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RESONANT INTERFACIAL CAPILLARY-GRAVITY WAVES IN THE PRESENCE OF DAMPING EFFECTS

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Abstract: An investigation is made into the waves of small and moderate amplitude which may occur at the interface of two inviscid fluids of different densities. The external forces are those of gravity and surface tension and the waves are due to the resonant interaction between the *Mth* and *Nth* harmonics of the fundamental mode. In contrast to previous studies, damping effects are taken into account. Important parameters in the problem are the velocity and density ratios. A pair of coupled nonlinear Schrodinger-type partial differential equations for the wave amplitudes is derived which model the evolution of the waves, correct up to third order. A wide variety of sinusoidal solutions to the equations is shown to exist, irrespective of the values assigned to the parameters. The stability of these solutions to small modulational perturbations is considered. It is found that when the damping is due to dissipation then the waves are stabilized.

Keywords: Capillary-gravity waves, nonlinear Schrodinger equations, stability, damping

1. Introduction

In this paper we continue an investigation into the evolution, nature and stability of the resonant capillary-gravity waves which occur on the interface of two ideal fluids, each of semi-infinite vertical extent. One of the earliest pieces of work on resonant water-waves was that of Wilton [51]. He considered the waves which occur on the free surface of a fluid and arise due to the interaction between the fundamental mode and its second harmonic. These waves now bear his name. The topic then, perhaps surprisingly, lay relatively dormant until the 1960's when Pierson and Fife [46] reconsidered the Wilton ripple phenomenon by employing the method of multiple scales to derive some power series expansions of the possible wave profiles. In a series of papers Nayfeh used similar methods to study both second and third harmonic resonances. In Nayfeh [38-40] he looked at second and third harmonic resonant waves in the cases of both deep water and finite depth while in [41] he studied second harmonic resonances between an air stream and a train of capillary-gravity waves. At around the same time McGoldrick [33, 34] published a pair of papers which used similar techniques to study second harmonic resonances and also considered briefly the more general interaction between the fundamental and higher order modes. The first study of perfectly general *M-N* resonances appears to be that of Chen & Saffman [3] who employed weakly nonlinear methods to give a fairly complete description of the kinds of waves which may occur on a free surface. Later Jones & Toland [30] cast the problem as one in bifurcation theory and used methods of functional analysis to obtain results which largely confirmed Chen & Saffman's conclusions, a similar analysis was conducted by Okamoto [43]. Subsequently Jones went on to consider the resonant waves which occur at the interface of two fluids. In [22] and [28] he regarded the problem as one in bifurcation theory and deduced results concerning the existence and multiplicity of solutions. In [23] he employed the method of multiple scales to derive a pair of coupled nonlinear Schrodinger equations which describe the evolution of the interface and presented sinusoidal-type solutions to these equations whose stability was analysed.

The stability of wavetrains has a long history but one of the major results of the last fifty years is that of Benjamin & Feir [2] who showed that a nearly monochromatic gravity plane wavetrain of moderate amplitude on deep water is unstable in the presence of small sideband perturbations. Shortly afterwards Zakharov [53] showed how the evolution of such a wavetrain may be described by the nonlinear Schrodinger equation and how this may be used to deduce similar results concerning its stability. In a very interesting paper [48], Segur and his co-workers conducted a careful re-appraisal of the conclusions of Benjamin & Feir. They examined uniform trains consisting of capillary-gravity waves and took into account the effects of dissipation. Their results showed that the Benjamin & Feir instability may be stabilized by any finite amount of dissipation, however small, see also [52].

Other noteworthy studies of interfacial capillary-gravity waves are to be found in [4,5,8]. In [4] a study is made of the waves which arise between fluids of finite and unequal depths. Initially a variational approach is employed in order to derive the Lagrangian formulation of the problem. Both travelling waves, standing waves and mixed waves (ie those which are formed by an interaction between standing and travelling waves) are considered. Subsequently the authors employ the method of multiple scales by means of which they derive an equation of the Davey-Stewartson type in order to analyse the stability of the waves. The topic of [5] is a study of the resonant waves which are formed at the interface of two fluids of infinite vertical extent. It deals with the case which arises when the second harmonic and the fundamental mode are at near resonance and also employs a Lagrangian formulation to derive the normal form of the problem. However, this particular resonance is specifically excluded from this report. A more geometrical approach was taken in [8] which exploits the symmetries inherent in the problem and derives a new Hamiltonian formulation. Most of this work is devoted to non-resonant waves but a singularity leading to nonlinear resonance is identified and investigated numerically.

In this paper we extend these studies and investigate the resonant interfacial capillary-gravity waves which are formed by the interaction of the *Mth* and *Nth* harmonics of the fundamental wave where $M>N$, (for technical reasons the cases $M=2N$ and $M=3N$ are excluded). It is shown how the evolution of these interfaces may be modelled by a pair of coupled nonlinear partial differential equations of a nonlinear Schrodinger type. These generalise the equations appearing contained in [23] which considered the same problem at precise resonance. In this report imperfections or damping effects are taken into account. These imperfections can be caused in a variety of ways. As considered in [47], they could be as a result of dissipation, or since we are considering resonant waves they could be as a result of waves which are close to, but not equal to, the critical wavelength.

A very wide class of solutions to these equations are exhibited, generalising the somewhat restricted set given in [23]. In contrast with that report, no restrictions are imposed on the values of the parameters present in the problem. We then proceed to consider the stability of the interfaces. In the case when the resonance is exact many differing stability portraits were found to be possible, depending on the chosen values of the parameters. However, when dissipation is present the waves are always stable. The presence of damping in general and dissipation in particular would hence seem to be very significant and important in the study of water waves in nature. For instance in [50], Snodgrass *et al* successfully tracked ocean swell across the whole expanse of the Pacific ocean, whereas according to the nondissipative theory of [2] and [53], such waves are intrinsically unstable.

2. Setting the Scene

We shall be concerned with the irrotational motion of two inviscid and incompressible fluids, each of infinite horizontal and vertical extent. We shall choose a three dimensional Cartesian co-ordinate system so that when the system is in its unexcited state the interface of the fluids is given by $z=0$, the lower fluid is moving everywhere with constant velocity V_1 in the x-direction and the upper fluid is moving everywhere with constant velocity V_2 in the x-direction. (Note that we do not assume the V_i's to be positive so the flows need not be uni-directional.) When the system is in its disturbed state the interface is given by $z = H(x, y, t)$. Throughout the paper the subscript 1 is used to denote quantities associated with the lower fluid (that occupying $z \leq 0$) and 2 is used to denote those associated with the upper fluid (that occupying $z \geq 0$). We shall denote the densities of the fluids by ρ_i where $\rho_1 > \rho_2$ so that the heavier fluid is the lower. We define the relative density ρ to be ρ_2 / ρ_1 which clearly lies between zero and unity. Since the motion is irrotational we may introduce the velocity potentials for the sinusoidal disturbances $\varphi_i(x, y, z, t)$ in each fluid. The forces on the fluid are gravity g which acts in the negative z direction and the surface tension S which acts at the interface.

We shall be interested in the motion arising from the interaction of the Mth and Nth modes of the fundamental where M and N are fixed but arbitrary integers. To this end we introduce the notation $E(n) = \exp in(x - \omega t)$, where n is any integer, ω is the fundamental frequency and we have normalized the wavenumber to unity. We also introduce a small positive parameter ε which acts as a measure of the interface steepness and the 'slow variables' $X = \varepsilon x, Y = \varepsilon y, T_0 = \varepsilon t$ together with the 'very slow variable' $T = \varepsilon^2 t$. The notation T_0 might look slightly ugly but it turns out that T_0 drops out of the analysis at a fairly early stage and we work mainly with *T, X* and *Y*. The method of multiple scales is now employed in order to derive the governing equations of the motion. For more detailed descriptions of this procedure, see [6,10,13,19,20,23,26,31,42].

