Turán Problems in Graphs and Hypergraphs

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Declaration

I, Adam Sanitt, hereby confirm that the work presented in this thesis is my own, except for those parts stated below. Where information has been derived from other sources, I confirm this has been indicated in the thesis.

Chapter 2 is joint work with John Talbot.

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To my wife Julia and my children Ethan, Leo and Jacqueline

Abstract

Mantel's theorem says that among all triangle-free graphs of a given order the balanced complete bipartite graph is the unique graph of maximum size. In Chapter 2, we prove an analogue of this result for 3-graphs (3-uniform hypergraphs) together with an associated stability result. Let K_4^-, F_5 and F_6 be 3-graphs with vertex sets $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4, 5, 6\}$ respectively and edge sets $E(K_4^-) = \{123, 124, 134\}$, $E(F_5) = \{123, 124, 345\}$, $E(F_6) = \{123, 124, 345, 156\}$ and $\mathcal{F} = \{K_4, F_6\}$. For $n \neq 5$ the unique $\mathcal{F}$-free 3-graph of order n and maximum size is the balanced complete tripartite 3-graph $S_3(n)$. This extends an old result of Bollobas that $S_3(n)$ is the unique 3-graph of maximum size with no copy of K_4^- or F_5.

In 1941, Turán generalised Mantel's theorem to cliques of arbitrary size and then asked whether similar results could be obtained for cliques on hypergraphs. This has become one of the central unsolved problems in the field of extremal combinatorics. In Chapter 3, we prove that the Turán density of $K_5^{(3)}$ together with six other induced subgraphs is 3/4. This is analogous to a similar result obtained for $K_4^{(3)}$ by Razborov.

In Chapter 4, we consider various generalisations of the Turán density. For example, we prove that, if the density in G of $\bar{P}_3$ is x and G is K_3-free, then $|E(G)| / \binom{n}{2} \leq 1/4 + (1/4)\sqrt{1 - (8/3)x}$. This is motivated by the observation that the extremal graph for K_3 is $\bar{P}_3$-free, so that the upper bound is a natural extension of a stability result for K_3.

The question how many edges can be deleted from a blow-up of H before it is H-free subject to the constraint that the same proportion of edges are deleted from each connected pair of vertex sets has become known as the *Turán density problem.* In Chapter 5, using entropy compression supplemented with some analytic methods, we derive an upper bound of $1 - 1/(\gamma(\Delta(H) - \beta))$, where $\Delta(H)$ is the maximum degree of H, $3 \leq \gamma < 4$ and $\beta \leq 1$. The new bound asymptotically approaches the existing best upper bound despite being derived in a completely different way.

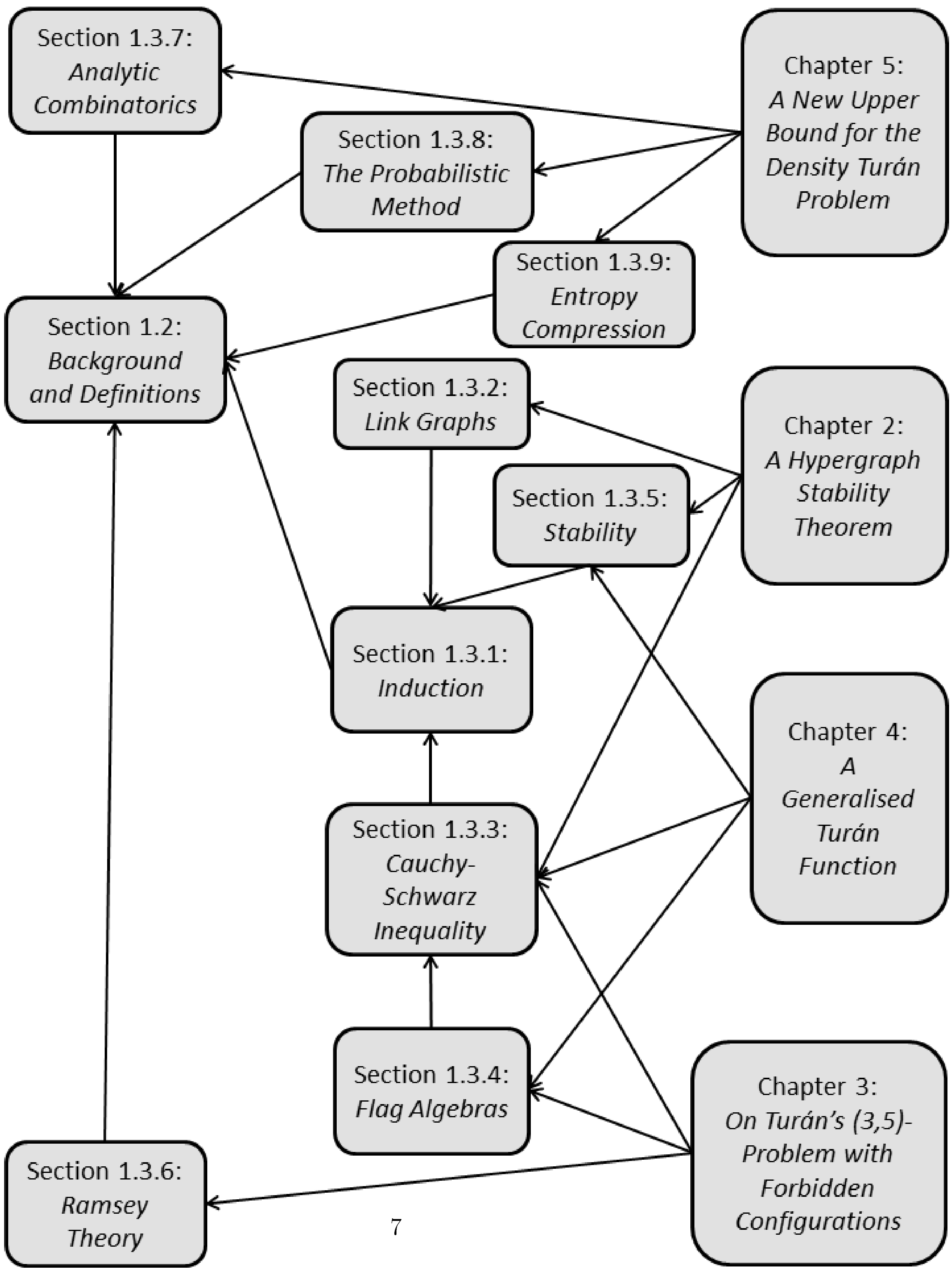
The techniques used in these results, illustrating their breadth and connections between them, are set out in Chapter 1.

Contents

[1] This chapter has been published in slightly amended form as **[28]**

Contents

Dependency Graph



Chapter 1

Techniques

1.1. Introduction

Extremal graph theory seeks to determine the extremal values of certain invariants in graphs (or related entities such as hypergraphs) that have a particular property. The most commonly studied invariant is the number of edges in a graph, although we will also consider similar invariants such as the number of copies of a given small sub-graph. The most commonly studied property is the absence of a particular graph as a sub-graph or induced sub-graph. In general, the property is a local feature of the graph whereas the invariant depends on the graph as a whole. Therefore, extremal graph theory often involves reasoning from local properties to infer features that apply globally.

In this thesis, we show how a variety of different techniques may be employed to answer questions in a particular branch of extremal graph theory: Turán problems. Using a wide variety of methods enables progress in areas that are difficult to tackle directly. The methods employed in this thesis include:

- combinatorial arguments, including induction, link graphs and use of Cauchy-Schwartz Theorem;
- stability methods;
- flag algebra;

- Ramsey theory;
- analytic arguments;
- the probabilistic method; and
- entropy compression.

In this introductory chapter, we first set out some standard background and definitions and then give further details of the variety of techniques to be used. The dependency graph on the preceding page shows the principal connections between the various sections of this chapter and the remainder of the thesis.

The methods set out here are not of merely abstract interest. They may be applied to produce results such as those set out in the following chapters.

So, in Chapter 2, we use Induction, Link Graphs and Stability to prove the Turán density and an associated stability result for a much studied hypergraph. Specifically, Mantel's theorem says that among all triangle-free graphs of a given order the balanced complete bipartite graph is the unique graph of maximum size. In Chapter 2, we prove an analogue of this result for 3-graphs (3-uniform hypergraphs) together with an associated stability result. Let K_4^-, F_5 and F_6 be 3-graphs with vertex sets $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4, 5, 6\}$ respectively and edge sets $E(K_4^-) = \{123, 124, 134\}$, $E(F_5) = \{123, 124, 345\}$, $E(F_6) = \{123, 124, 345, 156\}$ and $\mathcal{F} = \{K_4, F_6\}$. For $n \neq 5$ the unique $\mathcal{F}$-free 3-graph of order n and maximum size is the balanced complete tripartite 3-graph $S_3(n)$. This extends an old result of Bollobas that $S_3(n)$ is the unique 3-graph of maximum size with no copy of K_4^- or F_5.

In Chapter 3, we use elements of Ramsey Theory and Flag Algebra to supplement Induction with an analytic presentation to prove the Turán density for a family of hypergraphs including $K_5^{(3)}$. So, in 1941, Turán generalised Mantel's theorem to cliques of arbitrary size and then asked whether similar results could be obtained for cliques on hypergraphs. This has become one of the central unsolved problems in the field of extremal combinatorics. In Chapter 3, we prove that the Turán density of $K_5^{(3)}$ together with six other induced subgraphs is $3/4$. This is analogous to a similar result obtained for $K_4^{(3)}$ by Razborov.

In Chapter 4, we use Flag Algebra and analytic techniques to consider various generalisations of the Turán density. For example, we prove that, if the density in G of $\bar{P}_3$ is x and G is K_3-free, then $|E(G)|/\binom{n}{2} \leq 1/4 + (1/4)\sqrt{1-(8/3)x}$. This is motivated by the observation that the extremal graph for K_3 is $\bar{P}_3$-free, so that the upper bound is a natural extension of a stability result for K_3.

The question how many edges can be deleted from a blow-up of H before it is H-free subject to the constraint that the same proportion of edges are deleted from each connected pair of vertex sets has become known as the *Turán density problem.* In Chapter 5, using Entropy Compression and Analytic Combinatorics, we derive an upper bound of $1 - 1/(\gamma(\Delta(H) - \beta))$, where $\Delta(H)$ is the maximum degree of H, $3 \leq \gamma < 4$ and $\beta \leq 1$. The new bound asymptotically approaches the existing best upper bound despite being derived in a completely different way.

1.2. Background and Definitions

A uniform r-graph H is a set of r-tuples, $E(H)$ (known as edges if $r = 2$ or hyperedges otherwise), defined on a base set, $V(H)$ (known as vertices). A 2-graph is simply a graph, although we also use graph as an abbreviation for all r-graphs, not just 2-graphs. The number of vertices in a graph is the *order* of the graph. The number of edges in a graph is the *size* of the graph. Let $[n] = \{1, 2 \ldots n\}$.

Given an r-graph H and $W \subseteq V(H)$, we use the lower case w to denote the proportion of vertices of H in W; that is, $|W| = w\,|V(H)|$. For subsets of vertices $A, B \subseteq V(H)$, $H[A]$ refers to the subgraph of H restricted to the vertices of A and $H[A, B]$ refers to the subgraph of H on the vertices of $A \cup B$ consisting of all edges with vertices in both A and B; that is, $E(H[A, B]) = \{e \in E(H) : x, y \in e \,\&\, x \in A \,\&\, y \in B\}$. The subset of vertices that are neighbours of u in H is referred to as $\Gamma_H(u)$, where the subscript may be dropped if no ambiguity would result. The *degree* of a vertex $v \in H$ is $d(v) = |\Gamma_H(v)|$.

The field of extremal combinatorics starts with Mantel's Theorem.

THEOREM 1.1 (Mantel's Theorem). *A graph of order n that contains no triangles contains at most $\lfloor n^2/4 \rfloor$ edges.*

In terms of the scheme set out above, the invariant in Mantel's Theorem is the number of edges in the graph and the graph property is the absence of any triangles. Another way to approach this is to define $\mathscr{T}$ as the class of all graphs that do not contain a triangle. Then Mantel's Theorem states that any graph $T \in \mathscr{T}$ of order n has at most $n^2/4$ edges. We will illustrate the techniques used in this thesis by giving a number of different proofs of Mantel's Theorem.

Given a family of hypergraphs $\mathcal{F}$, a hypergraph is *$\mathcal{F}$-free* if it does not contain a (not necessarily induced) subgraph that is isomorphic to any member of $\mathcal{F}$. For any integer $n \geq r$, the *Turán number* of $\mathcal{F}$ is

$$\mathrm{ex}(n, \mathcal{F}) \quad = \quad \max\{|E(H)| \,:\, H \text{ is an } \mathcal{F}\text{-free, } r\text{-graph}, |V(H)| = n\}$$

and the related asymptotic *Turán density* is the following limit (an averaging argument due to Katona, Nemetz and Simonovits [**17**] shows that it always exists)

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

The problem of determining the Turán density is essentially solved for all 2-graphs by the Erdös-Stone-Simonovits Theorem:

Theorem 1.2 (Erdös and Stone [**9**], Erdös and Simonovits [**8**]). *Let $\mathcal{F}$ be a family of 2-graphs. If $t = \min\{\chi(F) \,:\, F \in \mathcal{F}\} \geq 2$, then*

$$\pi(\mathcal{F}) = \frac{t-2}{t-1}.$$

It follows that the set of all Turán densities for 2-graphs is $\{0, 1/2, 2/3, 3/4 \ldots\}$.

There are two directions in which Turán-type problems can be extended. The first is from graphs to hypergraphs. There is no analogous result for $r \geq 3$ and most progress has been made through determining the Turán densities of individual graphs or families of graphs. The second still considers graphs, but replaces edge densities with densities of other subgraphs and replaces the property of absence of a particular subgraph with more complex properties.

1.3. Techniques

1.3.1. Induction

This is the fundamental technique of combinatorics, as a generalisation of the process of counting.

The power of this technique comes from the fact that the inductive step focuses on only a small part of the graph but enables conclusions to be made about the graph as a whole. Its limitation is that, in the inductive step, no assumptions can be made about the structure of the remainder of the graph. That is, it is necessary to assume both that the remainder of the graph is maximal with respect to the invariant and that it is structured in a way independent from maximising that invariant.

We illustrate induction by our first proof of Mantel's theorem:

First Proof of Mantel's theorem. A graph consisting of two vertices has at most one edge. Assume that the theorem is true for any graph of order k. We aim to show that the theorem is then true for a graph G of order $k+2$. The result will then follow by mathematical induction. Take any pair of vertices, x and y, in G, connected by an edge. Then (noting that there can be only one edge between x and y and any other vertex in G):

$$\begin{aligned} |E(G)| &= |E(G\backslash\{x,y\})| + |E(G[xy, G\backslash\{x,y\}])| + 1 \\ &\leq \frac{k^2}{4} + k + 1 \\ &= \frac{(k+2)^2}{4} \end{aligned}$$

□

Induction is used throughout Chapter 2 – for simple examples very similar to the proof above, see Proposition 2.26 and Proposition 2.27.

Induction generally requires a base case, but an asymptotic version can be used even in the absence of a base case, where the relevant functions are positive and increasing, as is generally the case for combinatorial applications.

Take a positive increasing function $f(x)$. If there is another function $g(x)$ and x_0 such that for all $x \geq x_0$, $\frac{df}{dx} \leq \frac{dg}{dx}$, then $g(x)$ is eventually an upper bound for $f(x)$, in the sense that $\frac{f(x)}{g(x)} \leq 1 + o(1)$. We refer to this as progressive induction and it is the basis of the stability argument in Chapter 2 – see, specifically, Lemma 2.16 – and also forms part of the basic argument justifying entropy compression in Chapter 5.

In a proof by induction, the inductive step allows a specific conclusion about any strict subgraph - namely, that it satisfies the inductive hypothesis. This applies to every subgraph, so that there is a free choice of the additional element to be used to complete the induction. However, it limits the information that can be used about the rest of the graph - only that it satisfies the inductive hypothesis, without any further knowledge of its structure, even though it must in fact have a certain structure in order to achieve the maximum implied by the inductive hypothesis. So, for instance, the maximal graph for Mantel's theorem is the complete bipartite graph, but we cannot use this information in the proof by induction above.

One method to overcome this limitation is to incorporate additional information into the inductive hypothesis. For instance, if the extremal example is essentially unique then the hypothesis may state not only the relevant maximum but also the structure of the extremal example. This structure is then available to be used to complete the induction. This method can be seen in the proof of the Turán number for F_6 in the first part of Chapter 2.

Where further information about both parts of the graph is required to complete the argument, the technique of induction can be extended. The graph is still split into two sections and the aim is still to count the edges in both sections and between the two sections. But additional structure is available in both sections that may be more useful than the inductive hypothesis. We illustrate this with another proof of Mantel's theorem.

Second Proof of Mantel's theorem. Given any graph G of order n consider any vertex x of maximal degree in G and split G into $A = \Gamma(x)$ and $B = G \backslash A$. Then (noting that $G[A]$ is the empty graph):

$$\begin{aligned} E(G) &= \frac{1}{2}\sum_{z \in G} d(z) \\ &= \frac{1}{2}\left(\sum_{z \in A} d(z) + \sum_{z \in B} d(z)\right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2}\left(\sum_{z\in A}|B| + \sum_{z\in B}|A|\right) \\ &= |A|\,(n - |A|) \end{aligned}$$

which is maximised when $|A| = n/2$, giving the statement of the theorem. $\square$

This is the strategy used to count the edges in both the exact and stablity parts of the proof in Chapter 2.

Note that, in the last step of this proof, it was necessary to maximise the function $a(n-a)$ subject to $a \leq n$ and where a and n are both integers. Although this is of course trivial, it is worth drawing attention to the reasoning in more detail as it is demonstrates a widely-used approach. So, let $a = kn$ where $k \in \mathbb{R}$. The aim is to maximise $kn(n-kn) = n^2k(1-k)$. Differentiating with respect to k reveals a maximum at $k = 1/2$. By translating the problem into the realm of real variables, it is possible to apply analytical techniques. Of course, here, this is all done implicitly, but, in more complex cases, this is often accomplished by expressing the problem in terms of weighted graphs with real vertex and/or edge weightings. The results can usually be converted back into statements about large discrete graphs, although care is needed when irrational weights are used. An analytic approach is used in Subsection 4.3.2.

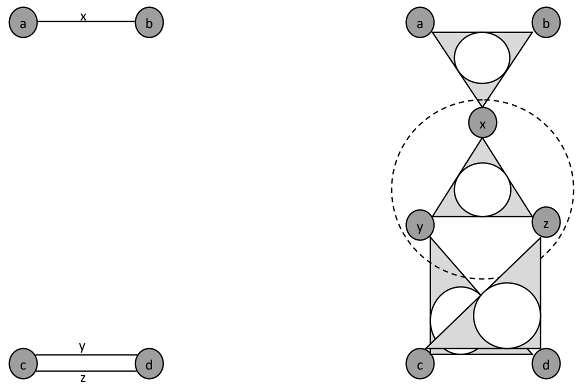
1.3.2. Link Graphs

A variant of induction that applies specifically to hypergraphs uses link graphs. For any 3-graph H containing an edge xyz, we define a number of link graphs. The link graph L_x is defined as follows:

$$\begin{aligned} V(L_x) &= V(H) - \{x\} \\ E(L_x) &= \{ab \,:\, abx \in E(H)\} \end{aligned}$$

The link graph L_{xyz} is the multigraph that is the subgraph of the union $L_x \cup L_y \cup L_z$ on $V(H) - \{x,y,z\}$. The label of an edge $ab \in E(L_{xyz})$ is $l(ab) = \{q \in \{x,y,z\} : abq \in E(G)\}$. The *weight* of an edge $ab \in L_{xyz}$ is $|l(ab)|$ and the weight of L_{xyz} is $w(L_{xyz}) = \sum_{ab\in L_{xyz}} |l(ab)|$.

For instance, here is the 3-graph $\{xyz, xab, ycd, zcd\}$ and the associated link graph L_{xyz}:



The link graph construction can be used for 3-graph proofs. The standard method is by a double induction. First, there is an induction on a single vertex or a single edge of the hypergraph. The inductive step is accomplished by considering the link graph of the relevant vertex or edge. This transforms a statement about hypergraphs into a statement about graphs. This statement may then in turn be solved by an induction on the link graph.

This technique is used extensively in Chapter 2 - see, for example, Subsection 2.2.1.

1.3.3. Cauchy-Schwarz Inequality

The second proof of Mantel's theorem used the convexity of x^2. This can be extended to proofs that count edges or vertex-degrees in different ways and then use the Cauchy-Schwarz Inequality to derive a helpful inequality. We generally use the following simple form of the Inequality:

Proposition (Cauchy-Schwarz Inequality). *For any positive sequence a_n:*

$$\sum_{i=1}^{n} a_n^2 \quad \geq \quad \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2$$

Use of the Cauchy-Schwarz Inequality is demonstrated by another proof of Mantel's theorem.

Third Proof of Mantel's theorem. First note that

$$\sum_{xy \in E(G)} d(x) + d(y) \quad \leq \quad |E(G)|\, n$$

because x and y are not both incident with any other vertex. Then note that

$$\sum_{xy \in E(G)} d(x) + d(y) \quad = \quad \sum_{x \in V(G)} d(x)^2$$

because the sum $d(x)$ is computed for each vertex $d(x)$ times.

As $\frac{1}{2} \sum_{x \in V(G)} d(x) = E(G)$, the Cauchy-Schwarz Inequality gives

$$\begin{aligned} \sum_{x \in V(G)} d(x)^2 \quad &\geq \quad \frac{\left(\sum_{x \in V(G)} d(x)\right)^2}{n} \\ &= \quad \frac{4\,|E(G)|^2}{n}. \end{aligned}$$

Putting these together gives

$$\begin{aligned} \frac{4\,|E(G)|^2}{n} \quad &\leq \quad |E(G)|\, n \\ |E(G)| \quad &\leq \quad \frac{n^2}{4}. \end{aligned}$$

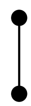
□

Cauchy-Schwarz is used extensively throughout the thesis whenever a convex function is to be maximised. For a specific example, see the end of Subsection 2.3.1.

1.3.4. Flag Algebras

The flag algebra method developed by Razborov allows Cauchy-Schwarz Inequality arguments to be vastly extended. Firstly, it enables a systematic treatment that permits problems to be expressed in the form of optimisation problems involving semi-definite matrices that may be solved computationally. Secondly, it allows algebraic reasoning about extremal problems at a high level of generality. We set out here a simplified annotated usage that demonstrates how it is applied in practice and then sufficient definitions to motivate the use of flag algebras in Chapter 4 (and for one specific use in Chapter 3). For a fuller description, see [**24**].

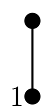
We build up to a definition of Mantel's Theorem using a slight simplification of the original algebra. Assume that we are given a large arbitrary graph G of order n and let G^* be a copy of G where one vertex is labelled 1. We make the following definitions:

 = the probability that a pair of vertices in G chosen uniformly at random comprise an edge

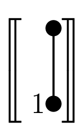
= the average degree of a vertex in G (expressed as a proportion)

$$= \frac{1}{n}\sum_{v\in G}\frac{d(v)}{n-1}$$

and

 = the probability that vertex 1 in G^* and another vertex chosen uniformly at random comprise an edge.

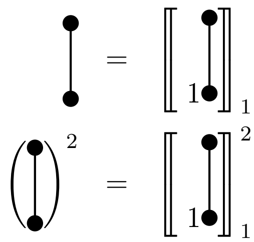
We will use the partially labelled graph to derive statements about the unlabelled graph, so we need some way of relating the two. This is given by the downward or averaging operator which, broadly speaking, expresses the probability of finding the relevant labelled subgraph starting with the unlabelled graph and labelling it uniformly at random:

 = the probability that a vertex chosen uniformly at random in G and labelled 1 together with another vertex chosen uniformly at random comprise an edge

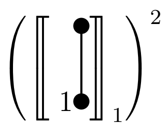
= the average degree of a vertex in G (expressed as a proportion)

$$= \frac{1}{n}\sum_{v\in G}\frac{d(v)}{n-1}.$$

This immediately gives the equality



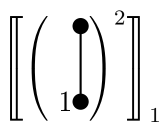
Next, it follows from the existing definitions that



$$= \left(\frac{1}{n}\sum_{v\in G}\frac{d(v)}{n-1}\right)^2$$

$$\frac{1}{n^2}\left(\sum_{v\in G}\frac{d(v)}{n-1}\right)^2$$

and



= the probability that, taking a vertex chosen

uniformly at random and labelled G,

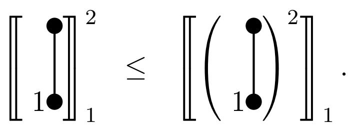
the following event occurs twice: another vertex

chosen uniformly at random comprises an edge

with vertex 1

$$= \frac{1}{n}\sum_{v\in G}\left(\frac{d(v)}{n-1}\right)^2.$$

An application of Cauchy's theorem then gives

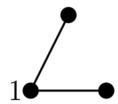


Also, compare these two events in the labelled graph G^*:

= the probability that the following event occurs twice:

a vertex chosen uniformly at random

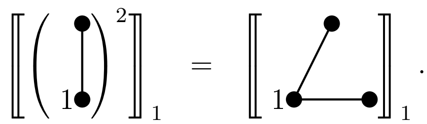
comprises an edge with vertex 1

 = the probability that two vertices

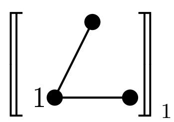
chosen uniformly at random are both

adjacent to vertex 1 (but not each other) .

In a triangle-free graph, these events are equivalent except for sampling with and without replacement, so that they differ only by $O(1)$. The flag algebra allows these two two be treated as asymptotically equivalent, so that the following formal statement is permitted:



Finally, we apply the averaging definition to this graph:

 = the probability that a vertex chosen uniformly

at random in G and labelled 1 together with

two other vertices chosen uniformly at random form

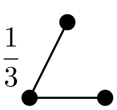
a graph consisting of two edges connected to vertex 1

= the probability that three vertices chosen uniformly

at random in G form a graph consisting of two edges

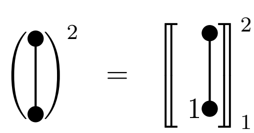
and that a vertex from that triple chosen uniformly

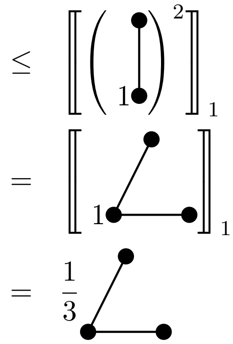
at random is connected to both edges

= $\frac{1}{3}$ 

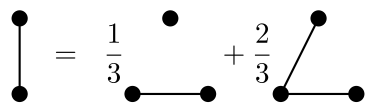
Putting all these elements together, a proof of Mantel's theorem using the flag algebra is as follows.

Fourth Proof of Mantel's theorem . We work in the class of graphs missing K_3 as a subgraph:

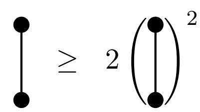




Also,



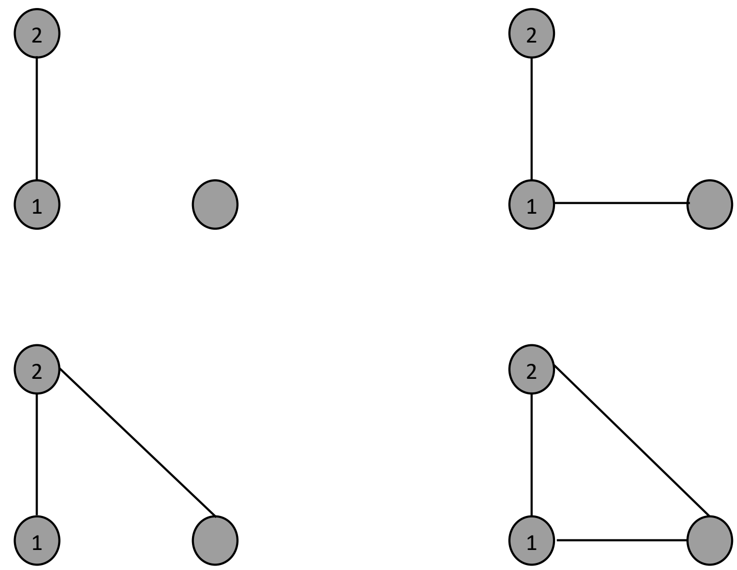
and so



which implies that the density is less than half. $\square$

A flag algebra is defined in the context of a particular class of objects, generally a class of graphs $\mathscr{T}$ where each graph $T \in \mathscr{T}$ does not contain a copy of any of a set of forbidden graphs $\mathscr{F}$ as a graph (or alternatively an induced suubgraph). This corresponds to our example above where the objects are the class of triangle-free graphs. Within a particular flag algebra, a *type* is a graph $\sigma \in \mathscr{T}$ of order s with vertices labelled $1 \dots s$. A σ-*flag* is a pair (F, θ) where $F \in \mathscr{T}$ and θ is a function $\theta : [s] \to V(F)$ such that σ is isomorphic to the labelled subgraph of F induced by $\mathrm{Im}(\theta)$. So a flag is a partially labelled graph; an unlabelled graph may be seen as a σ-flag where σ is the empty type; a type may be seen as a flag with no unlabelled vertices. So, in our example, we dealt with the type consisting of a single labelled vertex and the flags were the graphs on three vertices containing a single labelled vertex.

To consider another example, let σ be a labelled edge. Then the σ-flags on three vertices are the following four graphs:



Let $\mathcal{F}_m^\sigma$ be the set of σ-flags on m vertices and let $\mathcal{F}^\sigma = \cup_m \mathcal{F}_m^\sigma$. Define the following probabilities:

- For $F \in \mathcal{F}_m^\sigma, G \in \mathcal{F}_n^\sigma$, $p(F, G)$ is the probability that an $m-s$-set $V \in G \backslash \theta(\sigma)$ chosen uniformly at random together with $\theta(\sigma)$ induces a subgraph that is isomorphic to F via an isomorphism that preserves the embedding of σ. Note that $p(F, G) = 0$ if $m > n$.

- For $F_1 \in \mathcal{F}_m^\sigma, F_2 \in \mathcal{F}_n^\sigma, G \in \mathcal{F}_p^\sigma$, $p(F_1, F_2, G)$ is the probability that two $m-s$-sets $V_1, V_2 \in G\backslash\theta(\sigma)$ chosen uniformly at random subject to $V_1 \cap V_2 = \oslash$ (that is, V_1 is an $m-s$ set chosen uniformly at random from $G\backslash\theta(\sigma)$ and then V_2 is an $m-s$ set chosen uniformly at random from $G\backslash(\theta(\sigma) \cup V_1)$) together with $\theta(\sigma)$ induce subgraphs that are isomorphic to F_1, F_2 respectively via isomorphisms that preserve the embedding of σ. Note that $p(F_1, F_2, G) = 0$ if $m + n - s > p$.

A key result is that asymptotically $p(F_1, G)p(F_2, G)$ approaches $p(F_1, F_2, G)$. Formally:

THEOREM 1.3 (Razborov). *For $F_1, F_2, G \in \mathcal{F}^\sigma$, $p(F_1, G)p(F_2, G) = p(F_1, F_2, G) + o(1)$, where the $o(1)$ term tends to 0 as $|V(G)|$ tends to infinity*

Another key tool is the chain rule. For $m < n < p$, given $F \in \mathcal{F}^\sigma_m$, $G \in \mathcal{F}^\sigma_p$:

$$p(F,G) \quad = \quad \sum_{H \in \mathcal{F}^\sigma_n} p(F,H), p(H,G).$$

Let $\mathbb{R}\mathcal{F}^\sigma$ be the set of formal linear combinations of elements of $\mathcal{F}^\sigma$, let $\mathcal{K}^\sigma$ be the linear subspace generated by

$$H - \sum_{G \in \mathcal{F}^\sigma_m} p(H,G)G$$

and let $\mathcal{A}^\sigma = \mathbb{R}\mathcal{F}^\sigma/\mathcal{K}^\sigma$. The zero element of $\mathcal{A}^\sigma$ is $\mathcal{K}^\sigma$. The product of elements in $\mathcal{A}^\sigma$ is defined as follows. For $F \in \mathcal{F}^\sigma_m, G \in \mathcal{F}^\sigma_n$, choose an arbitrary $p \geq m+n-s$, then

$$F.G = \sum_{H \in \mathcal{F}^\sigma_p} p(F,G,H)H$$

This is then extended to all of $\mathcal{A}^\sigma$ by linearity. The product is well-defined with unit 1_σ - in particular, it does not depend on the choice of p (see [**24**] for details). This construction can be extended to the asymptotic case using Theorem 1.3. Intuitively, elements of the flag algebra represent the densities of the corresponding subgraphs in large arbitrary graphs of the relevant class. In our triangle-free example, we adopted the formalism by appealing to a large arbitrary graph G and treating the subgraphs as densities within that graph. The flag algebra allows these calculations to be treated as exact without having to consider the lower order terms separately.

