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Prime ideals of the enveloping algebra of the Euclidean algebra and a classification of its simple weight modules

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A classification of the simple *weight* modules is given for the (6-dimensional) Euclidean Lie algebra $\mathfrak{e}(3) = \mathfrak{sl}_2 \ltimes V_3$. As an intermediate step, a classification of *all* simple modules is given for the centralizer C of the Cartan element H (in the universal enveloping algebra $\mathcal{U} = U(\mathfrak{e}(3))$). Generators and defining relations for the algebra C are found (there are three quadratic relations and one cubic relation). The algebra C is a Noetherian domain of Gelfand-Kirillov dimension 5. Classifications of prime, primitive, completely prime, and maximal ideals are given for the algebra $\mathcal{U}$. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4973378]

I. INTRODUCTION

The semidirect product of groups $E(3) \simeq SO(3) \ltimes \mathbb{R}^3$ is called the *Euclidean group*. The *Euclidean algebra* $\mathfrak{e}(3)$ is the complexification of the Lie algebra of $E(3)$. Various classes of modules over the Euclidean algebra have been constructed and studied by many authors.[11,19,26,28] In particular, in Ref. 11 the simple Whittaker and quasi-Whittaker $\mathfrak{e}(3)$-modules were classified. In Refs. 12 and 13, families of indecomposable representations of $\mathfrak{e}(3)$ are constructed by embedding the Euclidean algebra $\mathfrak{e}(3)$ into the simple Lie algebra $\mathfrak{sl}(4,\mathbb{C})$ and using the irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$, respectively. The Euclidean algebra $\mathfrak{e}(3)$ is a member of a more general class of Lie algebras, the so-called conformal Galilei algebras. The representation theory for these algebras was developed in Refs. 1, 2, 23, 24, and 27.

In this paper, $\mathbb{K}$ is a field of characteristic zero unless stated otherwise. The Euclidean algebra $\mathfrak{e}(3)$ is a 6-dimensional Lie algebra with basis H, E, F, X, Y, Z, and Lie bracket as follows:

$$\begin{array}{lllll} [H,E]=2E, & [H,F]=-2F, & [E,F]=H, & [H,X]=2X, & [H,Y]=-2Y, \\ [H,Z]=0, & [E,Y]=2Z, & [E,Z]=2X, & [E,X]=0, & [F,X]=Z, \\ [F,Z]=Y, & [F,Y]=0, & [X,Y]=[Y,Z]=[X,Z]=0. & & \end{array}$$

The Lie algebra $\mathfrak{e}(3)$ is neither semisimple nor solvable. It is the semidirect product $\mathfrak{e}(3) = \mathfrak{sl}_2 \ltimes V_3$ of Lie algebras where $\mathfrak{sl}_2 = \mathbb{K}H \oplus \mathbb{K}E \oplus \mathbb{K}F$ and $V_3 = \mathbb{K}X \oplus \mathbb{K}Y \oplus \mathbb{K}Z$ is an abelian Lie algebra which is the three dimensional simple $\mathfrak{sl}_2$-module. Let $\mathcal{U} := U(\mathfrak{e}(3))$ be the universal enveloping algebra of $\mathfrak{e}(3)$. Then $\mathcal{U}$ is a Noetherian domain of Gelfand-Kirillov dimension 6. A quantum analog of $\mathcal{U}$, the *quantum Euclidean algebra*, was defined and studied in Ref. 6 where its prime, completely prime, primitive, and maximal ideals were classified.

Classification of prime ideals of $\mathcal{U}$. The centre of the algebra $\mathcal{U}$ is a polynomial algebra $\mathcal{Z} = \mathbb{K}[C_1, C_2]$, where $C_1 = XY - \frac{1}{2}Z^2$ and $C_2 = EY + HZ - 2FX$ (Proposition 2.4.(2)). By a different method, this result was also obtained in Ref. 11. The vector space V_3 is a Lie ideal of $\mathfrak{e}(3)$. Hence, (V_3) is an ideal of the algebra $\mathcal{U}$ such that $\mathcal{U}/(V_3) \simeq U := U(\mathfrak{sl}_2)$ and $\text{Spec}\,(U) \subseteq \text{Spec}\,(\mathcal{U})$. Furthermore, $(V_3) = (X) = (Y) = (Z)$ (Lemma 2.6.(1)). In Section II, the following classification of prime ideals of the algebra $\mathcal{U}$ is obtained.

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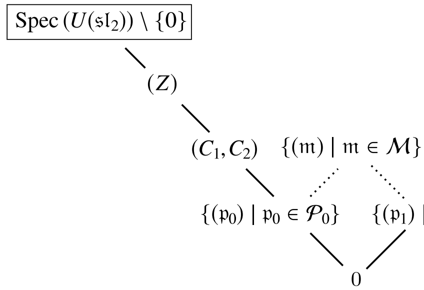
Theorem 1.1. *1.* $\mathrm{Spec}\,(\mathcal{U}) = \{(Z, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,\big(U(\mathfrak{sl}_2)\big)\} \sqcup \{(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[C_1, C_2])\}$.

2. The inclusions of prime ideals are given in the following diagram:



, (1)

where $\mathcal{M} := \mathrm{Max}(\mathbb{K}[C_1, C_2]) \setminus \{(C_1, C_2)\}$, $\mathcal{P}_0 := \{\mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[C_1, C_2]) \mid \mathrm{ht}\,(\mathfrak{p}) = 1, \mathfrak{p} \subset (C_1, C_2)\}$, *and* $\mathcal{P}_1 := \{\mathfrak{p}_1 \in \mathrm{Spec}\,(\mathbb{K}[C_1, C_2]) \mid \mathrm{ht}\,(\mathfrak{p}_1) = 1, \mathfrak{p}_1 \not\subset (C_1, C_2)\}$.

The idea of the proof is to use localizations of the algebra $\mathcal{U}$ and repeated application of Proposition 2.8. As a corollary of Theorem 1.1, the sets of maximal, primitive, and completely prime ideals of the algebra $\mathcal{U}$ are described (Corollary 2.9, Corollary 2.10, and Theorem 2.11). The algebra $\mathcal{U}$ is a free (left and right) module over the polynomial subalgebra $\mathbb{K}[C_1, C_2, H, Z]$ (Proposition 2.5). In particular, it is a free module over its centre $\mathbb{K}[C_1, C_2]$.

The prime or/and primitive ideals of various quantum algebras (and their classification) are considered in Refs. 9, 10, 14–18, and 20–22.

The centralizer $C_{\mathcal{U}}(H)$, its generators and defining relations, a classification of simple $C_{\mathcal{U}}(H)$-modules. In Section III, it is proved that, as an abstract algebra, the centralizer $C_{\mathcal{U}}(H) := \{u \in \mathcal{U} | uH = Hu\}$ of the element H in $\mathcal{U}$ is generated by elements C_1, C_2, H, Z, θ, and ϕ subject to the defining relations (Theorem 3.2) as follows:

$$[\phi, Z] = Z^2 + 2C_1, \qquad [\theta, Z] = 2\phi + (H-2)Z - C_2,$$
$$[\theta, \phi] = 2(\theta + H)Z - H\phi, \qquad \phi(\phi + HZ - C_2) = (\theta + H)(Z^2 + 2C_1),$$

where the elements C_1, C_2, and H are central. The algebra $C_{\mathcal{U}}(H)$ is a Noetherian domain of Gelfand-Kirillov dimension 5 (Theorem 3.2). An $\mathcal{U}$-module M is called a *weight module* if $M = \oplus_{\mu \in \mathbb{K}} M_\mu$, where $M_\mu = \{m \in M \mid Hm = \mu m\}$. An element $\mu \in \mathbb{K}$ such that $M_\mu \neq 0$ is called a *weight* of M. Every weight space M_μ is a module over the centralizer $C_{\mathcal{U}}(H)$. If the weight $\mathcal{U}$-module M is *simple*, then necessarily each nonzero M_μ is a *simple* $C_{\mathcal{U}}(H)$-module. Therefore, as the first step in classifying simple weight $\mathcal{U}$-modules we have to classify *all* simple $C_{\mathcal{U}}(H)$-modules. This is done in Sections V and IV, respectively, whether a simple $C_{\mathcal{U}}(H)$-module is annihilated by the element C_1 or not. These results are too technical to describe in the Introduction. Briefly, the problem of classification of simple $C_{\mathcal{U}}(H)$-modules is reduced to one but for the factor algebras $C^{\lambda_1, \lambda_2, \mu} := C_{\mathcal{U}}(H)/C_{\mathcal{U}}(H)(C_1 - \lambda_1, C_2 - \lambda_2, H - \mu)$ where $\lambda_1, \lambda_2, \mu \in \mathbb{K}$ (we assume that $\mathbb{K}$ is an algebraically closed field). It turns out that the cases $\lambda_1 \neq 0$; $\lambda_1 = 0, \lambda_2 \neq 0$; and $\lambda_1 = 0, \lambda_2 = 0$ are very different and different techniques are used in each of them. In each case, localizations of the algebra $C^{\lambda_1, \lambda_2, \mu}$ are used to partition its simple modules into torsion and torsionfree classes. A "generic" simple module depends on arbitrarily large number of independent parameters.

A classification of simple, finite dimensional $C_{\mathcal{U}}(H)$-modules is given (Theorem 3.13 and Theorem 5.3.(1)). Theorem 3.12 and Theorem 5.4 give a semisimplicity criterion for the algebra $C^{\lambda_1, \lambda_2, \mu}$.

Theorem 1.2. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Then the algebra $C^{\lambda_1, \lambda_2, \mu}$ is simple if and only if either*

1. $\lambda_1 \neq 0$ *and* $\frac{1}{2}\left(\mu \pm \frac{\lambda_2}{\sqrt{-2\lambda_1}}\right) \notin \mathbb{Z} \setminus \{0\}$ *or*
2. $\lambda_1 = 0$, $\lambda_2 \neq 0$.

Classification of simple weight $\mathcal{U}$-modules. Briefly, the problem of classification of simple weight $\mathcal{U}$-modules comprises three steps as follows: Step 1: To classify *all* simple $C_{\mathcal{U}}(H)$-modules. Step 2: How to reassemble some of the simple $C_{\mathcal{U}}(H)$-modules into a simple weight $\mathcal{U}$-module? Step 3: Isomorphism problem for simple weight $\mathcal{U}$-modules.

Step 1 was done in Sections IV and V. In Section VI, simple weight $\mathcal{U}$-modules are classified. The main idea in finding the set $\widehat{\mathcal{U}}$ (weight) of simple weight $\mathcal{U}$-modules is to use certain explicit localizations of the algebra $\mathcal{U}$ to partition the set $\widehat{\mathcal{U}}$ (weight) into various classes and then to classify modules of each class. In more detail,

$$\widehat{\mathcal{U}}\,(\text{weight}) = \widehat{\mathcal{U}}\,(\text{weight}, X\text{-torsion}) \sqcup \widehat{\mathcal{U}}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsion}) \sqcup \widehat{\mathcal{U}}\,(\text{weight}, (X,Y)\text{-torsionfree})$$

and the simple weight modules from first two sets are described in Theorem 6.13 and Proposition 6.15, respectively. The third set is a disjoint union of two subsets $\widehat{\mathcal{U}}\,(\boxed{1})$ and $\widehat{\mathcal{U}}\,(\boxed{2})$, see (56). The modules from $\widehat{\mathcal{U}}\,(\boxed{1})$ (respectively, $\widehat{\mathcal{U}}\,(\boxed{2})$) are described in Theorem 6.20 (respectively, Theorem 6.23).

In Section VI, simplicity criteria are given for the Verma modules and their dual analogs (Proposition 6.1.(3) and Proposition 6.2.(3)). Simple highest/lowest weight $\mathcal{U}$-modules are classified (Proposition 6.3 and Proposition 6.4). The *finite-infinite dimension dichotomy* was proved for simple $\mathcal{U}$-modules (Corollary 6.8): *For each simple weight $\mathcal{U}$-module, all its (nonzero) weight spaces are either finite or infinite dimensional*. Theorem 6.7 classifies all the simple weight $\mathcal{U}$-modules with finite dimensional weight spaces.

II. PRIME IDEALS OF THE ALGEBRA $\mathcal{U}$

In this section, it is proved that the centre of the algebra $\mathcal{U}$ is a polynomial algebra $\mathbb{K}[C_1, C_2]$ where C_1 and C_2 are quadratic elements of $\mathcal{U}$ (Proposition 2.4.(2)) and that the algebra $\mathcal{U}_X$ is a tensor product of three explicit algebras (Proposition 2.4.(1)). This fact is a key in finding the prime spectrum of the algebra $\mathcal{U}$ (Theorem 1.1). Explicit descriptions of the sets of maximal, primitive, and completely prime ideals of the algebra $\mathcal{U}$ are obtained (Corollary 2.9, Corollary 2.10, and Theorem 2.11).

Recall that an *involution* $*$ on a $\mathbb{K}$-algebra is a $\mathbb{K}$-algebra anti-automorphism ($(ab)^* = b^*a^*$) such that $a^{**} = a$ for all $a \in A$. The algebra $\mathcal{U}$ admits an involution $*$ defined by the rule

$$F^* = -E, \quad H^* = H, \quad E^* = -F, \quad Y^* = 2X, \quad Z^* = Z, \quad X^* = \frac{1}{2}Y. \tag{2}$$

The automorphism ι: The algebra $\mathcal{U}$ admits automorphisms

$$\iota: \quad E \mapsto F, \quad H \mapsto -H, \quad F \mapsto E, \quad X \mapsto -\frac{1}{2}Y, \quad Z \mapsto -Z, \quad Y \mapsto -2X, \tag{3}$$

$$\gamma: \quad E \mapsto E, \quad H \mapsto H, \quad F \mapsto F, \quad X \mapsto -X, \quad Z \mapsto -Z, \quad Y \mapsto -Y, \tag{4}$$

$$\iota\gamma: \quad E \mapsto F, \quad H \mapsto -H, \quad F \mapsto E, \quad X \mapsto \frac{1}{2}Y, \quad Z \mapsto Z, \quad Y \mapsto 2X. \tag{5}$$

Clearly, $\iota\gamma = \gamma\iota$ and $\iota^2 = \gamma^2 = (\iota\gamma)^2 = \mathrm{id}_{\mathcal{U}}$. The universal enveloping algebra $\mathcal{U} = U(\mathfrak{e}(3))$ admits the canonical involution κ given by the rule $\kappa(e) = -e$ for all $e \in \mathfrak{e}(3)$. Clearly,

$$\iota = \kappa \circ *. \tag{6}$$

Recall that the nth *Weyl algebra* $A_n = A_n(\mathbb{K})$ is an associative algebra generated by elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to the defining relations: $[x_i, x_j] = 0$, $[y_i, y_j] = 0$, and $[y_i, x_j] = \delta_{ij}$, where $[a, b] := ab - ba$ and δ_{ij} is the Kronecker delta function. The Weyl algebra A_n is a central, simple Noetherian domain of Gelfand-Kirillov dimension $2n$. For an algebra R, we denote by $Z(R)$ its centre. For a subset $S \subset R$, we denote by (S) the ideal of R generated by the elements of S. An element $a \in R$ is called a *normal element* if $aR = Ra$.

Lemma 2.1. [Ref. 25, Lemma 14.6.5] Let B be a $\mathbb{K}$-algebra, $S = B \otimes A_n$ be the tensor product of the algebra B and the Weyl algebra A_n, δ be a $\mathbb{K}$-derivation of S, and $T = S[t;\delta]$. Then there exists an element $s \in S$ such that the algebra $T = B[t';\delta'] \otimes A_n$ is a tensor product of algebras where $t' = t + s$ and $\delta' = \delta + \mathrm{ad}_s$.

Next, we consider two subalgebras $\mathcal{E}$ and $\mathcal{A}$ of $\mathcal{U}$ that are used in finding the prime spectrum of $\mathcal{U}$.

The subalgebra $\mathcal{E}$. Let $\mathcal{E}$ be the subalgebra of $\mathcal{U}$ generated by the elements E, X, Y, and Z. Then $\mathcal{E}$ is an iterated Ore extension $\mathcal{E} = \mathbb{K}[X,Z][E;\delta_1][Y;\delta_2]$, where $\delta_1(X) = 0$, $\delta_1(Z) = 2X$, $\delta_2(X) = \delta_2(Y) = 0$, and $\delta_2(E) = -2Z$. Clearly, X is a central element of $\mathcal{E}$ and the subalgebra $\mathbb{K}[X,Z][E;\delta_1]$ is isomorphic to the enveloping algebra of the three dimensional Heisenberg Lie algebra. Let $\mathcal{E}_X$ be the localization of $\mathcal{E}$ at the powers of the element X. Then

$$\mathcal{E}_X = \left(\mathbb{K}[X^{\pm 1}] \otimes A_1\right)[Y;\delta_2],$$

where $A_1 = \mathbb{K}\langle EX^{-1}, \frac{1}{2}Z\rangle$ is the first Weyl algebra since $[EX^{-1}, \frac{1}{2}Z] = 1$. Now, using Lemma 2.1 we can "delete" the derivation δ_2. Specifically, the element $s = -\frac{1}{2}Z^2X^{-1}$ satisfies the conditions of Lemma 2.1, and the element $Y' := Y + s = Y - \frac{1}{2}Z^2X^{-1}$ commutes with the elements of A_1. Notice that Y' also commutes with X, we have

$$\mathcal{E}_X = \mathbb{K}[X^{\pm 1}, Y'] \otimes A_1 = \mathbb{K}[X^{\pm 1}, C_1] \otimes A_1, \tag{7}$$

where $C_1 := Y'X = XY - \frac{1}{2}Z^2$. Note that C_1 belongs to the centre of $\mathcal{U}$.

Lemma 2.2. $Z(\mathcal{E}) = \mathbb{K}[X,C_1]$.

Proof. By (7), $Z(\mathcal{E}_X) = \mathbb{K}[X^{\pm 1}, C_1]$. Then $Z(\mathcal{E}) = \mathcal{E} \cap Z(\mathcal{E}_X) = \mathcal{E} \cap \mathbb{K}[X^{\pm 1}, C_1] = \mathbb{K}[X, C_1]$. □

The subalgebra $\mathcal{A}$. Let $\mathcal{A}$ be the subalgebra of the $\mathcal{U}$ generated by the elements H, E, X, Y, and Z. Then $\mathcal{A}$ is isomorphic to the enveloping algebra of the Lie subalgebra $\mathfrak{a} := \mathbb{K}H \oplus \mathbb{K}E \oplus \mathbb{K}X \oplus \mathbb{K}Y \oplus \mathbb{K}Z$ of $\mathfrak{e}(3)$. Notice that $\mathfrak{a}$ is a solvable Lie algebra, thus every prime ideal of $\mathcal{A}$ is completely prime [Ref. 25, Corollary 14.5.5]. Clearly, $\mathcal{A}$ is an Ore extension $\mathcal{A} = \mathcal{E}[H;\delta]$, where δ is a derivation of $\mathcal{E}$ defined by $\delta(E) = 2E$, $\delta(X) = 2X$, $\delta(Y) = -2Y$, and $\delta(Z) = 0$. The element X is a normal element of the algebra $\mathcal{A}$ since X is central in $\mathcal{E}$ and $XH = (H-2)X$. Let $\mathcal{E}_X$ be the localization of $\mathcal{E}$ at the powers of the element X. Then $\mathcal{A}_X = \mathcal{E}_X[H;\delta]$, by (7), $\mathcal{A}_X = \left(\mathbb{K}[X^{\pm 1}, C_1] \otimes A_1\right)[H;\delta]$. Since H commutes with the elements of A_1, the algebra $\mathcal{A}_X$ is a tensor product of algebras

$$\mathcal{A}_X = \mathbb{K}[C_1] \otimes \mathbb{K}[X^{\pm 1}][H;\delta] \otimes A_1. \tag{8}$$

In particular, $\mathcal{A}_X$ is a Noetherian domain of Gelfand-Kirillov dimension 5. The algebra $\mathbb{K}[X^{\pm 1}][H;\delta]$ where $\delta(X) = 2X$ and the Weyl algebra A_1 are central simple algebras. Hence, $Z(\mathcal{A}_X) = \mathbb{K}[C_1]$.

Lemma 2.3. $Z(\mathcal{A}) = \mathbb{K}[C_1]$.

Proof. Since $\mathbb{K}[C_1] \subseteq Z(\mathcal{A}) \subseteq \mathcal{A} \cap Z(\mathcal{A}_X) = \mathbb{K}[C_1]$, we have $Z(\mathcal{A}) = \mathbb{K}[C_1]$. □

Centre of $\mathcal{U}$. By the defining relations of $\mathcal{U}$, we see that the algebra $\mathcal{U}$ is a skew polynomial algebra

$$\mathcal{U} = \mathcal{A}[F;\sigma,\delta], \tag{9}$$

where σ is the automorphism of $\mathcal{A}$ defined by $\sigma(H) = H + 2$, $\sigma(E) = E$, $\sigma(X) = X$, $\sigma(Y) = Y$, $\sigma(Z) = Z$, and δ is the σ-derivation of $\mathcal{A}$ defined by $\delta(H) = \delta(Y) = 0$, $\delta(E) = -H$, $\delta(X) = X$, and $\delta(Z) = Y$. Let $\mathcal{U}_X$ be the localization of $\mathcal{U}$ at the powers of the element X. Then $\mathcal{U}_X = \mathcal{A}_X[F;\sigma,\delta]$.

Proposition 2.4. 1. $\mathcal{U}_X = \mathbb{K}[C_1,C_2] \otimes \mathbb{K}[X^{\pm 1}][H;\delta] \otimes A_1$ is a tensor product of algebras where $C_2 := EY + HZ - 2FX$ and $\delta(X) = 2X$.

2. $Z(\mathcal{U}) = \mathbb{K}[C_1,C_2]$.

3. $\mathcal{U}_{Z,X} \simeq \mathcal{U}_{X,Z} = \mathbb{K}[C_1, C_2] \otimes \mathbb{K}[X^{\pm 1}][H;\delta] \otimes B_1$, *where* $B_1 = A_{1,Z}$.
4. $C_1^* = \kappa(C_1) = \iota(C_1) = C_1$ *and* $C_2^* = \kappa(C_2) = \iota(C_2) = C_2$.

Proof. 1. Let $F' := FX$. By (8) and (9),

$$\mathcal{U}_X = \mathcal{A}_X[F;\sigma,\delta] = \mathcal{A}_X[F';\delta'] = \Big(\mathbb{K}[C_1] \otimes \mathbb{K}[X^{\pm 1}][H;\delta] \otimes A_1\Big)[F';\delta'],$$

where δ' is a derivation of $\mathcal{A}_X$ such that $\delta'(C_1) = 0$, $\delta'(X) = XZ$, $\delta'(H) = 0$, $\delta'(EX^{-1}) = -H - EX^{-1} \cdot Z$, and $\delta'(Z) = YX$. Using Lemma 2.1, we can "delete" the derivation δ'. In more detail, the element $s = -\frac{1}{2}HZ - \frac{1}{2}EY$ satisfies the conditions of Lemma 2.1, and the element $\tilde{F} = F' + s = FX - \frac{1}{2}HZ - \frac{1}{2}EY$ commutes with the elements of A_1. Moreover, $\tilde{F}$ commutes with X and H, hence $\tilde{F}$ is central in $\mathcal{U}_X$. Let $C_2 := -2\tilde{F} = EY + HZ - 2FX$. Then $\mathcal{U}_X = \mathbb{K}[C_1, C_2] \otimes \mathbb{K}[X^{\pm 1}][H;\delta] \otimes A_1$, as required.

2. By statement 1, $Z(\mathcal{U}_X) = \mathbb{K}[C_1, C_2]$. Then the inclusions $\mathbb{K}[C_1, C_2] \subseteq Z(\mathcal{U}) \subseteq \mathcal{U} \cap Z(\mathcal{U}_X) = \mathbb{K}[C_1, C_2]$ yield the equality $Z(\mathcal{U}) = \mathbb{K}[C_1, C_2]$.

3. Statement 3 follows from statement 1.

4. Straightforward (see also (6)). □

Proposition 2.5. The set $\mathcal{B} := \{E^iF^j, E^iF^jY^k, E^iX^k \mid i, j \in \mathbb{N}$ and $k \in \mathbb{N}_+\}$ *is a free basis of the (left and right)* $\mathbb{K}[C_1, C_2, H, Z]$*-module* $\mathcal{U}$*. In particular, the algebra* $\mathcal{U}$ *is a free* $\mathbb{K}[C_1, C_2]$*-module.*

Proof. As a vector space, the algebra $\mathcal{U}$ is a tensor product $U \otimes P_3$ of the vector spaces $U = U(\mathfrak{sl}_2)$ and $P_3 = U(V_3) = \mathbb{K}[X, Y, Z]$. Since $XY = C_1 + \frac{1}{2}Z^2$, the polynomial algebra P_3 is a free $\mathbb{K}[C_1, Z]$-module with a free basis $\{1, X^k, Y^k \mid k \in \mathbb{N}_+\}$. Using the equality $FX = \frac{1}{2}(EY + HZ - C_2)$, and the fact that V_3 is an abelian ideal of the Lie algebra $\mathfrak{e}(3) = \mathfrak{sl}_2 \ltimes V_3$, the result follows. □

The prime ideals of the algebra $\mathcal{U}$. The next two lemmas are key facts that are used in the proof of Theorem 1.1.

Lemma 2.6. 1. $(X) = (Y) = (Z) = (X, Y, Z)$.
2. $\mathcal{U}/(Z) \simeq U(\mathfrak{sl}_2)$.
3. *For all* $i \geqslant 1$, $[X, F^i] = -iF^{i-1}Z + \frac{1}{2}i(i-1)F^{i-2}Y$.
4. *For all* $i \geqslant 1$, $[Y, E^i] = -2iE^{i-1}Z + 2i(i-1)E^{i-2}X$.

Proof. 1. Statement 1 follows immediately from the defining relations of $\mathcal{U}$.

2. Statement 2 follows from statement 1.

3. Statement 3 can be proved by induction on i.

4. Statement 4 follows from statement 3 by applying the automorphism ι. □

Lemma 2.7. 1. $(Z) = \mathcal{U}Z + \mathcal{U}Y + \mathcal{U}X$.
2. $(Z)^i = (Z^i)$ *for all* $i \geqslant 1$.

Proof. 1. The inclusion $Z\mathcal{U} \subseteq \mathcal{U}Z + \mathcal{U}Y + \mathcal{U}X$ holds in the algebra $\mathcal{U}$. This follows from the equalities $[Z, E^i] = -2iE^{i-1}X$, $[Z, F^i] = -iF^{i-1}Y$, and Lemma 2.6.(3). Then $(Z) \subseteq \mathcal{U}Z + \mathcal{U}Y + \mathcal{U}X \subseteq (X, Y, Z) = (Z)$. Hence, $(Z) = \mathcal{U}Z + \mathcal{U}Y + \mathcal{U}X$.

2. It is clear that $(Z^i) \subseteq (Z)^i$. We prove that $(Z)^i \subseteq (Z^i)$ by induction on i. The case $i = 1$ is obvious. Suppose that the inclusion holds for all $i' < i$. Then $(Z)^i = (Z)(Z)^{i-1} \subseteq (Z)(Z^{i-1}) = \mathcal{U}Z\mathcal{U}Z^{i-1}\mathcal{U} = (\mathcal{U}Z + \mathcal{U}Y + \mathcal{U}X)Z^{i-1}\mathcal{U} = (Z^i) + (YZ^{i-1}) + (XZ^{i-1})$, by statement 1. Notice that $YZ^{i-1} \in (Z^i)$ since $[F, Z^i] = iYZ^{i-1}$, and $XZ^{i-1} \in (Z^i)$ since $[E, Z^i] = 2iXZ^{i-1}$. Hence, $(Z)^i \subseteq (Z^i)$, as required. □

For an algebra R, let $\mathrm{Spec}\,(R)$ be the set of its prime ideals. The set $(\mathrm{Spec}\,(R), \subseteq)$ is a partially ordered set (poset) with respect to inclusion of prime ideals. Each element $r \in R$ determines two maps from R to R, $r\cdot : x \mapsto rx$ and $\cdot r : x \mapsto xr$, where $x \in R$.

