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The Optimal Consumption Function in a Brownian Model of Accumulation Part C: A Dynamical System Formulation

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Abstract

This Paper continues the study of the Optimal Consumption Function in a Brownian Model of Accumulation, see Part A [2001] and Part B [2014]; a further Part D, dealing with the effects of perturbations of the Brownian model, is in preparation.

We begin here with a review of the o.d.e. system *S* which was used in Part B for the proof of the existence of an optimal consumption function. This system is non-linear, two dimensional but bilaterally asymptotically autonomous with limiting systems as log-capital tends to plus/minus infinity, each of which has a unique saddle point. An important part is played in the existence proof by the sets of forward/backward 'special' solutions, i.e. solutions of *S* converging to the asymptotic saddle points, and by their representing functions *f* and *g*. A 'star' solution, which is both a forward and a backward special solution, corresponds to an optimal consumption function.

It is shown here that the sets of special solutions of *S* are **C**(1) sub-manifolds of **R**(3), hence that the functions *f* and *g* are continuously differentiable. The argument involves the construction of an imbedding of *S* in a 3-D autonomous dynamical system such that the asymptotic saddle points are mapped to saddle points of the 3-D system and the sets of forward/backward special solutions are mapped into stable/unstable manifolds. The usual Stable/Unstable Manifold Theorem for hyperbolic stationary points then yields the required **C**(1) properties locally (i.e. near saddle points), and these properties can be extended globally. A 'star' solution of *S* then corresponds to a saddle connection in the 3-D system. A stability result for the saddle connection is given for a special case.

Keywords: Consumption, capital accumulation, Brownian motion, optimisation, ordinary differential equations, boundary value problems.

MSC 2010 subject classifications: 34B40, 34C12, 34C45, 49J15, 49K15.

JEL subject classifications: D900, E130, O410.

Abbreviated Title: Optimal Consumption as a Boundary Value Problem, Part C.

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The Optimal Consumption Function in a Brownian Model of Accumulation Part C: A Dynamical System Formulation *

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2016

Abstract

This Paper continues the study of the Optimal Consumption Function in a Brownian Model of Accumulation, see Part A, Foldes [2001] and Part B, Foldes [2014]. It is shown here that the sets $\mathcal{M}^{\triangleright}$ and $\mathcal{M}^{\triangleleft}$ of forward and backward 'special' solutions of the system S are $\mathbf{C}^1$ sub-manifolds of $\Re^3$, hence that the functions $f(\theta, z_{\diamondsuit})$ and $g(\theta, z_{\diamondsuit})$ defined in Part B are continuously differentiable.

The argument involves the construction of an imbedding of the systems S and $S_{\pm\infty}$ into a three-dimensional autonomous system $\mathfrak{S}$ such that the saddle points $\pi^*_{\pm\infty}$ of $S_{\pm\infty}$ are mapped to saddle points $p^*_{\pm\infty}$ of $\mathfrak{S}$ and $\mathcal{M}^{\triangleright}$ and $\mathcal{M}^{\triangleleft}$ are mapped into differentiable manifolds $\mathfrak{M}^{\triangleright}$ and $\mathfrak{M}^{\triangleleft}$ (respectively 'stable' at p^*_{∞} and 'unstable' at $p^*_{-\infty}$). This procedure permits application of the usual Stable/Unstable Manifold Theorems for stationary points to obtain the required $\mathbf{C}^1$ properties. The 'connection in S' between $\pi^*_{-\infty}$ and π^*_{∞} then corresponds to a saddle connection between $p^*_{-\infty}$ and p^*_{∞} in $\mathfrak{S}$. A stability result for the saddle connection is given for a special case.

Key words: Consumption, capital accumulation, Brownian motion, optimisation, ordinary differential equations, boundary value problems.

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Introduction to Part C

This paper resumes the study of the Optimal Consumption Function in a Brownian Model of Accumulation begun in Foldes [2001], hereinafter Part A or simply [A], comprising Sections 1 and 2 of our study, and Foldes [2014], hereinafter Part B or simply [B], comprising an Introduction, Sections 3 and 4, and Figures.

In Part A, we formulated a Brownian model of accumulation and derived sufficient conditions for optimality of a plan generated by a logarithmic consumption function, i.e. a relation expressing log-consumption $\ln \bar{c}$ as a time invariant, deterministic function $H(z)$ of log-capital z for $z \in \Re$ (both variables being measured in 'intensive' units). Writing

$$h(z) = H'(z), \qquad \theta(z) = \exp\{H(z) - z\},$$

the sufficient conditions require that the pair (h, θ) satisfy a certain non-linear, non-autonomous system of o.d.e.s $S = (F, G)$ of the form

$$h'(z) = F(h, \theta, z), \quad \theta' = G(h, \theta) = (h - 1)\theta$$

defined for $z \in \Re$ and (h, θ) in a suitable domain $\mathbf{U} \subseteq \Re^2$ — usually $\{h \in \Re, \theta \geq 0\}$, $\{h \in \Re, \theta > 0\}$ or $\{h > 0, \theta > 0\}$ — and that $h(z)$ and $\theta(z)$ converge to certain limiting values (depending on parameters) as $z \to \infty$ and as $z \to -\infty$. The system S is asymptotically autonomous as $z \to \infty$ and as $z \to -\infty$ with limiting autonomous systems $S_\infty = [F_\infty, G]$ and $S_{-\infty} = (F_{-\infty}, G)$, where $F_{\pm\infty}(h, \theta) = F(h, \theta, \pm\infty)$, and it was found that the appropriate limiting values for (h, θ) are defined by saddle points

$$\pi^*_\infty = (h^*_\infty, \theta^*_\infty) \quad \text{and} \quad \pi^*_{-\infty} = (h^*_{-\infty}, \theta^*_{-\infty})$$

of the systems S_∞ and $S_{-\infty}$; see [B] Prop.5 and Table I, also Section 5(ii) below. It was shown in Part B that (for each set of parameter values consistent with our Standing Assumptions) the resulting bilateral boundary value problem (b.v.p.) has a unique solution

$$\phi^*(z) = (h^*(z), \theta^*(z) \colon z \in \Re),$$

and hence that there exists a log-consumption function

$$H^* = (H^*(z) \colon z \in \Re)$$

generating an optimal plan.

The solution of the b.v.p. may be characterised loosely as the 'connection in S' between the saddle points of the systems S_∞ and $S_{-\infty}$, determined by the intersection of an 'in-set' $\mathcal{M}^\triangleright$ at π^*_∞, comprising points (h, θ, z) with $h > 0, \theta > 0$ which are 'forward starts' of solutions of S converging to π^*_∞ as $z \to \infty$, and an 'out-set' $\mathcal{M}^\triangleleft$ at $\pi^*_{-\infty}$, comprising points (h, θ, z) with $h > 0, \theta > 0$ which are 'backward starts' of solutions of S converging to $\pi^*_{-\infty}$ as $z \to -\infty$. Specifically, it was shown that, for a suitable fixed $z_\diamondsuit$, the section $\mathcal{M}^\triangleright(z_\diamondsuit)$ is the graph of a function $h = f(\theta, z_\diamondsuit)$ which is defined, continuous and decreasing in θ on an interval $(0, \theta_+(z_\diamondsuit))$, while the section $\mathcal{M}^\triangleleft(z_\diamondsuit)$ is the graph of a function $h = g(\theta, z_\diamondsuit)$ which is defined, continuous and increasing in θ on an interval $(\theta_-(z_\diamondsuit), \infty)$. For each admissible set of parameter values and suitably chosen $z_\diamondsuit$, the graphs are so situated that there is a point

$$\pi^*(z_\diamondsuit) = (h^*(z_\diamondsuit), \theta^*(z_\diamondsuit))$$

satisfying $\theta_-(z_\diamondsuit) < \theta^*(z_\diamondsuit) < \theta_+(z_\diamondsuit)$ and

$$h^*(z_\diamondsuit) = f[\theta^*(z_\diamondsuit), z_\diamondsuit] = g[\theta^*(z_\diamondsuit), z_\diamondsuit],$$

so that the solution of S satisfying the initial value problem (i.v.p.) through this point yields the solution ϕ^* of the b.v.p.

In Part B, the Stable/Unstable Manifold Theorems for stationary points, as usually stated, were not applicable because S is not autonomous and $\pi^*_{\pm\infty}$ are not saddle points of S but of $S_{\pm\infty}$. (In fact, apart from degenerate cases, S has no stationary points.) The existence proof presented did not show that $\mathcal{M}^\triangleright$ and $\mathcal{M}^\triangleleft$ are *differentiable* manifolds, and the representing functions for fixed $z_\diamondsuit$, f and g, were shown only to be continuous and monotonic in θ, rather than $\mathbf{C}^1$ in (h, θ, z). The present Part C reformulates the b.v.p. to avoid these limitations, although at the cost of a slight restriction on the production function in the underlying growth model. We construct an imbedding of the systems S and $S_{\pm\infty}$ into a three-dimensional autonomous system $\mathfrak{S}$ such that $\pi^*_{\pm\infty}$ are mapped to saddle points $p^*_{\pm\infty}$ of $\mathfrak{S}$, $\mathcal{M}^\triangleright$ and $\mathcal{M}^\triangleleft$ are mapped into differentiable manifolds $\mathfrak{M}^\triangleright$and $\mathfrak{M}^\triangleleft$ (respectively 'stable' at p^*_∞ and 'unstable' at $p^*_{-\infty}$), and the solution of the b.v.p. corresponds to a saddle connection in $\mathfrak{S}$. Differentiability of the manifolds will be needed in Part D for the investigation of the effect upon optimal consumption of perturbations of parameters of the growth model. Also, use of Stable Manifold Theory yields new approximations to the optimal consumption function when $|z|$ is large (i.e. when the economy is very rich or very poor).

5. Review of the Model

As indicated above, the main aim of the present Part C is to extend the proof of the existence of an optimal logarithmic consumption function to the setting of a three-dimensional autonomous dynamical system $\mathfrak{S}$. The argument will rely on various results from previous Parts, relating in particular to geometric properties of solutions of S and $S_{\pm\infty}$, and proofs of these results will usually not be repeated. However, in order to limit cross-reference and to facilitate comparisons, we begin here by recalling some definitions and results, adding or modifying details where necessary. These particulars will also be required for Part D.

(i) Recapitulation and Modification of the system S.

As in [B](0.1), we consider the system $S = (F, G)$ of o.d.e.s

$$(5.1) \qquad h' = F(h,\theta,z) = bh^2 + (2/\sigma^2)h[\theta - n + m/b - \frac{1}{2}b\sigma^2 - A] - 2[m - M]/b\sigma^2$$
$$\theta' = G(h,\theta,z) = (h-1)\theta$$

defined for $z \in \Re$ and (h, θ) in a suitable subset $\mathbf{U} \subset \Re^2$, where $h' = dh(z)/dz$, $\theta' = d\theta(z)/dz$, $A = A(z)$, $M = M(z)$. Details of the constants and the functions A and M are recalled below.

The system (5.1) is here considered formally as just a given first order o.d.e. system with an 'independent' variable $z \in \Re$ and 'dependent' variables (h, θ), but of course we bear in mind the economic derivation of the system explained in [A]. Thus we interpret z as log-capital and consider the situation where there is given a logarithmic consumption $H(z)$ which generates a feasible plan in the underlying growth model,[1] so that log-consumption $\ln \bar{c}$ is a time-invariant, deterministic function $H(z)$ of $z = \ln \bar{k}$ for $z \in \Re$, (both variables being measured in 'intensive' units). In this situation we define

$$h(z) = dH(z)/dz, \qquad \theta(z) = \exp\{H(z) - z\},$$

[1]For H to generate a feasible plan it is sufficient that $H'(z) = h(z)$ and $\theta(z) = \exp\{H(z) - z\}$ be bounded for $z \in \Re$ and that the *random processes* $\bar{c}_t = \bar{c}(\omega, t)$ and $z_t = z(\omega, t)$, representing random consumption and capital in intensive units, satisfy $\ln \bar{c}_t = H(z_t)$ and $dz_t = [A(z_t) - \theta(z_t)]dt + dx_t$ for all $t \geq 0$ a.s. with $z_0 = \ln K_0$ a.s., $K_0 > 0$, where $A(z_t)$ represents the (random) average product of log-capital, $\theta(z_t) = \exp\{H(z_t) - z_t\}$ and $dx_t = \mu dt + \sigma dB_t$. See [A] for details.

so that

$$\ln\theta(z) = H(z)/z = \bar{c}/\bar{k}, \qquad \theta' = (h-1)\theta;$$

then $\ln\theta(z)$ represents the 'average propensity to consume out of capital' while $h(z)$ is the '(logarithmic) marginal propensity to consume' or 'elasticity of consumption with respect to capital'.

Returning to eq. (5.1), recall that $b > 0$ is a coefficient of relative risk aversion — see [A] (1.5) and (1.30) — and $\sigma^2 > 0$, n, m are 'compound' parameters which are defined in terms of the 'primitive' parameters of the underlying stochastic model, namely b and the means and variances μ_η, σ^2_η, $\eta = \alpha, \beta, \gamma, \rho$ of the four Brownian motions — see [A] (1.7) et seq. Note that, in the present Part C, the values of all these parameters are assumed to be fixed throughout. The functions A and M, representing average and marginal products of log-capital, are defined for $z \in \Re$ in terms of the 'intensive' production function ψ by

$$(5.2) \qquad A(z) = \psi(\kappa)/\kappa = a(\kappa), \qquad M(z) = \psi'(\kappa), \qquad z = \ln\kappa, \quad \kappa > 0.$$

Recall that ψ is defined for $\kappa \geq 0$, with $\psi(0) = 0$, and is (at least) $\mathbf{C}^2$ with

$$(5.3) \qquad \psi'(\kappa) > 0 > \psi''(\kappa) \text{ for } 0 < \kappa < \infty, \quad \text{and limits}$$

$$(5.4) \qquad 0 < \psi'_0 = \psi'(0) < \infty, \qquad \psi'(\infty) = 0,$$

see [A] (1.3–1.4). It is further implicit in the assumptions made in Part A about the production function Ψ that the limits $\psi''(0)$ and $\psi''(\infty) = 0$ exist. Of course, $\psi'(\kappa) < a(\kappa)$ for $0 < \kappa < \infty$, and using (2) we have $da(\kappa)/d\kappa = [\psi'(\kappa) - a(\kappa)]/\kappa$. Thus $0 < M(z)/A(z) < 1$ for $z \in \Re$, and — as noted in [A] (1.23) —

$$(5.5) \qquad A'(z) = da(\kappa)/d\ln\kappa = \psi'(\kappa) - a(\kappa) = M(z) - A(z) = [M(z)/A(z) - 1]A(z),$$

also

$$(5.5a) \quad M'(z) = d\psi'(\kappa)/d\ln\kappa] = \kappa\psi''(\kappa)$$

hence

$$(5.5b) \quad M'(z)/A'(z) = [d\psi'(\kappa)/d\kappa]/[da(\kappa)/d\kappa] = \kappa\psi''(\kappa)/[\psi'(\kappa) - a(\kappa)] > 0.$$

It follows from (5–5a) and $\psi'' < 0$ that both $A(z)$ and $M(z)$ are decreasing and at least $\mathbf{C}^1$ on $\Re$, and by (2–4) these functions have one-sided limits

$$(5.6) \qquad A(-\infty) = M(-\infty) = \psi_0' = a(0), \qquad A(\infty) = M(\infty) = \psi'(\infty) = a(\infty) = 0.$$

We now list some further remarks and minor new assumptions about the limiting behaviour of the average and marginal product functions which will be needed later.

As $z \to -\infty$, $\kappa \to 0$, we have by the preceding remarks

$$(5.7) \qquad \lim_{Z \to -\infty} M(z)/A(z) = \lim_{K \to 0} \psi'(\kappa)/a(\kappa) = 1; \qquad A'(-\infty) = 0.$$

Assuming further that

$$(5.8) \qquad 0 > \psi''(0) > -\infty,$$

a simple application of Taylor's Theorem yields the limit

$$(5.9) \qquad \lim_{K \to 0} \{\kappa\psi''(\kappa)/[\psi'(\kappa) - a(\kappa)]\} = 2.$$ [2]

For $z \to \infty$, $\kappa \to \infty$, we further assume the existence of the limit

$$(5.10) \qquad r_0 \doteq \lim_{Z \to \infty} M(z)/A(z) = \lim_{K \to \infty} \psi'(\kappa)/a(\kappa), \quad 0 \le r_0 < 1, \quad r_0 \ne b.$$

The last term in (5) is therefore defined for $z = \infty$, so that

$$(5.11) \qquad A'(\infty) = 0.$$

Further, assuming convergence of the expressions in (5b), L'Hôpital's Theorem yields

$$(5.12) \qquad r_0 = \lim_{Z \to \infty} M'(z)/A'(z) = \lim_{K \to \infty} \{\kappa\psi''(\kappa)/[\psi'(\kappa) - a(\kappa)]\} \ge 0.$$

(For production functions usually considered which satisfy (4), one often obtains $r_0 = 0$.)

[2] Write $\psi(\kappa) = \kappa\psi'(0) + \frac{1}{2}\kappa^2\psi''(\delta_0\kappa)$, $a(\kappa) = \psi(\kappa)/\kappa = \psi'(0) + \frac{1}{2}\kappa\psi''(\delta_0\kappa)$, $\psi'(\kappa) = \psi'(0) + \kappa\psi''(\delta_1\kappa)$, substitute for ψ' and a in the expression $\kappa\psi''(\psi' - a)$, cancel κ and let $\kappa \to 0$. The existence of the limit on the left of (5.9) is required later on, but the assumption that $\psi''(0) < 0$ can be relaxed. A weaker alternative is to assume that $\psi(\kappa)$ can be developed as a Taylor series in a right neighbourhood of 0 up to some order i, where i is the first integer > 1 for which the derivative $\psi^{(i)}(0) \ne 0$. Then the limit in (5.9) is obtained as i instead of 2, and appropriate replacements must be made in later calculations.

Returning to (1), recall also the definitions of the following constants:

(5.13) $$N = n + (b-1)\psi_0'/b, \qquad \nu = N + \psi_0'/b = n + \psi_0',$$

(5.14) $$q = n + (b-1)(m + \frac{1}{2}b\sigma^2)/b,$$

(5.15) $$Q = n - m/b + \frac{1}{2}b\sigma^2 = q - m + \frac{1}{2}\sigma^2,$$

cf. [A] (1.15–16), [B] (3.39)ff. As in [B], we impose throughout the following

STANDING ASSUMPTIONS.

(5.16) If $b > 1$, then $N > 0$ and {either $n > 0$ or $q > 0$}.

(5.17) If $b < 1$, then $n > 0$ and {either $N > 0$ or $q > 0$}.

(5.18) If $b = 1$, then $N = n = q > 0$.

Usually we leave aside without special mention cases with $n = 0$ or $N = 0$, regarding which see [B], S.2 fn.3 and Fig.5, also cases with $b = 1$.

As in [B], we often denote by $\pi = (h, \theta)$ a point of $\Re^2$ and use curly brackets $\{..\}$ to denote a set of $\Re^2$, while bold curly brackets $\boldsymbol{\{..\}}$ denote a set of $\Re^3$. Sometimes we write the system (1) in vector form as

(5.1a) $$\pi' = S[\pi, z], \qquad S = (F, G),$$

and regard $S\colon \mathbf{U} \times \Re \to \Re^2$ as a 'capital-dependent' $\mathbf{C}^1$ vector field, with a suitable domain $\mathbf{U} \subseteq \Re^2$.

The choice of domain will vary according to the question under discussion. The system S (and the systems $S_{\pm\infty}$ considered below) will always be taken as defined for at least one of the domains, $\{\theta \geq 0\}$, $\{\theta > 0\}$, here called *basic domains*; but for particular arguments it will sometimes be convenient to consider the restriction to a suitable sub-domain, which may again be denoted by $\mathbf{U}$ (with affixes if necessary). For brevity we sometimes leave the domain to be inferred from the context.[3]

[3]The symbols $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are used generically in this Part and the next to denote sets of $\Re^2$, $\Re^3$ and $\Re^4$ respectively which serve as domains or sub-domains for various vector fields; thus $\mathbf{U}$ may be $\{\theta \geq 0\}$, $\mathbf{U}_+ = \{h > 0, \theta \geq 0\}$, $\{\theta > 0\}$, $\mathbf{U}_{++} = \{h > 0, \theta > 0\}$, while $\mathbf{V}$ may be $\mathbf{U} \times \Re$, $\mathbf{U} \times [z, \infty)$, $\mathbf{U} \times (-\infty, z]$ or a suitable subset of such a product, e.g., $\mathbf{V}_+ = \mathbf{U}_+ \times \Re$, $\mathbf{V}_{++} = \mathbf{U}_{++} \times \Re$, etc.

Incidentally, 'domain' here means 'domain of definition' rather than 'open connected set'. In case a domain contains some of its boundary points, we consider (without special mention) only 'inward' derivatives at these points, e.g when referring to $\mathbf{C}^1$ properties of a vector field. Also, for a point p of

Given a basic domain $\mathbf{U}$ for S and an interval $I \subseteq \Re$, a *solution* ϕ of S on I is by definition a function $z \mapsto \phi(z) = (h(z), \theta(z))$ of class $\mathbf{C}^1$ from I into $\mathbf{U}$ satisfying (5.1), or equivalently $\phi'(z) = S[\phi(z), z]$, for $z \in I$. The corresponding curve in the (h, θ) plane $\Re^2$ — strictly, the image set $\breve{\phi}(I) = \{\phi(z) \colon z \in I\}$ parametrised and ordered by I — is called the *path* of ϕ (on I).

In particular, a given point $(\pi_\diamondsuit, z_\diamondsuit) \in \mathbf{U} \times \Re$ defines a unique local solution of S

$$\phi = \phi(z; \pi_\diamondsuit, z_\diamondsuit) = (h(z; \pi_\diamondsuit, z_\diamondsuit), \theta(z; \pi_\diamondsuit, z_\diamondsuit))$$

'through' that point, i.e. $\phi(z_\diamondsuit; \pi_\diamondsuit, z_\diamondsuit) = \pi_\diamondsuit$, and this solution may be continued on a maximal open interval $I(\pi_\diamondsuit, z_\diamondsuit) = (z_-(\pi_\diamondsuit, z_\diamondsuit), z_+(\pi_\diamondsuit, z_\diamondsuit))$. The corresponding path in $\Re^2$ is denoted $\breve{\phi}(\pi_\diamondsuit, z_\diamondsuit) = \{\phi(z; \pi_\diamondsuit, z_\diamondsuit) \colon z \in I(\pi_\diamondsuit, z_\diamondsuit)\}$. According to context we say that the solution (or the corresponding path) *starts* at $(\pi_\diamondsuit, z_\diamondsuit)$, or at $\pi_\diamondsuit$ (given $z_\diamondsuit$), or simply at $z_\diamondsuit$. If $z_\diamondsuit = 0$, we write $\phi(z; \pi_\diamondsuit, 0)$ as $\phi^0(z; \pi_\diamondsuit)$ and $I(\pi_\diamondsuit, z_\diamondsuit)$ as $I^0(\pi_\diamondsuit)$ etc. Sometimes we consider only the forward or backward solution and write $S^\triangleright$, $\phi^\triangleright$ or $S^\triangleleft, \phi^\triangleleft$ instead of S, ϕ. Given $z_\diamondsuit$, the *forward solution* is restricted to $z \geq z_\diamondsuit$ and its interval of definition is of the form $I^\triangleright(\pi_\diamondsuit, z_\diamondsuit) = [z_\diamondsuit, z_+)$; similarly the *backward solution* is restricted to $z \leq z_\diamondsuit$ with interval of definition $I^\triangleleft(\pi_\diamondsuit, z_\diamondsuit) = (z_-, z_\diamondsuit]$.

a given domain $\mathbf{V} \subseteq \Re^K$, the term 'neighbourhood of p' will be taken to mean a 'neighbourhood w.r.t. $\mathbf{V}$'; thus a neighbourhood of a boundary point of $\mathbf{V}$ will be a set of the form $\mathfrak{N} \cap \mathbf{V}$, where $\mathfrak{N}$ is a neighbourhood of p w.r.t. $\Re^K$.

(ii) *Asymptotic Systems* $S_{\pm\infty}$.

Although the system $S = (F, G)$ is not autonomous, it depends on z only indirectly via the functions A and M. Using the limits (6) of these functions, we define limiting autonomous systems $S_{\pm\infty} = (F_{\pm\infty}, G)$, where

$$(5.19\text{a}) \qquad F_\infty(h,\theta) = bh^2 + (2/\sigma^2)h[\theta - n + m/b - \frac{1}{2}b\sigma^2] - 2m/b\sigma^2$$

$$(5.19\text{b}) \qquad F_{-\infty}(h,\theta) = bh^2 + (2/\sigma^2)h[\theta - N + (m - \psi_0')/b - \frac{1}{2}b\sigma^2] - 2(m - \psi_0')/b\sigma^2$$

see [B] (3.1–2). The limiting systems possess saddle points[4]

$$(5.20) \qquad \pi_\infty^* = (h_\infty^*, \theta_\infty^*) \text{ and } \pi_{-\infty}^* = (h_\infty^*, \theta_{-\infty}^*)$$

with co-ordinates as follows:

(5.21a)	$\pi_\infty^* = (1, n)$	if $n > 0$,	(S_∞ is Type 1)
	$\pi_{-\infty}^* = (1, N)$	if $N > 0$,	($S_{-\infty}$ is Type 1)
(5.21b)	$\pi_\infty^* = (h_\infty^+, 0)$	if $b > 1,\ q > 0 > n,\ 1/b < h_\infty^+ < 1,$	(S_∞ is Type 0)
	$\pi_{-\infty}^* = (h_{-\infty}^-, 0)$	if $b < 1,\ q > 0 > N,\ 1/b > h_{-\infty}^- > 1,$	($S_{-\infty}$ is Type 0)

where h_∞^+ is the greater of the two real roots of $F_\infty(h, 0) = 0$,

$h_{-\infty}^-$ is the lesser of the two real roots of $F_{-\infty}(h, 0) = 0$.

At each saddle point there are two distinct, real, non-zero eigenvalues λ_i, one positive and one negative. Further details of parameter values for S_∞ and $S_{-\infty}$ are given in [B] Section 3 and Table 1, also in Section 6, Table 2 below.

We recall some further definitions and results (occasionally with amendments) which are needed later on. Terminology for $S_{\pm\infty}$ is analogous to that for autonomous systems

[4]Given a system $\bar{S}$ (here S_∞ or $S_{-\infty}$), see [B] S.3, defined on a domain $\mathbf{U} \subseteq \Re^2$, a point $\pi^* \in \mathbf{U}$ is by definition a *saddle point* if it is a stationary point of the system and the Jacobian matrix (3.15) at π^* has two real, non-zero eigenvalues of opposite sign. In case π^* is a boundary point of $\mathbf{U}$, the entries in the matrix are 'inward' derivatives, see fn.3 above. Consequently, extending [B] S.3, fn.13, if $\mathbf{U} = \{\theta \geq 0\}$ and $\bar{S} = S_\infty$ is Type 0, the stable manifold at π_∞^* is a manifold with boundary; similarly if $\bar{S} = S_{-\infty}$ is Type 0, the unstable manifold at $\pi_{-\infty}^*$ is a manifold with boundary.

