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Evaluation of Rocking and Coupling Rotational Linear Stiffness Coefficients of Adjacent Foundations

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Abstract

This paper presents closed form expressions for rocking spring stiffnesses and coupling-interaction rotational spring stiffnesses for a set of closely spaced footings. Sub-structuring is employed to derive analytically the exact reduced order spring models of the system. The stiffness coefficients of this reduced order model are determined by employing both (i) an extended, novel, application of Boussinesq's surface displacement of a point loaded half-space and (ii) an empirically derived formulation that makes use of both Finite Element and experimental results. Further validation suggests that, within the scope of epistemic uncertainty present in the physical world, the interaction formulae between two footings is sufficient for more general multi-footing interaction cases.

Keywords Structure-Soil-Structure Interaction; Discrete Lumped Parameter Model; Rocking Stiffness Coefficient; Coupling Rotational Stiffness Coefficient.

Introduction

Dynamic cross-interaction, also known as Structure-Soil-Structure Interaction (SSSI), among adjacent structures has received considerable attention in recent decades. Imperative works of Warburton (Warburton et al. 1971), Lee and Wesley (Lee and Wesley 1973), Luco and Contesse (Luco and Contesse 1973), Kobori et al (Kobori and Kusakabe 1980; Kobori and Minai 1974; Kobori et al. 1973; Kobori et al. 1977) and Qian and Beskos (Qian and Beskos 1995) have demonstrated the need to include cross-interaction effects in the seismic analysis of buildings located in close proximity. In fact, a Soil-Structure Interaction (SSI) analysis is not considered complete unless it takes into account the mutual interaction between adjacent structures via the underlying soil medium (Zaman 1982).

The analysis of problems involving ground and structure interaction, such as SSI and SSSI, are conducted predominantly via two approaches, (Stewart et al. 1998) and (Wolf 1985). The first is referred to as the *direct methodology* where the whole interacting system, i.e. structure and semi-infinite soil, is analysed in one step using numerical discretisation procedures such as the Finite Element Method (FEM) or Boundary Element Method (BEM) or a combination of both. One advantage of using such methods is the possibility to model complex geometries and system nonlinearities, especially that of the soil continuum. However, because of the large number of degrees of freedom (*dof*s) involved, these analyses are computationally costly and time consuming, and in addition are sensitive to changes in soil constitutive model parameters. The second and more popular technique is the *substructure* or *impedance method* where each interacting component is dealt with in a separate step then assembled to form the final solution taking advantage of the superposition principle. The method starts with the evaluation of the design input motion, *i.e.* kinematic interaction, followed by determination of the system's impedance function which is a complex valued function that describes the force/moment-displacement/rotation relationship. Next, dynamic analysis of the structure resting on the impedances from step two and subjected to the input motion from step one is conducted. The latter method is a convenient and reliable tool for both time and frequency domains analyses, (Wolf 1994), (Bowles 1996), (Barros and Luco 1990) and (Dutta and Roy 2002). This approach allows a swift calculation of system properties, conducting parametric studies, examining different design schemes and the appreciation of the essential features of the problem.

Although the substructure method has an advantage that is the ability of breaking down the complex SSI problem into more manageable components that could be easily verified, the analysis using this method is essentially linear and time invariant which is a simplification. The equivalent linear method, (Idriss and Seed 1967), is commonly used to approximate the soil nonlinearity during the site response analysis stage. On the other hand, using the direct method, time domain nonlinear and hysteretic soil models could be implemented which is in theory more rigorous representation. However, in addition to the computational expense, thorough understanding and expertise in using such soil models and parameter selection are required for engineering practice.

Results predicted using simplified models have been demonstrated to approximate physical observations, for example (Kobori et al. 1977) and (Aldaikh et al. 2016; Aldaikh et al. 2015) hence, such models could serve as a practical civil engineering analysis tool and provide preliminary estimates of the effects of complex interaction problems until the need for more sophisticated analyses is determined. Simplified discrete models with limited numbers of degrees of freedom have been well recognized and applied to the substructure method for the analysis of static and dynamic soil-structure interaction problems. In these mechanical models, a lumped parameter system treats all masses, springs and dashpots as if they were lumped into a single mass, single spring and single damping constant for each mode of vibration. Original works such as (Bycroft 1956) described how to define the characteristics of discrete models by matching the resulting impedance functions with those resulting from the use of continuum models, i.e. rigid foundations resting on an elastic half-space. Many imperative subsequent works on vertically loaded foundations were based on the same methodology, (Barkan 1962) and (Lysmer and Richart 1966).

Some numerical results, for example (Dobry and Gazetas 1986) showed that the impedance function of the discrete system, i.e. dynamic stiffness and damping characteristics, exhibited a dependency on excitation frequency. This dependency is a result of the influence that frequency has on inertia rather than on soil properties, particularly (Gazetas 1993). As a result, linear Soil-Structure Interaction calculations cannot be directly used in time domain analyses and are usually performed in the frequency domain. By choosing representative frequency independent parameter values, the frequency dependency of the dynamic properties of the springs and dashpots can be reasonably approximated. It is suggested that these properties remain nearly constant within the frequency range of interest for typical building structures subjected to earthquakes, (Jennings and Bielak 1973) and (ATC 1978). Lumped parameter models have been used by some researchers to model the adjacent structures problem, i.e. SSSI in a 3D representation as in the work described by (Lee and Wesley 1973). Of particular mention are the studies presented by Mulliken and Karabalis (Karabalis and Mulliken 1995; Mulliken and Karabalis 1998) where it has been illustrated that this kind of modelling with frequency independent lumped parameters can be successfully applied in the evaluation of interaction between rigid massive adjacent two and three identical surface foundations supported by a homogenous linear elastic half space subjected to various loadings including impulsive force, moment, sinusoidal and random signals. The coupling effect was incorporated into the solution by means of empirical stiffness and damping coupling coefficients which were calculated replacing numerical constants of static coefficients of stiffness and damping evaluated by Wolf (Wolf 1988) with functions of a dimensionless inter-foundation distance ratio. More recently, (Mykoniou et al. 2016) have used the same approach and utilised the coupling coefficients in (Mulliken and Karabalis 1998) to study the interaction of adjacent liquid-storage tanks.

Based on the above discussion, the aim for this paper is to introduce the theoretical background and mathematical formulations of the problem of adjacent surface footings within the linear elastic domain. The formulation is algebraically solved and simplified in order to obtain closed form solutions for the frequency-independent rotational foundation and coupling spring coefficients that could be used in recently developed simplified discrete analyses of SSSI problems (Aldaikh 2013). Only cases of two and three identical equispaced footings are considered. The paper also will examine if the proposed formulae are comparable to a novel application of Boussinesq's point loaded half-space solution and more sophisticated Finite Element analyses. In addition, an analogue experimental procedure examining the case of two adjacent foundations is described and results are used to validate the analytical and numerical analyses.

Objectives

The objectives of this paper are

1. To clarify the formulae for rotational coupling spring coefficients for the case of multi-footing interaction. These formulae are provided as an alternative to a full continuum model.
2. To theoretically demonstrate why the rotational coupling springs between adjacent and alternate footings must have negative values.
3. To derive a theoretical estimate based on a novel application of Boussinesq's surface displacement of a half-space subjected to a point load. The accuracy of this theoretical estimate is compared with an empirical numerical/experimental fits for rotational coupling springs between adjacent footings.
4. To determine the validity of a previous assumption, (Aldaikh et al. 2015), in which the rotational coupled-interaction springs between alternate footings were ignored in SSSI analyses.
5. To determine whether it is sufficiently accurate to make use of the coupled interaction formulae derived originally in (Alexander et al. 2013) for two adjacent structures for the case of multiple adjacent footings (i.e. greater than two structures)?

Model description

Prior to developing the analytical formulations the following simplifying assumptions are initially outlined:

1. The analyses are limited to the linear elastic domain for both rigid foundations and underlying half-space. Linear analysis is commonly adopted for the analysis of critical structures such as nuclear power plants, also for machine foundation problems, (Wolf 1991).
2. Only cases of two and three identical equispaced (i.e. equal inter-building spacing) footings are considered.
3. Foundation and coupling stiffnesses are independent of loading frequency, hence static analysis is justified.