Proposition 2.8. (Ref. 5.) Let R be a Noetherian ring and s be an element of R such that $\mathcal{S}_s := \{s^i \mid i \in \mathbb{N}\}$ is a left denominator set of the ring R and $(s^i) = (s)^i$ for all $i \geqslant 1$ (e.g., s is a normal element such that $\ker(\cdot s_R) \subseteq \ker(s_R \cdot)$). Then $\mathrm{Spec}\,(R) = \mathrm{Spec}(R,s) \sqcup \mathrm{Spec}_s(R)$, where $\mathrm{Spec}(R,s) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid s \in \mathfrak{p}\}$, $\mathrm{Spec}_s(R) = \{\mathfrak{q} \in \mathrm{Spec}\,(R) \mid s \notin \mathfrak{q}\}$ and

(a) the map $\mathrm{Spec}\,(R,s) \to \mathrm{Spec}\,(R/(s))$, $\mathfrak{p} \mapsto \mathfrak{p}/(s)$, is a bijection with the inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ where $\pi : R \to R/(s), r \mapsto r + (s)$.
(b) The map $\mathrm{Spec}_s(R) \to \mathrm{Spec}\,(R_s)$, $\mathfrak{p} \mapsto \mathcal{S}_s^{-1}\mathfrak{p}$, is a bijection with the inverse $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$ where $\sigma : R \to R_s := \mathcal{S}_s^{-1}R$, $r \mapsto \frac{r}{1}$.
(c) For all $\mathfrak{p} \in \mathrm{Spec}\,(R,s)$ and $\mathfrak{q} \in \mathrm{Spec}_s(R), \mathfrak{p} \not\subseteq \mathfrak{q}$.

In this paper, we identify the sets in the statements (a) and (b) via the bijections given there.

Proof of Theorem 1.1. The strategy of classifying the prime ideals of $\mathcal{U}$ is to use the following localizations of the algebra $\mathcal{U}$

$$\begin{array}{ccccc} \mathcal{U} & \longrightarrow & \mathcal{U}_Z & \longrightarrow & \mathcal{U}_{Z,X} \\ \downarrow & & & & \\ \mathcal{U}/(Z) \simeq U(\mathfrak{sl}_2) & & & & \end{array} \tag{10}$$

together with the fact that $(X^i)_Z = (X)_Z^i = \mathcal{U}_Z$ (which follows from the relation $[Z^{-1}F, X] = 1$; in more detail, for all $i \geqslant 1$, $1 = \frac{1}{i!}\mathrm{ad}(Z^{-1}F)^i(X^i) \in (X^i)$). By Proposition 2.8, $\mathrm{Spec}\,(\mathcal{U}_Z) = \mathrm{Spec}\,(\mathcal{U}_{Z,X})$ and $\mathrm{Spec}\,(\mathcal{U}) = \mathrm{Spec}\,(\mathcal{U}/(Z)) \sqcup \mathrm{Spec}\,(\mathcal{U}_Z) = \mathrm{Spec}\,(U(\mathfrak{sl}_2)) \sqcup \mathrm{Spec}\,(\mathcal{U}_Z)$. By Proposition 2.4.(3), $\mathrm{Spec}\,(\mathcal{U}_{Z,X}) = \mathrm{Spec}\,(\mathbb{K}[C_1, C_2])$ since the algebras $\mathbb{K}[X^{\pm 1}][H;\delta]$ and $A_{1,Z}$ are central simple algebras. By Proposition 2.5, the algebra $\mathcal{U}$ is a free (left and right) $\mathbb{K}[C_1, C_2, Z]$-module. Therefore, for all $\mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[C_1, C_2])$, $\mathcal{U} \cap \mathfrak{p}\mathcal{U}_Z = \mathfrak{p}\mathcal{U}$. Now, statement 1 is obvious. So all the prime ideals are presented in diagram (1) and the inclusions in (1) are obvious. Clearly, there are no additional inclusions in diagram (1). □

The next corollary describes the set of maximal ideals $\mathrm{Max}\,(\mathcal{U})$ of the algebra $\mathcal{U}$.

Corollary 2.9. $\mathrm{Max}\,(\mathcal{U}) = \mathrm{Max}\,(U(\mathfrak{sl}_2)) \sqcup \mathrm{Max}\,(\mathcal{Z}) \setminus \{(C_1, C_2)\}$.

Proof. The equality follows from (1). □

A prime ideal P of a ring R is said to be *locally closed* if the set $\{P\}$ is locally closed in the topological space $\mathrm{Spec}\,(R)$ where $\mathrm{Spec}\,(R)$ is equipped with Zariski topology [Ref. 9, II.1.1]. A prime ideal P of a Noetherian $\mathbb{K}$-algebra R is said to be *rational* if the field $Z\big(\mathrm{Frac}(R/P)\big)$ is algebraic over $\mathbb{K}$ where $\mathrm{Frac}(R/P)$ is the left (right) quotient ring of the Noetherian prime algebra R/P. We say that the *Dixmier-Moeglin equivalence* holds for a Noetherian $\mathbb{K}$-algebra A if for each prime ideal P of A we have the following equivalences:

$$P \text{ is locally closed} \iff P \text{ is primitive} \iff P \text{ is rational.}$$

The next corollary describes the set of primitive ideals $\mathrm{Prim}\,(\mathcal{U})$ of the algebra $\mathcal{U}$.

Corollary 2.10. $\mathrm{Prim}\,(\mathcal{U}) = \mathrm{Prim}\,(U(\mathfrak{sl}_2)) \sqcup \mathrm{Max}\,(\mathcal{Z})$.

Proof. Since $\mathcal{U}$ is a universal enveloping algebra of a finite dimensional Lie algebra, it satisfies the Dixmier-Moeglin equivalence. By Ref. 9 [Lemma II.7.7], a prime ideal P in a ring R is locally closed if and only if the intersection of all prime ideals properly containing P is also an ideal properly containing P. By (1), the set of locally closed prime ideals is $\mathrm{Prim}\,(U(\mathfrak{sl}_2)) \sqcup \mathrm{Max}\,(\mathcal{Z})$. Then the corollary follows from the Dixmier-Moeglin equivalence for $\mathcal{U}$. □

The next theorem describes the set of completely prime ideals $\mathrm{Spec}_c(\mathcal{U})$ of the algebra $\mathcal{U}$ (its proof is given at the end of Section III).

Theorem 2.11. *Let $\mathcal{F}$ be the set of annihilators of simple finite dimensional $U(\mathfrak{sl}_2)$-modules of dimension $\geqslant 2$. Then $\mathrm{Spec}_c(\mathcal{U}) = \mathrm{Spec}\,(\mathcal{U}) \setminus \mathcal{F}$.*

III. THE ALGEBRA $C_{\mathcal{U}}(H)$, ITS GENERATORS, AND DEFINING RELATIONS

The aim of this section is to find generators and defining relations for the centralizer $C_{\mathcal{U}}(H)$ of the element H in the algebra $\mathcal{U}$ (Theorem 3.2.(1)), to show that the centre of $C_{\mathcal{U}}(H)$ is a polynomial algebra $\mathbb{K}[C_1, C_2, H]$ (Lemma 3.1.(2)), to prove that the algebra $C_{\mathcal{U}}(H)$ is a free (left and right) module over its polynomial subalgebra $\Gamma = \mathbb{K}[C_1, C_2, H, Z]$, and to find an explicit free Γ-basis for $C_{\mathcal{U}}(H)$ (Theorem 3.2.(2)). We introduced and studied the factor algebras $C^{\lambda,\mu} = C^{\lambda_1,\lambda_2,\mu} := C_{\mathcal{U}}(H)/(C_1 - \lambda_1, C_2 - \lambda_2, H - \mu)$ of $C_{\mathcal{U}}(H)$ (where $\lambda_1, \lambda_2, \mu \in \mathbb{K}$) that play a key role in classifying simple weight $\mathcal{U}$-modules (in Section VI). The sets of prime, completely prime, maximal, and primitive ideals of the algebra $C^{\lambda,\mu}$ are found (Theorem 3.16). Simple finite dimensional $C^{\lambda,\mu}$-modules are classified where $\lambda_1 \neq 0$ (Theorem 3.13). We realize the algebra $C_{\mathcal{U}}(H)$ as an algebra of differential operators ((12) and (13)).

The next lemma describes the centre of the algebra $C_{\mathcal{U}}(H)$.

Lemma 3.1. 1. $C_{\mathcal{U}_X}(H) = \mathbb{K}[C_1, C_2, H] \otimes A_1$, *where* $A_1 = \mathbb{K}\langle \frac{1}{2}EX^{-1}, Z\rangle$.

2. $Z(C_{\mathcal{U}}(H)) = Z(C_{\mathcal{U}_X}(H)) = \mathbb{K}[C_1, C_2, H]$.

Proof. 1. The result follows from Proposition 2.4.(1).

2. Statement 2 follows from statement 1 since $Z(A_1) = \mathbb{K}$. □

Since $\iota(H) = -H$, the automorphism ι of the algebra $\mathcal{U}$ (see (3)) induces the automorphism ι of the algebra $C_{\mathcal{U}}(H)$ by the rule

$$\iota : C_{\mathcal{U}}(H) \to C_{\mathcal{U}}(H),\ \theta \mapsto \theta + H,\ \phi \mapsto -\phi - HZ + C_2,\ H \mapsto -H,\ Z \mapsto -Z,\ C_1 \mapsto C_1,\ C_2 \mapsto C_2. \tag{11}$$

Generators and defining relations of $C_{\mathcal{U}}(H)$. We embed the algebra $C_{\mathcal{U}}(H)$ into the first Weyl algebra over the polynomial algebra $\mathbb{K}[C_1, C_2, H]$ and use this fact in finding generators and defining relations of $C_{\mathcal{U}}(H)$ (Theorem 3.2). Let $\partial := \frac{1}{2}EX^{-1}$. The Weyl algebra $A_1 = \mathbb{K}\langle Z, \partial \mid [\partial, Z] = 1\rangle$ is the GWA $A_1 = \mathbb{K}[h][Z, \partial; \sigma, a = h]$, where $\sigma(h) = h - 1$ and $h := \partial Z$. The Weyl algebra $A_1 = \oplus_{i\in\mathbb{Z}} A_{1,i}$ is a $\mathbb{Z}$-graded algebra ($A_{1,i}A_{1,j} \subseteq A_{1,i+j}$ for all $i, j \in \mathbb{Z}$), where $A_{1,0} := \mathbb{K}[h]$ is a polynomial algebra in the variable h and, for $i \geqslant 1$, $A_{1,\pm i} = \mathbb{K}[h]v_{\pm i}$, where $v_i = Z^i$, $v_{-i} = \partial^i$ and $v_0 := 1$. As a $\mathbb{Z}$-graded algebra, the Weyl algebra A_1 has the ascending filtration $\mathcal{G} = \{A_{1,\leqslant i}\}_{i\in\mathbb{Z}}$ associated with the $\mathbb{Z}$-grading, where $A_{1,\leqslant i} := \oplus_{j\leqslant i} A_{1,j}$. The associated graded algebra $\mathrm{gr}_{\mathcal{G}}(A_1) = \oplus_{i\in\mathbb{Z}} A_{1,\leqslant i}/A_{1,\leqslant i-1}$ is isomorphic to the GWA $\mathbb{K}[h][Z, \partial; \sigma, 0]$. In particular, the algebra $\mathrm{gr}_{\mathcal{G}}(A_1)$ contains two skew polynomial rings, $\mathbb{K}[h][Z; \sigma]$ and $\mathbb{K}[h][\partial; \sigma^{-1}]$, as $\mathbb{Z}$-graded subalgebras. By Lemma 3.1, the centralizer $C_{\mathcal{U}_X}(H) = \oplus_{i\in\mathbb{Z}} C_{\mathcal{U}_X}(H)_i$ is a $\mathbb{Z}$-graded algebra where the $\mathbb{Z}$-grading is inherited from the Weyl algebra A_1, i.e., $C_{\mathcal{U}_X}(H)_i = \mathbb{K}[C_1, C_2, H] \otimes A_{1,i}$.

Clearly, the algebra $C_{\mathcal{U}}(H)$ is a subalgebra of $C_{\mathcal{U}_X}(H) = \mathbb{K}[C_1, C_2, H] \otimes A_1$ (Lemma 3.1). Let $\theta := FE$ and $\phi := EY$. Then $\theta, \phi \in C_{\mathcal{U}}(H)$ and

$$\phi = \partial(2C_1 + Z^2) = 2C_1\partial + hZ, \tag{12}$$

$$\theta = 2C_1\partial^2 - C_2\partial + (h + H)(h - 1) = (\phi + HZ - C_2)\partial = -\iota(\phi)\partial. \tag{13}$$

In more detail, $\phi = EY = EX^{-1} \cdot XY = 2\partial(C_1 + \frac{1}{2}Z^2) = 2C_1\partial + hZ$, since $\partial Z = h$. Similarly,

$$\begin{aligned}\theta &= FE = FX \cdot X^{-1}E = FX \cdot EX^{-1} = (EY + HZ - C_2) \cdot \partial = (\phi + HZ - C_2)\partial \\ &= \Big(2C_1\partial + (h + H)Z - C_2\Big)\partial = 2C_1\partial^2 - C_2\partial + (h + H)(h - 1),\end{aligned}$$

since $Z\partial = \sigma(h) = h - 1$. By (12), $[\partial, \phi] = 2\partial Z$. Then, by (12) and (13),

$$\theta = \partial\Big(\phi + (H - 2)Z - H - C_2\Big) = \partial\Big(\partial(2C_1 + Z^2) + (H - 2)Z - H - C_2\Big). \tag{14}$$

Theorem 3.2. *Recall that* $\theta = FE$ *and* $\phi = EY$. *Then*

1. *The algebra* $C_{\mathcal{U}}(H)$ *is generated by the elements* C_1, C_2, H, Z, θ, *and* ϕ *subject to the defining relations as follows:*

$$[\phi, Z] = Z^2 + 2C_1, \tag{15}$$

$$[\theta, Z] = 2\phi + (H-2)Z - C_2, \tag{16}$$

$$[\theta, \phi] = 2(\theta + H)Z - H\phi, \tag{17}$$

$$\phi(\phi + HZ - C_2) = (\theta + H)(Z^2 + 2C_1), \tag{18}$$

$$[C_1, \cdot] = 0, \ [C_2, \cdot] = 0, \text{ and } [H, \cdot] = 0, \tag{19}$$

where (19) means that the elements C_1, C_2, and H are central in $C_{\mathcal{U}}(H)$. In view of (15), the relation (18) can be replaced by relation

$$(\phi + HZ - C_2)\phi = \theta(Z^2 + 2C_1). \tag{20}$$

2. *The set $B = \{\theta^i \phi^j \mid i \in \mathbb{N}, j = 0, 1\}$ is a free basis of the (left and right) Γ-module $C_{\mathcal{U}}(H)$ where $\Gamma = \mathbb{K}[C_1, C_2, H, Z]$. The sets $\iota(B) = \{(\theta + H)^i \iota(\phi)^j \mid i \in \mathbb{N}, j = 0, 1\}$ and $B' = \{\theta^i \iota(\phi)^j \mid i \in \mathbb{N}, j = 0, 1\}$ are free bases of the (left and right) Γ-module $C_{\mathcal{U}}(H)$.*
3. *The algebra $C_{\mathcal{U}}(H)$ is a Noetherian algebra of Gelfand-Kirillov dimension 5.*

Proof. 1 and 2. The second part of statement 2 follows from the first one by applying the automorphism ι. By Proposition 2.5, the algebra $C = C_{\mathcal{U}}(H)$ is generated by the elements C_1, C_2, H, Z, θ, and ϕ. It is straightforward to check that they satisfy the relations (15)–(19). It remains to show that these relations are defining relations. By (15)–(19), the set B in statement 2 is a set of generators of the (left and right) Γ-module C. The fact that the set B is a free basis for the (right and left) Γ-module C follows from the claim below. Then statement 2 implies statement 1. In order to formulate the claim we need to introduce some notation. Let $\mathcal{K} = \mathbb{K}(C_1, C_2, H)$ be the field of rational functions in the variables C_1, C_2, and H. Let $A_1(\mathcal{K})$ be the Weyl algebra over the field $\mathcal{K}$. We have the inclusions of algebras $C \subseteq C_{\mathcal{U}_X}(H) = \mathbb{K}[C_1, C_2, H] \otimes A_1 \subseteq A_1(\mathcal{K}) := \mathcal{K} \otimes A_1$.

Claim. The elements $\{\theta^i \phi^j Z^k \mid i, k \in \mathbb{N}$ and $j = 0, 1\}$ of the algebra $A_1(\mathcal{K})$ are $\mathcal{K}$-linearly independent.

Suppose that this is not true. Then $r := \sum \lambda_{ijk} \theta^i \phi^j Z^k = 0$ for some elements $\lambda_{ijk} \in \mathcal{K}$, where $i, k \geqslant 0$ and $j = 0, 1$. The Weyl algebra $A_1(\mathcal{K})$ is a domain. By multiplying on the right the element r by Z^s, we can assume that all the elements $\theta^i \phi^j Z^k$ in the relation r belong to the skew polynomial algebra $A_{1,+}(\mathcal{K}) := \oplus_{i \geqslant 0} \mathcal{K} \otimes A_{1,i} = \mathcal{K}[h][Z; \sigma]$, where $\sigma(h) = h - 1$. The concept of Z-degree, $\deg_Z$, for $A_{1,+}(\mathcal{K})$ makes sense. Notice that, by (12) and (13),

$$\phi Z = 2C_1 h + hZ^2 = hZ^2 + \cdots, \tag{21}$$

$$\theta Z^2 = 2C_1(h+1)h - C_2 h + (h+H)(h-1)Z^2 = \alpha Z^2 + \cdots, \tag{22}$$

where $\alpha = (h + H)(h - 1)$ and the three dots denote smaller terms with respect to the Z-degree. Let $d := \max\{\deg_Z(\theta^i \phi^j Z^k) = j + k \mid \lambda_{ijk} \neq 0\}$. Then the leading term l of the element $r = 0$ must be equal to zero, i.e., $l = 0$. Notice that

$$\theta^i Z^k = \alpha^i Z^k + \cdots,$$

$$\theta^i \phi Z^k = \alpha^i h Z^k + \cdots.$$

Then

$$0 = l = \Big(\sum_{j+k=d, j=0,1} \lambda_{ijk} \alpha^i h^j \Big) Z^d = \Big(\sum_i (\lambda_{i0d} + \lambda_{i1,d-1} h) \alpha^i \Big) Z^d.$$

Since $\deg_h(\alpha) = 2$, the relation $l = 0$ implies that all $\lambda_{ijk} = 0$ (in the relation $l = 0$), a contradiction.

3. Since $\mathcal{U} = \oplus_{i \in \mathbb{Z}} \mathcal{U}_i$ is a $\mathbb{Z}$-graded Noetherian algebra where $\mathcal{U}_i = \ker_{\mathcal{U}}(H - i)$, the algebra $\mathcal{U}_0 = C_{\mathcal{U}}(H)$ is a Noetherian algebra. By statement 2, GK $(C_{\mathcal{U}}(H)) = 5$. ◻

Relation (18) can be written as

$$-\phi \iota(\phi) = \iota(\theta)(Z^2 + 2C_2). \tag{23}$$

Relation (20) can be written as

$$-\iota(\phi)\phi = \theta(Z^2 + 2C_2). \tag{24}$$

So, Relation (20) is obtained from relation (18) by applying the automorphism ι, and vice versa (since $\iota^{-1} = \iota$).

The algebras $C^{\lambda,\mu} = C^{\lambda_1,\lambda_2,\mu}$. Let $\lambda_1, \lambda_2, \mu \in \mathbb{K}$, and $\lambda = (\lambda_1, \lambda_2)$. Let $(C_1 - \lambda_1, C_2 - \lambda_2, H - \mu)$ be the ideal of the algebra $C_{\mathcal{U}}(H)$ generated by the elements in the brackets. The algebras $C^{\lambda,\mu} := C^{\lambda_1,\lambda_2,\mu} := C_{\mathcal{U}}(H)/(C_1 - \lambda_1, C_2 - \lambda_2, H - \mu)$ play an important role in the classification of simple weight $\mathcal{U}$-modules (see Section VI). Similarly, define $C^{\lambda,\mu}_{\mathcal{U}_X} := C^{\lambda,\mu}_{\mathcal{U}_X}(H) := C_{\mathcal{U}_X}(H)/(C_1 - \lambda_1, C_2 - \lambda_2, H - \mu)$. We use also notations $C^{\lambda_1,\lambda_2,\mu}_{\mathcal{U}_X}$ and $C^{\lambda_1,\lambda_2,\mu}_{\mathcal{U}_X}(H)$ to denote the algebra $C^{\lambda,\mu}_{\mathcal{U}_X}$. By Lemma 3.1.(1), $C^{\lambda,\mu}_{\mathcal{U}_X} \simeq A_1$ is the Weyl algebra.

Proposition 3.3. Let $\lambda_1, \lambda_2, \mu \in \mathbb{K}$. *Then*

1. *As an abstract algebra, the algebra* $C^{\lambda,\mu}$ *is generated by the elements* Z, θ *and* ϕ *that satisfy the defining relations as follows:*

$$[\phi, Z] = Z^2 + 2\lambda_1, \tag{25}$$
$$[\theta, Z] = 2\phi + (\mu - 2)Z - \lambda_2, \tag{26}$$
$$[\theta, \phi] = 2(\theta + \mu)Z - \mu\phi, \tag{27}$$
$$\phi(\phi + \mu Z - \lambda_2) = (\theta + \mu)(Z^2 + 2\lambda_1). \tag{28}$$

In view of (25), relation (28) can be replaced by the relation

$$(\phi + \mu Z - \lambda_2)\phi = \theta(Z^2 + 2\lambda_1). \tag{29}$$

2. *The set* $B = \{\theta^i\phi^j \mid i \in \mathbb{N}$ and $j = 0, 1\}$ *is a free basis of the (left and right)* $\mathbb{K}[Z]$*-module* $C^{\lambda,\mu}$.
3. *The algebra homomorphism* $C^{\lambda,\mu} \to C^{\lambda,\mu}_{\mathcal{U}_X} = A_1$, $Z \mapsto Z$, $\phi \mapsto 2\lambda_1\partial + hZ$, $\theta \mapsto 2\lambda_1\partial^2 - \lambda_2\partial + (h + \mu)(h - 1)$ *is a monomorphism. In particular,* $C^{\lambda,\mu}$ *is a domain.*
4. *The ideal* $(C_1 - \lambda_1, C_2 - \lambda_2, H - \mu)$ *of the algebra* $C_{\mathcal{U}}(H)$ *is equal to the intersection of the algebra* $C_{\mathcal{U}}(H)$ *and the ideal* $(C_1 - \lambda_1, C_2 - \lambda_2, H - \mu)$ *of the algebra* $C_{\mathcal{U}_X}(H)$. *In particular, the ideal* $(C_1 - \lambda_1, C_2 - \lambda_2, H - \mu)$ *is a completely prime ideal of* $C_{\mathcal{U}}(H)$.
5. $\mathrm{GK}\,(C^{\lambda,\mu}) = 2$ *and* $Z(C^{\lambda,\mu}) = \mathbb{K}$.

Proof. 1. Statement 1 follows from Theorem 3.2.(1).

2. Statement 2 follows from Theorem 3.2.(2).

3. In view of the inclusion $C_{\mathcal{U}}(H) \subseteq C_{\mathcal{U}_X}(H) = \mathbb{K}[C_1, C_2, H] \otimes A_1$ and the equalities (12) and (13), the homomorphism in statement 3 is well defined. The fact that it is a monomorphism follows from statement 2 and the claim below.

Claim: The images of the elements $\{\theta^i\phi^j Z^k \mid i, k \in \mathbb{N}$ and $j = 0, 1\}$ *in* A_1 *are* $\mathbb{K}$*-linearly independent*: Repeat the proof of the claim in the proof of Theorem 3.2 replacing $(\mathcal{K}, C_1, C_2, H)$ by $(\mathbb{K}, \lambda_1, \lambda_2, \mu)$ everywhere.

4. Statement 4 follows from statement 3.

5. The inclusion $\Lambda := \mathbb{K}\langle Z, \partial\rangle \subseteq C^{\lambda,\mu} \subseteq A_1$ yields the inequalities $2 = \mathrm{GK}\,(\Lambda) \leqslant \mathrm{GK}\left(C^{\lambda,\mu}\right) \leqslant \mathrm{GK}\,(A_1) = 2$, i.e., $\mathrm{GK}\left(C^{\lambda,\mu}\right) = 2$. Since $C_{A_1}(Z) = \mathbb{K}[Z]$, we must have $Z(C^{\lambda,\mu}) \subseteq \mathbb{K}[Z]$. Let $f \in Z(C^{\lambda,\mu})$. By (25), $0 = [\phi, f] = \frac{df}{dZ} \cdot (Z^2 + 2\lambda_1)$. Hence, $f \in \mathbb{K}$, i.e., $Z(C^{\lambda,\mu}) = \mathbb{K}$. □

The Weyl algebra A_1 admits a finite dimensional ascending filtration $S = \{S_i := \sum_{j+k\leqslant i} \mathbb{K}Z^j \partial^k\}_{i\in\mathbb{N}}$ by the total degree of the canonical generators Z and ∂ of A_1. The associated graded algebra $\mathrm{gr}(A_1) = \mathbb{K}[Z, \partial]$ is a polynomial algebra. The subalgebra $C^{\lambda,\mu} \subset A_1$ (Proposition 3.3.(3)) admits the induced filtration $\mathcal{F} = \{\mathcal{F}_i := C^{\lambda,\mu} \cap S_i\}_{i\in\mathbb{N}}$. It follows that the associated graded algebra $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is a subalgebra of the polynomial algebra $\mathrm{gr}(A_1) = \mathbb{K}[Z, \partial]$. In the algebra $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$, $\phi = Z^2\partial$ and $\theta = Z^2\partial^2$. In particular, $\phi^2 = \theta Z^2$.

Lemma 3.4. 1. $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu}) \simeq \mathbb{K}[Z, \phi, \theta]/(\phi^2 - \theta Z^2)$.

2. *The algebra* $\mathrm{gr}(A_1)$ *is not a finitely generated* $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$*-module.*
3. *The Weyl algebra* A_1 *is not a finitely generated left/right* $C^{\lambda,\mu}$*-module.*

Proof. 1. The algebra $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is an epimorphic image of the factor algebra $\mathbb{K}[Z,\phi,\theta]/(\phi^2-\theta Z^2)$ (since $\phi^2=\theta Z^2$ in $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$). In fact, $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})\simeq\mathbb{K}[Z,\phi,\theta]/(\phi^2-\theta Z^2)$, by Proposition 3.3.(2).

2. Clearly, the algebra $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is a subalgebra of the algebra $\mathbb{K}[Z,Z\partial]$ (since $\phi=Z^2\partial$ and $\theta=Z^2\partial^2$). The polynomial algebra $\mathbb{K}[Z,\partial]$ is not a finitely generated $\mathbb{K}[Z,Z\partial]$-module, and statement 2 follows.