For simplicity, we leave aside the case $b > 1$, $h_\infty^+ = 1$, $n = 0$, where π_∞^* is a saddle-node if $\mathbf{U} = \Re^2$ but a saddle if $\mathbf{U} = \{\theta \geq 0\}$, see [B] 5.6 fn.6 and Figure 5; similarly we leave aside the case $b < 1$, $h_{-\infty}^- = 1$, $N = 0$.

$\bar{S}$ considered in [B], S.3, see especially S.3 fn.5. We write $\pi' = S_\infty(\pi)$ and regard $S_\infty\colon \mathbf{U} \mapsto \Re^2$ as a vector field, with a basic domain $\mathbf{U} \subseteq \Re^2$.

Given $\mathbf{U}$ and an interval I, a *solution* of S_∞ on I is a $\mathbf{C}^1$ function $z \mapsto \phi_\infty(z) = (h_\infty(z), \theta_\infty(z))$ from I into $\mathbf{U}$ satisfying $\phi'_\infty(z) = S_\infty[\phi_\infty(z)]$ for $z \in I$. The corresponding *path* in $\Re^2$ is $\breve{\phi}_\infty(I) = \{\phi_\infty(z)\colon z \in I\}$. A point $(\pi_\diamondsuit, z_\diamondsuit)$ defines a unique solution $\phi_\infty(z; \pi_\diamondsuit, z_\diamondsuit)$ 'through' that point, i.e. $\phi_\infty(z_\diamondsuit; \pi_\diamondsuit, z_\diamondsuit) = \pi_\diamondsuit$; this solution may be continued on a maximal interval $I_\infty(\pi_\diamondsuit, z_\diamondsuit)$ containing $z_\diamondsuit$. If $z_\diamondsuit = 0$, we write this solution as $\phi^0_\infty(z; \pi_\diamondsuit)$ and the interval of definition as $I^0_\infty(\pi_\diamondsuit)$. Because of autonomy,

$$\phi_\infty(z; \pi_\diamondsuit, z_\diamondsuit) = \phi^0_\infty(\zeta; \pi_\diamondsuit), \text{ where } \zeta = z - z_\diamondsuit \in I^0_\infty(\pi_\diamondsuit) = I_\infty(\pi_\diamondsuit, z_\diamondsuit) - z_\diamondsuit. \tag{5.22}$$

The corresponding path is denoted $\breve{\phi}_\infty(\pi_\diamondsuit, z_\diamondsuit)$, or simply $\breve{\phi}^0_\infty(\pi_\diamondsuit)$ if $z_\diamondsuit = 0$. Forward and backward solutions and paths starting at a given $z_\diamondsuit$ are defined in the obvious way. Analogous conventions apply to $S_{-\infty}$, $\phi_{-\infty}$ etc.

Information about the phase picture and asymptotic behaviour of solutions of systems $S_{\pm\infty}$ may be obtained from the discussion of systems $\bar{S} = (\bar{F}, G)$ in [B] by setting $\bar{F} = F_\infty$ or $\bar{F} = F_{-\infty}$. In particular, every solution of S_∞ with $\theta > 0$ which remains bounded as z increases (decreases) can be continued as $z \to \infty$ ($z \to -\infty$) and converges to one of the stationary points of S_∞; similarly for $S_{-\infty}$. The appropriate limits are indicated in [B] Figures 2 (see also S.3, fns.11 and 13 regarding solutions with $\theta = 0$). Here we are mainly concerned with solutions of S_∞ which converge as $z \to \infty$ to the saddle point π^*_∞, and solutions of $S_{-\infty}$ which converge as $z \to -\infty$ to the saddle point $\pi^*_{-\infty}$. Details follow.

Given a set $\mathbf{U} \subseteq \Re^2$, the *stable set* for S_∞ (with respect to $\mathbf{U}$) at the saddle point π^*_∞ is by definition the set

$$\mathcal{M}^\triangleright(S_\infty, \mathbf{U}) = \mathcal{M}^\triangleright_\infty(\mathbf{U}) = \{\pi \in \mathbf{U}\colon \phi^0_\infty(\zeta; \pi) \to \pi^*_\infty \text{ as } \zeta \to \infty\} \tag{5.23}$$

(where the notation is understood to entail that $\phi^0_\infty(\zeta; \pi)$ is defined for all $\zeta \geq 0$).

Let $\mathbf{U} = \{\theta \geq 0\}$ and let S_∞ be either Type 1, or Type 0 with $b > 1$. Referring to [B] Section 3, Prop.6 and fn.13, also Figs 2, it is seen that $\mathcal{M}^\triangleright_\infty(\mathbf{U})$ is the graph of a $\mathbf{C}^1$ function $h = f_\infty(\theta)$, which is defined for $0 < \theta < \infty$ if S_∞ is Type 1, but for $0 \leq \theta < \infty$ if S_∞ is Type 0. Of course, the reason for this distinction is that the saddle point π^*_∞ must be considered as a point of the stable set. In order to avoid repeating such distinctions, we shall sometimes write

(5.24) $\quad \gtrsim^+$ to mean '$>$ if S_∞ is Type 1, $\geq$ if S_∞ is Type 0'.

$\gtrsim^-$ to mean '$>$ if $S_{-\infty}$ is Type 1, $\geq$ if $S_{-\infty}$ is Type 0.'

Now define

(5.25) $$\mathbf{U}_+ \doteq \{h > 0, \theta \geq 0\}, \quad \mathbf{U}_{++} = \{h > 0, \theta > 0\}.$$

Referring again to [B] Section 3 it is seen that (if S_∞ is Type 1, or Type 0 with $b > 1$)

(5.26) $\mathcal{M}^{\triangleright}_\infty(\mathbf{U}_+)$ is the graph of $h = f_\infty(\theta)$, where f_∞ is restricted to an interval of the form $0 \lesssim^+ \theta < \theta_+(\infty)$ with $\theta_+(\infty) \leq \infty$, and is positive, $\mathbf{C}^1$ and decreasing on this interval.

From now on, the domain of the function f_∞ (when defined) is taken to be the interval on which it is positive; thus

(5.26a) $$\text{dom } f_\infty = (\theta \colon 0 \lesssim^+ \theta < \theta_+(\infty)).$$

We usually write $\mathcal{M}^{\triangleright}_\infty(\mathbf{U}_+)$ simply as $\mathcal{M}^{\triangleright}_\infty$. It is clear from the discussion of systems $\bar{S} = (\bar{F}, G)$ in [B] S.3, see esp. fn.13, that $\mathcal{M}^{\triangleright}_\infty$ is an (embedded) $\mathbf{C}^1$ sub-manifold in $\Re^2$ (with boundary if S_∞ is Type 0). Also, when f_∞ is defined, $\mathcal{M}^{\triangleright}_\infty(\mathbf{U}_{++})$ is the graph of f_∞ restricted to $0 < \theta < \theta_+(\infty)$ and is a sub-manifold (without boundary).[5]

If S_∞ is Type 1, as in [B] Figs.2(i,ii), f_∞ satisfies

(5.26b) $$f_\infty(n) = 1, \quad f'_\infty(n) = \lambda_-/n < 0,$$

where $\lambda_- = \lambda_-(1, n)$ is the negative eigenvalue at $(1, n)$, see [B](3.19) and [B] Table 1.

If S_∞ is Type 0 with $b > 1$, as in [B] Figs.2(iii,iv), f_∞ satisfies

(5.26c) $$f_\infty(0) = h^+_\infty = 1 + \lambda_-, \quad f'_\infty(0) = 2(1 + \lambda_-)/\sigma^2(\lambda_- - \lambda_+) < 0,$$

[5] If $\mathbf{U} = \mathbf{U}_+$ and S_∞ is Type 0 with $b < 1$, $\mathcal{M}^{\triangleright}_\infty$ lies in the axis $\{\theta = 0\}$ so that f_∞ is undefined, cf.fig.2(v).

where $\lambda_\pm = \lambda_\pm(h^+_\infty, 0)$ — see [B](3.22) and [B] Table 1.[6]

Notation and definitions for $S_{-\infty}$ are analogous to those for S_∞ and we omit various details. The definitions of solution, interval of definition and path are like those above with ∞ replaced by $-\infty$. Given a set $\mathbf{U} \subseteq \Re^2$, the *unstable set* at $\pi^*_{-\infty}$ (i.e. the stable set for the backward motion) is

(5.27) $$\mathcal{M}^\triangleleft(S_{-\infty}, \mathbf{U}) = \mathcal{M}^\triangleleft_{-\infty}(\mathbf{U}) = \{\pi \in \mathbf{U} \colon \phi^0_{-\infty}(\zeta; \pi) \to \pi^*_{-\infty} \text{ as } \zeta \to -\infty\}.$$

Restricting to $\mathbf{U}_+$ (and assuming that $S_{-\infty}$ is either Type 1, or Type 0 with $b < 1$)

(5.28) $\mathcal{M}^\triangleleft_{-\infty}(\mathbf{U}_+)$ is the graph of $h = g_{-\infty}(\theta)$, where $g_{-\infty}$ is restricted to an interval $\theta_- \lesssim^- \theta < \infty$ with $\theta_- = \theta_-(-\infty) \geq 0$ and $\theta_- \cdot g(\theta_-) = 0$, and is positive, $\mathbf{C}^1$ and increasing on this interval.

From now on, the domain of the function $g_{-\infty}$ (when defined) is taken to be the interval on which it is positive. Thus

(5.28a) $$\text{dom } g_{-\infty} = (\theta \colon \theta_-(-\infty) \lesssim^- \theta < \infty).$$

We usually write $\mathcal{M}^\triangleleft_{-\infty}(\mathbf{U}_+)$ simply as $\mathcal{M}^\triangleleft_{-\infty}$. It is a $\mathbf{C}^1$ sub-manifold in $\Re^2$ (with boundary if $S_{-\infty}$ is Type 0). Also $\mathcal{M}^\triangleleft_{-\infty}(\mathbf{U}_{++})$ is a sub-manifold (without boundary) whichever the Type.

If $S_{-\infty}$ is Type 1, as in [B] Figs. 2(i,ii), $g_{-\infty}$ satisfies

(5.28b) $$g_{-\infty}(N) = 1, \quad g'_{-\infty}(N) = \lambda_+/N > 0,$$

where $\lambda_+ = \lambda_+(1, N)$, see [B] (3.19) and Table 1.

If $S_{-\infty}$ is Type 0 with $b < 1$, as in [B] Fig. 2(v), $g_{-\infty}$ satisfies

(5.28c) $$g_{-\infty}(0) = h^-_{-\infty}, \quad g'_{-\infty}(0) = 2(1 + \lambda_-)/\sigma^2(\lambda_- - \lambda_+) > 0$$

[6] Given $\mathbf{U} \subseteq \Re^2$, we may also define the *unstable* set $\mathcal{M}^\triangleleft(S_\infty, \mathbf{U})$ for S_∞ (w.r.t. $\mathbf{U}$) at π^*_∞ by replacing $\zeta \to \infty$ with $\zeta \to -\infty$ in (5.23).

If $\mathbf{U} = \mathbf{U}_+$ and S_∞ is Type 1, so that $\pi^*_\infty = (1, n)$, $\mathcal{M}^\triangleleft_\infty$ is the graph of a $\mathbf{C}^1$ function $h = g_\infty(\theta)$ which is defined, positive and increasing for $\theta_- = \theta_-(\infty) < \theta < \infty$, where $0 < \theta_-$ and $\theta_- \cdot g_\infty(\theta_-) = 0$. In this case $g_\infty(n) = 1$ and $g'_\infty(n) = \lambda_+/n > 0$, cf.(5.25b) and Figs.2(i,ii). If S_∞ is Type 0, $b > 1$, $\mathcal{M}^\triangleleft_\infty$ lies in the axis $\{\theta = 0\}$ and g_∞ is undefined cf. Figs.2(iii,iv).

where $\lambda_\pm = \lambda_\pm(h^-_{-\infty}, 0)$ — see [B] 3.23 and [B] Table 1.[7] [8]

[7]Unfortunately, the definitions in [B] imply that, for a saddle point $\pi^*_{-\infty} = (h^-_{-\infty}, 0)$ of $S_{-\infty}$, we have $\lambda_- > 0 > \lambda_+$.

[8]As in fn.6, we may also define the *stable* set $\mathcal{M}^\triangleright(S_{-\infty}, \mathbf{U})$ for $S_{-\infty}$ (w.r.t. $\mathbf{U}$) at $\pi^*_{-\infty}$ as in (5.23) with S_∞ replaced by $S_{-\infty}$, π^*_∞ by $\pi^*_{-\infty}$.

If $\mathbf{U} = \mathbf{U}_+$ and $S_{-\infty}$ is Type 1, so that $\pi^*_{-\infty} = (1, N)$, $\mathcal{M}^\triangleright_{-\infty}$ is the graph of a $\mathbf{C}^1$ function $h = f_{-\infty}(\theta)$ which is defined, positive and decreasing for $0 < \theta < \theta_+(-\infty)$, where $\theta_+(-\infty) \le \infty$, cf.Figs.2(i,ii). In this case $f_{-\infty}(N) = 1$ and $f'_{-\infty}(N) = \lambda_-/N < 0$. If $S_{-\infty}$ is Type 0, $b < 1$, $\mathcal{M}^\triangleright_{-\infty}$ lies in the axis $\{\theta = 0\}$ and $f_{-\infty}$ is undefined, cf.Fig.2(v).

(iii) Special Solutions of S

We return to the system S and recall some results relating to asymptotic behaviour of solutions which were obtained in [B]. According to Props.1 and 11 of [B], (see also S.4, fn.3), a solution of S with $\theta \geq 0$ which remains bounded (in both co-ordinates) as z increases (decreases) can be continued to $z = \infty$ ($z = -\infty$) and converges to a finite limit, which by Prop.2 must be a stationary point of S_∞ ($S_{-\infty}$). The stationary points available depend on the Types of S_∞ and $S_{-\infty}$; see [B] Prop.11 and Cor.11 for further details.

We now focus on solutions of S which converge to saddle points of S_∞ or $S_{-\infty}$. Let $\mathbf{U}$ be $\{\theta \geq 0\}$ or $\{\theta > 0\}$. A solution ϕ of S which is defined on a right (left) unbounded z-interval and converges to π^*_∞ ($\pi^*_{-\infty}$) is called a *forward (backward) special solution* of S (relative to $\mathbf{U}$), or *f.s.s.* (*b.s.s.*) for short; if $\phi(z_\diamondsuit) = \pi_\diamondsuit$ we call $(\pi_\diamondsuit, z_\diamondsuit)$ — or, depending on context, just $\pi_\diamondsuit$ or just $z_\diamondsuit$ — a *forward (backward) special start.* (Note that we reserve the expressions 'stable/unstable solutions' for autonomous systems.)[9] A solution which is defined for all $z \in \Re$ and is both a f.s.s. and a b.s.s. is called a *star solution* (or solution of the bilateral b.v.p.) and is denoted

(5.29) $$\phi^* = \phi^*(z; z \in \Re) = (h^*(z), \theta^*(z); z \in \Re).^{10}$$

It was shown in [B] that there is one and only one star solution. It will be useful to recall some of the main ideas involved in the proof given in [B].

Let $\mathbf{U} = \{\theta > 0\}$, $b > 1$ and S_∞ of either Type. According to [B] Prop.12(α), there is for each fixed $z_\diamondsuit \in \Re$ an interval $(0, \theta_+(z_\diamondsuit))$ and, for θ in this interval, a unique positive $h = f(\theta, z_\diamondsuit)$ such that $f(\cdot, z_\diamondsuit)$ is continuous and decreasing in θ and the point

[9]If $\mathbf{U} = \{\theta \geq 0\}$, we allow $\theta \geq 0$ in the definitions of f.s.s. and b.s.s., although solutions with $\theta = 0$ are of no economic interest. Of course, S has no f.s.s. with $\theta = 0$ if S_∞ is Type 1, and no b.s.s. with $\theta = 0$ if $S_{-\infty}$ is Type 1; thus the condition $\theta \geq 0$ could be replaced in the definitions of f.s.s. and b.s.s. by $\theta \gtrsim^+ 0$ and $\theta \gtrsim^- 0$ respectively.

[10]Recall that the b.v.p. is said to be of Type 1 if both $n > 0$ and $N > 0$, where $N \equiv n + (b-1)\psi_0'/b$, i.e. if both S_∞ and $S_{-\infty}$ are Type 1, cf.[B] Figs.2(i,ii) and 3. Otherwise the b.v.p. is of Type 0, the possible cases being

(i) $N > 0 > n$, with $b > 1$ and $q > 0$ according to [B](0,8), cf.Figs.4(i,ii,iii)

(ii) $n > 0 > N$, with $0 < b < 1$ and $q > 0$ according to [B](0,9), cf.Figs.4(iv,v).

$(f(\theta, z_\diamondsuit), \theta, z_\diamondsuit)$ is a forward special start, i.e.

$$\varphi(z; f(\theta, z_\diamondsuit), \theta, z_\diamondsuit) \to \pi^*_\infty, \quad z_\diamondsuit \le z \to \infty. \tag{5.30a}$$

Next, Prop.12(β) asserts that, for $b > 1$ and δ, z^δ chosen as in(3.43) and fixed $z_\diamondsuit \le z^\delta$ (i.e. $z_\diamondsuit$ *far left*), there is a θ-interval $(\theta_-(z_\diamondsuit), \infty)$ and, for θ in this interval, a unique positive $h = g(\theta, z_\diamondsuit)$ such that $g(\cdot, z_\diamondsuit)$ is continuous and increasing and the point $(g(\theta, z_\diamondsuit), \theta, z_\diamondsuit)$ is a backward special start, i.e.

$$\varphi(z; g(\theta, z_\diamondsuit), \theta, z_\diamondsuit) \to \pi^*_{-\infty}, \quad z_\diamondsuit \ge z \to -\infty. \tag{5.30b}$$

The proof of the Existence Theorem [B]T.4 for $b > 1$ then shows that, for fixed $z_\diamondsuit \le z^\delta$, there is a point

$$(h^*_\diamondsuit, \theta^*_\diamondsuit, z_\diamondsuit) = (h^*(z_\diamondsuit), \theta^*(z_\diamondsuit), z_\diamondsuit)$$

which is both a forward and a backward special start, i.e.

$$h^*_\diamondsuit = f(\theta^*_\diamondsuit, z_\diamondsuit) = g(\theta^*_\diamondsuit, z_\diamondsuit), \tag{5.31a}$$

i.e. the point is the start of a star solution

$$\varphi(z; h^*_\diamondsuit, \theta^*_\diamondsuit, z_\diamondsuit) \to \pi^*_\infty \text{ as } z \to \infty \text{ and } \to \pi^*_{-\infty} \text{ as } z \to -\infty. \tag{5.31b}$$

Also, since $f(\cdot, z_\diamondsuit)$ is decreasing and $g(\cdot, z_\diamondsuit)$ is increasing, the point $(h^*_\diamondsuit, \theta^*_\diamondsuit, z_\diamondsuit)$ is unique (for fixed $z_\diamondsuit$), as is the star solution.

Similarly, if $b \le 1$, $S_{-\infty}$ of either Type, Prop.13(β) says that there is for each fixed $z_\diamondsuit \in \Re$ a θ-interval $(\theta_-(z_\diamondsuit), \infty)$ and, for θ in this interval, a unique positive $h = g(\theta, z_\diamondsuit)$ such that $g(\cdot, z_\diamondsuit)$ is continuous and increasing in θ and the point $(g(\theta, z_\diamondsuit), \theta, z_\diamondsuit)$ is a backward special start, cf.(5.30b). Next, Prop.13(α) asserts that, for $b < 1$ and ϱ, z^ϱ chosen as in (3.51) and fixed $z_\diamondsuit \ge z^\rho$ (i.e. $z_\diamondsuit$ *far right*), there is an interval $(\theta_-(z_\diamondsuit), \infty)$ and, for θ in this interval, a unique positive $h = f(\theta, z_\diamondsuit)$ such that $f(\theta, z_\diamondsuit)$ is continuous and decreasing and the point $(f(\theta, z_\diamondsuit), \theta, z_\diamondsuit)$ is a forward special start, cf.(5.30a).

The proof of T.4 for $b \le 1$ then shows that, for $z_\diamondsuit \ge z^\varrho$, there is a point $(h^*_\diamondsuit, \theta^*_\diamondsuit, z_\diamondsuit) = (h^*(z_\diamondsuit), \theta^*(z_\diamondsuit), z_\diamondsuit)$ which is the start of a star solution, cf.(5.31a,b).

Now, extending slightly the notation of [B] (4.1), consider a set $\mathbf{V} \subseteq \mathbf{U} \times \Re$ and let

(5.33)
$$\mathcal{M}^{\triangleright}(\mathbf{V}) = \mathcal{M}^{\triangleright} = \{(\pi_\diamond, z_\diamond) = (h_\diamond, \theta_\diamond, z_\diamond) \in \mathbf{V} \colon \phi(z; \pi_\diamond, z_\diamond) \to \pi^*_\infty,\ z_\diamond \leq z \uparrow \infty\},$$
(5.34)
$$\mathcal{M}^{\triangleleft}(\mathbf{V}) = \mathcal{M}^{\triangleleft} = \{(\pi_\diamond, z_\diamond) = (h_\diamond, \theta_\diamond, z_\diamond) \in \mathbf{V} \colon \phi(z; \pi_\diamond, z_\diamond) \to \pi^*_{-\infty},\ z_\diamond \geq z \downarrow -\infty\},$$

where π^*_∞ and $\pi^*_{-\infty}$ are the saddle points of S_∞ and $S_{-\infty}$ (of either Type). In case $\mathbf{V} = \mathbf{U} \times \Re$, the sets (5.33–34) are sometimes written simply as $\mathcal{M}^{\triangleright}$ and $\mathcal{M}^{\triangleleft}$, cf.[B](4.1). The section of $\mathcal{M}^{\triangleright}$ at a fixed $z_\diamond$ is written $\mathcal{M}^{\triangleright}(z_\diamond)$ and the section at a fixed $(\theta_\diamond, z_\diamond)$ is $\mathcal{M}^{\triangleright}(\theta_\diamond, z_\diamond)$. For example,

(5.35a) $$\mathcal{M}^{\triangleright}(z_\diamond) = \{\pi_\diamond \in \mathbf{U} \colon \phi(z; \pi_\diamond, z_\diamond) \to \pi^*_\infty,\ z_\diamond \leq z \uparrow \infty\}.$$

Also $\mathcal{M}^{\triangleright}(\mathbf{U} \times \Re)$ is written as

(5.35b) $$\mathcal{M}^{\triangleright}_{+} \text{ if } \mathbf{U} = \mathbf{U}_+, \quad \mathcal{M}^{\triangleright}_{++} \text{ if } \mathbf{U} = \mathbf{U}_{++}.$$

Similarly the section of $\mathcal{M}^{\triangleleft}$ at $z_\diamond$ is

(5.36a) $$\mathcal{M}^{\triangleleft}(z_\diamond) = \{\pi_\diamond \in \mathbf{U} \colon \phi(z; \pi_\diamond, z_\diamond) \to \pi^*_{-\infty}, z_\diamond \geq z \downarrow -\infty\}, \text{ and}$$
(5.36b) $$\mathcal{M}^{\triangleleft}(\mathbf{U} \times \Re) \text{ is written as } \mathcal{M}^{\triangleleft}_{+} \text{ if } \mathbf{U} = \mathbf{U}_+, \text{ as } \mathcal{M}^{\triangleleft}_{++} \text{ if } \mathbf{U} = \mathbf{U}_{++}.$$

It follows that, for fixed $z_\diamond \in \Re$ such that the function $f(\theta, z_\diamond)$ is defined,

(5.37a) $$\mathcal{M}^{\triangleright}_{++}(z_\diamond) \text{ is the graph of } \{(h, \theta) \colon h = f(\theta, z_\diamond), 0 < \theta < \theta_+(z_\diamond)\},$$

Similarly, for fixed $z_\diamond \in \Re$ such that the function $g(\theta, z_\diamond)$ is defined,

(5.37b) $$\mathcal{M}^{\triangleleft}_{++}(z_\diamond) \text{ is the graph of } \{(h, \theta) \colon h = g(\theta, z_\diamond), \theta_-(z_\diamond) < \theta < \infty\}.$$

The Proof of [B] T.4 shows that, in all cases satisfying our Standing Assumptions, there exist, for fixed $z_\diamond$ such that both $f(\theta; z_\diamond)$ and $g(\theta; z_\diamond)$ are defined,

(5.38a) $$\theta^*_\diamond = \theta^*(z_\diamond) \in (\theta_-(z_\diamond), \theta_+(z_\diamond)) \text{ and } h^*_\diamond = h^*(z_\diamond) > 0 \text{ satisfying}$$
$$h^*_\diamond = f(\theta^*_\diamond; z_\diamond) = g(\theta^*_\diamond; z_\diamond).$$

The solution $\phi(z; h^*_\diamondsuit, \theta^*_\diamondsuit, z_\diamondsuit)$ of S 'through' the initial point $(h^*_\diamondsuit, \theta^*_\diamondsuit, z_\diamondsuit)$ may be continued for all $z \in \Re$ and defines a 'star' solution. Also, since $f(\cdot, z_\diamondsuit)$ is decreasing and $g(\cdot, z_\diamondsuit)$ is increasing, the point $(h^*_\diamondsuit, \theta^*_\diamondsuit, z_\diamondsuit)$ is unique, as is the star solution. It follows that (for fixed $z_\diamondsuit$ chosen as above)

$$\mathcal{M}^{\triangleright}_{++}(z_\diamondsuit) \cap \mathcal{M}^{\triangleleft}_{++}(z_\diamondsuit) = \{h^*(z_\diamondsuit), \theta^*(z_\diamondsuit)\}. \tag{5.38b}$$

Now note that, for $b > 1$, Prop.12(α) yields, for *every* fixed $z_\diamondsuit \in \Re$, a function $f(\cdot, z_\diamondsuit)$ which is continuous, decreasing and positive on a maximal interval $(0, \theta_+(z_\diamondsuit))$ such that, for θ in this interval, the point $(f(\theta, z_\diamondsuit), \theta, z_\diamondsuit)$ is a forward special start; but for $b \leq 1$, Prop.13(α) yields such a function only for $z_\diamondsuit \geq z^\rho$, where $\rho \in (0, \psi'_0)$ is to be chosen so small that $A(z) < \rho$ for $z \geq z^\rho$, as in [B](3.51). Again, for $b \leq 1$, Prop.13(β) yields, for *every* fixed $z_\diamondsuit \in \Re$, a function $g(\cdot, z_\diamondsuit)$ which is continuous, increasing and positive on a maximal interval $(\theta_-(z_\diamondsuit), \infty)$ such that, for θ in this interval, the point $(g(\theta, z_\diamondsuit), \theta, z_\diamondsuit)$ is a backward special start; but for $b > 1$, Prop.12(β) yields such a function only for $z_\diamondsuit \leq z^\delta$, where $\delta \in (0, \psi'_0)$ is to be chosen so small that $\psi'_0 - M(z) < \delta$ for $z \leq z^\delta$, as in [B](3.43). However, it can be shown that the functions $f(\cdot, z_\diamondsuit)$ and $g(\cdot, z_\diamondsuit)$ exist and have the stated properties for all $b > 0$ and all $z_\diamondsuit \in \Re$. [11]

Remark 1. The functions f and g considered here should not be confused with the functions $\bar{f}$, $\bar{g}$ appearing in [B] S.3, see esp. Props.6 and 7. The functions $\bar{f}$, $\bar{g}$ relate to systems $\bar{S}$, whereas f, g relate to S.

