{\nabla\varrho^j_\varepsilon}{\varrho^j_\varepsilon+\eta}\ \mathrm{d}x\le 0. \tag{4.40}$$

Next, let $\eta\to 0$. Due to the regularity of $\varrho^j_\varepsilon$ and $\mathbf{u}^j_\varepsilon$, the only problematic term is $-\int_\Omega \varrho^{j-1}_\varepsilon\log(\varrho^j_\varepsilon+\eta)\ \mathrm{d}x$ for $\varrho^j_\varepsilon<1-\eta$, but from the above inequality, we may deduce that

$$-\int_\Omega \varrho^{j-1}_\varepsilon\log(\varrho^j_\varepsilon+\eta)\ \mathrm{d}x\le -\int_\Omega \varrho^j_\varepsilon\log(\varrho^j_\varepsilon+\eta)\ \mathrm{d}x+\int_\Omega \varrho^j_\varepsilon\mathbf{u}^j_\varepsilon\cdot\frac{\nabla\varrho^j_\varepsilon}{\varrho^j_\varepsilon+\eta}\ \mathrm{d}x\le c(\varepsilon).$$

We thus use the Lebesgue Monotone Convergence Theorem to show that when $\eta\to 0$ we have

$$-\int_\Omega \varrho^{j-1}_\varepsilon\log(\varrho^j_\varepsilon+\eta)\ \mathrm{d}x\to -\int_\Omega \varrho^{j-1}_\varepsilon\log\varrho^j_\varepsilon\ \mathrm{d}x.$$

Integrating the second term in (4.40) by parts once more, we end up with

$$(\triangle t)^{-1}\int_\Omega (\varrho_\varepsilon-\varrho^{j-1}_\varepsilon)\log\varrho^j_\varepsilon\ \mathrm{d}x+\int_\Omega \operatorname{div}\mathbf{u}^j_\varepsilon\varrho^j_\varepsilon\ \mathrm{d}x\le 0, \tag{4.41}$$

so, after letting $\varepsilon\to 0$ we finally arrive at

$$(\triangle t)^{-1}\int_\Omega \overline{(\varrho^j-\varrho^{j-1})\log\varrho^j}\ \mathrm{d}x+\int_\Omega \overline{\operatorname{div}\mathbf{u}^j\varrho^j}\ \mathrm{d}x\le 0.$$

As a consequence, identity (4.39) may be transformed into:

$$\begin{aligned}&\int_\Omega \overline{\pi(\varrho^j,Y^j)\varrho^j+\delta(\varrho^j)^{\Gamma+1}}\ \mathrm{d}x+(2\mu+\nu)(\triangle t)^{-1}\int_\Omega \overline{(\varrho^j-\varrho^{j-1})\log\varrho^j}\ \mathrm{d}x\\ &\le \int_\Omega \overline{\pi(\varrho^j,Y^j)+\delta(\varrho^j)^\Gamma}\varrho^j\ \mathrm{d}x+(2\mu+\nu)(\triangle t)^{-1}\int_\Omega (\varrho^j-\varrho^{j-1})\log\varrho^j\ \mathrm{d}x.\end{aligned} \tag{4.42}$$

The convexity of $\varrho^j \ln(\varrho^j)$ and $-\varrho^{j-1}\ln(\varrho^j)$ as functions of $\varrho^j$ ensure lower semicontinuity of the functional $\int_\Omega (\varrho^j - \varrho^{j-1})\log\varrho^j \,\mathrm{d}x$. In other words,

$$\int_\Omega (\varrho^j - \varrho^{j-1})\log\varrho^j \,\mathrm{d}x \le \int_\Omega \overline{(\varrho^j - \varrho^{j-1})\log\varrho^j} \,\mathrm{d}x.$$

Therefore, due to the definition of $\pi$, we have from (4.42)

$$\int_\Omega \left( \overline{(\varrho^j)^{\gamma+1} + \delta(\varrho^j)^{\Gamma+1}} + \varrho^j \sum_{k=1}^n \overline{\frac{Y_k^j}{m_k}\varrho^j} \right) \mathrm{d}x \le \int_\Omega \left( \overline{(\varrho^j)^{\gamma} + \delta(\varrho^j)^{\Gamma}}\,\varrho^j + \varrho^j \sum_{k=1}^n \overline{\frac{Y_k^j}{m_k}\varrho^j} \right) \mathrm{d}x. \tag{4.43}$$

This inequality can be used to show strong convergence of density as soon as one justifies

$$\overline{(\varrho^j)^\gamma}\,\varrho^j \le \overline{(\varrho^j)^{\gamma+1}}, \quad \overline{(\varrho^j)^\Gamma}\,\varrho^j \le \overline{(\varrho^j)^{\Gamma+1}}, \quad \overline{Y_k^j\varrho^j}\,\varrho^j \le \overline{Y_k^j(\varrho^j)^2}. \tag{4.44}$$

To do this, we will use a well known result about weak convergence of monotone functions composed with weakly converging sequences, whose proof can be found e.g. in [10], Theorem 10.19.