3. Statement 3 follows from statement 2. □

The $C^{\lambda,\mu}$-modules $V^{\lambda,\mu}(\nu_1)$ and $W^{\lambda,\mu}(\nu_1)$. We introduce important $C^{\lambda,\mu}$-modules $V^{\lambda,\mu}(\nu_1)$ and $W^{\lambda,\mu}(\nu_1)$ that play an important role in the classification of simple $C^{\lambda,\mu}$-modules (especially, finite-dimensional ones). Generically, these modules are simple. For all $\lambda_1,\lambda_2,\mu\in\mathbb{K}$, $\iota\Big((C_1-\lambda_1,C_2-\lambda_2,H-\mu)\Big)=(C_1-\lambda_1,C_2-\lambda_2,H+\mu)$. So, the automorphism ι of $C_{\mathcal{U}}(H)$ induces the automorphism of the algebras

$$\iota: C^{\lambda,\mu}\to C^{\lambda,-\mu},\quad \theta\mapsto\theta-\mu,\ \ \phi\mapsto-\phi+\mu Z+\lambda_2,\ \ Z\mapsto -Z. \tag{30}$$

The polynomial subalgebra $\Gamma=\mathbb{K}[Z,C_1,C_2,H]$ is ι-invariant since $\iota(Z)=-Z$, $\iota(C_1)=C_1$, $\iota(C_2)=C_2$ and $\iota(H)=-H$. By Theorem 3.2.(2), the algebra $C_{\mathcal{U}}(H)$ is the tensor product of vector spaces

$$C_{\mathcal{U}}(H)=\Big(\mathbb{K}[\theta]\oplus\mathbb{K}[\theta]\phi\Big)\otimes\Gamma. \tag{31}$$

By applying the automorphism ι, the algebra $C_{\mathcal{U}}(H)$ is a tensor product of vector spaces

$$C_{\mathcal{U}}(H)=\Big(\mathbb{K}[\theta+H]\oplus\mathbb{K}[\theta+H]\iota(\phi)\Big)\otimes\Gamma. \tag{32}$$

Let $(\lambda_1,\lambda_2,\mu)\in\mathbb{K}^*\times\mathbb{K}\times\mathbb{K}$ and $\lambda=(\lambda_1,\lambda_2)$. For $\lambda_1\in\mathbb{K}^*$, the polynomial $Z^2+2\lambda_1\in\mathbb{K}[Z]$ has two *distinct*, *nonzero* roots ν_1 and $-\nu_1$. Let us fix a root, say ν_1, of $Z^2+2\lambda_1$, i.e., $\nu_1^2+2\lambda_1=0$. The maximal ideal $\mathfrak{m}=(Z-\nu_1,C_1-\lambda_1,C_2-\lambda_2,H-\mu)$ of the algebra Γ determines a simple 1-dimensional Γ-module $\Gamma/\Gamma_{\mathfrak{m}}\simeq\mathbb{K}$. Consider the induced $C_{\mathcal{U}}(H)$-module

$$C_{\mathcal{U}}(H)\otimes_\Gamma\Gamma/\Gamma_{\mathfrak{m}}\overset{(31)}{\simeq}\Big(\mathbb{K}[\theta]\oplus\mathbb{K}[\theta]\phi\Big)\otimes\Gamma/\Gamma_{\mathfrak{m}}.$$

This $C_{\mathcal{U}}(H)$-module is, in fact, $C^{\lambda,\mu}$-module

$$C^{\lambda,\mu}(\nu_1):=C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1)\simeq\mathbb{K}[\theta]\tilde{1}\oplus\mathbb{K}[\theta]\phi\tilde{1}, \tag{33}$$

where $\tilde{1}=1+C^{\lambda,\mu}(Z-\nu_1)$. The $\mathbb{K}[\theta]$-module $C^{\lambda,\mu}(\nu_1)$ is a free module of rank 2. By (32), we also have

$$C^{\lambda,\mu}(\nu_1)=\mathbb{K}[\theta]\tilde{1}\oplus\mathbb{K}[\theta]\iota(\phi)\tilde{1}=\mathbb{K}[\theta]\tilde{1}\oplus\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}. \tag{34}$$

The $\mathbb{K}[\theta]$-submodule $\mathbb{K}[\theta]\phi\tilde{1}$ of $C^{\lambda,\mu}(\nu_1)$ is a $C^{\lambda,\mu}$-submodule,

$$C^{\lambda,\mu}\phi\tilde{1}\overset{(34)}{=}\mathbb{K}[\theta]\phi\tilde{1}+\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\phi\tilde{1}\overset{(29)}{=}\mathbb{K}[\theta]\phi\tilde{1}+\mathbb{K}[\theta]\theta(\nu_1^2+2\lambda_1)\phi\tilde{1}=\mathbb{K}[\theta]\phi\tilde{1}.$$

Define the $C^{\lambda,\mu}$-modules

$$W^{\lambda,\mu}(\nu_1):=\mathbb{K}[\theta]\phi\tilde{1}\quad\text{and}\quad V^{\lambda,\mu}(\nu_1):=C^{\lambda,\mu}/W^{\lambda,\mu}(\nu_1)\simeq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1,\phi)=\mathbb{K}[\theta]\bar{1}, \tag{35}$$

where $\bar{1}=1+C^{\lambda,\mu}(Z-\nu_1,\phi)$. The $\mathbb{K}[\theta]$-modules $W^{\lambda,\mu}(\nu_1)$ and $V^{\lambda,\mu}(\nu_1)$ are free modules of rank 1. Since $\mathbb{K}[\theta]\iota(\phi)\phi\tilde{1}\overset{(29)}{=}\mathbb{K}[\theta]\theta(\nu_1^2+2\lambda_1)\tilde{1}=0$ and the $\mathbb{K}[\theta]$-module $W^{\lambda,\mu}(\nu_1)$ is free of rank 1, it follows from (34) that

$$W^{\lambda,\mu}(\nu_1)\simeq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1,\iota(\phi))\simeq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1,\phi+\mu\nu_1-\lambda_2)=\mathbb{K}[\theta]1', \tag{36}$$

where $1':=1+C^{\lambda,\mu}(Z-\nu_1,\phi+\mu\nu_1-\lambda_2)$. Similarly, the $\mathbb{K}[\theta]$-submodule $\mathbb{K}[\theta]\iota(\phi)\tilde{1}=\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}$ of $C^{\lambda,\mu}(\nu_1)$ (see (34)) is a $C^{\lambda,\mu}$-submodule,

$$\begin{aligned}C^{\lambda,\mu}(\phi+\mu\nu_1-\lambda_2)\tilde{1}&\overset{(33)}{=}\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}+\mathbb{K}[\theta]\phi(\phi+\mu\nu_1-\lambda_2)\tilde{1}\\&\overset{(28)}{=}\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}+\mathbb{K}[\theta](\theta+\mu)(\nu_1^2+2\lambda_1)\tilde{1}=\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}.\end{aligned}$$

It follows from the above arguments and the fact that the $\mathbb{K}[\theta]$-module $\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}$ is free that the $C^{\lambda,\mu}$-homomorphism

$$V^{\lambda,\mu}(\nu_1)=\mathbb{K}[\theta]\bar{1}\to\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1},\quad \bar{1}\mapsto(\phi+\mu\nu_1-\lambda_2)\tilde{1}$$

is an isomorphism. Summarizing, we have short exact sequence of $C^{\lambda,\mu}$-modules that follow from the equalities (33) and (34), respectively,

$$0\to W^{\lambda,\mu}(\nu_1)\to C^{\lambda,\mu}(\nu_1)\to V^{\lambda,\mu}(\nu_1)\to 0, \tag{37}$$

$$0\to V^{\lambda,\mu}(\nu_1)\to C^{\lambda,\mu}(\nu_1)\to W^{\lambda,\mu}(\nu_1)\to 0. \tag{38}$$

The next lemma shows that, generically, these short exact sequences split.

Lemma 3.5. Let $(\lambda_1,\lambda_2,\mu)\in\mathbb{K}^*\times\mathbb{K}\times\mathbb{K}$.

1. *If $\mu\nu_1-\lambda_2\neq 0$, then $C^{\lambda,\mu}(\nu_1)=V^{\lambda,\mu}(\nu_1)\oplus W^{\lambda,\mu}(\nu_1)$ and $V^{\lambda,\mu}(\nu_1)\simeq\mathbb{K}[\theta]\iota(\phi)\tilde{1}\simeq\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}$ and $W^{\lambda,\mu}(\nu_1)=\mathbb{K}[\theta]\phi\tilde{1}$.*
2. *If $\mu\nu_1-\lambda_2=0$, then $V^{\lambda,\mu}(\nu_1)\simeq W^{\lambda,\mu}(\nu_1)$ and there is a short exact sequence of $C^{\lambda,\mu}$-modules*

$$0\to V^{\lambda,\mu}(\nu_1)\to C^{\lambda,\mu}(\nu_1)\to V^{\lambda,\mu}(\nu_1)\to 0.$$

Proof. 1. Since $\mu\nu_1-\lambda_2\neq 0$, by (33) and (34), $C^{\lambda,\mu}(\nu_1)=\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}\oplus\mathbb{K}[\theta]\phi\tilde{1}=V^{\lambda,\mu}(\nu_1)\oplus W^{\lambda,\mu}(\nu_1)$.

2. Since $\mu\nu_1-\lambda_2=0$, then $V^{\lambda,\mu}(\nu_1)=\mathbb{K}[\theta](\phi+\mu\nu_1-\lambda_2)\tilde{1}=\mathbb{K}[\theta]\phi\tilde{1}=W^{\lambda,\mu}(\nu_1)$. Then, by (37) we have the short exact sequence in statement 2. □

If we identify the algebras $C^{\lambda,\mu}$ and $C^{\lambda,-\mu}$ via the isomorphism $\iota: C^{\lambda,\mu}\mapsto C^{\lambda,-\mu}$, see (30), then the isomorphism ι induces a $C^{\lambda,\mu}$-module isomorphism $\iota: C^{\lambda,\mu}(\nu_1)\to C^{\lambda,-\mu}(-\nu_1)$. Clearly,

$$\iota\Big(V^{\lambda,\mu}(\nu_1)\Big)=W^{\lambda,-\mu}(-\nu_1)\quad\text{and}\quad \iota\Big(W^{\lambda,\mu}(\nu_1)\Big)=W^{\lambda,-\mu}(-\nu_1). \tag{39}$$

The simple n-dimensional $C^{\lambda,\mu}$-module $F_n^{\lambda,\mu}(\nu_1)$: We assume that $\lambda_1\neq 0$. Let ν_1 is a root of the polynomial $Z^2+2\lambda_1$, i.e., $\nu_1^2=-2\lambda_1$. There are two distinct roots of $Z^2+2\lambda_1$: ν_1 and $-\nu_1$ (since $\lambda_1\neq 0$). Let us consider the A_1-module $\mathcal{V}(\nu_1):=A_1/A_1(Z-\nu_1)=\mathbb{K}[\partial]\bar{1}$, where $\bar{1}=1+A_1(Z-\nu_1)$. The A_1-module $\mathcal{V}(\nu_1)$ is simple and the set of elements $\{\partial^i\bar{1}\mid i\in\mathbb{N}\}$ is its $\mathbb{K}$-basis. In particular, $\mathcal{V}(\nu_1)$ is a free $\mathbb{K}[\partial]$-module of rank 1. Clearly,

$$Z\bar{1}=\nu_1\bar{1}\quad\text{and}\quad Z\partial^i\bar{1}=\nu_1\partial^i\bar{1}-i\partial^{i-1}\bar{1}\text{ for } i\geqslant 1. \tag{40}$$

We see that $\mathcal{V}(\nu_1)=\bigcup_{i\geqslant 0}\ker(Z-\nu_1)^{i+1}$ and $\ker(Z-\nu_1)^{i+1}=\mathbb{K}[\partial]_{\leqslant i}\bar{1}$, where $\mathbb{K}[\partial]_{\leqslant i}:=\oplus_{j=0}^{i}\mathbb{K}\partial^j$. It is straightforward to show (using Proposition 3.3.(3)) that the action of the elements ϕ and θ on the basis elements of the A_1-module $\mathcal{V}(\nu_1)$ are given below

$$\phi\bar{1}=0\quad\text{and}\quad \phi\partial^i\bar{1}=-2i\nu_1\partial^i\bar{1}+i(i-1)\partial^{i-1}\bar{1},\ i\geqslant 1, \tag{41}$$

$$\theta\partial^i\bar{1}=\theta_i\partial^{i+1}\bar{1}+\eta_i\partial^i\bar{1},\ i\geqslant 0, \tag{42}$$

where $\theta_i:=\nu_1\big(\mu-2(i+1)\big)-\lambda_2$ and $\eta_i:=-(\mu-i)(i+1)$. Since $C^{\lambda,\mu}\subseteq A_1$, $\mathcal{V}(\nu_1)$ is also a $C^{\lambda,\mu}$-module. The lemma below is a simplicity criterion for the $C^{\lambda,\mu}$-module $\mathcal{V}(\nu_1)$. It shows that in case when the $C^{\lambda,\mu}$-module $\mathcal{V}(\nu_1)$ is not simple, it contains a unique proper submodule which is a finite dimensional simple $C^{\lambda,\mu}$-module.

Lemma 3.6. Let $\lambda_1\neq 0$ and $\nu_1^2=-2\lambda_1$. Then the $C^{\lambda,\mu}$-module $\mathcal{V}(\nu_1)$ is not simple if and only if $\nu_1(\mu-2n)-\lambda_2=0$, i.e., $\theta_{n-1}=0$, for some $n\in\mathbb{N}_+$. In this case, $F_n^{\lambda,\mu}(\nu_1):=\bigoplus_{i=0}^{n-1}\mathbb{K}\partial^i\bar{1}$ is a unique proper $C^{\lambda,\mu}$-submodule of $\mathcal{V}(\nu_1)$. The $C^{\lambda,\mu}$-module $F_n^{\lambda,\mu}(\nu_1)$ is simple, $\dim F_n^{\lambda,\mu}(\nu_1)=n$, and $(Z-\nu_1)^nF^{\lambda,\mu}(\nu_1)=0$.

Proof. If $\theta_{n-1}\neq 0$ for all $n\in\mathbb{N}_+$, then the $C^{\lambda,\mu}$-module $\mathcal{V}(\nu_1)$ is simple by (40) and (42). If $\theta_{n-1}=0$ for some $n\in\mathbb{N}_+$, then the number n is unique and $F_n^{\lambda,\mu}(\nu_1)$ is a simple, n-dimensional

$C^{\lambda,\mu}$-submodule of $\mathcal{V}(\nu_1)$, by (40) and (42). By (40) and (42), the factor module $\mathcal{V}(\nu_1)/F_n^{\lambda,\mu}(\nu_1)$ is a simple $C^{\lambda,\mu}$-module and $F_n^{\lambda,\mu}(\nu_1)$ is an essential submodule of $\mathcal{V}(\nu_1)$. Therefore, $F_n^{\lambda,\mu}(\nu_1)$ is a unique proper submodule of the $C^{\lambda,\mu}$-module $\mathcal{V}(\nu_1)$. □

Theorem 3.13 shows that the modules $F_n^{\lambda,\mu}(\nu_1)$ and their 'partners' $G_n^{\lambda,\mu}(\nu_1)$ (if exist) are precisely finite dimensional simple $C^{\lambda,\mu}$-modules.

The next two corollaries describe the $C^{\lambda,\mu}$-module $F_n^{\lambda,\mu}(\nu_1)$ in terms of the algebra $C^{\lambda,\mu}$.

Corollary 3.7. We keep the assumptions and notation of Lemma 3.6. Then

$$F_n^{\lambda,\mu}(\nu_1) \simeq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1,\ \phi,\ f_{n,\nu_1}^{\lambda,\mu}),$$

where $f_{n,\nu_1}^{\lambda,\mu} := \prod_{i=0}^{n-1}(\theta-\eta_i)$ *and* $\eta_i = -(\mu-i)(i+1)$. *Furthermore,* $f_{n,\nu_1}^{\lambda,\mu}F_n^{\lambda,\mu}(\nu_1)=0$.

Proof. The set $\{\bar{1},\partial\bar{1},\dots,\partial^{n-1}\bar{1}\}$ is a $\mathbb{K}$-basis of the simple $C^{\lambda,\mu}$-module $F_n^{\lambda,\mu}(\nu_1)$. By (40) and (41), $(Z-\nu_1)\bar{1}$ and $\phi\bar{1}=0$. By (42), the matrix $[\theta]$ of the linear map $\theta\cdot : F_n^{\lambda,\mu}(\nu_1)\to F_n^{\lambda,\mu}(\nu_1)$, $u\mapsto \theta u$, in the basis above is a lower diagonal $n\times n$ matrix given below where the diagonal elements are $\eta_0,\eta_1,\dots,\eta_{n-1}$ and below the diagonal are elements $\theta_0,\theta_1,\dots,\theta_{n-2}$,

$$[\theta]=\begin{bmatrix}\eta_0 & & & & \\ \theta_0 & \eta_1 & & 0 & \\ 0 & \theta_1 & \eta_2 & & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \dots & \theta_{n-2} & \eta_{n-1}\end{bmatrix}. \tag{43}$$

Then $f_{n,\nu_1}^{\lambda,\mu}F_n^{\lambda,\mu}(\nu_1)=0$. Therefore, the $C^{\lambda,\mu}$-module $F_n^{\lambda,\mu}(\nu_1)$ is an epimorphic image of the $C^{\lambda,\mu}$-module $V := C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1,\ \phi,\ f_{n,\nu_1}^{\lambda,\mu})$. By (35), $\dim(V)=n=\dim F_n^{\lambda,\mu}(\nu_1)$. Therefore, $V\simeq F_n^{\lambda,\mu}(\nu_1)$. □

By (35), $V^{\lambda,\mu}(\nu_1)\simeq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1,\ \phi)=\mathbb{K}[\theta]\bar{1}$. Using the defining relations of the algebra $C^{\lambda,\mu}$ (Proposition 3.3.(1)) and induction on i, we obtain that

$$\phi\bar{1}=0 \quad\text{and}\quad \phi\theta^i\bar{1}=-2i\nu_1\theta^i\bar{1}+\cdots,\ i\geqslant 1, \tag{44}$$

$$Z\bar{1}=\nu_1\bar{1} \quad\text{and}\quad Z\theta^i\bar{1}=\nu_1\theta^i\bar{1}-i\theta_{i-1}\theta^{i-1}\bar{1}+\cdots,\ i\geqslant 1, \tag{45}$$

where $\theta_i=\nu_1\big(\mu-2(i+1)\big)-\lambda_2$ (see (42)) and the three dots means smaller terms.

Simplicity criteria for the $C^{\lambda,\mu}$*-modules* $V^{\lambda,\mu}(\nu_1)$ *and* $W^{\lambda,\mu}(\nu_1)$. The next proposition is a simplicity criterion for the $C^{\lambda,\mu}$-module $V^{\lambda,\mu}(\nu_1)$. It also describes the submodules and factor modules of $V^{\lambda,\mu}(\nu_1)$.

Proposition 3.8. Let $\lambda_1\in\mathbb{K}^*$ *and* $\nu_1^2=-2\lambda_1$. *Then the* $C^{\lambda,\mu}$*-module* $V^{\lambda,\mu}(\nu_1)$ *is not simple if and only if* $n:=\frac{1}{2}(\mu-\frac{\lambda_2}{\nu_1})\in\mathbb{N}_+$ *if and only if* $\theta_{n-1}=0$ *for some* $n\in\mathbb{N}_+$. *In this case,*

1. $f_{n,\nu_1}^{\lambda,\mu}V^{\lambda,\mu}(\nu_1)$ *is the only proper submodule of the* $C^{\lambda,\mu}$*-module* $V^{\lambda,\mu}(\nu_1)$ *where* $f_{n,\nu_1}^{\lambda,\mu}=\prod_{i=0}^{n-1}(\theta-\eta_i)$ *and* $\eta_i=-(\mu-i)(i+1)$ *(see Corollary 3.7).*
2. $F_n^{\lambda,\mu}(\nu_1)\simeq V^{\lambda,\mu}(\nu_1)/f_{n,\nu_1}^{\lambda,\mu}V^{\lambda,\mu}(\nu_1)$ *is the unique simple factor module of the* $C^{\lambda,\mu}$*-module* $V^{\lambda,\mu}(\nu_1)$, $\dim F_n^{\lambda,\mu}(\nu_1)=n$, *and* $\mathbb{K}[\theta]\cap \mathrm{ann}_{C^{\lambda,\mu}}(F_n^{\lambda,\mu}(\nu_1))=f_{n,\nu_1}^{\lambda,\mu}\mathbb{K}[\theta]$.

Proof. By Proposition 3.3.(1), the algebra $C^{\lambda,\mu}$ is generated by the elements Z and θ (see (26)). Any submodule U of $V^{\lambda,\mu}$ is equal to $f\mathbb{K}[\theta]\bar{1}$ for a unique monic polynomial $f\in\mathbb{K}[\theta]$. The submodule $U=f\mathbb{K}[\theta]\bar{1}$ of $V^{\lambda,\mu}$ is a proper submodule if and only if $f\in\mathbb{K}[\theta]\setminus\mathbb{K}$ and $Zf\bar{1}=\nu_1 f\bar{1}$, by (45). Let $n=\deg_\theta(f)$. Then necessarily $\theta_{n-1}=0$, by (45), and the number $n\in\mathbb{N}_+$ is *unique* with this property. So, the proper submodule U is *unique*. Hence, the n-dimensional $C^{\lambda,\mu}$-module $F_n := V^{\lambda,\mu}/fV^{\lambda,\mu}$ is a unique proper factor module of the $C^{\lambda,\mu}$-module $V^{\lambda,\mu}$. Now, the proposition follows from Corollary 3.7. □

The next corollary is a simplicity criterion for the $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$. It also describes the submodules and factor modules of $W^{\lambda,\mu}(\nu_1)$.

Corollary 3.9. Let $\lambda_1 \in \mathbb{K}^*$ and $\nu_1^2 = -2\lambda_1$. Then

1. *The $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ is isomorphic to the twisted by the isomorphism $\iota : C^{\lambda,\mu} \to C^{\lambda,-\mu}$ $C^{\lambda,-\mu}$-module $V^{\lambda,-\mu}(-\nu_1)$, i.e., $W^{\lambda,\mu}(\nu_1) \simeq {}^{\iota}V^{\lambda,-\mu}(-\nu_1) \simeq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1, \iota(\phi))$.*
2. *The $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ is not simple if and only if $m := \frac{1}{2}(-\mu + \frac{\lambda_2}{\nu_1}) \in \mathbb{N}_+$. In this case,*
   (a) *$\iota(f^{\lambda,-\mu}_{m,-\nu_1})W^{\lambda,\mu}(\nu_1)$ is the only proper submodule of the $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ where $f^{\lambda,-\mu}_{m,-\nu_1} = \prod_{i=0}^{m-1}(\theta - \eta_i')$ and $\eta_i' := -(-\mu - i)(i+1)$.*
   (b) *$G^{\lambda,\mu}_m(\nu_1) := {}^{\iota}F^{\lambda,-\mu}_m(-\nu_1) \simeq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1, \iota(\phi), \iota(f^{\lambda,-\mu}_{m,-\nu_1}))$ is the unique simple factor module of the $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ and $\dim G^{\lambda,\mu}_m(\nu_1) = m$.*
   (c) *$\mathbb{K}[\theta] \cap \mathrm{ann}_{C^{\lambda,\mu}}(G^{\lambda,\mu}_m(\nu_1)) = \iota(f^{\lambda,-\mu}_{m,-\nu_1})\mathbb{K}[\theta]$ and $(Z-\nu_1)^m G^{\lambda,\mu}_m(\nu_1) = 0$.*

Proof. 1. Statement 1 follows from (39) and (35).
2. Statement 2 follows from statement 1 and Proposition 3.8. □

Corollary 3.10. Let $\lambda_1 \in \mathbb{K}^*$ and $\nu_1^2 = -2\lambda_1$. If one of the $C^{\lambda,\mu}$-modules $V^{\lambda,\mu}(\nu_1)$ or $W^{\lambda,\mu}(\nu_1)$ is not simple, then the other is necessarily simple.

Proof. By Proposition 3.8, the $C^{\lambda,\mu}$-module $V^{\lambda,\mu}(\nu_1)$ is not simple if and only if $\frac{1}{2}(\mu - \frac{\lambda_2}{\nu_1}) \in \mathbb{N}_+$. By Corollary 3.9.(2), the $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ is not simple if and only if $-\frac{1}{2}(\mu - \frac{\lambda_2}{\nu_1}) \in \mathbb{N}_+$. Now, the result follows. □

Lemma 3.11. Let $\lambda \in \mathbb{K}^*$. Suppose that $n = \frac{1}{2}(\mu - \frac{\lambda_2}{\nu_1}) \in \mathbb{N}_+$. Then

1. *$0 \longrightarrow F^{\lambda,\mu}_n(\nu_1) \longrightarrow \mathcal{V}(\nu_1) \longrightarrow \mathcal{V}(\lambda_1)/F^{\lambda,\mu}_n(\nu_1) \simeq W^{\lambda,\mu}(\nu_1) \longrightarrow 0$ is a short exact sequence of $C^{\lambda,\mu}$-modules and $W^{\lambda,\mu}(\nu_1)$ is a simple $C^{\lambda,\mu}$-module.*
2. *$f^{\lambda,\mu}_{n,\nu_1}V^{\lambda,\mu}(\nu_1) \simeq W^{\lambda,\mu}(\nu_1)$.*

Proof. 1. Let us show that the isomorphism makes sense. Clearly, $V := \mathcal{V}(\lambda_1)/F^{\lambda,\mu}_n(\nu_1) = \bigoplus_{i\geqslant 0}\mathbb{K}\partial^{n+i}\tilde{1}$, where $\tilde{1} = 1 + F^{\lambda,\mu}_n(\nu_1)$. By (40) and (41), $(Z-\nu_1)\tilde{1} = 0$ and $0 = (\phi + 2n\nu_1)\tilde{1} = (\phi + \mu\nu_1 - \lambda_2)\tilde{1} = (\phi + \mu Z - \lambda_2)\tilde{1} = -\iota(\phi)\tilde{1}$. So, V is an epimorphic image of $W^{\lambda,\mu}(\nu_1)$ (Corollary 3.9.(1)). Since $n \in \mathbb{N}_+$, we have $m := \frac{1}{2}(-\mu + \frac{\lambda_2}{\nu_1}) = -n \notin \mathbb{N}_+$, and so the $C^{\lambda,\mu}$-module $W^{\lambda,\nu}(\nu_1)$ is simple (Corollary 3.9.(2)). Hence, $V \simeq W^{\lambda,\mu}(\nu_1)$. Now, statement 1 follows.

2. Let $f = f^{\lambda,\mu}_{n,\nu_1}$. We keep the notation of Proposition 3.8. Notice that $(Z-\nu_1)f\bar{1} = 0$ since $\deg_\theta((Z-\nu_1)f\bar{1}) < \deg_\theta(f\bar{1})$ and $(Z-\nu_1)f\bar{1} \in f\mathbb{K}[\theta]\bar{1}$. By (44), $\deg_\theta((\phi + 2n\nu_1)f\bar{1}) < \deg_\theta(f\bar{1}) = n$, hence, $(\phi + 2n\nu_1)f\bar{1} = 0$ since $(\phi + 2n\nu_1)f\bar{1} \in f\mathbb{K}[\theta]\bar{1}$. Using the equalities $n = \frac{1}{2}(\mu - \frac{\lambda_2}{\nu_1})$ and $(Z-\nu_1)f\bar{1} = 0$, we obtain that $\iota(\phi)f\bar{1} = 0$: $-\iota(\phi)f\bar{1} = (\phi + \mu Z - \lambda_2)f\bar{1} = (\phi + \mu\nu_1 - \lambda_2)f\bar{1} = (\phi + 2n\nu_1)f\bar{1} = 0$. By (36), there is an epimorphism $W^{\lambda,\mu}(\nu_1) \to fV^{\lambda,\mu}(\nu_1)$. Since $n \in \mathbb{N}_+$, the $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ is simple (Corollary 3.9.(2)). Hence, $W^{\lambda,\mu}(\nu_1) \simeq fV^{\lambda,\mu}(\nu_1)$. □

Simplicity criteria for the algebra $C^{\lambda,\mu}$ where $\lambda_1 \neq 0$. The next theorem is a simplicity criterion for the algebra $C^{\lambda,\mu}$ where $\lambda_1 \neq 0$.