Remark 2. Props. 12 and 13 were proved with $\mathbf{U} = \{\theta > 0\}$. If we let $\mathbf{U} = \{\theta \geq 0\}$, the preceding assertions are essentially unaltered, except that $f(\cdot, z_\diamondsuit)$ is defined for $0 \lesssim^+ \theta < \theta_+(z_\diamondsuit)$, and $g(\cdot, z_\diamondsuit)$ for $0 \lesssim^- \theta_- < \theta < \infty$ where $\theta_- = \theta_-(z_\diamondsuit)$ and $\theta_- \cdot g(\theta_-; z_\diamondsuit) = 0$, (see (5.24) for notation, also Figs.3 & 4.) Note that, as in (5.26a) and (5.28a), $f(\theta, z_\diamondsuit)$ and $g(\theta, z_\diamondsuit)$ are now regarded as defined only on the θ-intervals on which these functions are positive.

[11] With minor modifications, the proof in Prop.12(α) that functions $f(\cdot, z_\diamondsuit)$ exist applies whenever the functions $f^\wedge(\theta)$ and $f^\vee(\theta)$ and hence the 'tube' $\mathbf{C}^\triangleright$ in (3.11) are defined. See [B] Figures 3(i–vi) and 4(i–iii). Similarly the proof in Prop.13(β) that functions $g(\cdot, z_\diamondsuit)$ exist applies with some modifications whenever $g^\wedge(\theta)$ and $g^\vee(\theta)$, and hence the 'tube' $\mathbf{C}^\triangleleft$ in (3.42), are defined. See Figures [B] 3(i–vi) and 4(iv–v). However this approach does not yield the existence of $f(\cdot, z_\diamondsuit)$ for $z_\diamondsuit < z^\rho$ if $b < 1$ and S_∞ is Type 1 but $S_{-\infty}$ is Type 0, as in Figs.4(iv,v), because then $f^\vee$ is not defined. Similarly, this approach does not yield the existence of $g(\cdot, z_\diamondsuit)$ for $z_\diamondsuit > z^\delta$ if $b > 1$ and $S_{-\infty}$ is Type 1 but S_∞ is Type 0, as in Figs.4(i,ii,iii), because then $g^\vee$ is not defined.

An alternative, more general, argument is given in the Appendix to this Section for the case $b \leq 1$ and f, the argument for $b > 1$ and g being analogous.

Remark 3. The present discussion (taking into account fn.11 and the Appendix to this Section) yields a separate function $h = f(\theta, z_\diamondsuit)$ for each $z_\diamondsuit \in \Re$. Letting $z_\diamondsuit$ vary, we obtain a function $f(\theta, z)$ with domain

$$\text{dom}\, f = \{(\theta, z)\colon 0 \lesssim^+ \theta < \theta_+(z), z \in \Re\}. \tag{5.39}$$

At this stage we do not have continuity in the pair (θ, z) or any $\mathbf{C}^1$ properties. However, $f(\theta, z)$ is well defined for each (θ, z), and since $f(\cdot, z)$ is strictly decreasing in θ there is a θ-a.e. defined (but not necessarily continuous) negative derivative $f'_\theta(\cdot, z)$. Also referring to the definitions (5.35a,b), we have that

(5.39a) $\quad \mathcal{M}^{\triangleright}_{+}$ is the graph of $\{f(\theta, z)\colon 0 \lesssim^+ \theta < \theta_+(z), z \in \Re\}$,

(5.39b) $\quad \mathcal{M}^{\triangleright}_{++}$ is the graph of $\{f(\theta, z)\colon 0 < \theta < \theta_+(z), z \in \Re\}$.

Analogous remarks for $\mathcal{M}^{\triangleleft}_{+}$, $\mathcal{M}^{\triangleleft}_{++}$ and g; thus

(5.40) $\quad \text{dom}\, g = \{(\theta, z)\colon \theta_-(z) \lesssim^- \theta < \infty, z \in \Re\}$,

(5.40a) $\quad \mathcal{M}^{\triangleleft}_{+}$ is the graph of $\{g(\theta, z)\colon \theta_-(z) \lesssim^- \theta < \infty, z \in \Re\}$,

(5.40b) $\quad \mathcal{M}^{\triangleleft}_{++}$ is the graph of $\{g(\theta, z)\colon \theta_-(z) < \theta < \infty, z \in \Re\}$.

Letting $\mathcal{M}^* = \mathcal{M}^*_{++}$ denote the graph of the solution $\varphi^* = \varphi^*(z; z \in \Re)$ in $\mathbf{V}_{++} = \mathbf{U}_{++} \times \Re$, i.e. the set $\boldsymbol{\{}(\pi^*(z), z)\colon z \in \Re\boldsymbol{\}}$, we have,

$$\mathcal{M}^*_{++} = \mathcal{M}^{\triangleright}_{++} \cap \mathcal{M}^{\triangleleft}_{++}, \quad \mathcal{M}^*_{++}(z_\diamondsuit) = \mathcal{M}^{\triangleright}_{++}(z_\diamondsuit) \cap \mathcal{M}^{\triangleleft}_{++}(z_\diamondsuit) \text{ for } z_\diamondsuit \in \Re. \tag{5.41}$$

Similarly if $\mathbf{U}_{++}$, $\mathbf{V}_{++}$, $\mathcal{M}^*_{++}$ etc. are replaced by $\mathbf{U}_{+}$, $\mathbf{V}_{+}$, $\mathcal{M}^*_{+}$.

As was mentioned above, the sets of f.s.s. and b.s.s. are analogous to the 'stable' and 'unstable' manifolds of saddle points of dynamical systems, but with two differences: that S, being non-autonomous, does not directly define a dynamical system, and that π^*_∞ and $\pi^*_{-\infty}$ are not saddle points of S but of systems 'at $\pm\infty$'. We therefore do not know at this stage whether or how the usual stable/unstable manifold theorems apply to sets of f.s.s./b.s.s.. For the investigations in Part D we need these sets to be (at least) $\mathbf{C}^1$ manifolds, with suitable $\mathbf{C}^1$ convergence properties as $z \to \infty$ and $z \to -\infty$ as well as (local and global) $\mathbf{C}^1$ persistence properties with respect to parameters. In the next Section we shall construct an imbedding of the systems S and $S_{\pm\infty}$ in a three-dimensional autonomous system $\mathfrak{S}$ such that $\pi^*_{\pm\infty}$ are mapped to (3-D) saddle points

$p^*_{\pm\infty}$ of $\mathfrak{S}$, the sets $\mathcal{M}^{\triangleright}$ and $\mathcal{M}^{\triangleleft}$ etc., are mapped into differentiable manifolds to which stable/unstable manifold theorems apply, and ϕ^* corresponds to a saddle connection in $\mathfrak{S}$. This construction will be extended in Part D to allow perturbations of the parameters of the underlying stochastic growth model.

Appendix to Section 5.

We consider further the existence of the functions $f(\cdot, z)$ for fixed $z_\diamondsuit \in \Re$ and their domains, particularly in case $b < 1$. Here we leave aside the argument of fn.11 above.

Let $\mathbf{U} = \{\theta > 0\}$ and consider the usual phase diagrams with co-ordinates (h, θ). Since $\theta' = (h-1)\theta$, the motion of S as $z \uparrow$ is always to the left on $\{h < 1\}$, i.e. $\theta(z) \downarrow$, and to the right on $\{h > 1\}$, i.e. $\theta(z) \uparrow$. Also it follows from (5.1) that $F(0, \theta, z)$ has the sign opposite to that of $m - M(z)$ for $z \in \bar{\Re}$. Thus, for any solution path of S (or $S_{\pm\infty}$) there can be no downcrossing of the axis $\{h = 0\}$ at a given $z_\diamondsuit$ as $z \uparrow$ if $m < M(z_\diamondsuit)$; if $m \leq 0$, there can be no downcrossing at all. Again, there can be no upcrossing at $z_\diamondsuit$ if $m > M(z_\diamondsuit)$; if $m \geq \psi_0'$, there can be no upcrossings at all. If $0 < m < \psi_0'$, there is $z_m \in \Re$ for which $M(z_m) = m$, and for $z_\diamondsuit \in (-\infty, z_m)$ we have $m < M(z_\diamondsuit)$, hence no downcrossings, while for $z_\diamondsuit \in (z_m, \infty)$ we have $m > M(z_\diamondsuit)$, hence no upcrossings. If $F(0, \theta(z_\diamondsuit), z_\diamondsuit) = 0$ at some $z_\diamondsuit$ along a solution path of S, then $z_\diamondsuit = z_m$, $m = M(z_\diamondsuit)$. In this case $h(z_m) = h'(z_m) = 0$, $\theta'(z_m) = -\theta(z_m) < 0$, hence

$$h'' = F' = F_h \cdot h' + F_\theta \cdot \theta' + F_Z = F_Z = 2M'(2m)/b\sigma^2 < 0$$

since $M(z)$ is decreasing. So there is a local maximum of $h(z)$ at $z = z_m$, hence a tangency to the axis from below.

Consider next the solution paths of forward special solutions (f.s.s.). As in the proof of [B] Prop.12(α), the set $\mathcal{M}^\triangleright(z_\diamondsuit)$ of f.s.s. starts at a fixed $z_\diamondsuit$ — see (5.35a) —, if not empty, is a continuous simple curve, say of the form $\mathfrak{f}(h, \theta, z_\diamondsuit) = 0$, defined for $h \in \Re$ and θ in an interval of the form $(\theta_-(z_\diamondsuit), \theta^*)$. Each point $(h_\diamondsuit, \theta_\diamondsuit, z_\diamondsuit)$ on the curve defines an f.s.s. $\varphi(z; h_\diamondsuit, \theta_\diamondsuit, z_\diamondsuit)$, where $\varphi(z) = (h(z), \theta(z))$. For simplicity we shall assume that such a solution can be continued to $z = -\infty$ as $z \downarrow$, which implies that $h(z) \to \pm 0$ and $\theta \to \theta_-(z_\diamondsuit) = \infty$ as $z \to -\infty$; (if $b < 1$, this holds automatically, cf. [B], the para. following eq(3.30)); of course, $\varphi(z) \to \pi_\infty^*$, which may be of either Type, as $z \to \infty$. This further implies that, for every $z_\diamondsuit \in \Re$, $\mathcal{M}^\triangleright(z_\diamondsuit)$ is not empty; (however, $\mathcal{M}_{++}^\triangleright(z_\diamondsuit)$, the restriction of $\mathcal{M}^\triangleright(z_\diamondsuit)$ to $\mathbf{U}_{++} = \{h > 0, \theta > 0\}$ could be empty). If $\mathcal{M}_{++}^\triangleright(z_\diamondsuit)$ is not empty, it is the graph of the function $h = f(\theta, z_\diamondsuit)$ obtained by restricting $\mathfrak{f}(h, \theta, z_\diamondsuit)$ to $\mathbf{U}_{++}$; this function is continuous, positive and decreasing on a maximal θ-interval $(0, \theta_+(z_\diamondsuit))$, where (with the usual convention for the limit) $f(\theta_+(z_\diamondsuit), z_\diamondsuit) = 0$.

If $b > 1$, the existence for every $z_\diamondsuit \in \Re$ of a function $f(\cdot, z_\diamondsuit)$ follows from Prop.12(α). We now assume $b < 1$ and choose $\rho > 0$ small enough so that Prop.13(α) applies; then $\mathcal{M}_{++}(z^\rho)$ is the graph of a function $h = f(\theta, z^\rho)$, with domain $(0, \theta_+(z^\rho))$ and it follows from the 'co-operative' property of S on $\mathbf{U}_{++}$ that, as $z = z_\diamondsuit$ varies through $\Re$, the image of the curve $h = f(\theta, z^\rho)$ under the motion $z^\rho \mapsto z_\diamondsuit$, restricted to $\mathbf{U}_{++}$, is a curve $h = f(\theta, z_\diamondsuit)$ which is again continuous, positive and decreasing in θ on a maximal interval $(0, \theta_+(z_\diamondsuit))$, the order of points along the curve $h = f(\theta, z_\rho)$ being preserved under the motion of S. Explicitly, if $(h^{i\rho}, \theta^{i\rho}), i = 0, 1$, are points satisfying

$h^i(z^\rho) = f(\theta^i(z^\rho), z^\rho)$, then

$$0 < h^1(z^\rho) < h^0(z^\rho) \text{ and } \theta^1(z^\rho) > \theta^0(z^\rho) \text{ implies}$$
$$0 < h^1(z_\diamondsuit) < h^0(z_\diamondsuit) \text{ and } \theta^1(z_\diamondsuit) > \theta^0(z_\diamondsuit),$$

where $h^i(z_\diamondsuit) = f(\theta^i, z_\diamondsuit)$, at least as long as $h^1(z_\diamondsuit) > 0$. For $z_\diamondsuit > z^\rho$ this follows from [B] Prop.9(α), and for $z_\diamondsuit < z^\rho$ from Prop.9(β). It follows that, for each $z_\diamondsuit$ for which $\mathcal{M}_{++}(z_\diamondsuit)$ is not empty, there is a function $f(\cdot, z_\diamondsuit)$ which is continuous, decreasing and positive on an interval $(0, \theta_+(z_\diamondsuit))$ with $\theta_+(z_\diamondsuit) \leq \infty$.

Better results can be given if account is taken of the value of the parameter m. Until further notice, we now drop the assumption that $b < 1$. Referring to [B] Prop.7 and Figures 3–4, we note that

$$0 < f^\vee(\theta) < f^{\vee\rho}(\theta) < f_\infty(\theta) < f^{\wedge\rho}(\theta) < f^\wedge(\theta)$$

for $0 < \theta < \infty$ whenever the functions in question are defined.

We consider the functions $f(\cdot, z_\diamondsuit)$ under alternative conditions on m.

(i) If $m \geq \psi_0'$, then $m > M(z)$ for all $z \in \Re \cup \{\infty\}$ as in Figs.3(iii), 3(vi) and 4(iii). In this case, there are no upcrossings of the axis $\{h = 0\}$ as $z \uparrow$, so any f.s.s. must satisfy $h(z) > 0$ for all z. In this case, curves $f(\theta, z_\diamondsuit)$ exist for each $z_\diamondsuit$ and lie between $f^\vee(\theta)$ and $f^\wedge(\theta)$ (although $f^\vee$ is not drawn in 3(vi)). Since $\theta_+^\vee = \theta_+^\wedge = \infty$, we have $\theta_+(z_\diamondsuit) = \infty$ and the curves $f(\theta, z_\diamondsuit)$ are defined and positive for all $\theta \in \Re$. (The case $b < 1$, $n > 0 > N$, $q > 0$ cannot occur with $m \geq \psi_0'$, since then $m - M(z) + \frac{1}{2}b\sigma^2 < 0$, see [B](3.6a).)

(ii) If $m < 0$, we have $0 < \theta_+^\vee < \theta_+^\wedge < \infty$ provided $f^\vee$ is defined, as in Figs.3(i), 3(iv), 4(i) (although $f^\vee$ is not drawn in 3(iv)). In this case we know that, as in the proof of Prop.12(α), a curve $h = f(\theta, z_\diamondsuit)$ for $z_\diamondsuit \in \Re$ must lie in the 'tube' bounded by $f^\wedge$ and $f^\vee$, so that $\mathcal{M}_{++}(z_\diamondsuit)$ is not empty and $\theta_+(z_\diamondsuit) < \infty$; a diagram shows that θ_+ increases as $z_\diamondsuit$ decreases.

If $f^\vee$ is undefined, as in Fig.4(iv), i.e. if $b < 1$, $n > 0 > N$, $q > 0$, $m < 0$, we can still argue as follows. Note that $f^\wedge$ is defined and $\theta_+(f^\wedge) < \infty$ in this case. Let (h^ρ, θ^ρ) vary along the curve $h = (f(\theta, z^\rho)$, with h restricted to $(0, 1)$, and consider the f.s.s. $\varphi(z; h^\rho, \theta^\rho, z^\rho)$ for $z < z^\rho$. As z decreases, $\theta(z)$ increases and $h(z)$ decreases (because $F(h, \theta, z) > 0$ for $0 < h < 1$ and $\theta > h$). Eventually either $\varphi(z)$ crosses the curve $f^\wedge$ at some point with $h(z) > 0$, which is not permitted, or $h(z)$ crosses the axis $\{h = 0\}$ at some $z_\diamondsuit$ with $\theta_+(z_\diamondsuit) < \theta_+^\wedge$, and then $h(z_\diamondsuit) = f(\theta_+(z_\diamondsuit), z_\diamondsuit) = 0$, as required. A diagram shows that $\theta_+(z_\diamondsuit)$ increases as $h^\rho \downarrow$, $\theta^\rho \uparrow$ along $f(\cdot, z^\rho)$.

(iii) If $0 < m < \psi_0'$, we may assume that $z^\rho > z_m$. For $z > z_m$, a solution path of S cannot downcross the axis $\{h = 0\}$ as $z \downarrow$, so any f.s.s. must have $h(z) > 0$ for $z > z_m$. Consider again the paths of f.s.s. of the form $\varphi(z; h^\rho, \theta^\rho, z^\rho)$ restricted to $0 < h^\rho < 1$. For $z_\diamondsuit \in (z_m, z^\rho)$, there are no downcrossings of the axis as $z \downarrow$, so $h(z_\diamondsuit; h^\rho, \theta^\rho, z^\rho) > 0$ and

$h(z_\diamondsuit)$ decreases as h^ρ decreases. Also $h(z_m; h^\rho, \theta^\rho, z^\rho) > 0$; indeed a point $(0, \theta(z_m), z_m)$ cannot be an f.s.s. start because (as seen above) it defines a tangency to the axis from below. Now consider an f.s.s. defined for $z_\diamondsuit < z_m$ on $\{0 < h < 1\}$. Such a solution has the form $\phi(z; h_m, \theta_m, z_m)$ with (h_m, θ_m) a point of the curve $h = f(\theta, z_m)$, where $0 < h_m < 1$ and $0 < \theta_m < \infty$. For $z < z_m$, downcrossing of the axis as $z \downarrow$ is possible but upcrossing is not. Now we may argue as in (ii) above. If both $f^\vee$ and $f^\wedge$ are defined, with $\theta_+ < \theta_+^\wedge < \infty$ as in Figs.3(ii), 3(v), 4(ii), then $f(\cdot, z_\diamondsuit)$ for $z_\diamondsuit < z_m$ lies in the tube bounded by $f^\vee$ and $f^\wedge$, so $\theta_+(z_\diamondsuit) < \infty$. If $f^\vee$ is undefined, $\varphi(z; h^\rho, \theta^\rho, z^\rho)$ either crosses $f^\wedge$ as $z \downarrow$ (not allowed) or crosses the axis at some $h(z_\diamondsuit)$ with $\theta_+(z_\diamondsuit) < \theta^\wedge(z_\diamondsuit) < \infty$. But in any case $f^\wedge$ is defined with $\theta_+^\wedge < \infty$, so we obtain $\theta_+(z_\diamondsuit) < \infty$, and a diagram shows that $\theta_+(z_\diamondsuit) \downarrow$ as $h^\rho \downarrow$.

These remarks yield the required result, namely that, for $b < 1$ and every m and each $z_\diamondsuit \in \Re$, there is a function $f(\cdot, z_\diamondsuit)$ which is continuous, decreasing and positive on an interval $(0, \theta_+(z_\diamondsuit))$ with $\theta_+(z_\diamondsuit) \leq \infty$.

A similar argument applies for functions $g(\cdot, \theta)$ with $f^\vee$, $f^\wedge$, $f^{\vee\rho}$ replaced by $g^\vee$, $g^\wedge$, $g^{\vee\delta}$ with δ and z^δ as in (3.43).

6. **A Dynamical System Formulation**

Various methods have been proposed for extending the stability theory of autonomous o.d.e.s to the non-autonomous case, in particular to systems which can be regarded as in some sense perturbed autonomous systems; see for example Nemitskii and Stepanov [1960] Ch.iii, Sell [1967]. However I have not found definitions and statements of relevant results in a form convenient for direct application to the theory of the consumption function. I shall therefore adopt a homespun method, relying on properties of the economic model, of imbedding the non-autonomous two-dimensional systems S and $S_{\pm\infty}$ in an autonomous, three-dimensional system $\mathfrak{S}$.

(i) *Definition of the System* $\mathfrak{S}$.

The construction will be built up in steps, restricting attention in this Part to the case of fixed parameters:

First Step. The terms on the right of (5.1) which make the system S non-autonomous are those involving $A(z)$ and $M(z)$, functions which express the dependence of average and marginal products on log-capital. Both of these are decreasing functions of z whose range is the interval $(0, \psi_0')$. Moreover $M(z)$ can be expressed directly as a proportion of $A(z)$ for each z, say

$$(6.1) \qquad M/A = r(A), \text{ with } 0 < r(A) < 1.$$

The idea then is to replace $A(z)$ in the formula for F by a new variable α, and to replace $M(z)$ by $\alpha r(\alpha)$, the function $r(\alpha)$ being so chosen that

$$(6.2) \qquad \alpha r(\alpha) = M(z) = \psi'(\kappa) \quad \text{when} \quad \alpha = A(z) = a(\kappa), \quad z = \ln \kappa \in \Re.$$

More formally, we introduce a new variable α and define a new function $r(\alpha)$, initially for $\alpha \in (0, \psi_0')$, by

$$(6.3) \qquad \alpha r(\alpha) = \psi'[a^{-1}(\alpha)], \text{ or } \alpha r(\alpha) = M[A^{-1}(\alpha)],$$

where a^{-1}, A^{-1} are the functions inverse to a, A. This definition makes no explicit reference to z or κ, but on setting

$$(6.4) \qquad \alpha = A(z) = a(\kappa)$$

we obviously have (2). The variables α and $\alpha r(\alpha)$ may be thought of as (standardised) 'dummy' average and marginal products. Replacing $A'(z)$, $A(z)$ and $M(z)/A(z)$ in (5.5) by α', α and $r(\alpha)$, and defining

$$J(\alpha) = [r(\alpha) - 1]\alpha, \tag{6.5}$$

we obtain an o.d.e.

$$\alpha' = J(\alpha) = [r(\alpha) - 1]\alpha, \tag{6.6}$$

where as usual $\alpha = \alpha(z)$, $\alpha' = d\alpha/dz$ for $z \in \Re$, and $J(\alpha)$ is defined for $\alpha \in (0, \psi_0')$. Since $0 < r(\alpha) < 1$, every solution $\alpha(z)$ of (6) is decreasing in z. Note that the equation has the 'natural' economic solution

$$\alpha(z) = A(z) \tag{6.7}$$

but it also has the family of 'artificial' solutions

$$\alpha(z) = A(z + \gamma) = a(\kappa e^{\gamma}), \tag{6.8}$$

where γ is a real constant. Since each solution is obtained by shifting the graph of $A(z)$ horizontally by $-\gamma$, all solutions (8) are defined for all $z \in \Re$ and all have the same image set $(0, \psi_0')$. In particular, for every γ the solution $\alpha(z; \alpha_\diamondsuit)$ of (6) with initial condition $\alpha_\diamondsuit = A(z_\diamondsuit + \gamma)$ converges to zero as $z \to \infty$ and to ψ_0' as $z \to -\infty$.

Now let $\mathfrak{F}(h, \theta, \alpha)$ denote the expression obtained on replacing A by α and M by $\alpha r(\alpha)$ in the formula for $F(h, \theta, z)$ — see (5.1) — so that

$$\mathfrak{F}(h, \theta, \alpha) = F(h, \theta, z) \text{ when } \alpha = A(z), \quad z \in \Re. \tag{6.9}$$

On replacing F by $\mathfrak{F}$ in (5.1) and adjoining (6), we have a new, autonomous $\mathbf{C}^1$ system $\mathfrak{S} = (\mathfrak{F}, G, J)$. Explicitly, this system can be written as

$$\begin{aligned} h' &= \mathfrak{F}(h, \theta, \alpha) = bh^2 + (2/\sigma^2)h[\theta - Q - \alpha] - (2/b\sigma^2)[m - \alpha r(\alpha)] \\ \theta' &= G(h, \theta) \quad = (h - 1)\theta \\ \alpha' &= J(\alpha) \qquad = [r(\alpha) - 1]\alpha. \end{aligned} \tag{6.10}$$

We now state some further definitions, analogous to those used in the case of S, (but where the meaning of terms is obvious we sometimes abbreviate). We often denote by $p = (h, \theta, \alpha) = (\pi, \alpha)$ a point of $\Re^3$ and write (10) in vector form as

$$p' = \mathfrak{G}(p), \quad \mathfrak{G} = (\mathfrak{F}, G, J), \tag{6.10a}$$

regarding $\mathfrak{G}\colon \mathbf{V} \mapsto \Re^3$ as a $\mathbf{C}^1$ vector field on a suitable domain $\mathbf{V}$. At this stage we set $\mathbf{V} = \mathbf{U} \times (0, \psi_0')$, where $\mathbf{U}$ is a basic domain of S. We write $p > 0$ if all co-ordinates (h, θ, α) are (strictly) positive.