The final construction used in the triangle-free example was the *averaging operator*. This may be formally defined as follows. For $F \in \mathcal{A}^\sigma$, let $G \in \mathcal{A}^\emptyset$ be the graph obtained by unlabelling the vertices of σ in F. Let p^σ_F be the probability that a random injective mapping from $[s]$ to $V(G)$ is an embedding of σ in G that yields a σ-flag isomorphic to F. Then

$$[\![F]\!]_\sigma \quad = \quad p^\sigma_F G$$

Various forms of the Cauchy-Schwarz inequality may be developed in relation to the averaging operator. For instance, for every linear combination $A^\sigma \in \mathcal{A}^\sigma$:

$$\left[\!\!\left[(A^\sigma)^2\right]\!\!\right]_\sigma \quad \geq \quad 0$$

This formalism may be developed further using the semi-definite method: constructing optimisation problems involving positive semi-definite matrices that are amenable to solving by computational means. Here, we use it as a convenient abstraction mechanism to allow reasoning that could in theory be expressed without it but would be vastly more complex in its absence.

1.3.5. Stability

Where an extremal solution has been found, a stability result seeks to show that any graph that is close to the extremal limit is somehow close in structure to the extremal graph - that is, the graph that constitutes a lower or upper bound to the extremal solution. A stability result is often harder to prove than the corresponding exact result and there are few examples in the field of extremal hypergraphs. In part, this is because it often presupposes that there is a single extremal graph whereas, in many cases, there is a family of non-isomorphic extremal graphs.

A stability result will typically be of the following form. Let $\mathscr{T}$ be a class of graphs with some desired property. Assume that, for all n, there exists $T_n \in \mathscr{T}_n$ which is the unique extremal graph of order n - that is, the graph of maximal density of order n in $\mathscr{T}$ - and that T_n is of density kn^2. Then a typical stability result would assert that, for all ϵ there exists a δ such that for any graph $G \in \mathscr{T}$ with density $(1-\epsilon)kn^2$ there exists a set of vertices $W \in V(G)$ with $|W| \geq (1-\delta)\,|V(G)|$ such that $G[W]$ is isomorphic to $T_{|W|}$ or has some other similar structural property to $T_{|W|}$. Variants may exclude a set of 'bad' edges rather than a set of bad vertices.

Proof of a stability result also typically employs the exact result. In particular, assume that G has density $(1-\epsilon)kn^2$. Then, because the density of G has the upper bound kn^2, there exists a large subgraph of G with some desirable property - such as a minimum degree - and the remainder of the graph may be placed into the 'bad' category. This process is repeated until the required exact structure is obtained.

There exist few stability results for hypergraphs. A stability result for F_6 is set out in Section 2.3. The functions obtained by generalisation of the Turán function set out in Chapter 4 also embody much of the same information as may be obtained by a stability result.

1.3.6. Ramsey Theory

Ramsey Theory is concerned with the appearance of ordered substructures given a structure of sufficient size. It may be used in extremal combinatorics to obtain subgraphs with guarantees as to structure - local reasoning about these structures must then be translated into global reasoning about the graph as a whole to obtain an extremal result We will only use Ramsey Theory in one place: a version of Ramsey's Theorem is used in Lemma 3.10 below. The necessary statements are as follows:

PROPOSITION 1.4 (Razborov [**26**]). *For any $l > 0$ there exists $N > 0$ such that the following holds. Let a hypergraph B be such that $V(B) = B_1 \dot\cup \ldots \dot\cup B_l$, where $|B_i| = N$. Then there exist $A_i \subseteq B_i$ with $|A_i| = 2$ such that for any $E \in [A_1 \cup \ldots \cup A_l]^3$, whether or not $E \in E(B)$ depends only on the tuple of cardinalities $\langle |E \cap A_1|, \ldots, |E \cap A_l| \rangle$.*

PROPOSITION 1.5 (Razborov [**26**]). *For all $l, n, \epsilon > 0$ there exists $N_0 > 0$ such that if $|B_i| = N$ $(1 \leq i \leq l)$ with $N \geq N_0$ and $S \subseteq B_1 \times \cdots \times B_l$ has $|S| \geq \epsilon N^l$, then there exist $A_i \subseteq B_i$ $(A_i = n)$ such that $A_1 \times \cdots \times A_l \subseteq S$.*

1.3.7. Analytic Combinatorics

Analytic combinatorics is a technique for counting mathematical objects. It is not immediately applicable to extremal questions but is used in Chapter 5 to count certain classes of trees that form part of the proof. As it is peripheral to the main ideas of this thesis and a large area of study in itself, we undertake here only a brief excursion to set out the main ideas that lead to the particular results that are used in Chapter 5. For a comprehensive treatment, see [**13**].

The central idea of analytic combinatorics is to use generating functions as formal structures to encode information about the enumeration of a certain class of objects and then to employ analytic methods on those functions in order to obtain insight into their asymptotic behaviour.

The ordinary generating function of a sequence $\{a_n\}_{n \geq 0}$ is the formal power series

$$\phi_a(x) = \sum_{n \geq 0} a_n x^n.$$

Consider, for instance, the sequence t_n enumerating the number of planar trees of order n. A planar tree consists of a node attached to a sequence of one or more subtrees. This is represented by a generating function as

$$\begin{aligned}\phi_t(x) &= x(1+\phi_t(x)+(\phi_t(x))^2+\ldots) \\ &= \frac{x}{1-\phi_t(x)}.\end{aligned}$$

The notation $[x^n]\phi_a(x) = a_n$ is used to extract the coefficient of x^n from ϕ_a. So, for instance, $[x^{10}]\phi_t(x)$ means the number of planar trees of order 10. For recursive definitions of generating functions the coefficients may be extracted using Lagrange Inversion:

THEOREM 1.6 (Lagrange Inversion). *Let $y(z)$ be a generating function such that $y(z) = z\phi(y(z))$ for an analytic function $\phi(w)$ with $\phi(0) \neq 0$. Then*

$$[z^n]y(z) = \frac{1}{n}[w^{n-1}]\phi(w)^n.$$

The definition can be extended to properties of objects by introducing further variables. The ordinary generating function of a sequence $\{a_{n,k}\}_{n\geq 0,k\geq 0}$ is the formal power series

$$\phi_a(x,u) = \sum_{n\geq 0,m\geq 0} a_{n,m}x^n u^m.$$

Consider, for instance, the sequence $t_{n,m}$ enumerating the number of planar trees of order n with m nodes of out-degree 1. This is represented by a generating function as

$$\begin{aligned}\phi_t(x,u) &= x(1+u\phi_t(x,u)+(\phi_t(x,u))^2+\ldots) \\ &= \frac{x}{1-\phi_t(x)}+(u-1)\phi_t(x,u).\end{aligned}$$

Constructions of combinatorial objects correspond to manipulations of the power series in a systematic way - for further details, see the exploration in [**13**].

The behaviour of a generating function in the complex plane gives information about its coefficients. In particular, the rate of exponential growth of the coefficients is determined by the location of the singularities of the function. The generating functions of combinatorial interest are analytic at

0 and the asymptotic behaviour is determined by the singularity of smallest modulus. The basic property is given by the *Transfer Lemma*:

LEMMA 1.7 (Flajolet, Odlyzko, [**12**]). *Let*

$$\phi(x) = \sum_{n\geq 0} a_n x^n$$

be analytic in a region

$$\Delta(x_0, \eta, \delta) = \{x : |x| < x_0 + \eta, |\arg(x/x_0 - 1)| > \delta\}$$

in which x_0 and η are positive real numbers and $0 < \delta < \pi/2$. If there exists a real number α such that

$$\phi(x) = O((1 - x/x_0)^{-\alpha})$$

then

$$a_n = O(x_0^{-n} n^{\alpha-1}).$$

The proof uses Cauchy's formula with a carefully chosen path of integration around the origin. The Transfer Lemma can be used to characterise the asymptotic behaviour of many combinatorial objects. It can also be extended to multivariate generating functions to derive a combinatorial central limit theorem.

THEOREM 1.8 (Combinatorial Central Limit Theorem, (Drmota, [**7**])). *Suppose that X_n is a sequence of random variables such that*

$$\mathbb{E}u^{X_n} = \frac{[x^n]y(x,u)}{[x^n]y(x,1)}$$

where $y(x,u)$ is a power series, that is the (analytic) solution of the functional equation $y = F(x,y,u)$, where $F(x,y,u)$ is an analytic function in x, y, u around 0 such that $F(0,y,u) = 0$, that $F(x,0,u) \neq 0$, and that all coefficients of $F(x,y,1)$ are real and non-negative. Then the unique solution of the functional equation $y = F(x,y,u)$ with $y(0,u) = 0$ is analytic around 0. If the region of convergence of $F(x,y,u)$ is large enough such that there exist non-negative solutions $x = x_0$ and $y = y_0$ of the system of equations

$$\begin{aligned} y &= F(x,y,1) \\ 1 &= F_y(x,y,1) \end{aligned}$$

and setting

$$\begin{aligned}
\mu &= \frac{F_u}{x_0 F_x} \\
\sigma^2 &= \mu + \mu^2 + \frac{1}{x_0 F_x^3 F_{yy}} \Bigg(F_x^2 \left(F_{yy} F_{uu} - F_{yu}^2 \right) - \\
& \qquad 2 F_x F_u \left(F_{yy} F_{xu} - F_{yx} F_{yu} \right) + F_u^2 \left(F_{yy} F_{xx} - F_{yx}^2 \right) \Bigg)
\end{aligned}$$

where all partial derivatives are evaluated at the point $(x_0, y_0, 1)$*, then*

$$\begin{aligned}
\mathbb{E}(X_n) &= \mu n + O(1) \\
\mathbb{V}ar(X_n) &= \sigma^2 n + O(1)
\end{aligned}$$

and if $\sigma^2 > 0$ *then*

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}ar(X_n)}} \xrightarrow{d} N(0,1)$$

The theorem is proved using the Transfer Lemma and the Quasi Power Theorem by H.K. Hwang (as set out in [**7**]), which provides a general setting to prove central limit theorems for sequences of random variables. It is readily extended to the multivariate case:

REMARK 1.9. If we have several variables $\mathbf{u} = (u_1, \ldots u_k)$ and a sequence of random vectors $\mathbf{X_n}$ with

$$\mathbb{E}\mathbf{u}^{\mathbf{X_n}} = \frac{[x^n] y(x, \mathbf{u})}{[x^n] y(x, \mathbf{1})}$$

where $y(x, \mathbf{u})$ is a power series, which is the solution of the functional equation $y = F(x, y, \mathbf{u})$ then

$$\begin{aligned}
\mathbb{E}(\mathbf{X_n}) &= \mu n + O(1) \\
\mathbb{C}\text{ov}(\mathbf{X_n}) &= \mathbf{\Sigma} n + O(1)
\end{aligned}$$

where $\mu = (\mu_1, \ldots, \mu_k)$ and $\mathbf{\Sigma} = (\sigma_{ij})_{1 \le i,j \le k}$ can be calculated as follows

$$\begin{aligned}
\mu_i &= \frac{F_{u_i}}{x_0 F_x} \\
\sigma_{ij} &= \mu_i \mu_j + \mu_i \delta_{ij} + \frac{1}{x_0 F_x^3 F_{yy}} \Bigg(F_x^2 \left(F_{yy} F_{u_i u_j} - F_{y u_i} F_{y u_j} \right) \\
& \quad - F_x F_{u_i} \left(F_{yy} F_{x u_j} - F_{yx} F_{y u_j} \right) - F_x F_{u_j} \left(F_{yy} F_{x u_i} - F_{yx} F_{y u_i} \right)
\end{aligned}$$

$$+F_{u_iu_j}\left(F_{yy}F_{xx}-F_{yx}^2\right)\Bigg)$$

and there is a central limit theorem of the form

$$\frac{1}{\sqrt{n}}(\mathbf{X_n}-\mathbb{E}(\mathbf{X_n})) \quad \xrightarrow{d} \quad N(\mathbf{0},\mathbf{\Sigma}).$$

1.3.8. The Probabilistic Method

The probabilistic method is the name given to the use of techniques from probability theory to prove the existence of combinatorial structures. The probability distributions are often over finite structures and so could be recast as counting questions, but the ability to use concepts such as linearity of expectation and concentration inequalities allows greater expressive power. The method includes a wide range of tools and a full reference is [**1**] - we mention it briefly here as it is used in conjunction with analytic combinatorics as part of the argument to Chapter 5.

A typical example of the method is provided by our final proof of Mantel's theorem.

Fifth proof of Mantel's theorem. Given a graph of order n, define a probability distribution over the vertices of G, such that the random variable X takes the value ij with probability p_ip_j. Start with a uniform distribution such that $p_i = 1/n$ for all i. The probability that X samples an edge of G is

$$\mathbb{P}[X \in E(G)] \quad = \quad \sum_{i,j\,:\,ij\in E(G)} p_ip_j$$

which, with the uniform distribution, is equal to $\frac{2}{n^2}\,|E(G)|$. We then modify the distribution to maximise this probability. In particular, take any two non-adjacent vertices i, j with $p_i, p_j > 0$. Let

$$\begin{aligned} s_i &= \sum_{k\in\Gamma(i)} p_k \\ s_j &= \sum_{k\in\Gamma(i)} p_k. \end{aligned}$$

If $s_i > s_j$ then set p_i to $p_i + p_j$ and set p_j to 0. Otherwise, set p_i to 0 and p_j to $p_i + p_j$. This operation reduces the number of non-adjacent vertices allocated a positive probability and does not decrease the probability that X samples an edge. Repeating this operation leads to a situation where the probability is concentrated on a set of adjacent vertices. As G is triangle-free, it follows that there are precisely two vertices, i and j, with positive probability and that $\mathbb{P}[X \in E(G)] = p_i p_j + p_j p_i \leq 1/2$ as $p_i + p_j = 1$. As these operations have not decreased the probability, it follows that

$$\begin{aligned} \frac{2}{n^2}\,|E(G)| &\leq \frac{1}{2} \\ |E(G)| &\leq \frac{n^2}{4}. \end{aligned}$$

□

1.3.9. Entropy Compression

The previous proof was essentially algorithmic. It set out an algorithm that was guaranteed to terminate as each iteration increased a particular quantity (the number of non-adjacent vertices) that was bounded. A more involved technique that has become known as entropy compression employs a similar idea. It works with probabilistic algorithms that, at each stage, make a change to a combinatorial object G within a class $\mathcal{G}$ to produce another object G' that is also within $\mathcal{G}$ and locally satisfies some set property (although, overall, the property may not be better satisfied by the new graph). For instance, if the criterion is to construct a certain path within the graph, the algorithm might add a new edge xy but simultaneously remove a number of other edges. Accordingly, the algorithm will only terminate if no improvement is possible with respect to the set property – that is, it has been satisfied throughout the whole graph.

The object, then, is to show that the algorithm always terminates and so the relevant property has been satisfied. This is accomplished as follows. The algorithm makes a random choice at each stage, so requires a random number as input – it can be seen as effectively 'consuming' a string of random numbers that gets longer as the algorithm continues. At each stage, the algorithm also creates a 'log', a separate history of the algorithm, recording the action it took. The string of random numbers can be reconstituted from the object G' and from the log. The key is that the algorithm is designed to take advantage of the particular structure of the problem so that the log can be stored efficiently. If the information content of the log grows at a slower rate than that of the random input, then the algorithm must eventually terminate, or otherwise it would compress the information content of the random string, which gives a contradiction.

The final stage to this argument is similar to progressive induction, as set out above, in that the relevant statement is not true initially - because the information content of G' is some large but essentially fixed number - but it must become true eventually because the rate of growth is lower than the rate of growth of the quantity it is being measured against.

Entropy compression is used to prove the main result in Chapter 5.

1.4. Conclusion

Many problems in extremal combinatorics can be expressed using elementary concepts. However, solving these problems can require a wide variety of techniques taken from different branches of mathematics. In this introductory chapter, we have set out the principal ones used in the remainder of this thesis. The list is not comprehensive – there are various important areas mentioned only in passing, such as Ramsey Theory in Chapter 3 – but we have attempted to give an overview of the variety of mathematical subjects incorporated into the study of combinatorics.

Chapter 2

A Hypergraph Stability Theorem[1]

2.1. Introduction

A r-uniform hypergraph, or r-graph, is a pair $H = (V(H), E(H))$ where $E(H) \subseteq V(H)^{(r)}$. The elements of $V(H)$ are referred to as *vertices* and the elements of $E(H)$ are referred to as *edges*. A 2-graph is a simple graph. For any vertex subset X, we use the lower case x to denote the proportion of vertices in X; that is, $|X| = x\,|V(H)|$.

Given a family of hypergraphs $\mathcal{F}$, a hypergraph is $\mathcal{F}$*-free* if it does not contain a (not necessarily induced) subgraph that is isomorphic to any member of $\mathcal{F}$. For any integer $n \geq r$, the *Turán number* of $\mathcal{F}$ is

$$\mathrm{ex}(n, \mathcal{F}) \quad = \quad \max\left\{|E(H)| \,:\, H \text{ is an } \mathcal{F}\text{-free, } r\text{-graph}, |V(H)| = n\right\}$$

and the related asymptotic *Turán density* is the following limit (an averaging argument due to Katona, Nemetz and Simonovits [**17**] shows that it always exists)

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

[1]This chapter has been published in slightly amended form as [**28**]

The problem of determining the Turán density is essentially solved for all 2-graphs by the Erdös-Stone-Simonovits Theorem:

THEOREM 2.1 (Erdös and Stone [**9**], Erdös and Simonovits [**8**]). *Let $\mathcal{F}$ be a family of 2-graphs. If $t = \min\{\chi(F) : F \in \mathcal{F}\} \geq 2$, then*

$$\pi(\mathcal{F}) = \frac{t-2}{t-1}.$$

It follows that the set of all Turán densities for 2-graphs is $\{0, 1/2, 2/3, 3/4 \ldots\}$.

There is no analogous result for $r \geq 3$ and most progress has been made through determining the Turán densities of individual graphs or families of graphs. A central problem, originally posed by Turán, is to determine $\pi\left(K_4^{(3)}\right)$, where $K_4^{(3)} = \{123, 124, 134, 234\}$, the complete 3-graph on 4 vertices. This is a natural extension of determining the Turán density of the triangle for 2-graphs, a question answered by Mantel's Theorem. Turán conjectured that the density is 5/9 but this question remains unanswered despite a great deal of work, with the current best upper bound of 0.561666 given by Razborov [**25**].

A related problem due to Katona is given by extending the triangle to the family of cancellative hypergraphs. A cancellative hypergraph H has the property that, for any edges $a, b \in H$, there is no edge $c \in H$ such that $a \triangle b \subseteq c$ (where $\triangle$ is the symmetric difference). For 2-graphs, this amounts to forbidding all triangles. For a 3-graph, it is equivalent to forbidding the two non-isomorphic configurations $K_4^- = \{123, 124, 134\}$ and $F_5 = \{123, 124, 345\}$.

Let $S(n)$ be the complete balanced tripartite 3-graph on n vertices, that is, the 3-graph on n vertices divided into 3 sets of size as equal as possible and with edges consisting of all triples with one vertex from each set. Let $s(n)$ be the number of edges in $S(n)$.

THEOREM 2.2 (Bollobás [**4**]). *For $n \geq 3$, $S(n)$ is the unique cancellative 3-graph of order n and maximum size.*

This result was refined by Frankl and Füredi [**14**] and Keevash and Mubayi [**19**], who proved that $S(n)$ was the extremal configuration for the single forbidden graph F_5 for $n \geq 33$; that is, $\mathrm{ex}(n, F_5) = s(n)$ for $n \geq 33$.

The *blow up* of a k-graph H is the graph $H(t)$ obtained by replacing each vertex $a \in V(H)$ with a set of t vertices $V_a \in V(H(t))$ such that for any k vertices $\{p_1, \ldots, p_k\} \in \binom{V(H)}{k}$ and all sets of k vertices $\{q_1, \ldots q_k\} \in \binom{V(H(t))}{k}$ with q_i in V_{p_i}, $q_1 \ldots q_k$ is an edge in $H(t)$ iff $p_1 \ldots p_k$ is an edge in H. The following result is an invaluable tool in determining the Turán density of a graph that can be shown to be contained in the blow ups of other graphs:

THEOREM 2.3 (Brown and Simonovits [**5**],[**2**]). *If F is an r-graph that is contained in a blow up of every member of a family of r-graphs $\mathcal{G}$, then $\pi(F) = \pi(F \cup \mathcal{G})$.*

Baber and Talbot considered the 3-graph $F_6 = \{123, 124, 345, 156\}$, which is not contained in any blow up of F_5 (so that Theorem 2.3 does not guarantee that $\pi(F_6) \leq 2/9$ and so, by analogy with the case for 2-graphs, it might be expected that the Turán density was not $2/9$). Using Razborov's flag algebra framework[**24**], they gave a computational proof that in fact $\pi(F_6) = 2/9$. In this paper, we give two proofs of $\pi(F_6) = 2/9$ that do not rely on computational analysis, together with an associated stability result.

Note first that F_6 is contained in a blow up of K_4^-. Indeed, taking $K_4^-(2)$ as the blow up of $\{abc, abd, acd\}$, then $\{a_1b_1c_1, a_1b_1d_1, c_1d_1a_2, b_1a_2c_2\}$ is a copy of F_6. Theorem 2.3 implies that $\pi(F_6) = \pi(\mathcal{F})$, where $\mathcal{F} = \left\{K_4^-, F_6\right\}$. Accordingly, we work throughout with the family $\mathcal{F}$.

Our main result in this chapter is the following theorem which determines the exact Turán number for $\mathcal{F}$. Let $C_5^{(3)}$ be the tight cycle graph on 5 vertices; that is, $C_5^{(3)} = \{123, 234, 345, 451, 512\}$.

THEOREM 2.4. *If $n \geq 3$ then the unique $\mathcal{F}$-free 3-graph with $ex(n, \mathcal{F})$ edges and n vertices is $S_3(n)$ unless $n = 5$ in which case it is $C_5^{(3)}$.*

As noted above, F_6 is contained in $K_4^-(2)$, so that this Turán density result follows.

THEOREM 2.5. $\pi(F_6) = \frac{2}{9}$.

In the second part of this chapter we provide associated stability results as well as an alternative proof of the Turán density.

2.2. Turán Number

THEOREM 2.4. *If $n \geq 3$ then the unique $\mathcal{F}$-free 3-graph with $ex(n, \mathcal{F})$ edges and n vertices is $S_3(n)$ unless $n = 5$ in which case it is $C_5^{(3)}$.*

PROOF. We use induction on n. Note that the result holds trivially for $n = 3, 4$. For $n = 5$ it is straightforward to check that the only $\mathcal{F}$-free 3-graphs with 4 edges are $S_3(5)$, $\{123, 124, 125, 345\}$ and $\{123, 234, 345, 451\}$. Of these the first two are edge maximal while the third can be extended by a single edge to give $C_5^{(3)}$. Thus we may suppose that $n \geq 6$ and the theorem is true for $n - 3$.

Let G be $\mathcal{F}$-free with $n \geq 6$ vertices and $\mathrm{ex}(n, \mathcal{F})$ edges. Since $S_3(n)$ is $\mathcal{F}$-free we have $e(G) \geq s_3(n)$.

The inductive step proceeds as follows: select a special edge $abc \in E(G)$ (precisely how we choose this edge will be explained in Lemma 2.6 below). For $0 \leq i \leq 3$ let f_i be the number of edges in G meeting abc in exactly i vertices. Thus by our inductive hypothesis we have

(2.2.1)
$$\begin{aligned} e(G) &= f_0 + f_1 + f_2 + f_3 \\ &\leq \mathrm{ex}(n-3, \mathcal{F}) + f_1 + f_2 + 1. \end{aligned}$$

Note that unless $n - 3 = 5$ our inductive hypothesis says that $\mathrm{ex}(n - 3, \mathcal{F}) = s_3(n - 3)$ with equality iff $G - \{a, b, c\} = S_3(n - 3)$. For the moment we will assume that $n \neq 8$ and so we have the following bound

(2.2.2)
$$e(G) \leq s_3(n-3) + f_1 + f_2 + 1,$$

with equality iff $G - \{a, b, c\} = S_3(n - 3)$.

Let $V^- = V(G) - \{a, b, c\}$. For each pair $xy \in \{ab, ac, bc\}$ define $\Gamma_{xy} = \{z \in V^- : xyz \in E(G)\}$ and let $\Gamma_{abc} = \Gamma_{ab} \cup \Gamma_{ac} \cup \Gamma_{bc}$ be the *link-neighbourhood* of abc. Note that since G is K_4^--free and abc is an edge this is a disjoint union, so

$$f_2 = |\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}| = |\Gamma_{abc}|.$$

For $x \in \{a, b, c\}$ define $L(x)$ to be the *link-graph of* x, so $V(L(x)) = V^-$ and $E(L(x)) = \{yz \subset V^- : xyz \in E(G)\}$. The *link-graph of the edge* abc is the edge labelled graph L_{abc} with vertex set V^- and edge set $L(a) \cup L(b) \cup L(c)$. The label of an edge $yz \in E(L_{abc})$ is $l(yz) = \{x \in \{a,b,c\} : xyz \in E(G)\}$. The *weight* of an edge $yz \in L_{abc}$ is $|l(yz)|$ and the weight of L_{abc} is $w(L_{abc}) = \sum_{yz \in L_{abc}} |l(yz)|$. Note that $f_1 = w(L_{abc})$.

The following lemma provides our choice of edge abc.

LEMMA 2.6. If G is $\mathcal{F}$-free with $n \geq 6$ vertices and $\mathrm{ex}(n, \mathcal{F})$ edges then there is an edge $abc \in E(G)$ such that

$$w(L_{abc}) + |\Gamma_{abc}| \leq t_3(n-3) + n - 3,$$

with equality iff $L_{abc} = T_3(n-3)$ and $\Gamma_{abc} = V^-$.

Let $abc \in E(G)$ be a fixed edge given by Lemma 2.6.

By assumption $e(G) \geq s_3(n)$ so Lemma 2.11(i) and Lemma 2.6 together with the bound on $e(G)$ given by (2.2.2) imply that $e(G) = s_3(n)$ and hence $G - \{a,b,c\} = S_3(n-3)$, $L_{abc} = T_3(n-3)$ and $\Gamma_{abc} = V^-$. To complete the proof we need to show that $G = S_3(n)$. First note that since $L_{abc} = T_3(n-3)$ and $\Gamma_{abc} = V^-$, Lemma 2.8(i) and Lemma 2.7(F_6-3) imply that no vertex in Γ_{ab} is in an edge with label c and similarly for Γ_{ac}, Γ_{bc}. Hence L_{abc} is the complete tripartite graph with vertex classes Γ_{ab}, Γ_{ac} and Γ_{bc} and the edges between any two parts are labelled with the common label of the parts (e.g. all edges from Γ_{ab} to Γ_{ac} receive label a).

Finally we need to show that $G - \{a,b,c\} = S_3(n-3)$ has the same tripartition as L_{abc}. This is straightforward: any edge $xyz \in E(G - \{a,b,c\})$ not respecting the tripartition of L_{abc} meets one of the parts at least twice. But if $x, y, z \in \Gamma_{ab}$ then $|\Gamma_{ac}| \geq 2$ so let $u \in \Gamma_{ac}$. Setting $a = 1, b = 2, x = 3, y = 4, z = 5, u = 6$ gives a copy of F_6. If $x, y \in \Gamma_{ab}$ and $z \in \Gamma_{ac}$ then $a = 1, x = 3, y = 4, z = 2$ gives a copy of K_4^-.

Hence $G = S_3(n)$ and the proof is complete in the case $n \neq 8$.

For $n = 8$ we note that if $G - \{a,b,c\}$ is F_5 -free then Theorem 2.2 implies that the result follows as above, so we may assume that $G - \{a,b,c\}$ contains a copy of F_5. In this case it is sufficient to show that $e(G) \leq 17 < 18 = s_3(8)$.

If $V(G - \{a,b,c\}) = \{s,t,u,v,w\}$ then we may suppose that $stu, stv, uvw, abc \in G$. Since G is K_4^--free it does not contain suv or tuv.

Moreover it contains at most 3 edges from $\{u, v, w\}^{(2)} \times \{a, b, c\}$ and at most 5 edges from $\{s, t, u, v, w\} \times \{a, b, c\}^{(2)}$. Since G is F_6-free it contains no edges from $\{s, t\} \times \{w\} \times \{a, b, c\}$.

The only potential edges we have yet to consider are those in $\{st, su, tu, sv, tv\} \times \{w, a, b, c\}$. Since G is K_4^--free it contains at most 2 edges from std, sud, tud, svd, tvd for any $d \in \{w, a, b, c\}$. Moreover, since G is F_6-free, if it contains 2 such edges for a fixed d then it can contain at most 3 such edges in total for the other choices of d. Hence at most 5 such edges are present.

Thus in total $e(G) \leq 4 + 3 + 5 + 5 = 17$ as required. $\square$

2.2.1. Structure of Link Graphs

Our analysis of link graphs relies fundamentally on the following basic facts.

LEMMA 2.7. *For any 3-graph H containing an edge abc (and at least 3 other vertices), let L and L^* be respectively the link graph and weighted link graph of abc in H. If H is $\mathcal{F}$-free then the following configurations cannot appear as subgraphs of L^*. Moreover any configuration that can be obtained from one described below by applying a permutation to the labels $\{a, b, c\}$ must also be absent.*

- (F_6-1) The triangle xy, xz, yz with $l(xy) = l(xz) = a$ and $l(yz) = b$.
- (F_6-2) The pair of edges xy, xz with $l(xy) = ab$ and $l(xz) = c$.
- (F_6-3) A vertex $x \in \Gamma_{ab}$ and edges xy, yz with labels $l(xy) = c$ and $l(yz) = a$.
- (F_6-4) A vertex $x \in \Gamma_{ab}$ and edges xy, yz, zw with labels $l(xy) = l(zw) = a$ and $l(yz) = b$.
- (F_6-5) Vertices $x \in \Gamma_{ac}, y \in \Gamma_{bc}, z \in \Gamma_{ab}$ and the edge xy with label $l(xy) = b$.
- (K_4^--1) The triangle xy, xz, yz with $l(xy) = l(xz) = l(yz) = a$.
- (K_4^--2) The vertex $x \in \Gamma_{ab}$ and edge xy with label $l(xy) = ab$.
- (K_4^--3) The vertices $x, y \in \Gamma_{ab}$ and edge xy with label $l(xy) = a$.

In each case we describe a labelling of the vertices of the given configuration to show that if it is present then G is not $\mathcal{F}$-free.

- (F_6-1) $a = 1$, $b = 5$, $c = 6$, $x = 2$, $y = 3$, $z = 4$.
- (F_6-2) $a = 3$, $b = 4$, $c = 5$, $x = 1$, $y = 2$, $z = 6$.
- (F_6-3) $a = 1$, $b = 2$, $c = 3$, $x = 4$, $y = 5$, $z = 6$.
- (F_6-4) $a = 1$, $b = 3$, $x = 2$, $y = 4$, $z = 5$, $w = 6$.
- (F_6-5) $a = 5$, $b = 1$, $c = 3$, $x = 4$, $y = 2$, $z = 6$.
- (K_4^--1) $a = 1$, $x = 2$, $y = 3$, $z = 4$.
- (K_4^--2) $a = 3$, $b = 4$, $x = 1$, $y = 2$.
- (K_4^--3) $a = 1$, $b = 2$, $x = 3$, $y = 4$. □

LEMMA 2.8. *For any 3-graph H containing an edge abc (and at least 3 other vertices), let L_{abc} be the link graph of abc in H. If H is $\mathcal{F}$-free then:*

(1) The only K_4s in L_{abc} are rainbow (that is, each vertex is incident with all 3 colours).

(2) L_{abc} is K_5-free.

(3) If $xy \in E(L_{abc})$ has $l(xy) = abc$ then x, y meet no other edges in L_{abc} and $x, y \notin \Gamma_{abc}$.

(4) If $V^4_{abc} = \{x \in V^- :$ there is a K_4 containing $x\}$ then $\Gamma_{abc}(V^4_{abc}) = \emptyset$.

(5) There are no edges in L_{abc} between Γ_{abc} and V^4_{abc}.

(6) If $x \in V^4_{abc}$ then $|l(xy)| \leq 1$ for all $y \in V^-$.

(7) If $x \in \Gamma_{ac}$, $y \in \Gamma_{bc}$ and $l(xy) = ab$, then $\Gamma_{bc} = \emptyset$. Moreover, if $xz \in E(L_{abc})$ with $z \neq y$ then $z \notin \Gamma_{abc}$ and $l(xz) = a$, while if $yz \in E(L_{abc})$ with $z \neq x$ then $z \notin \Gamma_{abc}$ and $l(yz) = b$.

(8) If $xy, xz \in E(L_{abc})$, $l(xy) = ab$ and $z \in \Gamma_{abc}$ then $|l(xz)| \leq 1$.

PROOF. We will make repeated use of Lemma 2.7.

(1) This follows immediately from (F_6-1) and (K_4^--1): if $uvwx$ is a copy of K_4 then we may suppose $l(uv) = a, l(uw) = b, l(vw) = c$, thus $l(ux) = c$, continuing we see that $uvwx$ must be rainbow.

(2) This follows immediately from (1): if $xyzuv$ is a copy of K_5 then by (F_6-1) we may suppose that $l(xy), l(xz), l(xu), l(xv)$ are all distinct single colours but this is impossible since there are only 3 labels in total.