Theorem 3.12. *Let $\lambda_1 \in \mathbb{K}^*$ and $\nu_1^2 = -2\lambda_1$. The following statements are equivalent.*

1. *$C^{\lambda,\mu}$ is a simple algebra.*
2. *The $C^{\lambda,\mu}$-modules $V^{\lambda,\mu}(\pm\nu_1)$ and $W^{\lambda,\mu}(\pm\nu_1)$ are simple.*
3. *$\frac{1}{2}(\mu \pm \frac{\lambda_2}{\nu_1}) \notin \mathbb{Z} \setminus \{0\}$.*
4. *There is no finite dimensional simple $C^{\lambda,\mu}$-module.*

Proof. $2 \Leftrightarrow 3 \Leftrightarrow 4$. These implications follow from Proposition 3.8 and Corollary 3.9.
$(1 \Rightarrow 4)$ This implication is obvious since $\dim C^{\lambda,\mu} = \infty$.

(4 ⇒ 1) Since the localization $C^{\lambda,\mu}_{Z^2+2\lambda_1}$ of the algebra $C^{\lambda,\mu}$ is isomorphic to the localization $A_{1,Z^2+2\lambda_1}$ of the Weyl algebra A_1, the algebra $C^{\lambda,\mu}_{Z^2+2\lambda_1}$ is simple. Therefore, any nonzero ideal $\mathfrak{a}$ of $C^{\lambda,\mu}$ contains s^i for some $i \geqslant 1$ where $s = Z^2 + 2\lambda_1$. Hence, the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}/\mathfrak{a}$ contains a submodule, say M, which is an epimorphic image of the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}/C^{\lambda,\mu}s \simeq C^{\lambda,\mu}(\nu_1) \oplus C^{\lambda,\mu}(-\nu_1)$ since the $C^{\lambda,\mu}$-modules $V^{\lambda,\mu}(\pm\nu_1)$ and $W^{\lambda,\mu}(\pm\nu_1)$ are faithful (Lemma 3.15), the module M must be finite dimensional, by Proposition 3.8 and Corollary 3.9. □

Classification of simple finite dimensional $C^{\lambda,\mu}$-modules. The next theorem classifies the simple finite dimensional $C^{\lambda,\mu}$-modules.

Theorem 3.13. *Let $\lambda_1 \in \mathbb{K}^*$, $n = \frac{1}{2}(\mu - \frac{\lambda_2}{\nu_1})$, and $m = \frac{1}{2}(\mu + \frac{\lambda_2}{\nu_1})$, where $\nu_1^2 + 2\lambda_1 = 0$. Then*

$$\widehat{C^{\lambda,\mu}}\,(\text{fin. dim.}) = \begin{cases} \{F_n^{\lambda,\mu}(\nu_1),\ F_m^{\lambda,\mu}(-\nu_1)\} & \text{if } n,m \in \mathbb{N}_+,\\ \{F_n^{\lambda,\mu}(\nu_1),\ G_{-m}^{\lambda,\mu}(-\nu_1)\} & \text{if } n,-m \in \mathbb{N}_+,\\ \{G_{-n}^{\lambda,\mu}(\nu_1),\ F_m^{\lambda,\mu}(-\nu_1)\} & \text{if } -n,m \in \mathbb{N}_+,\\ \{G_{-n}^{\lambda,\mu}(\nu_1),\ G_{-m}^{\lambda,\mu}(-\nu_1)\} & \text{if } -n,-m \in \mathbb{N}_+,\\ \{F_n^{\lambda,\mu}(\nu_1)\} & \text{if } n \in \mathbb{N}_+, \pm m \notin \mathbb{N}_+,\\ \{G_{-n}^{\lambda,\mu}(\nu_1)\} & \text{if } -n \in \mathbb{N}_+, \pm m \notin \mathbb{N}_+,\\ \{F_m^{\lambda,\mu}(-\nu_1)\} & \text{if } \pm n \notin \mathbb{N}_+, m \in \mathbb{N}_+,\\ \{G_{-m}^{\lambda,\mu}(-\nu_1)\} & \text{if } \pm n \notin \mathbb{N}_+, -m \in \mathbb{N}_+,\\ \emptyset & \text{if } \pm n, \pm m \notin \mathbb{N}. \end{cases}$$

Proof. By Theorem 3.9, $\widehat{C^{\lambda,\mu}}\,(\text{fin. dim.}) = \emptyset$ if and only if $\pm n, \pm m \notin \mathbb{N}_+$. Let V be a simple finite dimensional $C^{\lambda,\mu}$-module. By Lemma 3.5, V is an epimorphic image of some of the $C^{\lambda,\mu}$-modules: $V^{\lambda,\mu}(\pm\nu_1)$, $W^{\lambda,\mu}(\pm\nu_1)$. Now, the equalities in the theorem follow from the description of factor modules of the modules $V^{\lambda,\mu}(\pm\nu_1)$ (Proposition 3.8) and $W^{\lambda,\mu}(\pm\nu_1)$ (Corollary 3.9). It remains to show that in the first four cases the two modules are not isomorphic. This follows from the fact that $(Z-\nu_1)^n F_n^{\lambda,\mu}(\nu_1) = 0$, $(Z-\nu_1)^t G_t^{\lambda,\mu}(\nu_1) = 0$ and $\nu_1 \neq 0$ (since $\nu_1^2 = -2\lambda_1 \neq 0$). □

Semisimplicity of the category of finite dimensional $C^{\lambda,\mu}$-modules where $\lambda_1 \neq 0$. The next theorem shows that the category of finite dimensional $C^{\lambda,\mu}$-modules is semisimple provided $\lambda_1 \neq 0$. As a corollary, the annihilator of every simple finite dimensional $C^{\lambda,\mu}$-module is an idempotent ideal.

Theorem 3.14. *Let $\lambda_1 \in \mathbb{K}^*$. Then the category of finite dimensional $C^{\lambda,\mu}$-modules is semisimple.*

Proof. The sets $S(\nu_1) = \{(Z-\nu_1)^i \mid i \in \mathbb{N}\}$ and $S(-\nu_1) = \{(Z+\nu_1)^i \mid i \in \mathbb{N}\}$ are Ore sets of the algebra $C^{\lambda,\mu}$. For a $C^{\lambda,\mu}$-module V, we denote by $\text{tor}_{S(\pm\nu_1)}(V) := \bigcup_{i\geqslant 1} \ker_V(Z \mp \nu_1)^i$ its $S(\pm\nu_1)$-torsion submodule. To prove the theorem, we have to show that every short exact sequence of $C^{\lambda,\mu}$-modules $0 \to F \to M \to \widetilde{F} \to 0$ splits where F, $\widetilde{F}$ are simple finite dimensional $C^{\lambda,\mu}$-modules. Recall that every simple finite dimensional $C^{\lambda,\mu}$-module is either $S(\nu_1)$- or $S(-\nu_1)$-torsion (but not both).

If F is $S(\nu_1)$-torsion and $\widetilde{F}$ is $S(-\nu_1)$-torsion, then $F = \text{tor}_{S(\nu_1)}(M)$ and $\text{tor}_{S(-\nu_1)}(M) \neq 0$ (since $(Z-\nu_1)^n(Z+\nu_1)^n M = 0$ for some $n \geqslant 1$ and $(Z-\nu_1)^i M \neq 0$ for all $i \geqslant 1$). Therefore, $\text{tor}_{S(-\nu_1)}(M) \simeq \widetilde{F}$ and $M = \text{tor}_{S(\nu_1)}(M) \oplus \text{tor}_{S(-\nu_1)}(M) \simeq F \oplus \widetilde{F}$, as required.

In view of Corollary 3.9.(2b) and Theorem 3.13, it suffices to consider the case when $F = \widetilde{F} = F_n^{\lambda,\mu}(\nu_1)$. Let $\bar{1}$ and $\tilde{1}$ be the canonical generators of the modules F and $\widetilde{F}$, respectively, see Lemma 3.6. We may assume that $M = F \oplus \widetilde{F}$ is a direct sum of vector spaces. By (41), $\phi\tilde{1} \in F$, $F = \text{im}(\phi) \oplus \ker(\phi)$ and $\ker(\phi) = \mathbb{K}\bar{1}$. So, $\phi\tilde{1} = \phi(f_1) + \phi_0\bar{1}$ for some $f_1 \in F$ and $\phi_0 \in \mathbb{K}$. So, replacing the generator $\tilde{1}$ by $\tilde{1} - f_1$, we can assume that $\phi\tilde{1} = \phi_0\bar{1}$.

(i) $(Z-\nu_1)\tilde{1}=0$: By (40), $v:=(Z-\nu_1)\tilde{1}\in F$. Relation (25) can be written as $[\phi, Z-\nu_1]=(Z+\nu_1)(Z-\nu_1)$. Since $(Z-\nu_1)\phi\tilde{1}=(Z-\nu_1)\phi_0\bar{1}=0$, we have $\phi v=\phi(Z-\nu_1)\tilde{1}=[\phi,Z-\nu_1]\tilde{1}=(Z+\nu_1)(Z-\nu_1)\tilde{1}=(Z+\nu_1)v$, i.e., $(\phi-Z-\nu_1)v=0$. By (40) and (41), the element $\phi-Z-\nu_1$ acts bijectively on the n-dimensional module F (since the linear map $\phi-Z-\nu_1: F\to F$ has $n=\dim(F)$ distinct nonzero eigenvalues: $-2\nu_1, -4\nu_1,\dots,-2n\nu_1$). Therefore, $v=0$.

(ii) $\phi\tilde{1}=0$: We have to show that $\phi_0=0$. By the statement (i) and (29), $0=\theta(Z+\nu_1)(Z-\nu_1)\tilde{1}=\theta(Z^2+2\lambda_1)\tilde{1}=(\phi+\mu Z-\lambda_2)\phi\tilde{1}=\phi_0(\phi+\mu Z-\lambda_2)\bar{1}=\phi_0(\mu Z-\lambda_2)\bar{1}=\phi_0(\mu\nu_1-\lambda_2)\bar{1}=\phi_0\nu_1(\mu-\frac{\lambda_2}{\nu_1})\bar{1}=\phi_0\nu_1 2n\bar{1}$, by Proposition 3.8.(2). Hence, $\phi_0=0$.

The theorem follows from the next statement.

(iii) *The $C^{\lambda,\mu}$-module M is isomorphic to the direct sum of $C^{\lambda,\mu}$-modules $F\oplus\widetilde{F}$*: By the statements (i) and (ii), the elements $\bar{1}$ and $\tilde{1}$ are $\mathbb{K}$-linearly independent elements of M that are annihilated by the elements $Z-\nu_1$ and ϕ. By (35), $V^{\lambda,\mu}(\nu_1)=C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\nu_1,\phi)=\mathbb{K}[\theta]\bar{1}$. By Proposition 3.8, the image, say F', of the $C^{\lambda,\mu}$-module homomorphism $V^{\lambda,\mu}(\nu_1)\to M$, $\bar{1}\mapsto\tilde{1}$, is isomorphic to the $C^{\lambda,\mu}$-module $F_n^{\lambda,\mu}(\nu_1)$. Since the intersection of the kernels $\ker_{F_n^{\lambda,\mu}(\nu_1)}(Z-\nu_1)\cap\ker_{F_n^{\lambda,\mu}(\nu_1)}(\phi)$ is a 1-dimensional vector space, $F\cap F'=0$ (since the elements $\bar{1}$ and $\tilde{1}$ are linearly independent), i.e., $M=F\oplus F'$, as required. □

Lemma 3.15. Let $\lambda_1\in\mathbb{K}^*$ and $\nu_1^2=-2\lambda_1$. Then $\mathrm{ann}_{C^{\lambda,\mu}}(V^{\lambda,\mu}(\nu_1))=\mathrm{ann}_{C^{\lambda,\mu}}(W^{\lambda,\mu}(\nu_1))=0$.

Proof. Let $V=V^{\lambda,\mu}(\nu_1)$ and $\mathfrak{a}:=\mathrm{ann}_{C^{\lambda,\mu}}(V)$. In view of (39), it suffices to show that $\mathfrak{a}=0$. If V is a simple $C^{\lambda,\mu}$-module, then $V\simeq\mathcal{V}(\nu_1)$ (Lemma 3.6) is a simple module over the Weyl algebra A_1. The algebra A_1 is simple, hence $0=\mathrm{ann}_{A_1}(\mathcal{V}(\nu_1))\supseteq\mathfrak{a}$, i.e., $\mathfrak{a}=0$.

If V is not a simple $C^{\lambda,\mu}$-module, then it contains a nonzero submodule $f_{n,\nu_1}^{\lambda,\mu}V^{\lambda,\mu}(\nu_1)$ (Proposition 3.8.(1)) which is isomorphic to the $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ (Lemma 3.11.(2)). By Corollary 3.9.(2), the $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ is simple, hence it is a faithful module, by Corollary 3.9.(1). Therefore, V is also a faithful module. □

The prime spectrum of $C^{\lambda,\mu}$ where $\lambda_1\neq 0$. The subalgebra $\Phi:=\mathbb{K}\langle Z,\phi\rangle$ of $C^{\lambda,\mu}$ is isomorphic to the algebra $\Phi=\mathbb{K}[Z][\phi; s\frac{d}{dZ}]$, where $s=Z^2+2\lambda_1$. We have the inclusions of algebras

$$\Phi\subset C^{\lambda,\mu}\subset A_1\subset\Phi_s=C_s^{\lambda,\mu}=A_{1,s}. \tag{46}$$

The next theorem together with the classification of finite dimensional simple $C^{\lambda,\mu}$-modules (Theorem 3.13) describes $\mathrm{Spec}\,(C^{\lambda,\mu})$.

Theorem 3.16. *Let $\lambda_1\in\mathbb{K}^*$.*

1. $\mathrm{Spec}\,(C^{\lambda,\mu})=\{0,\ \mathrm{ann}_{C^{\lambda,\mu}}(M)\mid M\in\widehat{C^{\lambda,\mu}}\,(\text{fin. dim.})\}$.
2. $\mathrm{Max}\,(C^{\lambda,\mu})=\begin{cases}\{0\} & \text{if } C^{\lambda,\mu} \text{ is simple,}\\ \{\mathrm{ann}_{C^{\lambda,\mu}}(M)\mid M\in\widehat{C^{\lambda,\mu}}\,(\text{fin. dim.})\} & \text{if } C^{\lambda,\mu} \text{ is not simple.}\end{cases}$
3. $\mathrm{Prim}\,(C^{\lambda,\mu})=\mathrm{Spec}\,(C^{\lambda,\mu})$.
4. $\mathrm{Spec}_c(C^{\lambda,\mu})=\{0,\ \mathrm{ann}_{C^{\lambda,\mu}}(M)\mid M\in\widehat{C^{\lambda,\mu}}\,(\text{fin. dim.}),\ \dim M=1\}$.

Proof. 1. Let P be a nonzero prime ideal of $C^{\lambda,\mu}$. We have to show that P is the annihilator of a finite dimensional simple $C^{\lambda,\mu}$-module. By (46), the algebra $C_s^{\lambda,\mu}=A_{1,s}$ is a simple Noetherian algebra. Hence, $s^i\in P$ for some $i\geqslant 1$. The left $C^{\lambda,\mu}$-module $C^{\lambda,\mu}/P$ is an epimorphic image of the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}/C^{\lambda,\mu}s^i$. By (31), for all $j\in\mathbb{N}$,

$$C^{\lambda,\mu}s^j/C^{\lambda,\mu}s^{j+1}\simeq C^{\lambda,\mu}/C^{\lambda,\mu}s\simeq C^{\lambda,\mu}(\nu_1)\oplus C^{\lambda,\mu}(-\nu_1)$$

since $\mathbb{K}[Z]/s\simeq\mathbb{K}[Z]/(Z-\nu_1)\times\mathbb{K}[Z]/(Z+\nu_1)$ and $\nu_1\neq-\nu_1$ (since $\lambda_1\neq 0$). By Lemma 3.5, the left $C^{\lambda,\mu}$-module $C^{\lambda,\mu}/P$ has a finite ascending chain of submodules with factors, say $F_1,\dots,F_k$, each of them is an epimorphic image of one of the $C^{\lambda,\mu}$-modules: $V^{\lambda,\mu}(\pm\nu_1), W^{\lambda,\mu}(\pm\nu_1)$. Since $PF_i=0$ for all $i=1,\dots k$, each factor F_i must be a proper epimorphic image, by Lemma 3.15. Since every proper epimorphic image of the modules $V^{\lambda,\mu}(\pm\nu_1)$, $W^{\lambda,\mu}(\pm\nu_1)$ is a finite dimensional simple $C^{\lambda,\mu}$-module, $\dim(C^{\lambda,\mu}/P)<\infty$ and all factors $F_1,\dots,F_k$ are finite dimensional simple

$C^{\lambda,\mu}$-modules. Then $\mathfrak{a}_1 \cdots \mathfrak{a}_k \subseteq P$ where $\mathfrak{a}_i = \mathrm{ann}_{C^{\lambda,\mu}}(F_i)$. Hence, $\mathfrak{a}_i \subseteq P$ for some i, and so $\mathfrak{a}_i = P$ (since $\mathfrak{a}_i$ is a maximal ideal of $C^{\lambda,\mu}$).

2. Statement 2 follows from statement 1.
3. Statement 3 follows from statement 1 and Lemma 3.15.
4. Statement 4 follows from statement 1. □

Proof of Theorem 2.11. Since $\mathcal{U}/(Z) \simeq U(\mathfrak{sl}_2)$, the ideal (Z) is a completely prime ideal. Let $\mathfrak{p} \in \mathrm{Spec}\,(\mathcal{Z})$ where $\mathcal{Z} = Z(\mathcal{U}) = \mathbb{K}[C_1, C_2]$. The factor algebra $\mathcal{Z}[Z]/\mathfrak{p}\mathcal{Z}[Z] \simeq \mathcal{Z}/\mathfrak{p}[Z]$ is a domain and $t = Z^2 + 2C_1 \notin \mathfrak{p}\mathcal{Z}[Z]$. Hence, we have the inclusion

$$\frac{\mathcal{Z}[Z]}{\mathfrak{p}\mathcal{Z}[Z]} \subset \frac{\mathcal{Z}[Z]_t}{\mathfrak{p}\mathcal{Z}[Z]_t}.$$

Since $C_t \simeq \mathbb{K}[C_1, C_2, H] \otimes A_{1,t}$, the factor algebra $C_t/\mathfrak{p}C_t \simeq \mathbb{K}[C_1, C_2]/\mathfrak{p} \otimes \mathbb{K}[H] \otimes A_{1,t}$ is a domain. By Proposition 2.5 and the equality $\mathcal{U}_t = C_t[X^{\pm 1}; \omega_X]$, we have the inclusion of algebras

$$\mathcal{U}/\mathfrak{p}\mathcal{U} \subseteq \mathcal{U}_t/\mathfrak{p}\mathcal{U}_t \simeq \Big(\frac{C_t}{\mathfrak{p}C_t}\Big)[X^{\pm 1}; \omega_X]$$

and the last one is a domain. Hence, the ideal $\mathfrak{p}\mathcal{U}$ is a completely prime ideal. □

IV. CLASSIFICATION OF SIMPLE $C^{\lambda,\mu}$-MODULES WHERE $\lambda_1 \neq 0$

In this section, a classification of simple $C^{\lambda,\mu}$-modules is given (Theorem 4.2 and Theorem 4.4) where $\lambda_1 \neq 0$. The case when $\lambda_1 = 0$ is treated in Section V. Despite the fact that the algebras $C^{\lambda,\mu}$ are more complicated algebras comparing to the Weyl algebra A_1, their simple modules are closely related.

As a corollary of Theorem 3.14 and Theorem 3.16, we obtain a classification of all the ideals of the algebra $C^{\lambda,\mu}$ provided $\lambda \neq 0$.

Corollary 4.1. Let $\lambda_1 \in \mathbb{K}^*$. Then

1. *Every nonzero ideal I of the algebra $C^{\lambda,\mu}$ is an annihilator of a finite dimensional $C^{\lambda,\mu}$-module. In particular, the factor algebra $C^{\lambda,\mu}/I$ is a finite dimensional semisimple algebra.*
2. *All ideals of the algebra $C^{\lambda,\mu}$ commute ($IJ = JI$).*
3. *All ideals of the algebra $C^{\lambda,\mu}$ are idempotent ideals ($I^2 = I$).*
4. *For all ideals I and J of the algebra $C^{\lambda,\mu}$, $I \cap J = IJ$.*
5. *Every nonzero ideal of the algebra $C^{\lambda,\mu}$ is a unique product (up to permutation) of distinct maximal ideals of $C^{\lambda,\mu}$. In particular, the number of ideals of $C^{\lambda,\mu}$ is at most 4.*
6. *Every ideal of the algebra $C^{\lambda,\mu}$ is a semiprime ideal.*

Proof. If the algebra $C^{\lambda,\mu}$ is simple, then there is nothing to prove. So, we may assume that the algebra $C^{\lambda,\mu}$ is not simple. Let P and Q be annihilators of simple finite dimensional $C^{\lambda,\mu}$-modules. By Theorem 3.14, $P^2 = P$ and $PQ = P \cap Q = QP$. By Theorem 3.16.(1), all prime ideals of the algebra $C^{\lambda,\mu}$ commute and are idempotent ideals. Let I be a nonzero ideal of $C^{\lambda,\mu}$. The algebra $C^{\lambda,\mu}$ is Noetherian. So, the set $\min(I)$ of minimal primes over I is a finite set and $\mathfrak{n}^i \subseteq I \subseteq \mathfrak{n} \subseteq C^{\lambda,\mu}$ for some $i \geqslant 1$, where $\mathfrak{n} = \cap_{P \in \min(I)} P$. By Theorem 3.16.(1), every element of $\min(I)$ is a maximal ideal of $C^{\lambda,\mu}$ of finite co-dimension. Hence, $\dim(C^{\lambda,\mu}/\mathfrak{n}) < \infty$, and so $\dim(C^{\lambda,\mu}/\mathfrak{n}^j) < \infty$ for all $j \geqslant 1$ (since $C^{\lambda,\mu}$ is a Noetherian algebra). Therefore, $\dim(C^{\lambda,\mu}/I) < \infty$. The finite dimensional algebra $C^{\lambda,\mu}/I$ is semisimple, by Theorem 3.14. This proves statement 1. Hence, $I = \mathfrak{n} = \prod_{P \in \min(I)} P$. Now, statements 2–6 follows. □

Classification of simple $C^{\lambda,\mu}$-modules where $\lambda_1 \neq 0$. The set $S_s = \{s^i \mid i \in \mathbb{N}\}$ (where $s = Z^2 + 2\lambda_1$) is an Ore set of the algebra $C^{\lambda,\mu}$ such that $C_s^{\lambda,\mu} = A_{1,s}$, see (46). Then

$$\widehat{C^{\lambda,\mu}} = \widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion}) \;\sqcup\; \widehat{C^{\lambda,\mu}}\,(S_s\text{-torsionfree}). \tag{47}$$

Descriptions of these two sets are given by Theorem 4.2 and Theorem 4.4, respectively.

The set $\widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion})$ *where* $s = Z^2 + 2\lambda_1$ *and* $\lambda_1 \neq 0$. Recall that $\pm\nu_1$ are the roots of the polynomial $Z^2 + 2\lambda_1$. By Proposition 3.8 and Corollary 3.9.(2), each of the $C^{\lambda,\mu}$-modules $V^{\lambda,\mu}(\nu_1)$ and $W^{\lambda,\mu}(\nu_1)$ admits a unique simple factor module denoted $\overline{V}^{\lambda,\mu}(\nu_1)$ and $\overline{W}^{\lambda,\mu}(\nu_1)$, respectively. Let $n = \frac{1}{2}(\mu - \frac{\lambda_2}{\nu_1})$ and $m = \frac{1}{2}(-\mu + \frac{\lambda_2}{\nu_1}) = -n$. Then

$$\overline{V}^{\lambda,\mu}(\nu_1) = \begin{cases} V^{\lambda,\mu}(\nu_1) \simeq \mathcal{V}(\nu_1) & \text{if } n \notin \mathbb{N}_+, \\ F_n^{\lambda,\mu}(\nu_1) & \text{if } n \in \mathbb{N}_+ \end{cases} \text{ and } \overline{W}^{\lambda,\mu}(\nu_1) = \begin{cases} W^{\lambda,\mu}(\nu_1) \simeq {}^{\iota}\mathcal{V}(-\nu_1) & \text{if } m \notin \mathbb{N}_+, \\ G_m^{\lambda,\mu}(\nu_1) & \text{if } m \in \mathbb{N}_+. \end{cases} \quad (48)$$

The next theorem is an explicit description of the set $\widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion})$.

Theorem 4.2. *Let* $\lambda_1 \neq 0$, $\nu_1^2 + 2\lambda_1 = 0$, *and* $s = Z^2 + 2\lambda_1$. *Then*

$$\widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion}) = \{\overline{V}^{\lambda,\mu}(\pm\nu_1),\ \overline{W}^{\lambda,\mu}(\pm\nu_1)\}$$

and the four modules in the bracket are not isomorphic unless

1. $\mu\nu_1 - \lambda_2 = 0$, $\overline{V}^{\lambda,\mu}(\nu_1) \simeq \overline{W}^{\lambda,\mu}(\nu_1)$,
2. $\mu(-\nu_1) - \lambda_2 = 0$, $\overline{V}^{\lambda,\mu}(-\nu_1) \simeq \overline{W}^{\lambda,\mu}(-\nu_1)$.

In particular, all four modules are isomorphic if and only if $\mu = 0$ *and* $\lambda_2 = 0$.

Proof. Let $M \in \widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion})$. Then M is a simple epimorphic image of the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}/C^{\lambda,\mu}s \simeq C^{\lambda,\mu}(\nu_1) \oplus C^{\lambda,\mu}(-\nu_1)$ and vice versa. By Lemma 3.5.(1), M is a simple epimorphic image of one of the $C^{\lambda,\mu}$-modules $V^{\lambda,\mu}(\pm\nu_1)$, $W^{\lambda,\mu}(\pm\nu_1)$, and vice versa. Therefore, $\widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion}) = \{\overline{V}^{\lambda,\mu}(\pm\nu_1),\ \overline{W}^{\lambda,\mu}(\pm\nu_1)\}$. It remains to sort out when some of these four modules are isomorphic or not. By Lemma 3.5.(2), statements 1 and 2 hold. Since $\overline{V}^{\lambda,\mu}(\pm\nu_1) = \bigcup_{i\geqslant 1}\ker(Z \mp \nu_1)^i$ and $\overline{W}^{\lambda,\mu}(\pm\nu_1) = \bigcup_{i\geqslant 1}\ker(Z \mp \nu_1)^i$, the only possible isomorphisms are of the type $\overline{V}^{\lambda,\mu}(\nu_1) \simeq \overline{W}^{\lambda,\mu}(\nu_1)$ or $\overline{V}^{\lambda,\mu}(-\nu_1) \simeq \overline{W}^{\lambda,\mu}(-\nu_1)$. To finish the proof of the theorem, it suffices to show that if $\overline{V}^{\lambda,\mu}(\nu_1) \simeq \overline{W}^{\lambda,\mu}(\nu_1)$ then $\mu\nu_1 - \lambda_2 = 0$. So, suppose that $\overline{V}^{\lambda,\mu}(\nu_1) \simeq \overline{W}^{\lambda,\mu}(\nu_1)$. By Corollary 3.10, if one of the modules $\overline{V}^{\lambda,\mu}(\nu_1)$ or $\overline{W}^{\lambda,\mu}(\nu_1)$ is finite dimensional then the other is necessarily infinite dimensional. Since we assume that the modules $\overline{V}^{\lambda,\mu}(\nu_1)$ and $\overline{W}^{\lambda,\mu}(\nu_1)$ are isomorphic, they must be both infinite dimensional. Then $\overline{V}^{\lambda,\mu}(\nu_1) \simeq V^{\lambda,\mu}(\nu_1)$ (Proposition 3.8) and $\overline{W}^{\lambda,\mu}(\nu_1) \simeq W^{\lambda,\mu}(\nu_1)$ (Corollary 3.9). By Lemma 3.6, $V^{\lambda,\mu}(\nu_1) \simeq \mathcal{V}(\nu_1)$. Then, by (41), the set of eigenvalues of the $\mathbb{K}$-linear map $\phi : \mathcal{V}(\nu_1) \to \mathcal{V}(\nu_1)$, $v \mapsto \phi v$, is $\mathrm{Ev}(\phi) = -2\mathbb{N}\nu_1$. By Corollary 3.9.(1), $W^{\lambda,\mu}(\nu_1) \simeq {}^{\iota}V^{\lambda,-\mu}(-\nu_1)$. Since the $C^{\lambda,\mu}$-module $W^{\lambda,\mu}(\nu_1)$ is simple and infinite dimensional, the $C^{\lambda,-\mu}$-module $V^{\lambda,-\mu}(-\nu_1)$ is so. Hence, $V^{\lambda,-\mu}(-\nu_1) \simeq \mathcal{V}(-\nu_1)$, and so $W^{\lambda,\mu}(\nu_1) \simeq {}^{\iota}\mathcal{V}(-\nu_1)$. The action of the element ϕ on $W^{\lambda,\mu}(\nu_1)$ is the same as the action of the element $\iota(\phi) = -\phi + \mu Z + \lambda_2$ on $\mathcal{V}(-\nu_1)$. By (40) and (41), $\mathrm{Ev}(\iota(\phi)) = \mu(-\nu_1) + \lambda_2 - 2\mathbb{N}\nu_1$. Therefore, if $V^{\lambda,\mu}(\nu_1) \simeq W^{\lambda,\mu}(\nu_1)$ then $-2\mathbb{N}\nu_1 = \mathrm{Ev}(\phi) = \mathrm{Ev}(\iota(\phi)) = \mu(-\nu_1) + \lambda_2 - 2\mathbb{N}\nu_1$, i.e., $\mu\nu_1 - \lambda_2 = 0$, as required. □

Let A be an algebra and M be an A-module. We denote by $l_A(M)$ the *length* of the A-module M.