The equations governing the motion are then

$$\nabla^2 \varphi_1 = 0, \quad z \le H, \tag{1}$$

$$\nabla^2 \varphi_2 = 0, \quad z \ge H, \tag{2}$$

$$\nabla \varphi_1 \to 0, \quad z \to -\infty, \tag{3}$$

$$\nabla \varphi_2 \to 0, \quad z \to \infty, \tag{4}$$

$$H_t - \varphi_{jz} + (V_j + \omega) H_x + \varphi_{jx} H_x + \varphi_{jy} H_y = 0, \quad z = H \qquad j = 1,2 \tag{5}$$

$$\begin{aligned} &\rho\, \varphi_{2t} - \varphi_{1t} + \rho\, (V_2 + \omega)\varphi_{2x} - (V_1 + \omega)\varphi_{1x} - \frac{MN}{M+N}(\rho V_2^2 + V_1^2) H + \\ &\frac{\rho}{2}(\varphi_{2x}^2 + \varphi_{2y}^2 + \varphi_{2z}^2) - \frac{1}{2}(\varphi_{1x}^2 + \varphi_{1y}^2 + \varphi_{1z}^2) + \\ &\frac{(\rho V_2^2 + V_1^2)(H_{xx}(1 + H_y^2) + H_{yy}(1 + H_x^2) - 2 H_x H_y H_{xy})}{(M+N)(1 + H_x^2 + H_y^2)^{3/2}} + \rho \varepsilon^2 \delta\, \varphi_2 - \varepsilon^2 \delta\, \varphi_1 = 0, \quad z = H. \end{aligned} \tag{6}$$

In the governing equations the coefficients in the boundary conditions (5) and (6) have been chosen to ensure that the linearised forms are satisfied by the *N*th and *M*th harmonics. Details may be found in [19,23,26].

In these equations, the parameter δ represents damping effects. This damping can be caused by a variety of mechanisms, for instance slight variations in the frequency of the wave maker; the presence of cross waves or the Raman effect. In [37] Miles considered damping due to dissipation. In this case δ is positive and he developed analytic forms for δ based on the various types of dissipation which can occur in deep water systems. It is of course possible to perturb the equations in many other ways to account for damping effects. For instance Dias *et al* [9] present a formulation in which both Bernoulli's and the kinematic boundary conditions are perturbed and justify it rigorously for free surface gravity waves. The work of Zhang & Vinals [54] included a perturbation term of the form $\delta\phi_{zz}$ while in [38] Nayfeh perturbed the wave number. In [24,27] Jones perturbed the term involving H_{xx} in (6) while in [26] he replaced the leading order term C_N in the expansion of H (see (9) below) with $C_N e^{-i\varepsilon\delta t}$ and similarly for C_M.

The solutions $\phi = 0$ and $H = 0$ to the system (1-6) represent horizontal laminar flow with velocities V_1 and V_2 in the *x*-direction in the lower and upper fluids respectively. The interface is given by *z*=0.

The next step in the analysis is to expand the velocity potentials and the interface profile in ascending powers of ε in order to take into account the disturbance due to the two harmonics. Up to appropriate order, the relevant expansions are

$$\varphi_1 = [\ \varepsilon i V_1 C_N + \varepsilon^2 (A_N^{(2)} + z V_1 C_{NX}) + \varepsilon^3 (A_N^{(3)} - iz A_{NX}^{(2)} - \frac{iz^2}{2} V_1 C_{NXX} - \frac{iz}{2} V_1 C_{NYY})\ ] E(N) e^{Nz} +$$

$$[\ \varepsilon i V_1 C_M + \varepsilon^2 (A_M^{(2)} + z V_1 C_{MX}) + \varepsilon^3 (A_M^{(3)} - iz A_{MX}^{(2)} - \frac{iz^2}{2} V_1 C_{MXX} - \frac{iz}{2} V_1 C_{MYY})\ ] E(M) e^{Mz} +$$

$$\varepsilon^2 A(2N)E(2N)e^{2Nz} + \varepsilon^2 A(2M)E(2M)e^{2Mz} +$$

$$\varepsilon^2 A(M+N)E(M+N)e^{(M+N)z} + \varepsilon^2 A(M-N)E(M-N)e^{(M-N)z} + \text{(cc)} \tag{7}$$

$$\varphi_2 = \Big[-\varepsilon i V_2 C_N + \varepsilon^2 (B_N^{(2)} + z V_2 C_{NX}) +$$

$$\varepsilon^3 (B_N^{(3)} + izB_{NX}^{(2)} + \frac{iz^2}{2} V_2 C_{NXX} - \frac{iz}{2} V_1 C_{NYY}) \Big] E(N)e^{-Nz} +$$

$$\Big[-\varepsilon i V_2 C_M + \varepsilon^2 (B_M^{(2)} + z V_2 C_{MX}) +$$

$$\varepsilon^3 (B_M^{(3)} + izB_{MX}^{(2)} + \frac{iz^2}{2} V_2 C_{MXX} - \frac{iz}{2} V_2 C_{MYY}) \Big] E(M)e^{-Mz} +$$

$$\varepsilon^2 B(2N)E(2N)e^{-2Nz} + \varepsilon^2 B(2M)E(2M)e^{-2Mz} +$$

$$\varepsilon^2 B(M+N)E(M+N)e^{-(M+N)z} + \varepsilon^2 B(M-N)E(M-N)e^{-(M-N)z} + cc \tag{8}$$

$$H = \varepsilon\, C_N E(N) + \varepsilon\, C_M E(M) + \varepsilon^2 C(2N)E(2N) + \varepsilon^2 C(2M)E(2M) +$$

$$\varepsilon^2 C(M+N)E(M+N) + \varepsilon^2 C(M-N)E(M-N) + cc. \tag{9}$$

In these expressions it is assumed that the coefficients $C_N, A_N^{(2)}, B_N^{(2)}$ *etc* are functions of the slow variables *X,Y, T* and T_0 only and *cc* stands for complex conjugate. It is assumed that $M \neq 2N, 3N$ since in these cases the quantities *2M, 2N, M+N, M − N* are not all distinct and the expansion (7-9) take on a different form. The next step consists of substituting these expansions into the boundary conditions (5-6) and matching like terms. The linear terms are matched already due to the choice of coefficients in (5) and (6). Matching terms of the form $\varepsilon^2 E(2N)$ yields

$$C(2N) = \frac{N(N+M)(V_1^2 - \rho V_2^2)}{(M-2N)(V_1^2 + \rho V_2^2)} C_N^2 \tag{10}$$

as well as similar expressions for *A(2N)* and *B(2N)*.

Similarly we obtain

$$C(2M) = \frac{M(N+M)(V_1^2 - \rho V_2^2)}{(N-2M)(V_1^2 + \rho V_2^2)} C_M^2 \tag{11}$$

$$C(M+N) = \frac{2(N+M)(\rho V_2^2 - V_1^2)}{(M-2N)(V_1^2 + \rho V_2^2)} C_N C_M \tag{12}$$

$$C(M-N) = \frac{2(M^2 - N^2)(\rho V_2^2 - V_1^2)}{(2N-M)(V_1^2 + \rho V_2^2)} C_N^* C_M, \tag{13}$$

where the asterisk stands for complex conjugate.

When the terms of $\varepsilon^2 E(N)$ are considered the kinematic conditions present us with

$$NA_N^{(2)} = C_{NT_0} + \omega C_{NX} \tag{14}$$

$$NB_N^{(2)} = -C_{NT_0} - \omega C_{NX}. \tag{15}$$

(Note that here and in certain other formulae, subscripts are used to denote both differentiation and also to identify the coefficients. This should cause no confusion.)

Then Bernoulli's condition gives us

$$i\rho V_2 C_{NT_0} + i\rho\omega NB_N^{(2)} + iV_1 C_{NT_0} - i\omega NA_N^{(2)} + i\rho(\omega + V_2)V_2 C_{NX} - i\rho(\omega + V_2)NB_N^{(2)} +$$

$$i(\omega + V_1)V_1 C_{NX} + i(\omega + V_1)NA_N^{(2)} - 2iN\frac{(V_1^2 + \rho V_2^2)}{M+N} C_{NX} = 0. \tag{16}$$

These can only be compatible if

$$C_{NT_0} = s(N,M)C_{NX} \tag{17}$$

where

$$s(N,M) = \frac{(N-M)(V_1^2 + \rho V_2^2)}{2(M+N)(V_1 + \rho V_2)} - \omega \tag{18}$$

which implies

$$A_N^{(2)} = -B_N^{(2)} = \frac{(N-M)(V_1^2+\rho V_2^2)}{2N^2(M+N)(V_1+\rho V_2)} C_{NX} . \tag{19}$$

We now proceed to the consideration of the terms of order $\varepsilon^3 E(N)$.

