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Extremal and probabilistic results for regular graphs

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Declaration

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Statement of conjoint work

The contents of Chapter 1 are well-known results due to various authors not including myself.

I confirm that Chapters 2, 3, and 4 contain work jointly co-authored with Matthew Jenssen, Will Perkins, and Barnaby Roberts and I contributed 25% of this work.

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Abstract

In this thesis we explore extremal graph theory, focusing on new methods which apply to different notions of *regular graph*. The first notion is *d-regularity*, which means each vertex of a graph is contained in exactly d edges, and the second notion is *Szemerédi regularity*, which is a strong, approximate version of this property that relates to pseudorandomness.

We begin with a novel method for optimising observables of Gibbs distributions in sparse graphs. The simplest application of the method is to the hard-core model, concerning independent sets in d-regular graphs, where we prove a tight upper bound on an observable known as the *occupancy fraction*. We also cover applications to matchings and colourings, in each case proving a tight bound on an observable of a Gibbs distribution and deriving an extremal result on the number of a relevant combinatorial structure in regular graphs. The results relate to a wide range of topics including statistical physics and Ramsey theory.

We then turn to a form of Szemerédi regularity in sparse hypergraphs, and develop a method for embedding complexes that generalises a widely-applied method for counting in pseudorandom graphs. We prove an inheritance lemma which shows that the neighbourhood of a sparse, regular subgraph of a highly pseudorandom hypergraph typically inherits regularity in a natural way. This shows that we may embed complexes into suitable regular hypergraphs vertex-by-vertex, in much the same way as one can prove a counting lemma for regular graphs.

Finally, we consider the multicolour Ramsey number of paths and even cycles. A well-known density argument shows that when the edges of a complete graph on kn vertices are coloured with k colours, one can find a monochromatic path on n vertices. We give an improvement to this bound by exploiting the structure of the densest colour, and use the regularity method to extend the result to even cycles.

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Introduction

Extremal combinatorics is a large area of study that is primarily concerned with maximising the value of some parameter over a collection of objects. The field is naturally connected to optimisation; a simple example of an extremal problem is when one has a family of objects and a real-valued function on the objects, and one asks for the maximum value attained by the function. In fact the range of questions considered in extremal combinatorics is more diverse than this example suggests, and a host of specialised techniques exist to investigate such problems. In this thesis we consider three topics with an extremal flavour, each of which concerns some notion of *regular graph*. The rest of this chapter is organised as an introduction to the three topics.

A graph G is a pair (V, E), where V is a set of *vertices* and E is a set of pairs of vertices known as *edges*. Graphs are fundamental discrete structures that model connections or associations between objects. In the pioneering work of Euler [31], a graph was used to represent bridges between islands, and in modern science there are applications as diverse as algorithms, neural connections, and interacting regions of physical systems. A useful generalisation which permits associations between more than two objects is that of a *hypergraph* which, for example, can be useful when enumerating or studying solutions to equations. Given some equation $f(x, y, z) = 0$, one may consider a hypergraph whose vertices are the integers and where edges are triples $\{x, y, z\}$ that form a solution. For more of the basic definitions and notation used here, see [27].

In Section 1.1 we introduce probability distributions known as *Gibbs distri-*

butions on graphs and discuss a variety of relevant extremal problems. We consider maximising and minimising the number of three kinds of substructure in *d-regular* graphs, which are graphs where every vertex is contained in exactly d edges. This topic has a strong connection to statistical physics and it is natural to state some classical extremal problems on graphs in the language of statistical physics. This language also suggests valuable probabilistic interpretations of the results. Section 1.1 serves as an introduction to Chapters 2, 3, and 4.

In Section 1.2 we turn to *Szemerédi regularity* and counting, an important pair of concepts in extremal combinatorics. Szemerédi regularity is a property of a graph (or hypergraph) that captures a form of *pseudorandomness*: it describes how closely the graph behaves to a probabilistic model in which edges are chosen independently at random. The concept is useful primarily because a *counting lemma* states that the number of small subgraphs in a Szemerédi-regular graph is approximately equal to the expected number given by the corresponding random model. We develop an approach to proving a counting lemma for hypergraphs in Chapter 5, and apply a method which relies on a counting lemma for graphs to a Ramsey problem in Chapter 6.

Section 1.3 contains a brief introduction to *Ramsey theory* and covers the necessary preliminaries for Chapter 6. Essentially, Ramsey theory is the study of structure that cannot be avoided by large systems. Ramsey-type theorems state that every large enough example of a system contains a substructure of some prescribed kind. Here we are interested in monochromatic subgraphs of large complete graphs whose edges have been assigned colours.