Reduced order models and mechanical analogue systems

The static analysis of any linearly elastic mechanical system can be defined by the following algebraic equations:

$$\begin{bmatrix} \mathbf{f}_s \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sm} \\ \mathbf{K}_{sm}^T & \mathbf{K}_{mm} \end{bmatrix} \begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_m \end{bmatrix} \quad (1)$$

where $\mathbf{u}_m$ is the vector of 'master' degrees of freedom (in this paper these will be the rotations at footings) and $\mathbf{u}_s$ is the vector of 'slave' degrees of freedom (which are all other displacement and rotation *dofs*). Similarly $\mathbf{f}_m$ is the vector actions applied at the 'master' *dofs* (in this paper these will be applied moments at footings) and $\mathbf{f}_s$ is vectors actions at all other *dofs*. Block matrices $\mathbf{K}_{mm}, \mathbf{K}_{sm}, \mathbf{K}_{ss}$ are classical stiffness matrices. Eq.(1) can be condensed, (by *partitioning* or *sub-structuring* see Guyan (Guyan 1965.)) to achieve the following reduced order model which is a reduced rank system:

$$\mathbf{f} = \mathbf{K}\mathbf{u} \quad (2)$$

where matrices are defined as follows

$$\mathbf{K} = \mathbf{K}_{mm} - \mathbf{K}_{sm}^T \mathbf{K}_{ss}^{-1} \mathbf{K}_{sm}, \quad \mathbf{f} = \mathbf{f}_m - \mathbf{K}_{sm}^T \mathbf{K}_{ss}^{-1} \mathbf{f}_s, \quad \mathbf{u} = \mathbf{u}_m \quad (3)$$

Note that if actions are only applied at 'master' degrees of freedom then $\mathbf{f}_s = 0$ and the action vector $\mathbf{f} = \mathbf{f}_m$. If the displacement/rotations at 'slave' *dof*s are required then the following equation, Eq.(4), could be employed although this equation would be equivalent to solving Eq.(1) directly.

$$\mathbf{u}_s = \mathbf{K}_{ss}^{-1}\left(\mathbf{f}_s - \mathbf{K}_{sm}\mathbf{u}_m\right) \tag{4}$$

From energy considerations (Zienkiewicz et al. 2013) the global stiffness matrix of the system in Eq.(1) is symmetric, hence the block matrices $\mathbf{K}_{mm}$ and $\mathbf{K}_{ss}$ must also be symmetric. Matrix $\mathbf{K}_{sm}$ is not, in general, symmetric.

A question arises as to whether the reduced order model stiffness matrix $\mathbf{K}$ is necessarily symmetric. It may be assumed from energy considerations that this should be true. Nevertheless, the following simple proof demonstrates this. Two matrix theorems are employed (Petersen and Pedersen 2008), first $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ which states that the inverse of a symmetric matrix is symmetric; hence $\mathbf{K}_{ss}^{-1}$ is symmetric. Second, using $(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$ it can be concluded that $\mathbf{K}_{sm}^T\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}$ is also symmetric. Hence it is known, without any loss of generality, that any reduced order model stiffness matrix $\mathbf{K}$ is symmetric.

While the reduced order system in Eq.(2) has been obtained from a condensed system in Eq.(1) it can also be obtained from an independent system of three dofs interconnected with three springs. Fig.1(a) displays a system of three static moments applied to a linear elastic half-space. This can be analysed using the finite element method; which generally results in a large set of linear algebraic equations. In the case at hand here it is desirable to define 'master' degrees of freedom as $\mathbf{u}_m = \left[\theta_1, \theta_2, \theta_3\right]^T$. The reduced order model of this system has the form of Eq.(2) and in this particular case is a set of three linear algebraic equations in terms of just the rotational degrees of freedom θ_1, θ_2 and θ_3.

It is clear mathematically that the mechanical system in Fig.1(b) is a *completely identical analogue* to the condensed version of the system in Fig.1(a). If appropriate stiffness coefficients are assigned to the springs in Fig.1(b) then its stiffness matrix (which is a general diagonal matrix) mathematically equals the condensed stiffness matrix $\mathbf{K}$ of the system in Fig.1(a). This is because both stiffness matrices are arbitrary symmetric matrices.

However, while the reduced order model and mechanical analogue system have identical stiffness matrices it may not be possible to ensure that the stiffness coefficients of all springs in the mechanical analogue system are positive. In the case herein it turns out that all the coupled interaction springs k_{12}, k_{23} & k_{13} that cross-couple the footings must be negative. By physical reasoning, (i.e. by considering applied moments at the surface) it is clear that an anticlockwise rotation of a footing is likely to produce a clockwise rotation of an adjacent footing. Therefore a 'spring' connecting these two footings must have a negative stiffness. Thus, it is not easy to envisage a physical incarnation of the mechanical analogue system Fig.1(b). It exists principally as a mathematical abstraction.

The potential energy of the system Fig.1(b) is given in Eq.(5) and its Euler-Lagrange equations are given in Eq.(6)

$$U = -\sum_{i=1}^{3} M_i \theta_i + \tfrac{1}{2}\sum_{i=1}^{3} k_i \theta_i^2 + \tfrac{1}{2}k_{12}\left(\theta_2 - \theta_1\right)^2 + \tfrac{1}{2}k_{23}\left(\theta_3 - \theta_2\right)^2 + \tfrac{1}{2}k_{13}\left(\theta_3 - \theta_1\right)^2 \quad (5)$$

$$\begin{aligned} M_1 &= k_1\theta_1 - k_{12}\left(\theta_2 - \theta_1\right) - k_{13}\left(\theta_3 - \theta_1\right) \\ M_2 &= k_2\theta_2 + k_{12}\left(\theta_2 - \theta_1\right) - k_{23}\left(\theta_3 - \theta_2\right) \\ M_3 &= k_3\theta_3 + k_{23}\left(\theta_3 - \theta_2\right) + k_{13}\left(\theta_3 - \theta_1\right) \end{aligned} \quad (6)$$

Using Eq.(6) (which are $\partial U / \partial \theta_i = 0$) for any given set of moments (and their associated surface rotation field) the stiffness coefficients k_i and k_{ij} can be evaluated. Castigliano's theorem states that more than one load regime may be required to determine all stiffness coefficients in a general case. However, not all combinations of load cases result in a rank sufficient system in terms of the stiffness coefficients k_i and k_{ij} as variables, so care is required. Here, an analysis of the system in Fig.1(a) is used to obtain the associated surface moments M_1, M_2 & M_3 and rotations θ_1, θ_2 & θ_3. Thus, the spring stiffnesses for the mechanical analogue system can be derived.

Surface displacement field caused by applied surface moments

To determine the stiffness coefficients in Eq.(6) the surface moment-rotation relationship must be determined. In this paper, two approaches are presented: (i) an analytic approximation based on a combination of the application of the Boussinesq solution (Poulos and Davis 1974) and (M I Gorbunov-Possadov et al. 1961) results and (ii) an empirical fit of finite element and experimental results.

For small deflections, the surface displacement field $U(x)$ is defined in Eq.(7) in terms of a decay function $\Delta(x)$ (see Fig.2), where *x* is an arbitrary horizontal coordinate in the free surface plane

$$U(x)=\tfrac{b}{2}\phi\Delta(x), \quad \text{for} \quad x\geq\tfrac{b}{2}, \quad \Delta\left(\tfrac{b}{2}\right)=1 \tag{7}$$

where ϕ is the rotation of the rigid footing and *b* is the actual width of the footing. This equation is non-dimensionalised by the introduction of the non-dimensional length $x=\xi b$ (where ξ is a non-dimensional horizontal coordinate) and non-dimensional surface vertical displacement $u(x)$ (where $U(x)=u(x)b$). Hence Eq. (7) becomes.

$$u(\xi)=\tfrac{1}{2}\phi\Delta(\xi), \quad |\xi|\geq\tfrac{1}{2}, \quad \Delta\left(\tfrac{1}{2}\right)=1 \tag{8}$$

By differentiating Eq.(8), an expression for the surface rotation field is obtained.

$$\theta(\xi)=u'(\xi)=\tfrac{1}{2}\phi\Delta'(\xi) \ , |\xi|\geq\tfrac{1}{2}, \tag{9}$$

The prime notation in this equation is defined as $\Delta' = \mathrm{d}\Delta/\mathrm{d}\xi$.