The kinematic terms give us

$$C_{NT} + iA_{NX}^{(2)} - NA_N^{(3)} + nlt = 0 \tag{20}$$

and

$$C_{NT} - iB_{NX}^{(2)} + NB_N^{(3)} + nlt = 0, \tag{21}$$

where *nlt* stands for nonlinear terms.

Bernoulli's condition yields

$$i\rho NV_2 B_N^{(3)} - iNV_1 A_N^{(3)} - i(V_1+\rho V_2)C_{NT} + \rho(V_2+\omega)B_{NX}^{(2)} - (V_1+\omega)A_{NX}^{(2)} +$$

$$+ \rho B_{NT_0}^{(2)} - A_{NT_0}^{(2)} + \frac{(V_1^2+\rho V_2^2)}{M+N} C_{NXX} + i\delta\rho V_2 C_N + i\delta V_1 C_N + nlt = 0. \tag{22}$$

Consistency within (20-22) then dictates

$$-2i(V_1+\rho V_2)C_{NT} + \rho\omega B_{NX}^{(2)} - \omega A_{NX}^{(2)} + \rho B_{NT_0}^{(2)} - A_{NT_0}^{(2)} +$$

$$\frac{(V_1^2+\rho V_2^2)}{M+N} C_{NXX} + \frac{(M+3N)(V_1^2+\rho V_2^2)}{2N(M+N)} C_{NYY} - i\delta(V_1+\rho V_2)C_N + nlt = 0. \tag{23}$$

We can simplify this using (17-19) and it becomes

$$2i(V_1+\rho V_2)C_{NT} + \left\{ \frac{(\rho+1)(V_1^2+\rho V_2^2)^2(M-N)^2}{4N(V_1+\rho V_2)^2(M+N)^2} - \frac{(V_1^2+\rho V_2^2)}{M+N} \right\} C_{NXX} -$$

$$\frac{(M+3N)(V_1^2+\rho V_2^2)}{2N(M+N)} C_{NYY} + i\delta(V_1+\rho V_2)C_N + nlt = 0. \tag{24}$$

Clearly an analogous equation may be obtained by considering the term of the form *E(M)*.

The resulting equations may be placed in a more congenial form by means of the scalings

$$T \to 2(V_1 + \rho V_2)T,\;\; 2(V_1 + \rho V_2)(M+N)X \to (V_1^2 + \rho V_2^2)X,\;\; \sqrt{2(M+N)}Y \to \sqrt{(V_1^2 + \rho V_2^2)}Y$$

$(V_1 + \rho V_2)\delta \to \delta$ and we also introduce the velocity ratio parameter V which is defined to be $V = V_2/V_1$.

The final result consists of the two equations

$$iC_{NT} + r(N,M)C_{NXX} + u(N,M)C_{NYY} + p(N,M)\left|C_N\right|^2 C_N + q(N,M)\left|C_M\right|^2 C_N + i\delta C_N = 0, \quad \mathfrak{D}a$$

$$iC_{MT} + r(M,N)C_{MXX} + u(M,N)C_{MYY} + p(M,N)\left|C_M\right|^2 C_M + q(N,M)\left|C_N\right|^2 C_M + i\delta C_M = 0 \quad \mathfrak{D}b$$

where

$$r(N,M) = \frac{(\rho+1)(M-N)^2(1+\rho V^2) - 4N(M+N)(1+\rho V)^2}{N} \quad (25)$$

$$u(N,M) = -\frac{3N+M}{N} \quad (26)$$

$$p(N,M) = \frac{\left\{N^3\left\{(8M^2 + MN + 2N^2)(1+\rho^2 V^4) - 6\rho V^2 N(5M+2N)\right\}\right\}}{2(M-2N)} \quad (27)$$

$$q(N,M) = \frac{\left\{N^2\left[M(2N^2 + 10MN - M^2)(1+\rho^2 V^4) - 2\rho V^2(8N^3 + 14MN^2 - 2M^2 N + M^3)\right]\right\}}{(M-2N)}. \quad (28)$$

The pair of equations $\mathfrak{D}$ describes up to third order the nonlinear evolution of an interface between two layered fluids of different but uniform densities which are caused by the interaction of the *Mth* and *Nth* modes of the fundamental wave in the presence of damping effects. They are not valid when *M=2N* or *3N*. For details of these cases see [4,7,25,27,29,38,40,41] .They are generalisations and simplifications of the equations to be found in [23] and should be compared with those in [26] where the perturbations take a slightly different form. A similar pair of equations appears in [17] which contains a study of the stability of two dimensional surface waves on deep water. The equations presented here may be compared with a single perturbed nonlinear Schrodinger equation which was used in [48] in order to model one dimensional dissipative non-resonant waves in a single fluid but these reports provide no derivation. Mei & Hancock [35] derived a similar single equation in the context of a completely different problem in fluid mechanics. Of course it is possible to modify the nonlinear Schrodinger equations in other ways to that given here, for instance by taking the analysis to fourth order , see [11,21].

The single equation has also been used to study problems in other physical systems such as optics, see Ostrovsky [45] and Hasegawa & Kodama [15] or plasmas, see Hasegawa [14]. A number of other authors have made studies of coupled systems of non-linear Schrodinger equations, notably Forrest *et al* [12] who employed them to investigate the propagation of light pulses in nonlinear optical fibre. However they made the restrictive assumption that $p(N,M)=p(M,N)=q(N,M)$ and in addition they only considered one space dimension.

3. Solutions and Wavetrains

When $\delta = 0$ a spatially constant solution of $\mathfrak{D}$ is

$$C_N = D_N \exp[ip(N,M])\,|\,D_N\,|^2\, T + iq(N,M)\,|\,D_M\,|^2\, T] \qquad (29)$$

$$C_M = D_M \exp[ip(M,N)\,|\,D_M\,|^2\, T + iq(N,M)\,|\,D_N\,|^2\, T]\,. \qquad (30)$$

This is the counterpart to the classical Stokes wave solution of the single nonlinear Schrodinger equation representing a spatially uniform train of plane waves.

The corresponding solution for nonzero δ is

$$C_N = D_N e^{-\delta T} \exp\left[ip(N,M)\,|\,D_N\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) + iq(N,M)\,|\,D_M\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) \right] \quad (31)$$

$$C_M = D_M e^{-\delta T} \exp\left[ip(M,N)\,|\,D_M\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) + iq(N,M)\,|\,D_N\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) \right]. \quad (32)$$

Here D_N and D_M are arbitrary constants which can assume any values, positive or negative, real or complex.

The interface disturbance *H* may now be obtained by using (9) and turns out to take the form, up to order ε^2 :

$$H = 2\varepsilon\,|\,D_N\,|\,e^{-\delta T}\cos(Nx - N\omega t + \arg(D_N) + \chi_N) +$$

$$2\varepsilon\,|\,D_M\,|\,e^{-\delta T}\cos(Mx - M\omega t + \arg(D_M) + \chi_M) +$$

$$2\varepsilon^2 \frac{N(M+N)(1-\rho V^2)}{(M-2N)(1+\rho V^2)}\,|\,D_N\,|^2\, e^{-2\delta T}\cos(2Nx - 2N\omega t + 2\arg(D_N) + 2\chi_N) +$$

$$2\varepsilon^2 \frac{M(M+N)(1-\rho V^2)}{(N-2M)(1+\rho V^2)}\,|\,D_M\,|^2\, e^{-2\delta T}\cos(2Mx - 2M\omega t + 2\arg(D_M) + 2\chi_M) +$$

$$4\varepsilon^2 \frac{(M+N)(\rho V^2 - 1)}{(1+\rho V^2)}\,|\,D_N D_M\,|\; e^{-2\delta T}\cos((M+N)x - (M+N)\omega t + \Psi(M+N)) +$$

$$4\varepsilon^2 \frac{(M^2-N^2)(1-\rho V^2)}{(M-2N)(1+\rho V^2)}\,|\,D_N D_M\,|\; e^{-2\delta T}\cos((M-N)x - (M-N)\omega t + \Psi(M-N)). \quad (33)$$

In this expression

$$\chi_N = p(N,M)\,|\,D_N\,|^2\left(\frac{1-e^{-2\delta T}}{2\delta}\right) + q(N,M)\,|\,D_M\,|^2\left(\frac{1-e^{-2\delta T}}{2\delta}\right), \tag{34}$$

$$\chi_M = p(M,N)\,|\,D_M\,|^2\left(\frac{1-e^{-2\delta T}}{2\delta}\right) + q(N,M)\,|\,D_N\,|^2\left(\frac{1-e^{-2\delta T}}{2\delta}\right), \tag{35}$$

and

$$\Psi(M \pm N) = \arg(D_M) \pm \arg(D_N) + (\chi_M \pm \chi_N), \tag{36}$$

together with the appropriate limiting modifications in the case $\delta = 0$.

