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PARTITION REGULARITY OF A SYSTEM OF DE AND HINDMAN

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Abstract

We prove that a certain matrix, which is not image partition regular over $\mathbb{R}$ near zero, is image partition regular over $\mathbb{N}$. This answers a question of De and Hindman.

1. Introduction

Let A be an integer matrix with only finitely many non-zero entries in each row. We call A *kernel partition regular* (*over* $\mathbb{N}$) if, whenever $\mathbb{N}$ is finitely colored, the system of linear equations $Ax = 0$ has a monochromatic solution; that is, there is a vector x with entries in $\mathbb{N}$ such that $Ax = 0$ and each entry of x has the same color. We call A *image partition regular* (*over* $\mathbb{N}$) if, whenever $\mathbb{N}$ is finitely colored, there is a vector x with entries in $\mathbb{N}$ such that each entry of Ax is in $\mathbb{N}$ and has the same color. We also say that the system of equations $Ax = 0$ or the system of expressions Ax is *partition regular*.

The finite partition regular systems of equations were characterised by Rado [4]. Let A be an $m \times n$ matrix and let $c^{(1)}, \dots, c^{(n)}$ be the columns of A. Then A has the *columns property* if there is a partition $[n] = I_1 \cup I_2 \cup \dots \cup I_t$ of the columns of A such that $\sum_{i \in I_1} c^{(i)} = 0$, and, for each s,

$$\sum_{i \in I_s} c^{(i)} \in \langle c^{(j)} : j \in I_1 \cup \dots \cup I_{s-1} \rangle,$$

where $\langle \cdot \rangle$ denotes (rational) linear span and $[n] = \{1, 2, \dots, n\}$.

Theorem 1 ([4]). *A finite matrix A with integer coefficients is kernel partition regular if and only if it has the columns property.*

The finite image partition regular systems were characterised by Hindman and Leader [3].

In the infinite case even examples of partition regular systems are hard to come by: see [1] for an overview of what is known. De and Hindman [2, Q3.12] asked whether the following matrix was image partition regular.

$$\begin{pmatrix} 1 & & & & & & & \cdots \\ & 1 & 1 & & & & & \cdots \\ 2 & 1 & & & & & & \cdots \\ 2 & & 1 & & & & & \cdots \\ & & & 1 & 1 & 1 & 1 & \cdots \\ 4 & & & 1 & & & & \cdots \\ 4 & & & & 1 & & & \cdots \\ 4 & & & & & 1 & & \cdots \\ 4 & & & & & & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where we have omitted zeroes to make the block structure of the matrix more apparent. De and Hindman's matrix corresponds to the following system of linear expressions.

$$\begin{array}{lcl} x_{21} + x_{22} & \qquad x_{21} + 2y & \qquad y \\ & x_{22} + 2y & \\ & & \\ x_{41} + x_{42} + x_{43} + x_{44} & x_{41} + 4y & \\ & x_{42} + 4y & \\ & x_{43} + 4y & \\ & x_{44} + 4y & \\ & \vdots & \\ x_{2^n1} + \cdots + x_{2^n2^n} & x_{2^n1} + 2^n y & \\ & \vdots & \\ & x_{2^n2^n} + 2^n y & \\ & \vdots & \end{array}$$

A matrix A is called *image partition regular over* $\mathbb{R}$ *near zero* if, for every $\delta > 0$, whenever $(-\delta, \delta)$ is finitely colored, there is a vector x with entries in $\mathbb{R} \setminus \{0\}$ such that each entry of Ax is in $(-\delta, \delta)$ and has the same color. De and Hindman sought a matrix that was image partition regular but not image partition regular over $\mathbb{R}$ near zero. It is easy to show that the above matrix is not image partition regular over $\mathbb{R}$ near zero, so showing that it is image partition regular would provide an example.

The main result of this paper is that De and Hindman's matrix is image partition regular.

Theorem 2. *For any sequence (a_n) of integer coefficients, the system of expressions*

$$\begin{array}{lll}
x_{11} & x_{11}+a_1y & y \\
 & & \\
x_{21}+x_{22} & x_{21}+a_2y & \\
 & x_{22}+a_2y & \\
 & & \\
x_{31}+x_{32}+x_{33} & x_{31}+a_3y & \\
 & x_{32}+a_3y & \\
 & x_{33}+a_3y & \\
 & \vdots &
\end{array} \tag{1}$$

is partition regular.

Taking $a_n = n$ implies that De and Hindman's matrix is image partition regular.