Given $\mathbf{V}$ and an interval $I \subseteq \Re$, a *solution* of $\mathfrak{G}$ on I is by definition a function $z \mapsto \Phi(z) = (h(z), \theta(z), \alpha(z)) = (\pi(z), \alpha(z))$ from I into $\mathbf{V}$ satisfying (10), or equivalently $\Phi'(z) = \mathfrak{G}[\Phi(z)]$, for $z \in I$. The corresponding curve in $\Re^3$ — strictly, the image set $\breve{\Phi} = \{\Phi(z)\colon z \in I\}$ parametrised and ordered by I — is here called the *trajectory* of Φ (on I). In particular, a given point $(p_\diamond, z_\diamond)$ defines a unique solution $\Phi(z; p_\diamond, z_\diamond)$ 'through' that point, i.e. $\Phi(z_\diamond; p_\diamond, z_\diamond) = p_\diamond$, and this solution may be continued on a maximal interval $I(p_\diamond, z_\diamond) = (z_-(p_\diamond, z_\diamond), z_+(p_\diamond, z_\diamond))$. Sometimes we call $z_\diamond$ the *start* of the solution through $p_\diamond$. If $z_\diamond = 0$, we write $\Phi(z; p_\diamond, 0)$ as $\Phi^0(z; p_\diamond)$ and $I(p_\diamond, 0)$ as $I^0(p_\diamond)$. Since $\mathfrak{G}$ is autonomous, we have for $z_\diamond \in \Re$,

$$\Phi(z; p_\diamond, z_\diamond) = \Phi^0(\zeta; p_\diamond) \text{ where } \zeta = z - z_\diamond \in I^0(p_\diamond) = I(p_\diamond, z_\diamond) - z_\diamond. \tag{6.11}$$

The corresponding trajectory is denoted $\breve{\Phi}(p_\diamond, z_\diamond)$, or simply $\breve{\Phi}^0(p_\diamond)$ if $z_\diamond = 0$; cf.(5.22) above, also [B] Section 3, fn.5.[1]

The function (or family of functions)

$$\Phi^0(z; p) = (\Phi^0_Z p\colon z \in I^0(p), p \in \mathbf{V}) \tag{6.12}$$

is usually called the (global) *flow* defined on $\mathbf{V}$ by $\mathfrak{G}$ and the pair $(\mathbf{V}, (\Phi^0_Z p))$ is a *dynamical system*; (an *incomplete* system, meaning that not all solutions can be continued to the whole of $\Re$; cf. Hirsch [1984], p.27). The definition of flow in terms of solutions starting with $z_\diamond = 0$ is customary, but an alternative parametrisation is often useful here. Writing $p_\diamond = (\pi_\diamond, \alpha_\diamond)$, we choose for $z_\diamond$ the value

$$z_\diamond = A^{-1}(\alpha_\diamond) \text{ — or simply } A^{-1}\alpha_\diamond \text{ —} \tag{6.13}$$

[1] Of course, the 'unparametrised' image set $\Phi(z; p_\diamond, z_\diamond)$ does not depend on $z_\diamond$; however we need terminology which keeps track of intervals of definition because not all solutions can be continued to the whole of $\Re$. This difficulty could be avoided by modifying our systems, but that would complicate the geometric picture.

and define, using (11)

$$\Phi^{\diamondsuit}(z;p_{\diamond}) = \Phi(z;p_{\diamond},A^{-1}\alpha_{\diamond}) = \Phi^{0}(z-A^{-1}\alpha_{\diamond};p_{\diamond}),$$
$$z \in I^{\diamondsuit}(p_{\diamond}) = I(\pi_{\diamond},A^{-1}\alpha_{\diamond}) = I^{0}(p_{\diamond}) + A^{-1}\alpha_{\diamond}, \tag{6.14}$$

so that

$$\Phi^{\diamondsuit}(A^{-1}\alpha_{\diamond};p_{\diamond}) = p_{\diamond} \tag{6.14a}$$

i.e. $\Phi^{\diamondsuit}(z;p_{\diamond})$ is the solution of $\mathfrak{S}$ 'through' $(p_{\diamond},z_{\diamond}) = (\pi_{\diamond},\alpha_{\diamond},A^{-1}\alpha_{\diamond})$. The flow on $\mathbf{V} = \mathbf{U}\times(0,\psi_0')$ may therefore also be represented by

$$\Phi^{\diamondsuit}(z;p) = (\Phi^{\diamondsuit}_{Z}p\colon z \in I^{\diamondsuit}(p), p \in \mathbf{V}). \tag{6.14b}$$

Now let $\mathbf{U}$ be $\{\theta \geq 0\}$ or $\{\theta > 0\}$ and $p_{\diamond} = (\pi_{\diamond},\alpha_{\diamond}) \in \mathbf{U}\times(0,\psi_0')$. Note that the solution of (6) with the initial condition $\alpha(z_{\diamond}) = \alpha_{\diamond} = A(z_{\diamond})$ is just $\alpha(z) = A(z)$ for all $z \in \Re$, i.e. $\gamma = 0$ in (8). Referring to the notation of Section 5(i) and comparing definitions for $\mathfrak{S}$ and S, it is seen that the maximal interval of definition $I^{\diamondsuit}(p_{\diamond})$ for $\Phi^{\diamondsuit}(z;p_{\diamond})$ coincides with the maximal interval $I(\pi_{\diamond},z_{\diamond})$ for $\phi(z;\pi_{\diamond},z_{\diamond})$ and that, for z in this interval, the solution $\Phi^{\diamondsuit}(z;p_{\diamond}) = \Phi(z;p_{\diamond},A^{-1}\alpha_{\diamond})$ coincides with the vector of functions whose components are (i) the solution $\phi(z;\pi_{\diamond},z_{\diamond})$ and (ii) the solution $\alpha(z;z_{\diamond}) = A(z)$ of (6). We write this relation for short as

(6.15a)
$$\Phi^{\diamondsuit}(z;p_{\diamond}) = \Phi^{\diamondsuit}(z;\pi_{\diamond},\alpha_{\diamond}) = \langle\phi(z;\pi_{\diamond},A^{-1}\alpha_{\diamond}),A(z)\rangle, \quad z \in I^{\diamondsuit}(p_{\diamond}) = I(\pi_{\diamond},A^{-1}\alpha_{\diamond}),$$

or just

$$\breve{\Phi}^{\diamondsuit}(p) = \breve{\Phi}^{\diamondsuit}(\pi,\alpha) = \langle\breve{\phi}(\pi,A^{-1}\alpha),A\rangle. \tag{6.15b}$$

Obviously the map

$$\Xi\colon p = (\pi,\alpha) \mapsto (\pi,A^{-1}\alpha), \quad \mathbf{U}\times(0,\psi_0') \mapsto \mathbf{U}\times\Re, \tag{6.16}$$

each space being equipped with its Euclidian metric, is a $\mathbf{C}^1$-diffeomorphism and induces a bijection

$$\breve{\Phi}^{\diamondsuit}(p) = \breve{\Phi}^{\diamondsuit}(\pi,\alpha) \leftrightarrow \breve{\phi}(\pi,A^{-1}\alpha), \quad I^{\diamondsuit}(\pi,\alpha) \leftrightarrow I(\pi,A^{-1}\alpha), \tag{6.17}$$

from trajectories of $\mathfrak{S}$ to paths of S. Indeed, the projection of the trajectory $\breve{\Phi}^{\diamondsuit}(\pi, \alpha)$ into the plane of the variables $\pi = (h, \theta)$ coincides with the path $\breve{\phi}(\pi, A^{-1}(\alpha))$. Thus the geometric analysis of S in [B] translates to $\mathfrak{S}$ with only minor changes of notation.

Second Step. Given the system (10) or (10a) defined on a domain $\mathbf{V} = \mathbf{U} \times (0, \psi_0')$, we wish to extend the definition to $\mathbf{U} \times [0, \psi_0']$ in such a way that

(6.18a) $$\mathfrak{F}(h, \theta, 0) = F_{\infty}(h, \theta), \qquad \mathfrak{F}(h, \theta, \psi_0') = F_{-\infty}(h, \theta),$$

(6.18b) $$J(0) = J(\psi_0') = 0,$$

and that the extended system is $\mathbf{C}^1$, i.e. the triple $\mathfrak{S} = (\mathfrak{F}, G, J)$ together with its first-order partial derivatives is continuous (with 'inward' continuity at boundary points belonging to $\mathbf{U} \times [0, \psi_0']$).

Starting with (18b), we note that the definition (2) of $r(\alpha)$ together with $a^{-1}(\psi_0') = 0$ yields the limit $r(\psi_0') = 1$. For $\alpha = 0$ we have

$$\begin{aligned} \lim r(\alpha) &= \lim[\psi'[a^{-1}(\alpha)]/\alpha] && \alpha \to 0 \\ &= \lim[\psi'(\kappa)/a(\kappa)] && \kappa = a^{-1}(\alpha) \to \infty \\ &= r_0 \end{aligned}$$

as defined by (5.10), and by assumption $0 \le r_0 < 1$. Thus, setting

(6.19) $$\begin{aligned} &r(0) = r_0, \quad r(\psi_0') = 1, \quad \text{hence} \\ &\alpha r(\alpha) = 0 \text{ when } \alpha = 0, \quad \alpha r(\alpha) = \psi_0' \text{ when } \alpha = \psi_0', \end{aligned}$$

we have $J(0) = J(\psi_0') = 0$, with inward continuity at these points. The same calculations together with the values of $A(\pm\infty)$ and $M(\pm\infty)$ given by (5.6) yield (18a) and the inward continuity of $\mathfrak{F}$ along $\alpha = 0$ and $\alpha = \psi_0'$. As regards the partial derivatives, these are set out at (23) below. It is seen that inward continuity of the partials requires — in addition to the preceding results — the existence of limits for

$$(\alpha r)' = (d/d\alpha)(\alpha r) = r(\alpha) + \alpha r'(\alpha) \qquad \text{as } \alpha \downarrow 0 \text{ and as } \alpha \uparrow \psi_0'.$$

Now, using the definition (2) and (5.5–5.5a) we have

$$\begin{aligned}(\alpha r)' &= (d/d\alpha)\psi'[a^{-1}(\alpha)] \\ &= \psi''(\kappa)/a'(\kappa) \\ &= \kappa\psi''(\kappa)/[\psi'(\kappa) - a(\kappa)] \quad \text{evaluated at } \kappa = a^{-1}(\alpha). \\ &= M'(z)/A'(z) \quad \text{evaluated at } z = \ln\kappa.\end{aligned}$$

By (5.12) and (5.9), the last expression converges to r_0 as $\alpha \to 0$, $\kappa \to \infty$, and converges to 2 as $\alpha \to \psi_0'$, $\kappa \to 0$. Thus, setting

$$(6.20) \qquad \begin{aligned}&(\alpha r)' = r_0 \quad \text{when } \alpha = 0 \\ &(\alpha r)' = r(\psi_0') + \psi_0' \cdot r'(\psi_0') = 1 + \psi_0' \cdot r'(\psi_0') = 2 \quad \text{when } \alpha = \psi_0',\end{aligned}$$

the system $\mathfrak{S}$ is $\mathbf{C}^1$ on $\mathbf{U} \times [0, \psi_0']$.[2]

The previous definitions of solution, trajectory and flow up to and including (12), as well as the definition of dynamical system following (12), continue to apply with $\mathbf{V} = \mathbf{U} \times [0, \psi_0']$. Now let $\mathbf{U}$ be $\{\theta \geq 0\}$ or $\{\theta > 0\}$. Bear in mind that a solution of $\mathfrak{S}$ has $\alpha(z) = 0$ for all $z \in \Re$ or for none, similarly with $\alpha(z) = \psi_0'$. The formula $z = A^{-1}(\alpha)$ makes sense for $\alpha = 0$ and $\alpha = \psi_0'$ if we allow $z = \pm\infty$. Thus the map Ξ in (16) extends to a map from $\mathbf{U} \times [0, \psi_0']$ to $\mathbf{U} \times [-\infty, \infty]$.

The definition of $\Phi^\diamondsuit(z; p_\diamondsuit)$ does not make sense for points of the form $p_\diamondsuit = (\pi_\diamondsuit, 0)$ or $p_\diamondsuit = (\pi_\diamondsuit, \psi_0')$, so that the map $\breve{\Phi}^\diamondsuit(\pi, \alpha) \leftrightarrow \breve{\phi}(\pi, A^{-1}(\alpha))$ in (17) does not extend. However, the solution of $\mathfrak{S}$ through a point $p_\diamondsuit = (\pi_\diamondsuit, 0)$, with start $z_\diamondsuit \in \Re$, agrees with the vector function whose components are the solution of S_∞ through $(\pi_\diamondsuit, z_\diamondsuit)$ and the zero function $\mathbf{0}$. Equivalently, choosing $z_\diamondsuit = 0$, we have

$$(6.21a) \qquad \Phi^0(z; \pi_\diamondsuit, 0) = \langle \phi_\infty^0(z; \pi_\diamondsuit), \mathbf{0} \rangle, \quad z \in I^0(\pi_\diamondsuit, 0) = I_\infty^0(\pi_\diamondsuit), \quad \pi_\diamondsuit \in \mathbf{U},$$

or simply

$$(6.21b) \qquad \breve{\Phi}^0(\pi_\diamondsuit, 0) = \langle \breve{\phi}_\infty^0(\pi_\diamondsuit), \mathbf{0} \rangle.$$

Similarly, the solution of $\mathfrak{S}$ through $(\pi_\diamondsuit, \psi_0')$ with start $z_\diamondsuit$ corresponds to the solution

[2]Remarks: (i) Regarding the value 2, see eq.(5.9) and S.5 fn.2 above.
(ii) Note that (6.20) does not define a value for $r'(0)$.

of $S_{-\infty}$ through $(\pi_\diamond, z_\diamond)$. With notation analogous to (21a–b) we have

$$\text{(6.22a)} \qquad \Phi^0(z;\pi_\diamond,\psi_0') = \langle \phi^0_{-\infty}(z;\pi_\diamond), \boldsymbol{\psi}_0'\rangle, \quad z \in I^0(\pi_\diamond,\boldsymbol{\psi}_0') = I^0_{-\infty}(\pi_\diamond), \quad \pi_\diamond \in \mathbf{U},$$

where $\boldsymbol{\psi}_0'$ is the function constant at ψ_0', or

$$\text{(6.22b)} \qquad \breve{\Phi}^0(\pi_\diamond,\psi_0') = \langle \breve{\phi}^0_{-\infty}(\pi_\diamond), \boldsymbol{\psi}_0'\rangle.$$

The Jacobian matrix of $\mathfrak{S}$ at an *arbitrary* point $p = (h,\theta,\alpha) = (\pi,\alpha)$ in $\mathbf{U} \times [0,\psi_0']$ can now be written as follows, (using 'inward' derivatives in case p is a boundary point cf.S.5, fn.3):

$$\text{(6.23)} \qquad \Delta_{\mathfrak{S}}(p) = \begin{bmatrix} \mathfrak{F}_h & \mathfrak{F}_\theta & \mathfrak{F}_\alpha \\ G_h & G_\theta & 0 \\ 0 & 0 & J_\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (2/\sigma^2)(b\sigma^2 h + \theta - Q - \alpha) & (2/\sigma^2)h & (2/\sigma^2)(-h + (\alpha r(\alpha))'/b) \\ \theta & h-1 & 0 \\ 0 & 0 & -1 + (\alpha r(\alpha))' \end{bmatrix}.$$

We write this matrix simply as $\Delta = [a_{ij}]$ if there is no ambiguity. It should be compared with the expression in [B](3.15) for the Jacobian matrix of $\bar{S} = (\bar{F}, G)$ at a point π in $\mathbf{U}$; the first two entries in the first two columns of (23) are the same as in (3.15), except that $\mathfrak{F}_h$ is obtained from $\bar{F}_h$ on replacing $\bar{Q}$ by $Q + \alpha$; thus values of $\mathfrak{F}_h$ with $\alpha = 0$ ($\alpha = \psi_0'$) correspond to values of F_h with $z = \infty$ ($z = -\infty$).[3] Two eigenvalues of Δ can be obtained from [B](3.16), according to Type, on replacing $\bar{Q}$ there by $Q + \alpha$, $\bar{m}$ by $m - \alpha$. We denote these by

$$\text{(6.24)} \qquad \lambda_\pm = \lambda_\pm(p) = \lambda_\pm(\pi,\alpha) = \lambda_\pm(\pi,z), \qquad \text{where } z = A^{-1}(\alpha).$$

(We omit details, but explicit expressions in the case of saddle points are given in the text and in Table 1 of [B], Section 3, also in Table 2 below). A third eigenvalue is given

[3]Explicitly: Here $\bar{F}(h,\theta) = bh^2 + (2/\sigma^2)h[\theta - \bar{Q}] - (2/b\sigma^2)\bar{m}$, as defined in [B](3.3a), should be read as the function obtained from

$$F(h,\theta,z) = bh^2 + (2/\sigma^2)h[\theta - Q - A] - (2/b\sigma^2)[m - M],$$

where $A = A(z)$, $M = M(z)$, by fixing $z = \bar{z}$, hence $A = A(\bar{z})$, $M = M(\bar{z})$, and setting $\bar{Q} = Q + A(\bar{z})$, $\bar{m} = m - M(\bar{z})$ cf.[B](3.6–3.8); here $-\infty \le \bar{z} \le \infty$. These expressions for $\bar{Q}$ and $\bar{m}$ also apply to the calculations of other parameters, including eigenvalues, in the lines from [B](3.15) to (3.23).

by

$$\lambda_3 = \lambda_3(p) = [\alpha r(\alpha)]' - 1.$$

In particular, using (20) we have, for arbitrary $\pi \in \mathbf{U}$,

$$(6.25) \qquad \lambda_3(\pi, 0) = r_0 - 1 < 0, \qquad \lambda_3(\pi, \psi_0') = +1 > 0.$$

Third Step. The preceding definitions have been stated only for $\alpha \in [0, \psi_0']$, but sometimes it is useful to extend $\mathfrak{S}$ so that it is a $\mathbf{C}^1$ system in full neighbourhoods of points of the form $p = (\pi, 0)$ or $p = (\pi, \psi_0')$. Referring to (10), it is seen that both $\mathfrak{F}$ and J contain terms in α and $\alpha r(\alpha)$, so that for this purpose we need both $\alpha r(\alpha)$ and $(\alpha r(\alpha))'$ to be continuous at $\alpha = 0$ and at $\alpha = \psi_0'$. The simplest way to achieve this is to prolong linearly $\alpha r(\alpha)$ by using the values of its derivatives given by (20). Thus we set

(6.26a)

$$(\alpha r(\alpha))' = r_0,$$

$$\text{hence } \alpha r(\alpha) = \alpha r_0, \quad J(\alpha) = \alpha(r_0 - 1), \quad \text{for } \alpha < 0;$$

(6.26b)

$$(\alpha r(\alpha))' = 2,$$

$$\text{hence } \alpha r(\alpha) = \psi_0' + 2(\alpha - \psi_0') = 2\alpha - \psi_0', \ J(\alpha) = \alpha - \psi_0', \ \text{for } \alpha > \psi_0';$$

and we similarly prolong $\mathfrak{F}$. With this extension, $\mathfrak{S}$ may be regarded as defined on the whole of $\Re^3$ or on a conveniently chosen subset.[4]

Given a point $p_\diamond \in \mathbf{V}$, the linearisation of $\mathfrak{S}$ about $p_\diamond$ is the system $\mathfrak{L}(p_\diamond)$ defined

[4]If $\mathfrak{S}$ is to be (say) a $\mathbf{C}^2$ system in neighbourhoods of stationary points with $\alpha = \psi_0'$ and $\alpha = 0$, it is necessary to extend $(\alpha r)'' = \alpha r'' + 2r'$ as well as αr and $(\alpha r)'$. Assuming $\psi''(0) \neq 0$ and ψ'', ψ''' both continuous at and near $\kappa = 0$, the procedure based on Taylor series expansion outlined above yields

$$(\alpha r(\alpha))'' = 4[3\psi_0'' - 2\psi_0''']/(\psi_0'')^3 \text{ at } \alpha = \psi_0'.$$

(If $\psi''(0) = 0$ but $\psi'''(0) \neq 0$, an expression involving ψ''' and ψ^{iv} is obtained, and so forth.)
On the other hand, if the limits of r, $(\alpha r)'$ and $(\alpha r)''$ exist as $\alpha \to 0$, one obtains, via l'Hôpital's Theorem, that

$$\begin{aligned} r_0 &= \lim[\psi'(\kappa)/a(\kappa)] = \lim[\psi''(\kappa)/a'(\kappa)] = \lim[\kappa\psi''/(\psi' - a)] = [\alpha r(\alpha)]'_{\alpha=0} \\ &= \lim[\psi'''(\kappa)/\alpha''(\kappa)] = \lim[\kappa^2\psi'''/(\kappa\psi'' - 2\psi' + 2a)] \end{aligned}$$

as $k \to \infty$, $\alpha \to 0$, hence by a calculation that $(\alpha r)'' = 0$ at $\alpha = 0$.
One then extends $\mathfrak{F}$ and J by setting $(\alpha r)''$ for $\alpha > \psi_0'$ equal to the value for $\alpha = \psi_0'$, similarly $(\alpha r)'' = 0$ for $\alpha < 0$. This procedure can be extended to a $\mathbf{C}^j$ system with $j > 2$.

for $p \in \mathbf{V}$ (in matrix form) by

$$(\delta p)' = \Delta(p_\diamondsuit) \cdot (p - p_\diamondsuit) = \Delta(p_\diamondsuit) \cdot \delta p \tag{6.27}$$

where $\delta p = p - p_\diamondsuit$, $d(\delta p)/dz = (\delta p)' = p'$, $\delta p = (\delta h, \delta\theta, \delta\alpha) = (h - h_\diamondsuit, \theta - \theta_\diamondsuit, \alpha - \alpha_\diamondsuit)$ and the partial derivatives $\mathfrak{F}_h$ etc. in (23) are evaluated at $p_\diamondsuit$.

The co-ordinates of *stationary* points of $\mathfrak{S}$ in the plane $\{\alpha = 0\}$ are as for stationary points of S_∞, with 0 adjoined as a third co-ordinate, while stationary points of $\mathfrak{S}$ in the plane $\{\alpha = \psi_0'\}$ are as for stationary points of $S_{-\infty}$ with ψ_0' adjoined. We confine attention to those stationary points p^* of $\mathfrak{S}$ which correspond to 2-dimensional saddle points of S_∞ or $S_{-\infty}$. These may be denoted by

$$p^*_\infty = (\pi^*_\infty, 0) \quad \text{and} \quad p^*_{-\infty} = (\pi^*_{-\infty}, \psi_0') \tag{6.28}$$

— cf. (5.21). They are 3-dimensional saddle points of $\mathfrak{S}$ in the sense that for each p^*, $\Delta(p^*)$ has only real, non-zero eigenvalues, not all of the same sign, cf.fn.7 below and S.5, fn.4, and are said to be of Type 1 or 0 according to the Type of the corresponding points of S_∞ or $S_{-\infty}$. (Note that they are finite points of the domain of $\mathfrak{S}$, not points 'at $\pm\infty$'.) For both Types there are two negative eigenvalues at p^*_∞ and one positive, indicating a two-dimensional stable manifold and a one-dimensional unstable manifold; at $p^*_{-\infty}$, there are two positive eigenvalues and one negative, indicating a one-dimensional stable manifold and a two-dimensional unstable manifold[5]. Details of the Jacobian matrices and eigenvalues at saddle points are set out in Table 2, which extends [B] Table I. The Jacobian entries are obtained by substituting in (23), while the eigenvalues are calculated as in (24–25). For simplicity, it is assumed (in each of the cases displayed) that the three eigenvalues are distinct.

[5]These are not quite the usual definitions for 3-dimensional saddle points, as considered, for example, in Bonatti and Dufraine [2003]. If S_∞ is Type 0, the domain of S_∞ is usually taken here to be $\{\theta \geq 0\}$ — cf. fn.4 of S.5 — and then the stable manifold at π^*_∞ is considered to be a manifold with boundary, therefore so is the stable manifold at p^*_∞; and similarly, if $S_{-\infty}$ is Type 0, for $\pi^*_{-\infty}$ and $p^*_{-\infty}$. Also, *all* the saddle points (6.28), of either Type, are boundary points unless the definition of $\mathfrak{S}$ is extended as in the 'third step' above.

$$\begin{bmatrix} (2/\sigma^2)(m/b + \frac{1}{2}b\sigma^2) & (2/\sigma^2) & (2/\sigma^2)(r_0/b - 1) \\ n & 0 & 0 \\ 0 & 0 & r_0 - 1 \end{bmatrix}$$

$$\lambda_\pm \sigma^2 = m/b + \frac{1}{2}b\sigma^2 \pm [(m/b + \frac{1}{2}b\sigma^2)^2 + 2n\sigma^2]^{\frac{1}{2}}$$

$$\lambda_+ > 0 > \lambda_-, \quad \lambda_3 = r_0 - 1 < 0$$

Data for Saddle Point at $p^*_\infty = (1, n, 0)$, $n > 0$ (Type 1)

$$\begin{bmatrix} (2/\sigma^2)[(m - \psi'_0)/b + \frac{1}{2}b\sigma^2] & (2/\sigma^2) & (2/\sigma^2)(2/b - 1) \\ N & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda_\pm \sigma^2 = \frac{1}{2}b\sigma^2 + [(m - \psi'_0)/b] \pm [\{\frac{1}{2}b\sigma^2 + (m - \psi'_0)/b\}^2 + 2N\sigma^2]^{\frac{1}{2}}$$

$$\lambda_+ > 0 > \lambda_-, \quad \lambda_3 = \psi'_0 r'(\psi'_0) = 1$$

Data for Saddle Point at $p^*_{-\infty} = (1, N, \psi'_0)$, $N > 0$ (Type 1)

$$\begin{bmatrix} (2/\sigma^2)(b\sigma^2 h^+_\infty - Q) & (2/\sigma^2)h^+_\infty & (2/\sigma^2)(r_0/b - h^+_\infty) \\ 0 & h^+_\infty - 1 & 0 \\ 0 & 0 & r_0 - 1 \end{bmatrix}$$

$$\lambda_+ = (2/\sigma^2)(b\sigma^2 h^+_\infty - Q) > 0; \quad b\sigma^2 h^+_\infty = Q + [Q^2 + 2m\sigma^2]^{1/2}$$

$$\lambda_- = h^+_\infty - 1 < 0 \qquad \lambda_3 = r_0 - 1 < 0$$

Data for Saddle Point at $p^*_\infty = (h^+_\infty, 0, 0)$, $q > 0 > n$, $b > 1$ (Type 0)

$$\begin{bmatrix} (2/\sigma^2)(b\sigma^2 h^-_{-\infty} - Q - \psi'_0) & (2/\sigma^2)h^-_{-\infty} & (2/\sigma^2)(2/b - h^-_{-\infty}) \\ 0 & h^-_{-\infty} - 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda_+ = (2/\sigma^2)(b\sigma^2 h^-_{-\infty} - Q - \psi'_0) < 0; \; b\sigma^2 h^-_{-\infty} = Q + \psi'_0 - [(Q + \psi'_0)^2 + 2(m - \psi'_0)\sigma^2]^{1/2}$$

$$\lambda_- = h^-_{-\infty} - 1 > 0$$

$$\lambda_3 = \psi'_0 r'(\psi'_0) = 1$$

Data for Saddle Point at $p^*_{-\infty} = (h^-_{-\infty}, 0, \psi'_0)$, $q > 0 > N$, $b < 1$ (Type 0)

TABLE 2: DATA FOR SADDLE POINTS OF $\mathfrak{S}$

If p^* is one of the saddle points p^*_∞ or $p^*_{-\infty}$, the linear o.d.e. system (6.27) may be solved using the data in Table 2, and equations obtained in the variables $(\delta h, \delta\theta, \delta\alpha)$ which characterise the stable and unstable *subspaces* $\mathfrak{L}^{\triangleright}(p^*)$ and $\mathfrak{L}^{\triangleleft}(p^*)$ of $\mathfrak{L}(p^*)$, i.e. the invariant subspaces corresponding respectively to the negative and positive eigenvalues at p^*.[6]

[6]Further symmetries between results given in Table 2 for p^*_∞ and $p^*_{-\infty}$ may be obtained as follows: for Type 1, recall that $\bar\theta_1$ is defined as the solution of $\bar F(1,\theta) = 0$, see [B](3.4), so that $\bar\theta_1 = n$ for p^*_∞ and $\bar\theta_1 = N$ for $p^*_{-\infty}$, see [B](3.4) and (3.7–8). In each of the first two blocks of the Table, we have $\lambda_+ > 0 > \lambda_-$; the leading entry a_{11} on the diagonal is equal to $\lambda_+ + \lambda_-$; also $\lambda_+\lambda_- = -2\bar\theta_1/\sigma^2$, or, since $\bar\theta_1 = a_{21}$ and $2/\sigma^2 = a_{12}$, $\lambda_+\lambda_- = -a_{21}a_{12}$. [Cf. below, eqns(6.48) and $(6.48)^{\triangleleft}$.] Further, according to (6.20), $(\alpha r)' = r_0$ when $\alpha = 0$, $z = \infty$, while $(\alpha r)' = 2$ when $\alpha = \psi_0'$, $z = -\infty$, so that $\lambda_3 = (\alpha r)' - 1$ in each block of Table 2. Both matrices for Type 1 Saddle Points may therefore be written

$$\begin{bmatrix} \lambda_+ + \lambda_- & 2/\sigma^2 & (2/\sigma^2)[(1+\lambda_3)/h - 1] \\ \bar\theta_1 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

For Type 0 Saddle Points, corresponding to the third and fourth blocks in Table 2, $a_{11} = \lambda_+$, $a_{22} = \lambda_-$ and $\lambda_- = h - 1$ where $h = h^+_\infty$ for p^*_∞, $h = h^-_{-\infty}$ for $p^*_{-\infty}$, so that both matrices are of the form

$$\begin{bmatrix} \lambda_+ & (2/\sigma^2)(1+\lambda_-) & (2/\sigma^2)[(1+\lambda_3)/b - (1+\lambda_-)] \\ 0 & \lambda_- & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Unfortunately the conventions established in [B], see (3.23a, b), lead to $\lambda_+ > 0 > \lambda_-$ for p^*_∞ but $\lambda_+ < 0 < \lambda_-$ for $p^*_{-\infty}$.