The solutions should be compared with those given in [20] for the unperturbed equations and Jones [23] for the perturbed ones. The solutions presented in those reports were more restrictive than those which appear here. For certain values of the parameters they sometimes gave no solutions or the solutions which they did give were of a particular nature, for instance in various cases only symmetric or symmetric profiles might exist which could then be destroyed by varying the perturbation parameter. In contrast, the solutions presented here are quite general in the sense that both symmetric and asymmetric profiles exist for all values of the parameters M, N, ρ, V, δ . Some of the wave profiles are depicted in Fig.1.

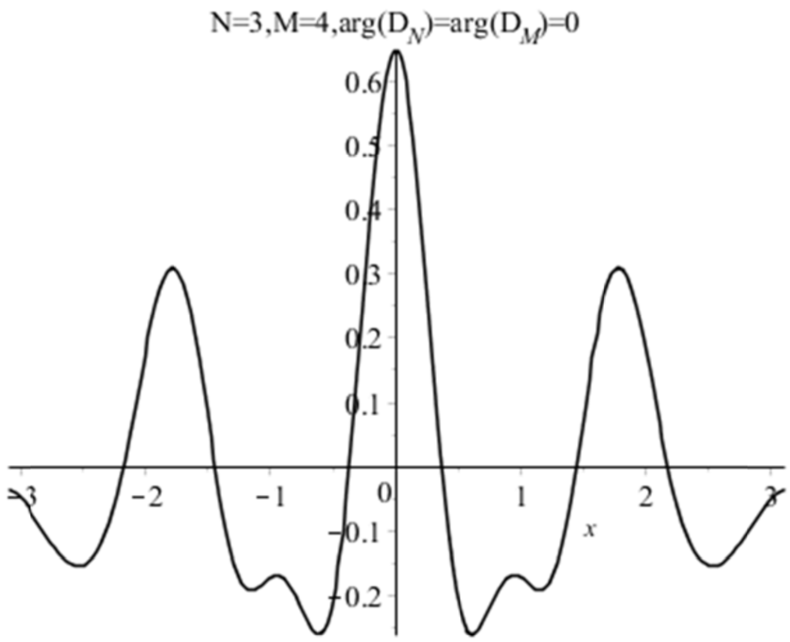

Fig.1(a) Some wave profiles.
In all cases ε=0.1, δ=0, ρ=0.5, V=2, t=0, $|D_N| = |D_M| = 1$.
The horizontal axis is the *x*-variable, the vertical is the expression (33).

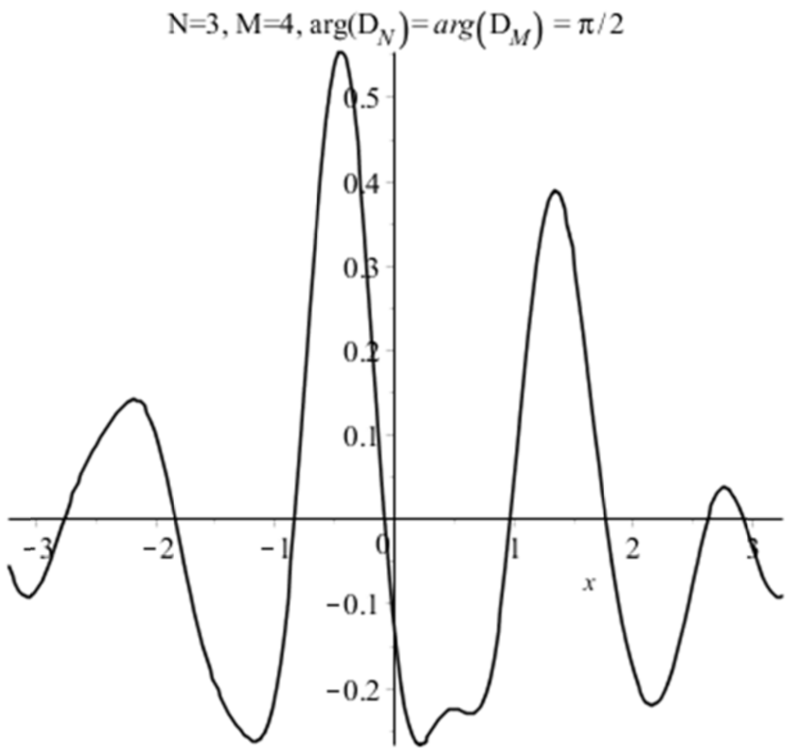

Fig. 1(b)

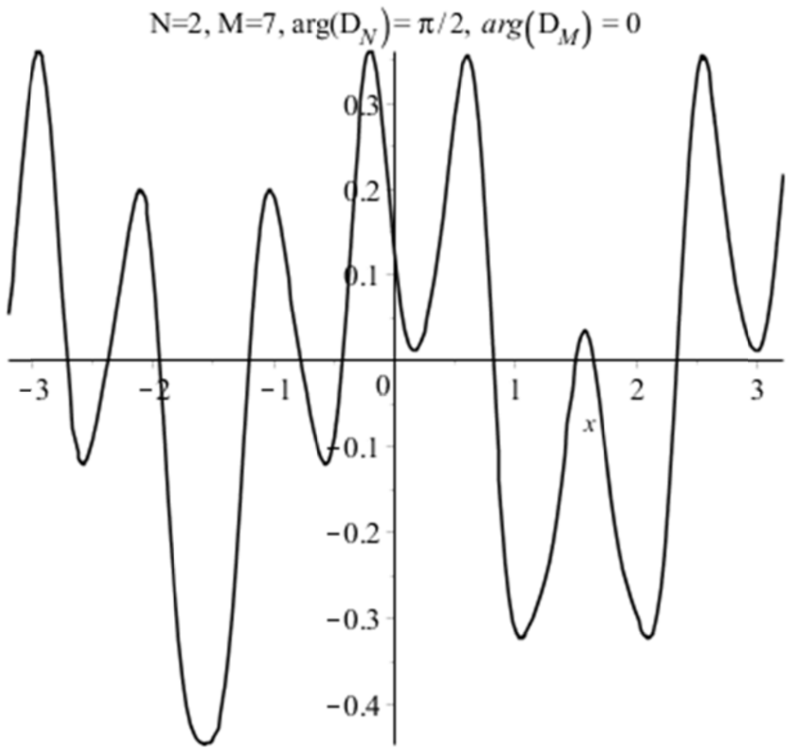
N=2, M=7, arg(D_N)= π/2, arg(D_M) = 0
0.3
0.2
0.1
−3
−2
−1
0
1
2
3
x
−0.1
−0.2
−0.3
−0.4

Fig. 1(c)

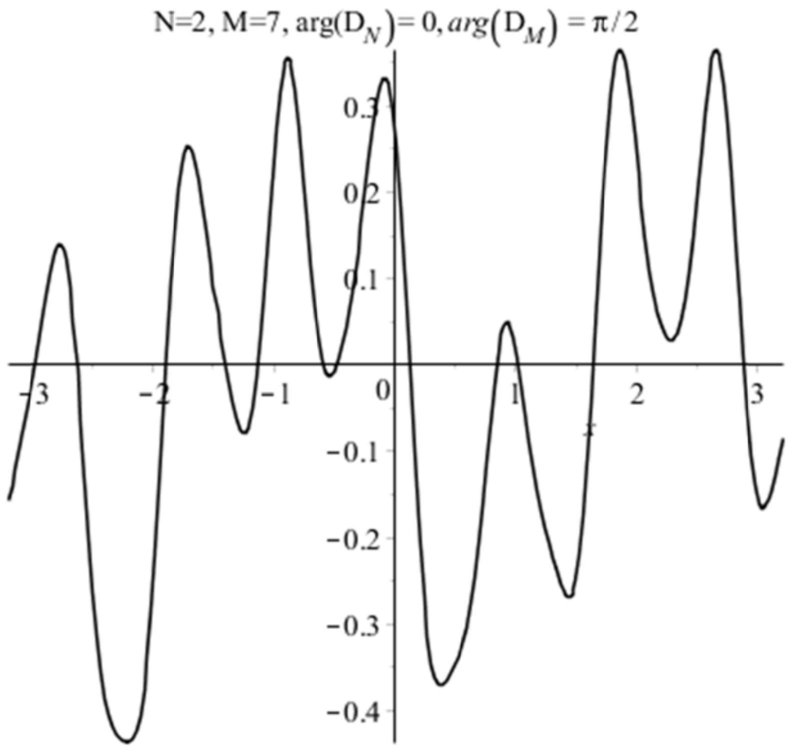
N=2, M=7, arg(D_N)= 0, arg(D_M) = π/2
0.3
0.2
0.1
−3
−2
−1
0
1
2
3
x
−0.1
−0.2
−0.3
−0.4