Boussinesq approximation for surface rotation field

Boussinesq (Poulos and Davis 1974) suggested that the vertical surface displacement field ρ_z due to a vertical point load *P* applied to a linear elastic half-space is given by the following equation

$$\rho_z(x)=\frac{P}{\pi}\frac{\left(1-\nu^2\right)}{E}\frac{1}{|x|}: \quad |x|\gg 0 \tag{10}$$

where *v* is the Poison's ratio and *E* is the elastic modulus of the half space. The ordinate *x* is any radial distance from the point load in the surface plane. If this formula, Eq.(10), is applied to the case of a couple of equal and opposite forces one located at *x* = -*b*/2 and the other at *x* = +*b*/2 an estimate of the surface vertical displacement function *U*(*x*) due to an applied moment *m* = *Pb* can be obtained by superposition, as follows:

$$\begin{aligned} U(x) &= \rho_z\left(x-\tfrac{b}{2}\right)-\rho_z\left(x+\tfrac{b}{2}\right)=\frac{P}{\pi}\frac{\left(1-\nu^2\right)}{E}\left(\frac{1}{\left|x-\frac{b}{2}\right|}-\frac{1}{\left|x+\frac{b}{2}\right|}\right) \\ &= \frac{m}{\pi}\frac{\left(1-\nu^2\right)}{Eb^2}\frac{\operatorname{sgn}(x)}{\left(\left(\frac{x}{b}\right)^2-\frac{1}{4}\right)}:\quad |x|\gg\tfrac{b}{2} \end{aligned} \tag{11}$$

The simplification above is easily obtained by assuming two cases one where $x<-b/2$ and the other when $x>b/2$. These two cases can be combined into equation (11) by employing the Signum function. The Signum function used above is defined as $\operatorname{sgn}(x)=x/|x|$. Note that this expression is not valid in the range $b/2>x>-b/2$ where we assume that the footing imposes a linear displacement field. Introducing the non-dimensional coordinate $x=\xi b$ and displacements $U(x)=u(x)b$

$$u(\xi)=\frac{m}{k_s}\frac{1}{\pi}\frac{\operatorname{sgn}(\xi)}{\xi^2-\frac{1}{4}},\qquad k_s=\frac{Eb^3}{1-\nu^2}:\quad |\xi|\gg\tfrac{1}{2}, \tag{12}$$

The rotation of the surface is given by differentiation for the case of small deflection theory, (Boas 2006)

$$u'(\xi)=\theta(\xi)=-\frac{m}{k_s}\frac{1}{\pi}\frac{2\xi\operatorname{sgn}(\xi)}{\left(\xi^2-\frac{1}{4}\right)^2} \tag{13}$$

Note that the derivative of $\operatorname{sgn}(\xi)$ is a Dirac delta $\delta(\xi)$ hence we would expect to see this in equation (13). However, since the range of analysis here is limited to $|\xi|\gg\frac{1}{2}$ the derivate terms involving $\delta(\xi)$ can be safely neglected as it is zero for $\forall\xi\neq 0$. This result is reasonably accurate away from the application of the point loads but is, unfortunately, singular (infinite) at the edge of the foundation, i.e. $\xi = \frac{1}{2}$, due to the limitation of Boussinesq's conjecture. So the formula suggested by (M I Gorbunov-Possadov et al. 1961) is used instead for the rotation ϕ at the footing itself.

$$\phi = \frac{m}{k_s} = m\frac{1-v}{0.5Gb^3} \tag{14}$$

The term k_s in Eq.(12) is assumed to be the rotational stiffness of the footing; *G* is the elastic shear modulus of the half-space. Therefore expressing m/k_s in terms of ϕ and recalling the form of Eq.(8), an estimate of the surface displacement at any point a non-dimensional distance ξ away from a footing subject to a rotation ϕ is obtained:

$$\begin{gathered} u(\xi) = \tfrac{1}{2}\phi\,\Delta(\xi) \\ \Delta(\xi) = \tfrac{1}{2\pi}\frac{\operatorname{sgn}(\xi)}{\xi^2 - \frac{1}{4}}: \quad |\xi| \gg \tfrac{1}{2} \end{gathered} \tag{15}$$

It should be noted that this formula (15) gives $\Delta\left(\frac{1}{2}\right) = \infty$ rather than 1. This is a consequence of the singularity embedded in Boussinesq's result. By differentiation an estimate of the surface rotation function $\Delta'(\xi)$ is obtained:

$$\Delta'(\xi) = -\tfrac{1}{\pi}\frac{\xi\operatorname{sgn}(\xi)}{\left(\xi^2 - \frac{1}{4}\right)^2}: \quad |\xi| \gg \tfrac{1}{2} \tag{16}$$

Empirical fit surface decay function using finite element analysis (FEA)

The weakness of Eq.(16) is that its accuracy is likely to reduce as ξ reduces i.e. as the footings get closer together, and this is when it needs to be most accurate. Additionally, it does not include the constraining effects of the footing itself, that is a footing applies a moment but also constrains displacements locally. Finally, Eq.(16) is only applicable for a very simple case of a linearly elastic, homogeneous isotropic half-space. For more complex cases finite element analysis is required.

From a finite element (PLAXIS2D, (PLAXIS-BV 2012)) solution of the problem of this single moment applied to an isotropic linear elastic half-space, (Aldaikh 2013), a good least squares match (R^2=0.99) to the decay function $\Delta(\xi)$ is obtained by using the following inverse square relationship

$$\Delta(\xi)=\frac{\operatorname{sgn}(\xi)}{\left(2.83|\xi|-0.415\right)^{2}} \quad : \quad |\xi|\geq \tfrac{1}{2} \qquad (17)$$

The FE model is a two-dimensional (2-D) plane strain model (i.e. results are represented per unit length in the out of plane direction) with linear elastic underlying material conditions which have the elastic properties of the Polyurethane foam hereinafter described, Fig.3. Adjacent footings were modelled using 2-D plate elements of 1m unit width, composed of beam elements with three degrees of freedom: two translational *dofs* and one rotational *dof* in the *x-y* plane. The beam elements are perfectly rigid and based on Mindlin's beam theory (See PLAXIS2D reference manual). The soil was modelled using an unstructured mesh of 15 node triangular elements with finer mesh coarseness in regions close to the foundation plates. It has been recommended that finite element mesh for shallow foundations of width *r* on isotropic homogeneous soil usually includes an area extending to about 5*r* laterally and 8*r* vertically, an area within most of the stresses variation are expected to occur, (Azizi 2000).

Thus, by differentiating Eq.(17) we obtain an estimate of the surface rotation function $\Delta'(\xi)$

$$\Delta'(\xi)=-\frac{5.66}{\left(2.83|\xi|-0.415\right)^{3}} \quad : \quad |\xi|\geq \tfrac{1}{2} \qquad (18)$$

This empirical curve-fit in Eq.(17) is an inverse quadratic and as such is of the same order as Eq.(15). It should be noted, however, that this equation, Eq.(17), is also constrained to give $\Delta\left(\tfrac{1}{2}\right)=1$ which is the correct value and so it differs at small ξ from the Boussinesq derived Eq. (15) which is singular. Fig.4(a) and Fig.4(b) respectively display comparisons between the surface decay function and surface rotation function using the Boussinesq results, Eq.(15) and Eq.(16), with the FEA fitted functions, Eq.(17) and Eq.(18). It can be seen that the form of both functions in Eq.(15) and Eq.(17) is very similar.

Example applications

In this section, two example cases are considered: (i) two identical footings, and (ii) three equispaced footings. These are considered here to conjecture whether simple formulae for rotational spring stiffnesses can be determined, that are sufficiently accurate (for practising engineering analyses/design) for a range of different system geometries, (i.e. for a different number of footings and non-identical ones).

Analysis case 1: two identical rigid footings with interaction

Consider Fig.1(b), where $k_3 = k_{23} = k_{13} = 0$ and $k_1 = k_2$. For a load case it is assumed that a single moment $M_1 = m$ is applied to rigid footing 1, and $M_2 = M_3 = 0$. According to Eq.(9) the rotations of the footings are, for this load case, $\theta_1 = \phi$, $\theta_2 = \frac{1}{2}\phi\Delta'(\xi)$ and $\theta_3 = \frac{1}{2}\phi\Delta'(2\xi)$. Hence, Eq.(6) can be solved to determine the unknown stiffness coefficients k_1 and k_{12}

$$k_1 = k_2 = k_s \frac{2}{\Delta'(\xi) + 2}$$

$$k_{12} = -k_1 \frac{\Delta'(\xi)}{\Delta'(\xi) - 2} \tag{19}$$

$$k_s = \frac{m}{\phi}$$

It should be noted that k_s would be the rotational spring stiffness of a single, completely isolated, rigid footing; that is to say, the k_s value could be obtained directly from Eq.(14) (M I Gorbunov-Possadov et al. 1961). The rotational spring stiffness $k_1 \neq k_s$ as it includes the additional stiffening effect of the adjacent footing.