Lemma 4.8 (Weak convergence of monotone functions). *Let $\Omega$ be a domain in $\mathbb{R}^N$ and $(P,G)\in C(\mathbb{R})\times C(\mathbb{R})$ be a couple of non-decreasing functions. Assume that $u_n$ is a sequence of functions from $L^1(\Omega)$ with values in $\mathbb{R}$ such that*

$$\left.\begin{array}{r} P(u_n)\to\overline{P(u)}, \\ G(u_n)\to\overline{G(u)}, \\ P(u_n)G(u_n)\to\overline{P(u)G(u)} \end{array}\right\} \textit{weakly in } L^1(\Omega).$$

*(i) Then*

$$\overline{P(u)}\;\overline{G(u)} \le \overline{P(u)G(u)} \quad \textit{a.e. in } \Omega.$$

*(ii) If, in addition*

$$G(z)=z, \quad P\in C(\mathbb{R}), \quad P \textit{ is non-decreasing}$$

*and*

$$\overline{P(u)G(u)} = \overline{P(u)}\;\overline{G(u)},$$

*then*

$$\overline{P(u)} = P(u).$$

Applying this lemma to (4.44), we see that the first inequality is evidently true since $P(\varrho^j_\varepsilon)=(\varrho^j_\varepsilon)^\gamma$ and $G(\varrho^j_\varepsilon)=\varrho^j_\varepsilon$ are increasing. Regarding the second inequality, by (4.25) we know that $\overline{Y_k^j\varrho^j}=Y_k^j\varrho^j$. Thus $\varrho^j\overline{Y_k^j\varrho^j}=Y_k^j(\varrho^j)^2$, and the r.h.s. satisfies $\overline{\varrho^jY_k^j\varrho^j}=Y_k^j\overline{(\varrho^j)^2}\ge Y_k^j(\varrho^j)^2$. Here we applied Lemma 4.8 with $P(\varrho^j_\varepsilon)=G(\varrho^j_\varepsilon)=\varrho^j_\varepsilon$. Hence, by comparing (4.43) with (4.44) we obtain, using statement (ii) of Lemma 4.8, that

$$\overline{(\varrho^j)^\gamma} = (\varrho^j)^\gamma \quad \text{a.e. in } \Omega.$$

This in turn implies the strong convergence of the density as $L^\gamma(\Omega)$ is a uniformly convex Banach space. This completes the proof of Theorem 3.3.

## 5. Back to the continuous system

The aim of this section is to prove Theorem 2.2. The first part is devoted to the derivation of estimates uniform with respect to $\triangle t$. Some of them can be deduced from the previous section after letting $\varepsilon$ go to 0. Then we let the time-discretization parameter $\triangle t$ go to zero. Finally, we discuss the limit passage with the last parameter $\delta$.

**5.1. Limit passage to a continuous system with artificial pressure.** Before we let $\triangle t \to 0$, we turn back to (4.11) and rewrite it in a slightly changed form. First, we add and subtract $\frac{(\triangle t)^{-1}}{2}\int_\Omega \varrho^{j-1}|\mathbf{u}^{j-1}|^2 \, \mathrm{d}x$ and $\frac{(\triangle t)^{-1}}{\gamma-1}\int_\Omega (\varrho^{j-1})^\gamma \, \mathrm{d}x + \frac{(\triangle t)^{-1}\delta}{\Gamma-1}\int_\Omega (\varrho^{j-1})^\Gamma \, \mathrm{d}x$ to (4.11), to get

$$\begin{aligned} &\frac{(\triangle t)^{-1}}{2}\int_\Omega \left(\varrho^j|\mathbf{u}^j|^2 - \varrho^{j-1}|\mathbf{u}^{j-1}|^2\right) \mathrm{d}x + \frac{(\triangle t)^{-1}}{2}\int_\Omega \varrho^{j-1}|\mathbf{u}^j - \mathbf{u}^{j-1}|^2 \, \mathrm{d}x \\ &\quad + \int_\Omega \mathbf{S}(\mathbf{u}^j):\nabla \mathbf{u}^j \, \mathrm{d}x + \frac{(\triangle t)^{-1}}{\gamma-1}\int_\Omega \left((\varrho^j)^\gamma - (\varrho^{j-1})^\gamma\right) \mathrm{d}x \\ &\quad + \frac{(\triangle t)^{-1}\delta}{\Gamma-1}\int_\Omega \left((\varrho^j)^\Gamma - (\varrho^{j-1})^\Gamma\right) \mathrm{d}x \\ &\quad + \varepsilon\gamma\int_\Omega (\varrho^j)^{\gamma-2}|\nabla\varrho^j|^2 \, \mathrm{d}x + \varepsilon\delta\Gamma\int_\Omega (\varrho^j)^{\Gamma-2}|\nabla\varrho^j|^2 \, \mathrm{d}x \\ &\quad + \frac{(\triangle t)^{-1}}{\gamma-1}\int_\Omega \left((\gamma-1)(\varrho^j)^\gamma + (\varrho^{j-1})^\gamma - \gamma(\varrho^j)^{\gamma-1}\varrho^{j-1}\right) \mathrm{d}x \\ &\quad + \frac{(\triangle t)^{-1}\delta}{\Gamma-1}\int_\Omega \left((\Gamma-1)(\varrho^j)^\Gamma + (\varrho^{j-1})^\Gamma - \Gamma(\varrho^j)^{\Gamma-1}\varrho^{j-1}\right) \mathrm{d}x \\ = &\int_\Omega \varrho^j \mathbf{f}^j \cdot \mathbf{u}^j \, \mathrm{d}x + \sum_{k=1}^n \int_\Omega \frac{Y_k^j}{m_k}\varrho^j \operatorname{div}\mathbf{u}^j \, \mathrm{d}x. \end{aligned} \tag{5.1}$$