Fig. 1(d)

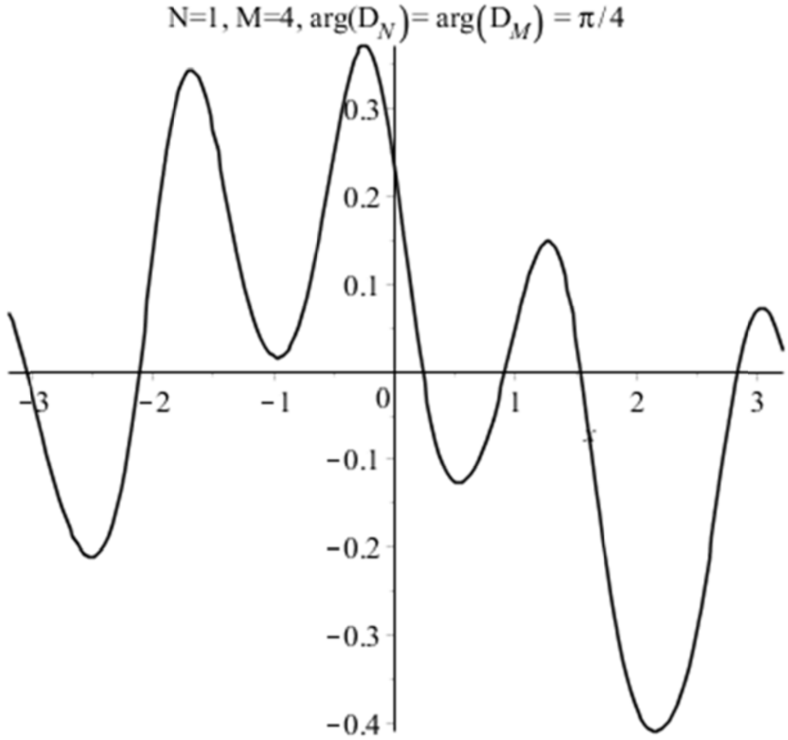
N=1, M=4, arg(D_N)= arg(D_M) = π/4
0.3
0.2
0.1
−0.1
−0.2
−0.3
−0.4
−3
−2
−1
0
1
2
3
x

Fig. 1(e)

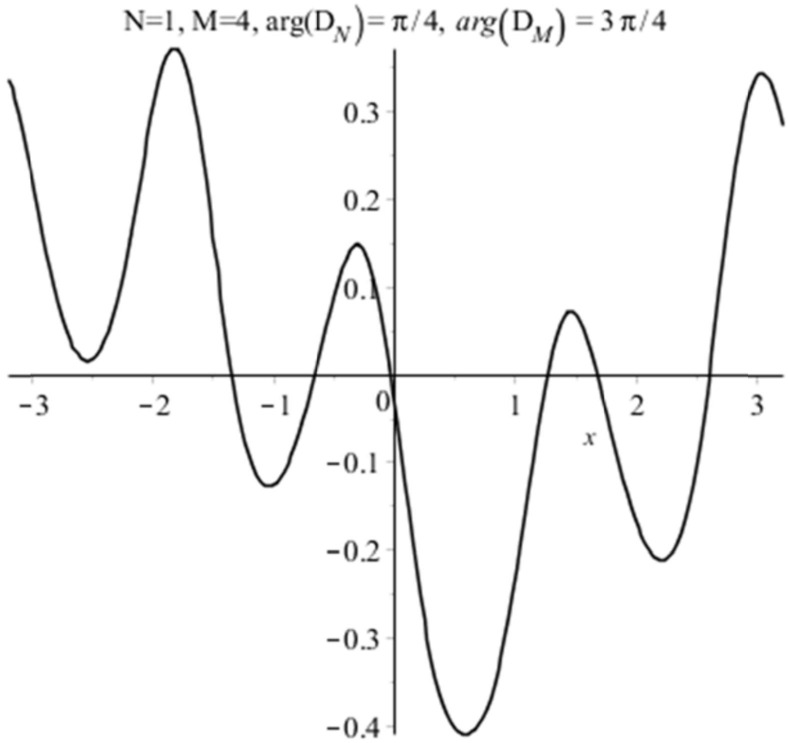
N=1, M=4, arg(D_N)= π/4, arg(D_M) = 3π/4
0.3
0.2
0.1
−0.1
−0.2
−0.3
−0.4
−3
−2
−1
0
1
2
3
x

Fig. 1(f)

4. Interface Stability

Before proceeding to a stability analysis it will prove convenient to recast the system $\mathfrak{D}$ by means of the transformations (see [48]) eq(2.4)):

$$C_N = G_N e^{-\delta T}, \quad C_M = G_M e^{-\delta T}. \tag{37}$$

The equations $\mathfrak{D}$ then become

$$iG_{NT} + r(N,M)G_{NXX} + u(N,M)G_{NYY} + p(N,M)\left|G_N\right|^2 G_N e^{-2\delta T} + q(N,M)\left|G_M\right|^2 G_N e^{-2\delta T} = 0 \tag{38}$$

$$iG_{MT} + r(M,N)G_{MXX} + u(M,N)G_{MYY} + p(M,N)\left|G_M\right|^2 G_M e^{-2\delta T} + q(N,M)\left|G_N\right|^2 G_M e^{-2\delta T} = 0. \tag{39}$$

When $\delta = 0$ the solutions to these are the same as $\mathfrak{D}$ of course, while when $\delta \neq 0$ they take the form

$$G_N = D_N \exp\left(ip(N,M)\,|\,D_N\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) + iq(N,M)\,|\,D_M\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) \right) \tag{40}$$

$$G_M = D_M \exp\left(ip(M,N)\,|\,D_M\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) + iq(N,M)\,|\,D_N\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) \right) \tag{41}.$$

We now perturb these solutions by writing

$$G_N = D_N (1 + a + i\theta) \exp\left(ip(N,M)\,|\,D_N\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) + iq(N,M)\,|\,D_M\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) \right) \tag{42}$$

$$G_M = D_M (1 + b + i\psi) \exp\left(ip(M,N)\,|\,D_M\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) + iq(N,M)\,|\,D_N\,|^2 \left(\frac{1-e^{-2\delta T}}{2\delta} \right) \right) \tag{43}.$$

Here the quantities *a, b, θ* and *ψ* are functions of *X, Y* and *T* and are assumed to be so small that their products can be ignored.

Then substituting (42,43) into (38,39) and taking real and imaginary parts leads to

$$\theta_T - r(N,M)a_{XX} - u(N,M)a_{YY} - 2p(N,M)|D_N|^2 e^{-2\delta T} a - 2q(N,M)|D_M|^2 e^{-2\delta T} b = 0 \tag{45}$$

$$a_T + r(N,M)\theta_{XX} + u(N,M)\theta_{YY} = 0 \tag{46}$$

$$\psi_T - r(M,N)b_{XX} - u(M,N)b_{YY} - 2p(M,N)|D_M|^2 e^{-2\delta T} b - 2q(N,M)|D_N|^2 e^{-2\delta T} a = 0 \tag{47}$$

$$b_T + r(M,N)\psi_{XX} + u(M,N)\psi_{YY} = 0\,. \tag{48}$$

The analysis now proceeds somewhat differently according as *δ* is zero or non-zero *i.e.* whether the interactions are at precise or near –resonance. We deal with these cases separately.