Analysis case 2: three identical, equispaced, rigid footings with interaction

In this symmetrical case with identical rigid footings, $k_1 = k_3$ and $k_{12} = k_{23}$. For this problem, there are four unknown stiffness coefficients $k_1, k_2, k_{12} \,\&\, k_{13}$. Hence two load cases are required. First, a moment $M_1 = m$ is applied to rigid footing 1 and $M_2 = M_3 = 0$. According to Eq. (9) the rotations of the footings are, for this load case, $\theta_2 = \phi$, $\theta_2 = \frac{1}{2}\phi\Delta'(\xi)$ and $\theta_3 = \frac{1}{2}\phi\Delta'(2\xi)$. In the second load case, a moment $M_2 = m$ is applied to rigid footing 2 and $M_1 = M_3 = 0$. According to Eq. (9) the rotations of the footings are, for this load case, $\theta_2 = \phi$, $\theta_1 = \theta_3 = \frac{1}{2}\phi\Delta'(\xi)$. Hence Eq. (6) can be solved to determine the unknown stiffness coefficients $k_1, k_2, k_{12} \,\&\, k_{13}$.

$$k_1 = k_3 = k_s \frac{\Delta'(\xi) - 2}{\Delta'(\xi)^2 - \Delta'(2\xi) - 2}$$

$$k_2 = k_1 \frac{2\Delta'(\xi) - \Delta'(2\xi) - 2}{\Delta'(\xi)^2 - 2} \tag{20}$$

$$k_{12} = k_{23} = -k_1 \frac{\Delta'(\xi)}{\Delta'(\xi) - 2}$$

$$k_{13} = -k_1 \frac{\Delta'(\xi)^2 - \Delta'(2\xi),}{\left(\Delta'(\xi) - 2\right)\left(\Delta'(2\xi) - 2\right)}, \tag{21}$$

Experimental Evaluation of Spring Coefficients

To physically validate the theoretical expressions proposed for the rotational coupling and foundation springs, a simple experiment was performed for the case of two identical adjacent rigid foundations as described in the following paragraphs. The aim here is to produce physical similitude of the analytical method used to evaluate the rotational springs stiffnesses, i.e. k_i and k_{ij}.

Setup and Procedure

The two foundations were modelled with square Perspex plates (width *B*= 80 mm and *t*=5 mm thick) and were firmly glued using an epoxy adhesive to the surface of a Polyurethane foam block (dimensions: 1000x1000x750 mm, Young's modulus 120 kN/m^2; Poisson's ratio 0.11 and density 50 kg/m^3). The foam block proved suitable as a representation of the linear elastic half-space, (Aldaikh et al. 2015), (Aldaikh et al. 2016) and (Soubestre et al. 2012). The experiment setup is depicted in Fig.5. A moment was applied at the centre of one plate (active plate) and the resulting rotations of the active plate itself and at the second plate (passive plate) were measured. This procedure was followed for different spacing intervals *z,* as shown in Table A.1, between the two plates to eventually derive a function between rotational springs stiffnesses and spacing. It was not, however, experimentally straightforward to apply a moment at the centre of the active plate, hence, an aluminium rod of negligible weight was fixed at the middle of the active plate which was pulled by a wire running through a pulley. The wire carried weights which would generate a tension force pulling the aluminium bar and creating a moment at the centre of the first plate.

The moment was equivalent to the tension force *T* multiplied by the lever arm *l*. Vertical displacements at the edges of each plate were recorded using Linear Variable Differential Transformer (LVDT) transducers, two per plate as shown in Fig. 5. Values of rotations ϕ_1 and ϕ_2 (Appendix A) at the centre of each plate were calculated as follows:

$$\phi_1 = \frac{y_2 - y_1}{B}$$
$$\phi_2 = \frac{y_4 - y_3}{B} \quad (22)$$

where y_1 and y_2 are the vertical displacements at the edges of the active plate (ends 1 and 2) where the moments were applied while y_3 and y_4 are the vertical displacements at the edges of the second plate (ends 3 and 4). By rearranging Eq.(6) the formulae for k_1 and k_{12} as functions of ϕ_2 and ϕ_1 are as follows:

$$k_1 = k_s \frac{\phi_1}{\phi_1 + \phi_2}$$

$$k_{12} = k_1 \frac{\phi_2}{\phi_1 - \phi_2} \qquad (23)$$

$$k_s = \frac{m_1}{\phi_1}$$

where k_s is the experimentally determined foundation stiffness of an isolated footing (with no neighbouring footing).

Results

Analysis case 1: results

Fig.6(a) and Fig.6(b) respectively present the variation of foundation rotational stiffness (normalised by k_s) and interaction (coupling) rotational stiffness (normalised by k_1) with the non-dimensional inter-footing spacing for the case of two identical adjacent footings. It can be seen from Fig.6(a) and Fig.6(b) that the increase in the rotational stiffness of a single foundation (i.e. separation distance independent) could reach up to 25% when there is a negligible distance between the edges of the adjacent foundations. Similarly, it can be seen that as the inter-foundation spacing increases the interaction effect diminishes. At a spacing of approximately 2.5 times the foundation's width, the rotational coupling stiffness is negligible. It can also be observed that results from the proposed formulation for both individual foundation and coupling interaction stiffness coefficients agree very well with both FEA and experimental data. Moreover, the current results for the coupling coefficients, Fig6.(b), are compared to those resulted from the logarithmic curve fitting formula proposed by Mulliken and Karabalis (Mulliken and Karabalis 1998). However, using the Boussinesq approximate Eq.(15) resulted in a slightly stiffer estimate of stiffness coefficients. It should be noted that the experimental stiffness ratios shown in Fig.6(a) and Fig.6(b) are the average values resulted from all applied bending moment levels.

Analysis case 2: results

In this section, the following questions are considered: (i) in the case where there are more than two adjacent foundations, would adjacent footing coupling springs k_{12} and k_{23} be sufficient to model the mutual interaction i.e. is the additional alternate footing coupling spring k_{13} necessary? (ii) are the resultant numerical values for k_{12} significantly different in the two and three footings case? (iii) are stiffness coefficients k_1 and k_2 significantly different in the two and three footings case?

These questions are examined in Fig.7(a) and Fig.7(b) where they respectively present the variation of foundation rotational stiffness and interaction (coupling) rotational stiffness with the inter-footing centre-to-centre spacing for the case of two adjacent footings in comparison to that where a third foundation is present.

The value of the alternate footing coupling spring coefficient k_{13} decreases as the footing spacing increases and it approximately equals one-quarter of that of the adjacent footing coupling k_{12} at spacing where footings touch, i.e. at $\xi = 1$, (see Fig.7(b)). Given other epistemic uncertainty present in the application of this theory to physical problems (e.g. due to the site characterisation of soil) it appears that the alternate footing coupling spring coefficient k_{13} may be neglected without significant error, as was done in (Aldaikh et al. 2015).

The values of the adjacent coupling spring coefficient k_{12} are almost identical for the case of two and three footings; i.e. formulae Eq.(19) and Eq.(21) for k_{12} produce almost identical results regardless of centre-to-centre footing spacing ξ. This suggests that Eq.(19) for adjacent coupling spring coefficients is a reasonable and simple approximation for a more general case.

Finally, the values of spring coefficients k_1 and k_2 for the two and three footing cases show very similar qualitative forms. However, these coefficients in the three footing case are slightly stiffer than the two footing case. Fig.8 displays these relative stiffening effects graphically when moving from two to three closely spaced footings. The central spring k_2 is generally greater than the outer spring k_1 in this case. It should be noted here that these small relative stiffening effects were neglected in (Aldaikh et al. 2015).