(ii) *Representation of Stable and Unstable Solutions and Sets.*

We turn now to a discussion of the stable and unstable sets/manifolds at saddle points of $\mathfrak{S}$, and their representation as graphs of differentiable functions. We begin with notation and representation of manifolds by functions, then turn to a discussion of smoothness in (iii) below.[7]

Let p^* be p^*_∞ or $p^*_{-\infty}$. A solution $\Phi^0(z; p_\diamondsuit)$ of $\mathfrak{S}$, defined for $z \geq 0$ and converging to p^* as $z \to \infty$, is called a *stable solution at* p^*. Similarly a solution $\Phi^0(z; p_\diamondsuit)$ of $\mathfrak{S}$ defined for $z \leq 0$ and converging to p^* as $z \to -\infty$ is called an *unstable solution at* p^*. Given a set $\mathbf{V} \subset \Re^3$ we define

$$\text{(6.29a)} \qquad \mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathbf{V}, p^*) = \mathfrak{M}^{\triangleright}(p^*) = \{p_\diamondsuit \in \mathbf{V} \colon \Phi^0(z; p_\diamondsuit) \to p^* \text{ as } 0 \leq z \uparrow \infty\},$$

$$\text{(6.29b)} \qquad \mathfrak{M}^{\triangleleft}(\mathfrak{S}, \mathbf{V}, p^*) = \mathfrak{M}^{\triangleleft}(p^*) = \{p_\diamondsuit \in \mathbf{V} \colon \Phi^0(z; p^*_\diamondsuit) \to p^* \text{ as } 0 \geq z \downarrow -\infty\};$$

[7] Some slightly informal reminders concerning stable and unstable manifolds at a saddle point of a dynamical system (autonomous system of o.d.e.s) may be useful. To avoid introducing new notation, the system is again called $\mathfrak{S}$ and is taken to be 3-dimensional, and related notation is also kept; but essentially the statements which follow are quite general for n-dimensional systems.

Let p^* be a stationary point of $\mathfrak{S}$, and assume that $\mathfrak{S}$ is defined in a neighbourhood $\mathfrak{N}$ of p^* in $\Re^3$. If the Jacobian matrix of $\mathfrak{S}$ at p^* has no complex eigenvalues with real part zero, p^* is called an elementary stationary (or critical) point. If the matrix has only real, non-zero eigenvalues λ_i, not all of the same sign, p^* is called a saddle point of $\mathfrak{S}$, and according to the Stable/Unstable Manifold Theorem (SMT) there exist local stable and unstable manifolds $\mathfrak{M}^{\triangleright}$ and $\mathfrak{M}^{\triangleleft}$ whose dimensions are equal respectively to the number of negative and positive eigenvalues λ_i. Here 'local' means 'restricted to some (possibly smaller) neighbourhood' which we still call $\mathfrak{N}$, while the local stable and unstable manifolds comprise those points $p \in \mathfrak{N}$ which are starts of solutions of $\mathfrak{S}$ converging respectively to p^* as $z \uparrow \infty$ and as $z \downarrow -\infty$. These manifolds are as smooth as $\mathfrak{S}$, say $\mathbf{C}^1$ for present puposes.

To simplify notation, it is often convenient to assume that the saddle point p^* is the origin $\mathbf{0}$ of $\Re^3$, or equivalently to replace p by $\delta p = p - p^*$ and $\mathfrak{N}$ by $\mathfrak{N} - p^*$. Consider the system $\mathfrak{L} = \mathfrak{L}(p^*)$ obtained by linearising $\mathfrak{S}$ about p^*, i.e. the system defined for (say) $\delta p \in \Re^3$ by $\delta p' = \Delta(p^*) \cdot \delta p$, cf.(6.27), and let $\mathfrak{L}^{\triangleright}$ and $\mathfrak{L}^{\triangleleft}$ be the linear subspaces associated with the negative and positive λ_i. The SMT asserts that the local stable and unstable manifolds at $\mathbf{0}$ are graphs of $\mathbf{C}^1$ functions $\psi^{\triangleright}(\delta p)$ and $\psi^{\triangleleft}(\delta p)$, mapping respectively a neighbourhood of the origin in $\mathfrak{L}^{\triangleright}$ ($\mathfrak{L}^{\triangleleft}$) into $\mathfrak{L}^{\triangleleft}$ ($\mathfrak{L}^{\triangleright}$), vanishing at the origin, tangent there to $\mathfrak{L}^{\triangleright}$ ($\mathfrak{L}^{\triangleleft}$), and satisfying $D\psi(\mathbf{0}) = 0$.

The local manifolds at p^* are $\mathbf{C}^1$ sub-manifolds of $\Re^3$ (in fact, embedded open disks through p^* of the appropriate dimensions). Given a domain $\mathbf{V} = \mathbf{U} \times \Re$, e.g. $\mathbf{V} = \Re^3$, there are also so-called global stable/unstable manifolds comprising all points $p \in \mathbf{V}$ which are starts of solutions converging to p^* as $z \uparrow \infty$ and as $z \downarrow -\infty$. Standard theorems assert only that these are 'injectively immersed' $\mathbf{C}^1$ submanifolds (of the appropriate dimensions);see Chillingworth [1976] for definitions.

In the situations of interest here, the saddle point p^* will often be a boundary point of the domain $\mathbf{V}$ of $\mathfrak{S}$, so that the manifolds may be 'with boundary' and the preceding results are to be modified as indicated in [B]S.3, fn.13 and in S.6,fn.5 above. Subject to this, our manifolds will turn out to be true (not just injectively immersed) submanifolds.

For more on SMTs, see for example Chillingworth [1976], Abraham and Robbin [1967] Section 27 and Appendix C by Al Kelley, also Palis and de Melo [1982] Section 6, Ruelle [1988] Section 6, Shub [1987] Chapter 5. Also Hirsch [1976], Ch 1, S.4 re manifolds with boundary.

$\mathfrak{M}^{\triangleright}(p^*)$ is called a *stable set*, $\mathfrak{M}^{\triangleleft}(p^*)$ an *unstable set* (at p^* w.r.t. $\mathbf{V}$) — (a *stable/unstable manifold* if it is a differentiable manifold, possibly with boundary). If $p^* = p^*_\infty$, we write $\mathfrak{M}^{\triangleright}(p^*)$ as $\mathfrak{M}^{\triangleright}_\infty$, $\mathfrak{M}^{\triangleleft}(p^*)$ as $\mathfrak{M}^{\triangleleft}_\infty$, similarly for $p^* = p^*_{-\infty}$.

Now let $p^* = p^*_\infty$ until further notice (so that the subscript ∞ is usually omitted); we set out changes for $p^*_{-\infty}$ in (v) below. Recall that

$$\mathbf{U}_+ = \{h > 0, \theta \geq 0\} \text{ and } \mathbf{U}_{++} = \{h > 0, \theta > 0\}$$

see (5.25), we write $\mathfrak{M}^{\triangleright}_\infty(\mathbf{V})$ as

(6.30a) $\quad \mathfrak{M}^{\triangleright}_{\oplus} \quad$ if $\mathbf{V} = \mathbf{U}_+ \times [0, \psi'_0)$,

(6.30b) $\quad \mathfrak{M}^{\triangleright}_{+} \quad$ if $\mathbf{V} = \mathbf{U}_+ \times (0, \psi'_0)$,

$\quad \mathfrak{M}^{\triangleright}_{++} \quad$ if $\mathbf{V} = \mathbf{U}_{++} \times (0, \psi'_0)$,

(6.30c) $\quad \mathfrak{M}^{\triangleright}_{0} \quad$ if $\mathbf{V} = \mathbf{U}_+ \times [\mathbf{0}]$.

Definition (6.29) should be compared with (5.33–34). If p^*_∞ is Type 1, $\mathfrak{M}^{\triangleright}_+ = \mathfrak{M}^{\triangleright}_{++}$. When the meaning is clear, we sometimes omit one or several of the following: the symbol $\mathfrak{S}$; the clause 'as $z \to \infty$'; the subscripts on $p_\diamondsuit$, p_∞, $\mathfrak{M}^{\triangleright}_\infty$; the superscripts on $\mathfrak{M}^{\triangleright}$, $\gtrsim^+$.

If $\mathfrak{N}$ is a neighbourhood of p^* in $\Re^3$, we sometimes replace $\mathbf{V}$ by $\mathfrak{N} \cap \mathbf{V}$, $\mathfrak{M}$ by $\mathfrak{M}(\mathfrak{N}) = \mathfrak{M} \cap \mathfrak{N}$ with appropriate subscripts in the notation (29–30), e.g.

(6.30d) $$\mathfrak{N}_\oplus = \mathfrak{N} \cap \mathbf{V}_\oplus, \quad \mathfrak{M}(\mathfrak{N}_\oplus) = \mathfrak{M} \cap \mathfrak{N}_\oplus \text{ etc.}$$

The condition $\Phi^0(z;p_\diamondsuit) \to p^*_\infty$ can be replaced in (29) by $\Phi^\diamondsuit(z;p_\diamondsuit) \to p^*_\infty$ if $\mathbf{V}$ is as in (30b); this follows from (14) since $0 < \alpha_\diamondsuit$ is equivalent to $z_\diamondsuit = A^{-1}(\alpha_\diamondsuit) < \infty$. On making this replacement and taking into account (14a) and (16), it is seen that $\mathfrak{M}^{\triangleright}_+$ is the inverse image under Ξ of $\mathcal{M}^{\triangleright}_+$ — see (5.35) and (5.39a), and we know that the latter set is the graph of the function f. Consequently

(6.31) $$\mathfrak{M}^{\triangleright}_+(p^*_\infty) = \{p = (h, \theta, \alpha) \colon h = f(\theta, A^{-1}\alpha),\ 0 \lesssim^+ \theta < \theta_+(A^{-1}\alpha),\ 0 < \alpha < \psi'_0\};$$

similarly

(6.31a) $$\mathfrak{M}^{\triangleright}_{++}(p^*_\infty) = \{p = (h, \theta, \alpha) \colon h = f(\theta, A^{-1}\alpha),\ 0 < \theta < \theta_+(A^{-1}\alpha),\ 0 < \alpha < \psi'_0\}$$

is the inverse image under Ξ of $\mathcal{M}^{\triangleright}_{++}$, see (5.35) and (5.39b).

Now consider $\mathfrak{M}^{\triangleright}_{0}$. According to (21a), $\Phi^0(z;\pi,0) = \langle \varphi^0_\infty(z;\pi), \mathbf{0}\rangle$, hence

$$\Phi^0(z;\pi,0) \to p^*_\infty \quad \text{iff} \quad \varphi^0_\infty(z;\pi) \to \pi^*_\infty .$$

Taking into account (29a), (30c), (21a), also (5.23), (5.26a) we obtain

$$\begin{aligned} \mathfrak{M}^{\triangleright}_{0} &= \boldsymbol{\{} p = (h,\theta,0)\colon\, h > 0,\ \theta \ge 0,\ \varphi^0_\infty(z;h,\theta) \to \pi^*_\infty \boldsymbol{\}} \\ &= \boldsymbol{\{} p = (h,\theta,0)\colon\, h = f_\infty(\theta),\ 0 \preceq^+ \theta < \theta_+(\infty) \boldsymbol{\}} \\ &= \boldsymbol{\{} p = (h,\theta,0)\colon\, h = f_\infty(\theta),\ \theta \in \operatorname{dom} f_\infty \boldsymbol{\}}, \end{aligned} \tag{6.32}$$

or (with some abuse of notation) $\mathfrak{M}^{\triangleright}_{0} = \langle \mathcal{M}^{\triangleright}_{\infty}, \mathbf{0}\rangle$.

Combining the definitions of $f(\theta,z)$ and $f_\infty(\theta)$, we now write

$$\begin{aligned} &\tilde f(\theta,\alpha) \doteq f(\theta, A^{-1}\alpha) \text{ for } (\theta,\alpha) \in \operatorname{dom} \tilde f, \\ &\text{where } A^{-1}0 = \infty,\ f(\theta, A^{-1}0) = f_\infty(\theta), \\ &\operatorname{dom} \tilde f \doteq \{(\theta,\alpha)\colon 0 \lesssim^+ \theta < \theta_+(A^{-1}\alpha),\ 0 \le \alpha < \psi_0'\}. \end{aligned} \tag{6.33}$$

Combining (31) and (32) yields

$$\mathfrak{M}^{\triangleright}_{\oplus} = \mathfrak{M}^{\triangleright}_{+} \cup \mathfrak{M}^{\triangleright}_{0} = \boldsymbol{\{} (h,\theta,\alpha)\colon\, h = \tilde f(\theta,\alpha),\ (\theta,\alpha) \in \operatorname{dom} \tilde f \boldsymbol{\}}. \tag{6.34}$$

(iii) *Local* $\mathbf{C}^1$ *Properties of Stable Sets and Representing Functions.*

We now consider smoothness properties of stable sets at $p^* = p^*_\infty$ (of either Type) and of the representing function $\tilde{f}$ — first 'locally' (i.e. on suitable neighbourhoods of p^*) and then 'globally' (i.e. on the whole domain of definition of each stable set).

According to the 'local' stable manifold theorem (s.m.t.) for elementary critical points — see fn.7 — there exists a neighbourhood $\mathfrak{N}$ of p^* w.r.t. $\mathbf{V}$ such that $\mathfrak{M}(\mathfrak{N}) = \mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N})$ is a two-dimensional $\mathbf{C}^1$ sub-manifold of $\Re^3$ (a $\mathbf{C}^1$ surface), possibly with boundary. However, our main concern here is not with the local manifolds as such, but with the properties of $\tilde{f}$ (and hence of f) which are needed for the theory of the boundary value problem and the consumption function. We want to show that $\tilde{f}$ is a $\mathbf{C}^1$ function of (θ, α), so that in particular we may use its partial derivatives to approximate the function near p^*, and hence obtain asymptotic approximations of f with small $\theta - \theta^*$ and large positive z. Differentiability will also be required in Part D for the discussion of perturbations of the boundary value problem.

The local s.m.t. is usually proved for a system of o.d.e.s defined on a neighbourhood of a critical point situated at the *origin of co-ordinates* with the matrix of the linear part of the vector field in *real canonical form.* We start by setting out the transformations required to display $\mathfrak{S}$ and the s.m.t. in the form which we require. To be specific, we assume that the domain of $\mathfrak{S}$ is $\mathbf{V} = \mathbf{U}_+ \times [0, \psi'_0)$.

Let $\mathfrak{L} = \mathfrak{L}(p^*)$ denote the linearisation of $\mathfrak{S}$ about $p^* = p^*_\infty$, i.e. the system (27) with $p_\diamondsuit = p^*_\infty$. We write this for short as[8]

(6.35) $$\delta' = \Delta \cdot \delta, \delta \in \mathbf{V}, \text{ where}$$
$$\delta = (\delta h, \delta\theta, \delta\alpha) = (h - h^*, \theta - \theta^*, \alpha - \alpha^*) = p - p^*,$$

and $\Delta = \Delta_{\mathfrak{S}}(p^*) = [a_{ij}]$ is the matrix appearing in the first or the third block of Table 2, depending on the Type of p^*; of course, $\alpha^* = \alpha^*_\infty = 0$ for both Types. Note that, for $p = p^* + \delta$ in a neighbourhood $\mathfrak{N}$ (w.r.t. $\mathbf{V}$) of p^*, we may write $\mathfrak{S}(p) = \Delta(p^*)\delta + H(\delta)$, where $H = (H^h, H^\theta, H^\delta)$ is a vector-valued $\mathbf{C}^1$ function which vanishes together with its first order partial derivative at $\delta = 0$ (Taylor's Theorem).

[8]The prime stands for differentiation with respect to z, as usual. The notation will not distinguish explicitly between column and row vectors.

For each Type, the matrix Δ has real, non-zero eigenvalues, one positive and two negative, so that the real canonical form is a diagonal matrix

$$\Lambda = \mathrm{diag}[\lambda_+, \lambda_-, \lambda_3], \tag{6.36}$$

which is uniquely determined up to the order of the diagonal elements, see Palis and de Melo [1982] Ch.2 S.2, Gantmacher [1958] Ch.VI. Thus there exists a non-singular transformation matrix T (of order 3) such that

$$T^{-1}\,\Delta\,T = \Lambda; \tag{6.37a}$$

suitable matrices for the two Types are exhibited in Table 3.[9]

[9] To calculate T^{-1}, write $\Delta = T\,\Lambda\,T^{-1}$ as $T^{-1}\Delta - \Lambda T^{-1} = 0$, and consider this matrix equation as a system of 9 linear homogeneous equations with 9 unknowns; in general, the solution is not unique and one has to choose a solution with $|T^{-1}| \neq 0$, cf. Gantmacher [1958]. Normally this method is laborious, but here there are enough zeros to make it straightforward.

$$T_1^{-1} = [\tau_{ij}]_1 = \begin{bmatrix} 1 & a_{12}/\lambda_+ & a_{13}/\left(\lambda_+ - \lambda_3\right) \\ \lambda_-/a_{12} & 1 & a_{13}\lambda_-/a_{12}\left(\lambda_- - \lambda_3\right) \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_0^{-1} = [\tau_{ij}]_0 = \begin{bmatrix} 1 & -a_{12}\left(\lambda_- - \lambda_+\right) & -a_{13}/\left(\lambda_3 - \lambda_+\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

TABLE 3: TRANSFORMATION MATRICES T^{-1} TO REAL CANONICAL FORM — see (6.37a)

NOTATION:

$\hat{\delta} = T^{-1}\delta$, $\hat{\delta} = (\eta, \xi, \chi)$, $\delta = p - p^*$, $p = (h, \theta, \alpha)$.

T_1^{-1} and T_0^{-1} apply to Type 1 and Type 0 saddle points respectively.

Values of a_{ij} and λ_k for T_1^{-1} are obtained from the first block of Table 2 if $p^* = p^*_\infty = (1, n, 0)$, from the second block if $p^* = p^*_{-\infty} = (1, N, \psi'_0)$.

Values of a_{ij} and λ_k for T_0^{-1} are obtained from the third block of Table 2 if $p^* = p^*_\infty = (h^+_\infty, 0, 0)$, from the fourth block if $p^* = p^*_{-\infty} = (h^-_{-\infty}, 0, \psi'_0)$.

The a_{ij} denote matrix entries and the λ_k eigenvalues in the relevant block of Table 2.

For completeness we also set out the inverse matrices $T_1 = [t_{ij}]_1$ and $T_0 = [t_{ij}]_0$.

$$T_1 = [t_{ij}]_1 = \begin{bmatrix} \lambda_+/(\lambda_+ - \lambda_-) & -a_{12}/(\lambda_+ - \lambda_-) & a_{13}\lambda_3/(\lambda_- - \lambda_3)(\lambda_+ - \lambda_3) \\ -\lambda_+\lambda_-/a_{12}(\lambda_+ - \lambda_-) & \lambda_+/(\lambda_+ - \lambda_-) & -a_{13}\lambda_+\lambda_-/a_{12}(\lambda_- - \lambda_3)(\lambda_+ - \lambda_3) \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_0 = [t_{ij}]_0 = \begin{bmatrix} 1 & a_{12}/(\lambda_- - \lambda_+) & a_{13}/(\lambda_3 - \lambda_+) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Writing

(6.37b) $$T^{-1}\delta = \hat{\delta} = (\eta, \xi, \chi), \quad T^{-1}\mathbf{V} = \hat{\mathbf{V}}$$

induces a linear transformation of co-ordinates such that the system $\mathfrak{L}$ becomes

(6.38) $$\hat{\mathfrak{L}}\colon \hat{\delta}' = \Lambda \cdot \hat{\delta}, \quad \hat{\delta} \in \hat{\mathbf{V}} \text{ or}$$
$$\eta' = \lambda_+\eta, \quad \xi' = \lambda_-\xi, \quad \chi' = \lambda_3\chi.$$

The vectors of the form $(\eta, 0, 0)$ span the unstable linear subspace $\hat{\mathfrak{L}}^{\triangleleft}$ for $\hat{\mathfrak{L}}$, while the vectors $(0, \xi, \chi)$ span the stable linear subspace $\hat{\mathfrak{L}}^{\triangleright}$. The system $\mathfrak{S}$ referred to the new co-ordinates (with origin $\hat{\delta}^* = T^{-1}p^* = \mathbf{0}$) is denoted $\hat{\mathfrak{S}}$ and has the form

(6.39a) $$\hat{\delta}' = \Lambda \cdot \hat{\delta} + \hat{H}(\hat{\delta}), \quad \hat{\delta} \in \hat{\mathbf{V}},$$

where $\hat{H} = (\hat{H}^\eta, \hat{H}^\xi, \hat{H}^\chi)$ is a vector-valued $\mathbf{C}^1$ function which vanishes together with its first order derivatives at $\hat{\delta}^* = \mathbf{0}$. Explicitly, this system has the form

(6.39b) $\eta' = \lambda_+\eta + \hat{H}^\eta(\hat{\delta}),\ \xi' = \lambda_-\xi + \hat{H}^\xi(\hat{\delta}),\ \chi' = \lambda_3\chi + \hat{H}^\chi(\hat{\delta})$ where $\hat{\delta} = (\eta, \xi, \chi) \in \hat{\mathbf{V}}$;

it is clear that $\chi = \delta\alpha = \alpha - \alpha^*_\infty = \alpha$ and $\hat{H}^\chi$ depends only on χ.

If $\hat{\mathfrak{N}}$ is a neighbourhood of $\hat{\delta}^*$ (w.r.t. $\hat{\mathbf{V}}$) in the new co-ordinates, then $\mathfrak{N} = \{p = p^* + T \cdot \hat{\delta}, \hat{\delta} \in \hat{\mathfrak{N}}\}$ defines a neighbourhood of p^* (w.r.t. $\mathbf{V}$) in the old co-ordinates. Extending our notation in an obvious way, let

(6.40a) $$\hat{\mathfrak{M}}^{\triangleright}(\hat{\mathfrak{S}}, \hat{\mathfrak{N}}) \doteq \{\hat{\delta} \in \hat{\mathfrak{N}}\colon \hat{\Phi}^0(z; \hat{\delta}) \to \mathbf{0} \text{ as } z \to \infty\},$$

where $\hat{\Phi}^0(z, \hat{\delta})$ is the flow on $\hat{\mathfrak{N}}$ defined by $\hat{\mathfrak{S}}$. Then

(6.40b)
$$\mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N}) = \{p = p^* + T\hat{\delta}, \hat{\delta} \in \hat{\mathfrak{M}}^{\triangleright}(\hat{\mathfrak{S}}, \hat{\mathfrak{N}})\} = \{p \in \mathfrak{N}\colon \Phi^0(z; p) \to p^* \text{ as } z \to \infty\}.$$

The local s.m.t. for the critical point of the system $\hat{\mathfrak{S}}$ at the origin in $\Re^3$ can now be stated as follows (cf.fn.7 above and Abraham and Robbin [1967] S.27):

(6.40c) PROPOSITION 14(α)(i) (Local Stable Manifold for $\hat{\mathfrak{S}}$ at $\hat{\delta}^* = \mathbf{0}$).

There exist

(1) a number $\hat{\gamma} \in (0, \psi_0')$ and a neighbourhood of (ξ^*, χ^*) in $\Re^2$ of the form $\hat{\mathbf{N}} = \hat{\mathbf{N}}(\hat{\gamma}) \doteq \{(\xi, \chi)\colon \xi^2 + \chi^2 < \hat{\gamma}^2, \xi \geq 0, 0 \leq \chi < \psi_0'\}$,

(2) a neighbourhood $\hat{\mathfrak{N}} = \mathfrak{N}(\hat{\gamma})$ (w.r.t. $\hat{\mathbf{V}}$) of $\hat{\delta}^*$ in $\Re^3$,

(3) a function of the form $\eta = w(\xi, \chi)$, defined and $\mathbf{C}^1$ for (ξ, χ) in $\hat{\mathbf{N}}$, with values η such that $\hat{\delta} = (\eta, \xi, \chi)$ is in $\hat{\mathfrak{N}}$, w vanishing together with its partial derivatives w_ξ and w_χ at $\xi = \chi = 0$, such that

$$\hat{\mathfrak{M}}^{\triangleright}(\hat{\mathfrak{S}}, \hat{\mathfrak{N}}) = \boldsymbol{\{}\hat{\delta} = (\eta, \xi, \chi)\colon \eta = w(\xi, \chi), (\xi, \chi) \in \hat{\mathbf{N}}\boldsymbol{\}}.$$

In other words, the stable manifold of $\hat{\mathfrak{S}}$ at $\mathbf{0}$, restricted to $\hat{\mathfrak{N}}$, is the graph of a $\mathbf{C}^1$ function $\eta = w(\xi, \chi)$ and so is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$ (with boundary), and it is tangent at the origin to $\hat{\mathfrak{L}}^{\triangleright}$. Since $\mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N}) - p^*$ is obtained from $\hat{\mathfrak{M}}^{\triangleright}(\hat{\mathfrak{S}}, \hat{\mathfrak{N}})$ by a linear change of co-ordinates, this set also is a two-dimensional $\mathbf{C}^1$ submanifold and is tangent at the origin (of the old co-ordinates) to $\mathfrak{L}^{\triangleright} = T \cdot \hat{\mathfrak{L}}^{\triangleright}$, the stable subspace of $\mathfrak{L}$. Substituting in $\eta = w(\xi, \chi)$ the linear expressions for (η, ξ, χ) in terms of (δh, $\delta\theta$, $\delta\alpha$) given by (37b) and Table 3, one obtains an equation involving the latter variables which holds for points of $\mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N}) - p^*$, and hence an equation in (h, θ, α) which holds for points of $\mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N})$. The programme then is to compare this equation with the equation $h = \tilde{f}(\theta, \alpha)$ for points $p = (h, \theta, \alpha) \in \mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N})$ such that $(\theta, \alpha) \in \mathrm{dom}\tilde{f}$ in order to show that $\tilde{f}$ is locally $\mathbf{C}^1$, and to obtain estimates of its derivatives at (θ^*, α^*). For this purpose it is convenient to consider the two Types separately.