4.1 The undamped case

In this case the equations (45-48) possess constant coefficients so it is logical to assume solutions have the form

$$\begin{pmatrix} a \\ \theta \\ b \\ \psi \end{pmatrix} = \begin{pmatrix} \bar{a} \\ \bar{\theta} \\ \bar{b} \\ \bar{\psi} \end{pmatrix} \exp\{imX + ilY + i\kappa T\}. \tag{49}$$

Substituting this into (45-48) we find the following set of simultaneous equations for $\bar{a}, \bar{\theta}, \bar{b}$ *and* $\bar{\psi}$:

$$i\kappa\bar{\theta} + P_N\bar{a} - 2p(N,M)|D_N|^2\bar{a} - 2q(N,M)|D_M|^2\bar{b} = 0 \tag{50}$$

$$i\kappa\bar{a} - P_N\bar{\theta} = 0 \tag{51}$$

$$i\kappa\overline{\psi} + P_M \overline{b} - 2q(N,M)\,|\,D_N\,|^2\,\overline{a} - 2p(M,N)\,|\,D_M\,|^2\,\overline{b} = 0 \tag{52}$$

$$i\kappa\overline{b} - P_M \overline{\psi} = 0\,. \tag{53}$$

In these equations $P_N = r(N,M)m^2 + u(N,M)l^2$, $P_M = r(M,N)m^2 + u(M,N)l^2$.

The system (50-53) can only be consistent if the following determinant is zero:

$$\begin{vmatrix} \kappa & 0 & P_N & 0 \\ 0 & \kappa & 0 & P_M \\ P_N - 2p(N,M)\,|\,D_N\,|^2 & -2q(N,M)\,|\,D_M\,|^2 & \kappa & 0 \\ -2q(N,M)\,|\,D_N\,|^2 & P_M - 2p(M,N)\,|\,D_M\,|^2 & 0 & \kappa \end{vmatrix}. \tag{54}$$

Expanding this presents us with the equation

$$\kappa^4 + A\kappa^2 + B = 0 \tag{55}$$

where

$$A = P_M\left\{2p(M,N)\,|\,D_M\,|^2 - P_M\right\} + P_N\left\{2p(N,M)\,|\,D_N\,|^2 - P_N\right\} \tag{56}$$

$$B = P_M P_N\left\{[P_N - 2p(N,M)\,|\,D_N\,|^2][P_M - 2p(M,N)\,|\,D_M\,|^2] - 4q(N,M)^2\,|\,D_N D_M\,|^2\right\}. \tag{57}$$

We also introduce the *discriminant* Δ which is defined to be $A^2 - 4B$ and equals

$$\left[P_M\left\{P_M - 2p(M,N)\,|\,D_M\,|^2\right\} - P_N\left\{P_N - 2p(N,M)\,|\,D_N\,|^2\right\}\right]^2 \;+$$

$$16 P_M P_N q(N,M)^2\,|D_N D_M\,|^2\,. \tag{58}$$

Clearly the interfaces can only be stable if all roots of (55) are real and this only happens if either B=0 and $A \le 0$ or $A \le 0$, $B > 0$ and $\Delta \ge 0$. Since there are at least six parameters in the problem: M, N, ρ, V, D_N and D_M, many different stability portraits will be found. However, as we shall see, much useful information can be found. Let us begin with some general observations.

For large l and m:

$$A \approx -P_M^2 - P_N^2,\; B \approx P_M^2 P_N^2 \quad \text{and} \quad \Delta \approx (P_M^2 - P_N^2)^2 \tag{59}$$

which are negative, positive and positive respectively. We thus conclude that for such values the interfaces are *stable*. This is to be expected because we would not expect disturbances with wavenumbers far away from the main flow to have much of an impact. (By the main flow we mean that given by (33)). This is similar to the classical result of Benjamin & Feir [2] which states that wavetrains are unstable if the wavenumbers of the sidebands are not too large. Note however though, we tend to be most interested in interfaces which are stable to small wavenumbers since these are the ones most likely to appear in nature or to be demonstrable in a laboratory.

In our following discussions of stability we shall confine ourselves to the case when $|D_N|$ and $|D_M|$ are both equal to unity. This seems reasonable since we are primarily interested the motion caused by the interactions between the *M th* and the *N th* harmonics rather than either of the harmonics dominating the other. An inspection of the expressions (56-58) shows that the signs of the quantities P_M and P_N will be of some importance. It is clear from (26) that *u(N,M)* and *u(M,N)* are both always negative and so the signs of *r(N,M)* and *r(M,N)* will be of relevance. Let us now consider the cases *V>0* and *V<0* separately.

Consider first *V>0*. A calculation shows that *r(M,N)* is always negative. However *r(N,M)* does change sign and another quantity whose sign is important is $p(M,N)p(N,M)-q(N,M)^2$ since this will have a significant influence on the signs of B and Δ when *l* and *m* (ie the wavenumbers of the perturbations) are small. The signs of the latter two quantities are depicted in Fig. 2 for the case when the density ratio ρ is equal to 0.5. (In fact further calculations show that this picture remains much the same, both qualitatively and quantitatively, for other values of ρ.) We observe that the curves divide the plane into a number of regions some of which we have labelled (I) to (V). We shall choose values of the parameters in the different regions in order to demonstrate the different stability portraits which may arise.

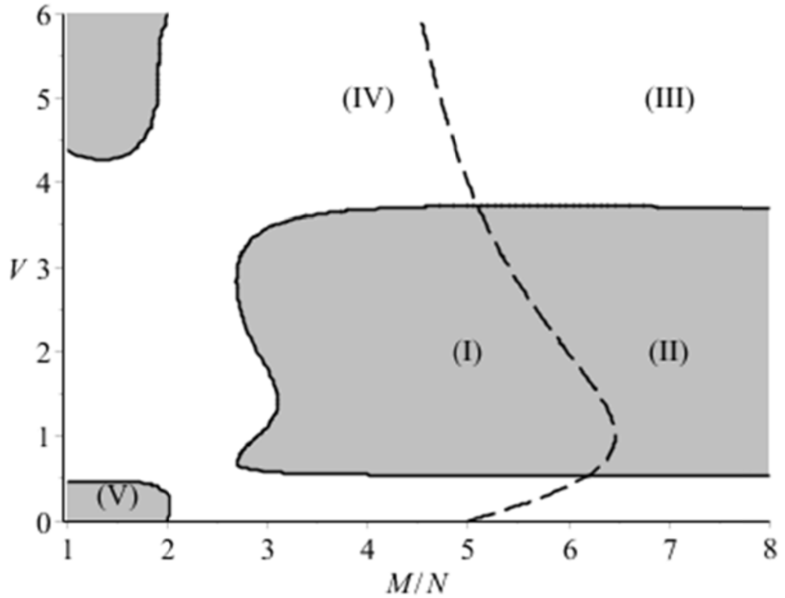

Fig. 2. The solid line shows the zeros of $p(M,N)p(N,M)-q(N,M)^2$. This quantity is positive in all the shaded regions. The dashed line shows the zeros of $r(N,M)$ which is positive everywhere to the right of it.

(i) *M=9, N=2, V=1.5*

Here we are in region (I). Within this region *r(N,M)* is negative and $p(M,N)p(N,M)-q(N,M)^2$ is positive. Calculations further show that the quantity A is negative while B and Δ are both positive. We therefore conclude that the interfaces are *stable* to *all* perturbations. In fact we repeated our calculations for a number of other values in region (I), specifically *M=17, N=3, V=1*; *M=19, N=4, V=3; M=7, N=2, V=1; M=10, N=3, V=3* and in all cases precisely the same conclusions were reached. These particular values of the parameters were chosen because they do not lie close to each other within this region.

It thus appears that all the interfaces within region (I) are stable. This is a most interesting conclusion, essentially identifying a region of stable interfaces which may very roughly be described as $3 < M / N < 5, 1 < V < 4$.

(ii) *M=17, N=3, V=3*

Here we are in region (II) where *r(N,M)* is positive and so P_M will change sign. In addition the expression $p(M,N)p(N,M) - q(N,M)^2$ is positive. Calculations now show that the quantity A is negative while Δ is positive. However B changes sign and its zeroes are shown in Fig. 3. This quantity is negative and hence the interfaces are unstable in the shaded region. The conclusion is that near the origin the interfaces are stable for longitudinal perturbations but unstable for transverse ones. Depending on the precise values of the wavenumbers, the interfaces may or may not be stable for oblique perturbations. As the wavenumbers of the perturbations move away from the origin the interfaces become more and more stable.