Relative errors in employing the two footing formulation more generally

Eq.(19) along with the surface slope decay function, Eq.(18) are simple and easy to adopt for a more general case of multiple footings (greater than 2). The results in Analysis case 2 section suggest that the Eq.(19) estimate of adjacent footing coupling rotational springs $k_{i,i+1}$ are almost exactly the same as the more complex and accurate Eq.(21). Additionally, these results suggest that there is an argument to completely neglect alternate footing coupling rotational springs $k_{i,i+2}$. However, the same results also suggest that if the estimate of foundation springs k_i from Eq.(19) is employed for a more general case of multiple footings (greater than 2) then it tends to underestimate the stiffnesses (see Fig.8).

Therefore the question remains if formulation in Eq.(19) is used for three footings (with $k_1 = k_2 = k_3$ and $k_{13} = 0$) rather than Eq.(20) and Eq.(21), what errors would be introduced?

For any given rotations of footings, θ_i the resulting norm of moments $\|M_i\|$ can be evaluated using Eq.(6). This analysis is performed for both cases (a) stiffness from Eq.(19) with $k_1 = k_2 = k_3$ and $k_{13} = 0$ and (b) stiffness from Eq.(20) and Eq.(21). Therefore the relative percentage error ε of using formulation Eq.(19) in expressed as follows.

$$\varepsilon = 100 \frac{\|M_i\|_{\text{case (a)}} - \|M_i\|_{\text{case (b)}}}{\|M_i\|_{\text{case (b)}}} \tag{24}$$

The relative percentage error ε must be evaluated for a random set of footing rotations $\phi_i \in [-1,1]$, i.e. a range of different load cases. Fig.9 displays the results of such an analysis, plotting the relative error, Eq.(24), as a function of the centre to centre footing spacing ξ .The mean error $\mu(\varepsilon)$ at a touching distance ($\xi = 1$) is approximately -7%. Given other epistemic uncertainties present (e.g. in site soil classification) in the application of this theory to physical problems, this is a small error.

Conclusions

The current study presented a simplified analytical formulation for the evaluation of frequency-independent stiffness coefficients for the problem of adjacent identical footings resting on a linear elastic half-space. A derivation of the formulae was presented for the case of two and three adjacent foundations. Boussinesq's solution for the surface displacement field caused by a point load is extended to the case of a moment and combined with the Gorbunov-Possadov moment-rotation relationship for an isolated footing.

The extended Boussinesq's solution, along with a rigorous finite element model and analogue physical model, showed excellent agreement with the proposed formulae for both foundation rotational and coupling spring stiffness coefficients. Contrary to the common assumption in past literature, the dependency of rocking stiffness of individual foundations on the inter-foundation spacing has been demonstrated which indicates that reliance on such spacing-independent rocking stiffness could lead to over-conservative analyses. Results have also shown that there exists only a small difference in the value of adjacent footing rotational stiffnesses when more than two foundations are considered in the analysis. Hence, omitting springs connecting alternate footings is permissible given the other epistemic uncertainties in a physical setting. Bearing in mind this limiting assumption, the formulae proposed in Eq. (18) and Eq.(19) are simple and straightforward to adopt for a more general case of multiple footings (greater than two). These can be directly used in the straight-forward implementation of discrete lumped parameter modelling of adjacent structure interaction problems which could save considerable computational effort in preliminary design.

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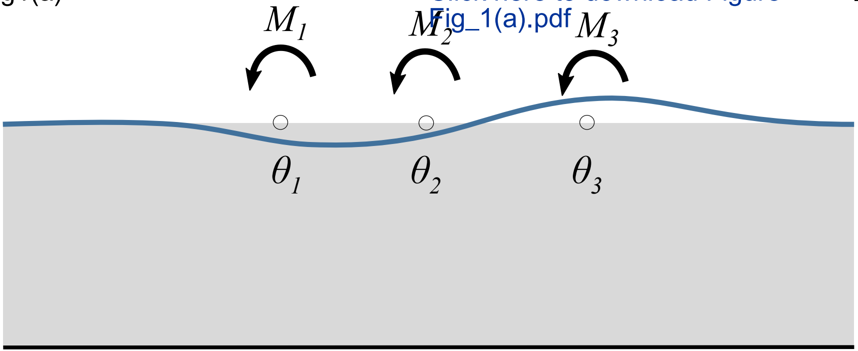
Appendix A

Table A.1 Foundations rotations in radians

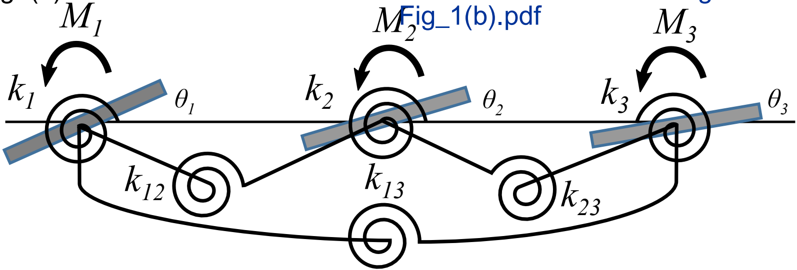
Moment m_1 [N.mm] \ *Spacing z* [mm]		8	16	24	32	40	48	56	64	72	80
68	ϕ_1	0.00355	0.0053	0.0057	0.0058	0.006113	0.00465	0.00485	0.0049	0.00497	0.005
	ϕ_2	-0.0006	-0.00042	-0.00042	-0.0005	-0.00017	-0.00013	-0.00013	-0.00013	-0.00017	-0.00013
98	ϕ_1	0.0071	0.0079	0.0083	0.0083	0.0091	0.0071	0.00751	0.0069	0.00764	0.0072
	ϕ_2	-0.0009	-0.00066	-0.00063	-0.00063	-0.0003	-0.0002	-0.0003	-0.00013	-0.00023	-0.00017
173	ϕ_1	0.0126	0.0144	0.0149	0.0153	0.0185	0.0128	0.0132	0.012	0.0128	0.0130
	ϕ_2	-0.0016	-0.00126	-0.00099	-0.00099	-0.00053	-0.0004	-0.0005	-0.00037	-0.00027	-0.00023
248	ϕ_1	0.0197	0.0215	0.0239	0.0261	0.0287	0.0191	0.0185	0.0186	0.0186	0.0191
	ϕ_2	-0.0026	-0.00126	-0.00143	-0.00133	-0.0008	-0.00059	-0.00063	-0.0006	-0.00043	-0.00036
323	ϕ_1	0.027	0.031	0.0356	0.0389	0.0399	0.0257	0.0254	0.0255	0.0255	0.0257
	ϕ_2	-0.0034	-0.00253	-0.00193	-0.00169	-0.00106	-0.0008	-0.00086	-0.00076	-0.00063	-0.00049
473	ϕ_1	0.0413	0.046	0.0539	0.0596	0.0597	0.0385	0.0392	0.0385	0.0385	0.0406
	ϕ_2	-0.0049	-0.0038	-0.0026	-0.00243	-0.00159	-0.00133	-0.00123	-0.0011	-0.001	-0.00083
773	ϕ_1	0.0777	0.088	0.0931	0.0912	0.0906	0.0744	0.0769	0.0750	0.0760	0.077
	ϕ_2	-0.008	-0.00624	-0.0042	-0.0035	-0.00238	-0.00229	-0.00199	-0.00175	-0.00149	-0.0015

Fig1(a)



(a)

Fig1(b)



(a)

Fig2

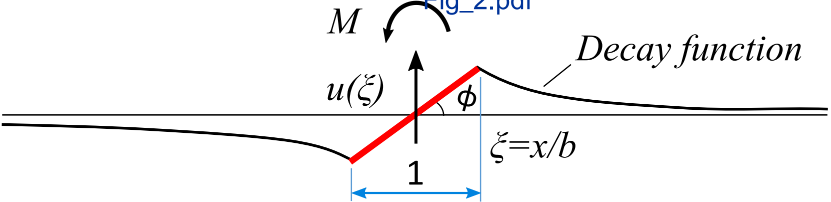


Fig4(a)