If $p^* = p^*_\infty$ is Type 0, i.e. $p^* = (h^+_\infty, 0, 0)$, $b > 1$, we have

(6.41a) $$\delta h = h - h^+_\infty, \quad \delta\theta = \theta \geq 0, \quad \delta\alpha = \alpha - \alpha^*_\infty = \alpha \in [0, \psi_0').$$

Table 3 yields

(6.41b) $$\eta = \delta h - \delta\theta \cdot a_{12}/(a_{22} - a_{11}) - \delta\alpha \cdot a_{13}/(a_{33} - a_{11}),$$

the a_{ij} being entries in the third block of Table 2; also note that $\xi = \delta\theta = \theta$, $\chi = \delta\alpha = \alpha$. Combining these results with $\eta = w(\xi, \chi)$ yields

(6.41c) $$h = h^+_\infty + \theta \cdot a_{12}/(a_{22} - a_{11}) + \alpha \cdot a_{13}/(a_{33} - a_{11}) + w(\theta, \alpha).$$

Denote the expression on the right of (41c) by $\tilde{h}(\theta, \alpha)$. We may choose $\gamma \leq \hat{\gamma}$ so small that $h > 0$ for $p \in \mathfrak{N}$. Then, for (θ, α) satisfying $\theta \geq 0$, $\alpha \geq 0$, the couple (θ, α) belongs to dom $\tilde{f}$ — see (33). Since both $\tilde{h}$ and $\tilde{f}$ are uniquely defined there, we have $\tilde{f}(\theta, \alpha) = \tilde{h}(\theta, \alpha)$ on the common domain of the two functions. Moreover $\tilde{h}$ is $\mathbf{C}^1$, and setting the derivative of $\tilde{f}$ equal to the derivative of $\tilde{h}$[10] we conclude that

(6.42a) $$\tilde{f} \text{ is } \mathbf{C}^1 \text{ in } (\theta, \alpha) \quad \text{for} \quad \theta \geq 0,\ \alpha \geq 0 \text{ such that } \theta^2 + \alpha^2 < \gamma^2.$$

This argument yields further useful results for the derivatives of $\tilde{f}$ and hence of f. By the definition of $\tilde{f}$,

(6.42b) $$f \text{ is } \mathbf{C}^1 \text{ in } (\theta, z) \quad \text{for} \quad \theta \geq 0,\ z \in \Re \text{ such that } \theta^2 + (A(z))^2 < \gamma^2.$$

Further, setting $\tilde{f}$ equal to (41c), differentiating partially both sides — first w.r.t. θ, then w.r.t. α, evaluating at p^*, i.e. at $\theta = \theta^* = 0$, $\alpha = \alpha^* = 0$, and taking into account that the first order partials of w vanish at this point, we obtain, using the Tables,

(6.43) $$\tilde{f}^*_\theta = a_{12}/(a_{22} - a_{11}) = f'_\infty(0) < 0, \text{ cf.[B] Table 1},$$

(6.44) $$\tilde{f}^*_\alpha = a_{13}/(a_{33} - a_{11}),$$

and $w(\theta, \alpha) = o(\theta, \alpha)$ in (41c),

the coefficients being taken from the third block of Table 2. Also, differentiating $\tilde{f}(\theta, \alpha) = \tilde{f}(\theta, A(z)) = f(\theta, z)$ w.r.t. z and letting $z \to \infty$, $A(z) \to 0$, $A'(z) \to 0$, we get

(6.45a) $$\tilde{f}_\alpha[\theta, A(z)] \cdot A'(z)) = f_Z(\theta, z) \to 0 \text{ as } z \to \infty,$$

at least for $\theta \in [0, \gamma)$, in particular

(6.45b) $$\lim_{Z \to \infty} f_Z(\theta^*, z) = 0.$$

Now suppose that $p^* = p^*_\infty$ is Type 1, i.e. $p^* = (1, n, 0)$. We have

(6.46a) $$\delta h = h - 1, \quad \delta\theta = \theta - n \text{ with } \theta > 0, \quad \delta\alpha = \alpha - \alpha^*_\infty = \alpha \in [0, \psi'_0).$$

[10] We omit qualifications to the effect that continuous versions of the derivatives of $\tilde{f}$, f, $\tilde{g}$, g are to be chosen.

Table 3 yields

(6.46b)
$$\eta = \delta h + \delta\theta \cdot a_{12}/\lambda_+ + \delta\alpha \cdot a_{13}/(\lambda_+ - \lambda_3),$$
$$\xi = \delta h \cdot \lambda_-/a_{12} + \delta\theta + \delta\alpha \cdot a_{13}\lambda_-/a_{12}(\lambda_- - \lambda_3),$$

and $\chi = \delta\alpha = \alpha$, the parameters being obtained from the first block of Table 2.

Substituting into $0 = -\eta + w(\xi, \chi)$ yields

(6.46c)
$$0 = -(h-1) - (\theta - n)a_{12}/\lambda_+ - \alpha \cdot a_{13}/(\lambda_+ - \lambda_3)$$
$$+ w[(h-1)\lambda_-/a_{12} + (\theta - n) + \alpha \cdot a_{13}\lambda_-/a_{12}(\lambda_- - \lambda_3), \alpha].$$

Since w_ξ and w_χ vanish at $\xi = \chi = 0$, i.e. at $(h-1) = (\theta - n) = \alpha = 0$, the derivative of the right-hand side of (46c) w.r.t. $h-1$ at this point is -1, and it follows from the Implicit Function Theorem that (46c) may be solved uniquely in the form

(6.46d)
$$h = \tilde{h}(\theta, \alpha) = 1 - (\theta - n)a_{12}/\lambda_+ - \alpha \cdot a_{13}/(\lambda_+ - \lambda_3) + o(\theta - n, \alpha),$$

where $\tilde{h}$ is a $\mathbf{C}^1$ function defined for $(\theta - n)^2 + \alpha^2 < \gamma^2$, $\theta > 0$, $0 \le \alpha < \psi_0'$, with some $\gamma \le \hat{\gamma}$; we further choose γ so small that $\tilde{h}(\theta, \alpha) > 0$. Then the couple (θ, α) belongs to dom $\tilde{f}$, and noting that $\tilde{f} = \tilde{h}$ on the common domain of the two functions and equating the derivatives we conclude as above that

(6.47a)
$$\tilde{f} \text{ is } \mathbf{C}^1 \text{ in } (\theta, \alpha) \quad \text{for} \quad \theta > 0,\ \alpha \ge 0 \text{ such that } (\theta - n)^2 + \alpha^2 < \gamma^2.$$

From (47a) we also infer that

(6.47b)
$$f \text{ is } \mathbf{C}^1 \text{ in } (\theta, z) \quad \text{for} \quad \theta > 0,\ z \in \Re \text{ such that } (\theta - n)^2 + (A(z))^2 < \gamma^2.$$

Again, setting $\tilde{f}(\theta, \alpha)$ equal to (46d), differentiating partially, first w.r.t. θ, then w.r.t. α, evaluating at p^* and noting that the first order partials of w vanish, yields

(6.48)
$$\tilde{f}_\theta^* = -a_{12}/\lambda_+ = -2/\sigma^2\lambda_+ = \lambda_-/n = f_\infty'(n) < 0,$$

taking into account that $\lambda_+\lambda_- = -2n\sigma^2$ and $\lambda_-/n = f_\infty'(n)$ cf.[B] Table 1. Also

(6.49)
$$\tilde{f}_\alpha^* = -a_{13}/(\lambda_+ - \lambda_3) = -2(r_0 - b)/b\sigma^2(\lambda_+ - r_0 + 1),$$

which has the sign of $b - r_0$ where $r_0 < 1$. Once again, we have (45a), at least for $\theta > 0$

and $(\theta - n)^2 < \gamma^2$, also (45b). The coefficients in (48–49) are to be taken from the first block of Table 2.

Combining the calculations for the two Types, we obtain the local s.m.t. for $\mathfrak{S}$ in the following form:

(6.50) PROPOSITION 14(α)(ii) (Local Stable Manifold for $\mathfrak{S}$ at $p^* = p^*_\infty = (h^*_\infty, \theta^*_\infty, \alpha^*_\infty)$).

Let $\mathbf{V} = \mathbf{U}_+ \times [0, \psi'_0)$. There exist

(1) a number $\gamma \in (0, \psi'_0)$ and a neighbourhood of $(\theta^*_\infty, \alpha^*_\infty)$ in $\Re^2$ of the form $\mathbf{N} = \mathbf{N}(\gamma) \doteq \{(\theta, \alpha)\colon (\theta - \theta^*_\infty)^2 + (\alpha - \alpha^*_\infty)^2 < \gamma^2, \theta \gtrsim^+ 0, 0 \leq \alpha < \psi'_0\}$,

(2) a neighbourhood $\mathfrak{N} = \mathfrak{N}(\gamma)$ (w.r.t. $\mathbf{V}$) of p^*_∞ in $\Re^3$,

(3) a function $\tilde{h}(\theta, \alpha)$, defined and $\mathbf{C}^1$ for $(\theta, \alpha) \in \mathbf{N}$ satisfying $\tilde{h}(\theta^*_\infty, \alpha^*_\infty) = h^*_\infty$, with values h such that (h, θ, α) is in $\mathfrak{N}$, such that

$$\mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N}) = \boldsymbol{\{} p = (h, \theta, \alpha)\colon h = \tilde{h}(\theta, \alpha), (\theta, \alpha) \in \mathbf{N} \boldsymbol{\}}.$$

This set is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$ (with boundary). Consequently the set $\mathcal{M}^{\triangleright}_+$ — see (5.35a,b) — is a submanifold of $\Re^3$ (with boundary if S_∞ is Type 0).

(6.51) The functions $\tilde{h}$ and $\tilde{f}$ (and their derivatives) coincide on $\mathbf{N}$,

so that $\tilde{f}$ is $\mathbf{C}^1$ on this set (allowing one-sided limits on the boundary), and

$$\mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N}) = \boldsymbol{\{} p = (h, \theta, \alpha)\colon h = \tilde{f}(\theta, \alpha), (\theta, \alpha) \in \mathbf{N} \boldsymbol{\}}$$

is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$ with boundary.

If in the preceding statements we replace $\mathbf{V}$, $\mathbf{N}$, $\mathfrak{N}$, by $\mathbf{V}_{++} = \mathbf{U}_{++} \times (0, \psi'_0)$, $\mathbf{N}_{++} = \mathbf{N} \cap \{\theta > 0, \alpha > 0\}$, $\mathfrak{N}_{++} = \mathfrak{N} \cap \mathbf{V}_{++}$, then for γ small enough,

(6.52)
$$\begin{gathered}\tilde{f}(\theta, \alpha) = f(\theta, A^{-1}\alpha) \text{ is defined and } \mathbf{C}^1 \text{ on } \mathbf{N}_{++}(\gamma),\\ \text{with values } h = \tilde{f}(\theta, \alpha) \text{ such that } (h, \theta, \alpha) \in \mathfrak{N}_{++},\\ \mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathfrak{N}_{++}) = \{p = (\tilde{f}(\theta, \alpha), \theta, \alpha)\colon (\theta, \alpha) \in \mathbf{N}_{++}(\gamma)\},\end{gathered}$$

and this set is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$ (an embedded open disk, or surface). Consequently, $\mathcal{M}^{\triangleright}_{++}$ — see (5.30b) — is also a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$.

The partial derivatives of $\tilde{f}$ evaluated at $(\theta^*_\infty, 0)$ are given (according to Type) by (6.43–44) and (6.48–49) respectively.

The preceding argument also establishes the following results:

Corollary 14(i)(α). The stable manifold of $\mathfrak{S}$ at p^*_∞ (rich economy) may be approximated to first order 'locally' — i.e. for $(\theta, \alpha) \in N_\oplus(\gamma)$ — by an equation of the form

$$
\begin{aligned}
(6.53) \qquad & h = \tilde{f}(\theta, \alpha) = h^*_\infty + (\theta - \theta^*_\infty)\tilde{f}^*_\theta + (\alpha - \alpha^*_\infty)\tilde{f}^*_\alpha + o(\theta - \theta^*_\infty, \alpha - \alpha^*_\infty), \\
& \text{where } h^*_\infty = h^+_\infty,\ \theta^*_\infty = 0,\ \alpha^*_\infty = 0 \text{ for Type 0}, \\
& h^*_\infty = 1,\ \theta^*_\infty = n,\ \alpha^*_\infty = 0 \text{ for Type 1.}
\end{aligned}
$$

Corollary 14(ii)(α). The function $f(\theta, z)$ is $\mathbf{C}^1$ for

$$
(6.54) \qquad \theta \succeq^+ 0,\ z \in \Re \text{ such that } (\theta - \theta^*_\infty)^2 + (A(z))^2 < \gamma^2
$$

for γ small enough. For sequences (θ_n, z_n) satisfying (54),

$$
\theta_n \to \theta \succeq^+ 0,\ z_n \to \infty
$$

implies

$$
f(\theta_n, z_n) \to f_\infty(\theta), \quad f_\theta(\theta_n, z_n) \to f'_\infty(\theta), \quad f_Z(\theta_n, z_n) \to 0.
$$

(iv) *Global* $\mathbf{C}^1$ *Properties of Stable Sets/Manifolds and Representing Functions.*

PROPOSITION 15(α) (Global Stable Manifold).

(i) The function $\tilde{f}$ is $\mathbf{C}^1$ on its domain $\{(\theta,\alpha)\colon 0 \lesssim^{+} \theta < \theta_+(A^{-1}\alpha), 0 \le \alpha < \psi_0'\}$ — see (6.33).

(ii) The function $\tilde{f}$ is $\mathbf{C}^1$ on the interior of its domain i.e. on $\{(\theta,\alpha)\colon 0 < \theta < \theta_+(A^{-1}\alpha), 0 < \alpha < \psi_0'\}$ — see (6.31a).

PROOF. For brevity, we prove only (ii), the proof of (i) being similar if allowance is made for boundary points. Let $\mathbf{V} = \mathbf{V}_{++} = \mathbf{U}_{++} \times (0, \psi_0')$ and write $\mathfrak{M}^{\triangleright}_{++}$ as $\mathfrak{M}$ for short. Let $\bar{p}_1 = (\bar{\pi}_1, \bar{\alpha}_1) = (\bar{h}_1, \bar{\theta}_1, \bar{\alpha}_1) \in \mathfrak{M}$; then $\bar{p}_1 > 0$, and $\bar{p}_1$ is the start of a solution stable at $p^* = p^*_\infty$ and satisfies $\bar{h}_1 = \tilde{f}(\bar{\theta}_1, \bar{\alpha}_1)$, or equivalently $\bar{h}_1 = f(\bar{\theta}_1, \bar{z}_1)$ with $\bar{z}_1 = A^{-1}\bar{\alpha}_1$.

Now let $\gamma \in (0, \psi_0')$, $\mathbf{N} = \mathbf{N}_{++}(\gamma) = \{(\theta,\alpha)\colon (\theta - \theta^*_\infty)^2 + (\alpha - \alpha^*_\infty)^2 < \gamma^2, \theta > 0, \alpha > 0\}$ be a neighbourhood of $(\theta^*_\infty, \alpha^*_\infty)$ in $\Re^2$, $\mathfrak{N} = \mathfrak{N}(\gamma)$ a neighbourhood of p^* in $\Re^3$, such that γ, $\mathbf{N}$, $\mathfrak{N}$ satisfy the conditions of Prop. 14(α)(ii). Then $\tilde{f}(\theta,\alpha)$ is defined on $\mathbf{N}$ with $(\tilde{f}(\theta,\alpha), \theta, \alpha) \in \mathfrak{N}$, and $\mathfrak{M} \cap \mathfrak{N}$ is a two-dimensional submanifold of $\Re^3$.

Since $\bar{p}_1$ is the start of a solution $\Phi^0(\zeta; \bar{p}_1)$ converging to p^*_∞ as $\zeta \to \infty$, we may choose $\bar{\zeta}$ large enough so that $\Phi^0(\bar{\zeta}; \bar{p}_1) = \bar{p}_0 = (\bar{h}_0, \bar{\theta}_0, \bar{\alpha}_0)$ is in the 'local' stable manifold $\mathfrak{M} \cap \mathfrak{N}$. Then $\bar{p}_0 > 0$, $(\bar{\theta}_0, \bar{\alpha}_0) \in \mathbf{N}$ and $\bar{h}_0 = \tilde{f}(\bar{\theta}_0, \bar{\alpha}_0) = f(\bar{\theta}_0, \bar{z}_0)$ with $\bar{\alpha}_0 = A(\bar{z}_0) = A(\bar{z}_1 + \bar{\zeta})$.

Choose points (θ_0, α_0) in a neighbourhood $\mathbf{N}_0$ of $(\bar{\theta}_0, \bar{\alpha}_0)$ in $\Re^2$, small enough so that the points $p_0 = (h_0, \theta_0, \alpha_0)$ with $h_0 = \tilde{f}(\theta_0, \alpha_0)$ and $(\theta_0, \alpha_0) \in \mathbf{N}_0$ form a set $\mathfrak{N}_0 \subset \mathfrak{M} \cap \mathfrak{N}$ in $\Re^3$. The points $p_0 \in \mathfrak{N}_0$ are 'stable' starts for $\mathfrak{S}$, (or equivalently the corresponding points $(\pi_0, z_0) = (h_0, \theta_0, z_0)$ with $h_0 = f(\theta_0, z_0)$, $A(z_0) = \alpha_0$ are forward special starts for S). Evidently the parametrisation $(\theta_0, \alpha_0) \mapsto p_0 = (\tilde{f}(\theta_0, \alpha_0), \theta_0, \alpha_0)$, $\mathbf{N}_0 \mapsto \mathfrak{N}_0$, is $\mathbf{C}^1$ and bijective.

Consider next the points of the form

$$\mathfrak{N}_1 = \{p_1 = \Phi^0(-\bar{\zeta}, p_0), p_0 \in \mathfrak{N}_0\}, \quad p_1 = (h_1, \theta_1, \alpha_1).$$

If γ, hence $\mathbf{N}_0$, is small enough, these points will be close enough to $\bar{p}_1$ so that $h_1 > 0$ (and of course $\theta_1 > 0$ and $0 < \alpha_1 < \psi_0'$). Also, since $\mathfrak{M}$ is invariant, the points of $\mathfrak{N}_1$ belong to $\mathfrak{M}$, hence belong to the domain of $\tilde{f}$ and satisfy $h_1 = \tilde{f}(\theta_1, \alpha_1)$, (or equivalently $h_1 = f(\theta_1, z_1)$, $z_1 = A^{-1}(\alpha_1)$).

Since $p_0 \mapsto p_1$ is $\mathbf{C}^1$ and bijective, $(\theta_0, \alpha_0) \mapsto p_1$ defines a $\mathbf{C}^1$ parametrisation of $\mathfrak{N}_1$ by means of $\mathbf{N}_0$. Write this as $p_1 = \tau(\theta_0, \alpha_0)$, or in more detail

$$h_1 = \tau^h(\theta_0, \alpha_0), \quad \theta_1 = \tau^\theta(\theta_0, \alpha_0), \quad \alpha_1 = \tau^\alpha(\alpha_0); \tag{6.55}$$

all the τ^i are $\mathbf{C}^1$, and the first equation is dependent because $h_1 = \tilde{f}(\theta_1, \alpha_1)$.

Now recall that, according to [B] Prop. 9(β) (with slightly different notation), if (h_0^i, θ_0^i, z_0), $i = a, b$, are points of $\mathbf{U}_{++} \times \Re$ with $0 < h_0^b \leq h_0^a$ and $0 < \theta_0^a \leq \theta_0^b$ (with the same z_0), then on any interval of the form $z_0 > z > z_1 > -\infty$ we have, for the solutions

$$\phi^i(z) = \phi(z; h_0^i, \theta_0^i, z_0) = (h^i(z), \theta^i(z)) \text{ of } S,$$

that $h^b(z) < h^a(z)$ and $\theta^a(z) < \theta^b(z)$, provided that both solutions exist on $[z_1, z_0]$ and the h-co-ordinate remains positive for at least one of the solutions. (This 'order-preserving' result follows from the 'co-operative' property of S, i.e. $F_\theta > 0$ and $G_h > 0$, in the positive quadrant; but note that in general $\mathfrak{S}$ is not co-operative, because J_α need not have definite sign.)

In the present situation, this result implies that $\partial\theta_1/\partial\theta_0 = \partial\tau^\theta(\theta_0, \alpha_0)/\partial\theta_0 > 0$. Also, $\partial\alpha_1/\partial\alpha_0 = \partial\tau^\alpha(\alpha_0)/\partial\alpha_0 > 0$, while $\partial\alpha_1/\partial\theta_0 = 0$ since α does not depend on θ. Consequently the Jacobian determinant

$$\frac{\partial(\theta_1, \alpha_1)}{\partial(\theta_0, \alpha_0)} = \begin{vmatrix} \partial\theta_1/\partial\theta_0 & \partial\theta_1/\partial\alpha_0 \\ \partial\alpha_1/\partial\theta_0 & \partial\alpha_1/\partial\alpha_0 \end{vmatrix} = \frac{\partial\theta_1}{\partial\theta_0} \cdot \frac{\partial\alpha_1}{\partial\alpha_0} \text{ is positive,}$$

so that we may solve the system

$$\theta_1 = \tau^\theta(\theta_0, \alpha_0), \quad \alpha_1 = \tau^\alpha(\alpha_0),$$

uniquely in the form

$$\theta_0 = \sigma^\theta(\theta_1, \alpha_1), \quad \alpha_0 = \sigma^\alpha(\alpha_1), \tag{6.56}$$

with $\mathbf{C}^1$ functions σ^θ and σ^α, cf. Courant [1957] Vol.II, Ch.III, Goetz [1970] Ch.4. But then, substituting into $h_1 = \tau^h(\theta_0.\alpha_0)$, we obtain h_1 as a $\mathbf{C}^1$ function of (θ_1, α_1) locally near (θ_1, α_1), and this solution must coincide with $\tilde{f}(\theta_1, \alpha_1)$. It follows that $\tilde{f}$ is a $\mathbf{C}^1$ function of (θ, α) at each point (θ_1, α_1) near $(\bar{\theta}_1, \bar{\alpha}_1)$ in the domain of $\tilde{f}$ i.e. $\tilde{f}$ is $\mathbf{C}^1$ in a (θ, α)-neighbourhood of each point of its domain, hence f is $\mathbf{C}^1$ in a (θ, z)-neighbourhood of each point of its domain.‖

Corollary 15(i)(α). $\mathfrak{M}^{\triangleright}_{++}$ is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$, and $\mathfrak{M}^{\triangleright}_{\oplus}$ is a two-dimensional submanifold with boundary.

These results follow from (6.31a) and (6.34) because the representation $h = \tilde{f}(\theta, \alpha)$ expresses one of the co-ordinates of the stable set at p^*_∞ as a $\mathbf{C}^1$ function of the two others.

Corollary 15(ii)(α), The function $h = f(\theta, z)$ is $\mathbf{C}^1$ on its domain

$$\{(\theta, z)\colon 0 \lesssim^{+} \theta < \theta_{+}(z), z \in \Re\}.$$

Hence $\mathcal{M}^{\triangleright}_{++}$ is a submanifold of $\Re^3$ and $\mathcal{M}^{\triangleright}_{+}$ is a submanifold (with boundary if S_∞ is Type 0); see (5.35a,b) and (6.16), (6.31–31a).

(v) *Formulas for* $p^*_{-\infty}$ *and* g.

In sub-sections (ii), (iii) and (iv) we have considered mainly solutions of $\mathfrak{S}$ converging to, or evaluated at, p^*_∞ and the corresponding stable sets and representing functions f. We now consider some of the corresponding formulas for $p^*_{-\infty}$ and g.

Starting with sub-section (ii), relevant results are obtained, loosely speaking, on replacing ∞ by $-\infty$, $\triangleright$ by $\triangleleft$, thus

$$p^* = p^*_\infty = (h^*_\infty, \theta^*_\infty, \alpha^*_\infty) \text{ with } \alpha^*_\infty = 0, \quad \delta\alpha = \alpha,$$

is replaced by

$$p^* = p^*_{-\infty} = (h^*_{-\infty}, \theta^*_{-\infty}, \alpha^*_{-\infty}) \text{ with } \alpha^*_{-\infty} = \psi'_0, \quad \delta\alpha = \alpha - \psi'_0,$$

also f, $\tilde{f}$ is replaced by g, $\tilde{g}$, as well as the replacements according to Type indicated in Table 2, i.e. data in the first block of the Table are replaced by the second block and data in the third block by the fourth. However, it will be useful to set out some of the more important points explicitly.

In the definitions (6.30 a–c), $\mathfrak{M}^\triangleright_\infty(\mathbf{V})$ is replaced by $\mathfrak{M}^\triangleleft_{-\infty}(\mathbf{V})$ and $[0, \psi'_0)$ by $(0, \psi'_0]$, also $[\mathbf{0}]$ by $[\boldsymbol{\psi}'_0]$, and the resulting expressions are denoted (6.30a–c)$^\triangleleft$.

The condition $\Phi^0(z; p_\Diamond) \to p^*$ in (6.29b), with $p^* = p^*_{-\infty}$, can be replaced by $\Phi^\Diamond(z; p_\Diamond) \to p^*_{-\infty}$ when $\mathbf{V} = \mathbf{U}_+ \times (0, \psi'_0)$, and then it is seen that $\mathfrak{M}^\triangleleft_+$ is the inverse image under Ξ of $\mathcal{M}^\triangleleft_+$, the latter set being the graph of the function g, see (5.36a). Thus

(6.31)$^\triangleleft$

$$\mathfrak{M}^\triangleleft_+(p^*_{-\infty}) = \{p = (h, \theta, \alpha)\colon h = g(\theta, A^{-1}\alpha),\ \theta_-(A^{-1}\alpha) \lesssim^- \theta < \infty,\ 0 < \alpha < \psi'_0\}.$$

Similarly $\mathfrak{M}^\triangleleft_{++}$ is the inverse image under Ξ of $\mathcal{M}^\triangleleft_{++}$, see (5.37b), so

(6.31a)$^\triangleleft$

$$\mathfrak{M}^\triangleleft_{++}(p^*_{-\infty}) = \{p = (h, \theta, \alpha)\colon h = g(\theta, A^{-1}\alpha),\ \theta_-(A^{-1}\alpha) < \theta < \infty,\ 0 < \alpha < \psi'_0\}.$$

Also $\Phi^0(z; \pi, \psi'_0) \to p^*_{-\infty}$ if $\varphi^0_{-\infty}(z; \pi) \to \pi^*_{-\infty}$, so that

(6.32)$^\triangleleft$

$$\begin{aligned} \mathfrak{M}^\triangleleft_0 &= \{p = (h, \theta, \psi'_0)\colon h > 0,\ \theta \geq 0,\ \varphi^0_{-\infty}(z; h, \theta) \to \pi^*_{-\infty}\} \\ &= \{p = (h, \theta, \psi'_0)\colon h = g_{-\infty}(\infty),\ \theta_-(-\infty) \lesssim^- \theta < \infty\} \\ &= \{p = (h, \theta, \psi'_0)\colon h = g_{-\infty}(\theta),\ \theta \in \operatorname{dom} g_{-\infty}\}, \end{aligned}$$

or (with some abuse of notation) $\mathfrak{M}_0^{\triangleleft} = \langle \mathcal{M}^{\triangleleft}_{-\infty}, \boldsymbol{\psi}'_0 \rangle$.