Note that since $\varrho^j$, $\varrho^{j-1} \geq 0$, $\gamma$, and $\Gamma > 1$, the two last integrals from the l.h.s. are nonnegative. Let us introduce the following notation:

$$\left.\begin{aligned} &\hat\phi(x,t) = \phi^k(x) \\ &\tilde\phi(x,t) = \phi^k(x) + (t-t_k)\left(\tfrac{\phi^{k+1}-\phi^k}{\triangle t}\right)(x) \end{aligned}\right\} \quad \text{if } t_k \leq t < t_{k+1},\ k \in \{0,\dots,\tilde N\}, \tag{5.2}$$

and let us define the shift operator

$$\sigma\phi^k = \phi^{k-1}, \quad k \in \{1,\dots,\tilde N\}.$$

We can then rewrite the system as

$$\begin{aligned} &\partial_t\tilde\varrho + \operatorname{div}(\hat\varrho\hat{\mathbf{u}}) = 0, \\ &\partial_t\widetilde{\varrho\mathbf{u}} + \operatorname{div}(\hat\varrho\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}) - \operatorname{div}\mathbf{S}(\hat{\mathbf{u}}) + \nabla\pi(\hat\varrho,\hat Y) + \delta\nabla\hat\varrho^\Gamma = \mathbf{0}, \\ &\partial_t\widetilde{\varrho Y_k} + \operatorname{div}(\hat\varrho\hat Y_k\hat{\mathbf{u}}) + \operatorname{div}\mathbf{F}_i(\hat\varrho,\hat Y) = \hat\varrho\omega_k(\hat Y), \quad k \in \{1,\dots,n\}, \end{aligned} \tag{5.3}$$

and keeping in mind (5.2), we can use (5.1) to deduce that

$$\hat\varrho,\ \tilde\varrho \text{ are bounded in } L^\infty(0,T;L^\gamma(\Omega)), \tag{5.4}$$

$$\delta^{1/\Gamma}\hat{\varrho},\ \delta^{1/\gamma}\tilde{\varrho} \text{ are bounded in } L^\infty(0,T;L^\Gamma(\Omega)) \tag{5.5}$$

$$\hat{\varrho}|\hat{\mathbf{u}}|^2, \widetilde{\varrho|\mathbf{u}|^2} \text{ are bounded in } L^\infty(0,T;L^1(\Omega)) \tag{5.6}$$

$$\hat{\mathbf{u}},\ \tilde{\mathbf{u}} \text{ are bounded in } L^2(0,T;W^{1,2}(\Omega)) \tag{5.7}$$

$$\hat{\varrho}\hat{\mathbf{u}}, \widetilde{\varrho\mathbf{u}} \text{ are bounded in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}) \cup L^2(0,T;L^r(\Omega)) \tag{5.8}$$

for $1 \le r \le \frac{6\gamma}{6+\gamma}$, where the last one holds since

$$\|\hat{\varrho}\hat{\mathbf{u}}\|_{L^{2\gamma/(\gamma+1)}(\Omega)} \le \|\hat{\varrho}\|^{1/2}_{L^\gamma(\Omega)} \|\hat{\varrho}|\hat{\mathbf{u}}|^2\|^{1/2}_{L^1(\Omega)} \quad \text{and} \quad \|\hat{\varrho}\hat{\mathbf{u}}\|_{L^r(\Omega)} \le \|\hat{\varrho}\|_{L^\gamma(\Omega)} \|\hat{\mathbf{u}}\|_{W^{1,2}(\Omega)}.$$

Furthermore, (5.1) gives rise to two more estimates which are of crucial importance for the limit passage. Namely,

$$\|\hat{\varrho} - \sigma\hat{\varrho}\|^\gamma_{L^\gamma(0,T;L^\gamma(\Omega))} + \delta\|\hat{\varrho} - \sigma\hat{\varrho}\|^\Gamma_{L^\Gamma(0,T;L^\Gamma(\Omega))} \le c\triangle t, \tag{5.9}$$

and

$$\|\hat{\varrho}|\hat{\mathbf{u}} - \sigma\hat{\mathbf{u}}|^2\|_{L^1(0,T;L^1(\Omega))} \le c\triangle t. \tag{5.10}$$

The first one is due to the fact that, for $\gamma$, $\Gamma > 1$, there exists a positive constant $c$ such that

$$(\gamma-1)(\varrho^j)^\gamma + (\varrho^{j-1})^\gamma - \gamma(\varrho^j)^{\gamma-1}\varrho^{j-1} \ge c|\varrho^j - \varrho^{j-1}|^\gamma,$$

$$\delta\left((\Gamma-1)(\varrho^j)^\Gamma + (\varrho^{j-1})^\Gamma - \Gamma(\varrho^j)^{\Gamma-1}\varrho^{j-1}\right) \ge c\delta|\varrho^j - \varrho^{j-1}|^\Gamma.$$

Repeating the steps leading to (4.12), one can verify that $\hat{\varrho}\hat{Y_k}^2$ and $\widetilde{\varrho Y_k^2}$ are bounded in $L^\infty(0,T;L^1(\Omega))$, and $\hat{Y_k}$ and $\tilde{Y_k}$ are bounded in $L^2(0,T;W^{1,2}(\Omega))$, and

$$\left\|\hat{\varrho}\left(\hat{Y_k} - \sigma\hat{Y_k}\right)^2\right\|_{L^1(0,T;L^1(\Omega))} \le c\triangle t. \tag{5.11}$$

And by (4.5) we deduce that $\hat{Y_k}$ and $\tilde{Y_k}$ are bounded in $L^\infty((0,T)\times\Omega)$. Finally, an estimate similar to (4.16) can be performed to get

$$\|\hat{\varrho}\|^{\gamma+\beta}_{L^{\gamma+\beta}((0,T)\times\Omega)} + \delta\|\hat{\varrho}\|^{\Gamma+\beta}_{L^{\Gamma+\beta}((0,T)\times\Omega)} \le c(T,\Omega). \tag{5.12}$$