Theorem 4.3. *Let* $\lambda_1 \in \mathbb{K}^*$. *For each* $a \in C^{\lambda,\mu} \setminus \{0\}$, *the* $C^{\lambda,\mu}$*-module* $C^{\lambda,\mu}/C^{\lambda,\mu}a$ *has finite length.*

Proof. Recall that the algebra $C^{\lambda,\mu}$ is a Noetherian domain of Gelfand-Kirillov dimension 2 (Proposition 3.3.(3,5)). By Lemma 3.4, the algebra $C^{\lambda,\mu}$ is a somewhat commutative algebra. Hence, $\mathrm{GK}\,(M) \leqslant 1$ where $M = C^{\lambda,\mu}/C^{\lambda,\mu}a$. If $\mathrm{GK}\,(M) = 0$ then the module M is finite dimensional, and the result is obvious. It remains to consider the case when $\mathrm{GK}\,(M) = 1$. Suppose that the $C^{\lambda,\mu}$-module has infinite length, we seek a contradiction. Then there is a descending chain of submodules of M, $M = M_0 \supset M_1 \supset \cdots \supset M_i \supset M_{i+1} \supset \cdots$, with simple factors $\overline{M}_i := M_i/M_{i+1}$. By the additivity of the multiplicity, there is a natural number n such that the factors $\overline{M}_i$ are finite dimensional for all $i \geqslant n$. Hence, the algebra $C^{\lambda,\mu}$ is not simple. Let I be the least nonzero ideal of $C^{\lambda,\mu}$. By Corollary 4.1.(1), the algebra $\overline{C} := C^{\lambda,\mu}/I$ is a semisimple finite dimensional algebra. Let $d = \dim(\overline{C})$ and m be the number of generators of the $C^{\lambda,\mu}$-module M. By Theorem 3.14, for all $i > n$, the $C^{\lambda,\mu}$-module M_n/M_i is also a $\overline{C}$-module. Hence, $\dim(M_n/M_i) \leqslant md$ for all $i > n$, a contradiction. □

The sets $\widehat{C^{\lambda,\mu}}$ $(S\text{-torsion})$ *and* $\widehat{C^{\lambda,\mu}}$ $(S\text{-torsionfree})$ *where* $\lambda_1 \neq 0$. The set $S = \mathbb{K}[Z] \setminus \{0\}$ is an Ore set of the Weyl algebra A_1 and the algebra $B_1 := S^{-1}A_1 = \mathbb{K}(Z)[\partial; \frac{d}{dZ}]$ is an Ore extension where $\mathbb{K}(Z)$ is the field of rational functions in the variable Z. The algebra B_1 is a left and right principle ideal domain, i.e., every left/right ideal of B_1 is generated by a single element. When $\mathbb{K}$ is an algebraically closed field of characteristic zero, a classification of simple A_1-modules was given by Ref. 8 (see also Refs. 3 and 4 for an alternative approach),

$$\widehat{A_1} = \widehat{A_1}\,(S\text{-torsion}) \ \sqcup\ \widehat{A_1}\,(S\text{-torsionfree}),$$

$$\widehat{A_1}\,(S\text{-torsion}) = \{[\mathcal{V}(\gamma)] \mid \gamma \in \mathbb{K}\} \ \text{where}\ \mathcal{V}(\gamma) = A_1/A_1(Z-\gamma),$$

$$\widehat{A_1}\,(S\text{-torsionfree}) = \{[M_b := A_1/A_1 \cap B_1 b] \mid b \in B_1 \ \text{is irreducible and good}\},$$

where the element $b \in B_1$ is called good if it satisfies the conditions of Ref. 8 [Theorem 1]. The set S is also an Ore set of the algebra $C^{\lambda,\mu}$ and $S^{-1}C^{\lambda,\mu} = B_1$. Then

$$\widehat{C^{\lambda,\mu}} = \widehat{C^{\lambda,\mu}}\,(S\text{-torsion}) \ \sqcup\ \widehat{C^{\lambda,\mu}}\,(S\text{-torsionfree}).$$

Clearly, $\widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion}) \subseteq \widehat{C^{\lambda,\mu}}\,(S\text{-torsion})$ since $S_s \subseteq S$.

Theorem 4.2 and Theorem 4.4 classify the set of simple $C^{\lambda,\mu}$-modules where $\lambda_1 \neq 0$. Theorem 4.4 shows a close connection between the sets of simple $C^{\lambda,\mu}$-modules and A_1-modules.

Theorem 4.4. *Let $\lambda_1 \in \mathbb{K}^*$ and $S = \mathbb{K}[Z] \setminus \{0\}$. Suppose that $\mathbb{K}$ is an algebraically closed field. Then*

1. $\widehat{C^{\lambda,\mu}}\,(S\text{-torsion}) \setminus \widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion}) = \widehat{A_1}\,(S\text{-torsion}) \setminus \{\mathcal{V}(\pm\nu_1)\} = \{[\mathcal{V}(\gamma)] \mid \gamma \in \mathbb{K} \setminus \{\pm\nu_1\}\}$, *where $\nu_1 = \sqrt{-2\lambda_1}$, i.e., every simple S-torsion A_1-module which is not isomorphic to $\mathcal{V}(\pm\nu_1)$ is a simple S-torsion $C^{\lambda,\mu}$-module which is not S_s-torsion.*
2. *The map* $\widehat{A_1}\,(S\text{-torsionfree}) \to \widehat{C^{\lambda,\mu}}\,(S\text{-torsionfree})$, $[M] \mapsto [\mathrm{soc}_{C^{\lambda,\mu}}(M)]$, *is a bijection with the inverse* $[N] \mapsto [\mathrm{soc}_{A_1}(N_s)]$.
3. *For each* $[M] \in \widehat{A_1}\,(S\text{-torsionfree})$, *i.e., $M \simeq M_b := A_1/A_1 \cap B_1 b$, where b is an irreducible and good element of B_1,* $\mathrm{soc}_{C^{\lambda,\mu}}(M_b) \simeq N_{bs^{-i}} := C^{\lambda,\mu}/C^{\lambda,\mu} \cap B_1 bs^{-i}$ *for all $i \gg 0$.*

Proof. 1. Notice that $\widehat{A_1}\,(S\text{-torsion}) = \{[\mathcal{V}(\gamma)] \mid \gamma \in \mathbb{K}\}$ and $\widehat{A_1}\,(S_s\text{-torsion}) = \{[\mathcal{V}(\pm\nu_1)]\}$.

(i) $\mathcal{A} := \widehat{A_1}\,(S\text{-torsion}) \setminus \{[\mathcal{V}(\pm\nu_1)]\} \subseteq \mathcal{C} := \widehat{C^{\lambda,\mu}}\,(S\text{-torsion}) \setminus \widehat{C^{\lambda,\mu}}\,(S_s\text{-torsion})$: We have to show that each A_1-module $\mathcal{V}(\gamma)$ where $\gamma \in \mathbb{K} \setminus \{\pm\nu_1\}$ is a simple $C^{\lambda,\mu}$-module and that two such modules are isomorphic as $C^{\lambda,\mu}$-modules $\mathcal{V}(\gamma) \simeq \mathcal{V}(\gamma')$ if and only if $\gamma = \gamma'$. Since $\mathcal{V}(\gamma) = \bigcup_{i\geqslant 0} \ker(Z-\gamma)^i$ and $\gamma \neq \pm\nu_1$, the map $s = Z^2 + 2\nu_1 = (Z-\nu_1)(Z+\nu_1) : \mathcal{V}(\gamma) \to \mathcal{V}(\gamma)$, $v \mapsto sv$ is a bijection. Therefore, $\mathcal{V}(\gamma) = \mathcal{V}(\gamma)_s$. Since $C_s^{\lambda,\mu} = A_{1,s}$, the $C^{\lambda,\mu}$-module $\mathcal{V}(\gamma)$ is simple. Clearly, the $C^{\lambda,\mu}$-modules $\mathcal{V}(\gamma)$ and $\mathcal{V}(\gamma')$ (where $\gamma, \gamma' \in \mathbb{K} \setminus \{\pm\nu_1\}$) are isomorphic if and only if $\gamma = \gamma'$.

(ii) $\mathcal{A} = \mathcal{C}$: Given $[N] \in \mathcal{C}$. Then N_s is a simple $C_s^{\lambda,\mu}$-module, i.e., N_s is a simple $\mathbb{K}[Z]_s$-torsion $A_{1,s}$-module (since $C_s^{\lambda,\mu} = A_{1,s}$). Therefore, $N_s = A_{1,s}/A_{1,s}(Z-\gamma) = \mathcal{V}(\gamma)_s$ for some $\gamma \in \mathbb{K} \setminus \{\pm\nu_1\}$. Now, the statement (ii) follows from statement (i).

2. By Theorem 4.3, the map

$$\widehat{C^{\lambda,\mu}}\,(S\text{-torsionfree}) \to \widehat{C_s^{\lambda,\mu}}\,(S\text{-torsionfree}),\ [N] \mapsto [N_s]$$

is a bijection with the inverse $[N_s] \mapsto [\mathrm{soc}_{C^{\lambda,\mu}}(N_s)]$. Similarly, the map

$$\widehat{A_1}\,(S\text{-torsionfree}) \to \widehat{A_{1,s}}\,(S\text{-torsionfree}),\ [M] \mapsto [M_s]$$

is a bijection with the inverse $[M_s] \mapsto [\mathrm{soc}_{A_1}(M_s)]$. Since $C_s^{\lambda,\mu} = A_{1,s}$, we have the inclusions $\mathrm{soc}_{C^{\lambda,\mu}}(M_s) \subseteq M \subseteq M_s$, for all M as above, and so $\mathrm{soc}_{C^{\lambda,\mu}}(M_s) = \mathrm{soc}_{C^{\lambda,\mu}}(M)$ and statement 2 follows.

3. The $C^{\lambda,\mu}$-module M_b contains the $C^{\lambda,\mu}$-module $N_b := C^{\lambda,\mu}/C^{\lambda,\mu}\cap B_1 b = C^{\lambda,\mu}\bar{1}$ (where $\bar{1} = 1 + C^{\lambda,\mu}\cap B_1 b$) which has finite length, by Theorem 4.3. The simple $C^{\lambda,\mu}$-module $L := \mathrm{soc}_{C^{\lambda,\mu}}(M_b)$ is an essential submodule of M_b. Hence, $L \subseteq N_b$. If $L = N_b$ then $C^{\lambda,\mu} = C^{\lambda,\mu}s^i + C^{\lambda,\mu}\cap B_1 b$ for all $i \geqslant 0$, and so

$$L = N_b = \frac{C^{\lambda,\mu}s^i + C^{\lambda,\mu}\cap B_1 b}{C^{\lambda,\mu}\cap B_1 b} \simeq \frac{C^{\lambda,\mu}s^i}{C^{\lambda,\mu}s^i\cap B_1 b} \simeq \frac{C^{\lambda,\mu}}{C^{\lambda,\mu}\cap B_1 bs^{-i}} = N_{bs^{-i}},$$

and we are done. If $L \subsetneq N_b$ then the $C^{\lambda,\mu}$-module N_b/L is S-torsion, and so there is a $C^{\lambda,\mu}$-module epimorphism $f : N_b/L \to U$, where U is a simple, S-torsion $C^{\lambda,\mu}$-module.

Claim: *U is S_s-torsion*: If not then, by statement 1, $U \simeq \mathcal{V}(\gamma)$ for some $\gamma \neq \pm\nu_1$. Then $\mathcal{V}(\gamma) = \mathcal{V}(\gamma)_s$ is simple $C_s^{\lambda,\mu}$-module/$A_{1,s}$-module since $C_s^{\lambda,\mu} = A_{1,s}$. There is a commutative diagram of $C^{\lambda,\mu}$-homomorphisms,

$$\begin{array}{ccc} N_b & \hookrightarrow & N_{b,s} \\ \downarrow & & \downarrow g \\ N_b/L & \xrightarrow{f} & \mathcal{V}(\gamma) = \mathcal{V}(\gamma)_s, \end{array}$$

where g is an epimorphism. The $C^{\lambda,\mu}$-module/$A_{1,s}$-module $N_{b,s}$ is a nonzero one which is an $A_{1,s}$-submodule of the simple $A_{1,s}$-module $M_{b,s}$. Hence, $M_{b,s} = N_{b,s} \simeq \mathcal{V}(\gamma)_s$, a contradiction. Therefore, U is S_s-torsion.

By Theorem 4.3, the $C^{\lambda,\mu}$-module N_b has finite length. Therefore, the descending chain $\{L_i := C^{\lambda,\mu}s^i\bar{1} \mid i \in \mathbb{N}\}$ of $C^{\lambda,\mu}$-modules of N_b stabilizes, say, at jth step: $N_0 = L_0 \supseteq L_1 \supseteq \ldots \supseteq L_j = L_{j+1} = \cdots$. For all $i \in \mathbb{N}$,

$$L_i \simeq \frac{C^{\lambda,\mu}s^i + C^{\lambda,\mu}\cap B_1 b}{C^{\lambda,\mu}\cap B_1 b} \simeq N_{bs^{-i}}.$$

By the claim and the choice of j, we have $L = L_j = L_{j+1} = \cdots$, and so $L \simeq N_{bs^{-i}}$ for all $i \geqslant j$, as required. □

Corollary 4.5. Let $\lambda_1 \in \mathbb{K}^*$ and $\nu_1 = \sqrt{-2\lambda_1}$. Then

1. *The set $\widehat{C^{\lambda,\mu}}$ (S_s-torsionfree) is a disjoint union of the sets in statements 1 and 2 of Theorem 4.4.*
2. *For each $\gamma \in \mathbb{K}\setminus\{\pm\nu_1\}$, $\mathcal{V}(\gamma) \simeq C^{\lambda,\mu}(\gamma) := C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\gamma)$.*

Proof. 1. Statement 1 follows from Theorem 4.4.

2. Since $\gamma \neq \pm\nu_1$ and $\mathcal{V}(\gamma) = \bigcup_{i\geqslant 0}\ker(Z-\gamma)^i$, the map $s_{\lambda_1} := Z^2 + 2\lambda_1 : \mathcal{V}(\gamma) \to \mathcal{V}(\gamma)$, $v \mapsto (Z^2+2\lambda_1)v$ is a bijection. Hence, $\mathcal{V}(\gamma) = \mathcal{V}(\gamma)_s$. Similarly, since $\gamma \neq \pm\nu_1$ and $C^{\lambda,\mu}(\gamma) = \bigcup_{i\geqslant 0}\ker(Z-\gamma)^i$, the map $s_{\lambda_1} = Z^2 + 2\lambda_1 : C^{\lambda,\mu}(\gamma) \to C^{\lambda,\mu}(\gamma)$, $v \mapsto (Z^2+2\lambda_1)v$ is a bijection. Hence, $C^{\lambda,\mu}(\gamma) = C^{\lambda,\mu}(\gamma)_s \simeq \mathcal{V}(\gamma)_s = \mathcal{V}(\gamma)$. □

V. CLASSIFICATION OF SIMPLE $C^{\lambda,\mu}$-MODULES WHERE $\lambda_1 = 0$

In this section, the following notation is fixed: $\lambda = -\lambda_2$, $C^{\lambda,\mu} := C^{0,-\lambda_2,\mu}$, and $C^{0,\mu} = C^{0,0,\mu}$. The simple $C^{\lambda,\mu}$-modules were classified in Ref. 7 [Section 4]. In this section, we recall this classification. The cases when $\lambda_2 \neq 0$ and $\lambda_2 = 0$ are quite different. We assume that the field $\mathbb{K}$ is an algebraically closed field of characteristic zero.

By Proposition 3.3, the algebra $C^{\lambda,\mu} = C^{0,-\lambda_2,\mu}$ is generated by the elements Z,θ, and ϕ that satisfy the defining relations,

$$[\phi,Z] = Z^2, \qquad [\theta,Z] = 2\phi + (\mu-2)Z + \lambda,$$

$$[\theta,\phi] = 2\theta Z + (-\phi+2Z)\mu, \qquad \theta Z^2 = (\phi+\mu Z+\lambda)\phi,$$

and it is a subalgebra of the Weyl algebra A_1 via a monomorphism $C^{\lambda,\mu} \to A_1$, $Z \mapsto Z$, $\phi \mapsto hZ$, $\theta \mapsto \lambda\partial + (h+\mu)(h-1)$. Furthermore, $C^{\lambda,\mu} \subset A_1 \subset A_{1,Z} = C_Z^{\lambda,\mu}$.

Classification of simple $C^{\lambda,\mu}$-modules where $\lambda \neq 0$. The Weyl algebra A_1 is a subalgebra of the skew Laurent polynomial algebra $B = \mathbb{K}(h)[Z, Z^{-1}; \sigma]$, where $\sigma(h) = h - 1$. The algebra B is the localization $S^{-1}A_1$ of the Weyl algebra A_1 at the (left and right) Ore set $S := \mathbb{K}[h] \setminus \{0\}$. The algebra B is a Euclidean ring with left and right division algorithms. In particular, the algebra B is a principle left and right ideal domain. Each simple B-module is isomorphic to B/Bb where b is an irreducible (indecomposable) element of B. B-modules B/Bb and B/Bc are isomorphic if and only if the elements b and c are *similar*, i.e., there exists an element $d \in B$ such that 1 is the greatest common right divisor of c and d, and bd is their least common left multiple.

Let $\alpha, \beta \in S = \mathbb{K}[h] \setminus \{0\}$. We write $\alpha < \beta$ if there are no roots λ and μ of the polynomials α and β, respectively, such that $\lambda - \mu \in \mathbb{N}$.

Definition, [Ref. 4]. An element $b = \partial^m \beta_m + \partial^{m-1}\beta_{m-1} + \cdots + \beta_0$, where $m > 0$, $\beta_i \in \mathbb{K}[h]$ and $\beta_0, \beta_m \neq 0$, is called *normal* if $\beta_0 < \beta_m$ and $\beta_0 < h$.

For a simple A_1-module M there are two options either $S^{-1}M = 0$ or $S^{-1}M \neq 0$. Accordingly, we say that the simple module is $\mathbb{K}[h]$-*torsion* or $\mathbb{K}[h]$-*torsionfree*, respectively.

Theorem 5.1. *[Refs. 3 and 4].* $\widehat{A_1} = \widehat{A_1}(\mathbb{K}[h]\text{-torsion}) \sqcup \widehat{A_1}(\mathbb{K}[h]\text{-torsionfree})$ *where*

1. $\widehat{A_1}(\mathbb{K}[h]\text{-torsion}) = \{A_1/A_1Z,\ A_1/A_1\partial,\ A_1/A_1(h - \lambda_O) \mid O \in \mathbb{K}/\mathbb{Z} \setminus \{\mathbb{Z}\}\}$ *where* λ_O *is any fixed element of* $O = \lambda_O + \mathbb{Z}$.
2. *Each simple* $\mathbb{K}[h]$*-torsionfree* A_1*-module is isomorphic to* $M_b := A_1/A_1 \cap Bb$ *for a normal, irreducible element* b*. Simple* A_1*-modules* M_b *and* $M_{b'}$ *are isomorphic if and only if the elements* b *and* b' *are similar.*

The following theorem gives a classification of simple $C^{\lambda,\mu}$-modules where $\lambda \neq 0$. It shows that there is a tight connection between the sets of simple $C^{\lambda,\mu}$-modules and A_1-modules. The theorem gives an explicit construction for each simple $C^{\lambda,\mu}$-module as a factor module $C^{\lambda,\mu}/I$ where I is a left maximal ideal of $C^{\lambda,\mu}$. For a $C^{\lambda,\mu}$-module M, we denote by $l_{C^{\lambda,\mu}}(M)$ its *length*.

Theorem 5.2. *[Ref. 7]. Let* $\lambda \in \mathbb{K}^*$ *and* $\mu \in \mathbb{K}$*. Then*

1. *The map* $\mathrm{soc} = \mathrm{soc}_{C^{\lambda,\mu}} : \widehat{A_1} \longrightarrow \widehat{C^{\lambda,\mu}}$, $[M] \mapsto [\mathrm{soc}_{C^{\lambda,\mu}}(M)]$*, is an injection, and* $\widehat{C^{\lambda,\mu}} = \mathrm{soc}(\widehat{A_1}) \sqcup \{N^{\lambda,\mu}\}$*. Furthermore,*
   - (a) *the map* $\mathrm{soc}^{tf} : \widehat{A_1}(Z\text{-torsionfree}) \longrightarrow \widehat{C^{\lambda,\mu}}(Z\text{-torsionfree})$, $[M] \mapsto [\mathrm{soc}_{C^{\lambda,\mu}}(M)]$*, is a bijection, but*
   - (b) *the map* $\mathrm{soc}^{tt} : \widehat{A_1}(Z\text{-torsion}) = \{A_1/A_1Z\} \longrightarrow \widehat{C^{\lambda,\mu}}(Z\text{-torsion}) = \{M^{\lambda,\mu}, N^{\lambda,\mu}\}$, $[A_1/A_1Z] \mapsto [M^{\lambda,\mu}]$*, is an injection which is not a bijection where* $M^{\lambda,\mu} = C^{\lambda,\mu}/C^{\lambda,\mu}(Z, \phi)$ *and* $N^{\lambda,\mu} = C^{\lambda,\mu}/C^{\lambda,\mu}(Z, \phi + \lambda)$*. In particular, the simple* $C^{\lambda,\mu}$*-modules* $M^{\lambda,\mu}$ *and* $N^{\lambda,\mu}$ *are not isomorphic.*
2. *For each* $[M] \in \widehat{A_1}(\mathbb{K}[h]\text{-torsion})$*, the* $C^{\lambda,\mu}$*-module* M *is simple, i.e.,* $\mathrm{soc}_{C^{\lambda,\mu}}(M) = M$.
3. *For each* $[M] \in \widehat{A_1}(\mathbb{K}[h]\text{-torsionfree})$*, i.e.,* $M = M_b = A_1/A_1 \cap Bb$*, where* $b \in B$ *is as in Theorem 5.1.(2),* $N_b := C^{\lambda,\mu}/C^{\lambda,\mu} \cap Bb \subseteq M_b$ *and* $\mathrm{soc}_{C^{\lambda,\mu}}(M_b) = \mathrm{soc}_{C^{\lambda,\mu}}(N_b) \simeq N_{bt^{-n}}$ *for all* $n \gg 0$.

Classification of simple $C^{0,\mu}$-modules. The subalgebra R of the Weyl algebra A_1 which is generated by the elements Z and h is a skew polynomial algebra $R = \mathbb{K}[h][Z; \sigma]$, where $\sigma(h) = h - 1$. The algebra R is a homogeneous subalgebra of the $\mathbb{Z}$-graded algebra A_1, it is the non-negative part of the $\mathbb{Z}$-grading of A_1. For all $\mu \in \mathbb{K}$, $C^{0,\mu} \subset R \subset A_1$ and the subalgebra $C^{0,\mu}$ of R is generated by the elements $Z, \phi = hZ$ and $\theta = (h + \mu)(h - 1)$. Clearly, $\mathbb{K}[\theta] \subseteq \mathbb{K}[h]$ and $\mathbb{K}[h] = \mathbb{K}[\theta] \oplus \mathbb{K}[\theta]h$. The element Z is a normal element of the algebra R and $(Z) = \oplus_{i \geqslant 1}\mathbb{K}[h]Z^i$. The set $S = \mathbb{K}[h] \setminus \{0\}$ is a (left and right) Ore set of the domain $C^{0,\mu}$ and $B := S^{-1}C^{0,\mu} = \mathbb{K}(h)[Z; \sigma]$ is a skew polynomial algebra where $\sigma(h) = h - 1$. The algebra B is a principle (left and right) ideal domain. Let $\mathrm{Irr}(B)$ be the set of irreducible elements of B.

Theorem 5.3. *[Ref. 7].*

1. $\widehat{C^{0,\mu}}(Z\text{-torsion}) = \{[M] \in \widehat{C^{0,\mu}} \mid (Z)M = 0\} = \widehat{C^{0,\mu}/(Z)} = \{[C^{0,\mu}/C^{0,\mu}(\theta - \nu, Z, \phi)] \mid \nu \in \mathbb{K}\}$. *The set* $\widehat{C^{0,\mu}}(Z\text{-torsion})$ *contains precisely finite dimensional simple* $C^{0,\mu}$*-modules (all of them are 1-dimensional).*
2. $\widehat{C^{0,\mu}}(Z\text{-torsionfree}) = \widehat{R}(Z\text{-torsionfree}) = \widehat{R}(\mathbb{K}[h]\text{-torsionfree}) = \{[M_b = R/R \cap Bb] \mid b \in \mathrm{Irr}(B), R = RZ + R \cap Bb\}$.

The next theorem is a simplicity criterion for the algebra $C^{0,\lambda_2,\mu}$.

Theorem 5.4. *[Ref. 7]. The algebra* $C^{0,\lambda_2,\mu}$ *is simple if and only if* $\lambda_2 \neq 0$.

VI. A CLASSIFICATION OF SIMPLE WEIGHT $\mathcal{U}$-MODULES

In this section, a classification of simple weight $\mathcal{U}$-modules is given. They are partitioned into several classes that are dealt with separately (see the Introduction for details).

Weight $\mathcal{U}$*-modules.* An $\mathcal{U}$-module M is called a *weight module* if $M = \oplus_{\mu\in\mathbb{K}} M_\mu$, where $M_\mu = \{m \in M \mid Hm = \mu m\}$. An element $\mu \in \mathbb{K}$ such that $M_\mu \neq 0$ is called a *weight* of M. Let $\mathrm{Wt}(M)$ be the set of all weights of the module M.