Combining the definitions of $g(\theta, z)$ and $g_{-\infty}(\theta)$ we write

$$(6.33)^{\triangleleft} \qquad \begin{aligned} &\tilde{g}(\theta, \alpha) = g(\theta, A^{-1}\alpha) \text{ for } 0 < \alpha \le \psi'_0 \\ &\text{where } A^{-1}\psi'_0 = -\infty,\ g(\theta, A^{-1}\psi'_0) = g_{-\infty}(\theta), \\ &\text{dom } \tilde{g} \doteq \big\{(\theta, \alpha)\colon \theta_-(A^{-1}\alpha) \lesssim^{\triangleleft} \theta < \infty,\ 0 < \alpha \le \psi'_0\big\}. \end{aligned}$$

Combining $(31)^{\triangleleft}$ and $(32)^{\triangleleft}$ yields

$$(6.34)^{\triangleleft} \qquad \mathfrak{M}^{\triangleleft}_{\oplus} = \mathfrak{M}^{\triangleleft}_{+} \cup \mathfrak{M}^{\triangleleft}_{0} = \big\{(h, \theta, \alpha)\colon h = \tilde{g}(\theta, \alpha) \in \text{dom } \tilde{g}\big\}.$$

Turning to sub-section (iii), we note that all preceding calculations are based on the forward motion for $\mathfrak{S}$. However, arguments involving convergence to $p^*_{-\infty}$ are more conveniently stated for the backward motion $\mathfrak{S}^{\leftarrow}$, which yields a closer symmetry with the forward case. Briefly, for the backward flow and its linearisation, the matrix $\Delta = [a_{ij}]$ is replaced by $\Delta^{\leftarrow} = -\Delta$, so that each eigenvalue λ shown in Table 2 changes sign. Thus, with respect to the backward motion, there is for each Type at $p^*_{-\infty}$ a two-dimensional stable manifold and a one-dimensional unstable manifold, with characterisations analogous to those for manifolds at p^*_{∞}.[11]

[11] Notation for the backward motion becomes complicated; the following conventions (which apply only in the present sub-section) are set out for completeness. A superscript arrow ($\rightarrow$ or $\leftarrow$) distinguishes concepts relating to the forward from those relating to the backward motion, but the forward arrow is usually omitted when the meaning is clear.

Concepts for the forward motion of $\mathfrak{S}$ are defined in 6(i). To define the backward motion briefly, set

$$\begin{aligned} &v = -z, \quad \mathfrak{S}^{\leftarrow}(p) = dp/dv = -dp/dz = -\mathfrak{S}^{\rightarrow}p, \\ &\Phi_v^{\leftarrow 0} = \Phi_{-Z}^{\rightarrow 0}p \text{ for } v \in I^0(p),\ \text{cf.}(6.12), \end{aligned}$$

so that, for a solution $\Phi^0(p; z)$ of $\mathfrak{S}$ defined for all $z \in \Re$, $\Phi^{\leftarrow 0}(v; p)$ with $v = -z$ is the same solution considered w.r.t. $\mathfrak{S}^{\leftarrow}$. Hence

$$\Phi^{\leftarrow 0}(v; p_{\Diamond}) = \Phi^0(-z; p_{\Diamond}) = \Phi^{\Diamond}(-z + z_0; p_{\Diamond}, z_{\Diamond}) = \Phi^{\leftarrow\Diamond}(v + z_{\Diamond}; p_{\Diamond}, z_{\Diamond}),\ \text{cf.}(6.14).$$

Recall that, if p^* is a stationary point of $\mathfrak{S}$ (say p^*_{∞} or $p^*_{-\infty}$), $\mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathbf{V}, p^*)$, abbreviated to $\mathfrak{M}^{\triangleright}(\mathbf{V})$ or just $\mathfrak{M}^{\triangleright}$, denotes a stable set at p^*; similarly $\mathfrak{M}^{\triangleleft}(\mathfrak{S}, \mathbf{V}, p^*)$ denotes an unstable set at p^*, abbreviated to $\mathfrak{M}^{\triangleleft}(\mathbf{V})$ or just $\mathfrak{M}^{\triangleleft}$, (all w.r.t. the forward motion), cf.fn.7 above. Now stable sets at p^* for the forward motion are unstable sets at p^* for the backward motion and vice versa, so $\mathfrak{M}^{\triangleright}(\mathfrak{S}^{\leftarrow}, \mathbf{V}, p^*) = \mathfrak{M}^{\triangleleft}(\mathfrak{S}^{\rightarrow}, \mathbf{V}, p^*)$, or briefly $\mathfrak{M}^{\leftarrow\triangleright} = \mathfrak{M}^{\rightarrow\triangleleft}$; similarly $\mathfrak{M}^{\triangleleft}(\mathfrak{S}^{\leftarrow}, \mathbf{V}, p^*) = \mathfrak{M}^{\triangleright}(\mathfrak{S}^{\rightarrow}, \mathbf{V}, p^*)$, or $\mathfrak{M}^{\leftarrow\triangleleft} = \mathfrak{M}^{\rightarrow\triangleright}$. See also Hirsch [1984] p.28.

In more detail, the argument of 6(iii) is adapted to the backward flow as follows. We assume that the domain of $\mathfrak{S}^{\leftarrow}$ is $\mathbf{V}^{\leftarrow} = \mathbf{U} \times (0, \phi'_0)$ with $\mathbf{U} = \{\theta \ge 0\}$. Writing $\delta^{\leftarrow} = (h - h^*_{-\infty}, \theta - \theta^*_{-\infty}, \alpha - \alpha^*_{-\infty})$, The argument for the backward flow corresponding to (6.35) – (6.50) is much the same as in (iii), with tedious but routine substitution of notation — see fn.11. Here we retain the notation for the forward motion, and skip to the main result:

(6.50)$^{\triangleleft}$ PROPOSITION 14(β)(ii): Local Unstable Manifold for $\mathfrak{S}$ at
$p^* = p^*_{-\infty} = (h^*_{-\infty}, \theta^*_{-\infty}, \alpha^*_{-\infty})$.

Let $\mathbf{V} = \mathbf{U}_+ \times (0, \psi_0']$. There exist

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the linearisation $\mathfrak{L}^{\leftarrow}$ of $\mathfrak{S}^{\leftarrow}$ about $p^* = p^*_{-\infty}$, cf.(6.35), is

$$d\delta^{\leftarrow}/dv = \Delta^{\leftarrow} \cdot \delta^{\leftarrow}, \quad \delta^{\leftarrow} \in \mathbf{V}^{\leftarrow},$$

where $v = -z$, $\Delta^{\leftarrow} = [a_{ij}^{\leftarrow}] = -[a_{ij}] = -\Delta$ and Δ is the matrix appearing in the second or the fourth block of Table 2, depending on the Type of $p^*_{-\infty}$; now $\alpha^*_{-\infty} = \psi_0'$ for both Types.

For each Type, the matrix Δ^* has real, non-zero eigenvalues $\lambda_i^{\leftarrow} = -\lambda_i$, i.e.

$$\lambda_+^{\leftarrow} = -\lambda_+, \quad \lambda_-^{\leftarrow} = -\lambda_-, \quad \lambda_3^{\leftarrow} = -\lambda_3 = -1,$$

so that the real canonical form is $\Lambda^{\leftarrow} = \mathrm{diag}(\lambda_+^{\leftarrow}, \lambda_-^{\leftarrow}, \lambda_3^{\leftarrow})$, cf.(6.36). For the transformation matrices we may choose the same matrices as in Table 3, say $(T_i^{\leftarrow})^{-1} = T_i^{-1}$, $i = 0, 1$, with entries a_{ij} replaced by $-a_{ij} = a_{ij}^{\leftarrow}$ and λ_i replaced by $\lambda_i^{\leftarrow}$. Then (6.37 a–b) are replaced by

$$(T^{\leftarrow})^{-1} \cdot \Delta^{\leftarrow} = \Lambda^{\leftarrow} \cdot (T^{\leftarrow})^{-1}$$
$$(T^{\leftarrow})^{-1}\delta^{\leftarrow} = \hat{\delta}^{\leftarrow} = (\eta^{\leftarrow}, \xi^{\leftarrow}, \chi^{\leftarrow}), \quad (T^{\leftarrow})^{-1} = \hat{\mathbf{V}}^{\leftarrow}.$$

The system $\mathfrak{L}^{\leftarrow}$ becomes — cf.(6.38) —

$$\hat{\mathfrak{L}}^{\leftarrow}\colon d\hat{\delta}^{\leftarrow}/dv = \Lambda^{\leftarrow} \cdot \hat{\delta}^{\leftarrow}, \quad \hat{\delta}^{\leftarrow} \in \hat{\mathbf{V}}^{\leftarrow}, \text{ or}$$
$$d\eta^{\leftarrow}/dv = \lambda_+^{\leftarrow} \cdot \eta^{\leftarrow}, \quad d\xi^{\leftarrow}/dv = \lambda_-^{\leftarrow} \cdot \xi^{\leftarrow}, \quad d\chi^{\leftarrow}/dv = \lambda_3^{\leftarrow} \cdot \chi^{\leftarrow}.$$

The vectors of the form $(\eta^{\leftarrow}, 0, 0)$ span the unstable subspace for $\hat{\mathfrak{L}}^{\leftarrow}$, while the vectors $(0, \xi^{\leftarrow}, \chi^{\leftarrow})$ span the stable subspace. The system $\mathfrak{S}^{\leftarrow}$ referred to the transformed co-ordinates (with origin $\hat{\delta}^{*\leftarrow} = \mathbf{0}$) is denoted $\hat{\mathfrak{S}}^{\leftarrow}$ and has the form

$$d\delta^{\leftarrow}/dv = \Lambda^{\leftarrow} \cdot \hat{\delta}^{\leftarrow} + \hat{H}^{\leftarrow}(\hat{\delta}^{\leftarrow})$$

and so forth, with $\hat{\chi}^{\leftarrow} = \alpha - \alpha^*_{-\infty}$, cf.(6.39).

Also, if $\hat{\mathfrak{N}}$ is a neighbourhood of $\hat{\delta}^*$ (w.r.t. $\hat{\mathbf{V}}^{\leftarrow}$) in the new co-ordinates, then $\mathfrak{N} = \{p^* + T \cdot \hat{\delta}^{\leftarrow}, \hat{\delta}^{\leftarrow} \in \hat{\mathfrak{N}}\}$ defines a neighbourhood of p^* (w.r.t. $\mathbf{V}^{\leftarrow}$) in the old co-ordinates. Let

$$\hat{\mathfrak{M}}^{\triangleright}(\hat{\mathfrak{S}}^{\leftarrow}, \mathfrak{N}) \doteq \{\hat{\delta}^{\leftarrow} \in \hat{\mathfrak{N}}\colon \hat{\Phi}^{\leftarrow 0}(v, \hat{\delta}^{\leftarrow}) \to \mathbf{0} \text{ as } v \to \infty\},$$

where $\hat{\Phi}^{\leftarrow 0}(v, \hat{\delta}^{\leftarrow})$ is the flow on $\hat{\mathfrak{N}}$ defined by $\hat{\mathfrak{S}}^{\leftarrow}$, cf.(6.40a). Then

$$\mathfrak{M}^{\triangleright}(\mathfrak{S}^{\leftarrow}, \mathfrak{N}) = \{p = p^* + T \cdot \hat{\delta}^{\leftarrow}, \hat{\delta}^{\leftarrow} \in \hat{\mathfrak{M}}^{\triangleright}(\hat{\mathfrak{S}}^{\leftarrow}, \mathfrak{N})\} = \{p \in \mathfrak{N}\colon \Phi^{\leftarrow 0}(v, p) \to p^*_{-\infty}\}.$$

As in (6.40c), $\hat{\mathfrak{M}}^{\triangleright}(\hat{\mathfrak{S}}^{\leftarrow}, \mathfrak{N})$ is the stable manifold of $\hat{\mathfrak{S}}^{\leftarrow}$ at $\mathbf{0}$, restricted to $\mathfrak{N}$, and is the graph of a $\mathbf{C}^1$ function $\eta^{\leftarrow} = w^{\leftarrow}(\xi^{\leftarrow}, \chi^{\leftarrow})$. Hence $\mathfrak{M}^{\triangleright}(\mathfrak{S}^{\leftarrow}, \mathfrak{N})$ is a (local) stable manifold at $p^*_{-\infty}$ for $\mathfrak{S}^{\leftarrow}$ and is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$ (with boundary). Equivalently, $\mathfrak{M}^{\triangleleft}(\mathfrak{S}, \mathfrak{N})$ is a local unstable manifold for $\mathfrak{S}$ at $p^*_{-\infty}$, and is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$ (with boundary).

(1) a number $\gamma \in (0, \psi_0')$ and a neighbourhood of $(\theta^*_{-\infty}, \alpha^*_{-\infty})$ in $\Re^2$ of the form

$$\mathbf{N} = \mathbf{N}(\gamma) = \{(\theta, \alpha)\colon (\theta - \theta^*_{-\infty})^2 + (\alpha - \alpha^*_{-\infty})^2 < \gamma^2, \theta \gtrsim^- 0, 0 < \alpha \leq \psi_0'\},$$

(2) a neighbourhood $\mathfrak{N} = \mathfrak{N}(\gamma)$ (w.r.t. $\mathbf{V}$) of $p^*_{-\infty}$ in $\Re^3$,

(3) a function $\hat{h}(\theta, \alpha]$, defined and $\mathbf{C}^1$ for $(\theta, \alpha) \in \mathbf{N}$, satisfying $\tilde{h}(\theta^*_{-\infty}, \alpha^*_{-\infty}) = h^*_{-\infty}$, with values h such that (h, θ, α) is in $\mathfrak{N}$ and

$$\mathfrak{M}^{\triangleleft}(\mathfrak{S}, \mathfrak{N}) = \{p = (h, \theta, \alpha)\colon h = \tilde{h}(\theta, \alpha), (\theta, \alpha) \in \mathfrak{N}\}.$$

This set is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$ (with boundary). Consequently the set $\mathcal{M}_+^{\triangleleft}$ — see (5.36b) — is a submanifold of $\Re^3$ (with boundary if $S_{-\infty}$ is Type 0).

(6.51)$^{\triangleleft}$ The functions $\hat{h}$ and $\tilde{g}$ (and their derivatives) coincide on $\mathbf{N}$, so that $\tilde{g}$ is $\mathbf{C}^1$ on this set (allowing one-sided limits on the boundary) and $\mathfrak{M}^{\triangleleft}(\mathfrak{S}, \mathfrak{N}) = \{p = (h, \theta, \alpha)\colon h = \tilde{g}(\theta, \alpha), (\theta, \alpha) \in \mathbf{N}\}$ is a two-dimensional submanifold with boundary.

If we replace $\mathbf{V}$, $\mathbf{N}$, $\mathfrak{N}$ by $\mathbf{V}_{++} = \mathbf{U}_{++} \times (0, \psi_0')$, $\mathbf{N}_{++}(\gamma) = \mathbf{N}(\gamma) \cap \{\theta > 0, \alpha < \psi_0'\}$, $\mathfrak{N}_{++} = \mathfrak{N} \cap \mathbf{V}_{++}$, then for γ small enough,

(6.52)$^{\triangleleft}$ $\tilde{g}(\theta, \alpha) = g(\theta, \alpha)$ is defined and $\mathbf{C}^1$ on $\mathbf{N}_{++}(\gamma)$,
with values $h = \tilde{g}(\theta, \alpha)$ such that $(h, \theta, \alpha) \in \mathfrak{N}_{++}$,
$\mathfrak{M}^{\triangleleft}(\mathfrak{S}, \mathfrak{N}_{++}) = \{p = \tilde{g}(\theta, \alpha), \theta, \alpha)\colon (\theta, \alpha) \in \mathbf{N}_{++}(\gamma)\}$,

and this set is a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$. Consequently the set $\mathcal{M}_{++}^{\triangleleft}$ — see (5.37b) — is also a two-dimensional $\mathbf{C}^1$ submanifold of $\Re^3$.

The partial derivatives of $\tilde{g}$ evaluated at $(\theta^*_{-\infty}, \psi_0')$ are given, for Type 0, by

(6.43)$^{\triangleleft}$ $\tilde{g}^*_\theta = a_{12}/(a_{22} - a_{11}) = g'_{-\infty}(0) > 0,$ cf.[B] Table 1.

(6.44)$^{\triangleleft}$ $\tilde{g}^*_\alpha = a_{13}/(a_{33} - a_{11}).$

(6.45)$^{\triangleleft}$ $\tilde{g}_\alpha(\theta, A(z)) \cdot A'(z) = g_Z(\theta, z) \to 0$ as $z \to -\infty$,

the coefficients a_{ij} being chosen from the fourth block of Table 2.

For Type 1, we have

$$(6.48)^{\triangleleft} \qquad \hat{g}^*_\theta = a_{12}/\lambda_+ = 2/\sigma^2\lambda^+ = -\lambda_-/a_{21} = g'_{-\infty}(N) > 0, \text{ cf.[B] Table 1,}$$

$$(6.49)^{\triangleleft} \qquad \hat{g}^*_\alpha = a_{13}/(\lambda_+ - \lambda_3) = a_{13}/(\lambda_+ - 1)$$

with coefficients from the third block of Table 2.

Cor.14(i)(α) is replaced by

Corollary 14(i)(β). The unstable manifold of $\mathfrak{S}$ at $p^*_{-\infty}$ (poor economy), i.e. the stable manifold for the backward motion, may be approximated to first order 'locally' by an expression of the form

$$(6.53)^{\triangleleft} \qquad h = \tilde{g}(\theta, \alpha) = h^*_{-\infty} + (\theta - \theta^*_{-\infty})\tilde{g}^*_\theta + (\alpha - \alpha^*_{-\infty})\tilde{g}^*_\alpha + o(\theta - \theta^*_{-\infty}, \alpha - \alpha^*_{-\infty}),$$
$$\text{where } h^*_{-\infty} = h^-_{-\infty}, \quad \theta^*_{-\infty} = 0, \quad \alpha^*_{-\infty} = \psi'_0 \text{ for Type 0,}$$
$$h^*_{-\infty} = 1, \quad \theta^*_{-\infty} = N, \quad \alpha^*_{-\infty} = \psi'_0 \text{ for Type 1,}$$

and the partial derivatives of $\tilde{g}$ at $(\theta^*_{-\infty}, \psi'_0)$ are given by $(6.43\text{–}44)^{\triangleleft}$ and $(6.48\text{–}49)^{\triangleleft}$ respectively.

There are also results analogous to Cor.14(ii)(α).

Turning to sub-section (iv), we have

Proposition 15(β)(Global Unstable Manifold).

(i) The function $\tilde{g}$ is $\mathbf{C}^1$ on its domain

$$\{(\theta, \alpha) \colon \theta_-(A^{-1}\alpha) \lesssim^- \theta < \infty, 0 < \alpha \leq \psi'_0\}$$

— see $(6.31)^{\triangleleft}$, $(6.33)^{\triangleleft}$.

(ii) The function $\tilde{g}$ is $\mathbf{C}^1$ on the interior of its domain, i.e. on

$$\{(\theta, \alpha) \colon \theta_-(A^{-1}\alpha) < \theta < \infty, 0 < \alpha < \psi'_0\}$$

— see $(6.31a)^{\triangleleft}$, $(6.33)^{\triangleleft}$.

Corollary 15(i)(β). $\mathfrak{M}^{\triangleleft}_{++}$ is a $\mathbf{C}^1$ submanifold of $\Re^3$ and $\mathfrak{M}^{\triangleleft}_{\oplus}$ is a $\mathbf{C}^1$ submanifold with boundary.

Corollary 15(ii)(β). The function $h = g(\theta, z)$ is $\mathbf{C}^1$ on its domain

$$\{(\theta, z)\colon \theta_-(z) \lesssim^- \theta < \infty, z \in \Re\}.$$

Hence $\mathcal{M}^{\triangleleft}_{++}$ is a submanifold of $\Re^3$ and $\mathcal{M}^{\triangleleft}_{+}$ is a submanifold (with boundary if $S_{-\infty}$ is Type 0); — see (5.37a,b) and (6.16), $(6.31)^{\triangleleft}$, $(6.31\text{a})^{\triangleleft}$.

Appendix to Section 6

This Appendix sets out calculations of the stable and unstable *subspaces* determined by the matrices in Table 2 without making use of the transformation to real canonical form; details aside, these subspaces are tangent at the origin to the corresponding stable and unstable *local manifolds* of $\mathfrak{S} - p^*$. The calculations thus serve as a check on the formulas stated in Section 6 for the local manifolds. For brevity we consider only $p^* = p^*_\infty$, of both Types, but similar calculations can be carried out for $p^*_{-\infty}$.

Consider the linearisation of $\mathfrak{S}$ about p^* as in (6.35), i.e. the system

$$\begin{aligned}&\delta' = \Delta \cdot \delta, \text{ where } \Delta = \Delta_{\mathfrak{S}}(p^*),\\ &\delta = (\delta h, \delta\theta, \delta\alpha) = (h - h^*, \theta - \theta^*, \alpha - \alpha^*) = p - p^* \in \Re^3.\end{aligned} \tag{1}$$

For brevity we write

$$y = h - h^*, \quad x = \theta - \theta^*, \quad u = \alpha - \alpha^*$$

and denote by a_{ij} the elements of the matrix Δ, distinguishing between p^* of Types 1 and 0.

Thus for Type 1, $\mathfrak{L}$ has the form

$$\begin{aligned} y' &= a_{11}y + a_{12}x + a_{13}u\\ x' &= a_{21}y\\ u' &= \qquad\qquad\quad a_{33}u\end{aligned} \tag{2}$$

where

$$y = h - 1, \quad x = \theta - n, \quad u = \alpha.$$

The a_{ij} and the eigenvalues of Δ are those in the first block of Table 2.

We now write the eigenvalues as λ_1, λ_2, λ_3 instead of λ_+, λ_-, λ_3, and assume that $\lambda_2 \neq \lambda_3$.

For Type 0, $\mathfrak{L}$ has the form

$$\begin{aligned} y' &= a_{11}y + a_{12}x + a_{13}u\\ x' &= \qquad\quad a_{22}x\\ u' &= \qquad\qquad\quad a_{33}u\end{aligned} \tag{3}$$

where $y = h - h^*_\infty$, $x = \theta$, $u = \alpha$. The a_{ij} and eigenvalues are those in the third block of Table 2, $\lambda_+ = \lambda_1$, $\lambda_- = \lambda_2$, and we assume that $h^+_\infty \neq r_0$ so that the eigenvalues are distinct.

Now, for either Type, the system $\mathfrak{L}$ has, for each λ_i, $i = 1, 2, 3$, linearly independent solutions of the form

$$y_i = P_{y_i} e^{\lambda_i z}, \quad x_i = P_{x_i} e^{\lambda_i z}, \quad u_i = P_{u_i} e^{\lambda_i z}, \tag{4}$$

the coefficients P being real constants in the present case because the λ_i are simple and real. The solutions (4) may be found by inserting the expressions (4) into (1), (i.e. into (2) or (3)) as trial solutions. A collection of three such linearly independent solutions comprises a Fundamental System of solutions, and every solution of $\mathfrak{L}$ may be written as a sum of such solutions, see for example Kamke [1943] Section 13 or Kaplan [1958] Chs.6-1 and 12-8. Thus the general solution of $\mathfrak{L}$ is given by

$$\begin{aligned} y &= AP_{y_1}e^{\lambda_1 z} + BP_{y_2}e^{\lambda_2 z} + CP_{y_3}e^{\lambda_3 z} \\ x &= AP_{x_1}e^{\lambda_1 z} + BP_{x_2}e^{\lambda_2 z} + CP_{x_3}e^{\lambda_3 z} \\ u &= AP_{u_1}e^{\lambda_1 z} + BP_{u_2}e^{\lambda_2 z} + CP_{u_3}e^{\lambda_3 z} \end{aligned} \tag{5}$$

where A, B, C are arbitrary constants. Equations characterising the stable and unstable subspaces of $\Re^3$ may be obtained by considering separately the terms with negative and those with positive eigenvalues, i.e. setting respectively $A = 0$ and $B = C = 0$.

Next, some calculations, starting with p^*_∞ of Type 1. For the solutions (4), we have

$$y'_i = \lambda_i y_i, \quad x'_i = \lambda_i x_i, \quad u'_i = \lambda_i u_i,$$

so that on substituting into (2) the exponential terms cancel and there remains

$$\begin{aligned} \lambda_i y_i &= a_{11}y_i + a_{12}x_i + a_{13}u_i \\ \lambda_i x_i &= a_{21}y_i \\ \lambda_i u_i &= \qquad\qquad a_{33}u_i \end{aligned}$$

Note that, from Table 2, $a_{21} = n$, $a_{12} = 2/\sigma^2$, $\lambda_1 + \lambda_2 = a_{11}$, $\lambda_1\lambda_2 = -a_{12}a_{21} = -2n/\sigma^2$, also $\lambda_3 = r_0 - 1 = a_{33}$.

Now the equations for $\lambda_i = \lambda_1$ yield

$$\begin{aligned} \lambda_1 P_{y_1} &= a_{11}P_{y_1} + a_{12}P_{x_1} + a_{13}P_{u_1} \\ \lambda_1 P_{x_1} &= a_{21}P_{y_1} \\ \lambda_1 P_{u_1} &= \qquad\qquad a_{33}P_{u_1}, \end{aligned}$$

hence $P_{x_1} = (a_{21}/\lambda_1)P_{y_1}$, $P_{u_1} = 0$ because $\lambda_1 > 0 > \lambda_3 = a_{33}$. This leaves

$$0 = P_{y_1}(a_{11} - \lambda_1 + a_{12}a_{21}/\lambda_1) \text{ or } 0 = P_{y_1}(\lambda_1^2 - \lambda_1 a_{11} + \lambda_1\lambda_2),$$

and the bracket vanishes since $a_{11} = \lambda_1 + \lambda_2$, leaving P_{y_1} undetermined.

Similarly, the equations for $\lambda_i = \lambda_2$ yield

$$\begin{aligned} \lambda_2 P_{y_2} &= a_{11}P_{y_2} + a_{12}P_{x_2} + a_{13}P_{u_2} \\ \lambda_2 P_{x_2} &= a_{21}P_{y_2} \\ \lambda_2 P_{u_2} &= \qquad\qquad a_{33}P_{u_2}, \end{aligned}$$

hence $P_{x_2} = (a_{21}/\lambda_2)P_{y_2}$, $P_{u_2} = 0$ because (by assumption) $\lambda_2 \neq \lambda_3 = a_{33}$, leaving

$$0 = P_{y_2}(a_{11} - \lambda_2 + a_{12}a_{21}/\lambda_2) \text{ or } 0 = P_{y_2}(\lambda_2^2 - \lambda_2 a_{11} + \lambda_1\lambda_2),$$

and the bracket vanishes leaving P_{y_2} undetermined.

Last, the equations for $\lambda_i = \lambda_3$ yield

$$\begin{aligned}
\lambda_3 P_{y_3} &= a_{11}P_{y_3} + a_{12}P_{x_3} + a_{13}P_{u_3} \\
\lambda_3 P_{x_3} &= a_{21}P_{y_3} \\
\lambda_3 P_{u_3} &= \qquad\qquad\qquad a_{33}P_{u_3},
\end{aligned}$$

hence $P_{x_3} = (a_{21}/\lambda_3)P_{y_3}$, so that

$$0 = P_{y_3}(a_{11} - \lambda_3 + a_{12}a_{21}/\lambda_3) + a_{13}P_{u_3}, \text{ or } 0 = P_{y_3}(\lambda_3^2 - a_{11}\lambda_3 + \lambda_1\lambda_2) - a_{13}\lambda_3 P_{u_3}.$$

Note that $P_{u_3}$is not determined by the u-equation because $\lambda_3 = a_{33}$, however P_{u_3}/P_{y_3} is determined by the y-equation.