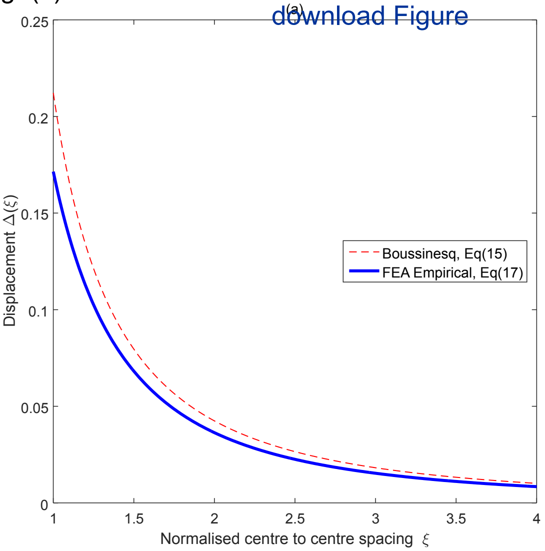

(a)
0.25
0.2
0.15
0.1
0.05
0
1
1.5
2
2.5
3
3.5
4
Displacement $\Delta(\xi)$
Normalised centre to centre spacing ξ
Boussinesq, Eq(15)
FEA Empirical, Eq(17)

Fig4(b)

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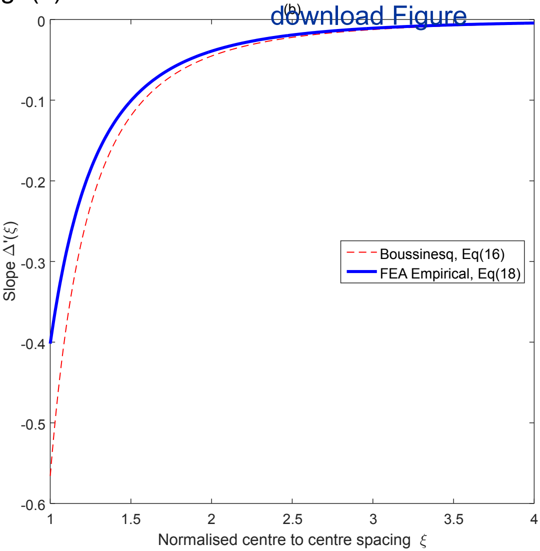

Fig5

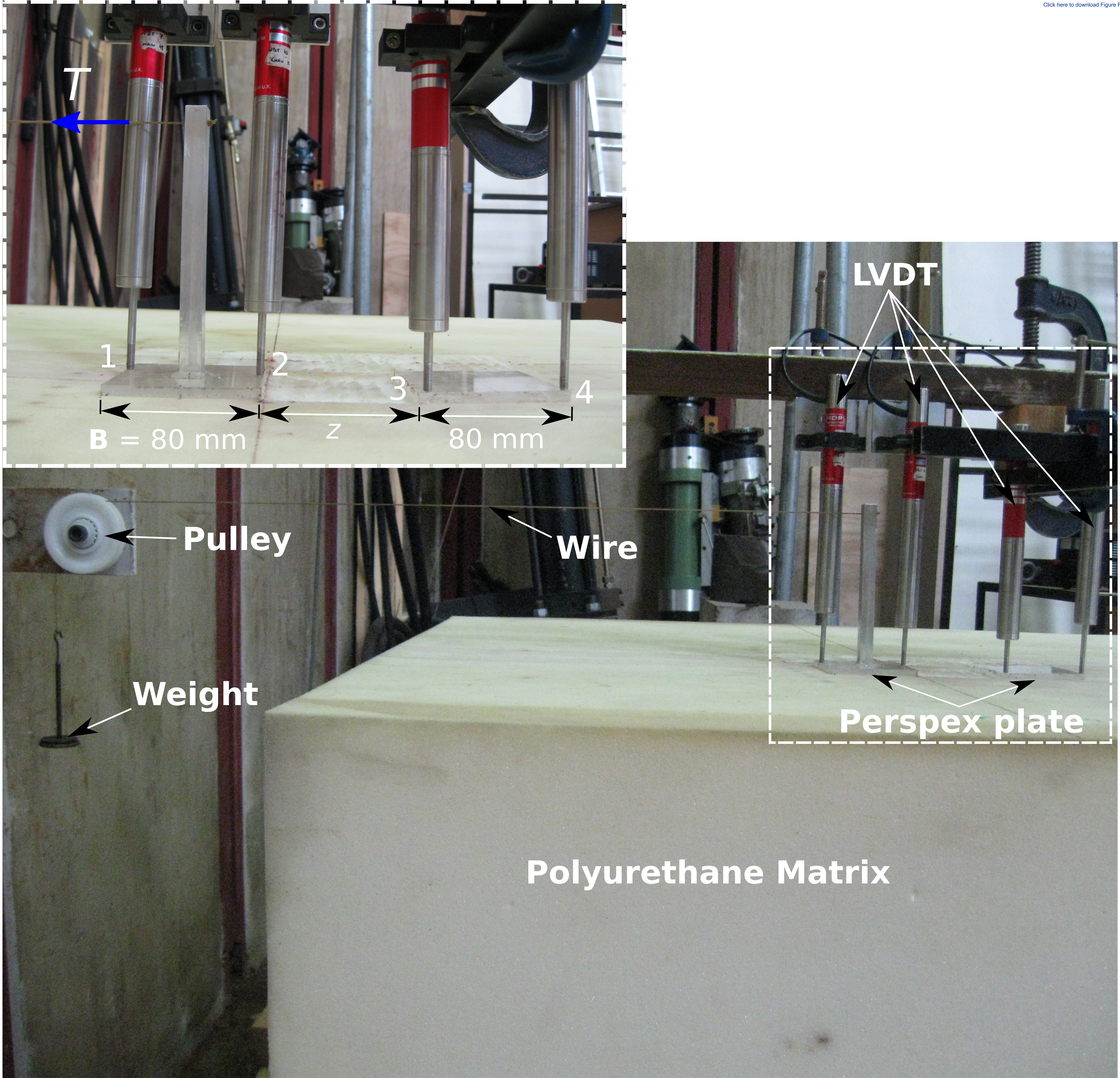
T
1
2
3
4
B = 80 mm
z
80 mm
LVDT
Pulley
Wire
Weight
Perspex plate
Polyurethane Matrix

Fig6(a)

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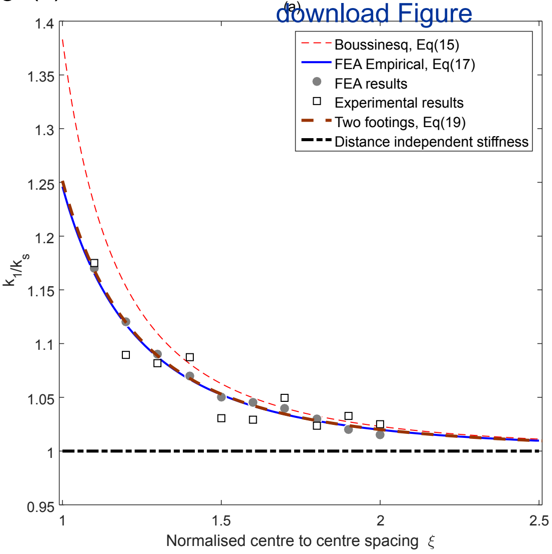

Fig6(b)

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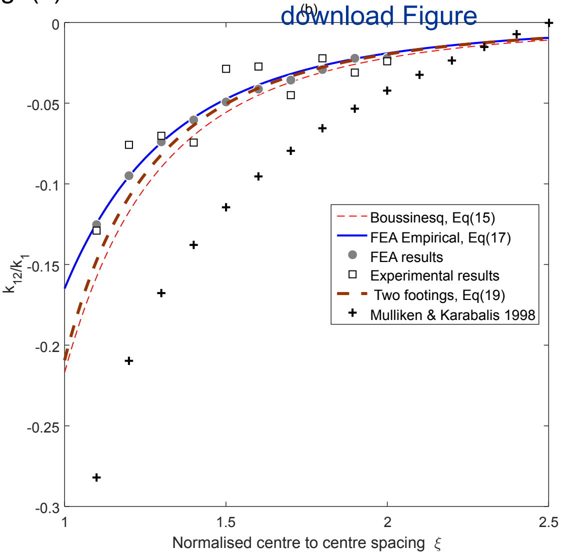

Fig7(a)