This is the last estimate needed to perform the limit passage $\triangle t \to 0$ in all the terms except the pressure. Indeed, passing to a subsequence, it can be shown, combining (5.4)–(5.11), that the following convergences hold:

$$[\hat{\varrho} - \sigma\hat{\varrho}], [\hat{\varrho} - \tilde{\varrho}] \to 0 \quad \text{in } L^q(0,T;L^\gamma(\Omega)) \tag{5.13}$$

for $q \in [1,\infty)$,

$$[\hat{\varrho}\hat{\mathbf{u}} - \sigma\hat{\varrho}\hat{\mathbf{u}}],\ [\hat{\varrho}\hat{\mathbf{u}} - \widetilde{\varrho\mathbf{u}}] \to 0 \quad \text{in } L^q(0,T;L^r(\Omega)), \tag{5.14}$$

for $\{q \in [1,\infty),\ r \in [1, \frac{2\Gamma}{\Gamma+1}]\} \cup \{q \in [1,2),\ r \in [1, \frac{6\Gamma}{6+\Gamma}]\}$,

$$[\hat{\varrho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} - \widetilde{\varrho\mathbf{u}} \otimes \hat{\mathbf{u}}] \to 0 \quad \text{in } L^1(0,T;L^r(\Omega)) \cup L^q(0,T;L^1(\Omega)), \tag{5.15}$$

for $q \in [1,\infty)$ $r \in [1, \frac{3\Gamma}{3+\Gamma}]$,

$$[\hat{\varrho}\hat{Y_k} - \sigma\hat{\varrho}\hat{Y_k}],\ [\hat{\varrho}\hat{Y_k} - \widetilde{\varrho Y_k}] \to 0 \quad \text{in } L^q(0,T;L^r(\Omega)), \tag{5.16}$$

for $\{q \in [1,\infty),\ r \in [1, \frac{2\Gamma}{\Gamma+1}]\} \cup \{q \in [1,2),\ r \in [1, \frac{6\Gamma}{6+\Gamma}]\}$.

From what has already been written, we deduce that

$$\hat{\varrho},\ \tilde{\varrho} \rightharpoonup \varrho \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^\Gamma(\Omega)), \text{ weakly in } L^{\Gamma+\beta}((0,T)\times\Omega), \tag{5.17}$$

$$\hat{\mathbf{u}} \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{5.18}$$

$$\hat{Y}_k \rightharpoonup Y_k \quad \text{weakly}^* \text{ in } L^\infty((0,T)\times\Omega), \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{5.19}$$

REMARK 5.1. Since $\tilde{\varrho}$, $\hat{\varrho}$, and $\hat{\mathbf{u}}$ satisfy the continuity equation $\partial_t\tilde{\varrho} + \operatorname{div}(\hat{\varrho}\hat{\mathbf{u}}) = 0$ in the sense of distributions, the sequence of functions $f(t) = \left(\int_\Omega \tilde{\varrho}\phi \, \mathrm{d}x\right)(t)$ is bounded and equicontinuous in $C([0,T])$ for all $\phi \in C^\infty(\overline{\Omega})$, and $\frac{\partial\phi}{\partial\mathbf{n}} = 0$ at $\partial\Omega$. Therefore, the Arzelà–Ascoli theorem, the density argument, and the convergence established in (5.13) yield

$$\tilde{\varrho} \to \varrho \quad \text{in } C_{weak}(0,T;L^\Gamma(\Omega)).$$

We now focus on the corresponding convergence of the products $\hat{\varrho}\hat{\mathbf{u}}$, $\hat{\varrho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}$, $\hat{\varrho}\hat{Y}_k$, and $\hat{\varrho}\hat{Y}_k\hat{\mathbf{u}}$. This can be done by repeated application of the following lemma.

LEMMA 5.2. *Let $g^n$ and $h^n$ converge weakly to $g$ and $h$ respectively in $L^{p_1}(0,T;L^{p_2}(\Omega))$ and $L^{q_1}(0,T;L^{q_2}(\Omega))$ where $1 \le p_1, p_2 \le \infty$, and*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

*Let us assume, in addition, that*

$$\partial_t g^n \text{ is bounded in } L^1(0,T;W^{-m,1}) \text{ for some } m \ge 0 \text{ independent of } n \text{ and} \tag{5.20}$$

$$\|h^n - h^n(\cdot+\xi,t)\|_{L^{q_1}(0,T;L^{q_2}(\Omega))} \to 0 \text{ as } |\xi| \to 0, \text{ uniformly in } n. \tag{5.21}$$

*Then $g^n h^n$ converges to $gh$ in the sense of distributions on $\Omega\times(0,T)$.*

For the proof, we refer the reader to [18].

Since $\partial_t\tilde{\varrho}$ is bounded in $L^\infty(0,T;W^{-1,2\Gamma/(\Gamma+1)}(\Omega))$, $\partial_t\widetilde{\varrho\mathbf{u}}$ is bounded in $L^2(0,T;W^{-1,1+\beta})$, and $\partial_t\widetilde{\varrho Y_k}$ is bounded in $L^2(0,T;W^{-1,1})$, condition (5.20) is satisfied for $g^n = \tilde{\varrho}$, $\widetilde{\varrho\mathbf{u}}$, $\widetilde{\varrho Y_k}$, and $m = 1$, respectively. Additionally, $h^n = \hat{\mathbf{u}}$ and $\hat{Y}_k$, which is bounded in $L^2(0,T;W^{1,2}(\Omega))$, satisfies condition (5.21).

Therefore, $\tilde{\varrho}\hat{\mathbf{u}}$ converges weakly* in $L^\infty(0,T;L^{2\Gamma/(\Gamma+1)}(\Omega))$ and weakly in $L^2(0,T;L^{6\Gamma/(\Gamma+6)}(\Omega))$ to $\varrho\mathbf{u}$, and $\tilde{\varrho}\hat{Y}_k$ converges weakly* in $L^\infty(0,T;L^\Gamma(\Omega))$ to $\varrho Y_k$. So, in view of (5.13), (5.14), and (5.16)

$$\widetilde{\varrho\mathbf{u}},\ \hat{\varrho}\hat{\mathbf{u}} \rightharpoonup \varrho\mathbf{u} \quad \text{weakly in } L^q(0,T;L^r(\Omega)) \tag{5.22}$$

and

$$\widetilde{\varrho Y_k},\ \hat{\varrho}\hat{Y}_k \rightharpoonup \varrho Y_k \quad \text{weakly in } L^q(0,T;L^r(\Omega)) \tag{5.23}$$

for $\{q \in [1,\infty),\ r \in [1, \frac{2\Gamma}{\Gamma+1}]\} \cup \{q \in [1,2),$ and $r \in [1, \frac{6\Gamma}{6+\Gamma}]\}$.