Barber, Hindman and Leader [1] recently found a different matrix that is image partition regular but not image partition regular over $\mathbb{R}$ near zero. Their argument proceeded via the following result on kernel partition regularity.

Theorem 3 ([1]). *For any sequence (a_n) of integer coefficients, the system of equations*

$$\begin{aligned}
x_{11}+a_1y &= z_1 \\
x_{21}+x_{22}+a_2y &= z_2 \\
&\vdots \\
x_{n1}+\cdots+x_{nn}+a_ny &= z_n \\
&\vdots
\end{aligned}$$

is partition regular.

In Section 2 we show that Theorem 2 can almost be deduced directly from Theorem 3. The problem we encounter motivates the proof of Theorem 2 that appears in Section 3.

2. A Near Miss

In this section we show that Theorem 2 can almost be deduced directly from Theorem 3.

Let $\mathbb{N}$ be finitely colored. By Theorem 3 there is a monochromatic solution to the system of equations

$$\begin{aligned} \tilde{x}_{11} - a_1 y &= z_1 \\ \tilde{x}_{21} + \tilde{x}_{22} - 2a_2 y &= z_2 \\ &\vdots \\ \tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - na_n y &= z_n \\ &\vdots \end{aligned}$$

For each n and i, set $x_{ni} = \tilde{x}_{ni} - a_n y$. Then

$$x_{n1} + \cdots + x_{nn} = \tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - na_n y = z_n,$$

and

$$x_{ni} + a_n y = \tilde{x}_{ni},$$

so we have found a monochromatic image for System (1). The problem is that we have not ensured that the variables $x_{ni} = \tilde{x}_{ni} - a_n y$ are positive. In Section 3 we look inside the proof of Theorem 3 to show that we can take (most of) the $\tilde{x}_{ni}$ to be as large as we please.

3. Proof of Theorem 2

The proof of Theorem 3 used a density argument. The (*upper*) *density* of a set $S \subseteq \mathbb{N}$ is

$$d(S) = \limsup_{n\to\infty} \frac{|S \cap [n]|}{n}.$$

The density of a set $S \subseteq \mathbb{Z}$ is $d(S \cap \mathbb{N})$. We call S *dense* if $d(S) > 0$. We shall use three properties of density.

1. If $A \subseteq B$, then $d(A) \le d(B)$.
2. Density is unaffected by translation and the addition or removal of finitely many elements.
3. Whenever $\mathbb{N}$ is finitely colored, at least one of the color classes is dense.

We will also use the standard notation for sumsets and difference sets

$$\begin{aligned} A+B &= \{a+b : a \in A, b \in B\} \\ A-B &= \{a-b : a \in A, b \in B\} \\ kA &= \underbrace{A+\cdots+A}_{k \text{ times}}, \end{aligned}$$

and write $m \cdot S = \{ms : s \in S\}$ for the set obtained from S under pointwise multiplication by m.

We start with two lemmas from [1].

Lemma 4 ([1]). *Let $A \subseteq \mathbb{N}$ be dense. Then there is an m such that, for $n \geq 2/d(A)$, $nA - nA = m \cdot \mathbb{Z}$.*

Lemma 5 ([1]). *Let $S \subseteq \mathbb{Z}$ be dense with $0 \in S$. Then there is an $X \subseteq \mathbb{Z}$ such that, for $n \geq 2/d(S)$, we have $S - nS = X$.*

The following consequence of Lemmas 4 and 5 is mostly implicit in [1]. The main new observation is that the result still holds if we insist that we use only large elements of A. Write $A_{>t} = \{a \in A : a > t\}$.

Lemma 6. *Let A be a dense subset of $\mathbb{N}$ that meets every subgroup of $\mathbb{Z}$, and let m be the least common multiple of $1, 2, \ldots, \lfloor 1/d(A) \rfloor$. Then, for $n \geq 2/d(A)$ and any t,*

$$A_{>t} - nA_{>t} \supseteq m \cdot \mathbb{Z}.$$

Proof. First observe that, for any t, $d(A_{>t}) = d(A)$. Let $n \geq 2/d(A)$, and let $X = A_{>t} - nA_{>t}$. For any $a \in A_{>t}$, we have by Lemma 5 that

$$(A_{>t} - a) - n(A_{>t} - a) = (A_{>t} - a) - (n+1)(A_{>t} - a),$$

and so

$$X = X - A_{>t} + a.$$

Since $a \in A_{>t}$ was arbitrary it follows that $X = X + A_{>t} - A_{>t}$, whence $X = X + l(A_{>t} - A_{>t})$ for all l. By Lemma 4 there is an $m_t \in \mathbb{Z}$ such that, for $l \geq 2/d(A)$, $l(A_{>t} - A_{>t}) = m_t \cdot \mathbb{Z}$. Hence $X = X + m_t \cdot \mathbb{Z}$, and X is a union of cosets of $m_t \cdot \mathbb{Z}$. Since A contains arbitrarily large multiples of m_t, one of these cosets is $m_t \cdot \mathbb{Z}$ itself.