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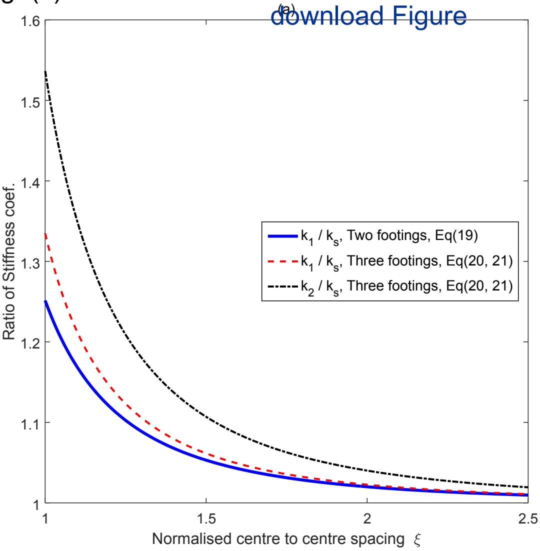

Fig7(b)

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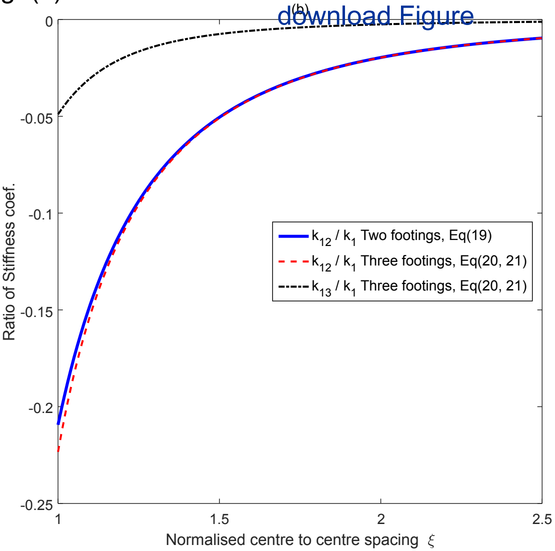
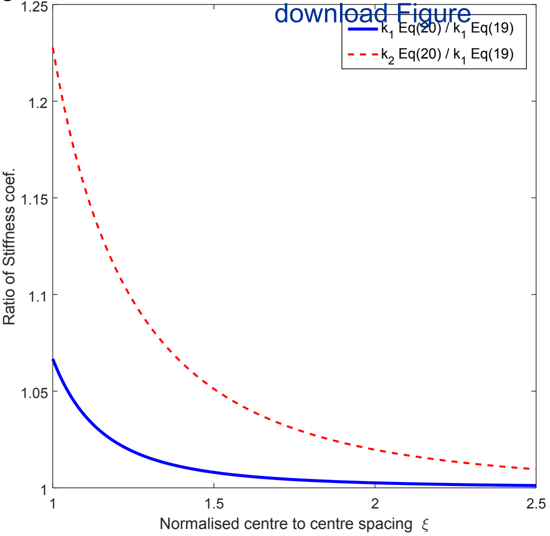
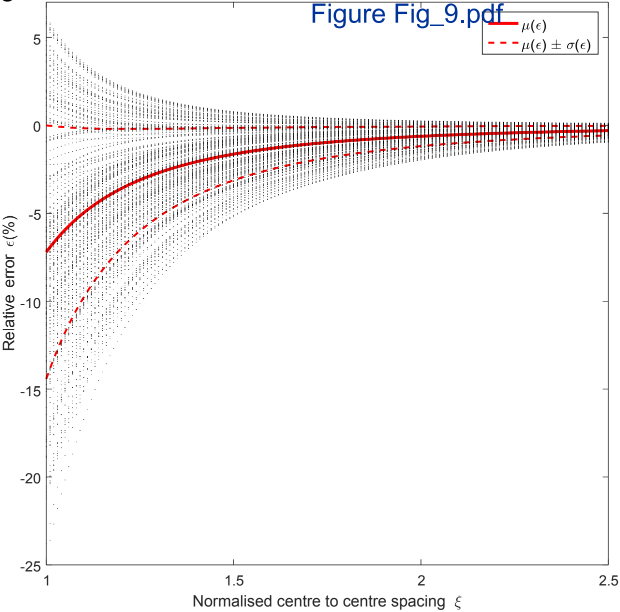


Fig8

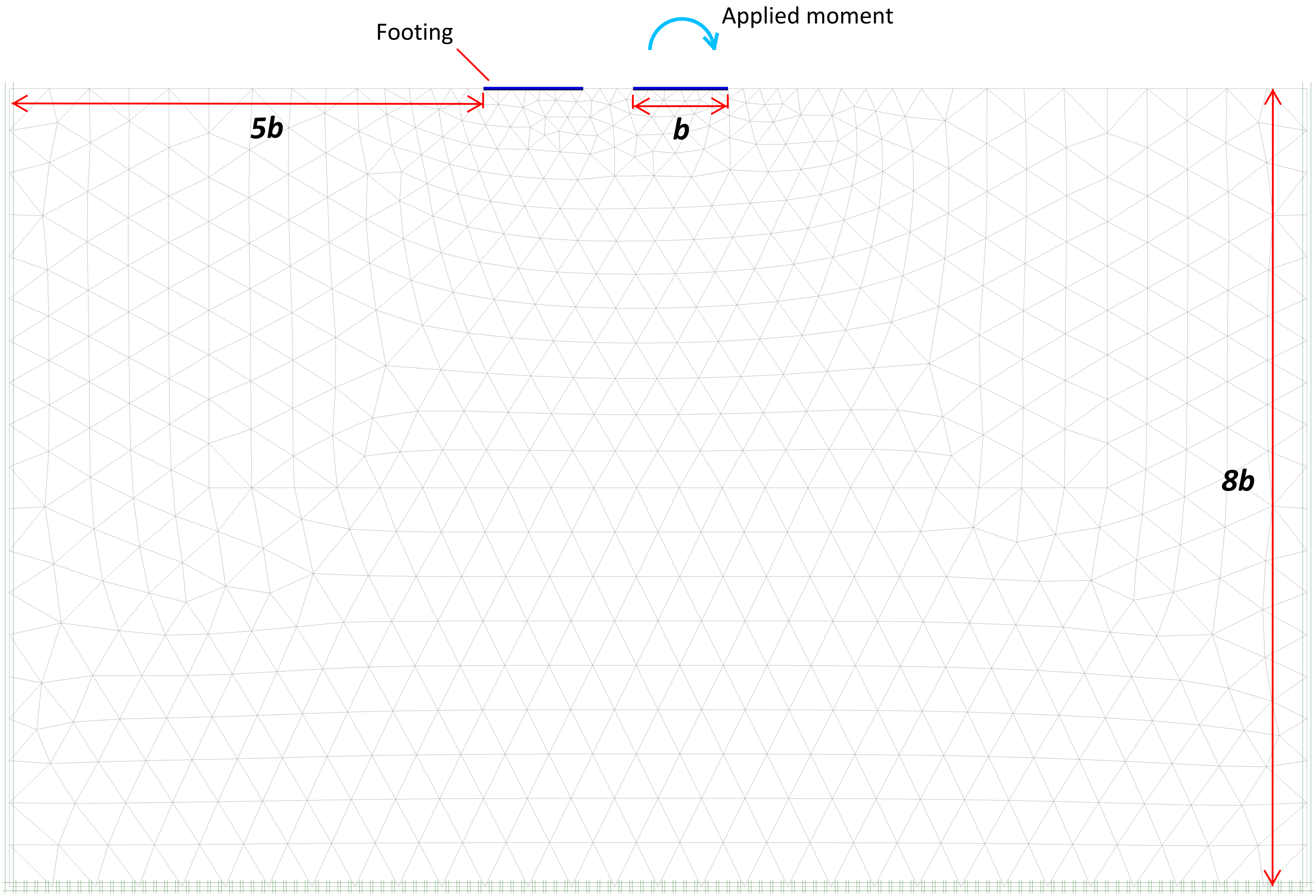


k1 Eq(20) / k1 Eq(19)
k2 Eq(20) / k1 Eq(19)
Ratio of Stiffness coef.
1.25
1.2
1.15
1.1
1.05
1
1.5
2
2.5
Normalised centre to centre spacing ξ

Fig9



$\mu(\epsilon)$
$\mu(\epsilon) \pm \sigma(\epsilon)$
5
0
-5
-10
-15
-20
-25
Relative error ϵ(%)
1
1.5
2
2.5
Normalised centre to centre spacing ξ


Footing
Applied moment
5b
b
8b

Figures Captions List

Fig.1. Idealisation of three adjacent foundation: **(a)** Complete system **(b)** Mechanical analogue system