(3) This follows immediately from (F_6-2) and (K_4^--2).

(4) If x is in a K_4 then by (1) it lies in edges with labels a, b, c, and (F_6-3) implies that $x \notin \Gamma_{abc}$.

(5) If $x \in \Gamma_{abc}$, say $x \in \Gamma_{ab}$, and $y \in V_{abc}^4$ with $xy \in E(L_{abc})$ then (F_6-3) implies that $l(xy) \neq c$, while (F_6-4) implies that $l(xy) \neq a, b$ (since there are t, u, v, w such that $l(yt) = b, l(tu) = a$ and $l(yv) = a, l(vw) = b$).

(6) This follows immediately from the fact that all $v \in V_{abc}^4$ meet edges with labels a, b, c and (F_6-2).

(7) (F_6-5) implies that $\Gamma_{bc} = \emptyset$. If $xz \in E(L_{abc})$ then (F_6-3) implies that $l(xz) = a$. Now (K_4-3) implies that $z \notin \Gamma_{ac}$ while (F_6-3) implies that $z \notin \Gamma_{bc}$. Hence $z \notin \Gamma_{abc}$. Similarly if $yz \in E(L_{abc})$ then $l(yz) = b$ and $z \notin \Gamma_{abc}$.

(8) If x or y belong to Γ_{abc} then this follows directly from (F_6-3) so suppose that $x, y \notin \Gamma_{abc}$, $l(xy) = ab$ and $|l(xz)| = 2$. Now (F_6-2) implies that $l(xz) = ab$ so (K_4-2) implies that $z \in \Gamma_{ac} \cup \Gamma_{bc}$. But then ($F_6$-3) is violated. Hence $|l(xz)| \leq 1$.

$\square$

Lemma 2.8(5) allows us to partition the vertices of L_{abc} as $V^- = \Gamma_{abc} \cup V_{abc}^4 \cup R_{abc}$, where $V_{abc}^4 = \{x \in V :$ there is a K_4 containing $x\}$ and $R_{abc} = V - \Gamma_{abc} \cup V_{abc}^4$

2.2.2. Lemmas for Turán Number

LEMMA 2.6. If G is $\mathcal{F}$-free with $n \geq 6$ vertices and $\mathrm{ex}(n, \mathcal{F})$ edges then there is an edge $abc \in E(G)$ such that

$$w(L_{abc}) + |\Gamma_{abc}| \leq t_3(n-3) + n - 3,$$

with equality iff $L_{abc} = T_3(n-3)$ and $\Gamma_{abc} = V^-$.

PROOF. Let G be $\mathcal{F}$-free with $n \geq 6$ vertices and $\mathrm{ex}(n, \mathcal{F})$ edges. By Lemma 2.33 we can choose an edge $abc \in E(G)$ such that $|\Gamma_{abc}| \geq n - \lfloor n/3 \rfloor - 3$. Let $V^- = \Gamma_{abc} \cup R_{abc} \cup V^4_{abc}$ be the partition of V^- given by Lemma 2.8(5). If $s = |V^-|$, $j = |\Gamma_{abc}|$, $k = |R_{abc}|$ and $l = |V^4_{abc}|$ then $n - 3 = s = j + k + l$ and $j \geq s - \lfloor s/3 \rfloor - 1 \geq j + k - \lfloor (j+k)/3 \rfloor - 1$. We can apply Lemma 2.9 to $H = L_{abc}[\Gamma_{abc} \cup R_{abc}]$, to deduce that

$$w(L_{abc}[\Gamma_{abc} \cup R_{abc}]) + |\Gamma_{abc}| \leq t_3(j+k) + j + k,$$

with equality iff $R_{abc} = \emptyset$ and $L_{abc}[\Gamma_{abc}] = T_3(j+k)$. Now if L_{abc} is K_4-free then $V^4_{abc} = \emptyset$ and the proof is complete, so suppose there is a K_4 in L_{abc}. In this case $4 \leq |V^4_{abc}| \leq n - 3 - |\Gamma_{abc}| \leq \lfloor n/3 \rfloor$, so $n \geq 12$.

We now need to consider the edges in L_{abc} meeting V^4_{abc}. By Lemma 2.8(2) we know that L_{abc} is K_5-free, while Lemma 2.8(6) says that V^4_{abc} meets no edges of weight 2 or 3, so by Turán's theorem $w(L_{abc}[V^4_{abc}]) \leq t_4(l)$.

Lemma 2.8(5) implies that there are no edges from Γ_{abc} to V^4_{abc} so the total weight of edges between $\Gamma_{abc} \cup R_{abc}$ and V^4_{abc} is at most kl. Thus

$$w(L_{abc}) + |\Gamma_{abc}| \leq t_3(j+k) + j + k + t_4(l) + kl.$$

Finally Lemma 2.10 with $s = n - 3$ implies that

$$w(L_{abc}) + |\Gamma_{abc}| \leq t_3(n-3) + n - 3,$$

with equality iff $R_{abc} = V^4_{abc} = \emptyset$ and $L_{abc} = T_3(n-3)$ as required. $\square$

LEMMA 2.9. *Let H be a subgraph of L_{abc} with $s \geq 3$ vertices satisfying $V(H) \cap V^4_{abc} = \emptyset$. If $H_\Gamma = V(H) \cap \Gamma_{abc}$ and $|H_\Gamma| \geq s - \lfloor s/3 \rfloor - 1$ then*

$$w(H) + |H_\Gamma| \leq t_3(s) + s,$$

with equality iff $H_\Gamma = V(H)$ *and* $H = T_3(s)$.

Proof. We prove this by induction on $s \geq 3$. The result holds for $s = 3, 4$ (see the end of this proof for the tedious details) so suppose that $s \geq 5$ and the result holds for $s - 2$.

Let H be a subgraph of L_{abc} with $s \geq 5$ vertices satisfying $V(H) \cap V_{abc}^4 = \emptyset$. Let $H_\Gamma = V(H) \cap \Gamma_{abc}$ and suppose that $|H_\Gamma| \geq s - \lfloor s/3 \rfloor - 1$.

Note that if H contains no edges of weight 2 or 3 then the result follows directly from Turán's theorem, so we may suppose there are edges of weight 2 or 3. With this assumption it is sufficient to show that

$$w(H) + |H_\Gamma| \leq t_3(s) + s - 1.$$

By Lemma 2.11 (iii) this is equivalent to showing that the following inequality holds:

$$w(H) + |H_\Gamma| \leq t_3(s-2) + 2s - 2 + \lfloor s/3 \rfloor \tag{2.2.3}$$

Case (i): There exists an edge of weight 3, $l(xy) = abc$.

Lemma 2.8 (3) implies that $x, y \notin H_\Gamma$ and x, y meet no other edges in H, so we can apply the inductive hypothesis to $H' = H - \{x, y\}$ to obtain

$$w(H) + |H_\Gamma| \leq w(H') + |H'_\Gamma| + 3 \leq t_3(s-2) + s - 2 + 3.$$

Hence (2.2.3) holds as required. So we may suppose that H contains no edges of weight 3.

Case (ii): The only edges of weight 2 are contained in H_Γ

Let $xy \in E(H)$ have weight 2, say $l(xy) = ab$. Now Lemma 2.7 (K_4^--2) implies that $x, y \notin \Gamma_{ab}$, while Lemma 2.7 (K_4^--3) implies that x, y cannot both belong to Γ_{ac} or Γ_{bc} so we may suppose that $x \in \Gamma_{ac}$ and $y \in \Gamma_{bc}$. Lemma 2.8 (8) implies that x, y have no more neighbours in H_Γ. If $H_\Gamma = V(H)$ then we can apply the inductive hypothesis to $H' = H - \{x, y\}$ to obtain

$$w(H) + |H_\Gamma| \leq t_3(s-2) + s - 2 + 2 + 2,$$

in which case (2.2.3) holds, so suppose $V(H) \neq H_\Gamma$.

Let $z \in V(H) - H_\Gamma$ be a neighbour of x in H if one exists otherwise let z be any vertex in $V(H) - H_\Gamma$. By our assumption that all edges of weight 2 are contained in H_Γ, z meets no edges of weight 2. Moreover, by Lemma 2.8 (7), all edges containing x (except xy) have label b, so x is not in any triangles in H. Hence x and z have no common neighbours in H and so the total weight of edges meeting $\{x, z\}$ is at most $2+1+s-3$ (if xz is an edge) and at most $2 + s - 2$ otherwise. Applying our inductive hypothesis to $H' = H - \{x, z\}$ we have

$$w(H) + |H_\Gamma| \leq t_3(s-2) + s - 2 + 1 + s,$$

and (2.2.3) holds.

Case (iii): There is an edge of weight 2 meeting $V(H) - H_\Gamma$.

So suppose that $xy \in E(H)$, $l(xy) = ab$ and $y \notin H_\Gamma$. Lemma 2.8 (8) implies that for any $z \in H_\Gamma$ we have $|l(xz)|, |l(yz)| \leq 1$. Let $\gamma_{xy} = |\{x, y\} \cap H_\Gamma| \leq 1$. Thus, since xy is not in any triangles, the total weight of edges meeting $\{x, y\}$ is at most

$$2 + s - 2 + |V(H) - H_\Gamma| - (2 - \gamma_{xy}).$$

Applying the inductive hypothesis to $H' = H - \{x, y\}$ we have

$$w(H) + |H_\Gamma| \leq t_3(s-2) + s - 2 + s + s - |H_\Gamma| - 2 + 2\gamma_{xy},$$

with equality holding only if $|H'_\Gamma| = s - 2$. Now $|H_\Gamma| \geq s - \lfloor s/3 \rfloor - 1$ implies that

$$w(H) + |H_\Gamma| \leq t_3(s-2) + 2s - 3 + \lfloor s/3 \rfloor + 2\gamma_{xy}, \tag{2.2.4}$$

with equality only if $|H'_\Gamma| = s - 2$ and $|H_\Gamma| = s - \lfloor s/3 \rfloor - 1$. If $\gamma_{xy} = 0$ then (2.2.3) holds as required, so suppose $\gamma_{xy} = 1$. In this case (2.2.3) holds, unless (2.2.4) holds with equality. But if (2.2.4) is an equality then $|H_\Gamma| = |H'_\Gamma| + 1 = s - 1$, while $|H_\Gamma| = s - \lfloor s/3 \rfloor - 1$, which is impossible for $s \geq 3$.

We finally need to verify the cases $s = 3, 4$. It is again sufficient to prove that if H contains edges of weight 2 or 3 then $w(H) + |H_\Gamma| \leq t_3(s) + s - 1$, thus we need to show that $w(H) + |H_\Gamma|$ is at most 5 if $s = 3$ and at most 8 if $s = 4$.

We note that argument in Case (i) above implies that if H contains an edge of weight 3 then $|H_\Gamma| \leq s-2$ and $w(H) \leq 3+3\binom{s-2}{2}$, so if $s=3$ then $w(H)+|H_\Gamma| \leq 4$ and if $s=4$ then $w(H)+|H_\Gamma| \leq 8$ so the result holds. So we may suppose there are no edges of weight 3.

Now let xy be an edge of weight 2. Using the fact that xy is not in any triangles and Lemma 2.8 (7) and (8) we find that for $s=3$ we have $w(H)+|H_\Gamma| \leq 2+3-|H_\Gamma|$, while for $s=4$ we have $w(H)+|H_\Gamma| \leq 2+6-|H_\Gamma|$, so the result holds. $\square$

LEMMA 2.10. *If $j,k,l \geq 0$ are integers satisfying $j+k+l=s \geq 5$ and $j \geq s-\lfloor s/3 \rfloor - 1$ then*

$$t_3(j+k)+t_4(l)+j+k+kl \leq t_3(s)+s, \tag{2.2.5}$$

with equality iff $l=0$.

PROOF. If $l=0$ then the result clearly holds, so suppose that $l \geq 1$, $j+k+l=s \geq 5$ and $j \geq s-\lfloor s/3 \rfloor - 1$. Let $f(j,k,l)$ be the LHS of (2.2.5) we need to check that $\Delta(j,k,l)=f(j,k+1,l-1)-f(j,k,l)>0$. Using Lemma 2.11 (4) we have

$$\begin{aligned}\Delta(j,k,l) &= j-\lceil (j+k+1)/3 \rceil + \lceil l/4 \rceil + 1 \\ &= j+\lceil l/4 \rceil - \lfloor (j+k)/3 \rfloor.\end{aligned}$$

So it is sufficient to check that $j+l/4>(j+k)/3$. This follows easily from $j \geq s-\lfloor s/3 \rfloor - 1$, $k \leq \lfloor s/3 \rfloor + 1$, $l \geq 1$ and $s \geq 5$. $\square$

The following identities are easy to verify.

LEMMA 2.11. *If $n \geq k \geq 3$ then*

(1) $s_3(n)=s_3(n-3)+t_3(n-3)+n-2$.

(2) $t_3(n)=t_3(n-3)+2n-3$.

(3) $t_3(n)=t_3(n-2)+n-1+\lfloor n/3 \rfloor$.

(4) $t_k(n)=t_k(n-1)+n-\lceil n/k \rceil$.

2.3. Turán Density and Stability

We now move on to the stability version of the Turán density and also provide an alternative proof of the Turán density using similar methods to those used in the stability result. The stability version is as follows:

THEOREM 2.12. *For any $\epsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds: if H is an $\mathcal{F}$-free 3-graph of order $n \geq n_0$ with at least $(1 - \delta)\, s(n)$ edges, then there is a partition of the vertex set of H as $V(H) = U_1 \cup U_2 \cup U_3$ so that all but at most ϵn^3 edges of H have one point in each U_i.*

The second proof of Theorem 2.5 given below uses the techniques similar to those needed for the stability result. It is essentially an induction argument based on the degrees of each vertex in the 3-graph. The induction itself provides a lower bound for the degree of each vertex. Using this lower bound we derive an upper bound on the degree of each vertex by examining the link (multi-)graph of a vertex. We show that the link graph of an edge in an $\mathcal{F}$-free 3-graph with vertices satisfying this lower bound does not contain a copy of K_4 and has no more edges than a simple graph: this bounds the number of edges in this link graph.

The necessary properties will follow from these lemmas:

LEMMA 2.13. Let H be a $\mathcal{F}$-free 3-graph of order $n+7$ such that each vertex in H has degree at least $(1 - 10\gamma)\left(n^2/9\right)$, where $\gamma \leq 10^{-4}$. Let $E = \{abc\}$ be any edge in H. Then the link graph of E does not contain a copy of K_4.

LEMMA 2.14. Let H be a $\mathcal{F}$-free 3-graph of order $n+3$ such that each vertex in H has degree at least $(n+3)^2/9$. Let $E = \{abc\}$ be any edge in H. If the link graph of E is K_4-free, then it has a maximum of $n^2/3$ edges.

The stability version starts with a similar argument, except that the link graph may include a small number of vertices incident with edges of weight 2. This requires a different version of Lemma 2.14:

LEMMA 2.15. Let H be an $\mathcal{F}$-free 3-graph of order $n+3$ such that every vertex in H has degree $(1 - 10\gamma)\left((n+3)^2/9\right)$, where $\gamma < 1/619520$, that contains an edge abc with total double neighbourhoods at least $(1 - \delta)\,(2n/3) -$

$\left[\frac{7}{3} + \frac{2}{3}\delta\right]$. Then the link graph of abc has at most $31\gamma n$ vertices incident with an edge of weight 2.

To prove the Turán density of $\mathcal{F}$ we use the following lemma.

LEMMA 2.16. *There is a constant N such that, if H is an $\mathcal{F}$-free 3-graph of order n, H has no more than $F(n)$ edges, where:*

$$F(n) \quad = \quad \sum_{x=1}^{n} f(x)$$

and

$$f(x) = \begin{cases} \frac{1}{2}x^2 & x \leq N \\ \frac{1}{9}x^2 & x > N \end{cases}$$

PROOF. For $n \leq N$, $F(n)$ is the number of edges in the complete 3-graph, so the statement is trivially true. For $n > N$, we proceed by induction. First, take the case where there is a vertex q in H that is incident with fewer than $\frac{1}{9}n^2$ edges. Then, by induction, the 3-graph $H - \{q\}$ has no more than $F(n-1)$ edges and so $e(H) \leq F(n-1) + \frac{1}{9}n^2 = F(n-1) + f(n) = F(n)$. Next, take the case where every vertex in H is incident with at least $\frac{1}{9}n^2$ edges. Take any edge $\{abc\}$ in H. Lemma 2.13 implies that the link graph of $\{abc\}$ is K_4-free (take $\gamma = 0$ in the statement of the Lemma). It follows that the preconditions of Lemma 2.14 are satisfied so that $\{abc\}$ is incident with at most $\frac{1}{3}(n-3)^2 + n + 1$ edges. Then, by induction, the 3-graph $H - \{abc\}$ has no more than $F(n-3)$ edges and so $e(H) \leq F(n-3) + \frac{1}{3}(n-3)^2 + n + 1 \leq F(n-3) + f(n-2) + f(n-1) + f(n) = F(n)$. □

The main theorem then follows immediately:

THEOREM 2.5. $\pi(F_6) = \frac{2}{9}$.

PROOF. The graph $S(n)$ demonstrates that $\pi(F_6) \geq 2/9$. Let N and $F(n)$ be as defined in Lemma 2.16 and define $K = F(N) - N^3/27$. Then, for all $n \geq N$, $F(n) = n^3/27 + K$. Accordingly, by the definition of the Turán density

$$\begin{aligned}\pi(\mathcal{F}) &= \lim_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{3}} \\ &\leq \lim_{n\to\infty} \frac{\frac{n^3}{27}+K}{\binom{n}{3}} \\ &= \frac{2}{9}\end{aligned}$$

□

2.3.1. Lemmas for Turán Density

We are now able to prove Lemma 2.14 regarding edges of weight 2 in the link graph used in the proof of the exact Turán density. Hereafter, we make liberal use of the convention set out in Section 1.2 that lower case is used to denote the proportion of vertices in the upper class vertex set (so that there are qn vertices in $Q \subset V(H)$).

Proof of Lemma 2.14. Let L_{abc} be the link graph of E. Note that at least $2/3$ of the vertices of L_{abc} are incident with each colour (so that each particular type of edge of weight 2 is only incident with at most $1/3$ of the vertices of L_{abc}). For instance, by Lemma 2.7(K_4^--1), L_x, the link graph of L_{abc} restricted to colour x, is triangle-free, so that if L_x has at least $n^2/9$ edges it has at least $2n/3$ vertices.

Let C_x be the set of vertices incident with colour x and $D_x = L - C_x$. We construct a series of disjoint vertex sets that together comprise $V(L)$. First, let M_{xy} be a set of vertices consisting of a maximal matching of edges of weight 2 and colours x and y; that is, choose an edge coloured xy contained in $L - M_{xy}$ and add the endpoints of that edge to M_{xy}, then repeat until there is no edge coloured xy contained in $L - M_{xy}$. Then let M_{abc} be a set of vertices incident with a maximal matching of edges of weight 3 constructed in the same way. Finally, let $R = L - \bigcup M_{xy} - M_{abc}$. The M_{xy} are disjoint, $e(M_{abc}) \leq (3/2)m_{abc}n$, there are no edges between M_{abc} and any other set and, by Turán's Theorem, $e(R) \leq (1/3)\,(rn)^2$. The following lemmas provide all the remaining densities in and between sets.

Proposition 2.17. *The maximum number of edges in* M_{xy} *is* $\frac{1}{4}(m_{xy}n)^{2} + \alpha_{xy}n^2$*, where there are* $\alpha_{xy}n^2$ *edges of weight 2 in* M_{xy}*, so that* $\alpha_{xy} \leq \frac{1}{4}m_{xy}^2$.

Proof. Ignoring the mulitiplicity of edges in M_{xy} gives a simple graph that is triangle-free, as M_{xy} is incident only with edges of colour x and y and there are no monochromatic or two-colour triangles. If there are no edges of weight 2 in M_{xy} then the maximum number of edges is $\frac{1}{4}(m_{xy}n)^2$. Accordingly, any edges above this number must consist of edges of weight 2. Given a total of $\frac{1}{4}(m_{xy}n)^2 + \alpha_{xy}n^2$ edges, it follows immediately that there must be at least $\alpha_{xy}n^2$ edges of weight 2 and that $\alpha_{xy} \leq \frac{1}{4}m_{xy}^2$. □

Corollary 2.18. *The maximum number of edges in* M_{xy} *is* $\frac{1}{2}(m_{xy}n)^2$*.*

Proposition 2.19. *Let* xy *be an edge of weight 2 with* $\{x, y\} \subset D_p$*. Then:*

(1) there is at most one edge between xy *and any vertex in* C_p*;*

(2) there are at most two edges between xy *and any vertex in* D_p*;*

(3) the maximum number of edges between xy *and any set of vertices* $Q \subset D_p$ *is* $qn + \alpha_q n$*, where* $\alpha_q n$ *is the number of edges of weight 2 between* xy *and* Q*.*

Proof. Assume, without loss of generality, that the colour of xy is ab and let the third vertex be z. Then x and y are only incident with edges of colour a and b. If z is incident with c it cannot be incident with an edge coloured ab and so there can only be edges of weight 1 between x or y and z. As xyz is triangle-free, this gives a maximum of 2 edges where z is not incident with c and 1 edge where z is incident with c. Given a total of $qn+\alpha_q n$ edges and a maximum of qn edges of weight 1, it follows immediately that there must be at least $\alpha_q n$ edges of weight 2. □

We form the partition of L consisting of $M_{ab}, M_{bc}, M_{ac}, M_{abc}$ and R. Note that these sets are pairwise disjoint and that the maximum size of each M_{xy} is $n/3$. Let $P = \{ab, ac, bc\}$. For $x \in \{a, b, c\}$, let D_x be the set of vertices disjoint from colour x. Let $|C_x| = \left(\frac{2}{3} + \delta_x\right) n$ and so $|D_x| = \left(\frac{1}{3} - \delta_x\right) n$, where δ_x is a non-negative number. Note that $M_{xy} \subseteq D_z$. We derive expressions for the upper bound of the total number of edges in L and ultimately show that this upper bound is no more than the lower bound of $n^2/3$. We form the upper bound for the number of edges in L by calculating an upper bound for the number of edges within each subset in L and for the number of edges between each pair of subsets in L.

For R, the maximum density is $1/3$ as it is K_4-free and contains only edges of weight 1. For each M_{xy}, Proposition 2.17 states that $e(M_{xy}) \leq \frac{1}{2}m_{xy}^2 n^2$. We then calculate the maximum number of edges between each M_{xy} and the other subsets of L. First, take the subset of C_z excluding vertices incident with a matching, which is of size $\left(\frac{2}{3} + \delta_z - \sum_{T \in P, T \neq xy} m_T\right) n$. By Proposition 2.19 each matched pair in M_{xy} sends at most one edge of weight 1 to each vertex in this subset, giving a maximum of $\frac{1}{2}m_{xy}\left(\frac{2}{3} + \delta_z - \sum_{T \in P, T \neq xy} m_T\right) n^2$ edges. By similar reasoning, considering the subset of D_z excluding M_{xy}, which is of size $\left(\frac{1}{3} - \delta_z - m_{xy}\right)$, each matched pair in M_{xy} sends at most one edge of weight at most 2 to each vertex in this subset, giving a maximum of $m_{xy}\left(\frac{1}{3} - \delta_z - m_{xy}\right) n^2$ edges. Finally, each matched pair in M_{xy} sends at most one edge to each vertex in M_{xz}, for a total of $\frac{1}{2}m_{xy}m_{xz}n^2$ edges: note that, as we sum over every M_{xy}, an additional factor $1/2$ is inserted to avoid double-counting.

The total number of edges in L is at most:

$$
\begin{aligned}
e(L) \leq n^2 \Bigg[& \frac{1}{3}\left(1 - m_{ab} - m_{ac} - m_{bc}\right)^2 + \frac{1}{2}\left(m_{ab}^2 + m_{ac}^2 + m_{bc}^2\right) \\
& + \sum_{S \in P} m_S \left\{ \frac{1}{2}\left(\frac{2}{3} + \delta_{abc-S} - \sum_{T \in P, T \neq S} m_T\right) + \left(\frac{1}{3} - \delta_{abc-S} - m_S\right) \right. \\
& \left. + \frac{1}{4} \sum_{T \in P, T \neq S} m_T \right\} \Bigg]
\end{aligned}
$$

and, using the same partition of L, we can express $n^2/3$ as:

$$\frac{1}{3}n^2 = n^2\left[\frac{1}{3}\left(1-m_{ab}-m_{ac}-m_{bc}\right)^2+\frac{1}{3}\left(m_{ab}^2+m_{ac}^2+m_{bc}^2\right)\right.$$
$$+\sum_{S\in P} m_S\left\{\frac{2}{3}\left(\frac{2}{3}+\delta_{abc-S}-\sum_{T\in P, T\neq S} m_T\right)+\frac{2}{3}\left(\frac{1}{3}-\delta_{abc-S}-m_S\right)\right.$$
$$\left.\left.+\frac{1}{3}\sum_{T\in P, T\neq S} m_T\right\}\right].$$

Taking the difference between the two gives

$$e(L)-\frac{1}{3}n^2 \leq n^2\left[\frac{1}{6}\left(m_{ab}^2+m_{ac}^2+m_{bc}^2\right)\right.$$
$$+\sum_{S\in P} m_S\left\{-\frac{1}{6}\left(\frac{2}{3}+\delta_{abc-S}-\sum_{T\in P, T\neq S} m_T\right)\right.$$
$$\left.\left.+\frac{1}{3}\left(\frac{1}{3}-\delta_{abc-S}-m_S\right)-\frac{1}{12}\sum_{T\in P, T\neq S} m_T\right\}\right]$$

$$\begin{aligned}
&= n^2\left[\sum_{S\in P}\frac{1}{6}m_S^2+m_S\left\{-\frac{1}{9}-\frac{1}{6}\delta_{abc-S}+\frac{1}{6}\sum_{T\in P, T\neq S} m_T+\right.\right.\\
&\qquad \left.\left.\frac{1}{9}-\frac{1}{3}\delta_{abc-S}-\frac{1}{3}m_S-\frac{1}{12}\sum_{T\in P, T\neq S} m_T\right\}\right]\\
&= n^2\left[\sum_{S\in P}\frac{1}{6}m_S^2+m_S\left\{-\frac{1}{2}\delta_{abc-S}+\frac{1}{12}\sum_{T\in P, T\neq S} m_T-\frac{1}{3}m_S\right\}\right]\\
&= n^2\left[\sum_{S\in P} m_S\left\{-\frac{1}{6}m_S-\frac{1}{2}\delta_{abc-S}+\frac{1}{12}\sum_{T\in P, T\neq S} m_T\right\}\right]\\
&\leq \frac{1}{6}n^2\left[\sum_{S\in P} m_S\left\{\frac{1}{2}\sum_{S\in P, T\neq S} m_T-m_S\right\}\right]\\
&= \frac{1}{6}n^2\left[\frac{1}{2}\left(m_{ab}+m_{bc}+m_{ac}\right)^2-\frac{3}{2}\left(m_{ab}^2+m_{bc}^2+m_{ac}^2\right)\right]
\end{aligned}$$

which for $m_{xy} \in [0, 1/3]$ reaches its maximum when all m_{xy} are equal, by Cauchy-Schwarz, and this maximum is 0. This shows that the upper bound for the number of edges in L is $n^2/3$. $\square$

2.3.2. Stability Lemmas

Now we prove the remaining lemmas concerning the structure of link graphs. These lemmas are presented in their stability versions. The version of Lemma 2.13 used to prove the exact Turán density follows immediately from the stability result presented here. The version of Lemma 2.15 used to prove the exact Turán density is proved separately above as Lemma 2.14.

LEMMA 2.13. Let H be a $\mathcal{F}$-free 3-graph of order $n+7$ such that each vertex in H has degree at least $(1-10\gamma)\left(n^2/9\right)$, where $\gamma \leq 10^{-4}$. Let $E = \{abc\}$ be any edge in H. Then the link graph of E does not contain a copy of K_4.

PROOF. Assume that the link graph of E does contain a copy of K_4 with vertices p, q, r and s and edges $\{apq, ars, bpr, bqs, cps, cqr\}$. Let L be the link graph of $\{a, b, c, p, q, r, s\} = Q$; that is, $V(L) = V(G) - Q$ and $E(L) = \{xy : \exists z \in V(G) - Q \text{ and } xyz \in E(G)\}$. By the given assumptions, L contains at least $(1-10\gamma)\, 7\frac{2}{9}\frac{n^2}{2} - \binom{7}{2}n - \binom{7}{3} = (1-10\gamma)\,\frac{7n^2}{9} - 21n - 35$ edges and is a multigraph containing edges of multiplicity up to 7.

We note the following facts about the subgraph Q:

(1) Every vertex is incident with exactly three edges.

(2) Every pair of vertices is included in exactly one edge.

(3) No two edges are entirely disjoint.

We note the following facts about L:

(4) L contains no monochromatic or two-colour triangles (because every pair of colours is part of an edge).

(5) To every pair of colours, there corresponds a third colour, such that a pair of vertices connected by edges with that pair of colours is not incident with the third colour – again, this follows from every pair of colours being part of an edge.

(6) There are exactly three pairs of colours corresponding to each colour, which satisfy the conditions of 2 above: this follows from each vertex being incident with three different edges within Q.

Next, we are able to characterise edges of weight 3:

PROPOSITION 2.20. *Let $\alpha\beta$ be an edge in L containing colours xyz. If xyz is an edge in H then α and β are incident only with edges of weight 1 (excluding the edge $\alpha\beta$). If xyz is not an edge then α and β are incident only with colours xyz and exactly one other, that together form the complement to an edge in X.*

PROOF. For $xyz \in E(H)$, each pair from xyz excludes the third of those colours, so α and β are not incident with any of xyz, other than in the edge $\alpha\beta$. But each pair outside xyz excludes one of these colours, or else there would be two disjoint edges in X, so α and β are not incident with any edges of weight 2 outside xyz. For $xyz \notin E(H)$, each pair from xyz excludes a different colour, or else there would be two edges that overlap in two colours, and none of these colours are x, y or z, as xyz is not an edge, so this leaves only one available colour outside xyz. This fourth colour together with xyz cannot contain an edge, or else one of xyz would be excluded, so it must consist of the complement to an edge. $\square$

These properties enable us to classify certain small structures that appear in L:

PROPOSITION 2.21. *L does not contain any triangles of total weight 7 or greater and the only triangles of weight 6 contain 3 edges of weight 2 and are disjoint from a particular colour.*

PROOF. In any triangle, all edges are different colours, or else there would be a two-colour triangle. Accordingly, there can be no triangle of weight more than 7. Also, any edge of weight two is not incident with at least one other colour, so at least one colour must be excluded from any triangle with edges of multiple weight, which contradicts any triangle of weight 7. Next, considering triangles of weight 6, there are two possibilities: 2-2-2 and 3-2-1. But any edge of weight 3 is either an edge, in which case it is incident only with edges of weight 1, or it is not an edge, in which case it is not incident with edges of weight 2 consisting wholly of colours different from those in the edge; in both cases this follows from Lemma 2.20. This leaves triangles of the form 2-2-2, where each pair excludes the same colour. There are seven possibilities, one corresponding to each colour (for instance, $ab - ps - qr$ for c), and that colour is not incident with any vertex of the triangle. $\square$

Corollary 2.22. *The sets of vertices incident with a particular triangle of weight 6 are disjoint.*

Definition 2.23. An edge is *degenerate* if it contains colours that form an edge in Q (that is, it is of weight greater than 4 or if it is of weight 4 but is not one of the following colours: $abps$, $abqr$, $acpr$, $acqs$, $bcpq$ or $bcrs$); and *non-degenerate* is defined correspondingly, so a non-degenerate edge of weight 4 contains colours that are the complement of an edge in X.

Proposition 2.24. *A vertex incident with a non-degenerate edge of weight 4 is not incident with any colours other than those forming part of that edge and a vertex incident with a degenerate edge is incident only with edges of weight 1 (excluding the degenerate edge), so that there are at most $n/2$ degenerate edges in L.*

Proof. A degenerate edge contains colours that constitute an edge in Q, so by Lemma 2.20 is incident only with edges of weight 1. Also by Lemma 2.20 any 4 colours that do not contain an edge form the complement to an edge and are incident exactly with those colours.

To calculate the total density of degenerate edges in L, take a maximal matching of degenerate edges in L. As degenerate edges are only incident with edges of weight 1, no two degenerate edges are incident, and so this matching includes all degenerate edges. Accordingly, there are a maximum of $n/2$ degenerate edges. $\square$

Corollary 2.25. *The sets of vertices incident with a particular type of non-degenerate edge of weight 4 are disjoint.*

Let dn be the number of degenerate edges in L of maximum size $7n/2$. We temporarily remove these edges from L. We now set out certain densities that apply to different sets of vertices within L.