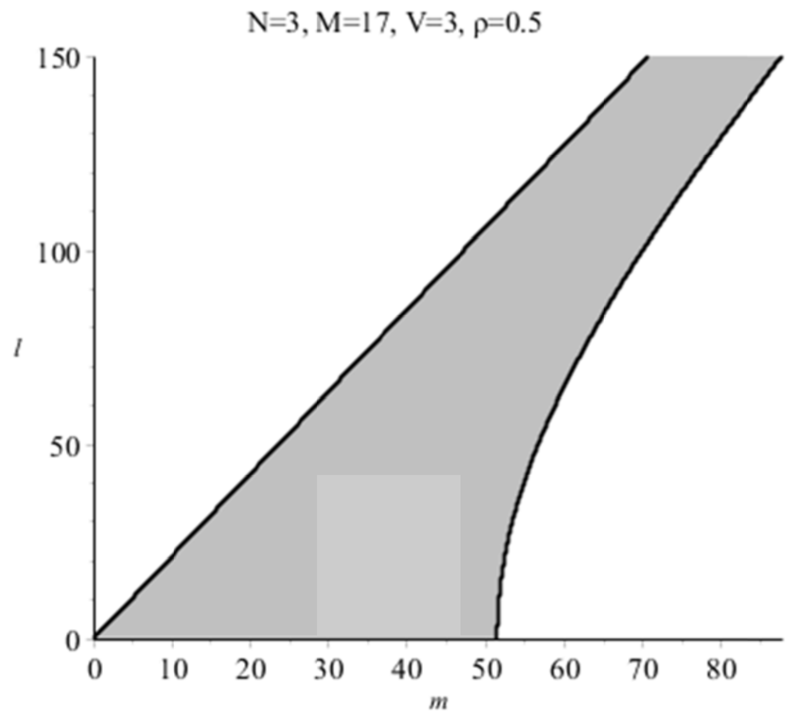

Fig. 3 The quantity B is negative and the waves are unstable in the shaded region.

(iii) $M=7, N=3, V=2$

Here we are in region (IV) where a calculation shows that the quantities $r(N,M)$, $p(M,N)p(N,M)-q(N,M)^2$ and A are all negative while Δ is positive. The zeros of B are depicted in Fig 4.

It is negative in the shaded region and the interfaces are unstable there. We conclude from this figure that the waves are unstable for all perturbations with wavenumbers close to the origin and stable for those with wavenumbers further away.

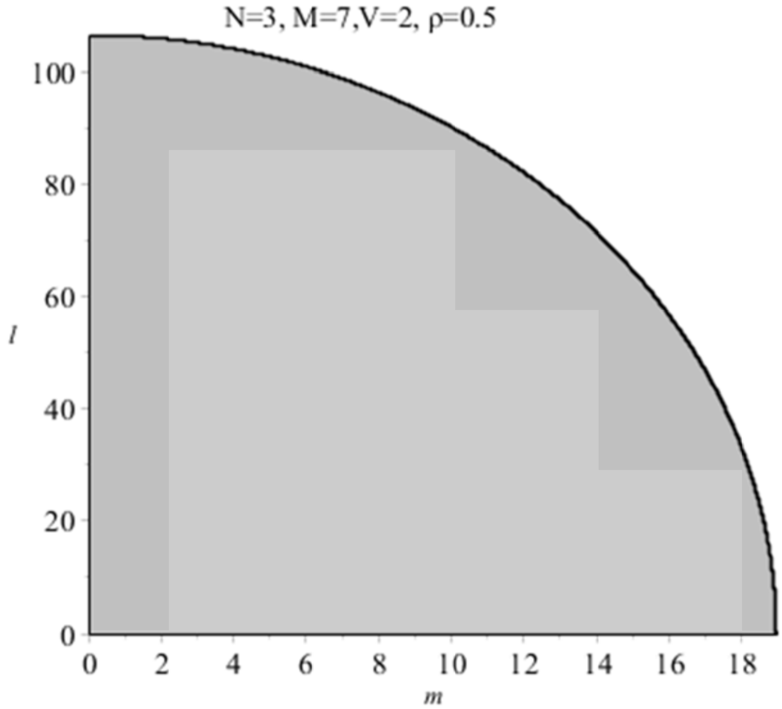

Fig. 4.The quantity B is negative and the waves are unstable in the shaded region.

(iv) M=13, N=2, V=6

This is region (III). Calculations now show that in this region the quantities *r(N,M)* and Δ are both positive while $p(M,N)p(N,M)-q(N,M)^2$ is negative and A and B both change sign. The sign combinations are more complicated than in the previous examples but their configurations are such that the interfaces are always unstable for perturbations with wavenumbers close to the origin so that the stability portrait is actually similar Fig.4.

(v) M=3, N=2, V=0.1

Here we are in the very small shaded region (V) near the origin roughly described by $1 < M/N < 2, 0 < V < 0.5.$ Calculations show that in this region the quantities $p(M,N)p(N,M)-q(N,M)^2$ and Δ are both positive while *r(N,M)* is negative. Further calculations show that A is positive near the origin and negative further away while B is positive everywhere except in a small annulus type region . The conclusions concerning stability are therefore broadly the same as those for Fig. 4.

Further calculations showed that although the precise configurations of the signs of A,B and Δ may be different, the conclusions concerning stability are similar to Fig.4 for other values of the parameters in the unshaded areas of Fig.2 meaning that the interfaces are unstable for perturbations with wavenumbers close to the carrier wave and stable for those further away.

We now briefly deal with the case when $V<0$. For this range of values of the velocity ratio, both the quantities $r(N,M)$ and $r(M,N)$ change sign and thus usually A, B and Δ will as well. The configuration of zeros of these quantities therefore tends to be quite complicated. However in all the cases examined, the conclusion was that the interfaces are unstable for perturbations close to the origin. Hence these are unlikely to be observable in practice and we do not present these results in detail.

An especially interesting and important case is when the density ratio ρ takes the value 0.0012 since this is the value for an air/water system. The diagram analogous to Fig. 2 for this case is given in Fig.5. As can be seen the configurations are qualitatively the same as those for Fig. 2 and the stability results are similar and not pursued in detail here .

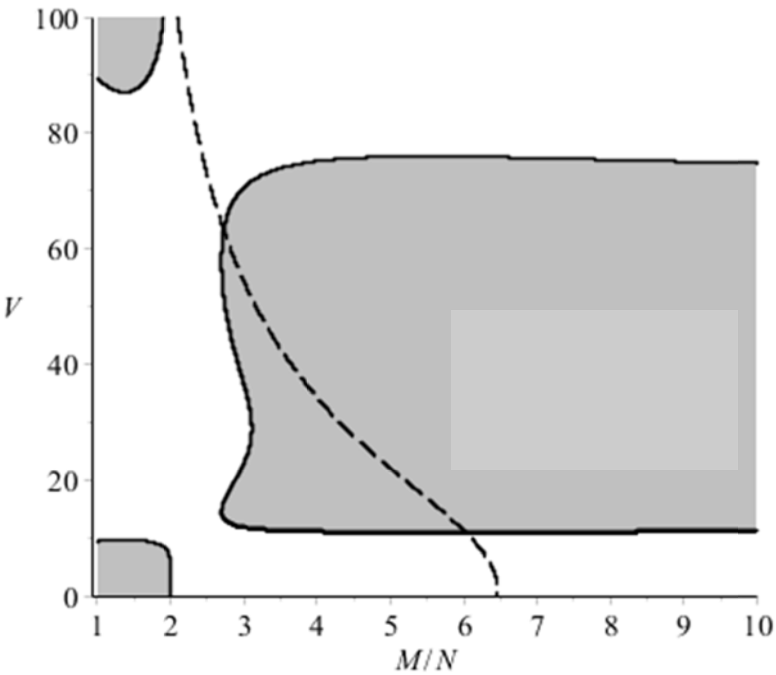

Fig. 5.
This depicts the same quantities as Fig. 2 except the density ρ is now 0.0012 rather than 0.5.

4.2 The dissipative case

In this section we shall assume that the parameter δ is positive. This is certainly the case when the damping is due to dissipation, see [37,47-49]

In this case the equations (45-48) do not have constant coefficients. Therefore we seek solutions of the form

$$\begin{pmatrix} a \\ \theta \\ b \\ \psi \end{pmatrix} = \begin{pmatrix} \hat{a} \\ \hat{\theta} \\ \hat{b} \\ \hat{\psi} \end{pmatrix} \exp\{imX + ilY\}, \tag{60}$$

where $\hat{a}$ *etc* are not constants as before but instead functions of *T*.