Since $lA_{>t} - lA_{>t}$ contains a translate of $A_{>t}$,

$$1/m_t = d(m_t \cdot \mathbb{Z}) \geq d(A),$$

and $m_t \leq 1/d(A)$. So m_t divides m and

$$A_{>t} - nA_{>t} \supseteq m \cdot \mathbb{Z}. \qquad \square$$

Lemma 6 will allow us to find a monochromatic image for all but a finite part of System (1). The remaining finite part can be handled using Rado's theorem, provided we take care to ensure that it gives us a solution inside a dense color class.

Lemma 7 ([1]). *Let $\mathbb{N}$ be finitely colored. For any $l \in \mathbb{N}$, there is a $c \in \mathbb{N}$ such that $c \cdot [l]$ is disjoint from the non-dense color classes.*

We can now show that System (1) is partition regular.

Proof of Theorem 2. Let $\mathbb{N}$ be r-colored. Suppose first that some color class does not meet every subgroup of $\mathbb{Z}$; say some class contains no multiple of m. Then $m \cdot \mathbb{N}$ is $(r-1)$-colored by the remaining color classes, so by induction on r we can find a monochromatic image. So we may assume that every color class meets every subgroup of $\mathbb{Z}$.

Let d be the least density among the dense color classes, and let m be the least common multiple of $1, 2, \ldots, \lfloor 1/d \rfloor$. Then for any dense color class A, any t and $n \geq 2/d$,

$$A_{>t} - nA_{>t} \supseteq m \cdot \mathbb{Z}.$$

Now let $N = \lceil 2/d \rceil - 1$. We will find a monochromatic image for the the expressions containing only y and x_{ni} for $n \leq N$ using Rado's theorem. Indeed, consider the following system of linear equations.

$$\begin{aligned}
u_1 &= x_{11} & v_{11} &= x_{11} + a_1 y \\
u_2 &= x_{21} + x_{22} & v_{21} &= x_{21} + a_2 y \\
& & v_{22} &= x_{22} + a_2 y \\
& & &\vdots \\
u_N &= x_{N1} + \cdots + x_{NN} & v_{N1} &= x_{N1} + a_N y \\
& & &\vdots \\
& & v_{NN} &= x_{NN} + a_N y
\end{aligned} \tag{2}$$

The matrix corresponding to these equations has the form

$$\begin{pmatrix} B & -I \end{pmatrix}$$

where B is a top-left corner of the matrix corresponding to the expressions of System (1) and I is an appropriately sized identity matrix. It is easy to check that this matrix has the columns property, so by Rado's theorem there is an l such that, whenever a progression $c \cdot [l]$ is r-colored, it contains a monochromatic solution to the equations of System (2).

Apply Lemma 7 to get c with $c \cdot [ml]$ disjoint from the non-dense color classes. Then $mc \cdot [l] \subseteq c \cdot [ml]$ is also disjoint from the non-dense color classes, and by the choice of l there is a dense color class A such that $A \cap (mc \cdot [l])$ contains a solution to System (2). Since the u_n, v_{ni} and y are all in A, y and the corresponding x_{ni} make the first part of System (1) monochromatic.

Now y is divisible by m, so for $n > N$ we have that

$$-na_n y \in A_{>a_n y} - nA_{>a_n y},$$

so there are $\tilde{x}_{ni}$ and z_n in $A_{>a_n y}$ such that

$$-na_n y = z_n - \tilde{x}_{n1} - \cdots - \tilde{x}_{nn}.$$

Set $x_{ni} = \tilde{x}_{ni} - a_n y$. Then

$$x_{n1} + \cdots + x_{nn} = \tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - na_n y = z_n,$$

and

$$x_{ni} + a_n y = \tilde{x}_{ni},$$

for each $n > N$ and $1 \leq i \leq n$. Since $\tilde{x}_{ni}$ and z_n are in A it follows that the whole of System (1) is monochromatic.

It remains only to check that all of the variables are positive. But for y and x_{ni} with $n \leq N$ this is guaranteed by Rado's theorem; for $n > N$ it holds because $\tilde{x}_{ni} > a_n y$. □

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