Proposition 2.26. *The maximum density of any set K of vertices of order k in L is k^2.*

PROOF. If $k = 2$, the maximum number of edges is $2^2 = 4$. We proceed by induction. Let $m(x)$ be the maximum number of edges in K, where K is of order x. Take an edge in K of maximum multiplicity. There are a maximum of four edges between the pair of vertices forming this edge and any other vertex (one edge of weight four or two edges of weight two). So, given that $m(k-2) \leq (k-2)^2$,

$$\begin{aligned} m(k) &\leq 4(k-2) + 4 + (k-2)^2 \\ &= k^2. \end{aligned}$$

□

PROPOSITION 2.27. *The maximum density of any set K of vertices of order k in L that does not contain an edge of weight 4 or more or a triangle of weight 6 is* $(3/4)k^2$

PROOF. If $k = 2$, the maximum number of edges is $3 = (3/4)2^2$. We proceed by induction. Let $m(x)$ be the maximum number of edges in K, where K is of order x. Take an edge in K of maximum multiplicity. There are a maximum of three edges between the pair of vertices forming this edge and any other vertex (one edge of weight three or two edges of weight one and two). So, given that $m(k-2) \leq \frac{3}{4}(k-2)^2$,

$$\begin{aligned} m(k) &\leq 3(k-2) + 3 + \frac{3}{4}(k-2)^2 \\ &= \frac{3}{4}k^2. \end{aligned}$$

□

PROPOSITION 2.28. *The maximum number of edges between any 2-2-2 triangle and any other vertex is 6; the maximum number of edges between any 2-2-2 triangle and any other vertex incident with the colour not part of that triangle is 3.*

PROOF. Let xyz be a triangle with edges coloured ab - ps - qr (the other cases are similar) and consider the edges between xyz and another vertex w. We have already seen that a degenerate edge cannot be incident with an edge of weight 2. We have also seen that there are no 3-2-1 triangles, so if there is an edge of weight 3 (or a non-degenerate edge of weight 4) between w and xyz, there are no other edges between w and xyz. Therefore, the maximum that can be achieved is three edges of weight 2.

Consider now the case where w is incident with an edge coloured c. Any edge of weight 4 between w and xyz must include c, as w is incident with c, but this is impossible as xyz is not incident with c. Similarly, any edge of weight 3 between w and xyz cannot include any of the pairs ab, ps or qr, so it must consist of one colour from each edge: say, for instance apr, but this then excludes q, s and b, so that it cannot be incident with the triangle. Finally, consider an edge of weight 2. It cannot contain any of the pairs forming edges of xyz, as these are not incident with c. Assume, for instance, there is an edge of weight 2 between w and x, where x is incident with edges coloured ab and ps. There are then two cases. The edge xw could take one colour from each of these pairs. But then there would be no edge between w and either of y and z, or else there would be a two-colour triangle, giving a total of two edges between xyz and w. Or the edge xw could consist of one colour from the edge yz and one colour from the edges incident with x: say qs. But each of these pairs excludes one of the colours incident with x, and so is not allowed. The maximum is achieved if we allow edges of weight 1 between xyz and w. $\square$

Proposition 2.29. *Let xy be a non-degenerate edge of weight 4. The maximum number of edges between xy and any other vertex is 4. The maximum number of edges between xy and any vertex incident with at least one colour not in that edge is 2.*

Proof. Let xy be a non-degenerate edge of weight 4 coloured $abps$ (the other cases are similar) and let z be any other vertex. As xyz only contains these four colours and as xy contains all four colours, xyz is triangle free, so the greatest number of edges between xy and z is achieved by an edge of weight 4 from x or y to z.

Now take the case where z is incident with a colour other than $abps$. Note that by Lemma 2.20 any triple of these colours excludes all other colours. Therefore, the maximum multiplicity of an edge between xy and z is 2. As xyz is triangle free, the greatest number of edges between xy and z is achieved by an edge of weight 2 from x or y to z. $\square$

Proposition 2.30. *Each colour is incident with at least $(2/3)(1-5\gamma)$ of the vertices of L.*

Proof. Let C_x be defined as the set of vertices incident with colour x in L. Note that the subgraph of colour x is triangle-free. So, by Mantel's Theorem,

$$\begin{aligned} \frac{(c_x n)^2}{4} &\geq (1-10\gamma)\frac{n^2}{9} \\ c_x &\geq (1-10\gamma)^{\frac{1}{2}}\frac{2}{3} \\ &\geq (1-5\gamma)\frac{2}{3}. \end{aligned}$$

□

Definition 2.31. Let M_x be a subset of the vertices of L defined as follows: M_x is the union of a maximal matching of edges of weight 4 that are not incident with colour x and a maximal matching of 2-2-2 triangles that are not incident with colour x.

We form the partition of L consisting of $M_a, M_b \dots M_s$ and R: all the remaining vertices. More precisely, choose a maximal matching of edges of weight 4 and 2-2-2 triangles that are not incident with colour a. Then, from the remaining vertices, choose a maximal matching of edges of weight 4 and 2-2-2 triangles that are not incident with colour b, and so on. So, for instance, a matching of edges $pqrs$ would be inside M_a.

For $x \in Q$, let D_x be the set of vertices disjoint from colour x. Let $|C_x| = (2/3 - \delta_x)\,n$ and so $|D_x| = (1/3 + \delta_x)\,n$, where δ_x is less than $(2/3)5\gamma = (10/3)\gamma$, as guaranteed by Proposition 2.30 (note that δ_x is also permitted to be negative). Let $|D_x \cap M_y| = d_{x,y}$ and $|C_x \cap M_y| = c_{x,y}$. We derive an upper bound for the total number of edges in L and show that this upper bound is always less than our lower bound of $(7/9)\,(1-10\gamma)$.

We form the upper bound of the number of edges in L by calculating an upper bound for the number of edges in each subset of L and between each pair of subsets in L. The maximum density of R is $\frac{3}{4}$, by Proposition 2.27, and the maximum density of each M_x is 1, by Proposition 2.26, so this gives terms of $\frac{3}{4}\left[\left(1 - \sum_{x\in Q} m_x\right) n\right]^2$ and $(m_x n)^2$ for each $x \in Q$ for all the densities of subsets of L. We then calculate the density of edges between each M_x and the rest of L.

For each M_x, consider first the set of vertices incident with colour x, other than those forming part of any M_y. We label this subset temporarily C

and note that it contains $\left((2/3) - \delta_x - \sum_{y \neq x \in Q} c_{x,y}\right) n$ vertices. Let the subset of M_x consisting of matched pairs of edges of weight 4 be labelled $M_{x,P}$ and the subset consisting of matched triangles of total weight 6 be labelled $M_{x,T}$. By Propositions 2.28 and 2.29, each matched pair in $M_{x,P}$ sends at most one edge of weight 2 to any vertex in C, giving a maximum of $2(m_{x,P}n/2)cn = m_{x,P}cn^2$ edges between those two subsets and each matched triangle in M_x sends at most three edges of weight 1 to any vertex in C giving a maximum of $3(m_{x,T}n/3)cn = m_{x,T}cn^2$. The overall maximum is therefore $m_x cn^2$.

For each M_x, consider next the set of vertices not incident with colour x, other than those forming M_x or any part of M_y. We label this subset temporarily D and note that it contains $\left((1/3) + \delta_x - m_x - \sum_{y \neq x \in Q} d_{x,y}\right) n$ vertices. By similar reasoning to above, the maximum number of vertices between M_x and D is $4(m_{x,P}n/2)dn + 6(m_{x,T}n/3)dn = 2m_x dn^2$.

Next, we consider the maximum number of edges between M_x and each M_y. Using similar reasoning to above, we can fix a maximum of $m_x c_{x,y} n^2$ edges between M_x and the subset of each M_y that is incident with x and a maximum of $2m_x d_{x,y} n^2$ edges between M_x and the subset of each M_y that is not incident with x. Note that, as we are summing over all M_x below, we introduce a factor 1/2 to avoid double counting.

Finally, we add the dn degenerate edges.

Accordingly, the total number of edges in L is at most:

$$
\begin{aligned}
e(L) \leq n^2 \Bigg[\frac{3}{4} \left(1 - \sum_{x \in Q} m_x \right)^2 &+ \sum_{x \in Q} \Bigg\{ m_x^2 + m_x \left(\frac{2}{3} - \delta_x - \sum_{y \neq x \in Q} c_{x,y} \right) \\
&+ 2m_x \left(\frac{1}{3} + \delta_x - m_x - \sum_{y \neq x \in Q} d_{x,y} \right) + \frac{1}{2} \sum_{y \neq x \in Q} \{ m_x c_{x,y} + 2m_x d_{x,y} \} \Bigg\} \\
&+ \frac{d}{n} \Bigg].
\end{aligned}
$$

We can express the lower bound for the number of edges in L using the same partition of vertex sets:

$$e(L) \geq \frac{7}{9}(1-10\gamma)n^2 - 21n - 35 = \frac{3}{4}n^2 + \left[\frac{1}{36} - \frac{70}{9}\gamma\right]n^2 - 21n - 35$$

$$= n^2\left[\frac{3}{4}\left(1-\sum_{x\in Q} m_x\right)^2 + \sum_{x\in Q}\left\{\frac{3}{4}m_x^2 + \frac{3}{2}m_x\left(\frac{2}{3} - \delta_x - \sum_{y\neq x\in Q} c_{x,y}\right)\right.\right.$$
$$\left. + \frac{3}{2}m_x\left(\frac{1}{3} + \delta - m_x - \sum_{y\neq x\in Q} d_{x,y}\right) + \frac{1}{2}\sum_{y\neq x\in Q}\left\{\frac{3}{2}m_x c_{x,y} + \frac{3}{2}m_x d_{x,y}\right\}\right\}$$
$$\left. + \left[\frac{1}{36} - \frac{70}{9}\gamma\right] - \frac{21}{n} - \frac{35}{n^2}\right].$$

Combining these inequalities gives:

$$0 \leq n^2\left[\sum_{x\in Q}\left\{\frac{1}{4}m_x^2 - \frac{1}{2}m_x\left(\frac{2}{3} - \delta_x - \sum_{y\neq x\in Q} c_{x,y}\right)\right.\right.$$
$$\left. + \frac{1}{2}m_x\left(\frac{1}{3} + \delta_x - m_x - \sum_{y\neq x\in Q} d_{x,y}\right) + \frac{1}{2}\sum_{y\neq x\in Q}\left\{-\frac{1}{2}m_x c_{x,y} + \frac{1}{2}m_x d_{x,y}\right\}\right\}$$
$$\left. - \left[\frac{1}{36} - \frac{70}{9}\gamma\right] + \frac{d+21}{n} + \frac{35}{n^2}\right]$$

$$= n^2\left[\sum_{x\in Q}\left\{\frac{1}{4}m_x^2 - \frac{1}{3}m_x + \frac{1}{2}m_x\delta_x + \frac{1}{2}m_x\sum_{y\neq x\in Q} c_{x,y} + \frac{1}{6}m_x + \frac{1}{2}m_x\delta_x\right.\right.$$
$$\left. - \frac{1}{2}m_x^2 - \frac{1}{2}m_x\sum_{y\neq x\in Q} d_{x,y} - \frac{1}{4}m_x\sum_{y\neq x\in Q} c_{x,y} + \frac{1}{4}m_x\sum_{y\neq x\in Q} d_{x,y}\right\}$$
$$\left. - \left[\frac{1}{36} - \frac{70}{9}\gamma\right] + \frac{d+21}{n} + \frac{35}{n^2}\right]$$

$$= n^2 \left[\sum_{x\in Q} \left\{ -\frac{1}{4}m_x^2 - \frac{1}{6}m_x + m_x\delta_x + \frac{1}{4}m_x \sum_{y\neq x\in Q} c_{x,y} - \frac{1}{4}m_x \sum_{y\neq x\in Q} d_{x,y} \right\} - \left[\frac{1}{36} - \frac{70}{9}\gamma\right] + \frac{d+21}{n} + \frac{35}{n^2} \right]$$

$$\begin{aligned}
&\leq n^2 \left[\sum_{x\in Q} \left\{ -\frac{1}{4}m_x^2 - \frac{1}{6}m_x + \frac{1}{4}m_x \left(\frac{2}{3} - \frac{10}{3}\gamma\right) + m_x\frac{10}{3}\gamma \right\} - \left[\frac{1}{36} - \frac{70}{9}\gamma\right] + \frac{49}{2n} + \frac{35}{n^2} \right] \\
&\leq n^2 \left[\sum_{x\in Q} \left\{ -\frac{1}{4}m_x^2 - \frac{1}{6}m_x + \frac{1}{4}m_x (1-m_x) + m_x\frac{10}{3}\gamma \right\} - \left[\frac{1}{36} - \frac{70}{9}\gamma\right] + \frac{49}{2n} + \frac{35}{n^2} \right] \\
&= n^2 \left[\sum_{x\in Q} \left\{ -\frac{1}{2}m_x^2 + \frac{1}{12}m_x + m_x\frac{10}{3}\gamma \right\} - \left[\frac{1}{36} - \frac{70}{9}\gamma\right] + \frac{49}{2n} + \frac{35}{n^2} \right] \\
&< n^2 \left[7\frac{1}{288} - \frac{1}{36} + \frac{100}{9}\gamma + \frac{49}{2n} + \frac{35}{n^2} \right] \\
&< 0,
\end{aligned}$$

where we have used the fact that, for each x, $\sum_{y\in Q} c_{x,y} \leq (2/3 - \delta_x) \leq (1-m_x)$, $\gamma < 10^{-4}$ and taken n sufficiently large.

This contradiction establishes the lemma. $\square$

Next we establish some preliminary stability results for link neighbourhoods that will be used in the next stability lemma.

PROPOSITION 2.32. *Let H be an $\mathcal{F}$-free 3-graph of order n with at least $(1-\delta)\,\frac{2}{9}\binom{n}{3}$ edges. Then there is at least one edge with link neighbourhood of size at least $(1-\delta)\frac{2}{3}(n-3) - \left[\frac{7}{3} + \frac{2}{3}\delta\right]$.*

PROOF. Note that, by Lemma 2.7(K_4^--3), given any edge abc, no vertex appears with multiplicity more than one in Γ_{abc}.

We employ the following equality:

$$\sum_{abc\,\in E(G)} (|\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}|) = \sum_{xy\in\binom{V(H)}{2}} |\Gamma_{xy}|\,(|\Gamma_{xy}|-1).$$

The left hand side measures the total size of all double neighbourhoods of all edges in H. The right hand side, for each pair of vertices that is part of an edge, provides the contribution of $\Gamma_{xy} - 1$ to the double neighbourhood of that edge from each of the other edges of which it is a part.

We then have

$$\sum_{\{abc\}\,\in E(G)} (|\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}|) \leq \sum_{abc\in E(G)} \max\left(|\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}|\right)$$
$$= |E(G)| \max\left(|\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}|\right)$$

and

$$\begin{aligned}\sum_{xy\in\binom{V(H)}{2}} |\Gamma_{xy}|\left(|\Gamma_{xy}| - 1\right) &\geq \frac{1}{\binom{V(H)}{2}}\left(\sum_{xy\in\binom{V(H)}{2}} |\Gamma_{xy}|\right)^2 - \sum_{xy\in\binom{V(H)}{2}} |\Gamma_{xy}| \\ &= \frac{1}{\binom{|V(H)|}{2}} 9\,|E(G)|^2 - 3\,|E(G)|\,,\end{aligned}$$

where the total number of neighbourhoods of every pair counts each edge three times. Putting these together gives

$$\begin{aligned}\max\left(|\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}|\right) &\geq \frac{2}{n(n-1)} \times 9 \times \frac{n(n-1)(n-2)}{6} \times \frac{2}{9}\,(1-\delta) - 3 \\ &= \frac{2}{3}(n-2)(1-\delta) - 3 \\ &= (1-\delta)\frac{2}{3}(n-3) - \left[\frac{7}{3} + \frac{2}{3}\delta\right].\end{aligned}$$

□

COROLLARY 2.33. *If G is a K_4^--free 3-graph of order n with $s(n)$ edges, then there is an edge $abc \in E(G)$ with $|\Gamma_{abc}| \geq n - \lfloor n/3 \rfloor - 3$.*

Note that, if abc is an edge, then $|\Gamma_{abc}| = |\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}| + 3$.

PROPOSITION 2.34. *Let H be an $\mathcal{F}$-free 3-graph of order $n+3$ that contains an edge abc with total link neighbourhoods at least $(1-\delta)\,2n/3 - 7/3 - 2\delta/3$. Then the link graph L_{abc} contains fewer than $(1+2\delta)^2\,n^2/36 + \zeta n$ edges of weight 2, where ζ is an arbitrary constant.*

PROOF. Let $P = \{ab, ac, bc\}$, let $D = \cup_{p \in P} \Gamma_p$ and let $R = L - D$, so that by assumption we have $|D| > (1-\delta)\, 2n/3 - \left[\frac{7}{3} + \frac{2}{3}\delta\right]$. Note that, by Lemma 2.7(K_4^--3), the Γ_{xy} are pairwise disjoint.

Without loss of generality, take any vertex x in Γ_{ab} and assume that x is incident with an edge of weight 2 in L: xy. Note that $y \notin \Gamma_{ab}$ and that xy is not coloured ab: in either case we have a copy of K_4^-. So assume, again without loss of generality, that xy is coloured ac.

Neither x nor y is incident with b, or else $\{xya, xyc, abc, xbq\}$ for example, would be a copy of F_6. Also, y is not incident with a (apart from the edge xy), or else $\{abc, abx, cxy, ayq\}$ for example, would be a copy of F_6. Therefore, y is not incident with any edge of weight 2 other than xy. Similarly, if $y \in \Gamma_{bc}$ (note that it is not possible for y to be contained in Γ_{ac}), then x is not incident with any edge of weight 2 other than xy.

Accordingly, there are two cases: if xy is contained entirely in the double neighbourhoods of abc, then neither x nor y is incident with any other edge of weight 2; if y is not contained in any double neighbourhoods, that is, $y \in R$, then y is not incident with any other edge of weight 2.

Let R_D be the set of vertices in R incident with an edge of weight 2, where the other vertex of this edge is contained in D, and let e be the number of edges of weight 2 in L. Define δ_r by $dn = (1-\delta_r)\, 2n/3 - [7/3 + 2\delta/3]$, so that $\delta_r \leq \delta$. We calculate an upper bound on the number of edges of weight 2 as follows: there are a maximum of $d/2$ such edges contained in D (take a matching of these edges); there are a maximum of $r_d n$ such edges incident with R_D; there are a maximum of $(r - r_d)^2\, n^2/4$ such edges in the remainder of R (because the graph of edges of weight 2 is triangle-free). So we have

$$\begin{aligned}
en^2 &\leq \frac{dn}{2} + r_d n + \frac{(r - r_d)^2\, n^2}{4} \\
&= (1-\delta_r)\,\frac{n}{3} - \left[\frac{7}{6} + \frac{\delta}{3}\right] + r_d n + \frac{\left(1 + 2\delta_r + \frac{7+2\delta}{n} - 3r_d\right)^2 \frac{n^2}{9}}{4} \\
&= \frac{n}{3}\left[1 - \delta_r - \frac{7/2+\delta}{n} + 3r_d + \frac{n}{12}\left(1 + 2\delta_r + \frac{7+2\delta}{n} - 3r_d\right)^2\right] \\
&\leq \frac{n^2}{36}\,[1+2\delta]^2 + \zeta n.
\end{aligned}$$

Hence

$$e \leq \frac{(1+2\delta)^2}{36} + \frac{\zeta}{n},$$

where we have assumed that n is sufficiently large and used the upper bound for δ_r, with ζ an arbitrary constant so that ζn is a term of order n. $\square$

These results enable us to prove the stability version of Lemma 2.14.

LEMMA 2.15. Let H be an $\mathcal{F}$-free 3-graph of order $n+3$ such that every vertex in H has degree $(1-10\gamma)\left((n+3)^2/9\right)$, where $\gamma < 1/619520$, that contains an edge abc with total double neighbourhoods at least $(1-\delta)(2n/3) - \left[\frac{7}{3} + \frac{2}{3}\delta\right]$. Then the link graph of abc has at most $31\gamma n$ vertices incident with an edge of weight 2.

PROOF. By Lemma 2.34, there are at most $(1+2\delta)^2 n^2/36 + \zeta n$ edges of weight 2 in L_{abc}. As the preconditions of Lemma 2.13 are satisfied, L does not contain a copy of K_4. Also, by Proposition 2.30, each colour is incident with at least $(2/3)(1-5\gamma)$ of the vertices in L_{abc}. Finally, any edge of weight 2 is not incident with any other colour; therefore, any edge of weight 3 is not incident with any other edge.

Let M_{xy} be the set of vertices consisting of a maximal matching of edges of weight 2 and colours x and y, let M_{abc} be the set of vertices incident with an edge of weight 3 and let $R = L_{abc} - \bigcup M_{xy} - M_{abc}$. From the facts above about L_{abc}, we have that the M_{xy} are disjoint, that $e(M_{abc}) \leq (3/2)m_{abc}n$, that there are no edges between M_{abc} and any other set and that $e(R) \leq (1/3)(rn)^2$. Define δ_r by $e(R) = (1/3)(1-\delta_r)(rn)^2$.

We form the partition of L_{abc} consisting of $M_{ab}, M_{bc}, M_{ac}, M_{abc}$ and R. Note that each of these sets is disjoint. For $x \in \{a,b,c\}$, let D_x be the set of vertices disjoint from colour x and let C_x be the set of vertices incident with colour x. Let $|C_x| = (2/3-\delta_x)n$ and so $|D_x| = (1/3+\delta_x)n$, where δ_x is less than $(10/3)\gamma$, as guaranteed by Proposition 2.30. Note that $M_{xy} \subseteq D_z$. We derive expressions for the upper bound of the total number of edges in L and ultimately show that this upper bound is less than the lower bound of $(1/3)(1-10\gamma)n^2$ unless the number of vertices incident with edges of weight 2 is less than $16\gamma n$. We form the upper bound for the number of edges in L_{abc} by calculating an upper bound for the number of edges within each subset in L_{abc} and for the number of edges between each pair of subsets in L_{abc}.

These upper bounds are calculated using Propositions 2.17 and 2.19, as in Lemma 2.14.

Let $P' = \{ab, ac, bc, abc\}$, $P = \{ab, ac, bc\}$ and $\alpha = \sum_{S \in P} (\alpha_S + \alpha_{abc-S})$. The total number of edges in L_{abc} is at most:

$$e(L_{abc}) \leq n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S \in P'} m_S\right)^2 + \frac{3m_{abc}}{2n} + \sum_{S \in P} \left[\frac{1}{4}m_S^2 + \alpha_S + \alpha_{abc-S} \right.\right.$$

$$\left.\left. + m_S \left\{ \frac{1}{2}\left(\frac{2}{3} - \delta_{abc-S} - \sum_{T \in P', T \neq S} m_T\right) + \frac{1}{2}\left(\frac{1}{3} + \delta_{abc-S} - m_S\right) + \frac{1}{4} \sum_{T \in P', T \neq S} m_T \right\} \right] \right]$$

$$\leq n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S \in P'} m_S\right)^2 + \alpha \right.$$

$$\left. + \sum_{S \in P} \left(\frac{1}{4}m_S + \frac{1}{3} - \frac{1}{2} \sum_{T \in P, T \neq S} m_T + \frac{1}{6} - \frac{1}{2}m_S + \frac{1}{4} \sum_{T \in P, T \neq S} m_T \right) \right]$$

$$\begin{aligned} &= n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S \in P'} m_S\right)^2 + \alpha + \sum_{S \in P} \left(\frac{1}{2} - \frac{1}{4} \sum_{T \in P} m_T \right) \right] \\ &= n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S \in P'} m_S\right)^2 + \alpha + \frac{1}{2} \sum_{S \in P} m_S - \frac{1}{4} \left(\sum_{S \in P} m_S \right)^2 \right] \end{aligned}$$

where, for the second inequality, we assume that n is sufficiently large that we may take $m_{abc} = 0$.

We express the lower bound for the number of edges in L_{abc} using the same partition:

$$\begin{aligned} e(L_{abc}) &\geq \frac{1}{3}(1-10\gamma)n^2 \\ &= n^2\left[\frac{1}{3}\left(1-\sum_{S\in P} m_S\right)^2 + \frac{1}{3}\sum_{S\in P} m_S^2\right. \\ &\qquad + \sum_{S\in P} m_S\left\{\frac{2}{3}\left(\frac{2}{3}-\delta_{abc-S}-\sum_{T\in P, T\neq S} m_T\right)\right. \\ &\qquad \left.\left. + \frac{2}{3}\left(\frac{1}{3}+\delta_{abc-S}-m_S\right) + \frac{1}{3}\sum_{T\in P, T\neq S} m_T\right\} - \frac{10}{3}\gamma\right] \end{aligned}$$

$$\begin{aligned} &= n^2\left[\frac{1}{3}\left(1-\sum_{S\in P} m_S\right)^2\right. \\ &\quad + \sum_{S\in P} m_S\left\{\frac{1}{3}m_S + \frac{4}{9} - \frac{2}{3}\sum_{T\in P, T\neq S} m_T + \frac{2}{9} - \frac{2}{3}m_S + \frac{1}{3}\sum_{T\in P, T\neq S} m_T\right\} \\ &\quad \left. - \frac{10}{3}\gamma\right] \end{aligned}$$

$$\begin{aligned} &= n^2\left[\frac{1}{3}\left(1-\sum_{S\in P} m_S\right)^2 + \sum_{S\in P} m_S\left\{\frac{2}{3}-\frac{1}{3}\sum_{T\in P} m_T\right\} - \frac{10}{3}\gamma\right] \\ &= n^2\left[\frac{1}{3}\left(1-\sum_{S\in P} m_S\right)^2 + \frac{2}{3}\sum_{S\in P} m_S - \frac{1}{3}\left(\sum_{S\in P} m_S\right)^2 - \frac{10}{3}\gamma\right]. \end{aligned}$$

First we show that $\sum_{S\in P} m_S < 1/4$. Combining the two inequalities gives

$$0 \leq n^2\left[-\frac{\delta_r}{3}\left(1-\sum_{S\in P} m_S\right)^2 + \alpha - \frac{1}{6}\sum_{S\in P} m_S + \frac{1}{12}\left(\sum_{S\in P} m_S\right)^2 + \frac{10}{3}\gamma\right].$$

Taking $\delta_r/3 = 0$, $(10/3)\gamma = (10/3)(1/480) = 1/144$ (weaker than our actual bound on γ) and $\alpha = (1+2\delta)^2/36 + \zeta/n$, we maximise $(\sum m_s)^2 - \sum m_s^2$ by taking $m_s = q$ for all S, so that $(\sum m_s)^2 = 9q^2$ and $\sum m_s = 3q$. Then the upper bound becomes:

$$n^2 \left[\frac{(1+2\delta)^2}{36} + \frac{\zeta}{n} + \frac{1}{144} + \frac{1}{12}\left(9q^2 - 6q\right) \right]$$

from which it follows that

$$3q^2 - 2q + \left[\frac{(1+2\delta)^2}{9} + \frac{1}{36} + \frac{4\zeta}{n} \right] \geq 0.$$

For $q \in [0, 1/3]$ and n sufficiently large this gives $3q = \sum_{S \in P} m_S < 1/4$, so that we have an approximate bound on the maximum number of vertices incident with an edge of weight 2.

Next we use the bound on $\sum_{S \in P} m_S$ to deduce an upper bound on δ_r. We use the following version of the upper bound for the number of edges in L_{abc} :

$$\begin{aligned} e(L_{abc}) \leq n^2 \Bigg[& \frac{1}{3}(1-\delta_r)\left(1 - \sum_{S \in P'} m_S\right)^2 + \frac{1}{2}\sum_{S \in P} m_S^2 + \frac{3m_{abc}}{2n} \\ & + \sum_{S \in P} m_S \left\{ \frac{1}{2}\left(\frac{2}{3} - \delta_{abc-S} - \sum_{T \in P, T \neq S} m_T\right) \right. \\ & \left. + \left(\frac{1}{3} + \delta_{abc-S} - m_S\right) + \frac{1}{4}\sum_{T \in P, T \neq S} m_T \right\} \Bigg] \end{aligned}$$

$$\begin{aligned} \leq n^2 \Bigg[& \frac{1}{3}(1-\delta_r)\left(1 - \sum_{S \in P} m_S\right)^2 + \frac{1}{2}\sum_{S \in P} m_S^2 \\ & + \sum_{S \in P} m_S \left\{ \frac{1}{2}\left(\frac{2}{3} - \delta_{abc-S} - \sum_{T \in P, T \neq S} m_T\right) + \left(\frac{1}{3} + \delta_{abc-S} - m_S\right) \right. \\ & \left. + \frac{1}{4}\sum_{T \in P, T \neq S} m_T \right\} \Bigg] \end{aligned}$$

$$= n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S\in P} m_S\right)^2 + \sum_{S\in P} m_S \left\{ \frac{1}{2}m_S + \frac{1}{3} - \frac{1}{2}\delta_{abc-S} \right.\right.$$

$$\left.\left. - \frac{1}{2}\sum_{T\in P, T\neq S} m_T + \frac{1}{3} + \delta_{abc-S} - m_S + \frac{1}{4}\sum_{T\in P, T\neq S} m_T \right\}\right]$$

$$\begin{aligned}
&= n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S\in P} m_S\right)^2 + \sum_{S\in P} m_S \left\{ -\frac{1}{2}m_S + \frac{2}{3} - \frac{1}{4}\sum_{T\in P, T\neq S} m_T + \frac{1}{2}\delta_{abc-S} \right\}\right] \\
&= n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S\in P} m_S\right)^2 + \sum_{S\in P} m_S \left\{ \frac{2}{3} - \frac{1}{4}m_S - \frac{1}{4}\sum_{T\in P} m_T + \frac{1}{2}\delta_{abc-S} \right\}\right] \\
&= n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S\in P} m_S\right)^2 + \frac{2}{3}\sum_{S\in P} m_S - \frac{1}{4}\sum_{S\in P} m_S^2 - \frac{1}{4}\left(\sum_{S\in P} m_S\right)^2 + \right. \\
&\left. \frac{1}{2}\sum_{S\in P} m_S \delta_{abc-S} \right]
\end{aligned}$$

where, in the second inequality, we assume that n is sufficiently large, so that $m_{abc} = 0$. Combining this with the inequality for the lower bound gives:

$$0 \le n^2 \left[-\frac{\delta_r}{3}\left(1 - \sum_{S\in P} m_S\right)^2 + \frac{1}{12}\left(\sum_{S\in P} m_S\right)^2 - \frac{1}{4}\sum_{S\in P} m_S^2 \right.$$

$$\left. + \frac{1}{2}\sum_{S\in P} m_S \delta_{abc-S} + \frac{10}{3}\gamma \right].$$

We have $\sum_{S\in P} m_S \in [0, 1/4]$, so $1 - \sum_{S\in P} m_S \ge 3/4$, and an application of the Cauchy-Schwarz Inequality gives

$$\left[(m_{ab} + m_{bc} + m_{ac})^2 - 3\left(m_{ab}^2 - m_{bc}^2 - m_{ac}^2\right) \right] \le 0.$$

Putting these together gives

$$\begin{aligned}
0 &\le n^2 \left[\frac{10}{3}\gamma + \frac{1}{2}\frac{1}{4}\frac{10}{3}\gamma - \frac{\delta_r}{3}\frac{9}{16} \right] \\
&= n^2 \left[\frac{15}{4}\gamma - \frac{3}{16}\delta_r \right]
\end{aligned}$$

which is negative if $\delta_r > 20\gamma$. This contradiction proves the lemma when $\delta_r > 20\gamma$.

Finally we consider the case where $\delta_r \leq 20\gamma$.

Assume that there are $\sqrt{\delta_r} rn$ vertices of degree less than $\left(1 - 2\sqrt{\delta_r}\right) 2rn/3$. Deleting them gives a graph on $\left(1 - \sqrt{\delta_r}\right) rn$ vertices with at least $\left(1 - \delta_r - 2\sqrt{\delta_r}\left(1 - 2\sqrt{\delta_r}\right)\right)(rn)^2/3$ edges. But

$$\begin{aligned} \left(1 - \sqrt{\delta_r}\right)^2 \frac{r^2}{3} &= \left(1 + \delta_r - 2\sqrt{\delta_r}\right) \frac{r^2}{3} \\ &< \left(1 - 2\sqrt{\delta_r} + 4\delta_r - \delta_r\right) \frac{r^2}{3} \end{aligned}$$

which violates Turán's theorem, as this subgraph is K_4-free. Therefore, there are a maximum of $\sqrt{\delta_r} rn$ vertices in R of degree less than $\left(1 - 2\sqrt{\delta_r}\right) \frac{2rn}{3}$. We label this set of vertices R_- and the remainder of R is labelled R_+.

We consider now the number of edges between any set M_{xy} and the set R. If there is an edge between x and any vertex in R_+, then there is no edge between x and any of the $\left(1 - 2\sqrt{\delta_r}\right) 2rn/3$ neighbours of this vertex. That is, x is connected to a maximum of $\left(1 + 4\sqrt{\delta_r}\right) rn/3$ vertices in R_+. Similar reasoning applies to y, so that the total number of edges between xy and R is $2\left(1 + 4\sqrt{\delta_r}\right) rn/3 + \sqrt{\delta_r} rn = \left(1 + (11/2)\sqrt{\delta_r}\right) 2rn/3$. Therefore, there are at least $\left(1 - 11\sqrt{\delta_r}\right) rn/3$ vertices in R_+ that are not connected to xy. We may assume, when evaluating the upper bound, that these vertices are in the set of vertices that may only be connected to xy by edges of weight 1.