Moreover, $\widetilde{\varrho\mathbf{u}}\otimes\hat{\mathbf{u}}$ converges weakly in $L^1(0,T;L^{3\Gamma/(\Gamma+3)}(\Omega))$ weakly* in $L^\infty(0,T;L^1(\Omega))$ to $\varrho\mathbf{u}\otimes\mathbf{u}$, and $\widetilde{\varrho Y_k}\hat{\mathbf{u}}$ converges weakly in $L^1(0,T;L^{3\Gamma/(\Gamma+3)}(\Omega))$ weakly* in $L^\infty(0,T;L^1(\Omega))$ to $\varrho Y_k\mathbf{u}$. Thus again, (5.15) and (5.16) can be used to show that

$$\hat{\varrho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}} \rightharpoonup \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{weakly in } L^1(0,T;L^r(\Omega)) \cup L^q(0,T;L^1(\Omega)), \tag{5.24}$$

and

$$\hat{\varrho}\hat{Y}_k\hat{\mathbf{u}} \rightharpoonup \varrho Y_k\mathbf{u} \quad \text{weakly in } L^1(0,T;L^r(\Omega))\cup L^q(0,T;L^1(\Omega)), \tag{5.25}$$

for $q\in[1,\infty)$ and $r\in[1,\frac{3\Gamma}{3+\Gamma}]$.

All these considerations allow us to let $\triangle t\to 0$ in the system (5.3), and we obtain

$$\begin{aligned} &\partial_t\varrho+\operatorname{div}(\varrho\mathbf{u})=0,\\ &\partial_t(\varrho\mathbf{u})+\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})-\operatorname{div}\mathbf{S}(\mathbf{u})+\nabla\overline{\pi(\varrho,Y)}+\delta\nabla\overline{\varrho^\Gamma}=\varrho\mathbf{f},\\ &\partial_t(\varrho Y_k)+\operatorname{div}(\varrho Y_k\mathbf{u})+\operatorname{div}\mathbf{F}_k(\varrho,Y_k)=\overline{\varrho\omega_k(Y)},\quad k\in\{1,\dots,n\}, \end{aligned} \tag{5.26}$$

which is now satisfied in the sense of distributions on $(0,T)\times\Omega$, together with boundary conditions (2.4). Regarding the initial conditions, we can repeat the argument from Remark 5.1 to verify that

$$\widetilde{\varrho\mathbf{u}}\to\varrho\mathbf{u} \quad \text{in } C_{weak}(0,T;L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)), \qquad \widetilde{\varrho Y_k}\to\varrho Y_k \quad \text{in } C_{weak}(0,T;L^\Gamma(\Omega)). \tag{5.27}$$

The last part of the proof is devoted to the issue of strong convergence of the density, which is necessary to identify the limits in the nonlinear terms. As previously, we seek to derive the effective viscous flux equality.

Note that the functions $\tilde{\varrho}$ and $\hat{\varrho}\hat{\mathbf{u}}$ extended by 0 outside $\Omega$ satisfy the continuity equation in all of $\mathbb{R}^3$. Next, one can check that

$$\hat{\varphi}(t,x)=\psi(t)\zeta(x)\tilde{\phi},\quad \tilde{\phi}=(\nabla\Delta^{-1})[1_\Omega\tilde{\varrho}],\quad \psi\in C_c^\infty((0,T)),\ \text{and}\ \zeta\in C_c^\infty(\Omega),$$

is an admissible test function for the momentum equation. After straightforward manipulations, we obtain

$$\begin{aligned} &\int_\Omega\psi\zeta\left(\left(\pi(\hat{\varrho},\hat{Y})+\delta\hat{\varrho}^\Gamma\right)\hat{\varrho}-\mathbf{S}(\hat{\mathbf{u}}):\mathcal{R}[1_\Omega\tilde{\varrho}]\right)\,\mathrm{d}x\\ =&-\int_\Omega\psi\zeta\hat{\varrho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}:\mathcal{R}[1_\Omega\tilde{\varrho}]\,\mathrm{d}x+\int_\Omega\psi\zeta\widetilde{\varrho\mathbf{u}}\cdot\mathcal{R}[1_\Omega\hat{\varrho}\hat{\mathbf{u}}]\,\mathrm{d}x\\ &-\int_\Omega\partial_t\psi\zeta\widetilde{\varrho\mathbf{u}}\cdot\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\,\mathrm{d}x-\int_\Omega\psi\hat{\varrho}(\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}):\nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\,\mathrm{d}x\\ &+\int_\Omega\psi\mathbf{S}(\hat{\mathbf{u}}):\nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\,\mathrm{d}x\\ &-\int_\Omega\psi\left(\pi(\hat{\varrho},\hat{Y})+\delta\hat{\varrho}^\Gamma\right)\nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\,\mathrm{d}x-\int_\Omega\zeta\varrho\mathbf{f}\cdot\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\,\mathrm{d}x\\ =&\sum_{i=1}^6 I_i, \end{aligned} \tag{5.28}$$

where we have used the approximate continuity equation to write

$$\partial_t\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]=-\nabla\Delta^{-1}[\operatorname{div}(1_\Omega\hat{\varrho}\hat{\mathbf{u}})].$$

Since $\Gamma>3$, we can use the analogous function to test the limit momentum equation

$$\varphi(t,x)=\psi(t)\zeta(x)\phi,\quad \phi=(\nabla\Delta^{-1})[1_\Omega\varrho],\quad \psi\in C_c^\infty((0,T)),\ \zeta\in C_c^\infty(\Omega). \tag{5.29}$$