Taking into account the relations among constants determined so far, the general solution (5) becomes

$$\begin{aligned}
y &= AP_{y_1}e^{\lambda_1 z} + BP_{y_2}e^{\lambda_2 z} + CP_{y_3}e^{\lambda_3 z} \\
x &= AP_{y_1}e^{\lambda_1 z}(a_{21}/\lambda_1) + BP_{y_2}e^{\lambda_2 z}(a_{21}/\lambda_2) + CP_{y_3}e^{\lambda_3 z}(a_{21}/\lambda_3) \\
u &= CP_{u_3}e^{\lambda_3 z} = CP_{y_3}e^{\lambda_3 z}(\lambda_3^z - a_{11}\lambda_3 + \lambda_1\lambda_2)/a_{13}\lambda_3.
\end{aligned}$$

In order to characterise the stable subspace, we set $A = 0$ and substitute the remaining expressions for x and u into the y-equation. This yields

$$y = (\lambda_2/a_{21})x + ua_{13}(\lambda_3 - \lambda_2)/(\lambda_3^2 - a_{11}\lambda_3 + \lambda_1\lambda_2).$$

Taking into account that $\lambda_2/a_{21} = -a_{12}/\lambda_1$, $a_{11} = \lambda_1 + \lambda_2$, $y = h - 1$, $x = \theta - n = \theta - a_{21}$, $u = \alpha - \alpha_\infty^* = \alpha$, $\lambda_1 = \lambda_+$, $\lambda_2 = \lambda_1$, this is found to agree with the expression

$$h = 1 - (\theta - n)a_{12}/\lambda_+ - \alpha \cdot a_{13}/(\lambda_+ - \lambda_3)$$

obtained from (6.46d) on omitting the term $o(\theta - n, \alpha)$ which distinguishes the local stable manifold from the stable subspace.

To characterise the unstable subspace, set $B = C = 0$ and note that we may replace APy_1 by A since both constants are arbitrary. Thus $y(z) = h(z) - 1 = Ae^{\lambda_1 z}$, so the unstable subspace is just a line through the point $(1, n, 0)$ parallel with the h-axis in $\Re^3$.

Turning to p_∞^* of Type 0, we consider the system (3) instead of (2), with eigenvalues $\lambda_+ = \lambda_1 = a_{11}$, $\lambda_- = \lambda_2 = a_{22}$, $\lambda_3 = a_{33}$. We now substitute solutions of the form (4) into (3).

For $\lambda_i = \lambda_1$, this yields

$$\begin{aligned}
\lambda_1 P_{y_1} &= a_{11}P_{y_1} + a_{12}P_{x_1} + a_{13}P_{u_1} \\
\lambda_1 P_{x_1} &= \qquad\qquad a_{22}P_{x_1} \\
\lambda_1 P_{u_1} &= \qquad\qquad\qquad\qquad a_{33}P_{u_1}.
\end{aligned}$$

The third equation implies $P_{u_1} = 0$ since $\lambda_1 \neq \lambda_3 = a_{33}$, and the second equation implies $P_{x_1} = 0$ since $\lambda_1 \neq \lambda_2 = a_{22}$. Since $\lambda_1 = a_{11}$ this leaves P_{y_1} undetermined.

For $\lambda_i = \lambda_2$, we have

$$\begin{aligned}
\lambda_2 P_{y_2} &= a_{11}P_{y_2} + a_{12}P_{x_2} + a_{13}P_{u_2} \\
\lambda_2 P_{x_2} &= \qquad\qquad a_{22}P_{x_2} \\
\lambda_2 P_{u_2} &= \qquad\qquad\qquad\qquad a_{33}P_{u_2}.
\end{aligned}$$

Since, by assumption, $\lambda_2 \neq \lambda_3 = a_{33}$, we have $P_{u_2} = 0$. The second equation leaves P_{x_2} undetermined since $\lambda_2 = a_{22}$ and, since $a_{11} = \lambda_1$, the first equation reduces to

$$0 = P_{y_2}(\lambda_1 - \lambda_2) + a_{12}P_{x_2}.$$

For $\lambda_i = \lambda_3$, we have

$$\begin{aligned}
\lambda_3 P_{y_3} &= a_{11}P_{y_3} + a_{12}P_{x_3} + a_{13}P_{u_3} \\
\lambda_3 P_{x_3} &= \qquad\qquad a_{22}P_{x_3} \\
\lambda_3 P_{u_3} &= \qquad\qquad\qquad\qquad a_{33}P_{u_3}.
\end{aligned}$$

Since $\lambda_3 = a_{33}$, the third equation leaves P_{u_3} undetermined. Since $\lambda_3 \neq \lambda_2 = a_{22}$, the second equation implies $P_{x_3} = 0$. The first equation then reduces to

$$0 = P_{y_3}(\lambda_1 - \lambda_3) + a_{13}P_{u_3}.$$

Taking into account the relations determined above, the general solution (5) reduces to

$$\begin{aligned}
y &= AP_{y_1}e^{\lambda_1 z} + BP_{y_2}e^{\lambda_2 z} + CP_{y_3}e^{\lambda_3 z} \\
x &= BP_{y_2}e^{\lambda_2 z}(\lambda_2 - \lambda_1)/a_{12} \\
u &= CP_{u_3}e^{\lambda_3 z}
\end{aligned}$$

For the stable subspace, set $A = 0$ and substitute into the y-equation. Using $Ce^{\lambda_3 z} = u/P_{u_3}$, this yields

$$y = x \cdot a_{12}/(\lambda_2 - \lambda_1) + u \cdot a_{13}/(\lambda_3 - \lambda_1)$$

which agrees with (6.41c) on dropping the term $w(\theta, \alpha)$ and noting that $y = h - h_\infty^+$, $x = \theta$, $u = \alpha$ and $\lambda_1 = \lambda_+ = a_{11}$, $\lambda_2 = \lambda_- = a_{22}$, $\lambda_3 = a_{33}$.

For the unstable subspace, set $B = C$ and replace AP_{y1} by A since both constants are arbitrary. This leaves $y = Ae^{\lambda_1 z}$, so the unstable subspace is the y-axis in (y, x, u)-space, or equivalently the h-axis in (h, θ, α)-space, in agreement with the geometric argument.

7. **Restatement of the Existence Theorem and a Perturbation Result.**

We can now give a three-dimensional restatement of Theorem 4.

Let $\mathbf{V} = \mathbf{U} \times [0, \psi_0']$, where $\mathbf{U} = \{\theta \geq 0\}$, be the domain of $\mathfrak{S}$. We say that a solution $\Phi^*(z) = (h^*(z), \theta^*(z), \alpha^*(z))$ defined for $z \in \Re$ is a *saddle connection* (from $p^*_{-\infty}$ to p^*_∞) if

(7.1) $$\Phi^*(z) \to p^*_{-\infty} \text{ as } z \to -\infty \text{ and } \Phi^*(z) \to p^*_\infty \text{ as } z \to \infty.$$

The trajectory corresponding to this solution — also called a saddle connection — is denoted $\breve{\Phi}^*$. Then Theorem 4B may be restated as

THEOREM 4C (Existence of a Saddle Connection).
In all cases consistent with the Standing Assumptions, the system $\mathfrak{S}$ admits a unique saddle connection Φ^*.

This statement calls for several comments.

(i) Our definition of saddle connection specifies a solution going from $p^*_{-\infty}$ to p^*_∞ as z increases, not a solution going in the opposite direction.

(ii) Let $\mathfrak{M}^\triangleleft = \mathfrak{M}(\mathfrak{S}, \mathbf{V}, p^*_{-\infty})$ and $\mathfrak{M}^\triangleright = \mathfrak{M}(\mathfrak{S}, \mathbf{V}, p^*_\infty)$ denote respectively the unstable manifold at $p^*_{-\infty}$ (i.e. the stable manifold for the backward motion) and the stable manifold at p^*_∞. Depending on the choice of domain $\mathbf{V}$, one or both of the saddle points may be boundary points of $\mathbf{V}$ — see S.6 fns.5 and 7 — so that the associated manifolds are properly manifolds with boundary.

(iii) An alternative statement of the Existence result is that for each $z_\Diamond \in \Re$ there is a unique point $p^*_\Diamond = (h^*_\Diamond, \theta^*_\Diamond, \alpha_\Diamond) = (\pi^*_\Diamond, \alpha_\Diamond)$ such that $\Phi^0(z; p^*_\Diamond) \to p^*_{-\infty}$ as $z \to -\infty$ and $\Phi^0(z; p^*_\Diamond) \to p^*_\infty$ as $z \to \infty$, i.e. that $\Phi^*(z) = \Phi^0(z; p^*_\Diamond)$, so that the solution of the *boundary* value problem is also the solution of the *initial* value problem for $p^*_\Diamond$. Also,

(7.2) $$\Phi^\Diamond(z; p^*_\Diamond) = \Phi^0(z - A^{-1}\alpha_\Diamond; p^*_\Diamond) \to p^*_{-\infty} \text{ as } z \to -\infty \text{ and } \to p^*_\infty \text{ as } z \to \infty, \text{ see (6.14)},$$

so that the conditions

$$\Phi^\Diamond(z; p^*_\Diamond) \to p^*_{\pm\infty} \text{ as } z \to \pm\infty \text{ and } \Phi^0(\zeta; p^*_\Diamond) \to p^*_{\pm\infty} \text{ as } \zeta \to \pm\infty$$

are equivalent; and since $\Phi^{\Diamond}(z; p^*_{\Diamond}) = \langle \phi(z; \pi^*_{\Diamond}, A^{-1}\alpha_{\Diamond}), A(z)\rangle$ — see (6.15a) — this is equivalent to $\phi(z; \pi^*_{\Diamond}, z_{\Diamond})$ with $z_{\Diamond} = A^{-1}\alpha_{\Diamond}$ being the 'star' solution $\phi^*(z)$ of S. This again amounts to saying that, for each $z_{\Diamond} \in \Re$, there is a unique point $\pi^*_{\Diamond} = (h^*_{\Diamond}, \theta^*_{\Diamond})$ such that

$$(7.3) \qquad \begin{aligned} &h^*_{\Diamond} = f(\theta^*_{\Diamond}, z_{\Diamond}) = g(\theta^*_{\Diamond}, z_{\Diamond}) \text{ where} \\ &\phi(z; f(\theta^*_{\Diamond}, z_{\Diamond}), \theta^*_{\Diamond}, z_{\Diamond}) \to \pi^*_{-\infty} \text{ as } z \to -\infty \text{ and } \to p^*_{\infty} \text{ as } z \to \infty. \end{aligned}$$

Yet again, using the definitions (6.33) and $(6.33)^{\triangleleft}$, this says that there is for each $z_{\Diamond}$ a unique point $p^*_{\Diamond}$ such that $h^*_{\Diamond} = \tilde{f}(\theta^*_{\Diamond}, \alpha_{\Diamond}) = \tilde{g}(\theta^*_{\Diamond}, \alpha_{\Diamond})$ and $\Phi^*(z) = \Phi^0(z; h^*_{\Diamond}, \theta^*_{\Diamond}, \alpha_{\Diamond})$.

(iv) Our saddle connection belongs, for large negative z (large positive $v = -z$), to the local unstable manifold at $p^*_{-\infty}$ for $\mathfrak{S}$ (the local stable manifold for $\mathfrak{S}^{\leftarrow}$), and for large positive z to the local stable manifold at p^*_{∞} for $\mathfrak{S}$, both of these manifolds being two-dimensional. The complementary one-dimensional manifolds (respectively unstable at $p^*_{-\infty}$ and stable at p^*_{∞}), sometimes called separatrices, do *not* belong to the connection. This is different from the situation considered in some papers, e.g. Bonatti and Dufraine [2003], DeBaggis [1952], where the connection is given by a path which is a separatrix at each of two saddle points.

(v) In Part D we shall consider the response of the system $\mathfrak{S}$ to perturbations of the parameters of the underlying stochastic growth model, in particular the effect of such perturbations on the saddle connection. This question is different from the usual formulation of the problem of structural stability of a dynamical system — see for example DeBaggis [1952], Smale [1960], Chillingworth [1976] — because we are not concerned with the effects of replacing $\mathfrak{S}$ by an arbitrary system which is 'close' according to some reasonable criterion but rather with replacement by a system which corresponds to a growth model which is close to the original one, and this question reduces essentially to the effect of varying the parameters which characterise the growth model. Nevertheless it is helpful to consider the question of the variation of the saddle connection in our model in the wider setting of structural stability as considered by Gordon [1974]. It turns out that in important cases one can obtain quite general results.

The following summary of some points of Gordon's paper uses his terminology, which follows Spivak [1965]; we refer to his paper for various technicalities. Gordon begins by defining the concept of a k-dimensional manifold M in the Euclidian space E^n, (which is what is usually called a submanifold). He then introduces the tangent space M_x of M at a point x, which in the case mainly of concern here, namely $n = 3$, $k = 2$, may be identified with the usual tangent plane. Next, two manifolds of dimensions n_1 and n_2 are said to have *normal intersection* if

(i) $M_1 \cap M_2$ is not void

(ii) $n_1 + n_2 - n = \dim(M_{1x} \cap M_{2x})$ for all $x \in M_1 \cap M_2$.

Next, let $\dot{x} = f(x)$ be a differential system, where $f\colon E^n \mapsto E^n$, and $\dot{x} = g(x)$, $g\colon E^n \mapsto E^n$ a perturbation of f. Assume that the system f has two distinct elementary critical points P_1 and P_2, and let s_1 be the set of all paths converging to P_1^* as $t \to -\infty$, s_2 the set of all paths converging to P_2^* as $t \to \infty$, and s be the set of all paths running from P_1 to P_2 as t runs from $-\infty$ to ∞. For present purposes we let P_1 and P_2 be saddle points and suppose that s is just a single path. Then s is said to be *persistent under perturbation* (Gordon Def.11) if for every $\epsilon > 0$ there exists a δ such that, if $\|f - g\|_{\mathbf{C}_1} < \delta$, the system g has critical points P_1^* and P_2^* and a connecting path s^*, and there exists a mapping ψ of E^n to E^n such that s^* is mapped to s and $\|y - \psi(y)\| \le \epsilon$ for any point $y \in s$. Clearly P_1^* and P_2^* will again be saddle points.[1]

According to Gordon's Theorem 2, (taking into account the uniqueness of s), if s is normal, i.e. if s_1 and s_2 have a normal intersection, then s is persistent under perturbation. It is also shown (his Prop.4, Cor.2) that, if there are normal paths running from P_1 to P_2, there cannot be normal paths running from P_2 to P_1.

For applications to our model we revert to our terminology and take $n = 3$, replace x by p, t by z, f by $\mathfrak{S}$ with domain (say) $\mathbf{V} = \{h > 0, \theta \ge 0, 0 \le \alpha \le \psi_0'\}$ and set $P_1 = p_{-\infty}^*$, $P_2 = p_{\infty}^*$ and $M_1 = \mathfrak{M}^{\triangleleft} = \mathfrak{M}^{\triangleleft}(\mathfrak{S}, \mathbf{V}, p_{-\infty}^*)$, $M_2 = \mathfrak{M}^{\triangleright} = \mathfrak{M}^{\triangleright}(\mathfrak{S}, \mathbf{V}, p_{\infty}^*)$. Strictly, $\mathfrak{M}^{\triangleleft}$ is defined as the set of *starts* of solutions stable at $p_{-\infty}^*$, but clearly this corresponds to the set of *trajectories* (restricted to $\mathbf{V}$) converging to $p_{-\infty}^*$, similarly $\mathfrak{M}^{\triangleright}$ corresponds to the set of trajectories converging to p_{∞}^*. So $n_1 = n_2 = 2$, $n_1 + n_2 - n = 1$. Also s corresponds to the 'star' trajectory $\breve{\Phi}^* = \breve{\Phi}^*(z; z \in \Re) = \mathfrak{M}^{\triangleleft} \cap \mathfrak{M}^{\triangleright}$, which by our Theorem 4C is not void. Next, according to our Prop.15(α,β), $\mathfrak{M}^{\triangleright}$ is the graph of the $\mathbf{C}^1$ function $\tilde{f} = \tilde{f}(\theta, \alpha)$ and $\mathfrak{M}^{\triangleleft}$ is the graph of the $\mathbf{C}^1$ function $\tilde{g} = \tilde{g}(h, \theta)$. Thus at a point $p_\Diamond \in \breve{\Phi}^*$, the tangent plane to $\mathfrak{M}^{\triangleright}$ has the form

$$\{(h, \theta, \alpha)\colon h - h_\Diamond = (\theta - \theta_\Diamond)\tilde{f}_{\Diamond\theta} + (\alpha - \alpha_\Diamond)\tilde{f}_{\Diamond\alpha}\}, \text{ cf.(6.31) and (6.33),}$$

[1] Here $\|\cdot\|_{\mathbf{c}_1}$ refers to distance for the $\mathbf{c}_1$-topology, see Gordon [1974] Def.10. Briefly, if $f\colon E^n \mapsto E^n$ is $\mathbf{C}^1$, let $\|f\| = \sum_i |f_i|$ and $\|f_J\| = \sum_{ij} |\partial f_i / \partial x_j|$ where $|J|$ is the $n \times n$ Jacobian matrix at $x \in E^n$, and set $\|f\|_{\mathbf{c}_1} = \|f\| + \|f_J\|$.

while the tangent plane to $\mathfrak{M}^{\triangleleft}$ has the form

$$\{(h,\theta,\alpha)\colon h-h_{\Diamond}=(\theta-\theta_{\Diamond})\tilde g_{\Diamond\theta}+(\alpha-\alpha_{\Diamond})\tilde g_{\Diamond\alpha}\},\ \text{cf.}(6.31)^{\triangleleft}\text{ and }(6.33)^{\triangleleft}.$$

Consequently the intersection $\mathfrak{M}^{\triangleright}_{p_\Diamond}\cap\mathfrak{M}^{\triangleleft}_{p_\Diamond}$ of the tangent plane s is given by

$$\{(\theta,\alpha)\colon(\theta-\theta_{\Diamond})(\tilde g_{\Diamond\theta}-\tilde f_{\Diamond\theta})+(\alpha-\alpha_{\Diamond})(\tilde g_{\Diamond\alpha}-\tilde f_{\Diamond\alpha})=0\},$$

and since $\tilde g_{\Diamond\theta}>0>\tilde f_{\Diamond\theta}$ this intersection is one-dimensional provided that $\tilde g_{\Diamond\alpha}\neq\tilde f_{\Diamond\alpha}$. If this inequality holds for all points $p_\Diamond\in\tilde\Phi^*$, then Gordon's conditions (i) and (ii) are satisfied and the 'star' trajectory is persistent under perturbation .

In general, we do not have information about the sign of $\tilde g_\alpha-\tilde f_\alpha$. However, along the saddle connection we have (dropping some affixes)

$$\begin{aligned}&h(z)\equiv f[\theta(z),\alpha(z)]\equiv g[\theta(z),\alpha(z)]\text{ for }z\in\Re,\ \text{hence}\\ &[g_\theta(\theta(z))-f_\theta(\theta(z))]\theta'(z)+[g_\alpha(\alpha(z))-f_\alpha(\alpha(z))]\alpha'(z)=0,\\ &\text{with }g_\theta-f_\theta>0,\ \alpha'(z)<0\text{ for }z\in\Re\text{ and }\theta'(z)=[h(z)-1]\theta(z)\neq0\\ &\text{unless }h(z)=1.\end{aligned}\tag{7.4}$$

If $b>1$, and both saddle points are Type 1, then $n<N$, $p^*_\infty=(1,n,0)$, $p^*_{-\infty}=(1,N,\psi_0')$ and in the (h,θ) plane the point $(1,n)$ lies to the left of $(1,N)$ see [B] Figs.3(i,ii,iii), so that overall $\theta(z)$ must decrease as z increases. Also, phase analysis shows that the 'star' path $\breve\phi^*(z)$ cannot leave$(1,N)$ in the 'wrong' direction or approach $(1,n)$ in the 'wrong' direction, i.e. $\breve\phi^*(z)$ must start and finish in the halfspace $\{h<1\}$ as z increases from $-\infty$ to ∞. Thus, if there are no excursions (loops) above the line $\{h=1\}$, then $\theta'(z)<0$ for all z. Contrariwise, at any upcrossing of $\{h=1\}$, $\theta(z)$ has a local minimum with $\theta'=(h-1)\theta=0$ and $\theta''=(h-1)\theta'+\theta\mathfrak{F}=\theta\mathfrak{F}\geq0$, which must be followed by a downcrossing with $\theta(z)$ at a local maximum with $\theta'=0$ and $\theta\mathfrak{F}\leq0$. A priori, there could be several successive such loops. Any conditions on $\mathfrak{F}$ which imply that an upcrossing of $\{h=1\}$ cannot be followed by a downcrossing, or perhaps a condition which implies that $h(z)$ cannot have a maximum with $h>1$, will suffice to exclude such a 'loopy' configuration and so imply that $h(z)<1$, $\theta'(z)<0$ for all z along the star path. We shall not investigate such conditions here, but simply consider directly the assumption that, if $b>1$, $\theta^*(z)$ decreases for all $z\in\Re$ as z increases. A similar discussion applies if p^*_∞ is Type 1 but $p^*_{-\infty}$ is Type 0. Alternatively, if $b<1$, $p^*_{-\infty}$ lies to the left of p^*_∞ in the (h,θ) plane, and in this case we consider the assumption that

$\theta^*(z)$ increases for all $z \in \Re$ as z increases. With either of these assumptions, $(g_\theta - f_\theta)\theta'$ in (7.1) has definite sign so that $(g_\alpha - f_\alpha)\alpha'$ must have the opposite sign, i.e. positive if $b > 1$ so that $g_\alpha - f_\alpha < 0$, negative if $b < 1$ so that $g_\alpha - f_\alpha > 0$. In either case, the possibility that $\tilde{g}_\alpha = \tilde{f}_\alpha$ is excluded, and Gordon's condition (ii) is satisfied with both sides of the equation equal to 1 for all $p \in \mathfrak{M}^\triangleright \cap \mathfrak{M}^\triangleleft$. To sum up, we have

PROPOSITION 16. If along the saddle connection $\breve{\Phi}^*$, there are no crossings of $\{h(z) = 1\}$ at any $z \in \Re$, then the connection is persistent under perturbation.

REMARK. The case $b = 1$ is special. In this case $n = N > 0$, so the points π^*_∞ and $\pi^*_{-\infty}$ coincide at (say) $\pi^* = (1, n)$, and $p^*_\infty = (\pi^*, 0)$, $p^*_{-\infty} = (\pi^*, \psi'_0)$. Now π^* is not a stationary point of S (unless the production function ψ is linear, which is excluded here) so any connection $\breve{\Phi}^*$ must involve a 'loop' of $\breve{\phi}^*$, *either* leaving π^* into $\{h < 1\}$ and passing into $\{h > 1\}$ before returning to π^* as $z \uparrow$ (and $\alpha \downarrow$), *or* leaving π^* into $\{h < 1\}$ and returning via $\{h > 1\}$, (and a priori the possibility of successive loops also exists). In any case, the condition for persistence is not satisfied, which is reasonable if one considers perturbing the value of b.

FINAL REMARK. It may be useful to review some of the main points of the argument so far. Our approach to the search for a solution of a bilateral b.v.p. has been to replace this problem by a pair of unilateral problems having common initial conditions. More precisely, given a suitable $z_\diamondsuit$, we have looked for an initial condition $p^\triangleright_\diamondsuit = (h^\triangleright_\diamondsuit, \theta^\triangleright_\diamondsuit, z_\diamondsuit)$ such that the 'forward' solution $\Phi(z; p^\triangleright_\diamondsuit)$ for $z \geq z_\diamondsuit$ converges to p^*_∞, also for $p^\triangleleft_\diamondsuit = (h^\triangleleft_\diamondsuit, \theta^\triangleleft_\diamondsuit, z_\diamondsuit)$ such that the 'backward' solution $\Phi(z; p^\triangleleft_\diamondsuit)$ for $z \leq z_\diamondsuit$ converges to $p^*_{-\infty}$, and further that these initial conditions may be chosen so as to coincide, say $p^\triangleright_\diamondsuit = p^\triangleleft_\diamondsuit = p^*_\diamondsuit$, so that the solution of the bilateral b.v.p. may be characterised as the solution $\Phi^*(z) = \Phi(z; p^*_\diamondsuit)$ of the initial value problem (i.v.p.) 'through' $p^*_\diamondsuit$. The choice of $p^\triangleright_\diamondsuit$ is equivalent to the choice of initial conditions such that $(h_\diamondsuit, \theta_\diamondsuit) \in \mathfrak{M}^\triangleright(z_\diamondsuit)$ or $h_\diamondsuit = f(\theta_\diamondsuit, z_\diamondsuit)$, while the choice of $p^\triangleleft_\diamondsuit$ is equivalent to $(h_\diamondsuit, \theta_\diamondsuit) \in \mathfrak{M}^\triangleleft(z_\diamondsuit)$ or $h_\diamondsuit = g(\theta_\diamondsuit, z_\diamondsuit)$. The problem is delicate because the manifolds $\mathfrak{M}^\triangleright$ and $\mathfrak{M}^\triangleleft$ (or equivalently $\mathcal{M}^\triangleright$ and $\mathcal{M}^\triangleleft$) are 'thin', i.e. the long-term behaviour of solutions starting in these manifolds is sensitive to initial conditions, see [B] Cor.11 and Remark (2) for details. The possibility of determining a point $(h^*_\diamondsuit, \theta^*_\diamondsuit) \in \mathcal{M}^\triangleright(z_\diamondsuit) \cap \mathcal{M}^\triangleleft(z_\diamondsuit)$ depends on the location of these manifolds (or equivalently of the graphs of the functions $h = f(\theta, z_\diamondsuit)$ and $h = g(\theta, z_\diamondsuit)$) in the plane, and in particular on the fact that $f(\cdot, z_\diamondsuit)$ is a decreasing and $g(\cdot, z_\diamondsuit)$ an increasing function.

While the representation of the solution of a bilateral b.v.p. as the 'common' solution of two i.v.p.s, one forward and one backward, is useful for the existence proof, the appropriate co-ordinates $(h^*_\diamondsuit, \theta^*_\diamondsuit)$ for given $z_\diamondsuit$ cannot in general be determined without first solving the b.v.p. by other means, e.g. by numerical simulation.

Nevertheless, approximations to the solution of the b.v.p., and hence to the optimal log-consumption function, can be given for large $|z|$, i.e. for situations where the economy is either very rich or very poor. Obviously, given the 'Type' of the b.v.p., the co-ordinates of π^*_∞ and $\pi^*_{-\infty}$ are known, and $(h^*(z), \theta^*(z))$ must be close to $(h^*_\infty, \theta^*_\infty)$ for large $z > 0$, similarly $(h^*(z), \theta^*(z))$ must be close to $(h^*_{-\infty}, \theta^*_{-\infty})$ for large $z < 0$ (i.e. $\bar{k}$ close to 0). The imbedding of S_∞ and $S_{-\infty}$ in $\mathfrak{S}$ allows the approximations to be expressed simply by a Taylor expansion to first order, see (6.53) and (6.53)$^\triangleleft$. Explicitly, setting $\alpha = A(z)$ and $\alpha^*_\infty = 0$, (6.53) yields an approximation to $h = f(\theta, z)$ for large $z > 0$; similarly setting $\alpha = A(z)$ and $\alpha^*_{-\infty} = \psi'_0$ yields an approximation to $h = g(\theta, z)$ for large $z < 0$. Incidentally, these results may be useful for numerical calculation of the optimum, since they allow a bilateral b.v.p. with boundary values at $\pm\infty$ to be replaced by a sequence of b.v.p.s on finite intervals.

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