This gives the following version of the upper bound (where we assume, as above, that n is sufficiently large so $m_{abc} = 0$):

$$\begin{aligned} e(L_{abc}) \leq n^2 \Bigg[\frac{1}{3}(1 - \delta_r) \left(1 - \sum_{S \in P} m_S\right)^2 + \sum_{S \in P} m_S \Bigg\{ \frac{1}{2} m_S \\ + \frac{1}{2} \left(\frac{2}{3} - \delta_{abc-S} - \sum_{T \in P, T \neq S} m_T - \left(1 - 11\sqrt{\delta_r}\right) \frac{\left(1 - \sum_{S \in P} m_S\right)}{3} \right) \\ + \left(\frac{1}{3} + \delta_{abc-S} - m_S \right) + \frac{1}{4} \sum_{T \in P, T \neq S} m_T \Bigg\} \Bigg] \end{aligned}$$

$$= n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S\in P} m_S\right)^2 + \sum_{S\in P} m_S \left\{ -\frac{1}{4}m_S + \frac{2}{3} + \frac{1}{2}\delta_{abc-S} \right.\right.$$

$$\left.\left. -\frac{1}{4}\sum_{T\in P} m_T - \frac{1}{2}\left(1 - 11\sqrt{\delta_r}\right)\frac{\left(1 - \sum_{S\in P} m_S\right)}{3} \right\}\right]$$

$$= n^2 \left[\frac{1}{3}(1-\delta_r)\left(1 - \sum_{S\in P} m_S\right)^2 - \frac{1}{4}\sum_{S\in P} m_S^2 + \frac{2}{3}\sum_{S\in P} m_S + \frac{1}{2}\sum_{S\in P} m_S\delta_{abc-S} \right.$$

$$\left. -\frac{1}{4}\left(\sum_{S\in P} m_S\right)^2 - \frac{1}{2}\left(\sum_{S\in P} m_S\right)\left(1 - 11\sqrt{\delta_r}\right)\frac{\left(1 - \sum_{S\in P} m_S\right)}{3} \right].$$

Therefore, using the same lower bound as above:

$$0 \le n^2 \left[\frac{1}{12}\left[\left(\sum_{S\in P} m_S\right)^2 - 3\left(\sum_{S\in P} m_S^2\right)\right] + \frac{1}{2}\sum_{S\in P} m_S\delta_{abc-S} + \frac{10}{3}\gamma \right.$$

$$\left. -\frac{1}{2}\left(\sum_{S\in P} m_S\right)\left(1 - 11\sqrt{\delta_r}\right)\frac{\left(1 - \sum_{S\in P} m_S\right)}{3} - \frac{\delta_r}{3}\left(1 - \sum_{S\in P} m_S\right)^2 \right]$$

$$< n^2 \left[\frac{10}{3}\gamma + \frac{1}{2}\frac{1}{4}\frac{10}{3}\gamma - \left(\sum_{S\in P} m_S\right)\left(1 - 11\sqrt{\delta_r}\right)\frac{1}{8} - \frac{3\delta_r}{16} \right],$$

taking $\left(1 - \sum_{S\in P} m_S\right) > 3/4$ and again using Cauchy-Schwarz to show $\left(\sum_{S\in P} m_S\right)^2 - 3\left(\sum_{S\in P} m_S^2\right) \le 0$.

It follows that

$$\begin{aligned} \sum_{S\in P} m_S &\le \left(30\gamma - \frac{3}{2}\delta_r\right)\left(\frac{1}{1 - 11\sqrt{\delta_r}}\right) \\ &< 31\gamma, \end{aligned}$$

because $\delta_r < (1/30976) = 20\gamma$. □

2.3.3. Stability For F_6

The approach of this section is substantially the same as the proof of Theorem 1.5, the stability result for F_5, in [**19**]. The principal difference is in the requirement for the stability lemmas of the previous section and in certain other details of the argument that we highlight below.

The following proposition is a slight variant of a case of the Simonovits stability theorem (see Proposition 5.1 in [**19**]).

THEOREM 2.35. *For any $\epsilon' > 0$ there exists $\delta' > 0$ and n_0 such that the following holds: if G is a K_4-free graph on $n > n_0$ vertices with at least $(1 - \delta')\, t_3(n)$ edges, then one can delete $\epsilon' n$ vertices from G so that the remaining graph is tripartite.*

The following theorem is the stability version of the Turán density result for F_6 (recall that $\mathcal{F} = \{F_6, K_4^-\}$).

THEOREM 2.12. *For any $\epsilon > 0$ there exists $\delta > 0$ and n_0 such that the following holds: if H is an $\mathcal{F}$-free 3-graph of order $n \geq n_0$ with at least $(1 - \delta)\, s(n)$ edges, then there is a partition of the vertex set of H as $V(H) = U_1 \cup U_2 \cup U_3$ so that all but at most ϵn^3 edges of H have one point in each U_i.*

PROOF. We use constants that satisfy the following hierarchy: $\delta \ll \gamma \ll \delta' \ll \epsilon' \ll \epsilon$. In particular:

- Let $\epsilon' < 10^{-8}\epsilon^2$.
- Let $\delta' < \epsilon'$ and be the result of applying Theorem 2.35 with ϵ'.
- Let $\gamma = \epsilon\delta'$.
- Let $\delta < 12\gamma^2$.

Define $U_0 \subset V(H)$. We add a small number of (bad) vertices to U_0 and show that all but a small number of hyperedges in $H - U_0$ respect the partition.

Assume, to derive a contradiction, that there are γn vertices of degree at most $(1 - 5\gamma)n^2/9$. Deleting them gives a 3-graph H' with $(1 - \gamma)n$ vertices and at least $(1-\delta-3\gamma(1-5\gamma))\,n^2/27$ edges. It follows that

$$\begin{aligned} e(H') &\geq (1-\delta-3\gamma(1-5\gamma))\,\frac{n^3}{27} \\ &= \left(1-\delta-3\gamma+15\gamma^2\right)\frac{n^3}{27} \\ &> \left(1-3\gamma+3\gamma^2\right)\frac{n^3}{27} \\ &= \frac{[(1-\gamma)\,n]^3}{27}+\frac{\gamma^3 n^3}{27}. \end{aligned}$$

But this is contrary to Theorem 2.5, for n sufficiently large. It follows that there are fewer than γn vertices of degree at most $(1-5\gamma)\,n^2/9$ and we add these to the set U_0.

Consider the 3-graph $H-U_0$. Every vertex in this 3-graph has degree at least $(1-5\gamma)\,(n^2/9)-(\gamma n^2/2)\geq(1-10\gamma)\,(n^2/9)$. As H has at least $(1-\delta)\,s(n)$ edges, by Proposition 2.32, there is at least one hyperedge abc in this graph with total link neighbourhoods greater than $(1-\delta)\,(2/3)n$. The preconditions of Lemma 2.15 are satisfied so that the link graph of abc is K_4-free and has a maximum of $31\gamma n$ vertices incident with an edge of weight 2. We add these to the set U_0.

Let J be the link graph of abc in H. This graph is K_4-free and has no edges of weight 2, that is, it is a simple graph.

Suppose that J has $10^{-1}\delta' n$ vertices with degree at most $\left(1-10^{-3}\epsilon\right)2n/3$. Then the graph $J'=J-\{x:d(x)\leq\left(1-10^{-3}\epsilon\right)2n/3\}$ has $\left(1-10^{-1}\delta'\right)n$ vertices and at least $\left(1-\delta-2\times10^{-1}\delta'\left(1-10^{-3}\epsilon\right)\right)n^2/3$ edges, but

$$\frac{\left[\left(1-10^{-1}\delta'\right)n\right]^2}{3}=\left[1-2\times10^{-1}\delta'+10^{-2}\delta'^2\right]\frac{n^2}{3}$$

and

$$\left(1-\delta-2\times10^{-1}\delta'\left(1-10^{-3}\epsilon\right)\right)\frac{n^2}{3}=\left(1-2\times10^{-1}\delta'+2\times10^{-4}\gamma-\delta\right)\frac{n^2}{3}$$

which gives a contradiction, because $2\times10^{-4}\gamma-\delta>10^{-2}\delta'^2$, which means that J', which is K_4-free, violates Turán's theorem.

Therefore, we can remove the at most $10^{-1}\delta' n$ vertices from H and J' and add them to U_0. Let L be the resulting link graph of abc in $H-U_0$. It has, trivially, at least $(1-\delta')\,n^2/3$ edges. This enables us to apply Proposition 2.35. So there are $\epsilon' n$ vertices which may be removed from H such that the remaining link graph L is tripartite. We add these vertices to U_0. L now has at least $(1-\delta'-3\epsilon')\,n^2/3$ edges, which is greater than $\left(1-10^{-7}\epsilon^2\right)n^2/3$ by the choice of δ' and ϵ'. It may be partitioned into three vertex sets V_1, V_2 and V_3, each of which contains no edges.

Note that $||V_i|-n/3|<10^{-3}\epsilon n$ for each i. Assume otherwise, so that V_1, say, violates this and L would have at most

$$\begin{aligned}|V_1|\,(n-|V_1|)+\frac{(n-|V_1|)^2}{4} &= \frac{n^2}{3}-\frac{(3\,|V_1|-n)^2}{12}\\ &< \frac{n^2}{3}-\frac{3}{4}10^{-6}\epsilon^2 n^2\\ &< \left(1-10^{-7}\epsilon^2\right)\frac{n^2}{3}\end{aligned}$$

edges, which gives a contradiction. It also follows that each vertex in V_i has degree at least $\left(1-10^{-3}\right)2n/3-\left(1/3+10^{-3}\epsilon\right)n>n/3-10^{-2}\epsilon n$ in both V_j, $j\neq i$.

Let $v_1v_2v_3$ be a triangle in L with v_i in V_i. For every vertex x in L, if $x\in V_i$ and it is not adjacent to both v_j, $j\neq i$, then add it to U_0. There are at most $6.10^{-2}\epsilon n$ such vertices. As all triangles are multicoloured, we may assume that v_iv_j has colour k for $\{i,j,k\}=\{1,2,3\}$. Then each vertex of V_k is joined to the vertices v_i, v_j by one edge of colour i and one of colour j. Let V_k^1 consist of those vertices v in V_k for which vv_i has colour i and vv_j has colour j and $V_k^2=V_k-V_k^1$.

All edges from v_1 to $V_2^1\cup V_3^1$ have colour 1. Therefore there are no edges between V_2^1 and V_3^1, and the same holds between V_i^1 and V_j^1 for any two distinct $i,j\in\{1,2,3\}$. If both V_i^1 and V_j^1 have size at least $10^{-2}\epsilon n$, then L has at most $n^2/3-\left(10^{-2}\epsilon n\right)^2<\left(1-10^{-7}\epsilon^2\right)n^2/3$ edges, which is impossible. It follows that there is at most one l for which $\left|V_l^1\right|\geq 10^{-2}\epsilon n$. Without loss of generality we assume that $l=1$. Thus both V_2^1 and V_3^1 have size at most $10^{-2}\epsilon n$, and we add their vertices to U_0.

Now take any edge pqr of H in $V-V_0$. Consider first the case where all 3 vertices are in one of the sets. Take $\{p,q,r\}\subset V_1^1$. Then x_2v_2p, x_2v_2q and pqr are all edges of H. Take a vertex s in V_2^2 which is incident with

r. The edge rs must be of colour 2 as rv_3 is colour 3 and v_3s is colour 1. But then the edge x_2rs completes a copy of F_6. The other cases are similar. Therefore, pqr is not contained in any one of the sets.

Next take the case where 2 vertices are in one of the sets. Take $\{p, q\} \subset V_1^1$ and $r \in V_2^2$. But then the edges x_3v_3p, x_3v_3q, pqr and x_3rv_1 form a copy of F_6. The other cases are similar. Therefore, pqr does not have exactly two vertices in any one set.

Finally, consider the case where $p \in V_1^1$, $q \in V_1^2$ and $r \in V_2^2$. If qr is an edge it must be of colour 3 as qv_3 is colour 2 and v_3r is colour 1. But then qrp, qrx_3, px_3v_3 and qv_3x_2 form a copy of F_6. Therefore, qr is not an edge of L. Since L has at least $\left(1 - 10^{-7}\epsilon^2\right)n^2/3$ edges respecting the partition of (V_1, V_2, V_3) out of at most $n^2/3$ possible edges, there are at most $10^{-7}\epsilon^2n^2/3$ choices for qr, so at most $10^{-7}\epsilon^2n^3/3$ such hyperedges pqr.

Similarly, there are at most $10^{-7}\epsilon^2n^3/3$ hyperedges pqr with $p \in V_1^1$, $q \in V_1^2$ and $r \in V_3^2$. All other edges have one point in each of $V_1^1 \cup V_1^2$, V_2^2 and V_3^2. Define a tripartition $V = U_1 \cup U_2 \cup U_3$ so that $V_1^1 \cup V_1^2 \subseteq U_1$, $V_2^2 \subseteq U_2$ and $V_3^2 \subseteq U_3$ and the bad vertices U_0 are distributed arbitrarily. Since $u_0 < (1/2)\epsilon n$ and there are fewer than $(1/2)\epsilon n^3$ exceptional edges all but at most ϵn^3 edges of H have one point in each U_i, so the theorem is proved. $\square$

2.4. Conclusion

We have shown that $\pi(\mathcal{F}) = \pi(F_6) = 2/9$ and that the extremal graph is $S(n)$. This is the first proof of $\pi(F_6) = 2/9$ that does not rely on computational methods. We have also proved an associated stability result. As $\pi(F_5) = 2/9$, both F_5 and F_6 have the same Turán density.

Contrary to the situation with 2-graphs, it does not follow from the fact that F_6 is not contained in a blow up of F_5 that $\pi(F_6)$ is greater than $\pi(F_5)$ - in fact, both have Turán density 2/9. It appears that there is no simple criterion for determining the distribution of Turán densities for 3-graphs in the same way as the chromatic number does for 2-graphs. More insight could be obtained by determining whether there are any other larger 3-graphs with similar properties.

Chapter 3

On Turán's (3,5)-Problem with Forbidden Configurations

3.1. Introduction

Mantel's Theorem states that a graph of order n that contains no triangles has at most $\lfloor n^2/4 \rfloor$ edges. In 1941, Turán generalised this result to cliques of arbitrary size and then asked whether similar results could be obtained for cliques on hypergraphs. This has become one of the central unsolved problems in the field of extremal combinatorics. Erdös offered a cash prize for determining the Turán density of $K_k^{(m)}$ for any pair k, m with $k > m \geq 3$. The prize remains unclaimed.

In 2012, in a series of papers (see principally [**25**] and [**26**]), Razborov considered the simplest unresolved case: the complete 3-graph on 4 vertices, $K_4^{(3)}$, for which Turán had conjectured the correct density was 5/9. The best result for the general case was given by a flag algebra calculation which suggested an upper bound of approximately 0.561. However, Razborov made further progress by considering Turán densities for families of graphs comprising the complete graph and certain other induced graphs. In [**25**], he showed that the Turán density of $\{K_4^{(3)}, E_4^{(3)}\}$ (where $E_4^{(3)}$ is a 3-edge on 4 vertices and is forbidden only as an induced subgraph) is 5/9. And, in [**26**], he showed that the (induced) Turán density of $\{K_4^{(3)}, H_1, H_2, H_3\}$ is 5/9, where the H_i

are the following 3-graphs on 5 vertices:

$$\begin{aligned} H_1 &= \{135, 145, 235, 245\} \\ H_2 &= \{125, 345\} \\ H_3 &= \{345\}. \end{aligned}$$

Notably, these subgraphs are missing from all known extremal configurations.

In this paper, we address the next complete 3-graph, $K_5^{(3)}$. Turán conjectured a density of $3/4$ for $K_5^{(3)}$ and a number of extremal configurations are known. The best upper bound computed using flag algebra is approximately 0.769533 (see [**11**]).

We prove for $K_5^{(3)}$ a counterpart to Razborov's result for $K_4^{(3)}$: that the Turán density of $K_5^{(3)}$ together with six other induced subgraphs is $3/4$. All of these subgraphs are missing from the known extremal configurations up to and including four equivalence classes (one is found in configurations with six or more equivalence classes; one is found in an extremal configuration of nine equivalence classes). All have a density lower than the conjectured extremal density. This can also be seen as an improvement on [**11**], where the (induced) Turán density for $K_5^{(3)}$ and another family of graphs was shown to be $3/4$, but the additional graphs were missing only from the extremal examples on two equivalence classes and also had densities higher than $3/4$.

In setting out the extremal configurations, we also add a slight generalisation to those previously described. Overall, this result reduces the problem of finding the Turán density of $K_5^{(3)}$ to consideration of hypergraphs that contain at least one of these other subgraphs with positive density.

3.2. Background and Definitions

Given a family of hypergraphs $\mathcal{F}$, a hypergraph is *$\mathcal{F}$-free* if it does not contain a (not necessarily induced) subgraph that is isomorphic to any member of $\mathcal{F}$. For any integer $n \geq r$, the *Turán number* of $\mathcal{F}$ is

$$\mathrm{ex}(n, \mathcal{F}) \quad = \quad \max\{|E(H)| \,:\, H \text{ is an } \mathcal{F}\text{-free, } r\text{-graph}, |V(H)| = n\}$$

and the related asymptotic *Turán density* is the following limit (an averaging argument due to Katona, Nemetz and Simonovits [**17**] shows that it always exists)

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{r}}$$

The problem of determining the Turán density is essentially solved for all 2-graphs by the Erdös-Stone-Simonovits Theorem:

THEOREM 3.1 (Erdös and Stone [**9**], Erdös and Simonovits [**8**]). *Let $\mathcal{F}$ be a family of 2-graphs. If $t = \min\{\chi(F) : F \in \mathcal{F}\} \geq 2$, then*

$$\pi(\mathcal{F}) = \frac{t-2}{t-1}$$

We will require a version of Ramsey's Theorem for the regularisation used in Lemma 3.10 below. The multi-partite version of Ramsey's theorem is as follows.

PROPOSITION 3.2 ([**15**], Theorem 5.1.4). *For any $l > 0$, $n > 0$ and $r_1, \ldots, r_l > 0$ there exists $N > 0$ such that if $|B_i| = N$ $(1 \leq i \leq l)$ and $[B_1]^{r_1} \times \cdots \times [B_l]^{r_l}$ is coloured in two colours then there exist $A_i \subseteq B_i$ $(|A_i| = n)$ such that $[A_1]^{r_1} \times \cdots \times [A_l]^{r_l}$ is monochromatic.*

Proposition 3.2 can be iterated to obtain the following (setting $n = 2$).

PROPOSITION 3.3 (Razborov [**26**]). *For any $l > 0$ there exists $N > 0$ such that the following holds. Let a 3-graph B be such that $V(B) = B_1 \dot\cup \ldots \dot\cup B_l$, where $|B_i| = N$. Then there exist $A_i \subseteq B_i$ with $|A_i| = 2$ such that for any $E \in [A_1 \cup \ldots \cup A_l]^3$, whether or not $E \in E(B)$ depends only on the tuple of cardinalities $\langle |E \cap A_1|, \ldots, |E \cap A_l| \rangle$.*

Proposition 3.3 follows from Proposition 3.2 by considering every partition of $3 = r_1 + \cdots + r_l$ $(r_i \geq 0)$. The 3-graph B corresponds to a two colouring of $[B]^3$ which induces a colouring of $[B_1]^{r_1} \times \cdots \times [B_l]^{r_l}$. Then Proposition 3.2 is applied (in an arbitrary order) to each of these partitions recursively.

Taking $r_1 = \ldots = r_l = 1$ gives a density version of Proposition 3.2, as follows.

PROPOSITION 3.4 (Razborov [**26**]). *For all $l, n, \epsilon > 0$ there exists $N_0 > 0$ such that if $|B_i| = N$ $(1 \leq i \leq l)$ with $N \geq N_0$ and $S \subseteq B_1 \times \cdots \times B_l$ has $|S| \geq \epsilon N^l$, then there exist $A_i \subseteq B_i$ $(A_i = n)$ such that $A_1 \times \cdots \times A_l \subseteq S$.*

We will work from now on exclusively with 3-graphs and oriented 2-graphs, referred to as graphs and oriented graphs respectively. For clarity, we suppress the superscript notation for named 3-graphs where no ambiguity would result (so we may refer to K_5 not $K_5^{(3)}$).

From the flag algebra formalism, we will require only the following definitions (see, for example, [**24**] for more details):

- let A and B be 3-graphs, then $p(A, B)$ is the probability that a set of $|V(A)|$ vertices in B chosen uniformly at random induce a copy of A; and
- ρ is the graph on 3 vertices consisting of an edge.

Accordingly, $p(\rho, A)$ is the edge density of A.

We have the following conjecture about the Turán density of K_5:

Conjecture 3.5 (Turán [**30**]). $\pi(K_5^{(3)}) = 3/4$.

There are a number of non-isomorphic graphs that demonstrate the lower bound of 3/4. These are all constructed from equivalence classes of vertices - that is, the adjacency of any three vertices is defined according to their membership of these equivalence classes. The constructions contain an even number of equivalence classes (apart from one that has nine equivalence classes but where one equivalence class contains a single vertex). These constructions are set out in [**29**] and [**18**]. For extremal configurations on two and four equivalence classes, both Sidorenko's and Keevash and Mubayi's constructions can be simply described as follows:

Example 3.6. Let V_1, V_2 be a balanced partition of a set V of n vertices. Let G be the 3-graph on V where the edges consist of all triples with two points in V_i and one point in V_j for $i \neq j$.

Example 3.7. Let V_1, V_2, W_1, W_2 be a balanced partition of a set V of n vertices. Let G be the 3-graph on V where the edges consist of all triples as follows:

(1) Two points in X_i and one point in X_j for $i \neq j$ and $X = V$ or $X = W$;

(2) Two points in the first and one point in the second of each of these pairs: $(V_1, W_1), (V_2, W_2), (W_1, V_2), (W_2, V_1)$;

(3) One point in each of three different vertex sets.

In fact, both Example 3.6 and Example 3.7 can be comprehended by a single configuration which extends Example 3.7:

EXAMPLE 3.8. Let $k \in [0, n/2]$ and let V_1, V_2, W_1, W_2 be a partition of a set V of n vertices with k vertices in each of V_1 and V_2 and $\frac{n}{2} - k$ vertices in each of W_1 and W_2. Let G be the 3-graph on V where the edges are as set out in Example 3.7.

The best upper bound obtained on $\pi(K_5)$ by a flag algebra computation is approximately 0.769533 (see [**11**]). Also, using the same flag algebra software, Falgas-Ravry and Vaughan [**11**] proved an upper bound of 3/4 for the family $\{K_5,\ 5:8\}$ where $5:8$ is the set of all 3-graphs on 5 vertices with 8 edges and are forbidden as induced subgraphs. These additional graphs are missing from Example 3.6, the conjectured extremal graph for K_5 on two equivalence classes, but not Example 3.7, the conjectured extremal graph for K_5 on four equivalence classes, or the conjectured extremal graphs on more than four equivalence classes. We will consider additional graphs that are missing from both Example 3.6 and Example 3.7. Most importantly, the $5:8$ graphs have density 4/5, higher than the conjectured extremal density of 3/4. We will consider additional graphs that all have density less than 3/4. In both these respects, our main theorem may be seen as an improvement of the result in [**11**].

Define the following hypergraphs:

$$\begin{aligned}
F_{2,2,1} &= \{135, 145, 235, 245\}, \\
F^{+}_{2,2,1} &= \{135, 145, 235, 245, 125\}, \\
F_{K4+} &= \{123, 124, 134, 234, 125, 345\}, \\
F_{e:4} &= 4:\{123\} \text{ (the single edge on 4 vertices)}, \\
T_{2,2,2} &= \{123, 124, 345, 346, 156, 256, 135, 136, 145, 146, 235, 236, 245, 246\}, \\
T_{extra} &= \{123, 124, 134, 234, 356, 456, 135, 136, 145, 146, 235, 236, 245, 246\}.
\end{aligned}$$

Let $\mathcal{F} = \{K_5, F_{2,2,1}, F^{+}_{2,2,1}, F_{K4+}, F_{e:4}, T_{2,2,2}, T_{extra}\}$ and describe a hypergraph as $\mathcal{F}$-free if it does not contain K_5 as a subgraph or any of the other graphs in $\mathcal{F}$ as induced subgraphs: note that this is different to the usual definition (where all the graphs are forbidden as subgraphs, whether induced or not).

Define the following classes of hypergraphs:

$$\begin{aligned}\mathcal{H}_n &= \{H\ :\ H \text{ is a 3-graph of order } n\}\\ \mathcal{G}_n &= \{H\ :\ H \in \mathcal{H}_n, \text{ is } \mathcal{F}\text{-free and } p(\rho, H) \leq \frac{3}{4}\}\\ \bar{\mathcal{G}}_n &= \{H\ :\ H \in \mathcal{H}_n, \text{ is } \mathcal{F}\text{-free and } p(\rho, H) > \frac{3}{4}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{G} &= \bigcup_n \mathcal{G}_n\\ \bar{\mathcal{G}} &= \bigcup_n \bar{\mathcal{G}}_n\end{aligned}$$

Our aim is to show that all $\mathcal{F}$-free hypergraphs are in $\mathcal{G}$ (in other words, that $\bar{\mathcal{G}}$ is empty). That is:

Theorem 3.9. $\pi(\mathcal{F}) = 3/4$.

3.3. Construction Using Regularisation

We first use the regularisation technique introduced by [**26**] to show that, if the theorem is false, there exists a counterexample with some additional helpful structure.

Specifically, Lemma 3.10 states that if there exists a counterexample then there also exists a blow-up of that counterexample with 2 vertices in each vertex set and a useful additional property. This property is that given any two vertex sets, $a = \{a_1, a_{2.}\}$ and $b = \{b_1, b_2\}$, $a_1a_2b_1$ is an edge if and only if $a_1a_2b_2$ is an edge.

Lemma 3.10. *Let $\{H_m\}$ be a sequence of 3-graphs of increasing order that are $\mathcal{F}$-free with $\liminf_{m\to\infty} p(\rho, H_m) > 3/4$. Then there exists an $\mathcal{F}$-free graph $G^* \in \bar{\mathcal{G}}_l$ and a graph $H \in \mathcal{H}_{2l}$ such that $H \subseteq X$ for some $X \in \{H_m\}$,$V(H) = \{a_i, b_i\ :\ 1 \leq i \leq l\}$ and for $c_i \in \{a_i, b_i\}$:*

$$\begin{aligned}\forall i \neq j \neq k \in [l]\ c_ic_jc_k \in E(H) \quad &\textit{iff} \quad ijk \in E(G^*)\\ \forall i \neq j \in [l]\ a_ib_ia_j \in E(H) \quad &\textit{iff} \quad a_ib_ib_j \in E(H_1).\end{aligned}$$

PROOF. Given an increasing sequence $\{H_m\}$ of 3-graphs that are $\mathcal{F}$-free with $\liminf_{m\to\infty} p(\rho, H_m) > \frac{3}{4}$, fix a subsequence such that with a suitable renumbering of m:

$$p(\rho, H_m) \geq \frac{3}{4} + \varepsilon$$

for a fixed $\varepsilon > 0$ and all m.

By the definition of $\mathcal{G}_n$, for any integer l, it is true that for every 3-graph $G \in \mathcal{G}$ with l vertices:

$$p(\rho, G) \leq \frac{3}{4}.$$

Now fix m such that $|V(H_m)| \geq l$ and define

$$R = \sum_{G \in \mathcal{G}_l} p(G, H_m).$$

Next we recall the chain rule from flag algebra

$$p(\rho, H_m) = \sum_{G \in \mathcal{H}_l} p(\rho, G) p(G, H_m)$$

so that

$$\begin{aligned} \frac{3}{4} + \varepsilon &\leq \sum_{G \in \mathcal{G}_l} p(\rho, G) p(G, H_m) + \sum_{G \in \mathcal{H}_l \setminus \mathcal{G}_l} p(\rho, G) p(G, H_m) \\ &\leq \frac{3}{4} R + (1 - R) = 1 - \frac{R}{4} \end{aligned}$$

and therefore

$$R \leq 1 - 4\epsilon < 1.$$

Accordingly, there is a positive constant $\delta \geq 4\epsilon / \binom{l}{3}$ which does not depend on m, such that there is a graph $G^* \in \mathcal{H}_l \setminus \mathcal{G}$ and $p(G^*, H_m) \geq \delta$ (for any $m \geq l$). Now we allow m to vary. Because there are only a finite number of graphs in $\mathcal{H}_l$ we may, by restricting to a subsequence again, assume that this graph G^* is the same for all m. Accordingly, we have shown that if $\liminf_{m\to\infty} p(\rho, H_m) > \frac{3}{4}$, there exists a particular $\mathcal{F}$-free graph $G^* \in \bar{\mathcal{G}}$ that exists in a subsequence of H_m with positive density at least δ that does not depend on m.

We are now in a position where it is possible to apply the 'regularisation machinery' employed by Razborov. The following argument is taken directly from [**26**]. In outline, given a positive density of G^* , we construct a blow-up of G^* using Proposition 3.4 with vertex sets sufficiently large that we can then apply Proposition 3.4 to find a subgraph of the blow-up with the additional property set out above.

In detail, firstly, apply Proposition 3.3 with $l = |V(G^*)|$ and let N_1 be the resulting bound. Next, apply Proposition 3.4 with $l = |V(G^*)|$, $n = N_1$ and

$$\epsilon = \frac{1}{2} l^{-l} \delta$$

and let N_0 be the resulting bound. Now fix m such that $|V(H_m)| > lN_0$. Without loss of generality, we may assume that $|V(H_m)|$ is divisible by l and let $N = \frac{1}{l}|V(H_m)|$. Note that $N \geq N_0$.

Let $V(G^*) = [l]$. Consider a random balanced partition $V(H_m) = B_1 \dot{\cup} \ldots \dot{\cup} B_l$ into N-sets. By a standard averaging argument, the expectation of the density of induced embeddings $\alpha : G^* \to H_m$ such that $\alpha(i) \in B_i$ for all $i \in [l]$ is at least ϵ. Fix an arbitrary balanced partition $V(H_m) = C_1 \dot{\cup} \ldots \dot{\cup} C_l$ with this property and let $S \subseteq [C_1] \times \ldots \times [C_l]$ consist of those tuples $(v_1, \ldots, v_l)$ for which the mapping $\beta : [l] \to V(H_m)$ given by $\beta(i) = v_i$ does define an induced embedding of G^*.

Applying Proposition 3.4 gives $D_i \subseteq C_i$ with $|D_i| = N_1$ and $D_1 \times \ldots \times D_l \subseteq S$. And applying Proposition 3.3 (with $B_i = D_i$) results in a graph $H \subseteq H_m$, where $|V(H)| = 2l$, $V(H) = \{a_i, b_i : 1 \leq i \leq l\}$ and the result of the regularisation is that, where $c_i \in \{a_i, b_i\}$:

$$\begin{aligned} \forall i \neq j \neq k \in [l] \; c_i c_j c_k \in E(H) \quad &\text{iff} \quad ijk \in E(G^*) \\ \forall i \neq j \in [l] \; a_i b_i a_j \in E(H) \quad &\text{iff} \quad a_i b_i b_j \in E(H_1). \end{aligned}$$

□

We regard G^* as interchangeable with the 3-graph defined on the equivalence sets $\langle i \rangle = \{a_i, b_i\}$ in H so that a vertex in G^* may be referred to as i or $\langle i \rangle$. Our aim is to determine the maximum density of the edges inside H that constitute G^* by taking advantage of the particular structure of all the edges in H. If we show that the maximum density of G^* is not greater than $3/4$, this contradiction can then be used to establish the main theorem.

Next we define an oriented graph O on the equivalence classes $\langle i \rangle$ (or, more simply, on the vertices i) by specifying that

$$ij \in E(O) \quad \text{iff} \quad a_i b_i a_j, a_i b_i b_j \in E(H).$$

If $\{12\} \in O, \{21\} \notin O$, we refer to this as a *single edge* and if $\{12, 21\} \in O$ we refer to this as a *double edge*.

3.4. Construction of O^*

We next use the additional structure of H to count the maximum number of edges in G^*. First, we construct an equivalence relation on vertices of O by the property of being non-connected.

LEMMA 3.11. *The property of non-adjacency defines an equivalence relation on the vertices of O.*

PROOF. Assume there are vertices $a, b, c \in O$ such that $ab, ba, ac, ca \notin E(O)$ but (without loss of generality) $bc \in E(O)$. Then $abc \in E(G^*)$ or else $a_1 b_1 b_2 c_1$ would be a copy of $F_{e:4}$. But then $a_1 a_2 b_1 b_2 c_1$ is a copy of $F^+_{2,2,1}$. $\square$

Define the vertices of a new oriented graph O^* as the equivalence classes of non-adjacent vertices in O. We determine the structure of O^* as follows.