We obtain

$$\begin{aligned}
&\int_\Omega \psi\zeta\left(\overline{\pi(\varrho,Y)+\delta\varrho^\Gamma}\varrho-\mathbf{S}(\mathbf{u}):\mathcal{R}[1_\Omega\varrho]\right)\,\mathrm{d}x\\
&=-\int_\Omega \psi\zeta\varrho\mathbf{u}\otimes\mathbf{u}:\mathcal{R}[1_\Omega\varrho]\,\mathrm{d}x+\int_\Omega\psi\zeta\varrho\mathbf{u}\cdot\mathcal{R}[1_\Omega\varrho\mathbf{u}]\,\mathrm{d}x\\
&\quad-\int_\Omega\partial_t\,\mathrm{d}x\psi\zeta\varrho\mathbf{u}\cdot\nabla\Delta^{-1}[1_\Omega\varrho]-\int_\Omega\psi\varrho(\mathbf{u}\otimes\mathbf{u}):\nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\varrho]\,\mathrm{d}x\\
&\quad+\int_\Omega\psi\mathbf{S}(\mathbf{u}):\nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\varrho]\,\mathrm{d}x-\int_\Omega\psi\overline{\pi(\varrho,Y)+\delta\varrho^\Gamma}\nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\varrho]\,\mathrm{d}x\\
&\quad-\int_\Omega\zeta\varrho\mathbf{f}\cdot\nabla\Delta^{-1}[1_\Omega\tilde{\varrho}]\,\mathrm{d}x=\sum_{i=1}^6 I_i.
\end{aligned}\tag{5.30}$$

Again, it is not difficult to check that comparison of (5.30) with the limit of (5.28) gives rise to the following equality:

$$\begin{aligned}
&\int_0^T\int_\Omega\psi\zeta\left(\left(\pi(\hat{\varrho},\hat{Y})+\delta\hat{\varrho}^\Gamma\right)\hat{\varrho}-\mathbf{S}(\hat{\mathbf{u}}):\mathcal{R}[1_\Omega\tilde{\varrho}]\right)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega\psi\zeta\,(\hat{\varrho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}:\mathcal{R}[1_\Omega\tilde{\varrho}]-\widetilde{\varrho\mathbf{u}}\cdot\mathcal{R}[1_\Omega\hat{\varrho}\hat{\mathbf{u}}])\,\mathrm{d}x\,\mathrm{d}t\\
\to&\int_0^T\int_\Omega\psi\zeta\left(\overline{\pi(\varrho,Y)+\delta\varrho^\Gamma}\varrho-\mathbf{S}(\mathbf{u}):\mathcal{R}[1_\Omega\varrho]\right)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega\psi\zeta\,(\varrho\mathbf{u}\otimes\mathbf{u}:\mathcal{R}[1_\Omega\varrho]-\varrho\mathbf{u}\cdot\mathcal{R}[1_\Omega\varrho\mathbf{u}])\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{5.31}$$

To show that the terms involving commutators cancel, we will again use Lemma A.2 with $\mathbf{V}_\varepsilon=\hat{\varrho}\hat{\mathbf{u}}$, $r_\varepsilon=\hat{\varrho}$, and we first check that

$$\widetilde{\varrho\mathbf{u}}\mathcal{R}[1_\Omega\tilde{\varrho}]-\tilde{\varrho}\mathcal{R}[1_\Omega\widetilde{\varrho\mathbf{u}}]\to\varrho\mathbf{u}\mathcal{R}[1_\Omega\varrho]-\varrho\mathcal{R}[1_\Omega\varrho\mathbf{u}]$$

in the sense of distributions on $\Omega$ ,but for all $t\in[0,T]$. The second property follows from Remark 5.1.

Next, by (5.18) and (5.22) and the density argument, we check that this convergence can be extended to a weak convergence in $L^{2\Gamma/(\Gamma+3)}(\Omega)$. However, we see that this space is compactly embedded into $W^{-1,2}(\Omega)$ only if $\Gamma>\frac{9}{2}$. This, together with (5.18), implies that

$$\begin{aligned}
&\lim_{\Delta\to 0}\int_0^T\int_\Omega\psi\zeta\hat{\mathbf{u}}\cdot(\widetilde{\varrho\mathbf{u}}\mathcal{R}[1_\Omega\tilde{\varrho}]-\tilde{\varrho}\mathcal{R}[1_\Omega\widetilde{\varrho\mathbf{u}}])\,\mathrm{d}x\,\mathrm{d}t\\
\to&\int_0^T\int_\Omega\psi\zeta\mathbf{u}\cdot(\varrho\mathbf{u}\mathcal{R}[1_\Omega\varrho]-\varrho\mathcal{R}[1_\Omega\varrho\mathbf{u}])\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{5.32}$$

Equality (5.32) is then almost what we need. To prove that

$$\begin{aligned}
&\lim_{\Delta\to 0}\int_0^T\int_\Omega\psi\zeta\,(\hat{\varrho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}:\mathcal{R}[1_\Omega\tilde{\varrho}]-\widetilde{\varrho\mathbf{u}}\cdot\mathcal{R}[1_\Omega\hat{\varrho}\hat{\mathbf{u}}])\,\mathrm{d}x\,\mathrm{d}t\\
&=\lim_{\Delta\to 0}\int_0^T\int_\Omega\psi\zeta\hat{\mathbf{u}}\cdot(\widetilde{\varrho\mathbf{u}}\mathcal{R}[1_\Omega\tilde{\varrho}]-\tilde{\varrho}\mathcal{R}[1_\Omega\widetilde{\varrho\mathbf{u}}])\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

we use properties (5.13) and (5.14). Then, repeating the steps leading from (4.36) to (4.39), we can transform (5.31) into

$$\begin{aligned}&\overline{(\pi(\varrho,Y)+\delta\varrho^{\Gamma})\,\varrho}-(2\mu+\nu)\overline{\operatorname{div}\mathbf{u}\varrho}\\ =&\overline{\pi(\varrho,Y)+\delta\varrho^{\Gamma}}\,\varrho-(2\mu+\nu)\operatorname{div}\mathbf{u}\varrho,\quad \text{a.e. in } \Omega.\end{aligned} \tag{5.33}$$