Verma module. Let $\alpha, \beta \in \mathbb{K}$, we define the *Verma modules* $M(\alpha,\beta) := \mathcal{U}/\mathcal{U}(H-\alpha, Z-\beta, E, X)$. Since the 4-dimensional space $\mathbb{K}H \oplus \mathbb{K}E \oplus \mathbb{K}Z \oplus \mathbb{K}X$ is a Lie subalgebra of $\mathfrak{e}(3)$, the $\mathcal{U}$-module $M(\alpha,\beta) = \mathbb{K}[F,Y]\bar{1}$ is a free $\mathbb{K}[F,Y]$-module where $\bar{1} = 1 + \mathcal{U}(H-\alpha, Z-\beta, E, X)$. Then

$$M(\alpha,\beta) = \bigoplus_{n=0}^{\infty} M(\alpha,\beta)_{\alpha-2n}, \text{ where } M(\alpha,\beta)_{\alpha-2n} := \ker_{M(\alpha,\beta)}(H-\alpha+2n) = \bigoplus_{i=0}^{n} \mathbb{K}F^iY^{n-i}\bar{1} \quad (49)$$

Hence, $\mathrm{Wt}\big(M(\alpha,\beta)\big) = \alpha - 2\mathbb{N}$ and $\dim M(\alpha,\beta)_{\alpha-2n} = n+1$ for all $n \geqslant 0$.

Proposition 6.1. 1. $\mathrm{Wt}\big(M(\alpha,\beta)\big) = \{\alpha - 2n \mid n \in \mathbb{N}\}$ *and* $\dim M(\alpha,\beta)_{\alpha-2n} = n+1$ *for all* $n \in \mathbb{N}$.

2. $M(\alpha,\beta) \simeq M(\alpha',\beta')$ *if and only if* $(\alpha,\beta) = (\alpha',\beta')$.
3. *The Verma module* $M(\alpha,\beta)$ *is a simple* $\mathcal{U}$*-module if and only if* $\beta \neq 0$.
4. *If* $\beta \neq 0$ *then* $\mathrm{ann}_{\mathcal{U}}\big(M(\alpha,\beta)\big) = \big(C_1 + \frac{1}{2}\beta^2, C_2 - (\alpha+2)\beta\big)$.

Proof. 1. Statement 1 follows from (49).

2. If the $\mathcal{U}$-module $M(\alpha,\beta)$ and $M(\alpha',\beta')$ are isomorphic, then $\alpha - 2\mathbb{N} = \mathrm{Wt}\big(M(\alpha,\beta)\big) = \mathrm{Wt}\big(M(\alpha',\beta')\big) = \alpha' - 2\mathbb{N}$, i.e., $\alpha = \alpha' - 2n$ for some $n \in \mathbb{N}$, and $1 = \dim M(\alpha,\beta)_\alpha = \dim M(\alpha',\beta')_{\alpha'-2n} = n+1$, i.e., $n = 0$ and $\alpha = \alpha'$. The vector space $M(\alpha,\beta)_\alpha = M(\alpha,\beta')_\alpha$ is one dimensional and is Z-invariant. Hence, $\beta = \beta'$.

3. Suppose that $\beta \neq 0$ and N is a nonzero submodule of $M(\alpha,\beta)$. We have to show that N contains the canonical generator $\bar{1}$ of the $\mathcal{U}$-module $M(\alpha,\beta)$. Clearly, $N = \oplus_{n=0}^{\infty} N_{\alpha-2n}$, where $N_{\alpha-2n} = N \cap M(\alpha,\beta)_{\alpha-2n}$. Since N is nonzero, $N_{\alpha-2n}$ is nonzero for some $n \in \mathbb{N}$. Let $0 \neq v = \sum_{i=0}^{m} \alpha_i F^i Y^{n-i}\bar{1} \in N_{\alpha-2n}$, where $\alpha_i \in \mathbb{K}$, $\alpha_m \neq 0$ and $0 \leqslant m \leqslant n$. Notice that $(Z-\mu)^m v = (-1)^m m!\,\alpha_m Y^n\bar{1} \in N$, hence $Y^n\bar{1} \in N$. Then $E \cdot Y^n\bar{1} = 2n\mu Y^{n-1}\bar{1}$, and so $E^n \cdot Y^n\bar{1} \in \mathbb{K}^*\bar{1}$, i.e., $\bar{1} \in N$, as required.

If $\beta = 0$ then the Verma module $M(\alpha, 0)$ is not a simple $\mathcal{U}$-module since the left ideal $\mathcal{U}(H-\alpha, Z, E, X)$ is properly contained in the left ideal $\mathcal{J} := \mathcal{U}(H-\alpha, Z, E, X, Y) = U(\mathfrak{sl}_2)(H-\alpha, E) + (Z)$ by Lemma 2.7.(1). This follows from the facts $\mathcal{U}/\mathcal{J} \simeq U(\mathfrak{sl}_2)/U(\mathfrak{sl}_2)(H-\alpha,E) \simeq \mathbb{K}[F]\tilde{1}$ and $M(\alpha,\beta) \simeq \mathbb{K}[F,Y]\bar{1}$. This means that $YM(\alpha,\beta)$ is a proper submodule of $M(\alpha,\beta)$.

4. Clearly, $\big(C_1 + \frac{1}{2}\mu^2, C_2 - (\lambda+2)\mu\big) \subseteq \mathrm{ann}_{\mathcal{U}}\big(M(\lambda,\mu)\big)$. Then the equality holds since the ideal $\big(C_1 + \frac{1}{2}\mu^2, C_2 - (\lambda+2)\mu\big)$ is maximal, by (1). □

Dual Verma module. For $\alpha, \beta \in \mathbb{K}$, we define the *dual Verma module* $M^\star(\alpha,\beta) := \mathcal{U}/\mathcal{U}(H-\alpha, Z-\beta, F, Y)$. Then $M^\star(\alpha,\beta) \simeq {}^{\iota}M(-\alpha,-\beta)$, where ${}^{\iota}M(-\alpha,-\beta)$ is the Verma $\mathcal{U}$-module $M(-\alpha,-\beta)$

twisted by the automorphism ι of the algebra $\mathcal{U}$. Notice that $M^\star(\alpha,\beta)=\mathbb{K}[E,X]\tilde{1}$ is a free $\mathbb{K}[E,X]$-module where $\tilde{1}=1+\mathcal{U}(H-\alpha,Z-\beta,F,Y)$. Then

$$M^\star(\alpha,\beta)=\bigoplus_{n=0}^{\infty}M^\star(\alpha,\beta)_{\alpha+2n},\ \text{where}\ M^\star(\alpha,\beta)_{\alpha+2n}:=\bigoplus_{i=0}^{n}\mathbb{K}E^iX^{n-i}\tilde{1}. \tag{50}$$

We summarize the properties of the dual Verma module $M^\star(\alpha,\beta)$ in the following proposition.

Proposition 6.2. 1. $\mathrm{Wt}\big(M^\star(\alpha,\beta)\big)=\{\alpha+2n\mid n\in\mathbb{N}\}$ *and* $\dim M^\star(\alpha,\beta)_{\alpha+2n}=n+1$ *for all* $n\in\mathbb{N}$.

2. $M^\star(\alpha,\beta)\simeq M^\star(\alpha',\beta')$ *if and only if* $(\alpha,\beta)=(\alpha',\beta')$.
3. *The dual Verma module* $M^\star(\alpha,\beta)$ *is a simple* $\mathcal{U}$*-module if and only if* $\beta\neq 0$.
4. *If* $\beta\neq 0$ *then* $\mathrm{ann}_{\mathcal{U}}\big(M^\star(\alpha,\beta)\big)=\big(C_1+\frac{1}{2}\beta^2, C_2-(\alpha-2)\beta\big)$.

Proof. The result follows from Proposition 6.1 since $M^\star(\alpha,\beta)\simeq {}^{\iota}M(-\alpha,-\beta)$. □

Classification of simple highest weight modules. Let V be a weight $\mathcal{U}$-module. A weight vector $v\in V$ is called a *highest weight vector* if $Ev=0$ and $Xv=0$. The $\mathcal{U}$-module V is called a *highest weight module* if V is generated by a highest weight vector. Clearly, the Verma modules $M(\alpha,\beta)$ are highest weight modules. The following proposition gives a classification of simple highest weight $\mathcal{U}$-modules.

Proposition 6.3. Let V be a simple highest weight $\mathcal{U}$-module. Then V is isomorphic to one of the following modules:

1. *the Verma modules $M(\alpha,\beta)$ where $\alpha\in\mathbb{K}$ and $\beta\in\mathbb{K}^*$, or*
2. *the simple highest weight $U(\mathfrak{sl}_2)$-modules.*

Proof. Let $v\in V_\alpha$ be a highest weight vector of V. Since V is a simple $\mathcal{U}$-module, the central element C_2 acts on V as a scalar, say λ_2. Then $\lambda_2v=C_2v=(\alpha+2)Zv$.

If $\alpha+2\neq 0$ then $Zv=\frac{\lambda_2}{\alpha+2}v$. So, V is an epimorphic image of the Verma module $M(\alpha,\frac{\lambda_2}{\alpha+2})$. If $\lambda_2\neq 0$ then, by Proposition 6.1.(3), $M(\alpha,\frac{\lambda_2}{\alpha+2})$ is a simple module and hence $V\simeq M(\alpha,\frac{\lambda_2}{\alpha+2})$. If $\lambda_2=0$ then V is isomorphic to a simple factor module of the Verma module $M(\alpha,0)$. But then $\mathrm{ann}_{\mathcal{U}}(V)\supset(Z)$, i.e., V is a simple (highest weight) $U(\mathfrak{sl}_2)$-module.

If $\alpha+2=0$ then $C_2v=0$. The central element C_1 acts on V as a scalar, say λ. Then $\lambda v=C_1V=-\frac{1}{2}Z^2v$. So, V is an epimorphic image of the $\mathcal{U}$-module $V(\lambda)=\mathcal{U}/\mathcal{U}(\frac{1}{2}Z^2+\lambda,H+2,E,X)$. If $\lambda\neq 0$ then $V(\lambda)$ has two largest submodules: $V(+)=\mathcal{U}v^+=\mathbb{K}[F,Y]v^+$, where $v^+=(Z+\sqrt{-2\lambda})\bar{1}$ and $V(-)=\mathcal{U}v^-=\mathbb{K}[F,Y]v^-$, where $v^-=(Z-\sqrt{-2\lambda})\bar{1}$ (where $\bar{1}=1+\mathcal{U}(\frac{1}{2}Z^2+\lambda,H+2,E,X)$). The two simple factor modules of $V(\lambda)$ are $L(+)=V(\lambda)/V(+)\simeq\mathcal{U}/\mathcal{U}(Z+\sqrt{-2\lambda},H+2,E,X)\simeq M(-2,-\sqrt{-2\lambda})$ and $L(-)=V(\lambda)/V(-)\simeq\mathcal{U}/\mathcal{U}(Z-\sqrt{-2\lambda},H+2,E,X)\simeq M(-2,\sqrt{-2\lambda})$, respectively. If $\lambda=0$ then V is isomorphic to a simple factor module of $V(0)=\mathcal{U}/\mathcal{U}(Z^2,H+2,E,X)$. Then it is clear that V is a simple (highest weight) $U(\mathfrak{sl}_2)$-module. □

Classification of simple lowest weight modules. Let V be a weight $\mathcal{U}$-module. A weight vector $v\in V$ is called a *lowest weight vector* if $Fv=0$ and $Yv=0$. The $\mathcal{U}$-module V is called a *lowest weight module* if V is generated by a lowest weight vector. Clearly, the dual Verma modules $M^\star(\alpha,\beta)$ are lowest weight modules. The following proposition gives a classification of simple lowest weight $\mathcal{U}$-modules.

Proposition 6.4. Let V be a simple lowest weight $\mathcal{U}$-module. Then V is isomorphic to one of the following modules:

1. *the dual Verma modules $M^\star(\alpha,\beta)$ where $\alpha\in\mathbb{K}$ and $\beta\in\mathbb{K}^*$, or*
2. *the simple lowest weight $U(\mathfrak{sl}_2)$-modules.*

Proof. The result follows from Proposition 6.3 by applying the automorphism ι, see (3). In particular, $M^\star(\alpha,\beta)\not\simeq M(\alpha',\beta')$ for all $\alpha,\alpha'\in\mathbb{K}$ and $\beta,\beta'\in\mathbb{K}^*$, by Proposition 6.1.(1) and Proposition 6.2.(1). □

Simple weight modules with a finite dimensional weight space. First, we give an example of simple weight $\mathcal{U}$-module with infinite dimensional weight spaces. For $\alpha, \beta \in \mathbb{K}$, we define the left $\mathcal{U}$-module $S(\alpha,\beta) = \mathcal{U}/\mathcal{U}(H-\alpha, Z-\beta, X, Y)$. Then $S(\alpha,\beta) = \sum_{i,j\in\mathbb{N}} \mathbb{K}E^iF^j\bar{1}$, where $\bar{1} = 1 + \mathcal{U}(H-\alpha, Z-\beta, X, Y)$.

Lemma 6.5. 1. The module $S(\alpha,\beta)$ is a simple $\mathcal{U}$-module if and only if $\beta \neq 0$.

2. $\mathrm{Wt}\big(S(\alpha,\beta)\big) = \{\alpha + 2n \mid n \in \mathbb{Z}\}$ *and each weight space is infinite dimensional. Moreover,*

$$S(\alpha,\beta)_{\alpha+2n} = \begin{cases} \bigoplus_{i\in\mathbb{N}} \mathbb{K}E^{n+i}F^i\bar{1}, & \text{if } n \geqslant 0, \\ \bigoplus_{i\in\mathbb{N}} \mathbb{K}E^iF^{i-n}\bar{1}, & \text{if } n \leqslant -1. \end{cases}$$

3. *If $\beta \neq 0$ then $\mathrm{ann}_{\mathcal{U}}(S(\alpha,\beta)) = (C_1 + \frac{1}{2}\beta^2, C_2 - \alpha\beta)$.*
4. *Let $(\alpha,\beta), (\alpha',\beta') \in \mathbb{K}\times\mathbb{K}^*$. Then $S(\alpha,\beta) \simeq S(\alpha',\beta')$ if and only if $(\alpha,\beta) = (\alpha',\beta')$.*

Proof. 1. Suppose that $\beta \neq 0$ we prove that $S(\alpha,\beta)$ is a simple $\mathcal{U}$-module. It suffices to prove that for any nonzero element $v = \sum_{j=0}^n f_j(E)F^j\bar{1} \in S(\alpha,\beta)$, where $f_j(E) \in \mathbb{K}[E]$ and $f_n(E) \neq 0$, there exists some element $u \in \mathcal{U}$ such that $uv \in \mathbb{K}^*\bar{1}$. By Lemma 2.6.(3), $X\cdot v = \sum_{j=1}^n f_j(E)[X, F^j]\bar{1}$ $= \sum_{j=1}^n f_j(E)(-jF^{j-1}Z + \frac{1}{2}j(j-1)F^{j-2}Y)\bar{1} = \sum_{j=1}^n f_j(E)(-j)\beta F^{j-1}\bar{1}$. Hence, $X^n\cdot v$ is a nonzero element in $\mathbb{K}[E]\bar{1}$. Thus we may assume that v is a nonzero element in $\mathbb{K}[E]\bar{1}$ and then v can be written as $v = \sum_{i=0}^n \alpha_i E^i\bar{1}$, where $\alpha_i \in \mathbb{K}$ and $\alpha_n \neq 0$. Since $Y\cdot v = \sum_{i=1}^n \alpha_i[Y, E^i]\bar{1} = \sum_{i=1}^n \alpha_i(-2)i\beta$ $E^{i-1}\bar{1}$, we have $Y^n\cdot v \in \mathbb{K}^*\bar{1}$, as required.

If $\beta = 0$ then, by Lemma 2.7.(1), the left ideal $\mathcal{U}(H-\alpha, Z, X, Y) = U(\mathfrak{sl}_2)(H-\alpha) + (Z)$. Then it is clear that $S(\alpha, 0) \simeq U(\mathfrak{sl}_2)/U(\mathfrak{sl}_2)(H-\alpha)$ is not a simple module.

2. The above argument also shows that $S(\alpha,\beta) = \oplus_{i,j\in\mathbb{N}}\mathbb{K}E^iF^j\bar{1}$. Hence, $\mathrm{Wt}(S(\alpha,\beta)) = \{\alpha + 2n \mid n \in \mathbb{Z}\}$. The rest is clear.

3. It is clear that $(C_1 + \frac{1}{2}\beta^2, C_2 - \alpha\beta) \subseteq \mathrm{ann}_{\mathcal{U}}(S(\alpha,\beta))$, the equality holds since $(C_1 + \frac{1}{2}\beta^2, C_2 - \alpha\beta)$ is a maximal ideal of $\mathcal{U}$, by (1).

4. Suppose that $S(\alpha,\beta) \simeq S(\alpha',\beta')$. Then by statement 3, $\frac{1}{2}\beta^2 = \frac{1}{2}\beta'^2$ and $\alpha\beta = \alpha'\beta'$. The case $\beta = -\beta'$ is not possible, since, otherwise, both the elements $Z-\beta$ and $Z+\beta$ act locally nilpotently on $S(\alpha,\beta)$. This implies that $\beta = 0$, a contradiction. So, $\beta = \beta'$ and then $\alpha = \alpha'$. □

Let $\mathcal{F}$ be the set of simple weight $\mathcal{U}$-modules with a finite dimensional weight space, and $\mathcal{B}$ be the set of simple highest weight and lowest weight modules. By Proposition 6.3 and Proposition 6.4, $\mathcal{B} \subseteq \mathcal{F}$. The next proposition describes the modules of the set $\mathcal{F}\setminus\mathcal{B}$. Recall that $\Delta := 4FE + H^2 + 2H$ is the Casimir element of $U(\mathfrak{sl}_2)$.

Proposition 6.6. Let $V \in \mathcal{F}\setminus\mathcal{B}$. Then

1. $\mathrm{Wt}(V) = \{\alpha + 2n \mid n \in \mathbb{Z}\}$ *for any* $\alpha \in \mathrm{Wt}(V)$ *and* $\dim V_\alpha = \dim V_{\alpha+2n}$ *for all* $n \in \mathbb{Z}$.
2. $\mathrm{ann}_{\mathcal{U}}(V) \supset (Z)$, *i.e., V is a simple $U(\mathfrak{sl}_2)$-module.*
3. $V \simeq V(\alpha,\lambda) := U(\mathfrak{sl}_2)/U(\mathfrak{sl}_2)(H-\alpha, \Delta-\lambda)$, *where* $\lambda \neq (\alpha+2i)(\alpha+2i-2)$ *for all* $i \in \mathbb{Z}$; $V(\alpha,\gamma) \simeq V(\alpha',\gamma')$ *if and only if* $\lambda = \lambda'$ *and* $\alpha - \alpha' \in 2\mathbb{Z}$ *and* $\dim V_{\alpha+2n} = 1$ *for all* $n \in \mathbb{Z}$.

Proof. 1. Since V is a simple module, $\mathrm{Wt}(V) \subseteq \{\alpha + 2n \mid n \in \mathbb{Z}\}$ for any $\alpha \in \mathrm{Wt}(V)$. Suppose that there exists $\alpha \in \mathrm{Wt}(V)$ such that $\dim V_\alpha > \dim V_{\alpha+2}$ then the maps $X : V_\alpha \to V_{\alpha+2}$ and $E : V_\alpha \to V_{\alpha+2}$ are not injections. Then the elements X and E act locally nilpotently on V. Since $XE = EX$, there exists a weight vector $v \in V$ such that $Xv = Ev = 0$. Then V is a highest weight module, a contradiction. Similarly, if $\dim V_\alpha < \dim V_{\alpha+2}$ for some $\alpha \in \mathrm{Wt}(V)$ then $Y : V_{\alpha+2} \to V_\alpha$ and $F : V_{\alpha+2} \to V_\alpha$ are not injections. Then the elements Y and F act locally nilpotently on V. Since $YF = FY$, there exists a weight vector v such that $Fv = Yv = 0$. Then V is a lowest weight module, a contradiction. Therefore, $\dim V_\alpha = \dim V_\beta$ for all $\alpha, \beta \in \mathrm{Wt}(V)$ and $\mathrm{Wt}(V) = \{\alpha + 2n \mid n \in \mathbb{Z}\}$ for any $\alpha \in \mathrm{Wt}(V)$.

2. Since V is a simple $\mathcal{U}$-module, in view of Lemma 2.7.(1), it suffices to show that there exists a weight vector $v \in V$ such that $Xv = Yv = Zv = 0$.

(i) *There exists a weight vector v such that $Xv = 0$*: Suppose this is not the case, then for all $\alpha \in \mathrm{Wt}(V)$, the map $X : V_\alpha \to V_{\alpha+2}$ is an injection and hence a bijection since all the weight spaces of V are finite dimensional and of the same dimension by statement 1. Hence, X acts bijectively on V, i.e., V is a simple module over the localized algebra $\mathcal{U}_X$. Notice that each weight space V_α of V is a simple $C_{\mathcal{U}_X}(H)$-module then $\dim V_\alpha = \infty$ (since $C_{\mathcal{U}_X}(H) = \mathbb{K}[C_1, C_2, H] \otimes A_1$, see Lemma 3.1), a contradiction.

(ii) *There exists a weight vector v such that $Yv = 0$*: the proof is similar to that of statement (i) by noticing that $C_{\mathcal{U}_Y}(H) = \mathbb{K}[C_1, C_2, H] \otimes \tilde{A}_1$, where $\tilde{A}_1 = \mathbb{K}\langle FY^{-1}, Z\rangle$ is the first Weyl algebra.

(iii) *There exists a weight vector v such that $Xv = Yv = Zv = 0$*: By statement (i) and statement (ii), the elements X and Y act locally nilpotently on V. By statement 1, each weight space V_α of V is finite dimensional. Hence the map $Z : V_\alpha \to V_\alpha$ has an eigenvector $v \in V_\alpha$ with eigenvalue, say β, i.e., $Zv = \beta v$. If $\beta = 0$ then Z acts locally nilpotently on V. Since the elements X, Y, and Z commute, there exists a weight vector $v \in V$ such that $Xv = Yv = Zv = 0$ and we are done. Now, suppose that there exists a weight vector $v' \in V$ such that $Zv' = \beta v'$ where $\beta \neq 0$, we seek a contradiction. Then there exists a weight vector $v \in V_\alpha$ such that $Xv = Yv = 0$ and $Zv = \beta v$, since X and Y act locally nilpotently on V. Then V is an epimorphic image of the module $S(\alpha, \beta) = \mathcal{U}/\mathcal{U}(H - \alpha, Z - \beta, X, Y)$. By Lemma 6.5.(1), $S(\alpha, \beta)$ is a simple module and hence $V \simeq S(\alpha, \beta)$. But by Lemma 6.5.(2), each weight space of $S(\alpha, \beta)$ is infinite dimensional, a contradiction.

3. $U(\mathfrak{sl}_2)$ is a GWA: $U(\mathfrak{sl}_2) = \mathbb{K}[\Delta, H][E, F; \sigma, a = \frac{1}{4}(\Delta - H(H-2))]$. Now, the result follows from Ref. 4 [Theorem 3.2] (the condition $\lambda \neq (\alpha + 2i)(\alpha + 2i - 2)$ is a necessary and sufficient condition that the $\mathcal{U}$-module $V(\alpha, \gamma)$ belongs to the modules in statement 1 of Ref. 4 [Theorem 3.2]). □

Let $\widehat{U(\mathfrak{sl}_2)}$ (weight) be the set of simple weight $U(\mathfrak{sl}_2)$-modules. The following theorem gives an explicit description of the set $\mathscr{F}$.

Theorem 6.7. $\mathscr{F} = \{M(\alpha, \beta) \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\} \sqcup \{M^\star(\alpha, \beta) \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\} \sqcup \widehat{U(\mathfrak{sl}_2)}$ (weight).

Proof. The theorem follows from Proposition 6.6.(2), Proposition 6.3, and Proposition 6.4. □

The following two corollaries follow from Theorem 6.7.

*Corollary 6.8. (**Finite-Infinite Dimension Dichotomy**). Let M be a simple weight $\mathcal{U}$-module. Then all its weight spaces are either finite or infinite dimensional.*

Corollary 6.9. $\widehat{\mathcal{U}}$ (fin. dim.) $= \widehat{U(\mathfrak{sl}_2)}$ (fin. dim.).

Our aim is to classify all the simple weight $\mathcal{U}$-modules. Notice that the set $\widehat{\mathcal{U}}$ (weight) of simple weight $\mathcal{U}$-modules is a disjoint union of two subsets

$$\widehat{\mathcal{U}}\text{ (weight)} = \widehat{\mathcal{U}}\text{ (weight, } X\text{-torsion)} \ \sqcup\ \widehat{\mathcal{U}}\text{ (weight, } X\text{-torsionfree)}. \quad (51)$$

Simple weight X-torsion $\mathcal{U}$-modules. Theorem 6.13 gives an explicit description of the set $\widehat{\mathcal{U}}$ (weight, X-torsion) of simple weight X-torsion modules. It is clear that

$$\widehat{\mathcal{U}}\text{ (weight, } X\text{-torsion)} = \widehat{\mathcal{U}}\text{ (weight, } X\text{-torsion, } Y\text{-torsion)}$$
$$\sqcup \widehat{\mathcal{U}}\text{ (weight, } X\text{-torsion, } Y\text{-torsionfree)}. \quad (52)$$

The set $\widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsion). The next proposition is an explicit description of the set $\widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsion).

Proposition 6.10. $\widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsion) $= \{[S(\alpha, \beta)] \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\} \ \sqcup\ \widehat{U(\mathfrak{sl}_2)}$ (weight).

Proof. Let $V \in \widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsion). Then the elements X and Y act locally nilpotently on the module V. Since $XY = YX$, there is a weight vector $v \in V$ such that $Xv = Yv = 0$.

Since V is a simple $\mathcal{U}$-module, the central element C_1 acts on V as a scalar, say λ_1. Then $\lambda_1 v = C_1 v = -\frac{1}{2}Z^2 v$, i.e., $Z^2 v = -2\lambda_1 v$. If $\lambda_1 = 0$ then $Z^2 v = 0$. We may assume that $Zv = 0$ (otherwise, we can replace v by $v' = Zv$). Now, $Xv = Yv = Zv = 0$ and hence $(Z) \subseteq \mathrm{ann}_{\mathcal{U}}(V)$, by Lemma 2.7.(1). So, V is a simple $U(\mathfrak{sl}_2)$-module. If $\lambda_1 \neq 0$ then there is a weight vector $v \in V_\alpha$ such that $Xv = Yv = 0$ and $Zv = \beta v$ for some $\beta \in \mathbb{K}^*$. (In more detail, notice that $(Z - \nu_1)(Z + \nu_1)v = 0$ where $\nu_1 = \sqrt{-2\lambda_1} \in \mathbb{K}^*$. If $(Z + \nu_1)v = 0$ then $Zv = -\nu_1 v$, otherwise let $v' := (Z + \nu_1)v$ then $Zv' = \nu_1 v'$.) Thus V is an epimorphic image of the module $S(\alpha, \beta)$. We must have $V \simeq S(\alpha, \beta)$ since $S(\alpha, \beta)$ is a simple module by Lemma 6.5. □

The set $\widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsionfree). For $\alpha, \beta \in \mathbb{K}$ and $\gamma \in \mathbb{K}^*$, we define the left $\mathcal{U}$-module $\mathfrak{X}^{\alpha,\beta,\gamma} := \mathcal{U}/\mathcal{U}(H - \alpha, Z - \beta, EY - \gamma, X)$. Then $\mathfrak{X}^{\alpha,\beta,\gamma} = \sum_{i\geqslant 1} \mathbb{K}[F]E^i\bar{1} + \mathbb{K}[F,Y]\bar{1}$, where $\bar{1} = 1 + \mathcal{U}(H - \alpha, Z - \beta, EY - \gamma, X)$. Clearly, $\mathfrak{X}^{\alpha,\beta,\gamma}$ is an X-torsion and Y-torsionfree weight $\mathcal{U}$-module.

Proposition 6.11. 1. If $\gamma \notin 2\mathbb{Z}\beta$ then $\mathfrak{X}^{\alpha,\beta,\gamma}$ is a simple $\mathcal{U}$-module and $\mathfrak{X}^{\alpha,\beta,\gamma} = \bigoplus_{i\geqslant 1} \mathbb{K}[F] E^i\bar{1} \oplus \bigoplus_{i\geqslant 0} \mathbb{K}[F]Y^i\bar{1}$.