PROPOSITION 3.12. *For $\alpha, \beta \in V(O^*)$ and $x \in \alpha, p \in \beta$, if $xp, px \in E(O)$ then $\forall y \in \alpha, q \in \beta\, yq, qy \in E(O)$. That is, if one pair of vertices is connected by a double edge, all vertices in those equivalence classes are connected by a double edge.*

PROOF. Take $x, y \in \alpha$ and $p, q \in \beta$ and $xp, px \in E(O)$. First, $pqx \in E(H)$, or else $p_1 p_2 x_1 q_1$ would be a copy of $F_{e:4}$ and, by similar reasoning, $pxy \in E(H)$. Then $qx \in E(O)$ or else $p_1 p_2 q_1 q_2 x_1$ would be a copy of $F^+_{2,2,1}$ and then $xq \in E(O)$ or else $p_1 p_2 q_1 q_2 x_1 x_2$ would be a copy of T_{extra}. By similar reasoning, $py, yp \in E(O)$. Next we consider edges between q and y: $pqy \in E(H)$ or else $p_1 p_2 q_1 y_1$ would be a copy of $F_{e:4}$ and then $qy \in E(O)$ or else $p_1 p_2 q_1 q_2 y_1$ would be a copy of $F^+_{2,2,1}$. By similar reasoning, $yq \in E(O)$. $\square$

Proposition 3.13. *For $\alpha, \beta \in V(O^*)$ and $x \in \alpha, p \in \beta$, if $xp \in E(O), px \notin E(O)$ then $\forall y \in \alpha, q \in \beta$ $yq \in E(O), qy \notin E(O)$. That is, if one pair of vertices is connected by a single edge, all vertices in those equivalence classes are connected by a single edge of the same orientation.*

Proof. Take $x, y \in \alpha$ and $p, q \in \beta$ and $xp \in E(O), px \notin E(O)$. First, $xyp \in E(G^*)$ or else $x_1x_2y_1p_1$ would be a copy of $F_{e:4}$ and $yp \in E(O)$ or else $x_1x_2y_1y_2p_1$ would be a copy of $F^+_{2,2,1}$. Next, $qx \notin E(O)$: otherwise, if $xpq \in E(G^*)$, then $p_1p_2q_1q_2x_1$ would be a copy of $F^+_{2,2,1}$ and, if $xpq \notin E(G^*)$, then $q_1q_2p_1x_1$ would be a copy of $F_{e:4}$. It follows that $xpq \notin E(G^*)$, or else $p_1p_2q_1q_2x_1$ would be a copy of $F_{2,2,1}$. And so $xq \in E(O)$, or else $x_1x_2p_1q_1$ would be a copy of $F_{e:4}$.

We have shown that $xq \in E(O)$ and $qx \notin E(O)$. So the same reasoning used for xpy can be applied to xqy and accordingly $xyq \in E(G^*)$ and $yq \in E(O)$. To show that $py, qy \notin E(O)$ – that is, that these are both single edges and not double edges – we rely on Lemma 3.12, which shows that there are no double edges between α and β. □

Lemma 3.14. *For any two equivalence classes α and β in $V(O^*)$, all edges between vertices in α and vertices in β are of the same type, ie, double edges or single edges of the same orientation.*

Proof. This follows directly from Lemma 3.11 and Propositions 3.12 and 3.13. □

We define the edges of O^* as the same as those between any two representatives of the relevant equivalence classes in O and, in accordance with Lemma 3.14, this is well-defined. We are now able entirely to characterise the graphs that may constitute O^*.

Lemma 3.15. *For any three vertex classes $\alpha\beta, \gamma \in V(O^*)$, only the following arrangements of edges are possible (up to permutation of the vertex classes):*

- T_S: $\alpha\beta, \alpha\gamma, \beta\gamma \in E(O^*)$
- T_{+-}: $\alpha\beta, \beta\alpha, \alpha\gamma, \gamma\beta \in E(O^*)$
- T_{--} : $\alpha\beta, \beta\alpha, \gamma\alpha, \gamma\beta \in E(O^*)$

and, in each of these cases, for all $a \in \alpha, b \in \beta, c \in \gamma$, $abc \in E(H)$.

PROOF. Take any triangle $\alpha\beta\gamma$ with representatives $a \in \alpha, b \in \beta, c \in \gamma$. We proceed by analysing the following cases:

Case 1. The triangle $\alpha\beta\gamma$ contains three double edges.

If $abc \in E(G^*)$ then $a_1a_2b_1b_2c_1$ is a copy of K_5^3. If $abc \notin E(H)$ then $a_1a_2b_1b_2c_1$ is a copy of K_{4+}. So there are no triangles of this type.

Case 2. The triangle $\alpha\beta\gamma$ contains two double edges.

Assume that the missing edge is $\gamma\beta$. If $abc \in E(G^*)$ then $a_1a_2b_1b_2c_1$ is a copy of K_5^3. If $abc \notin E(G^*)$ then $a_1a_2b_1b_2c_1$ is a copy of K_{4+}. So there are no triangles of this type.

Case 3. The triangle $\alpha\beta\gamma$ contains one double edge.

Case i. $\alpha\beta, \beta\alpha, \alpha\gamma, \alpha\beta \in E(O^*)$. Then the reasoning is identical to the two previous cases and so there are no triangles of this type.

Case ii. $\alpha\beta, \beta\alpha, \alpha\gamma, \gamma\beta \in E(O^*)$. If $abc \notin E(G^*)$ then $c_1c_2a_1b_1$ is a copy of $F_{e:4}$. So $abc \in E(G^*)$ and this is T_{+-} as set out above.

Case iii. $\alpha\beta, \beta\alpha, \gamma\alpha, \gamma\beta \in E(O^*)$. If $abc \notin E(G^*)$ then $a_1a_2b_1c_1$ is a copy of $F_{e:4}$. So $abc \in E(G^*)$ and this is T_{--} as set out above.

Case 4. The triangle $\alpha\beta\gamma$ contains no double edge.

Case i. $\alpha\beta, \beta\gamma, \gamma\alpha \in E(O^*)$. If $abc \notin E(G^*)$ then $a_1a_2b_1c_1$ is a copy of $F_{e:4}$. If $abc \in E(G^*)$ then $a_1a_2b_1b_2c_1c_2$ is a copy of $T_{2,2,2}$. So there are no triangles of this type.

Case ii. $\alpha\beta, \beta\gamma, \alpha\gamma \in E(O^*)$. If $abc \notin E(G^*)$ then $b_1b_2a_1c_1$ is a copy of $F_{e:4}$. So $abc \in E(G^*)$ and this is T_S as set out above.

□

Lemma 3.15 enables us to give an exhaustive characterisation of the graphs that constitute O^*.

LEMMA 3.16. *O^* is one of the following graphs:*

- *O_E: the graph on two vertices, consisting of a single edge or a double edge*
- *a graph on three vertices, consisting of T_S, T_{++} or T_{+-}.*
- *O_4: a graph on four vertices consisting of two double edges and four single edges, in the arrangement: $\alpha\beta, \beta\alpha, \gamma\delta, \delta\gamma, \alpha\gamma, \beta\delta, \gamma\beta, \delta\alpha$.*

PROOF. There cannot be more than four vertex classes, otherwise any selection of one vertex from each class would be a copy of K_5. For three vertex classes, the conclusion follows directly from Lemma 3.15.

For four vertex classes, as there are no triangles consisting of two or three double edges, there cannot be three or more double edges. No vertex can have out-degree three, otherwise there would be a copy of K_5. Using only T_{++} and T_{+-} and forbidding vertices of out-degree three, it is not possible to construct a four vertex graph with zero or one double edges. This leaves two double edges, and the only achievable arrangement is O_4. $\square$

3.5. Counting Edges in G^*

Having enumerated all the graphs that constitute O^*, it is necessary to count the maximum number of edges in the corresponding hypergraphs G^*. For any vertex set $\omega \in V(O^*)$, where $|V(G^*)| = n$, define $|\omega|$ such that there are $|\omega|\, n$ vertices in that vertex set. The appropriate formula is given by the following result.

LEMMA 3.17. *For any graph O^*, the number of edges in the corresponding graph G^* is given by*

$$|E(G^*)| = \sum_{\alpha\beta \in E(O^*)} \binom{|\alpha|\, n}{2} |\beta|\, n + \sum_{\alpha\beta\gamma \in \binom{V(O^*)}{3}} |\alpha|\, |\beta|\, |\gamma|\, n^3.$$

PROOF. Any three vertices p, q, r in a single vertex class do not form an edge, otherwise $p_1p_2q_1q_2r_1$ would be a copy of $F_{2,2,1}$. Given $p, q \in \alpha$ and $r \in \beta$, if $\alpha\beta \in E(O^*)$ then $pqr \in E(G^*)$, otherwise $p_1p_2q_1r_1$ would be a copy of $F_{e:4}$. But if $\alpha\beta \notin E(O^*)$, then $pqr \notin E(G^*)$, otherwise $p_1p_2q_1q_2r_1$ would be a copy of $F_{2,2,1}$. This constitutes the first sum. The second sum follows from the fact, set out in Lemma 3.15 that every three vertices from different vertex classes correspond to an edge. $\square$

Putting these elements together, we can count the maximum number of edges in G^*.

THEOREM 3.18. *There are no more than $\frac{1}{8}n^3 + O(n^2)$ edges in G^*.*

PROOF. We proceed by analysing all the possible graphs as set out in Lemma 3.16 using the formula in Lemma 3.17. For clarity, terms of order $O(n^2)$ and lower are suppressed in the formulae below.

Case 1. O_E

Let $\alpha, \beta \in O_E$. Edges of G^* are clearly maximised if there is a double edge. So the number of edges in G^* is

$$\begin{aligned} |E(G^*)| &= \left(\frac{\alpha^2\beta}{2} + \frac{\alpha\beta^2}{2}\right) n^3 \\ &= \frac{\alpha\beta n^3}{2}(\alpha+\beta) \\ &= \frac{\alpha\beta n^3}{2} \\ &\leq \frac{1}{8}. \end{aligned}$$

Case 2. T_S

Let $\alpha\beta, \alpha\gamma, \beta\gamma \in E(G^*)$. Then the number of edges in G^* is

$$\begin{aligned} |E(G^*)| &= \left(\frac{\alpha^2\beta}{2} + \frac{\alpha^2(1-\alpha-\beta)}{2} + \frac{\beta^2(1-\alpha-\beta)}{} + \alpha\beta(1-\alpha-\beta)\right) n^3 \\ &= \frac{n^3}{2}\left(\alpha^2\beta + \alpha^2 - \alpha^3 - \alpha^2\beta + \beta^2 - \alpha\beta^2 - \beta^3 + 2\alpha\beta - 2\alpha^2\beta - 2\alpha\beta^2\right) \\ &= \frac{n^3}{2}\left(-2\alpha^2\beta + \alpha^2 - \alpha^3 + \beta^2 - 3\alpha\beta^2 - \beta^3 + 2\alpha\beta\right) \\ &= \frac{n^3}{2}\left((\alpha+\beta)^2 - (\alpha+\beta)^3 + \alpha^2\beta\right). \end{aligned}$$

This is maximised when $\alpha = 12/23$, $\beta = 6/23$ and $\gamma = 5/23$ giving a density of $54/529 < 1/8$. So this arrangement does not achieve the highest possible density.

Case 3. T_{+-}

Let $\alpha\beta, \beta\alpha, \alpha\gamma, \gamma\beta \in E(G^*)$. Then the number of edges in G^* is

$$\begin{aligned}
|E(G^*)| &= \left(\frac{\alpha^2\beta}{2} + \frac{\alpha\beta^2}{2} + \frac{\alpha^2(1-\alpha-\beta)}{2} + \frac{\beta(1-\alpha-\beta)^2}{2} + \alpha\beta(1-\alpha-\beta)\right) n^3 \\
&= \frac{n^3}{2}\left(\alpha\beta(\alpha+\beta) + (1-\alpha-\beta)(\alpha^2 + \beta(1-\alpha-\beta) + 2\alpha\beta)\right) \\
&= \frac{n^3}{2}\left(\alpha\beta(\alpha+\beta) + (1-\alpha-\beta)(\alpha^2 + \beta - \beta^2 + \alpha\beta)\right) \\
&= \frac{n^3}{2}\left(\alpha\beta + (1-\alpha-\beta)(\alpha^2 + \beta - \beta^2)\right).
\end{aligned}$$

This is maximised when $\alpha = \beta = 1/2$: that is, it degenerates to Case 1.

Case 4. T_{--}

Let $\alpha\beta, \beta\alpha, \gamma\alpha, \gamma\beta \in E(G^*)$. Then the number of edges in G^* is

$$\begin{aligned}
|E(G^*)| &= \left(\frac{\alpha^2\beta}{2} + \frac{\alpha\beta^2}{2} + \frac{\alpha(1-\alpha-\beta)^2}{2} + \frac{\beta(1-\alpha-\beta)^2}{2} + \alpha\beta(1-\alpha-\beta)\right) n^3 \\
&= \frac{n^3}{2}\left(\alpha\beta(\alpha+\beta) + (1-\alpha-\beta)^2(\alpha+\beta) + 2\alpha\beta(1-\alpha-\beta)\right) \\
& \quad \frac{n^3}{2}\left(\alpha\beta(2-\alpha-\beta) + (1-\alpha-\beta)^2(\alpha+\beta)\right).
\end{aligned}$$

Again, this is maximised when $\alpha = \beta = 1/2$, which is identical to Case 1.

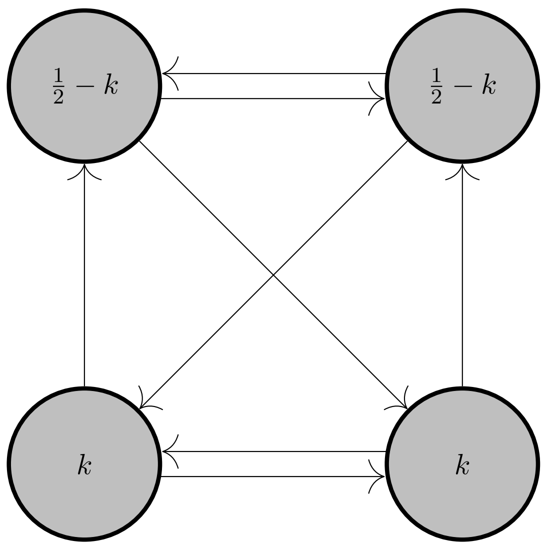
Case 5. O_4

Let $\alpha\beta, \beta\alpha, \gamma\delta, \delta\gamma, \alpha\gamma, \beta\delta, \gamma\beta, \delta\alpha \in E(G^*)$. Then the number of edges in G^* is

$$\begin{aligned}
|E(G^*)| &= \left(\frac{\alpha^2\beta}{2} + \frac{\alpha\beta^2}{2} + \frac{\gamma^2\delta}{2} + \frac{\gamma\delta^2}{2} + \frac{\alpha^2\gamma}{2} + \frac{\beta^2\delta}{2} + \right. \\
&\qquad \left. \frac{\gamma^2\beta}{2} + \frac{\delta^2\alpha}{2} + \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta\right) n^3 \\
&= \frac{n^3}{2}\left(\alpha^2(\beta+\gamma) + \beta^2(\alpha+\delta) + \gamma^2(\beta+\delta) + \right.
\end{aligned}$$

$$\delta^2(\alpha+\gamma)+\alpha\beta\gamma\delta\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)\Bigg).$$

This is maximised by a family of graphs that includes Case 1 as a special case. Let $0 \leq k \leq \frac{1}{2}$. Then $\alpha = \beta = k$, $\gamma = \delta = \frac{1}{2} - k$ gives a graph with a density of $\frac{1}{8}$:



Examination of the possible cases shows that there is a single family of maximal graphs of the form O_4, including the degenerate case O_E, with density $\frac{1}{8}$. $\square$

3.6. Proof of Main Theorem

Putting all the pieces together gives a proof of the main theorem.

Theorem 3.9. $\pi(\mathcal{F}) = 3/4$.

Proof. Assume, in order to establish a contradiction, that Theorem 3.9 is false. Then there exists a sequence $\{H_m\}$ satisfying the preconditions of Lemma 3.10. Applying Lemma 3.10 gives the graphs G^* and H, as set out in the statement of Lemma 3.10. Then, noting that $n^3/8 + O(n^2) = (3/4)\binom{n}{3} + O(n^2)$, Theorem 3.18 shows that G^* has density at most 3/4, contrary to the stipulation that it is not in $\mathcal{G}$. This contradiction establishes Theorem 3.9. $\square$

3.7. Conclusion

Our construction shows that the family of $\mathcal{F}$-free graphs with density greater than 3/4 is empty; that is, $\pi(\mathcal{F}) = 3/4$ (note that only induced versions of graphs apart from K_5 are forbidden). The extremal graphs correspond to the known extremal graphs for K_5 on any number of vertices but up to only four equivalence classes. Graphs with higher numbers of equivalence classes are known but are forbidden by K_{4+} (and there is one extremal configuration on eight equivalence classes forbidden by T_{extra}). It is possible that the result could be strengthened by removing some of these graphs from $\mathcal{F}$. Ultimately, Theorem 3.9 could lead to a proof of the Turán density for K_5 by considering only those graphs that contain a member of $\mathcal{F}$ as an induced subgraph with positive density.

Chapter 4

A Generalised Turán Function

4.1. Introduction

The balanced complete bipartite graph $K_{n,n}$ has the most edges of all K_3-free graphs. What can we say about the density of a K_3-free graph that is 'locally' different from $K_{n,n}$? The only graph on 3 vertices that does not appear as a subgraph of $K_{n,n}$ is $\bar{P}_3$, the graph with a single edge and an isolated vertex. So a logical first step is to consider K_3-free graphs that contain a certain positive density of $\bar{P}_3$. This may be expanded to consider the maximum density of a K_3-free graph as a function of the density of $\bar{P}_3$ in that graph. In this chapter, we derive the function, parameterised by the density of $\bar{P}_3$, that gives an exact bound for the maximum density of a K_3-free graph. We also derive linear functions that give upper bounds for K_n-free graphs parameterised by families of graphs that are natural generalisations of $\bar{P}_3$. These reults are analogous to stability results and, in general, even more informative: where the parameterised graphs are absent in the extremal graph, they reveal elements of the structure of those graphs that are close to the extremal graph in density.

The relationship between the possible densities of graphs on 3 vertices was considered in [**16**]. Specifically, they looked at the case of K_3-free graphs and considered the possible densities of $\bar{P}_3$, P_3 and $\bar{K}_3$ in such graphs – that is, the minimum and maximum densities in K_3-free graphs of each of these subgraphs as a function of the density of the others.

A converse problem fixes the density of the graph and determines the minimum density of a subgraph such as K_3 - that is, it seeks to determine the minimum density of K_3, for instance, as a function of the edge density. This problem was introduced by Erdös in [**30**] and has been studied extensively for K_3 and larger cliques – see [**23, 22, 27, 21**].

We conduct this study by exploring certain generalisations of the Turán function using a combination of flag algebra and analytical techniques. First we define $d(F;J)$ as the density of F as an induced subgraph of J; that is:

$$d(F;J) \quad = \quad \frac{\#\text{induced copies of } F \text{ in } J}{\binom{|V(J)|}{|V(F)|}}.$$

The Turán function $\mathsf{ex}(n,\mathcal{F})$ returns as a value the maximum number of edges of a graph of order n that does not contain any $F \subset \mathcal{F}$ as a subgraph. We consider various natural generalisations of the Turán function. The two parameters implicitly used in the definition of the Turán function are the subgraph which is being maximised – in the classical Turán function this subgraph is an edge – and the densities of subgraphs contained in the family $\mathcal{F}$ – in the classical Turán function these densities are all set to zero. The generalised Turán function expressly introduces these parameters:

$$\begin{aligned}\mathrm{ex}_{\mathrm{gen}}(n,\mathcal{F},\mathcal{G}=\{G_i\},K=\{k_i\},H) \quad = \quad & \max_J(\#\text{induced copies of } H \text{ in } J : \\ & |V(J)| = n \text{ and } J \text{ is } \mathcal{F}\text{-free and} \\ & \forall i \in [1,|\mathcal{G}|]\, d(G_i;J) \geq k_i)\end{aligned}$$

where $\mathcal{F}$ is the family of forbidden graphs, as before, $\mathcal{G}$ is another family of graphs and K is a set of real values, both indexed by i, such that each graph G_i is present with density at least k_i, and H is the subgraph whose density is being maximised. An alternative version parameterises the Turán function on the aggregate of the densities of the family $\mathcal{G}$, instead of a separate density for each member of that family:

$$\begin{aligned}\mathrm{ex}_{\mathrm{gen}^*}(n,\mathcal{F},\mathcal{G},k,H) \quad \equiv \quad & \max_J(\#\text{induced copies of } H \text{ in } J : \\ & |V(J)| = n \text{ and } J \text{ is } \mathcal{F}\text{-free and} \\ & \left(\sum_{G\in\mathcal{G}} d(G;J)\right) \geq k.\end{aligned}$$

The Turán density is defined in terms of the Turán function as follows

$$\pi(\mathcal{F}) \quad = \quad \lim_{n\to\infty} \frac{\mathsf{ex}(n,\mathcal{F})}{\binom{n}{2}}.$$

The generalised Turán density can similarly be defined in terms of the Turán function:

$$\pi_{gen}(\mathcal{F},\mathcal{G},K,H) \quad = \quad \lim_{n\to\infty} \frac{\mathrm{ex}_{\mathrm{gen}}(n,\mathcal{F},\mathcal{G},K,H)}{\binom{n}{|V(H)|}}$$

and in aggregate form:

$$\pi_{gen*}(\mathcal{F},\mathcal{G},k,H) \quad = \quad \lim_{n\to\infty} \frac{\mathrm{ex}_{\mathrm{gen}*}(n,\mathcal{F},\mathcal{G},k,H)}{\binom{n}{|V(H)|}}.$$

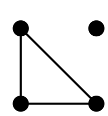
4.2. Results

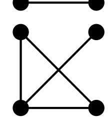
As set out above, the first graph to consider is the complete graph on n vertices, K_n. The extremal graph for K_n is the balanced complete $n-1$-partite graph. Starting with K_3, $\bar{P}_3$ is the only graph on three vertices that is absent from the complete bipartite graph, so it is natural to determine $\pi_{gen}(K_3, \bar{P}_3, k, K_2)$.

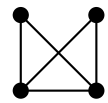
THEOREM 4.1. $\pi_{gen}\left(K_3, \bar{P}_3, k, K_2\right) = \frac{1+\sqrt{1-8k/3}}{4}$

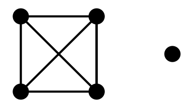
We provide two different proofs of this theorem, one using flag algebra and one using analytical techniques.

For K_n with $n > 3$, there are a larger number of graphs that are absent from the $n-1$-partite complete graphs. We consider one family of graphs which we label K_n^{-j}. Define $K_n^{-j} = K_{n-1} \cup \{x\}$, where $d(x) = n-1-j$. That is, K_n^{-j} is a graph of order n consisting of a K_{n-1} and a single vertex, with j edges missing between the vertex and the K_{n-1}. For $j > 1$, K_n^{-j} is not found as a subgraph of the complete $n-1$- partite graph. The first few members of this family are as follows

$K_4^{-3} \quad =$ 

$K_4^{-2} \quad =$ 

K_4^{-1} = 

K_5^{-4} = 

The same technique used in the proof of Theorem 4.1, gives a related upper bound for the density of triangles:

THEOREM 4.2. $\pi_{gen*}(K_4, \{K_4^{-2}, K_4^{-3}\}, k, K_3) \leq (1/6)(1 + \sqrt{1 - 3k})$.

With a different application of flag algebra, we prove a linear upper bound on the conditional Turan density of K_n with respect to the graphs K_{n-1}^{j}, for $j > 1$.

THEOREM 4.3. $\pi_{gen*}(K_t, \{K_t^{-j} \,:\, 1 < j < t\}, k, K_2) \leq \frac{t-2}{t-1} - \frac{(t-1)^{t-2}}{t!(t-2)}k$.

Finally, we are able to improve this result for the particular case of K_4:

THEOREM 4.4. $\pi_{gen}(K_4, (K_4^{-3}, K_4^{-2}), (x, y), K_2) \leq (2/3) - (3/8)x - (1/4)\,y$.

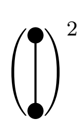
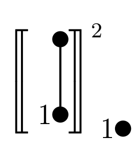
4.3. Proofs

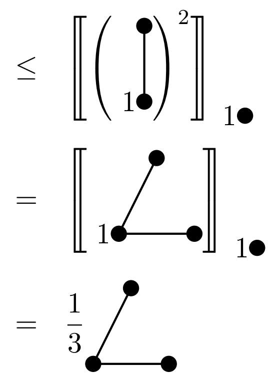
4.3.1. Proofs using Flag Algebras

An introduction to flag algebra and the particular constructions used in these proofs is set out in Chapter 1 in Section 1.3.4. In particular, we employ the algebra as 'syntactic sugar' to allow the expression of Cauchy-Schwarz type inequalities that would be infeasible otherwise.

First we provide the flag algebra proof for the conditional Turan density of K_3.

FIRST PROOF OF THEOREM 4.1. Working in the algebra of graphs missing K_3:

 = 



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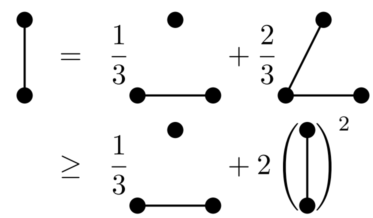
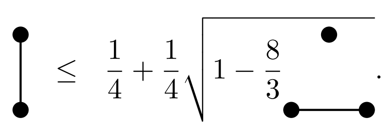


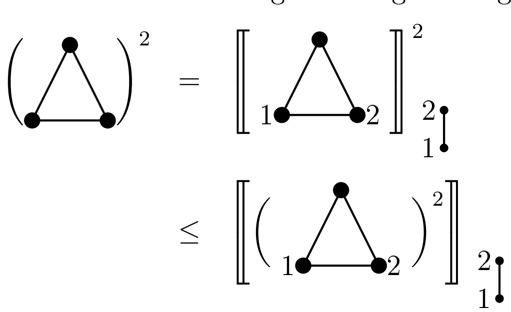
which implies that

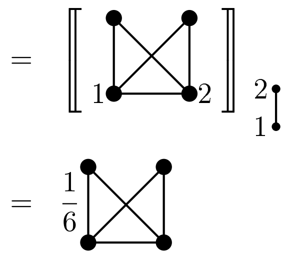


It follows from Lemma 4.6 that the balanced blow up of the bipartite graph, $K_2(p)$, where every edge exists with probability p, achieves this edge density, and so $\pi(K_3, \bar{P}_3, k, K_2) = 1/4 + 1/4\sqrt{1-(8/3)k}$. $\square$

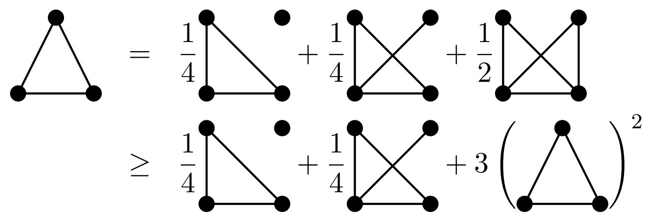
Next we use a similar argument to prove Theorem 4.2.

Proof of Theorem 4.2. Working in the algebra of graphs missing K_4:

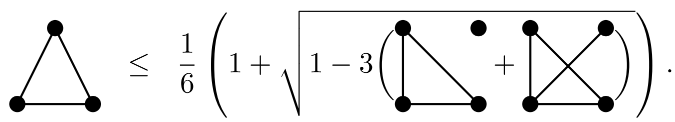




and



which implies that



$\square$

We introduce some additional nomenclature for Theorems 3 and 4. The type of order t isomorphic to the complete graph is labelled σ_t and the flag consisting of σ_t and v unlabelled vertices isomorphic to K_{t+v} is labelled $K_{t+v}^{\sigma_t}$.

The method used to prove Theorems 4.3 and 4.4 is illustrated first for the specific case of Theorem 4.4. The central idea is taken from a proof of Turan's Theorem due to Reiher.

Proof of Theorem 4.4. First, we show that, working in the flag algebra of graphs missing K_4, $K_3 + \frac{1}{4}\left(K_4^{-2} + K_4^{-3}\right) \leq \frac{1}{3}K_2$:

$$\begin{aligned}
\frac{1}{3}K_2 - K_3 &= 3\left[\!\left[\frac{1}{9}K_2^{\sigma_2} - \frac{1}{3}K_3^{\sigma_2}\right]\!\right]_{\sigma_2} \\
&= 3\left[\!\left[\left(\frac{1}{3} - K_3^{\sigma_2}\right)^2 + \frac{1}{3}K_3^{\sigma_2} - (K_3^{\sigma_2})^2\right]\!\right]_{\sigma_2} \\
&\geq 3\left[\!\left[\frac{1}{3}K_3^{\sigma_2} - (K_3^{\sigma_2})^2\right]\!\right]_{\sigma_2} \\
&= 3\left[\frac{1}{3}K_3 - \frac{1}{6}K_4^{-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= 3\left[\frac{1}{3}\left(\frac{1}{4}K_4^{-2}+\frac{1}{4}K_4^{-3}+\frac{1}{2}K_4^{-1}\right)-\frac{1}{6}K_4^{-1}\right] \\
&= \frac{1}{4}\left(K_4^{-2}+K_4^{-3}\right).
\end{aligned}
$$

This inequality for K_3 can then be substituted into the following:

$$
\begin{aligned}
\frac{2}{3}-K_2 &= \frac{3}{2}\left[\!\left[\frac{4}{9}-\frac{2}{3}K_2^{\sigma_1}\right]\!\right]_{\sigma_1} \\
&= \frac{3}{2}\left[\!\left[\left(\frac{2}{3}-K_2^{\sigma_1}\right)^2+\frac{2}{3}K_2^{\sigma_1}-(K_2^{\sigma_1})^2\right]\!\right]_{\sigma_1} \\
&\geq \frac{3}{2}\left[\!\left[\frac{2}{3}K_2^{\sigma_1}-(K_2^{\sigma_1})^2\right]\!\right]_{\sigma_1} \\
&= \frac{3}{2}\left[\frac{2}{3}K_2-K_3-\frac{1}{3}K_3^{-1}\right] \\
&\geq \frac{3}{2}\left[\frac{1}{2}\left(K_3+\frac{1}{4}\left(K_4^{-2}+K_4^{-3}\right)\right)+\right. \tag{4.3.1}\\
&\qquad \left.\frac{1}{2}\left(K_3+\frac{2}{3}K_3^{-1}+\frac{1}{3}K_3^{-2}\right)-K_3-\frac{1}{3}K_3^{-1}\right] \\
&= \frac{3}{2}\left[\frac{1}{8}\left(K_4^{-2}+K_4^{-3}\right)+\frac{1}{6}K_3^{-2}\right] \\
&\geq \frac{3}{2}\left[\frac{1}{8}\left(K_4^{-2}+K_4^{-3}\right)+\frac{1}{6}\left(\frac{1}{4}K_4^{-2}+\frac{3}{4}K_4^{-3}\right)\right] \tag{4.3.2}\\
&= \frac{3}{8}K_4^{-3}+\frac{1}{4}K_4^{-2}.
\end{aligned}
$$

At 4.3.1, K_2 is replaced by the substitution involving K_3 and is also expanded in terms of sub-graphs of order 3. At 4.3.2, K_3^{-2} is replaced by its expansion in terms of K_4^{-2} and K_4^{-3}. □

Finally, this technique is expanded to the general case.