Then substituting into (45-48) we obtain the set of equations

$$\hat{\theta}_T + (P_N - 2p(N,M)\,|\,D_N\,|^2\, e^{-2\delta T})\,\hat{a} - 2q(N,M)\,|\,D_M\,|^2\, e^{-2\delta T}\hat{b} = 0 \tag{61}$$

$$\hat{a}_T - P_N\hat{\theta} = 0 \tag{62}$$

$$\hat{\psi}_T + (P_M - 2p(M,N)\,|\,D_M\,|^2\, e^{-2\delta T})\hat{b} - 2q(N,M)\,|\,D_N\,|^2\, e^{-2\delta T}\,\hat{a} = 0 \qquad (63)$$

$$\hat{b}_T - P_M\hat{\psi} = 0. \qquad (64)$$

Then if we differentiate (62,64) *wrt T* and substitute into (61,63), we obtain the two coupled equations

$$\hat{a}_{TT} + P_N\left(P_N - 2p(N,M)\,|\,D_N\,|^2\, e^{-2\delta T}\right)\hat{a} - 2q(N,M)P_N\,|\,D_M\,|^2\, e^{-2\delta T}\,\hat{b} = 0, \qquad (65)$$

$$\hat{b}_{TT} + P_M\left(P_M - 2p(M,N)\,|\,D_M\,|^2\, e^{-2\delta T}\right)\hat{b} - 2q(N,M)P_M\,|\,D_N\,|^2\, e^{-2\delta T}\,\hat{a} = 0. \qquad (66)$$

These equations are of Sturm-Liouville type, see *eg* [1], and we have the following theorem.

Theorem 1 For all sufficiently large *T,* all solutions of the system (65, 66) are oscillatory.

Proof For all sufficiently large *T*, there is a constant c>0 such that the coefficient of $\hat{a}$ in (65) is greater than c^2. In fact since $P_N\left(P_N - 2p(N,M)\,|\,D_N\,|^2\, e^{-2\delta T}\right)$ tends to P_N^2 as *T* tends to infinity, we may take c^2 to be anything less than P_N^2. Clearly

$$-c^2 \sin cT + c^2 \sin cT = 0. \qquad (67)$$

(Another way of looking at this is to observe that $y = \sin cT$ satisfies the equation $y_{TT} + c^2 y = 0.$)

Now let $n\pi/c$ and $(n+1)\pi/c$ be consecutive zeros of $\sin cT$.

We shall assume that *n* is even so that $\sin cT$ is strictly positive between them. We shall prove the theorem by showing that $\hat{a}(T)$ changes sign between $n\pi/c$ and $(n+1)\pi/c$. Suppose that this is not so and that $\hat{a}(T)$ is positive everywhere there. Now multiply (65) by $\sin cT$ and (67) by $\hat{a}(T)$ and subtract.

Then integrating between the consecutive zeros yields

$$\int_{n\pi/c}^{(n+1)\pi/c} (\hat{a}\,sin\,cT + c^2 \hat{a}\,sin\,cT)\;dT +$$

$$\int_{n\pi/c}^{(n+1)\pi/c} \left\{ P_N \left(P_N - 2p(N,M)\,|\,D_N\,|^2\; e^{-2\delta T} \right) - c^2 \right\} \hat{a}\,sin\,cT\;\;dT -$$

$$\int_{n\pi/c}^{(n+1)\pi/c} q(N,M)\,|\,D_N\,|^2 e^{-2\delta T} \hat{b}\,sin\,cT\;\;dT = 0. \qquad (68)$$

Integrating the first integral by parts leads us to

$$\left[\hat{a}_T \sin cT\right]_{n\pi/c}^{(n+1)\pi/c} - \int_{n\pi/c}^{(n+1)\pi/c} c\,\hat{a}_T \cos cT\;\;dT - \left[c\,\hat{a}\cos cT\right]_{n\pi/c}^{(n+1)\pi/c} +$$

$$\int_{n\pi/c}^{(n+1)\pi/c} c\,\hat{a}_T \cos cT\;\;dT = c(\hat{a}((n+1)\pi/c) + \hat{a}(n\pi/c)) \qquad (69)$$

which is positive.

The second integral in (68) is clearly positive, while the third may be made arbitrarily small by choosing *T* sufficiently large. Hence the sum of the three integrals cannot be zero which yields a contradiction hence proving the theorem.

The significance of the preceding theorem is that it allows us to conclude that there is a finite time after which all solutions of (65-66) are oscillatory. Hence, none of the *(l,m)* perturbations in (60) grows forever. This illustrates the fundamental difference between the stability of dispersive and non-dispersive waves for prescribed values of the parameters: the former may or may not be stable according to the particular wave numbers of the perturbations (and their directions) while the latter are always stable.

5. Conclusions

A study has been made of the resonant waves which may occur at the interface of two stratified ideal fluids which are constrained by the twin forces of gravity and surface tension. A pair of coupled nonlinear partial differential equations have been derived for the amplitudes of the *Mth* and *Nth* harmonics. The pair of equations makes allowances for imperfections in the resonance, such as the effects of dissipation. Dissipation is important because it has been shown that in the theory of deep water waves [16,17, 48] any finite amount of dissipation (however small) has a stabilizing effect. This has been used to explain why it has been possible to track ocean waves propagating thousands of kilometres across the Pacific Ocean [18,50] which, in the non-dissipative theory, have been shown to be unstable. Dissipation has also been considered as a possible factor in the early development of freak or rogue waves, see [44,49].

There are five free parameters in the problem: the numbers of the wavemodes M and N; the damping parameter denoted by δ and the velocity and density ratios V and ρ respectively. Irrespective of the prescribed values of the parameters, a large number of solutions to the equations were always shown to exist (31,32). The solutions represent a spatially uniform train of plane waves (33). In all cases the wave profiles could assume a wide variety of configurations including both symmetric and asymmetric profiles. The wave profiles always existed regardless of the presence or absence of damping. It would be interesting to seek other types of solutions to the system $\mathfrak{D}$. For instance it is probably quite easy to generalise our solutions so that they have an X as well as a T dependence, something along these lines may be found in [14,23]. However it is also of interest to determines if there other classes of types of solutions such as solitons or breathers, see [32].

We then proceeded to consider the stability of the solutions to a small sideband perturbations of other waves whose frequency and direction is almost the same as the underlying wave train. When the resonance was exact the calculations were purely algebraic in nature and the question of stability was reduced to determining whether a certain quartic algebraic equation had real solutions or not. Despite the large number of parameters which were present in the problem, some useful conclusions were drawn. It was found that the waves were stable for a region roughly described by $3 < M / N < 5, 1 < V < 4$. Within this region small perturbations to the main wave train (in any direction) did not grow forever but remained bounded. For other values of the parameters (but V positive) the situation was more complicated. The interfaces could be stable or unstable and sometimes the stability was dependent on the direction of the perturbations: for instance they might be stable for transverse perturbations but unstable for longitudinal ones .

It was also found that the stability characteristics did not depend much on the density ratio ρ. However, when the values of the velocity ratio V were taken to be negative (that is when the underlying flows of the upper and lower fluids were are in the opposite directions), the wave profiles tended to be unstable.

The type of instability found here is modulational instability whereby the carrier wave (33) loses its energy to the sidebands which then grow exponentially. This growth continues until the perturbations themselves become so large that their nonlinear self interactions begin to be assume significance. Note that in the case when the carrier wave is stable, some energy is still lost from the original wavetrain, but only a finite amount. When dissipation is considered, (or more precisely when the damping parameter δ is positive) the situation as regards stability is somewhat different. In this case the sideband perturbation can only grow for a finite amount of time, whatever its wavenumbers might be. Theorem 1 shows that any perturbation is ultimately oscillatory in nature. Physically this means that the presence of dissipation ensures that the original wave train can only lose a finite amount of energy to the sideband, the consequence of this is that the sideband stops growing once it has absorbed enough energy to extinguish its instability. The nonlinear Schrodinger type equations derived in this report are only valid for wavetrains of small to moderate amplitude, typically this means a value of ε of less than about 0.15. For waves of large amplitude NLS- type models, with or without dissipation, do not apply, see Melville [36] or Henderson *et al* [16,17]. It would be very interesting to try and develop an analogous theory for waves of finite or large amplitude.

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