Fig.2. Anti-symmetric surface displacement field cause by an applied surface moment

Fig.3. Evaluation of surface deformation due to rotation of a rigid footing using plane-strain finite element formulation (PLAXIS2D, (2012))

Fig.4. Comparison among different approaches: **(a)** Comparison of FEA empirical fit, equation (17), and Boussinesq result (15) for surface decay function $\Delta(\xi)$, **(b)** Comparison of FEA empirical fit, equation (18), and Boussinesq result (16) for surface slope function $\Delta'(\xi)$

Fig.5. Overview of experimental setup

Fig.6. Comparison of proposed formulae, FEA empirical, Boussinesq approximation, FEA and experimental data: **(a)** individual footing stiffness (with a neighbouring footing) relative to a single footing (with no neighbouring footing), **(b)** cross coupling spring stiffness relative to individual footing stiffness (with a neighbouring footing)

Fig.7. A comparison of two and three rigid footing spring stiffness coefficients: **(a)** individual footing stiffness (with a neighbouring footing) relative to a single footing (with no neighbouring footing), **(b)** cross coupling spring stiffness relative to individual footing stiffness (with a neighbouring footing)

Fig.8. Comparison of stiffness coefficient estimates from two and three footings cases

Fig.9. Relative error of employing simplified formulae (19) over more complicated formulae (20) and (21)

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"Evaluation of Rocking and Coupling Rotational Linear Stiffness Coefficients of Adjacent Surface Foundations"

Authors: Hesham Aldaikh, PhD; Nicholas A Alexander, PhD, C. Math., C. S; Erdin Ibraim, PhD; Jonathan Knappett, PhD.

Dear *Editor/Editorial Team, ASCE's International Journal of Geomechanics*

The authors would like to thank you and the referees for the time and effort spent on reviewing the manuscript. The reviewers' comments are valuable and very constructive, the authors highly appreciate the contributions of the reviewers which helped improving the presentation, readability and overall quality of the paper. The authors have thoroughly addressed and revised all issues indicated by the referees' and believe that the revised version can meet the publication requirements. Please find below the authors' responses (highlighted normal font) below each of the reviewers' comments (italic font). All suggested changes have been highlighted in a revised annotated version of the manuscript (Response to Reviewers_RevisedManuscript).

Yours sincerely,

Dr Hesham Aldaikh

Dr Nick A. Alexander

Date: 26 April 2017

Response to Reviewers

Reviewer #1: *The paper deals with the evaluation of rocking and coupling rotational stiffness of adjacent surface foundations. The paper is really interesting and well written. The following minor comments might benefit the authors in the revised version of the paper.*

1) *In the introduction would be meritorious also to mention pros and cons of direct and impedance methods in the case of nonlinear cases.*

A paragraph has been added to the introduction briefly mentioning some of the pros and cons of direct and impedance methods in nonlinear analysis, (page 2, starting line 54).

2) *Model description. Is also necessary to mention that the footings are identical? It seems it is not a condition but both numerical and experimental analyses are dealing with identical footings.*

We agree with the reviewer. The word "identical" has been added to the text to indicate that the analysis is limited to the case of identical foundations. However, the formulation presented could in fact be extended to the case of two or more surface foundation of dissimilar widths.

3) *Figure 2 and Equation 7: "b" is not defined*

Thank you. A clarification has been added to the text, (page 8, line 198).

4) *Equation 7: does x needs to be in absolute value?*

Yes, please see Boussinesq's formula.

5) *Equation 11: the idea to determine the displacement function from the principle of superposition is interesting, but in the reviewer opinion comments are necessary regarding the difference between this solution unconstrained between -b/2 and b/2 and the case of the footing that will impose constraints in the displacement field.*

Thanks for the suggestion. The suggested solution (eqn 11) is only valid for the cases |x|>> b/2. So it cannot and does not provide a solution for the constrained displacements under the footing itself. This displacement field in range b/2>x>-b/2 for a rigid footing is linear so is easily obtained.

6) *Equation 11: few algebra might also help the readers to derived it*

Textual comments and a little further algebra have been added to explain this simplification more clearly, (highlighted in page 9).

7) *Equation 13: it might be pedantic but the derivative of the sign(x) should be also added. If it is defined through the Heaviside (Unit step function) it might will lead to a Dirac's delta function. Clearly it depends how the unit step function is defined.*

Some textual comments have been added to clarify the math, (highlighted in page 9).

8) *Again in the Conclusions it needs to be clearly state if the analytical formulation is valid for identical footings or if it can be extended to different ones.*

A note has been added to the conclusion.

Reviewer #2: *This paper presents closed form expressions for rocking spring stiffnesses and coupling-interaction rotational spring stiffnesses for closely spaced two or three surface footings. In general, this paper is expected to be beneficial to the geotechnical and structural engineering communities. In addition, the topic fits within the scope of this Journal. However, some concerns need to be addressed before acceptance for publication.*

1. *Line 242: As the reference Aldaikh (2013) [PhD dissertation] cannot be easily obtained by readers, it is suggested to add details of the finite element (FE) model in PLAXIS2D, e.g. structural geometry and material, ground stiffness (elastic modulus of the half-space), out-of-plain thickness of the plain-strain element, mesh properties, boundary conditions, etc.*

We thank the reviewer for pointing this out. More information on the FE model has been added, (page 11 lines 252 to 262).

2. *Line 243: Again a concern on the FE model. In the paper, the authors compare the FE analysis (FEA) results with the experimental results. So I guess, in the FE model, you adopted the same parameters as the experiment, isn't it? Also, it is suggested to address the influence of the*

structural and ground stiffnesses (or their ratio) on the empirical Equations (17) and (18)? Please clarify.

Yes, the same parameters as the experiment were used in the FE model. This has been clarified as per the preceding response to comment 1. Although not thoroughly, both issues (1) different ground (soil) stiffnesses and (2) structure to ground stiffness (represented through sets of frequency ratios) have been previously considered in other publications by the authors, (Alexander 2013 and Aldaikh 2015). It was found that SSSI is most pronounced on smaller structures and on weaker soils (i.e. loose sand). These issues will be further considered in future research by the authors.

3. *Line 254: At this stage, the FEA empirical equation has not been validated by experimental results. Therefore, it is not appropriate to say "accurate" when discussing the result at smaller inter-footing spacing. Please revise.*

We agree with the reviewer. The sentence has been omitted.

4. *In Figure 6: How to obtain the Experimental Results (square marks) based on the test data listed in Table 1? By which bending moment level? or in average? Please clarify.*

We thank the reviewer again for pointing this out. Yes, the experimental results of stiffness ratios (k_1/k_s) and (k_{12}/k_1) presented in Figure 6 are the average values of all stiffnesses ratios resulted from different moment levels. A clarification has been added, (page 15, line 343).

5. *Line 347: For the result of k_{13}/k_{1} in Fig. 7(b), the authors mentioned that it does not exceed "one quarter" of that of the adjacent footing coupling at spacing where footings touch. However, from Fig. 7(b), it seems to be 0.05 rather than "one quarter"*.

What is intended to be said here is that the value of (k_{13}/k_1) to that of (k_{12}/k_1) at zero separation distance approximately equals ≈0.05/0.2 = 0.25. This has been clarified in the text, (page 16, line 356).

6. In Section 5.3: The authors should detail the considered parameters of the analyzed cases, i.e., what is the considered range of the given rotations of footings; I believe it will impact the mean and standard deviation shown in Figure 9.

As the problem in question is a linear elastic one, the magnitude of the rotations is not deemed to be important. However, in the authors' opinion, exploring a range of positive and negative rotations is needed. A note has been added, (page 15, line 388).

7. *Line 380: Minor mistake. The authors mentioned "… for both case (a) stiffness Eqs(19) with k_1=k_2=k_3 and k_{13}=0 and …". Here, in case (a), it should be k_3=0 and k_{23}=0 as well.*

Figure 9 compares the case of 3 adjacent foundations when the outer (connecting alternate foundations) spring, k_{13} , is taken into account with that when only internal (connecting adjacent foundation) springs, k_{12} and k_{23} , are considered (i.e. only k_{13}=0).

Response to Editorial Comments

The authors thank the editorial team for their comments. All comments 1 to 5 by the Editorial Coordinator have been addressed in the revised version of the paper.