LEMMA 4.5. *Working in the algebra of graphs missing K_t, for all $i \in [2, t-1]$:*

$$
\frac{t-i}{t-1}K_{i-1} \quad \geq \quad K_i + E_i \sum_{j=2}^{t-1} K_t^{-j}
$$

where

$$
E_i \quad = \quad \frac{(t-1)^{t-i-1}\,(i-1)}{t(t-2)\,[(t-i)!\,]}.
$$

PROOF. We proceed downwards from $i = t-1$:

$$\begin{aligned}
\frac{1}{t-1}K_{t-2} - K_{t-1} &= (t-1)\left[\!\!\left[\left(\frac{1}{t-1}\right)^2 K_{t-2}^{\sigma_{t-2}} - \frac{1}{t-1}K_{t-1}^{\sigma_{t-2}}\right]\!\!\right]_{\sigma_{t-2}} \\
&= (t-1)\left[\!\!\left[\left(\frac{1}{t-1} - K_{t-1}^{\sigma_{t-2}}\right)^2 + \frac{1}{t-1}K_{t-1}^{\sigma_{t-2}} - \left(K_{t-1}^{\sigma_{t-2}}\right)^2\right]\!\!\right]_{\sigma_{t-2}} \\
&\geq (t-1)\left[\!\!\left[\frac{1}{t-1}K_{t-1}^{\sigma_{t-2}} - \left(K_{t-1}^{\sigma_{t-2}}\right)^2\right]\!\!\right]_{\sigma_{t-2}} \\
&= (t-1)\left[\frac{1}{t-1}K_{t-1} - \frac{2}{t(t-1)}K_t^{-1}\right] \\
&= (t-1)\left[\frac{1}{t-1}\left(\frac{2}{t}K_t^{-1} + \frac{1}{t}\sum_{j=2}^{t-1}K_t^{-j}\right) - \frac{2}{t(t-1)}K_t^{-1}\right] \\
&= \frac{1}{t}\sum_{j=2}^{t-1}K_t^{-j} \\
&= E_{t-1}\sum_{j=2}^{t-1}K_t^{-j}.
\end{aligned}$$

Next is $i = t-2$:

$$\begin{aligned}
\frac{2}{t-1}K_{t-3} - K_{t-2} &= \left(\frac{t-1}{2}\right)\left[\!\!\left[\left(\frac{2}{t-1}\right)^2 K_{t-3}^{\sigma_{t-3}} - \frac{2}{t-1}K_{t-2}^{\sigma_{t-3}}\right]\!\!\right]_{\sigma_{t-3}} \\
&\geq \left(\frac{t-1}{2}\right)\left[\!\!\left[\frac{2}{t-1}K_{t-2}^{\sigma_{t-3}} - \left(K_{t-2}^{\sigma_{t-3}}\right)^2\right]\!\!\right]_{\sigma_{t-3}} \\
&= \left(\frac{t-1}{2}\right)\left[\frac{2}{t-1}K_{t-2} - K_{t-1} - \frac{2}{(t-1)(t-2)}K_{t-1}^{-1}\right] \\
&= \left(\frac{t-1}{2}\right)\left[\frac{t-3}{t-2}\frac{1}{t-1}K_{t-2} + \frac{1}{t-2}K_{t-2} - K_{t-1} - \right. \\
&\qquad \left. \frac{2}{(t-1)(t-2)}K_{t-1}^{-1}\right] \\
&\geq \left(\frac{t-1}{2}\right)\left[\frac{t-3}{t-2}\left(K_{t-1} + \frac{1}{t}\sum_{j=2}^{t-1}K_t^{-j}\right) + \right. \\
&\qquad \left. \frac{1}{t-2}\left(K_{t-1} + \frac{2}{t-1}K_{t-1}^{-1}\right) - K_{t-1} - \frac{2}{(t-1)(t-2)}K_{t-1}^{-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(t-1)(t-3)}{2(t-2)t}\sum_{j=2}^{t-1} K_t^{-j} \\
&= E_{t-2}\sum_{j=2}^{t-1} K_t^{-j}.
\end{aligned}
$$

And we are then able to prove the general case by falling induction on i:

$$
\begin{aligned}
\frac{t-i}{t-1}K_{i-1} - K_i &= \left(\frac{t-1}{t-i}\right)\left[\!\!\left[\left(\frac{t-i}{t-1}\right)^2 K_{i-1}^{\sigma_{i-1}} - \frac{t-i}{t-1}K_i^{\sigma_{i-1}}\right]\!\!\right]_{\sigma_{i-1}} \\
&\geq \left(\frac{t-1}{t-i}\right)\left[\!\!\left[\frac{t-i}{t-1}K_i^{\sigma_{i-1}} - \left(K_i^{\sigma_{i-1}}\right)^2\right]\!\!\right]_{\sigma_{i-1}} \\
&= \left(\frac{t-1}{t-i}\right)\left[\frac{t-i}{t-1}K_i - K_{i+1} - \frac{2}{i(i+1)}K_{i+1}^{-1}\right] \\
&= \left(\frac{t-1}{t-i}\right)\left[\frac{t-(i+1)}{t-1}\frac{i-1}{i}K_i + \frac{1}{i}K_i - K_{i+1} - \frac{2}{i(i+1)}K_{i+1}^{-1}\right] \\
&\geq \left(\frac{t-1}{t-i}\right)\left[\frac{i-1}{i}\left(K_{i+1} + E_{i+1}\sum_{j=2}^{t-1}K_t^{-j}\right) + \frac{1}{i}\left(K_{i+1} + \frac{2}{i+1}K_{i+1}^{-1}\right)\right. \\
&\quad \left. -K_{i+1} - \frac{2}{i(i+1)}K_{i+1}^{-1}\right] \\
&= \frac{t-1}{t-i}\frac{i-1}{i}E_{i+1}\sum_{j=2}^{t-1}K_t^{-j} \\
&= E_i\sum_{j=2}^{t-1}K_t^{-j}.
\end{aligned}
$$

□

Lemma 4.5 leads quickly to a proof of Theorem 4.3.

Proof of Theorem 4.3. We wish to determine an upper bound for $\pi_{gen*}(K_t, \{K_t^{-j} : j \in [2, t-1]\}, k, K_2)$. Working in the algebra of graphs missing K_t, we apply Lemma 4.5 with $i = 2$:

$$
\frac{t-2}{t-1} - K_2 \geq E_2\sum_{j=2}^{t-1}K_t^{-j}
$$

$$
\begin{aligned}
&= \frac{(t-1)^{t-3}}{t(t-2)\,[(t-2)!\,]} \sum_{j=2}^{t-1} K_t^{-j} \\
&= \frac{(t-1)^{t-2}}{t!\,(t-2)} \sum_{j=2}^{t-1} K_t^{-j}.
\end{aligned}
$$

From this it follows that

$$
\pi_{gen*}(K_t, \{K_t^{-j} : j \in [2, t-1]\}, k, K_2) \leq \frac{t-2}{t-1} - \frac{(t-1)^{t-2}}{t!\,(t-2)} k.
$$

□

4.3.2. Proof using Analytic Techniques

Finally, we use a completely different analytical approach to provide an alternative proof of Theorem 4.1.

We prove the theorem for the family of weighted graphs $\mathcal{W}$, with both vertex and edge weights, which include graphs as a special case and derive Theorem 4.1 as a corollary. Define a weighted graph $G \in \mathcal{W}$ as a triple $(n, \mathbf{x}, \mathbf{A})$ subject to the following conditions:

- $|\mathbf{x}| = n$ and $\mathbf{A} = a_{ij}$ is a square matrix of order n;
- $\forall i \in [n]\; 0 < x_i \leq 1$;
- $\sum_{i \in [n]} x_i = 1$;
- $\forall i \in [n]\; a_{ii} = 0$;
- $\forall i, j \in [n]\; a_{ij} = a_{ji}$; and
- $\forall i, j \in [n]\; 0 \leq a_{ij} \leq 1$.

We interpret x_i as the proportion of vertices of the graph in vertex set i and a_{ij} as the density of edges between vertex sets i and j. Each vertex set i consists of independent vertices. The family $\mathcal{W}$ clearly includes all graphs as a (dense) subset: for any graph $G = (V(G), E(G))$, take $n = |V(G)|$, $x_i = 1/n$ for all $i \in n$ and either $a_{ij} = 1$ if $ij \in E(G)$ or $a_{ij} = 0$ if $ij \notin E(G)$.

Let $p_{ij} = 1 - a_{ij}$. For any weighted graph G, define:

$$d(G) = 2 \sum_{i,j \in \binom{V(G)}{2}} x_i a_{ij} x_j$$

$$d_{\bar{P}_3} = \sum_{i,j \in \binom{V(G)}{2}} x_i x_j a_{ij} \left[3\,(x_i + x_j)\left(2p_{ij} - 2p_{ij}^2\right) + 2 \sum_{k \in V(G), k \neq i,j} x_k p_{ik} p_{jk} \right]$$

so that $d(G)$ is the density of the underlying graph H and $d_{\bar{P}_3}(G)$ is the density of $\bar{P}_3$ in the underlying graph H.

Define $K_2(p)$ as the weighted graph consisting of two vertices of weight $1/2$ and one edge of weight p, so that the underlying graph is the blow up of K_2 where edges exist with probability p.

Lemma 4.6. $d(K_2(p)) = \frac{1+\sqrt{1-8d_{\bar{P}_3}(K_2(p))/3}}{4}$.

Proof. We have

$$\begin{aligned} d = d(K_2(p)) &= \frac{1}{2}p \\ e = d_{\bar{P}3}(K_2(p)) &= \frac{3}{2}p(1-p) \\ &= 3d(1-2d) \\ 0 &= -6d^2 + 3d - e \end{aligned}$$

so that by the quadratic formula

$$d = \frac{1+\sqrt{1-8e/3}}{4}.$$

□

From now on, we work with the family of weighted graphs $\mathcal{W}^*$ that model graphs that are K_3-free; in other words, for any weighted graph $G = (n, \mathbf{x}, \mathbf{A}) \in \mathcal{W}^*$, $\forall i,j,k \in [n]\ a_{ij} > 0 \,\&\, a_{jk} > 0 \Longrightarrow a_{ik} = 0$.

A weighted graph $G \in \mathcal{W}^*$ is *maximal* if $d(G) = d_1$ and $d_{\bar{P}3}(G) = d_2$ and there is no other graph $F \in \mathcal{W}^*$ such that $d(F) \geq d_1$ and $d_{\bar{P}_3}(F) > d_2$ or $d(F) > d_1$ and $d_{\bar{P}_3}(F) \geq d_2$. A weighted graph is *minimal* if there is no other graph $F \in \mathcal{W}^*$ with $d(F) = d(G)$ and $d_{\bar{P}_3}(F) = d_{\bar{P}_3}(G)$ and $|V(F)| < |V(G)|$.

Let $\mathcal{Z}$ be the family of weighted graphs that are both maximal and minimal. We show the following:

LEMMA 4.7. *For all $Z \in \mathcal{Z}$, there is no $i, j \in V(Z)$ such that $a_{ij} = 0$.*

PROOF. Assume that $a_{ij} = 0$. As Z is minimal there is at least one vertex k such that $a_{ik} \neq a_{jk}$ (otherwise, we could replace Z with a smaller graph Z^* with $V(Z^*) = V(Z) - \{j\}$ and $x_j^* = x_i + x_j$ and $a_{jq}^* = a_{jq}$ for all vertices q).

Consider the following transformation that leaves $d(Z)$ unchanged. Replace p_{ik} with $p_{ik} + \Delta$ and replace p_{jk} with $p_{jk} - \Delta \frac{x_i}{x_j}$. Call this new weighted graph Z'. Note that

$$\begin{aligned} d(Z') - d(Z) &= x_i \left(a_{ik} - \Delta\right) x_k + x_j \left(a_{jk} + \Delta \frac{x_i}{x_j}\right) x_k - x_i a_{ik} x_k - x_j a_{jk} x_k \\ &= 0. \end{aligned}$$

Next we define

$$D_{ijk}(\Delta) = d_{e_3}(Z') - d_{e_3}(Z)$$

and calculate

$$\begin{aligned} D_{ijk}(\Delta) &= 6x_i x_k \left(x_i + x_k\right) \left[\left(p_{ik} + \Delta\right) - \left(p_{ik} + \Delta\right)^2 - \left(p_{ik} - p_{ik}^2\right)\right] \\ &\quad +6x_j x_k \left(x_j + x_k\right) \left[\left(p_{jk} - \frac{x_i}{x_j}\Delta\right) - \left(p_{jk} - \frac{x_i}{x_j}\Delta\right)^2 - \left(p_{jk} - p_{jk}^2\right)\right] \\ &\quad +6x_i x_j x_k \Big[\left(p_{ik} + \Delta\right) + \left(p_{jk} - \frac{x_i}{x_j}\Delta\right) - 2\left(p_{ik} + \Delta\right)\left(p_{jk} - \frac{x_i}{x_j}\Delta\right) \\ &\quad - \left(p_{ik} + p_{jk} - 2p_{ik}p_{jk}\right)\Big] \\ &\quad +6x_i x_k \sum_{q \in V(G), q \neq i,j,k} x_q \Big[\left(p_{ik} + \Delta\right)\left(p_{iq} + p_{kq} - 2p_{iq}p_{kq}\right) \\ &\quad + \left(1 - p_{ik} - \Delta\right) p_{iq}p_{kq} - p_{ik}\left(p_{iq} + p_{kq} - 2p_{iq}p_{kq}\right) - \left(1 - p_{ik}\right) p_{iq}p_{kq}\Big] \\ &\quad +6x_j x_k \sum_{q \in V(G), q \neq i,j,k} x_q \Big[\left(p_{jk} - \frac{x_i}{x_j}\Delta\right)\left(p_{jq} + p_{kq} - 2p_{jq}p_{kq}\right) \\ &\quad + \left(1 - p_{jk} + \frac{x_i}{x_j}\Delta\right) p_{jq}p_{kq} - p_{jk}\left(p_{jq} + p_{kq} - 2p_{jq}p_{kq}\right) - \left(1 - p_{jk}\right) p_{jq}p_{kq}\Big] \\ &= 6x_i x_k \Delta \Big[x_i + x_k - 2p_{ik}x_i - 2p_{ik}x_k - x_i\Delta - x_k\Delta - x_j - x_k + 2p_{jk}x_j \end{aligned}$$

$$
\begin{aligned}
& +2p_{jk}x_k - x_i\Delta - x_k\frac{x_i}{x_j}\Delta + x_j - x_i - 2x_jp_{jk} + 2x_ip_{ik} + 2x_i\Delta \\
& + \sum_{q\in V(G), q\neq i,j,k} x_q\left(p_{iq} + p_{kq} - 3p_{iq}p_{kq}\right) \\
& + \sum_{q\in V(G), q\neq i,j,k} x_q\left(-p_{jq} - p_{kq} + 3p_{jq}p_{kq}\right)\Big] \\
= \; & 6x_ix_k\Delta\Big[2x_k\left(p_{jk} - p_{ik}\right) - x_k\frac{x_i + x_j}{x_j}\Delta \\
& + \sum_{q\in V(G), q\neq i,j,k} x_q\left(1 - 3p_{kq}\right)\left(p_{iq} - p_{jq}\right)\Big].
\end{aligned}
$$

Assume, without loss of generality, that $p_{jk} > p_{ik}$. Then it is possible to take Δ such that

$$
0 < \quad \Delta \quad < \frac{2x_j\left(p_{jk} - p_{ik}\right)}{x_i + x_j}.
$$

If $D_{ijk}(\Delta)$ is positive, then Z is not maximal. Therefore, $D_{ijk}(\Delta)$ is negative and so the sum $\sum_{q\in V(G), q\neq i,j,k} x_q\left(1 - 3p_{kq}\right)\left(p_{iq} - p_{jq}\right)$ must be negative. Let q be a vertex that gives a negative contribution to this sum. If $p_{kq} < \frac{1}{3}$ then $p_{jq} > p_{iq}$. But then p_{iq} and p_{ik} are both less than 1, so that ikq contains a copy of K_3. Therefore, $p_{kq} > \frac{1}{3}$ and $p_{iq} > p_{jq}$.

We define two sets of vertices in Z: $J = \{a : p_{ja} > p_{ia}\}$ and $I = \{b : p_{ib} > p_{jb}\}$. Note that both I and J are non-empty: $k \in J$ and $q \in I$. Note also that for all $a_1, a_2 \in J$, $p_{a_1a_2} = 1$ (or otherwise ia_1a_2 contains a copy of K_3) and that for all $b_1, b_2 \in I$, $p_{b_1b_2} = 1$ (or otherwise jb_1b_2 contains a copy of K_3). We next determine p_{ab} for all $a \in J$ and $b \in I$.

Let β_k be the weighted average of the factors $(1-3p_{kb})$ $(b \in I)$ for any vertex $k \in J$ and α_k the weighted average of the factors $(1 - 3p_{ka})$ $(a \in J)$ for any vertex $k \in I$, so

$$
\begin{aligned}
\beta_k &= \frac{\sum_{b\in I}\left(1 - 3p_{kb}\right)x_b\left(p_{ib} - p_{jb}\right)}{\sum_{b\in I} x_b\left(p_{ib} - p_{jb}\right)} \\
\alpha_k &= \frac{\sum_{a\in J}\left(1 - 3p_{ka}\right)x_a\left(p_{ia} - p_{ja}\right)}{\sum_{a\in I} x_a\left(p_{ia} - p_{ja}\right)}
\end{aligned}
$$

and $1 \geq \beta_k, \alpha_k \geq -2$.

Then, for any vertex $k \in J$

$$\sum_{a\in J} 2x_a\left(p_{ja}-p_{ia}\right)+\beta_k\sum_{b\in I} x_b\left(p_{ib}-p_{jb}\right)\leq 0$$

or else Z would not be maximal. Both of the summands are positive. And similarly for $q \in I$:

$$\sum_{b\in I} 2x_b\left(p_{jb}-p_{ib}\right)+\alpha_k\sum_{a\in J} x_b\left(p_{ia}-p_{ja}\right)\geq 0$$

and both of the summands are negative. Accordingly

$$\begin{aligned}\sum_{a\in J} 2x_a\left(p_{ja}-p_{ia}\right)+\beta_k\sum_{b\in I} x_b\left(p_{ib}-p_{jb}\right) &\leq \sum_{b\in I} 2x_b\left(p_{jb}-p_{ib}\right)+\alpha_k\sum_{a\in J} x_a\left(p_{ia}-p_{ja}\right)\\ \left(2+\alpha_k\right)\sum_{a\in J} x_a\left(p_{ja}-p_{ia}\right) &\leq -\left(2+\beta_k\right)\sum_{b\in I} x_b\left(p_{ib}-p_{jb}\right)\end{aligned}$$

and, as both summands are positive, it follows that $\alpha_k = \beta_k = -2$ (so that $p_{ab} = 1$ for all $a \in J$, $b \in I$) and also

$$\sum_{a\in J} x_a\left(p_{ja}-p_{ia}\right) = -\sum_{b\in I} x_b\left(p_{jb}-p_{ib}\right)$$

or

$$\sum_{a\in J\cup I} x_a\left(p_{ja}-p_{ia}\right) = 0.$$

It follows that, for any $k \in I \cup J$,

$$D_{ijk}(\Delta) = -6x_i x_k^2\Delta^2\frac{x_i+x_j}{x_j}.$$

We now consider a similar transformation applied to $k \in J$ and $q \in I$ simultaneously. Z' is identical to Z except that p_{ik} is replaced with $p_{ik}+\Delta_1$ and p_{jk} with $p_{jk}-\Delta_1\frac{x_i}{x_j}$ and Z'' is identical to Z' except that p_{iq} is replaced with $p_{iq}+\Delta_2$ and p_{jq} with $p_{jq}-\Delta_2\frac{x_i}{x_j}$. Note that, as per above, the edge densities of Z, Z' and Z'' are equal. We calculate the change in density of e_3 in the transformation from Z to Z'' as follows:

$$\begin{aligned}d_{e_3}(Z'')-d_{e_3}(Z) &= d_{e_3}(Z'')-d_{e_3}(Z')+d_{e_3}(Z')-d_{e_3}(Z)\\ &= D'_{ijq}(\Delta_2)+D_{ijk}(\Delta_1)\end{aligned}$$

where

$$D'_{ijq} = d_{e_3}(Z'')-d_{e_3}(Z')$$

and is calculated as follows

$$
\begin{aligned}
D'_{ijq}(\Delta_2) &= 6x_ix_q\left(x_i+x_q\right)\Big[p_{iq}+\Delta_2-\left(p_{iq}+\Delta_2\right)^2-\left(p_{iq}-p_{iq}^2\right)\Big] \\
&\quad +6x_jx_q\left(x_j+x_q\right)\Big[p_{jq}-\Delta_2\frac{x_i}{x_j}-\left(p_{jq}-\Delta_2\frac{x_i}{x_j}\right)^2-\left(p_{jq}-p_{jq}^2\right)\Big] \\
&\quad +6x_ix_jx_q\Big[\left(p_{iq}+\Delta_2\right)+\left(p_{jq}-\frac{x_i}{x_j}\Delta_2\right)-2\left(p_{iq}+\Delta_2\right)\left(p_{jq}-\frac{x_i}{x_j}\Delta_2\right) \\
&\quad -\left(p_{iq}+p_{jq}-2p_{iq}p_{jq}\right)\Big] \\
&\quad +6x_ix_qx_k\Big[\left(p_{iq}+\Delta_2+p_{ik}+\Delta_1-2\left(p_{iq}+\Delta_2\right)\left(p_{ik}+\Delta_1\right)\right) \\
&\quad -\left(p_{iq}+p_{ik}+\Delta_1-2p_{iq}\left(p_{ik}+\Delta_1\right)\right)\Big] \\
&\quad +6x_jx_qx_k\Big[\left(p_{jq}-\Delta_2\frac{x_i}{x_j}+p_{jk}-\Delta_1\frac{x_i}{x_j}-2\left(p_{jq}-\Delta_2\frac{x_i}{x_j}\right)\left(p_{jk}-\Delta_1\frac{x_i}{x_j}\right)\right) \\
&\quad -\left(p_{jq}+p_{jk}-\Delta_1\frac{x_i}{x_j}-2p_{jq}\left(p_{jk}-\Delta_1\frac{x_i}{x_j}\right)\right)\Big] \\
&\quad +6x_ix_q\Delta_2\sum_{m\in I\cup J\setminus\{k,q\}}2x_m\left(p_{jm}-p_{im}\right) \\
&= 6x_ix_q\Delta_2\Big[x_i+x_q-2x_ip_{iq}-2x_qp_{iq}-x_i\Delta_2-x_q\Delta_2-x_j-x_q \\
&\quad +2x_jp_{jq}+2x_qp_{jq}-x_i\Delta_2-\frac{x_qx_i}{x_j}\Delta_2+x_j-x_i-2x_jp_{jq}+2x_ip_{iq}+2x_i\Delta_2 \\
&\quad +x_k-2x_kp_{ik}-2x_k\Delta_1-x_k+2x_kp_{jk}-2\frac{x_kx_i}{x_j}\Delta_1 \\
&\quad +\sum_{m\in I\cup J\setminus\{k,q\}}2x_m\left(p_{jm}-p_{im}\right)\Big] \\
&= 6x_ix_q\Delta_2\Big[2x_q\left(p_{jq}-p_{iq}\right)+2x_k\left(p_{jk}-p_{ik}\right)+\sum_{m\in I\cup J\setminus\{k,q\}}2x_m\left(p_{jm}-p_{im}\right) \\
&\quad -x_q\frac{x_i+x_j}{x_j}\Delta_2-2x_k\frac{x_i+x_j}{x_j}\Delta_1\Big] \\
&= -6x_ix_q\frac{x_i+x_j}{x_j}\Big[x_q\Delta_2^2+2x_k\Delta_1\Delta_2\Big].
\end{aligned}
$$

Combining this with D_{ijq} gives

$$
\begin{aligned}
d_{e_3}(Z'')-d_{e_3}(Z) &= D'_{ijq}(\Delta_2)+D_{ijk}(\Delta_1) \\
&= -6x_i\frac{x_i+x_j}{x_j}\Big[x_q^2\Delta_2^2+2x_kx_q\Delta_1\Delta_2+x_k^2\Delta_1^2\Big]
\end{aligned}
$$

$$= \quad -6x_i \frac{x_i + x_j}{x_j} \left[x_k \Delta_1 + x_q \Delta_2\right]^2$$

which is zero if we set $\Delta_2 = -\frac{x_k}{x_q}\Delta_1$.

Set

$$\Delta_1 \quad = \quad \min\left(\frac{x_j\left(p_{jk} - p_{ik}\right)}{x_i + x_j}, \frac{x_q x_j\left(p_{iq} - p_{jq}\right)}{x_k\left(x_i + x_j\right)}\right)$$

and note that $\Delta_1 > 0$. Applying this transformation gives $Z'' \in \mathcal{Z}$ and either $p_{jk} = p_{ik}$ or $p_{iq} = p_{jq}$ (or both). This process can then be repeated until there are fewer than two vertices k where $p_{ik} \neq p_{jk}$. But, if there is one such vertex, the graph is not maximal (and therefore Z is not minimal). If there are zero such vertices, i and j are clones and the graph is not minimal (and therefore Z is not minimal). This contradiction establishes the lemma. □

The proof of Theorem 4.1 follows quickly from these lemmas.

Second proof of Theorem 4.1. Let H_1 be a graph with a fixed density of $\bar{P}_3$ and maximal edge density. Consider the weighted graph $G_1 \in \mathcal{W}^*$ with $V(H_1)$ vertices of weight $1/V(H_1)$ and with edge densities of 1 on the same edge-set as H_1. G_1 is clearly a model of H_1. If G_1 is not minimal, there is a graph $G_2 \in \mathcal{W}^*$ that is minimal with $d(G_2) = d(G_1)$ and $d_{\bar{P}_3}(G_1) = d_{e_3}(G_2)$ and which is also a model of H_1 (otherwise we set $G_2 = G_1$) . Apply Lemma 4.7 to G_2. It follows that the underlying graph of G_2 has no zero edge weights.

The only non-trivial weighted graph $G_2 \in \mathcal{W}^*$ with no zero edge weights has two vertices and a single edge. Let $G_2 = \{1, 2\}$ with $x_1 = k$, $x_2 = 1 - k$ and $a_{12} = q$. Then we have

$$\begin{aligned} d(H_1) = d(G_2) &= k(1-k)q \\ d_{\bar{P}3}(H_1) = d_{\bar{P}_3}(G_2) &= 6k(1-k)q(1-q) \end{aligned}$$

and, for any given $\bar{P}_3$ density, the edge density is maximised by taking $k = 1/2$. Accordingly, the extremal weighted graph H_1 is $K_2(p)$ and the result follows from Lemma 4.6. □

4.4. Discussion

We conjecture that the extremal graph for $\pi_{gen*}(K_t, \{K_t^{-j} : 2 \leq j \leq t-1\}, k, K_2)$ is $K_{t-1}(p)$, the blow up of K_{t-1} where edges are present with probability p. The linear inequalities generated using the flag algbra proofs are consistent with this, but the correct function is clearly not linear. Furthermore, the inequality is built up from a number of intermediate graphs - these are not all included in the single linear result, apart from in the equation for K_4 which is dealt with separately. If they were, it might be improved. Overall, the second approach - using analysis of weighted graphs - appears to be more promising as a route to the general result. As yet, we have only been able to apply it to K_3.

The functions π_{gen} and π_{gen*} embody information about the Turán density, extremal graphs, stability and the characteristics of a family of graphs. Where the result is a linear approximation, this is equivalent to a stability result: essentially, the approximation is close to the exact value for small values of the forbidden subgraph densities. Where the result includes non-linear terms, this is equivalent to a description of the entire family of graphs satisfying these criteria and correspondingly more informative than a stability result.

Chapter 5

A new upper bound for the density Turán problem

5.1. Introduction

Let H be a simple, connected graph. Define $\Delta(H)$ as the maximum degree of H. A blow-up of H, denoted as $H(N)$, is a graph that contains a set of N independent vertices corresponding to each vertex in H where vertices in different vertex sets are connected if the corresponding vertices in H are connected; that is, for all vertices $v, w \in V(H)$ there correspond sets of independent vertices $A_v, A_w \in V(H(N))$ such that $|V(A_v)| = |V(A_w)| = N$ and $\forall a \in A_v, b \in A_w \; vw \in E(H) \leftrightarrow ab \in E(H(N))$.

In this chapter, we will be considering subgraphs of $H(N)$. Accordingly, define the density between two sets of vertices A_i and A_j as

$$d(A_i, A_j) \equiv \frac{|E(A_i, A_j)|}{|A_i|\,|A_j|}$$

where $E(A_i, A_j)$ is the set of edges between A_i and A_j. Then for vertex sets A_v, A_w in $H(N)$, $vw \in E(H) \rightarrow d(A_v, A_w) = 1$.

A graph is $\mathcal{F}$-free if it does not contain a subgraph isomorphic to any member of $\mathcal{F}$. Turán-type problems study properties of graphs that are $\mathcal{F}$-free for certain fixed classes of graphs $\mathcal{F}$. The blow-up $H(N)$ is the paradigm example of a graph that is not H-free. The question naturally arises how many edges can be deleted from $H(N)$ before it is H-free. Adding the constraint that the same proportion of edges are deleted from each connected pair of vertex sets gives what has become known as the *Turán density problem* (see, for example, [**6**]).

Specifically, we define the family of graphs $\mathcal{H}_\alpha(N)$, where $G \in \mathcal{H}_\alpha(N)$ if:

- for each vertex $v \in V(H)$, G contains an independent set of N vertices A_v;
- for all vertices $a \in A_v$ and $b \in A_w$, $vw \notin E(H) \Rightarrow ab \notin E(G)$;
- $d(A_v, A_w) \geq \alpha$ for all $vw \in E(H)$.

A transversal of $G \in \mathcal{H}_\alpha(N)$ is a mapping $\varphi : V(H) \longmapsto V(G)$ such that $\varphi(v) \in A_v$ and $\forall v, w \in V(H)\ vw \in E(H) \to \varphi(v)\varphi(w) \in E(G)$. In other words, $\varphi(V(H))$ is isomorphic to H and so there is a transversal if H is an induced subgraph of G.

The Turán transversal number is the minimum value of α such that a transversal exists for all $G \in \mathcal{H}_\alpha(N)$:

$$\mathsf{ex}_d(H, N) = \min(\alpha : \text{there is a transversal of } G \in \mathcal{H}_\alpha(N))$$

and the Turán transversal density is defined accordingly:

$$\pi_d(H) = \limsup_N (\mathsf{ex}_d(H, N).$$

Note that, by virtue of Lemma 2.1 of [**20**], the Turán transversal number is a non-increasing function of N and so the Turán transversal density is well-defined.

Various bounds have been established for specific graphs, such as trees (see [**6**]) and certain unicyclic graphs (see [**3**]). The best upper bound for the general case, obtained in [**6**], is $1 - 1/(4(\Delta(H) - 1))$. Using the entropy compression technique supplemented with some analytic methods, we derive a different upper bound of $1-1/(\gamma(\Delta(H)-\beta))$, where $3 \leq \gamma < 4$ and $\beta \leq 1$. The new bound asymptotically approaches the existing best upper bound, but is derived in a completely different way.

5.2. Entropy Compression

The first main result is as follows:

THEOREM 5.1. *Given a graph H with maximum degree Δ, let $\alpha = x$ be the solution to $\sum_{i=0}^{\Delta}(i-1)x^i = 0$ where $0 < \alpha \leq 1$ and let $\gamma = \sum_{i=0}^{\Delta} \alpha^i/\alpha$. Then $\pi_d(H) \leq 1 - 1/\gamma\Delta$.*

This result can be refined using analytic techniques to produce the following:

THEOREM 5.2. *Given a graph H with maximum degree Δ, $\pi_d(H) \leq \eta$ where η is defined as follows:*

- *Set $\phi_\Delta(x) = 1 + \sum_{i=1}^{\Delta} x^i$.*
- *Set α as the solution to $x\phi'_\Delta(x) - \phi_\Delta(x) = 0$ with $0 < \alpha \leq 1$.*
- *Set*

$$\gamma = \frac{\phi_\Delta(\alpha)}{\alpha},$$

and define the vectors

$$\begin{aligned} \mu_i &= \left(\frac{\alpha^i}{\phi_\Delta(\alpha)}\right) \\ \mathbf{c_i} &= \left(\log\left(\frac{\Delta!}{(\Delta-i)!}\right)\right) \end{aligned}$$

and the (covariance) matrix

$$\mathbf{\Sigma_{ij}} = \mu_i\delta_{ij} - \mu_i\mu_j - \frac{\alpha^{i+j-2}(i-1)(j-1)}{\phi(\alpha)\phi''(\alpha)}$$

where, in each case, i, j run from 0 to Δ.

- *Set*

$$\beta = \Delta - e^{\mathbf{c_i}\mu_i + \mathbf{c_i}\mathbf{\Sigma_{ij}}\mathbf{c_j^T}}$$

and

$$\eta = 1 - \frac{1}{\gamma(\Delta - \beta)}.$$

The structure of the proof is as follows. An algorithm is given that builds up a mapping that, if the algorithm terminates, constitutes a transeversal.

The algorithm consumes a vector of random entries and keeps a record of its actions, from which the original vector can be reconstructed. We show that each record corresponds to a unique vector and that the number of possible records is eventually less than the number of possible vectors, so that there must be one vector for which the algorithm terminates. Another way of looking at this is that the record constitutes a compression of the original vector and that, if the algorithm does not terminate, eventually the compression has less entropy than the original random vector, which is impossible, hence the characterisation of this technique as *entropy compression*.

5.3. Algorithm

Consider the following algorithm, which takes as input a graph $G \in \mathcal{H}_\eta(N)$ and a large vector of random entries $\mathcal{Z}_t = (z_i)_{i \leq t}$, where each z_i is a random variable with integer values selected uniformly from $[1, N]$:

5.3.1. Step 1

Give a fixed ordering to the vertex classes A_v (or, equivalently, the vertices $V(H)$) and also give, for each vertex set A_i, separate fixed orderings to the individual vertices in that vertex set, labelling the vertices $A_i^1 \dots A_i^N$. Set $1 \to c$ and k as the vertex in H with the lowest index and create an empty vector $\mathcal{R}_t = (r_i)_{i \leq t}$ and the empty mapping φ_0.

5.3.2. Step 2

Set $\varphi_c = \varphi_{c-1} \cup (k \to A_k^{z_c})$; that is, the homomorphism φ_c is φ_{c-1} together with the mapping from k to the vertex in A_k with index z_c.

5.3.3. Step 3

Determine whether there are any missing edges in φ_c; that is whether there exists any vertex $v \in \mathrm{Dom}(\varphi_c)$ such that $vk \in E(H)$ and $\varphi_c(v)A_k^{z_c} \notin E(G)$. If there is no missing edge, set $r_c = 0$, set k as the vertex in H with the lowest index such that $k \notin \mathrm{Dom}(\varphi_c)$ and proceed to Step 6. Otherwise, proceed to step 4.