2. *If $\gamma \notin 2\mathbb{Z}\beta$ then* $\mathrm{Wt}(\mathfrak{X}^{\alpha,\beta,\gamma}) = \{\alpha + 2n \mid n \in \mathbb{Z}\}$ *and each weight space is infinite dimensional.*
3. $\mathrm{ann}_{\mathcal{U}}(\mathfrak{X}^{\alpha,\beta,\gamma}) = (C_1 + \frac{1}{2}\beta^2, C_2 - \gamma - \alpha\beta)$.
4. *Let (α, β, γ), $(\alpha', \beta', \gamma') \in \mathbb{K}^3$ such that $\gamma \notin 2\mathbb{Z}\beta$ and $\gamma' \notin 2\mathbb{Z}\beta'$. Then $\mathfrak{X}^{\alpha,\beta,\gamma} \simeq \mathfrak{X}^{\alpha',\beta',\gamma'}$ if and only if $(\alpha', \beta', \gamma') = (\alpha + 2i, \beta, \gamma - 2i\beta)$ for some $i \in \mathbb{Z}$.*

Proof. 1. It suffices to show that for any nonzero element $v \in \mathfrak{X}^{\alpha,\beta,\gamma}$ there exists some element $u \in \mathcal{U}$ such that $uv \in \mathbb{K}^*\bar{1}$. Notice that v can be written as $v = \sum_{i=1}^n g_i(F)E^i\bar{1} + h\bar{1}$, where $g_i(F) \in \mathbb{K}[F]$ and $h \in \mathbb{K}[F,Y]$. By Lemma 2.6.(4), $YE^i\bar{1} = (E^iY - 2iE^{i-1}Z + 2i(i-1)E^{i-2}X)\bar{1} = (\gamma - 2i\beta) E^{i-1}\bar{1}$ and the coefficient $\gamma - 2i\beta \neq 0$ since $\gamma \notin 2\mathbb{Z}\beta$. If $g_n(F) \neq 0$ then $Yv = \sum_{i=1}^n g_i(F)(\gamma - 2i\beta) E^{i-1}\bar{1} + Yh\bar{1}$. Hence, $Y^n v = P(F,Y)\bar{1}$ for some nonzero polynomial $P(F,Y) \in \mathbb{K}[F,Y]$. Therefore, we may assume that $v \in \mathbb{K}[F,Y]\bar{1}$ and $v = \sum_{j=0}^m a_j(Y)F^j\bar{1}$, where $a_j(Y) \in \mathbb{K}[Y]$ and $a_m(Y) \neq 0$. Notice that $(Z - \beta)F^j\bar{1} = -jYF^{j-1}\bar{1}$. Then $(Z - \beta)v = \sum_{j=1}^m a_j(Y)(-j)YF^{j-1}\bar{1}$. Hence, $(Z - \beta)^m v = Q(Y)\bar{1}$ for some nonzero polynomial $Q(Y) \in \mathbb{K}[Y]$. Therefore, we may assume that $v \in \mathbb{K}[Y]\bar{1}$ and $v = \sum_{i=0}^k c_iY^i\bar{1}$, where $c_i \in \mathbb{K}$ and $c_k \neq 0$. Since $HY^i\bar{1} = (\alpha - 2i)Y^i\bar{1}$ for all i and the eigenvalues $\{\alpha - 2i \mid i = 0, \dots, k\}$ are distinct. There exists a polynomial $f(H) \in \mathbb{K}[H]$ such that $f(H)v = Y^k\bar{1}$. Notice that $EY^k\bar{1} = (Y^kE + 2kY^{k-1}Z)\bar{1} = (\gamma + 2(k-1)\beta)Y^{k-1}\bar{1}$ and the coefficient $\gamma + 2(k-1)\beta \in \mathbb{K}^*$ since $\gamma \notin 2\mathbb{Z}\beta$. Then $E^kY^kv \in \mathbb{K}^*\bar{1}$, as required. The above argument also implies that $\mathfrak{X}^{\alpha,\beta,\gamma} = \bigoplus_{i\geqslant 1} \mathbb{K}[F]E^i\bar{1} \oplus \bigoplus_{i\geqslant 0} \mathbb{K}[F]Y^i\bar{1}$.

2. Statement 2 follows from the last equality in statement 1.

3. Clearly, $(C_1 + \frac{1}{2}\beta^2, C_2 - \gamma - \alpha\beta) \subseteq \mathrm{ann}_{\mathcal{U}}(\mathfrak{X}^{\alpha,\beta,\gamma})$. Then the equality holds since $(C_1 + \frac{1}{2}\beta^2, C_2 - \gamma - \alpha\beta)$ is a maximal ideal of $\mathcal{U}$, by (1).

4. ($\Rightarrow$) Notice that the element $Z - \beta$ acts locally nilpotently on the module $\mathfrak{X}^{\alpha,\beta,\gamma}$. If $\mathfrak{X}^{\alpha,\beta,\gamma} \simeq \mathfrak{X}^{\alpha',\beta',\gamma'}$ then we must have $\beta = \beta'$. By statement 2, $\{\alpha + 2i \mid i \in \mathbb{Z}\} = \mathrm{Wt}(\mathfrak{X}^{\alpha,\beta,\gamma}) = \mathrm{Wt}(\mathfrak{X}^{\alpha',\beta',\gamma'}) = \{\alpha' + 2i \mid i \in \mathbb{Z}\}$. Hence, $\alpha' = \alpha + 2i$ for some $i \in \mathbb{Z}$. Then, by statement 3, $\gamma + \alpha\beta = \gamma' + \alpha'\beta'$, i.e., $\gamma' = \gamma + (\alpha - \alpha')\beta = \gamma - 2i\beta$.

($\Leftarrow$) Suppose that $(\alpha', \beta', \gamma') = (\alpha + 2i, \beta, \gamma - 2i\beta)$ for some $i \in \mathbb{Z}$. Let $\bar{1}$ and $\bar{1}'$ be the canonical generators of the modules $\mathfrak{X}^{\alpha,\beta,\gamma}$ and $\mathfrak{X}^{\alpha',\beta',\gamma'}$, respectively. If $i \leqslant 0$, then the map $\mathfrak{X}^{\alpha',\beta',\gamma'} \to \mathfrak{X}^{\alpha,\beta,\gamma}$, $\bar{1}' \mapsto Y^{|i|}\bar{1}$ defines an isomorphism of $\mathcal{U}$-modules with the inverse defined by $\bar{1} \mapsto \frac{1}{g_i(\gamma,\beta)} E^{|i|}\bar{1}'$ where $g_i(\gamma,\beta) = \prod_{j=1}^{|i|}(\gamma - 2j\beta) \in \mathbb{K}^*$. If $i > 0$ then the map $\mathfrak{X}^{\alpha',\beta',\gamma'} \to \mathfrak{X}^{\alpha,\beta,\gamma}$, $\bar{1}' \mapsto E^i\bar{1}$ defines an isomorphism of $\mathcal{U}$-modules with the inverse defined by $\bar{1} \mapsto \frac{1}{f_i(\gamma,\beta)}Y^i\bar{1}'$ where $f_i(\gamma,\beta) = \prod_{j=1}^i \left(\gamma + 2(j-1)\beta\right) \in \mathbb{K}^*$. □

For any $\beta \in \mathbb{K}$, the subgroup $2\mathbb{Z}(1, -\beta)$ of $(\mathbb{K}^2, +)$ acts on $\mathbb{K}^2$ in a obvious way. For each $(\alpha, \gamma) \in \mathbb{K}^2$, we denote by $\mathcal{O}(\alpha, \gamma) := (\alpha, \gamma) + 2\mathbb{Z}(1, -\beta)$ the orbit of the element $(\alpha, \gamma) \in \mathbb{K}^2$ under the action of the subgroup $2\mathbb{Z}(1, -\beta)$. Clearly, the set of all $2\mathbb{Z}(1, -\beta)$-orbits can be identified with the factor group $\mathbb{K}^2/2\mathbb{Z}(1, -\beta)$. For each orbit $\mathcal{O} \in \mathbb{K}^2/2\mathbb{Z}(1, -\beta)$, we fix an element $(\alpha_{\mathcal{O}}, \gamma_{\mathcal{O}}) \in \mathcal{O}$. The next proposition is an explicit description of the set $\widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsionfree).

Proposition 6.12.

$$\widehat{\mathcal{U}}\,(\text{weight, } X\text{-torsion, } Y\text{-torsionfree}) = \left\{[M(\alpha,\beta)] \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\right\} \sqcup \left\{[\mathfrak{X}^{\alpha_O,\beta,\gamma_O}] \mid \beta \in \mathbb{K}, O \in \mathbb{K}^2/2\mathbb{Z}(1,-\beta), \gamma_O \notin 2\mathbb{Z}\beta\right\}.$$

Proof. Clearly, the Verma modules $M(\alpha,\beta) \in \widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsionfree). Now, let $V \in \widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsionfree) and V is not isomorphic to a Verma module. We show that $V \simeq \mathfrak{X}^{\alpha,\beta,\gamma}$ for some $\alpha,\beta,\gamma \in \mathbb{K}$ where $\gamma \notin 2\mathbb{Z}\beta$.

(i) *The module V is E-torsionfree*: Otherwise, E acts locally nilpotently on V and there is a nonzero weight vector $v \in V$ such that $Ev = 0$. Since V is X-torsion, X acts locally nilpotently on V. There is a weight vector $\tilde{v} \in V$ such that $E\tilde{v} = X\tilde{v} = 0$. Then V is a simple highest weight module and hence, by Proposition 6.3, V is isomorphic to a Verma module (since V is Y-torsionfree), a contradiction.

(ii) *There exists a weight vector $v \in V_\alpha$ such that $Zv = \beta v$, $EYv = \gamma v$ and $Xv = 0$ where $\beta \in \mathbb{K}, \gamma \in \mathbb{K}^*$*: The element X acts locally nilpotently on V, in particular, there is a nonzero weight vector $v' \in V$ such that $Xv' = 0$. The module V is a simple $\mathcal{U}$-module, so, the central elements C_1 and C_2 act on V as scalars, say λ_1 and λ_2, respectively. Then $\lambda_1 v' = C_1 v' = -\frac{1}{2}Z^2 v'$, i.e., $Z^2 v' = -2\lambda_1 v'$. So, there is a weight vector $v \in V_\alpha$ such that $Zv = \beta v$ and $Xv = 0$ (where $\beta = \nu_1$ or $-\nu_1$, $\nu_1 = \sqrt{-2\lambda_1}$ and λ_1 could be zero). Now, $\lambda_2 v = C_2 v = EYv + \alpha\beta v$, i.e., $EYv = \gamma v$, where $\gamma = \lambda_2 - \alpha\beta$. It remains to show that $\gamma \neq 0$. The element $w = Yv \in V$ is nonzero, since V is Y-torsionfree. If $\gamma = 0$ then $Ew = EYv = 0$, contradicts to the fact that V is E-torsionfree (see statement (i)).

(iii) $\gamma \notin 2\mathbb{Z}\beta$: Suppose that $\gamma = 2i\beta$ for some $i \in \mathbb{Z}$, we seek a contradiction. Then $i \neq 0$ and $\beta \neq 0$ since $\gamma \in \mathbb{K}^*$. If $i > 0$ we set $v' = E^i v$. Then $v' \in V$ is nonzero since V is E-torsionfree. By Lemma 2.6.(4), $Yv' = YE^i v = \left(E^i Y - 2iE^{i-1}Z + 2i(i-1)E^{i-2}X\right)v = (\gamma - 2i\beta)E^{i-1}v = 0$. This contradicts to the fact that V is Y-torsionfree. If $i < 0$ we set $v'' = Y^{-i+1}v$. Then $v'' \in V$ is nonzero since V is Y-torsionfree. But then $Ev'' = EY^{-i+1}v = \left(Y^{-i+1}E + 2(-i+1)Y^{-i}Z\right)v = (\gamma - 2i\beta)Y^{-i}v = 0$. This contradicts to the fact that V is E-torsionfree, by statement (i).

By statement (ii), V is an epimorphic image of the $\mathcal{U}$-module $\mathfrak{X}^{\alpha,\beta,\gamma}$ where $\alpha,\beta \in \mathbb{K}$ and $\gamma \in \mathbb{K}^*$. By statement (iii) and Proposition 6.11.(1), $\mathfrak{X}^{\alpha,\beta,\gamma}$ is a simple $\mathcal{U}$-module and hence, $V \simeq \mathfrak{X}^{\alpha,\beta,\gamma}$. Finally, Proposition 6.1.(2) and Proposition 6.11.(4) complete the proof. □

The following theorem is an explicit description of the set $\widehat{\mathcal{U}}$ (weight, X-torsion).

Theorem 6.13.

$$\widehat{\mathcal{U}}\,(\text{weight, } X\text{-torsion}) = \left\{[S(\alpha,\beta)] \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\right\} \sqcup \widehat{U(\mathfrak{sl}_2)}\,(\text{weight}) \sqcup \left\{[M(\alpha,\beta)] \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\right\} \sqcup \left\{[\mathfrak{X}^{\alpha_O,\beta,\gamma_O}] \mid \beta \in \mathbb{K}, O \in \mathbb{K}^2/2\mathbb{Z}(1,-\beta), \gamma_O \notin 2\mathbb{Z}\beta\right\}.$$

Proof. The theorem follows from (52), Proposition 6.10, and Proposition 6.12. □

Now, our goal is to describe the set $\widehat{\mathcal{U}}$ (weight, X-torsionfree). This set can be partitioned further into two disjoint union of subsets,

$$\widehat{\mathcal{U}}\,(\text{weight, } X\text{-torsionfree}) = \widehat{\mathcal{U}}\,(\text{weight, } X\text{-torsionfree, } Y\text{-torsion}) \sqcup \widehat{\mathcal{U}}\,(\text{weight, } X\text{-torsionfree, } Y\text{-torsionfree}). \quad (53)$$

The set $\widehat{\mathcal{U}}$ (weight, X-torsionfree, Y-torsion). For $\alpha,\beta \in \mathbb{K}$ and $\gamma \in \mathbb{K}^*$, we define the left $\mathcal{U}$-module $\mathcal{Y}^{\alpha,\beta,\gamma} = \mathcal{U}/\mathcal{U}(H-\alpha, Z-\beta, FX-\gamma, Y)$. Then $\mathcal{Y}^{\alpha,\beta,\gamma} \simeq {}^{\iota}\mathfrak{X}^{-\alpha,-\beta,-2\gamma}$, where ${}^{\iota}\mathfrak{X}^{-\alpha,-\beta,-2\gamma}$ is the $\mathcal{U}$-module $\mathfrak{X}^{-\alpha,-\beta,-2\gamma}$ twisted by the automorphism ι of $\mathcal{U}$, see (3).

Proposition 6.14. 1. If $\gamma \notin \mathbb{Z}\beta$ then $\mathcal{Y}^{\alpha,\beta,\gamma}$ is a simple $\mathcal{U}$-module.

2. *If $\gamma \notin \mathbb{Z}\beta$ then* $\mathrm{Wt}(\mathcal{Y}^{\alpha,\beta,\gamma}) = \{\alpha + 2i \mid i \in \mathbb{Z}\}$ *and each weight space is infinite dimensional.*

3. $\mathrm{ann}_{\mathcal{U}}(\mathcal{Y}^{\alpha,\beta,\gamma}) = (C_1 + \frac{1}{2}\beta^2, C_2 - \alpha\beta + 2\gamma)$.
4. *Let* (α, β, γ), $(\alpha', \beta', \gamma') \in \mathbb{K}^3$ *such that* $\gamma \notin \mathbb{Z}\beta$ *and* $\gamma' \notin \mathbb{Z}\beta'$. *Then* $\mathcal{Y}^{\alpha,\beta,\gamma} \simeq \mathcal{Y}^{\alpha',\beta',\gamma'}$ *if and only if* $(\alpha', \beta', \gamma') = (\alpha + 2i, \beta, \gamma + i\beta)$ *for some* $i \in \mathbb{Z}$.

Proof. The result follows from Proposition 6.11, since $\mathcal{Y}^{\alpha,\beta,\gamma} \simeq {}^{\iota}\mathfrak{X}^{-\alpha,-\beta,-2\gamma}$. □

For any $\beta \in \mathbb{K}$, the subgroup $\mathbb{Z}(2,\beta)$ of $(\mathbb{K}^2,+)$ acts on $\mathbb{K}^2$ in an obvious way. For each $(\alpha,\gamma) \in \mathbb{K}^2$, we denote by $O(\alpha,\gamma) := (\alpha,\gamma) + \mathbb{Z}(2,\beta)$ the orbit of the element $(\alpha,\gamma) \in \mathbb{K}^2$ under the action of the subgroup $\mathbb{Z}(2,\beta)$. Clearly, the set of all $\mathbb{Z}(2,\beta)$-orbits can be identified with the factor group $\mathbb{K}^2/\mathbb{Z}(2,\beta)$. For each orbit $O \in \mathbb{K}^2/\mathbb{Z}(2,\beta)$, we fix an element $(\alpha_O, \gamma_O) \in O$.

Proposition 6.15.

$$\widehat{\mathcal{U}}\,(\text{weight, } X\text{-torsionfree, } Y\text{-torsion}) = \left\{[M^\star(\alpha,\beta)] \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\right\} \sqcup \left\{[\mathcal{Y}^{\alpha_O,\beta,\gamma_O}] \mid \beta \in \mathbb{K}, O \in \mathbb{K}^2/\mathbb{Z}(2,\beta), \gamma_O \notin \mathbb{Z}\beta\right\}.$$

Proof. The result follows from Proposition 6.12 by applying the automorphism ι. □

Theorem 6.16.

$$\widehat{\mathcal{U}}\,(\text{weight}, Y\text{-torsion}) = \left\{[S(\alpha,\beta)] \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\right\} \sqcup \widehat{U(\mathfrak{sl}_2)}\,(\text{weight}) \sqcup \left\{[M^\star(\alpha,\beta)] \mid \alpha \in \mathbb{K}, \beta \in \mathbb{K}^*\right\} \sqcup \left\{[\mathcal{Y}^{\alpha_O,\beta,\gamma_O}] \mid \beta \in \mathbb{K}, O \in \mathbb{K}^2/\mathbb{Z}(2,\beta), \gamma_O \notin \mathbb{Z}\beta\right\}.$$

Proof. The theorem follows from Proposition 6.10 and Proposition 6.15, since $\widehat{\mathcal{U}}$ (weight, Y-torsion) = $\widehat{\mathcal{U}}$ (weight, X-torsion, Y-torsion) $\sqcup$ $\widehat{\mathcal{U}}$ (weight, X-torsionfree, Y-torsion). □

For λ_1, λ_2 and $\alpha \in \mathbb{K}$, we define the left $\mathcal{U}$-module $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha} = \mathcal{U}/\mathcal{U}(C_1 - \lambda_1, C_2 - \lambda_2, H - \alpha, Z)$.

Lemma 6.17. 1. If $\lambda_1 \in \mathbb{K}^*$ *then the module* $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$ *is a simple* $\mathcal{U}$*-module.*
2. *If* $\lambda_1 \in \mathbb{K}^*$ *then* $\mathrm{Wt}(\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}) = \{\alpha + 2i \mid i \in \mathbb{Z}\}$ *and each weight space is infinite dimensional.*
3. *If* $\lambda_1 \in \mathbb{K}^*$ *then* $\mathrm{ann}_{\mathcal{U}}(\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}) = (C_1 - \lambda_1, C_2 - \lambda_2)$.
4. *Let* $(\lambda_1, \lambda_2, \alpha)$, $(\lambda_1', \lambda_2', \alpha) \in \mathbb{K}^* \times \mathbb{K} \times \mathbb{K}$. *Then* $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha} \simeq \mathfrak{Z}^{\lambda_1',\lambda_2',\alpha'}$ *if and only if* $(\lambda_1', \lambda_2', \alpha') = (\lambda_1, \lambda_2, \alpha + 2i)$ *for some* $i \in \mathbb{Z}$.

Proof. 1. Let $\bar{1} = 1 + \mathcal{U}(C_1 - \lambda_1, C_2 - \lambda_2, H - \alpha, Z)$ be the canonical generator of the module $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$. Then $\lambda_1\bar{1} = C_1\bar{1} = XY\bar{1}$ and $\lambda_2\bar{1} = C_2\bar{1} = EY\bar{1} - 2FX\bar{1}$. So, $\lambda_2 X\bar{1} = \lambda_1 E\bar{1} - 2FX^2\bar{1}$, i.e., $E\bar{1} = \lambda_1^{-1}(2FX^2 + \lambda_2 X)\bar{1}$ since λ_1 is nonzero. Hence, $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha} = \sum_{i\geqslant 1}\mathbb{K}[F]X^i\bar{1} + \mathbb{K}[F,Y]\bar{1}$. To prove that $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$ is a simple $\mathcal{U}$-module, it suffices to prove that for any nonzero element $v = \sum_{i=1}^n a_i(F)X^i\bar{1} + g(F,Y)\bar{1} \in \mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$, where $a_i(F) \in \mathbb{K}[F]$ and g is a polynomial in $\mathbb{K}[F,Y]$, there exists some element $u \in \mathcal{U}$ such that $uv \in \mathbb{K}^*\bar{1}$. If $a_n(F) \neq 0$ then $Yv = \sum_{i=1}^n a_i(F)\lambda_1 X^{i-1}\bar{1} + Yg\bar{1}$. Hence, $Y^n v = P\bar{1}$ where P is a *nonzero* polynomial in $\mathbb{K}[F,Y]$. So, we may assume that v is a nonzero element in $\mathbb{K}[F,Y]\bar{1}$ and then v can be written as $v = \sum_{j=0}^m b_j(Y)F^j\bar{1}$, where $b_j(Y) \in \mathbb{K}[Y]$. If $b_m(Y) \neq 0$ then $Zv = \sum_{j=0}^m b_j(Y)ZF^j\bar{1} = \sum_{j=0}^m b_j(Y)j(-Y)F^{j-1}\bar{1}$. So, $Z^m v = Q\bar{1}$ where Q is a *nonzero* polynomial in $\mathbb{K}[Y]$. Now, we may assume that v is a nonzero element in $\mathbb{K}[Y]\bar{1}$ and v then can be written as $v = \sum_{i=0}^l c_i Y^i\bar{1}$ where $c_i \in \mathbb{K}$ and $c_l \neq 0$. Since $HY^i\bar{1} = (\alpha - 2i)Y^i\bar{1}$ for all i and the eigenvalues $\{\alpha - 2i \mid i = 0, \dots, l\}$ are distinct. There exists a polynomial $f(H) \in \mathbb{K}[H]$ such that $f(H)v = Y^l\bar{1}$. Then $X^lY^l\bar{1} = \lambda_1^l\bar{1} \in \mathbb{K}^*\bar{1}$, as required.

2. The proof of statement 1 implies that $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha} = \bigoplus_{i\geqslant 1}\mathbb{K}[F]X^i\bar{1} \oplus \bigoplus_{i\geqslant 0}\mathbb{K}[F]Y^i\bar{1}$. Then statement 2 follows.

3. Clearly, $(C_1 - \lambda_1, C_2 - \lambda_2) \subseteq \mathrm{ann}_{\mathcal{U}}(\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha})$. Then the equality holds since $(C_1 - \lambda_1, C_2 - \lambda_2)$ is a maximal ideal of $\mathcal{U}$, by (1).

4. It is clear that if $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha} \simeq \mathfrak{Z}^{\lambda'_1,\lambda'_2,\alpha'}$ then $(\lambda'_1,\lambda'_2,\alpha') = (\lambda_1,\lambda_2,\alpha+2i)$ for some $i \in \mathbb{Z}$. Now, suppose that $(\lambda'_1,\lambda'_2,\alpha') = (\lambda_1,\lambda_2,\alpha+2i)$ for some $i \in \mathbb{Z}$. Let $\bar{1}'$ and $\bar{1}$ be the canonical generators of the modules $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha'}$ and $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$, respectively. If $i \leqslant 0$ the map $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha'} \to \mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$, $\bar{1}' \mapsto Y^{|i|}\bar{1}$ defines an isomorphism of $\mathcal{U}$-modules. If $i > 0$ then the map $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha'} \to \mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$, $\bar{1}' \mapsto E^i\bar{1}$ defines an isomorphism of $\mathcal{U}$-modules. □

For any $\alpha \in \mathbb{K}$, the subgroup $2\mathbb{Z}$ of $(\mathbb{K},+)$ acts on $\mathbb{K}$ in a obvious way. For each $\alpha \in \mathbb{K}$, we denote by $O(\alpha) := \alpha + 2\mathbb{Z}$ the orbit of the element $\alpha \in \mathbb{K}$ under the action of the subgroup $2\mathbb{Z}$. Clearly, the set of all $2\mathbb{Z}$-orbits can be identified with the factor group $\mathbb{K}/2\mathbb{Z}$. For each orbit $O \in \mathbb{K}/2\mathbb{Z}$, we fix an element $\alpha_O \in O$.