Next, we take $\eta>0$ and multiply the discrete version of the continuity equation by $\log(\varrho^j+\eta)$. After integrating by parts over $\Omega$, one gets

$$(\triangle t)^{-1}\int_\Omega(\varrho^j-\varrho^{j-1})\log(\varrho^j+\eta)\,\mathrm{d}x-\int_\Omega\varrho^j\mathbf{u}^j\frac{\nabla\varrho^j}{\varrho^j+\eta}\,\mathrm{d}x=0.$$

By the Lebesgue monotone convergence theorem, we can pass with $\eta\to 0^+$ and then integrate by parts once more to find

$$(\triangle t)^{-1}\int_\Omega(\varrho^j-\varrho^{j-1})\log\varrho^j\,\mathrm{d}x+\int_\Omega\operatorname{div}\mathbf{u}^j\varrho^j\,\mathrm{d}x=0.$$

Recall that $\int_\Omega\varrho^j\,\mathrm{d}x=\int_\Omega\varrho^{j-1}\,\mathrm{d}x$. Thus since $x\log(x)$ is a convex function, the above equality may be changed into

$$(\triangle t)^{-1}\int_\Omega\left(\varrho^j\log\varrho^j-\varrho^{j-1}\log\varrho^{j-1}\right)\,\mathrm{d}x+\int_\Omega\operatorname{div}\mathbf{u}^j\varrho^j\,\mathrm{d}x\le 0. \tag{5.34}$$

Now, we sum (5.34) from $j=1$ to $j=\tilde{N}$, multiply by $\triangle$, and pass to the limit to get

$$\int_\Omega\overline{\varrho\log\varrho}(T)\,\mathrm{d}x+\int_0^T\int_\Omega\overline{\varrho\operatorname{div}\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\le\int_\Omega(\varrho\log\varrho)(0)\,\mathrm{d}x. \tag{5.35}$$

For the limit continuity equation, we take advantage of the fact that it is satisfied in the whole space in the sense of distributions. Thus the solution is automatically a renormalized solution; see for instance [10]. I.e. by an appropriate renormalization, we may get

$$\int_\Omega\varrho\log\varrho(T)\,\mathrm{d}x+\int_0^T\int_\Omega\varrho\operatorname{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t=\int_\Omega\varrho\log\varrho(0)\,\mathrm{d}x. \tag{5.36}$$

Consequently, the two results (5.35) and (5.36) give rise to

$$\int_\Omega\overline{\varrho\log\varrho}(T)\,\mathrm{d}x+\int_0^T\int_\Omega\overline{\varrho\operatorname{div}\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\le\int_\Omega\varrho\log\varrho(T)\,\mathrm{d}x+\int_0^T\int_\Omega\varrho\operatorname{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t,$$

which joined with (5.33) leads to the desired conclusion

$$\int_0^T\int_\Omega\left(\overline{\varrho^{\gamma+1}+\delta\varrho^{\Gamma+1}}+\varrho\overline{\sum_{k\in K}\frac{Y_k}{m_k}\varrho}\right)\,\mathrm{d}x\,\mathrm{d}t\le\int_0^T\int_\Omega\left(\overline{\varrho^{\gamma}+\delta\varrho^{\Gamma}}\varrho+\varrho\sum_{k\in K}\overline{\frac{Y_k}{m_k}\varrho}\right)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.37}$$

As in the stationary case, using Lemma 4.8, we easily verify that

$$\overline{\varrho^\gamma}\varrho\le\overline{\varrho^{\gamma+1}}\quad\text{and}\quad\overline{\varrho^\Gamma}\varrho\le\overline{\varrho^{\Gamma+1}}, \tag{5.38}$$

so to deduce the strong convergence of the density one should only check that $\overline{Y_k\varrho}\varrho \leq \overline{Y_k\varrho^2}$. It is easy to identify the l.h.s. due to (5.19), Remark 5.1, and the compact embedding of $L^\Gamma(\Omega)$ into $W^{-1,2}(\Omega)$. Identifying the limit from the r.h.s. is now a little more involved. However, we can show that

$$\hat{Y}_k \to Y_k \quad \text{a.e. on } \{(x,t) : \varrho(x,t) > 0\}. \tag{5.39}$$

Indeed, it is a consequence of the weak convergence of $\hat{Y}_k$ and a following convergence of norms:

$$\int_0^T \int_\Omega \varrho \hat{Y}_k^2 \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \varrho Y_k^2 \, \mathrm{d}x \, \mathrm{d}t, \tag{5.40}$$

where $\varrho$ is a positive density. On account of (5.19) and (5.27), we can repeat the argument from Remark 5.1 to verify that

$$\widetilde{\varrho Y_k}\hat{Y}_k \to \varrho Y_k^2 \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^\Gamma). \tag{5.41}$$

Note that $\nabla \hat{Y}_k^2$ is also uniformly bounded in $L^2(0,T;W^{1,2})$. Therefore,

$$(\tilde{\varrho} - \varrho)\hat{Y}_k^2 \to 0 \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^\Gamma). \tag{5.42}$$

Thus, the convergence of (5.40) follows from (5.16) combined with (5.41), (5.42), and the triangle inequality.

Having proven (5.39), we justify that $\overline{Y_k\varrho}\varrho = Y_k\varrho^2 \leq \overline{Y_k\varrho^2} = Y_k\overline{\varrho^2}$ on a set $\{\varrho > 0\}$. This is obvious on account of the weak lower semicontinuity of convex functions. For the set $\{\varrho = 0\}$, the l.h.s. becomes equal to 0 while the r.h.s. is always nonnegative, so the inequality is valid.