5.3.4. Step 4

Set as r_c the ordered pair (p, q) formed as follows. Pick any vertex $a = \varphi_c(v) \in A_v$ that was identified as part of a missing edge in Step 3. Index the neighbours of k in H from 1 to Δ using the order derived from the fixed ordering of the vertices of H. Then p is the derived index of v in H. Next we index all the missing edges between A_v and A_k using the fixed order of each of the vertices in A_v and A_k – say, by ordering the missing edges using the lexicographic ordering on A_v, A_k. Then q is the index of the missing edge between a and $A_k^{z_c}$. Note that there are exactly $(1-\eta)N^2$ missing edges between any two vertex classes.

5.3.5. Step 5

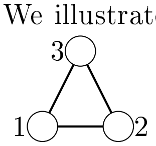
Delete from φ_c both the mappings from v and k.

5.3.6. Step 6

If the homomorphism φ_c is complete – that is, if $\mathrm{Dom}(\varphi_c) = V(H)$ – then set $\varphi = \varphi_c$ and terminate. Otherwise, increment c by 1 (k is unchanged) and go back to Step 2.

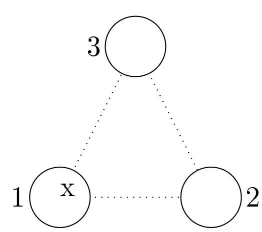
5.4. Example Run-Through of the Algorithm

We illustrate operation of the algorithm with a simple example. Let $H =$



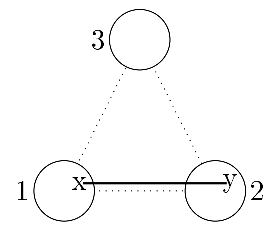
.

After the first iteration of the algorithm, G is



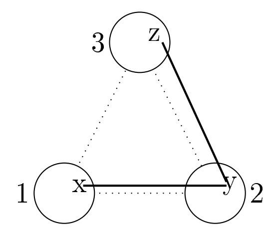
with $\varphi_1(1) = x$.

After the second iteration of the algorithm, G is

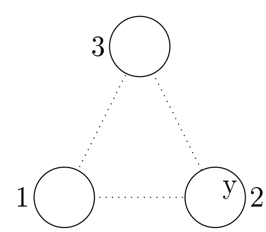


with $\varphi_2(1) = x$ and $\varphi_2(2) = y$.

In the third iteration, vertex z is added to give the graph



As $13 \in E(H)$ but $xz = (\varphi_3(1), \varphi_3(3)) \notin E(G)$, x and z are deleted from φ_3. The neighbours of 3 in H are 1 and 2, so the derived index of vertex 1 is 1. Using an ordering of edges between A_1 and A_3 gives an index for xz, say q. Then $r_3 = (1, q)$. After the third iteration of the algorithm, G is



and in the fourth iteration of the algorithm a vertex will be added from vertex set 3. Also, $\mathcal{R}_3 = (0, 0, (1, q))$.

5.5. Analysis of the Algorithm

If the algorithm terminates, then φ is a transversal of H. The algorithm only terminates when the domain of φ is all of $V(H)$ and, by construction, there are no required edges missing.

We now show that $\mathcal{R}_t$ (the record made at all stages up to and including t) and φ_t (φ after stage t) uniquely determine Z_t.

LEMMA 5.3. *For all i, the domain of φ_i and the vertex class to be considered at stage $i+1$ is uniquely determined by $\mathcal{R}_i$.*

PROOF. We proceed by induction. After stage 1, $r_1 = 0$, the domain of φ_1 consists of the lowest indexed vertex in $V(H)$ and the next vertex to be considered is the second lowest indexed vertex in $V(H)$. Now assume that, given $\mathcal{R}_{i-1}$, we have determined the domain of φ_{i-1} and the vertex to be considered at stage i, say v. There are two cases. If $r_i = 0$, $\mathrm{Dom}(\varphi_i) = \mathrm{Dom}(\varphi_{i-1}) \cup v$ and the next vertex to be considered is the lowest indexed vertex not in $\mathrm{Dom}(\varphi_i)$. If $r_i = (p, q)$, we determine the pth neighbour of v in H using the indexing derived from the underlying numbering of the vertices in H and denote this neighbour as w. Then $\mathrm{Dom}(\varphi_i) = \mathrm{Dom}(\varphi_{i-1}) \setminus w$ and the next vertex to be considered is v, ie, the same vertex as at stage i. $\square$

LEMMA 5.4. *The mapping from $\mathcal{Z}_t$ to $(\mathcal{R}_t, \varphi_t)$ is injective.*

PROOF. The aim is to show that the record $\mathcal{R}_t$ and the mapping φ_t uniquely determine $\mathcal{Z}_t$. We proceed by induction. After stage 1, $\mathrm{Dom}(\varphi_1) = v$ is a single vertex and z_1 is the index of $\varphi_1(v)$ in vertex set A_v.

Assume that $t \geq 2$ and that $\mathcal{Z}_{t-1}$ may be determined from $\mathcal{R}_{t-1}$ and φ_{t-1}. Given r_t and φ_t (but not φ_{t-1}), to complete the induction it is necessary to find z_t and φ_{t-1}. By Lemma 1, from $\mathcal{R}_{t-1}$, we know $\mathrm{Dom}(\varphi_{t-1})$ and the vertex v to be considered at stage t. If $r_t = 0$, then φ_{t-1} is φ_t with the removal of the single entry for v and z_t is the index of $\varphi_t(v)$ in A_v. Otherwise, $r_t = (p, q)$. Recall that p refers to a vertex class using the index derived from the neighbours of v and the underlying order on the vertices of H, say w, and then q refers to the non-edge between vertex sets A_v and A_w using the index derived from the underlying order on the vertices of those vertex sets. This gives sufficient information to determine the vertex in A_v that was selected at stage t, thereby determining z_t, and also the vertex in A_w that was removed at stage t, so that φ_{t-1} is φ_t with the addition of the mapping from w to that vertex. In both cases, for $t \geq 2$, we have determined z_t and φ_{t-1}. The induction is complete. $\square$

Let $\mathscr{S}_t$ be the set of vectors $\mathcal{Z}_t$ such that after step t of the algorithm the mapping is not complete and let $\mathscr{F}_t$ be the set of all vectors $\mathcal{Z}_t$. Clearly, $|\mathscr{F}_t| = N^t$ and $|\mathscr{S}_t| \leq |\mathcal{F}_t|$. If the inequality is strict, then there is a vector input that terminates by stage t – in other words, there is a transversal.

Let $\mathscr{R}_t$ be the family of all possible records at stage t that can be produced by an input from $\mathscr{S}_t$. Pairs from $(\mathscr{R}_t, \varphi_t)$ correspond to incomplete mappings and so, as a consequence of Lemma 2:

$$|\mathscr{S}_t| \quad \leq \quad (N+1)^{|V(H)|} \, |\mathscr{R}_t| \, .$$

Therefore, if, for large enough t, $(N+1)^{|V(H)|} \, |\mathscr{R}_t| < N^t$ then $\mathscr{S}_t < \mathscr{F}_t$ and there is a transversal. It remains to provide an upper bound for $|\mathscr{R}_t|$.

5.6. Computing $\mathscr{R}_t$

We define a series of mappings of a record $\mathcal{R}_t$. Recall that $\mathcal{R}_t = (0, 0, (p_3, q_3)..., 0, ...0, (p_t, q_t))$, a vector consisting of a series of entries consisting of either 0 or a pair of integers. Define $\mathcal{R}_t^*$ as the mapping that replaces each pair with the digit 1. So $\mathcal{R}_t^*$ looks like $(0, 0, 1, ...0, ..., 0, 1)$. Then define $\mathcal{R}_t^\bullet$ as the mapping that concatenates $\mathcal{R}_t^*$. So $\mathcal{R}_t^\bullet$ looks like 001...0...01.

Our first task is to count $|\mathscr{R}_t^\bullet|$, the family of all possible $\mathcal{R}_t^\bullet$. To do this, note that each 0 corresponds to addition of a vertex to the mapping and each 1 corresponds to deletion of a vertex from the mapping. It follows that, for each prefix of the sequence, there are at least as many 0s as 1s. This property defines the sequences known as *Dyck words*. In fact, they are partial Dyck words, in that the number of 0s and 1s in a complete sequence may not be equal, but will differ by a maximum of $|V(H)|$. Furthermore, there is an additional constraint in that the maximum descent – the maximum length of a consecutive sequence of 1s – is Δ. Let $C_{y,E}$ be the number of Dyck words with length $2y$ and all descents in E. We wish to determine $C_{t/2,[\Delta]}$.

Asymptotics for generalised Dyck words are considered in [**10**] and the following Lemma is a restatement of Lemma 8 of [**10**]:

LEMMA 5.5. *Let $E \neq \{1\}$ be a non-empty set of nonnegative integers. Define $\phi_E(x) = 1 + \sum_{i \in E} x^i$. If $\phi_E(x) - x\phi'_E(x) = 0$ has a solution $x = \alpha$ with $0 < \alpha < R$, where R is the radius of convergence of ϕ_E, then α is the unique* *solution of the equation in the open interval* $(0, R)$*. Moreover, there is a constant* c_E *such that* $C_{t,E} \leq c_E \gamma^t t^{-3/2}$*, where* $\gamma = \phi'_E(\alpha) = \phi_E(\gamma)/\gamma$*.*

In the case where $E = [\Delta]$,

$$\begin{aligned}
\phi_E(x) &= 1 + \sum_{i=1}^{\Delta} x^i \\
&= \frac{1 - x^{\Delta+1}}{1 - x} \\
x\phi'_E(x) &= \sum_{i=1}^{\Delta} ix^i \\
&= x \; \frac{-(1-x)(\Delta+1)x^{\Delta} + (1 - x^{\Delta+1})}{(1-x)^2} \\
&= \frac{\Delta x^{\Delta+2} - (\Delta+1)x^{\Delta+1} + x}{(1-x)^2} \\
\phi_\Delta(x) - x\phi'_\Delta(x) &= 1 + \sum_{i=1}^{\Delta}(1-i)x^i \\
&= \frac{(1 - x^{\Delta+1})(1-x) - \Delta x^{\Delta+2} + (\Delta+1)x^{\Delta+1} - x}{(1-x)^2} \\
&= \frac{1 - x^{\Delta+1} - x + x^{\Delta+2} - \Delta x^{\Delta+2} + (\Delta+1)x^{\Delta+1} - x}{(1-x)^2} \\
&= \frac{1 + (1-\Delta)x^{\Delta+2} + \Delta x^{\Delta+1} - 2x}{(1-x)^2}
\end{aligned}$$

and so the pre-conditions of the Lemma are satisfied and there is a constant γ. Furthermore, by Lemma 6 of [**10**], we may replace c_E with another constant c'_E to take account of the fact that R_t may be a partial Dyck word with a fixed maximum excess of 0s over 1s.

Next we determine the maximum size of the preimage of each element of $\mathcal{R}_t^\bullet$ in the mapping from $\mathcal{R}_t$ to $\mathcal{R}_t^\bullet$. For each pair (p_i, q_i), p_i is an integer from 1 to Δ, and q_i is an integer from 1 to the number of non-edges between the two relevant vertex classes, which is, by construction, exactly $(1-\eta)N^2$. Therefore there are $\Delta(1-\eta)N^2$ mappings from each (p, q) to 1. There are a maximum of $t/2$ entries equal to 1 in $\mathcal{R}_t^\bullet$. Accordingly, the multiplicity of the mapping from $\mathcal{R}_t$ to $\mathcal{R}_t^\bullet$ is $\left(\Delta(1-\eta)N^2\right)^{t/2}$ and $|\mathscr{R}_t| \leq c'_E \gamma^{t/2} t^{-3/2} \left(\Delta(1-\eta)N^2\right)^{t/2}$, where γ is the constant determined in accordance with Lemma 5.5.

This enables proof of Theorem 5.1:

THEOREM 5.1. *Given a graph H with maximum degree Δ, let $\alpha = x$ be the solution to $\sum_{i=0}^{\Delta}(i-1)x^i = 0$ where $0 < \alpha \leq 1$ and let $\gamma = \sum_{i=0}^{\Delta} \alpha^i/\alpha$. Then $\pi_d(H) \leq 1 - 1/\gamma\Delta$.*

PROOF. Given a graph $G \in \mathcal{H}_\eta(N)$ with $\eta > 1 - 1/\gamma\Delta$, where γ is determined according to the statement of the Theorem, ie, as in Lemma 5.5, we need to show that there is a vector $\mathcal{Z}_t \in \{N\}^t$ that yields a transversal of G. As above, let $\mathscr{S}_t$ be the set of vectors $\mathcal{Z}_t$ for which the transversal is incomplete and $\mathscr{F}_t$ the set of all vectors $\mathcal{Z}_t$. Then

$$|\mathscr{S}_t| \quad \leq \quad (N+1)^{|V(H)|} c_E' t^{-3/2} \left(\sqrt{\gamma\Delta(1-\eta)}N\right)^t$$

and $|\mathscr{F}_t| = N^t$, so that

$$\frac{|\mathscr{S}_t|}{|\mathscr{F}_t|} \quad \leq \quad (N+1)^{|V(H)|} c_E' t^{-3/2} \left(\sqrt{\gamma\Delta(1-\eta)}\right)^t.$$

This converges to 0 as $t \to \infty$ provided that

$$\begin{aligned} \sqrt{\gamma\Delta(1-\eta)} &< 1 \\ \eta &> 1 - \frac{1}{\gamma\Delta}. \end{aligned}$$

And so the theorem is proven given the constraint on η. □

5.7. Further Development

Entropy compression relies on finding an efficient method of storing a record of the algorithm that allows it to be reconstructed. The original Step 4 uses an index of size Δ to record the relevant neighbour when, in fact, only a certain subset of those neighbours, known at that time, need be indexed. and so it does not use all available information and could be made more efficient. Accordingly, Step 4 of the algorithm can be improved by only indexing neighbours that are included within $\mathrm{Dom}(\varphi_t)$ at Step 4 of stage t. At each stage t, $\mathrm{Dom}(\varphi_t)$ is known, and so all the neighbours of a vertex within $\mathrm{Dom}(\varphi_t)$ are known. We cannot state in generality the maximum number of neighbours of a vertex that are in $\mathrm{Dom}(\varphi_t)$ at stage t, but note that for any sequence of 1s, each 1 represents deletion of a vertex from $\mathrm{Dom}(\varphi_t)$ that is a neighbour of the vertex to be considered at stage $t+1$.

This puts an upper bound of $\Delta - \beta_t$ on the neighbours of the vertex to be considered at stage t, where β_t is the length of a maximal sequence of 1s ending at stage $t-1$ (and β_t is 0 if $r^{\bullet}_{t-1} = 0$).

In particular, substitute Stage 4 of the algorithm with the following:

5.7.1. Step 4

Set r_c as the ordered pair (p, q) formed as follows. Pick any vertex $a = \varphi(v) \in A_v$ that was identified as part of a missing edge in Step 3. Form the set of neighbours of k in $\mathrm{Dom}(\varphi_c)$: that is, define $N_k = \{w \in V(H) : w \in \Gamma(k) \text{ and } w \in \mathrm{Dom}(\varphi_c))$. Note that $|N_k| \leq \Delta$. Index N_k using the order derived from the fixed ordering of $V(H)$. Set p as the corresponding index of v in N_k. Next we index all the missing edges between A_v and A_k using the fixed order of each of the vertices – say, by ordering the missing edges using the lexicographic ordering on A_v, A_k. Then q is the index of the missing edge between a and $A_k^{z_c}$. Note that there are exactly $(1-\eta)N^2$ missing edges between any two vertex classes.

5.8. Analysis of Amended Algorithm

Lemma 5.4 applies to the amended algorithm. In particular, the reasoning for Lemma 5.3 proceeds as before, except that we use the domain of φ_t and the underlying ordering on $V(H)$ to reconstruct N_k instead of $\Gamma(k)$. Furthermore, for each pair (p_t, q_t), the maximum size of p_t is $\Delta - \beta_t$, where β_t is determined as above. So the multiplicity of each mapping from (p_t, q_t) to 1 is less than $(\Delta - \beta_t)(1-\eta)N^2$.

In order to calculate the product of the $(\Delta - \beta_t)(1-\eta)N^2$, recalling that β_t is the length of a maximal sequence of 1s ending at stage $t-1$, it is necessary to determine the distribution of lengths of maximal sequences of 1s in Dyck words. The necessary information is given by this Lemma:

LEMMA 5.6. *Define $\phi_\Delta(x) = 1 + \sum_{i=1}^{\Delta} x^i$. Let α be the solution to $\phi_\Delta(x) - x\phi'_\Delta(x) = 0$ with $0 < \alpha < R$, where R is the radius of convergence of ϕ_E. Take a Dyck word, W, of length $2n$ with no sequence of 1s greater than length Δ chosen uniformly at random from all such Dyck words of length $2n$. Define* *the random vector $(\mathbf{X_i})_n$ such that there are iX_in 1s contained in maximal sequences of length i in W. Define*

$$\begin{aligned}\mu_i &= \left(\frac{\alpha^i}{\phi(\alpha)}\right)\\ \Sigma_{ij} &= \mu_i\delta_{ij} - \mu_i\mu_j - \frac{\alpha^{i+j-2}(i-1)(j-1)}{\phi(\alpha)\phi''(\alpha)}\end{aligned}$$

and define the random variable

$$\mathbf{Z_i} \sim \mathcal{N}(0, \mathbf{\Sigma}).$$

Then $\mathbb{E}(nX_{i,n}) = n\mu_i + O(1)$ and $\mathbb{COV}(nX_{i,n}) = n\Sigma_{ij} + O(1)$ and $\sqrt{n}(\mathbf{X_{i,n}} - \mathbb{E}(\mathbf{X_{i,n}}))$ converges in distribution to $\mathbf{Z_i}$.

Remark. A sequence X_n converges in distribution to a random variable X if $\lim_{n\to\infty} \mathbb{P}\{X_n \le x\} = \mathbb{P}(X \le x)$ holds for all points of continuity (and the random variables considered here are continuous). It is denoted by $X_n \xrightarrow{d} X$. We have the following additional facts about convergence in distribution:

(1) The multivariate random variable $\mathbf{X_n}$ converges in distribution to $\mathbf{X}$ if $\mathbf{tX_n} \xrightarrow{d} \mathbf{tX}$ for all constant vectors $\mathbf{t}$.

(2) If X_n converges in distribution to X, then for any continuous bounded function F, $F(X_n) \xrightarrow{d} F(X)$ and, in particular, this follows for the exponential function.

(3) For any continuous bounded function F, if $X_n \xrightarrow{d} X$, then $\lim_{n\to\infty} \mathbb{E}(F(X_n) = \mathbb{E}(F(X))$.

(4) For any continous bounded function F, $\int F(z)dX_n(z) \to \int F(z)dX(z)$.

(5) If $\sqrt{n}(X_n - \mu) \xrightarrow{d} X$, then $\sqrt{n}(X_n - \mu) = X + o_p(1)$, where o_p is order in probability (X_n is $o_p(n^k)$ if $\forall \epsilon, \delta \, \exists n_0 \, \mathbb{P}(\left|X_n/n^k\right| > \epsilon) < \delta$ for all $n > n_0$).

Proof. Consider the following bijection from Dyck words to rooted planar trees. First, swap 1s and 0s and take the mirror image. Second, take each 1 terminated list of 0s (of possibly zero length) as the out-degree of the next vertex considered in depth-first order. The result of this composition is a bijection between the out-degrees of internal nodes of rooted planar trees and the length of sequences of 1s in Dyck words. For example, starting with the following Dyck word

$$0001101001001111$$

we swap 1s and 0s

$$1110010110110000$$

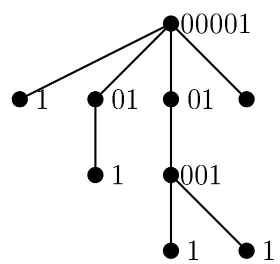
and take the mirror image

$$0000110110100111$$

giving 1 terminated lists of $0s$

$$00001\ 1\ 01\ 1\ 01\ 001\ 1\ 1$$

that corresponds to this tree



Accordingly, to prove the lemma, it is possible to consider the distribution of nodes in a random rooted planar tree of size n where no internal node has degree greater than Δ. The means to calculate this distribution is given by Theorem 2.23 in [**7**] and the statement of the Lemma is the result of those calculations. $\square$

Lemma 5.7. *Let*

$$\begin{aligned} \mathbf{c_i} &= \frac{\Delta!}{(\Delta - i)!} \\ \mathbf{lc_i} &= \log(\mathbf{c_i}) \end{aligned}$$

then

$$\mathbb{E}(\prod_{i=1}^{\Delta} c_i^{nX_i}) = e^{\mathbf{lc_i}\mu_i n + \frac{1}{2}\mathbf{lc_i}\mathbf{\Sigma_{ij}}\mathbf{lc_j} n + O(\sqrt{n})}.$$

Proof. Let $Z_i = \mathbf{lc_i}\mathbf{Z_i}$ and $W_{i,n} = \mathbf{lc_i}\mathbf{X_{i,n}}$. Then, by the facts about convergence in distribution listed above, $\sqrt{n}(W_{i,n} - \mathbb{E}(W_{i,n})) \xrightarrow{d} Z_i$. As a linear combination of normal variables forming part of a multivariate normal distribution, Z_i has the distribution $\mathcal{N}(0, \mathbf{lc_i}\mathbf{\Sigma_{ij}}\mathbf{lc_j})$. Also $\mathbb{E}(W_{i,n}n) = \mathbf{lc_i}\mu_i n + O(1)$. Next note that

$$\begin{aligned}
\prod_{i=1}^{\Delta} c_i^{nX_{i,n}} &= \prod_{i=1}^{\Delta} e^{lc_i X_{i,n} n} \\
&= e^{W_{i,n}n} \\
&= e^{\sqrt{n}\sqrt{n}(W_{i,n}-\mathbb{E}(W_{i,n}))+\mathbb{E}(W_{i,n}n)} \\
&= e^{\sqrt{n}(Z_i+o_p(1))+\mathbb{E}(W_{i,n})n} \\
&= e^{\mathbb{E}(W_{i,n}n)} e^{o_p(\sqrt{n})} e^{\sqrt{n}Z_i}.
\end{aligned}$$

Then, taking the expectation, and using the moment generating function of the normal distribution (for $Q \sim \mathcal{N}(\mu, \sigma^2)$, $\mathbb{E}(e^Q) = e^{\mu+\frac{1}{2}\sigma^2}$):

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{\Delta} c_i^{nX_{i,n}}\right) &= \mathbb{E}\left(e^{\mathbb{E}(W_{i,n})n} e^{o_p(\sqrt{n})} e^{\sqrt{n}Z_i}\right) \\
&= \mathbb{E}\left(e^{\mathbb{E}(W_{i,n}n)}\right) \mathbb{E}\left(e^{o_p(\sqrt{n})}\right) \mathbb{E}\left(e^{\sqrt{n}Z_i}\right) \\
&= e^{\mathbf{lc_i}\mu_i+O(1)} e^{O(\sqrt{n})} e^{\frac{1}{2}\mathbf{lc_i}\mathbf{\Sigma_{ij}}\mathbf{lc_j}n} \\
&= e^{\mathbf{lc_i}\mu_i n+\frac{1}{2}\mathbf{lc_i}\mathbf{\Sigma_{ij}}\mathbf{lc_j}n+O(\sqrt{n})}.
\end{aligned}$$

□

Lemmas 5.6 and 5.7 enable us to replace the β_t with a single overall average.

LEMMA 5.8. *Let $\{R_1, R_2, ...\}$ be an enumeration of $\mathscr{R}_n^{\bullet}$; that is, all Dyck words of length n with maximal sequences of 1s of length no more than Δ. Let $r_{a,b}$ be an enumeration of all the n entries in R_b. Let $\beta_{a,b}$ be the value of β_t corresponding to $r_{a,b}$ (if $r_{a,b} = 0$ then set $\beta_{a,b} = \Delta - 1$). Let $Y_{i,j}$ be such that there are $iY_{i,j}(n/2)$ 1s contained in a maximal sequence of length i in R_j. Adopt the definitions of μ_i and $\mathbf{\Sigma_{ij}}$ from Lemma 5.6 and define the following additional vectors:*

$$\begin{aligned}
\mathbf{c_i} &= \frac{\Delta!}{(\Delta-i)!} \\
\mathbf{lc_i} &= \log(c_i).
\end{aligned}$$

Let β_Δ be defined as follows:

$$\beta_\Delta = \Delta - \quad e^{\mathbf{lc_i}\mu_i+\frac{1}{2}\mathbf{lc_i}\mathbf{\Sigma_{ij}}\mathbf{lc_j}}.$$

Then, for all $\beta < \beta_\Delta$, there exists n_0 such that for all $n \geq n_0$,

$$\sum_{b=1}^{|\mathscr{R}_n^\bullet|} \prod_{a=1}^{n/2} (\Delta - \beta) \quad \geq \quad \sum_{b=1}^{|\mathscr{R}_n^\bullet|} \prod_{a=1}^{n} (\Delta - \beta_{a,b}).$$

.

Proof. Note that c_i is the geometric mean of the multiples associated with a series of 1s of length i. Then (because $n/2$ of the $\beta_{a,b}$ are equal to $\Delta - 1$)

$$\begin{aligned}\sum_{b=1}^{|\mathscr{R}_n^\bullet|} \prod_{a=1}^{n} (\Delta - \beta_{a,b}) &= \sum_{j=1}^{|\mathscr{R}_n^\bullet|} \prod_{i=1}^{\Delta} \left((c_i)^{1/i} \right)^{iY_{i,j}(n/2)} \\ &= \sum_{j=1}^{|\mathscr{R}_n^\bullet|} \prod_{i=1}^{\Delta} c_i^{Y_{i,j}(n/2)}.\end{aligned}$$

From the definition of $\mathbf{X_{i,n}}$ in Lemma 5.6, noting that $\mathbf{X_{i,n}}$ may also be seen as a random sample of $Y_{i,j}$ taken from the uniform distribution over j, it follows that

$$\sum_{j=1}^{|\mathscr{R}_n^\bullet|} \prod_{i=1}^{\Delta} c_i^{Y_{i,j}(n/2)} \quad = \quad |\mathscr{R}_n^\bullet| \, \mathbb{E} \left(\prod_{i=1}^{\Delta} c_i^{X_{i,n}(n/2)} \right).$$

Next, note that

$$\sum_{i=1}^{|\mathscr{R}_n^\bullet|} \prod_{j=1}^{n/2} (\Delta - \beta) \quad = \quad |\mathscr{R}_n^\bullet| \, (\Delta - \beta)^{n/2}$$

so that we wish to determine the β for which

$$\begin{aligned}|\mathscr{R}_n^\bullet| \, (\Delta - \beta)^{n/2} &\geq |\mathscr{R}_n^\bullet| \, \mathbb{E} \left(\prod_{i=1}^{\Delta} c_i^{X_{i,n}(n/2} \right) \\ \beta &\leq \Delta - \left(\mathbb{E} \left(\prod_{i=1}^{\Delta} c_i^{X_{i,n}(n/2)} \right) \right)^{2/n}.\end{aligned}$$

From the convergence of $\mathbf{X_{i,n}}$ to the normal distribution with parameters set out in Lemma 5.6 and applying Lemma 5.7, it follows that

$$\left(\mathbb{E} \left(\prod_{i=1}^{\Delta} c_i^{X_{i,n}(n/2)} \right) \right)^{2/n} \quad = \quad \left(e^{\mathbf{lc_i} \mu_i (n/2) + \frac{1}{2} \mathbf{lc_i} \mathbf{\Sigma_{ij}} \mathbf{lc_j} (n/2) + O(\sqrt{n})} \right)^{2/n}$$

$$= \quad e^{\mathbf{lc_i}\mu_i + \frac{1}{2}\mathbf{lc_i}\mathbf{\Sigma_{ij}}\mathbf{lc_j^T} + O(1/\sqrt{n})}$$

and so the statement is true for all β such that

$$\begin{aligned}\beta &\leq \Delta - e^{\mathbf{lc_i}\mu_i + \frac{1}{2}\mathbf{lc_i}\mathbf{\Sigma_{ij}}\mathbf{lc_j^T} + O(1/\sqrt{n})} \\ &= \beta_\Delta(1 - O(1/\sqrt{n}))\end{aligned}$$

which implies the Lemma. □

Theorem 5.2 follows from Lemma 5.8 using similar reasoning to Theorem 5.1.

THEOREM 5.2. *Given a graph H with maximum degree Δ, $\pi_d(H) \leq \eta$ where η is defined as follows:*

- *Set $\phi_\Delta(x) = 1 + \sum_{i=1}^{\Delta} x^i$.*
- *Set α as the solution to $x\phi'_\Delta(x) - \phi_\Delta(x) = 0$ with $0 < \alpha \leq 1$.*
- *Set*

  $$\gamma = \frac{\phi_\Delta(\alpha)}{\alpha},$$

  and define the vectors

  $$\begin{aligned}\mu_i &= \left(\frac{\alpha^i}{\phi_\Delta(\alpha)}\right) \\ \mathbf{c_i} &= \left(\log\left(\frac{\Delta!}{(\Delta - i)!}\right)\right)\end{aligned}$$

  and the (covariance) matrix

  $$\mathbf{\Sigma_{ij}} = \mu_i\delta_{ij} - \mu_i\mu_j - \frac{\alpha^{i+j-2}(i-1)(j-1)}{\phi(\alpha)\phi''(\alpha)}$$

  where, in each case, i, j run from 0 to Δ.
- *Set*

  $$\beta = \Delta - e^{\mathbf{c_i}\mu_i + \mathbf{c_i}\mathbf{\Sigma_{ij}}\mathbf{c_j^T}}$$

  and

  $$\eta = 1 - \frac{1}{\gamma(\Delta - \beta)}.$$

PROOF. Given a graph $G \in \mathcal{H}_\eta(N)$ with $\eta > 1 - 1/(\gamma(\Delta(H) - \beta_\Delta))$, where γ is determined as in Lemma 5.5 and β_Δ is determined as in the statement of the Theorem, we need to show that there exists a t_0 such that for all $t > t_0$, there is a vector $\mathscr{Z}_t \in \{N\}^t$ that yields a transversal of G.

There are $c'_E \gamma^{t/2} t^{-3/2}$ possible records (as set out in Lemma 5.5). Applying Lemma 5.8, for any $\beta > \beta_\Delta$, there is a t_0, such that for all $t > t_0$:

$$\begin{aligned} |\mathscr{R}_t| &\leq \sum_{j=1}^{c'_E \gamma^{t/2} t^{-3/2}} \prod_{i=1}^{t/2} \left((\Delta - \beta_{i,j})(1-\eta)N^2\right) \\ &\leq c'_E \gamma^{t/2} t^{-3/2} \left((\Delta - \beta)(1-\eta)N^2\right)^{t/2}. \end{aligned}$$

As previously, let $\mathscr{S}_t$ be the set of vectors $\mathscr{Z}_t$ for which the transversal is incomplete and $\mathscr{F}_t$ the set of all vectors $\mathscr{Z}_t$. Then

$$|\mathscr{S}_t| \leq (N+1)^{|V(H)|} c'_E t^{-3/2} \left(\sqrt{\gamma(\Delta - \beta)(1-\eta)}N\right)^t$$

and $|\mathscr{F}_t| = N^t$, so that

$$\frac{|\mathscr{S}_t|}{|\mathscr{F}_t|} \leq (N+1)^{|V(H)|} c'_E t^{-3/2} \left(\sqrt{\gamma(\Delta - \beta)(1-\eta)}\right)^t.$$

This converges to 0 as $t \to \infty$ provided that

$$\begin{aligned} \sqrt{\gamma(\Delta - \beta)(1-\eta)} &< 1 \\ \eta &> 1 - \frac{1}{\gamma(\Delta - \beta)}. \end{aligned}$$

And so the theorem is proven given the constraint on η. □

5.9. Values of the upper bound

Here we set out some values of the upper bound.

Δ	α	γ	$\Delta - \beta$	$4(\Delta(H) - 1)$	$\gamma(\Delta(H) - \beta)$
3	0.657	3.611	2.401	8	8.670
4	0.568	3.834	3.292	12	12.623
5	0.533	3.925	4.242	16	16.650
6	0.517	3.965	5.224	20	20.712
7	0.509	3.983	6.220	24	24.776
8	0.505	3.992	7.220	28	28.821
9	0.503	3.996	8.217	32	32.836
10	0.501	3.998	9.211	36	36.825
20	0.500	4.000	19.089	76	76.356
30	0.500	4.000	29.047	116	116.189

5.10. Conclusion

Entropy compression has been used here as a tool to solve an extremal Turan-type problem. It is a technique that is suited to such problems when they can be translated into an algorithmic form.

In this particular case, entropy compression leads to an upper bound for the density Turan problem that asymptotically approaches the existing best upper bound but is derived in a completely different fashion. There are two areas where this upper bound might be improved. Firstly, there is currently a free choice of vertex to be deleted when there is more than one missing edge. This suggests that the algorithm might be further compressed if this free choice were removed in a systematic way. Secondly, the proof relies only on the most basic characterstics of the graph (the maximum degree), whereas the previous proof relied on other characteristics - further exploration of those characteristics might yield an improvement to the compression algorithm.

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