Proposition 6.18.

$$\widehat{\mathcal{U}}\,(\text{weight, } X\text{-torsionfree, } Y\text{-torsionfree, } Z\text{-torsion}) = \{[\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha_O}] \mid \lambda_1 \in \mathbb{K}^*, \lambda_2 \in \mathbb{K}, O \in \mathbb{K}/2\mathbb{Z}\}.$$

Proof. Let $V \in \widehat{\mathcal{U}}$ (weight, X-torsionfree, Y-torsionfree, Z-torsion). Then there is a weight vector $v \in V_\alpha$ such that $Zv = 0$. Since V is a simple $\mathcal{U}$-module, the central elements C_1 and C_2 act on V as scalars, say λ_1 and λ_2, respectively. In particular, $\lambda_1 v = C_1 v = XYv$. This implies that λ_1 is nonzero since V is an X and Y-torsionfree $\mathcal{U}$-module. Therefore, V is an epimorphic image of the module $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$ where $\lambda_1 \in \mathbb{K}^*$. By Lemma 6.17.(1), $\mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$ is a simple module and, so, $V \simeq \mathfrak{Z}^{\lambda_1,\lambda_2,\alpha}$. Then Lemma 6.17.(4) completes the proof. □

The algebra $\mathcal{U}$ is a Noetherian domain. By Goldie's Theorem, its left/right quotient ring $Q(\mathcal{U})$ is a division ring. Each non-zero element $q \in Q(\mathcal{U})$ determines the *inner automorphism* $\omega_q : Q(\mathcal{U}) \to Q(\mathcal{U})$, $a \mapsto qaq^{-1}$. The inner automorphisms ω_X and ω_Y preserve the subalgebra $C_t = C_{\mathcal{U}}(H)_t$ of $Q(\mathcal{U})$,

$$\omega_X : C_t \to C_t, \ \ \theta \mapsto \theta - 2Z\phi t^{-1}, \ \phi \mapsto \phi, \ H \mapsto H-2, \ Z \mapsto Z, \ C_1 \mapsto C_1, \ C_2 \mapsto C_2,$$

$$\omega_Y : C_t \to C_t, \ \ \theta \mapsto \theta - 2Z\iota(\phi)t^{-1} - 2, \ \phi \mapsto \phi - 2Z, \ H \mapsto H+2, \ Z \mapsto Z, \ C_1 \mapsto C_1, \ C_2 \mapsto C_2.$$

In more detail, the action of ω_X on the elements ϕ, H, Z, C_1, and C_2 are obvious. Then the element $\omega_X(\theta)$ is found by applying ω_X to the equality (20) and using the equality $\omega_X(t) = t$ where $t = Z^2 + 2C_1$: $\omega_X(\theta) = \omega_X(\theta t)t^{-1} = \big(\phi + (H-2)Z - C_2\big)\phi t^{-1} = \theta t t^{-1} - 2Z\phi t^{-1} = \theta - 2Z\phi t^{-1}$. The equality $\iota(X) = -\frac{1}{2}Y$ implies the equality $\omega_Y = \iota\omega_X\iota$: $\omega_Y = \omega_{-\frac{1}{2}Y} = \omega_{\iota(X)} = \iota\omega_X\iota^{-1} = \iota\omega_X\iota$ since $\iota = \iota^{-1}$. Then the action of the automorphism ω_Y on the canonical generators of the algebra C_t is obvious (by using ω_X). The automorphisms ω_X and ω_Y of the algebra $C_t = \mathcal{Z}[H] \otimes A_1$ are $\mathcal{Z}$-automorphisms,

$$\omega_X(\partial) = \omega_X(\phi t^{-1}) = \phi t^{-1} = \partial, \quad \omega_X(Z) = Z, \quad \omega_X(H) = H-2,$$

$$\omega_Y(\partial) = \omega_Y(\phi t^{-1}) = \partial - 2Zt^{-1}, \quad \omega_Y(Z) = Z, \quad \omega_Y(H) = H+2.$$

In particular, the automorphism $\omega_X|_{C_t}$ is a $\mathbb{K}[C_1,C_2] \otimes A_1$-automorphism such that $\omega_X(H) = H-2$. Clearly,

$$\mathcal{U}_t = C_t[X^{\pm 1};\omega_X] = C_t[Y^{\pm 1};\omega_Y]. \tag{54}$$

The set $\widehat{\mathcal{U}}$ (weight, (X,Y)-torsionfree). Let M be a *simple, weight* (X,Y)*-torsionfree* $\mathcal{U}$*-module*. Then $(C_1-\lambda_1)M = (C_2-\lambda_2)M = 0$ for some $\lambda_1, \lambda_2 \in \mathbb{K}$. The $\mathcal{U}$-module M is a simple and weight module, hence $\mathrm{Wt}\,(M) \subseteq \mu + 2\mathbb{Z} = O(\mu)$ for some/any $\mu \in \mathbb{K}$ such that $M_\mu \neq 0$. So, $M = \bigoplus_{n\in\mathbb{Z}} M_{\mu+2n}$. The $\mathcal{U}$-module M is (X,Y)-torsionfree, i.e., the maps $X_M, Y_M : M \to M$ are injections. Therefore,

$$\mathrm{Wt}\,(M) = \mu + 2\mathbb{Z} \tag{55}$$

since $0 \neq X^n M_\mu \subseteq M_{\mu+2n}$ and $0 \neq Y^n M_\mu \subseteq M_{\mu-2n}$. Since $XY = \frac{1}{2}(Z^2+2C_1) = \frac{1}{2}t \in C$ and $S_t \subseteq S \subseteq C$, *every weight component* $M_{\mu+2n}$ *is a simple,* S_t*-torsionfree* $C^{\lambda,\mu+2n}$*-module*. The $\mathcal{U}$-module

M can be either S-torsion or, otherwise, S-torsionfree. Therefore, all the weight components of M are either S-torsion or, otherwise, S-torsionfree (since $S \subseteq C$). So,

$$\begin{aligned}\widehat{\mathcal{U}}\,(\text{weight}, (X,Y)\text{-torsionfree}) &= \widehat{\mathcal{U}}\,(\boxed{1}) \,\sqcup\, \widehat{\mathcal{U}}\,(\boxed{2}), \\ \widehat{\mathcal{U}}\,(\boxed{1}) &:= \widehat{\mathcal{U}}\,(\text{weight}, (X,Y)\text{-torsionfree}, S\text{-torsion}), \\ \widehat{\mathcal{U}}\,(\boxed{2}) &:= \widehat{\mathcal{U}}\,(\text{weight}, (X,Y)\text{-torsionfree}, S\text{-torsionfree})\end{aligned} \tag{56}$$

(since $S_t \subseteq S$). The simple, weight, (X,Y)-torsionfree $\mathcal{U}$-module $M = \bigoplus_{n\in\mathbb{Z}} M_{\mu+2n}$ belongs to $\widehat{\mathcal{U}}$ $(\boxed{1})$ (respectively, $\widehat{\mathcal{U}}\,(\boxed{2})$) if and only if, for all $n \in \mathbb{Z}$, $M_{\mu+2n} \in \widehat{C^{\lambda,\mu+2n}}\,(S\text{-torsion}, S_t\text{-torsionfree})$ (respectively, $M_{\mu+2n} \in \widehat{C^{\lambda,\mu+2n}}\,(S\text{-torsionfree})$).

Recall that $t = Z^2 + 2C_1 \in C$ and its image in the algebra $C^{\lambda,\mu}$ is $Z^2 + 2\lambda_1$. The sets $S_t = \{t^i \mid i \in \mathbb{N}\}$ and $S_{Z^2+2\lambda_1} = \{(Z^2+2\lambda_1)^i \mid i \in \mathbb{N}\}$ are Ore sets of the domains C and $C^{\lambda,\mu}$, respectively. Abusing the notations we define

$$\widehat{C^{\lambda,\mu}}\,(S\text{-torsion}, S_t\text{-torsionfree}) \coloneqq \widehat{C^{\lambda,\mu}}\,(S\text{-torsion}, S_{Z^2+2\lambda_1}\text{-torsionfree}).$$

For each $\lambda \in \mathbb{K}$, the $C^{\lambda,\mu}$-module

$$C^{\lambda,\mu}(\gamma) \coloneqq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\gamma) = \bigcup_{i\geqslant 1} \ker(Z-\gamma)^i$$

is S-torsion and, for each element $\gamma' \in \mathbb{K}$ such that $\gamma' \neq \gamma$, the map $(Z-\gamma)\cdot : C^{\lambda,\mu}(\gamma) \to C^{\lambda,\mu}(\gamma)$, $m \mapsto (Z-\gamma)m$ is a bijection. In particular, *the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}(\gamma)$ is $S_{Z^2+2\lambda_1}$-torsionfree if and only if $\gamma^2 + 2\lambda_1 \neq 0$.* Clearly, for $\gamma, \gamma' \in \mathbb{K}$, $C^{\lambda,\mu}(\gamma) \simeq C^{\lambda,\mu}(\gamma')$ if and only if $\gamma = \gamma'$.

The next lemma describes the elements of the set $\widehat{C^{\lambda,\mu}}\,(S\text{-torsion}, S_t\text{-torsionfree})$.

Lemma 6.19. $\widehat{C^{\lambda,\mu}}\,(S\text{-torsion}, S_t\text{-torsionfree}) = \{[C^{\lambda,\mu}(\gamma)] \mid \gamma \in \mathbb{K}, \gamma^2 + 2\lambda_1 \neq 0\}$ *and* $C^{\lambda,\mu}(\gamma) \simeq C^{\lambda,\mu}(\gamma')$ *if and only if* $\gamma = \gamma'$.

Proof. Since every module $M \in \widehat{C^{\lambda,\mu}}\,(S\text{-torsion}, S_t\text{-torsionfree})$ is an epimorphic image of $C^{\lambda,\mu}(\gamma)$ for a (unique) $\gamma \in \mathbb{K}$ such that $\gamma^2 + 2\lambda_1 \neq 0$ and the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}(\gamma)$ is S-torsion and S_t-torsionfree, it suffices to show that the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}(\gamma)$ is simple.

Since $C^{\lambda,\mu}(\gamma) = \bigcup_{i\geqslant 1}\ker(Z-\gamma)^i$, the map $t\cdot : C^{\lambda,\mu}(\gamma) \to C^{\lambda,\mu}(\gamma)$, $c \mapsto tc$ is a bijection (since $t = Z^2 + 2C_1$ and $\gamma^2 + 2C_1 \neq 0$). Since $C_t^{\lambda,\mu} \simeq A_{1,t}$, $C^{\lambda,\mu}(\gamma) = C^{\lambda,\mu}(\gamma)_t \simeq A_{1,t}/A_{1,t}(Z-\gamma)$ is a simple $A_{1,t}$-module, i.e., $C^{\lambda,\mu}(\gamma)$ is a simple $C^{\lambda,\mu}$-module, as required. □

For $\lambda_1, \lambda_2, \mu, \gamma \in \mathbb{K}$, let us consider the $\mathcal{U}$-module

$$\begin{aligned}\mathcal{U}(\lambda,\mu,\gamma) &:= \mathcal{U}/\mathcal{U}(C_1-\lambda_1, C_2-\lambda_2, H-\mu, Z-\gamma) \\ &= \mathcal{U}/\mathcal{U}(XY-\lambda_1-\frac{1}{2}\gamma^2, C_2-\lambda_2, H-\mu, Z-\gamma).\end{aligned}$$

The element $\bar{1} = 1 + \mathcal{U}(C_1-\lambda_1, C_2-\lambda_2, H-\mu, Z-\gamma)$ is called the canonical generator of the $\mathcal{U}$-module $\mathcal{U}(\lambda,\mu,\gamma)$. The next theorem is an explicit description of the elements of the set $\widehat{\mathcal{U}}\,(\boxed{1})$.

Theorem 6.20. $\widehat{\mathcal{U}}\,(\boxed{1}) = \{[\mathcal{U}(\lambda,\mu_O,\gamma)] \mid \lambda_1,\lambda_2,\gamma \in \mathbb{K}, \gamma^2+2\lambda_1 \neq 0 \text{ and } O \in \mathbb{K}/2\mathbb{Z}\}$, $\mathrm{Wt}\,(\mathcal{U}(\lambda,\mu_O,\gamma)) = O = \mu_O + 2\mathbb{Z}$, *and $\mathcal{U}$-modules $\mathcal{U}(\lambda,\mu_O,\gamma)$ and $\mathcal{U}(\lambda',\mu_{O'},\gamma')$ are isomorphic if and only if $O = O'$ and $(\lambda,\gamma) = (\lambda',\gamma')$. Furthermore, the maps $t\cdot$, $Y\cdot$, $X\cdot : \mathcal{U}(\lambda,\mu_O,\gamma) \to \mathcal{U}(\lambda,\mu_O,\gamma)$ are bijections,*

$$\begin{aligned}\mathcal{U}(\lambda,\mu_O,\gamma) = \mathcal{U}(\lambda,\mu_O,\gamma)_t &= \bigoplus_{n\in\mathbb{Z}} X^n\mathcal{U}(\lambda,\mu_O,\gamma)_{\mu_O} = \bigoplus_{n\in\mathbb{Z}} Y^n\mathcal{U}(\lambda,\mu_O,\gamma)_{\mu_O} \\ &= \bigoplus_{n\geqslant 1} Y^n\mathcal{U}(\lambda,\mu_O,\gamma)_{\mu_O} \oplus \bigoplus_{n\geqslant 0} X^n\mathcal{U}(\lambda,\mu_O,\gamma)_{\mu_O},\end{aligned}$$

for all $n \in \mathbb{Z}$, $\mathcal{U}(\lambda,\mu_O,\gamma)_{\mu+2n} = \mathcal{U}(\lambda,\mu_O,\gamma)_{\mu+2n,t}$ *is a* $C_t^{\lambda,\mu+2n}$*-module where* $\mu = \mu_O$ *and*

$$\begin{aligned}\mathcal{U}(\lambda,\mu_O,\gamma)_{\mu+2n} &= X^n\mathcal{U}(\lambda,\mu_O,\gamma)_\mu \simeq Y^{-n}\mathcal{U}(\lambda,\mu_O,\gamma)_\mu\\ &\simeq {}^{(\omega_X)^{-n}}\mathcal{U}(\lambda,\mu_O,\gamma)_\mu \simeq {}^{(\omega_Y)^{n}}\mathcal{U}(\lambda,\mu_O,\gamma)_\mu\\ &\simeq C^{\lambda,\mu+2n}/C^{\lambda,\mu+2n}(Z-\gamma) \in \widehat{C^{\lambda,\mu+2n}}\,(S\text{-torsion}, S_s\text{-torsionfree})\\ &\simeq A_{1,t}/A_{1,t}(Z-\gamma) \simeq A_1/A_1(Z-\gamma).\end{aligned}$$

Furthermore,

$$\mathcal{U}(\lambda,\mu_O,\gamma)_{\mu+2n} = \begin{cases} X^n\Big(\mathbb{K}[\theta]\oplus\mathbb{K}[\theta]\phi\Big)\,\bar{1}, & \text{if } n \geqslant 0,\\ Y^{|n|}\Big(\mathbb{K}[\theta]\oplus\mathbb{K}[\theta]\phi\Big)\,\bar{1}, & \text{if } n < 0,\end{cases} \tag{57}$$

where $\theta = FE$, $\phi = EY$ *and* $\bar{1}$ *is the canonical generator of the* $\mathcal{U}$*-module* $\mathcal{U}(\lambda,\mu_O,\gamma)$.

Proof. Let $M \in \widehat{\mathcal{U}}\,(\boxed{1})$. We keep the notation as above. In particular, the elements $C_1-\lambda_1$ and $C_2-\lambda_2$ annihilate the module M, $M = \bigoplus_{n\in\mathbb{Z}} M_{\mu+2n}$ where each weight component $M_{\mu+2n}$ belongs to the set $\widehat{C^{\lambda,\mu+2n}}\,(S\text{-torsion}, S_s\text{-torsionfree})$ where $s = Z^2+2\lambda_1$. By Lemma 6.19, $M_{\mu+2n} \simeq C^{\lambda,\mu+2n}/C^{\lambda,\mu+2n}(Z-\gamma_{\mu+2n})$ for some $\gamma_{\mu+2n}\in\mathbb{K}$ such that $\gamma_{\mu+2n}^2+2\lambda_1 \neq 0$.

(i) $\gamma := \gamma_\mu = \gamma_{\mu+2n}$ *for all* $n\in\mathbb{Z}$: The multiplicative set $T_\gamma := \{(Z-\gamma)^i \mid i\in\mathbb{Z}\}$ is an Ore set of the domain $\mathcal{U}$. The T_γ-torsion submodule of M is equal to $\text{tor}_{T_\gamma}(M) = \bigoplus_{\{n\in\mathbb{Z}|\gamma_{2n}=\gamma\}} M_{\mu+2n} \neq 0$ since $M_{\mu+2n} = \bigcup_{i\geqslant 1}\ker(Z-\gamma_{2n})^i$. The $\mathcal{U}$-module M is simple, hence $M = \text{tor}_{T_\gamma}(M)$, and so $\gamma = \gamma_{2n}$ for all $n\in\mathbb{Z}$.

(ii) $\gamma^2+2\lambda_1\neq 0$: This is obvious.

(iii) *The map* $t_M : M\to M$, $m\mapsto tm$, *is a bijection*: For all $n\in\mathbb{Z}$, the map $t_{M_{\mu+2n}} : M_{\mu+2n}\to M_{\mu+2n}$, $m\mapsto tm$ is a bijection, by the statement (ii) and the fact that $M_{\mu+2n} = \bigcup_{i\geqslant 1}\ker(Z-\gamma)^i$, and the result follows.

(iv) *The maps* X_M, Y_M *are bijections and* $X_M^{-1} = 2Y_M t_M^{-1}$: This follows from the statement (iii) and the equality $XY = YX = \frac{1}{2}t$.

(v) $M = M_t = \bigoplus_{n\in\mathbb{Z}} X^nM_\mu = \bigoplus_{n\in\mathbb{Z}} Y^nM_\mu$, $M_{\mu+2n} = X^nM_\mu \simeq {}^{(\omega_X)^{-n}}M_\mu$ and $M_{\mu+2n} = Y^{-n}M_\mu \simeq {}^{(\omega_Y)^{n}}M_\mu$: The statement (v) follows from the statement (iv) and the facts $XM_{\mu+2n}\subseteq M_{\mu+2(n+1)}$ and $YM_{\mu+2n}\subseteq M_{\mu+2(n-1)}$.

Notice that $C_t^{\lambda,\mu+2n}\simeq A_{1,t}$. By the statement (iii), we have the following chain of $C^{\lambda,\mu+2n}$-isomorphisms:

$$\frac{C^{\lambda,\mu+2n}}{C^{\lambda,\mu+2n}(Z-\gamma)} \simeq \left(\frac{C^{\lambda,\mu+2n}}{C^{\lambda,\mu+2n}(Z-\gamma)}\right)_t \simeq \frac{A_{1,t}}{A_{1,t}(Z-\gamma)} \simeq \frac{A_1}{A_1(Z-\gamma)}.$$

By Proposition 3.3.(2), $C^{\lambda,\mu+2n}/C^{\lambda,\mu+2n}(Z-\gamma)\simeq(\mathbb{K}[\theta]\oplus\mathbb{K}[\theta]\phi)\bar{1}$. Now, the equality (57) follows from the statement (v).

Given another module $M'\in\widehat{\mathcal{U}}\,(\boxed{1})$ with the parameters $\lambda_1',\lambda_2',\mu'$, and γ'. Let $O' = O(\mu') = \mu'+2\mathbb{Z}$.

(vi) *Then* $M\simeq M'$ *if and only if* $\lambda_1=\lambda_1', \lambda_2=\lambda_2', O=O'$, *and* $\gamma=\gamma'$: Suppose that $M\simeq M'$. Then $O = \text{Wt}(M) = \text{Wt}(M') = O'$. Clearly, $\lambda_1=\lambda_1'$ and $\lambda_2=\lambda_2'$. By the statement (i), $M = \bigcup_{i\geqslant 1}\ker(Z-\gamma)^i$ and $M' = \bigcup_{i\geqslant 1}\ker(Z-\gamma')^i$. Hence, $\gamma=\gamma'$. The implication ($\Leftarrow$) follows from the statements (iv) and (v).

In order to finish the proof of the theorem it suffices to prove the next statement.

(vii) $M\simeq\mathcal{U}(\lambda,\mu,\gamma)$ (*where* $\gamma^2+2\lambda_1\neq 0$): Let $M' := \mathcal{U}(\lambda,\mu,\gamma)$. By the very definition, $M' = \bigcup_{i\geqslant 1}\ker(Z-\gamma)^i$, and so the map $t_{M'}$ is a bijection. Then also the maps $X_{M'}$ and $Y_{M'}$ are bijections and $X_{M'}^{-1} = 2Y_{M'}t_{M'}^{-1}$. Hence, $M' = \bigoplus_{n\in\mathbb{Z}} X^nM_\mu'$. By the very definition of the module M', $M_\mu'\simeq C^{\lambda,\mu}/C^{\lambda,\mu}(Z-\gamma)\simeq M_\mu$ is a simple $C^{\lambda,\mu}$-module. By the statement (v), $M'\simeq M$. □

The set $\widehat{\mathcal{U}}\,(\boxed{2})$. Clearly,

$$\widehat{\mathcal{U}}\,(\boxed{2}) = \bigsqcup_{\lambda\in\mathbb{K}^2, O\in\mathbb{K}/2\mathbb{Z}} \widehat{\mathcal{U}}\,(\boxed{2},\lambda,O), \tag{58}$$

where $\widehat{\mathcal{U}}(\boxed{2},\lambda,\mathcal{O})$ contains $[M] \in \widehat{\mathcal{U}}(\boxed{2})$ such that $(C_1-\lambda_1)M = (C_2-\lambda_2)M = 0$ and $\mathrm{Wt}(M) = \mathcal{O}$.

Let $M \in \widehat{\mathcal{U}}(\boxed{2},\lambda,\mathcal{O})$. Then the simple $\mathcal{U}$-module M is an essential submodule of the $\mathcal{U}_t$-module M_t. Hence, $M = \mathrm{soc}_{\mathcal{U}}(M_t)$. Clearly, $M_t = \bigoplus_{n\in\mathbb{Z}} X^n M_{\mu,t}$, where $\mu = \mu_{\mathcal{O}}$, and $M_{t,\mu+2n} = X^n M_{\mu,t}$ for all $n \in \mathbb{Z}$. So, the simple $\mathcal{U}_t$-module M_t is uniquely determined by the simple $C_t^{\lambda,\mu}$-module $M_{\mu,t}$, and the last one is uniquely determined by its socle $M = \mathrm{soc}_C(M_{\mu,t})$, since $M_{\mu,t} = \mathrm{soc}_C(M_{\mu,t})_t$. So, the map

$$\widehat{\mathcal{U}}(\boxed{2},\lambda,\mathcal{O}) \to \widehat{C^{\lambda,\mu_{\mathcal{O}}}}(S\text{-torsionfree}),\ [M] \mapsto [M_{\mu_{\mathcal{O}}}] \tag{59}$$

is an *injection*.

Proposition 6.21. The map (59) is a bijection.

Proof. Since the map (59) is an injection, in order to finish the proof it suffices, for a given $[N] \in \widehat{C^{\lambda,\mu}}(S\text{-torsionfree})$, to construct a $\mathcal{U}$-module $[M] \in \widehat{\mathcal{U}}(\boxed{2},\lambda,\mathcal{O})$ with $M_{\mu_{\mathcal{O}}} \simeq N$. The induced $\mathcal{U}$-module $\mathcal{U}\otimes_C N$ is a weight module with $\mathrm{Wt}(\mathcal{U}\otimes_C N) = \mathcal{O}$ (since $S_t \subseteq S \subseteq C$, and N is an S-torsionfree C-module) and $(\mathcal{U}\otimes_C N)_{\mu_{\mathcal{O}}} = N$. It is annihilated by the elements $(C_1-\lambda_1)$ and $(C_2-\lambda_2)$. It contains the largest submodule, say L, with $L\cap N = 0$. The module L is the sum of all (weight) submodules that do not meet N. The $\mathcal{U}$-module L is weight.

Claim. $M := \mathcal{U}\otimes_C N/L \in \widehat{\mathcal{U}}(\boxed{2},\lambda,\mathcal{O})$ and $M_{\mu_{\mathcal{O}}} = N$: By the very definition, the $\mathcal{U}$-module M is simple, weight, $M_{\mu_{\mathcal{O}}} = N$ and annihilated by the elements $C_1-\lambda_1$ and $C_2-\lambda_2$. The inclusion $N \subset N_t$ yields the inclusion $\mathcal{U}\otimes_C N \subseteq \mathcal{U}\otimes_C N_t$ (since the algebra C is a direct summand of the C-bimodule $\mathcal{U}$). Since $S \subseteq C$, we have that $0 \neq S^{-1}N \subseteq S^{-1}M$, hence the $S^{-1}\mathcal{U}$-module $S^{-1}M$ is simple and $M \subseteq S^{-1}M$, and so M is an S-torsionfree $\mathcal{U}$-module. In particular, M is an S_t-torsionfree module (since $S_t \subseteq S$). Hence, M is an (X,Y)-torsionfree $\mathcal{U}$-module since $XY = 2t$. Therefore, $\mathrm{Wt}(M) = \mu_{\mathcal{O}} + 2\mathbb{Z} = \mathcal{O}$. This finishes the proof of the claim and the proposition. □

An explicit construction of modules in the class $\widehat{\mathcal{U}}(\boxed{2},\lambda,\mathcal{O})$. Let us consider the inverse map to (59),

$$\widehat{C^{\lambda,\mu_{\mathcal{O}}}}(S\text{-torsionfree}) \to \widehat{\mathcal{U}}(\boxed{2},\lambda,\mathcal{O}),\quad [N] \mapsto [M(\lambda,\mathcal{O},N)]. \tag{60}$$

In order to finish with classification of the modules in the class $\widehat{\mathcal{U}}(\boxed{2},\lambda,\mathcal{O})$, we give an explicit construction of them, i.e., we give a construction of the $\mathcal{U}$-module $M(\lambda,\mathcal{O},N)$ for each choice of N (Lemma 6.22). By (54), the $\mathcal{U}_t$-module

$$\mathcal{U}_t\otimes_{C_t} N_t = \left(C_t[X^{\pm 1};\omega_X]\right)\otimes_{C_t} N_t = \bigoplus_{n\in\mathbb{Z}} X^n N_t$$

is *simple and S-torsionfree*. Hence, $N \subseteq N_t \subseteq \mathcal{U}_t\otimes_{C_t} N_t$. The $\mathcal{U}$-module $\mathcal{U}_t\otimes_{C_t} N_t$ contains the $\mathcal{U}$-module $\mathcal{U}N$.

Lemma 6.22. $M(\lambda,\mathcal{O},N) \simeq \mathcal{U}N$ as $\mathcal{U}$-modules.

Proof. By the claim of the proof of Proposition 6.21, $M(\lambda,\mathcal{O},N) \simeq M$ where $M := \mathcal{U}\otimes_C N/L$ and L is the largest submodule $\mathcal{U}\otimes_C N$ such that $L \cap N = 0$. The kernel, say L', of the obvious $\mathcal{U}$-homomorphism $\mathcal{U}\otimes_C N \to \mathcal{U}N \subseteq \mathcal{M} := \mathcal{U}_t\otimes_{C_t} N_t$, $u\otimes n \mapsto un$, is contained in L. So, $\mathcal{U}\otimes_C N/L' \simeq \mathcal{U}N$.

Claim. $L' = L$: Suppose that $L' \neq L$, we seek a contradiction. Then $0 \neq L/L' \subseteq \mathcal{U}N$, and so $(L/L')_t = \mathcal{M} = (\mathcal{U}N)_t$, by simplicity of the $\mathcal{U}_t$-module $\mathcal{M}$. Hence,

$$0 \neq N_t \subseteq \left(\frac{\mathcal{U}\otimes_C N}{L}\right)_t \simeq \frac{\left(\mathcal{U}\otimes_C N/L'\right)_t}{\left(L/L'\right)_t} \simeq \mathcal{M}/\mathcal{M} = 0,$$

a contradiction. The proof of the claim is complete. By the claim, $M \simeq \mathcal{U}N$, as required. □

The next theorem is an explicit description of the elements of the set $\widehat{\mathcal{U}}(\boxed{2})$.

Theorem 6.23. *$\widehat{\mathcal{U}}(\boxed{2}) = \bigsqcup_{\lambda\in\mathbb{K}^2, O\in\mathbb{K}^2/2\mathbb{Z}} \widehat{\mathcal{U}}(\boxed{2}, \lambda, O)$ and $\widehat{\mathcal{U}}(\boxed{2}, \lambda, O) = \{[M(\lambda, O, N)] \mid [N] \in \widehat{C^{\lambda,\mu_O}}(S\text{-torsionfree})\}$ and $M(\lambda, \mu_O, N) \simeq M(\lambda, O, N')$ if and only if $N \simeq N'$.*

Proof. The theorem follows from (58), Proposition 6.21, and Lemma 6.22. □

Corollary 6.24. In view of (56), Theorems 6.20 and 6.23 classify the modules in $\widehat{\mathcal{U}}(\boxed{2})$.

Corollary 6.25. For each $[M] \in \widehat{\mathcal{U}}(\text{weight}\,(X,Y)\text{-torsionfree})$, $\text{ann}_{\mathcal{U}}(M) = (C_1 - \lambda_1, C_2 - \lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{K}$.

Proof. Clearly, $\mathfrak{a} := \text{ann}_{\mathcal{U}}(M) \supseteq \mathfrak{a}' := (C_1 - \lambda_1, C_2 - \lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{K}$. If $(\lambda_1, \lambda_2) \neq (0,0)$ then the ideal $\mathfrak{a}'$ is maximal (Theorem 1.1). Hence, $\mathfrak{a} = \mathfrak{a}'$. If $(\lambda_1, \lambda_2) = (0,0)$ and $\mathfrak{a} \supsetneqq \mathfrak{a}'$ then $\mathfrak{a} \supseteq (Z) = (X,Y,Z)$ (Theorem 1.1), a contradiction (since M is (X,Y)-torsionfree). Therefore, $\mathfrak{a} = \mathfrak{a}'$. □

Proof of Corollary 2.10. We use Theorem 1.1 and (1). By Corollary 6.25, $(C_1, C_2) \in \text{Prim}\,(\mathcal{U})$. Then $\text{Prim}\,(\mathcal{U}) \supseteq \text{Prim}\,(U(\mathfrak{sl}_2)) \sqcup \text{Max}\,(\mathcal{Z})$, by (1). Since $\mathcal{U}/(Z) \simeq U(\mathfrak{sl}_2)$ and $Z(U(\mathfrak{sl}_2)) = \mathbb{K}[\Delta]$, (Z) is not a primitive ideal of $\mathcal{U}$. Now, the result follows from (1). □

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