Recapitulating, the above considerations leads to the equality

$$\overline{\varrho^\gamma}\varrho = \overline{\varrho^{\gamma+1}} \quad \text{a.e. on } (0,T)\times\Omega.$$

Hence strong convergence follows.

The strong convergence of the density implies, together with (5.19), that

$$\overline{\varrho\omega(Y)} = \varrho\omega(Y).$$

**5.2. Limit passage $\delta \to 0$.** At this level, we lose the uniform estimate for $\varrho_\delta$ in $L^\infty(0,T;L^\Gamma(\Omega))$ following from the energy balance (5.1). Instead, we can show that $\varrho_\delta$ is uniformly bounded in $L^\infty(0,T;L^\gamma(\Omega))$ provided that the last term in (5.1) is bounded. This requires the assumption that $\gamma \geq 2$. This restriction follows from the fact that the variable in the elliptic operator appearing in the reaction-diffusion equations is concentration $Y_i$, not the species density $\varrho_i$.

Having this estimate, passage to limit with the last approximation parameter differs only in one step in comparison to the analysis performed in the previous subsection. Namely, at this level we cannot derive the effective viscous flux equality in the same way. Instead of testing the momentum equation with the function $\phi$ specified in (5.29), we have to use

$$\varphi(t,x) = \psi(t)\zeta(x)\phi, \quad \phi = (\nabla\Delta^{-1})[1_\Omega\varrho^\vartheta], \quad \psi \in C_c^\infty((0,T)), \ \zeta \in C_c^\infty(\Omega),$$

with $\vartheta < \frac{1}{6}$. Since we already know how to identify the limit in the molecular pressure term for the time-dependent case, the proof of strong convergence of the density would be just a repetition of the standard proof for the case of barotropic Navier–Stokes equations. Details of this procedure can be found e.g. in [10], Chapter 3 or in [31]. The proof of Theorem 2.2 is complete.

**Appendix A. The Riesz transform.** The inverse divergence operator $\mathcal{A}=\nabla\Delta^{-1}$ and the double Riesz transform $\mathcal{R}=\nabla\otimes\nabla\Delta^{-1}$ are defined as

$$\mathcal{A}_j[v]=\left(\nabla\Delta^{-1}\right)_j v=-\mathcal{F}^{-1}\left(\frac{i\xi_j}{|\xi|^2}\mathcal{F}(v)\right), \tag{A.1}$$

$$\mathcal{R}_{i,j}[v]=\partial_i\mathcal{A}_j[v]=\left(\nabla\otimes\nabla\Delta^{-1}\right)_{i,j}v=\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v)\right). \tag{A.2}$$

Here, the inverse Laplacian is identified through the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ as

$$(-\Delta)^{-1}(v)=\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2}\mathcal{F}(v)\right).$$

In what follows, we recall some of basic properties of these operators.

LEMMA A.1. *The operator $\mathcal{R}$ is a continuous linear operator from $L^p(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ for any $1<p<\infty$. In particular, the following estimate holds true:*

$$\|\mathcal{R}[v]\|_{L^p(\mathbb{R}^3)}\le c(p)\|v\|_{L^p(\mathbb{R}^3)} \quad \text{for all } v\in L^p(\mathbb{R}^3).$$

*The operator $\mathcal{A}$ is a continuous linear operator from $L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$, and from $L^p(\mathbb{R}^3)$ into $L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ for any $1<p<3$.*

*Moreover,*

$$\|\nabla\mathcal{A}[v]\|_{L^p(\mathbb{R}^3)}\le C(p)\|v\|_{L^p(\mathbb{R}^3)}, \quad 1<p<\infty.$$

The proof of this lemma can be found e.g. in [10], Section 10.16. In what follows, we present two important properties of commutators involving the Riesz operator. The first result is a straightforward consequence of the *Div-Curl* lemma. Its proof can be found in [7], Lemma 5.1.

LEMMA A.2. *Let*

$$\mathbf{V}_\varepsilon\rightharpoonup\mathbf{V} \text{ weakly in } L^p(\mathbb{R}^3), \quad r_\varepsilon\rightharpoonup r \text{ weakly in } L^q(\mathbb{R}^3),$$

*where*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{s}<1.$$

*Then*

$$\mathbf{V}_\varepsilon\mathcal{R}(r_\varepsilon)-r_\varepsilon\mathcal{R}(\mathbf{V}_\varepsilon)\rightharpoonup\mathbf{V}\mathcal{R}(r)-r\mathcal{R}(\mathbf{V}) \quad \text{weakly in } L^s(\mathbb{R}^3).$$

The next lemma can be deduced from the general results of Bajšanski and Coifman [1] and Coifman and Meyer [4].

LEMMA A.3. *Let $w\in W^{1,r}(\mathbb{R}^3)$ and $\mathbf{V}\in L^p(\mathbb{R}^3)$ be given, where $1<r<3$, $1<p<\infty$, and $\frac{1}{r}+\frac{1}{p}-\frac{1}{3}<\frac{1}{s}<1$. Then for all such $s$, we have*

$$\|\mathcal{R}[w\mathbf{V}]-w\mathcal{R}[\mathbf{V}]\|_{W^{\alpha,s}(\mathbb{R}^3)}\le c(s,p,r)\|w\|_{W^{1,r}(\mathbb{R}^3)}\|\mathbf{V}\|_{L^p(\mathbb{R}^3)},$$

*where* $\alpha$ *is given by* $\frac{\alpha}{3} = \frac{1}{s} + \frac{1}{3} - \frac{1}{p} - \frac{1}{r}$.

Here, $W^{\alpha,s}(\mathbb{R}^3)$ for $\alpha \in (0,\infty) \setminus \mathbb{N}$ denotes the Sobolev–Slobodeckii space (see [33]). The proof can be found in [10], Section 10.17.

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