1.1 Gibbs distributions

In Chapters 2, 3, and 4 we study examples from a family of probability distributions known as *Gibbs distributions* which arise in statistical physics. An expectation over a Gibbs distribution is known as an *observable*, as observing repeated samples from the distribution allows one to approximate such quantities, and we present a novel and general method for optimising observables of Gibbs distributions. In Chapter 2 we focus on applications to extremal problems on independent sets in d-regular graphs, and sketch the general method for optimising observables. In Chapters 3 and 4 we apply the method to matchings and colourings respectively, proving tight upper bounds on relevant observables and resolving several conjectures from extremal combinatorics as a consequence.

1.1.1 The hard-core model

The precise definition of a Gibbs distribution is somewhat technical and we leave the full definition to Chapter 2. Here we focus on an example, the hard-core model, which concerns collections of vertices in a graph that have no edges among them, known as *independent sets*.

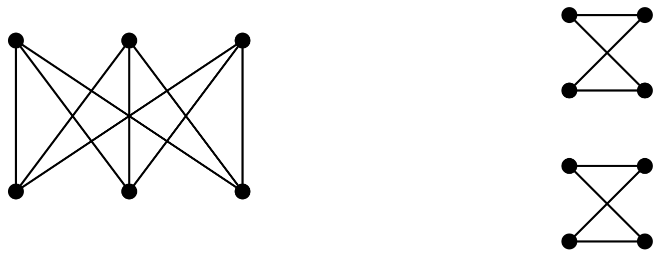
The hard-core model arises in statistical physics as a model of a gas whose particles have a non-negligible size (see [97]). If one constrains the particles to occupy the vertices of a lattice, then having non-negligible size corresponds to imposing that the set of vertices occupied by gas particles is an independent set. Though this constraint might seem rather unnatural, models with such assumptions still exhibit important behaviour such as *phase transitions* that, roughly, are order-disorder transitions akin to the freezing transition in liquids. Of course, the continuous analogue where particles occupy points of $\mathbb{R}^d$ is also of interest, but there much less is known than in the discrete case. The hard-core model we describe below was also rediscovered in the context of communication networks, see for example [60].

Given a graph G and *fugacity* parameter $\lambda > 0$, the *hard-core model* on G is a Gibbs distribution on the set $\mathcal{I}(G)$ of independent sets in G. Each independent set I occurs with probability proportional to $\lambda^{|I|}$, so that

$$\mathbb{P}(I) = \frac{\lambda^{|I|}}{Z_G^{\mathrm{ind}}(\lambda)}\,,$$

where we write $Z_G^{\mathrm{ind}}(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$ for the normalising constant which makes this a probability distribution. Note that we consider the empty set to be an independent set in G, and for $\lambda = 1$ we simply have the uniform distribution on $\mathcal{I}(G)$. The function $Z_G^{\mathrm{ind}}(\lambda)$ is the *partition function* of the model, which also corresponds to the *independence polynomial* from combinatorics.

In general, partition functions encode a wealth of information about Gibbs distributions, and computing or bounding these functions is a major topic in statistical physics [5, 91, 94]. The partition function is also important in extremal combinatorics, where we note for this example that $Z_G^{\mathrm{ind}}(1)$ counts the number of independent sets in G. The fact that many different phenomena can be represented by independent sets in graphs means that bounds on their number appear throughout mathematics. Here, and in Chapter 2, we are motivated by a question of Granville on the d-regular graphs which have the most independent sets (see [3]). We write $K_{d,d}$ for the complete bipartite graph with d vertices in each part, and when $2d$ divides n we let $H_{d,n}$ denote the d-regular, n-vertex graph that is the disjoint union of $n/2d$ copies of $K_{d,d}$, see Figure 1.1 for small examples.



(a) The complete bipartite graph $K_{3,3}$ (b) Two copies of $K_{2,2}$ making $H_{2,8}$

Figure 1.1: Important graphs for Chapters 2–4

Kahn [57] showed that $H_{d,n}$ maximises the total number of independent sets over all d-regular, n-vertex, bipartite graphs with an argument known as the entropy method. He then showed [58] that $K_{d,d}$ maximises $\frac{1}{|V(G)|}\log Z_G^{\mathrm{ind}}(\lambda)$ for $\lambda \geq 1$ over all d-regular bipartite graphs. Noting that

$$Z_{H_{d,n}}^{\mathrm{ind}}(\lambda) = Z_{K_{d,d}}^{\mathrm{ind}}(\lambda)^{n/2d},$$

we see that in Kahn's second result one can use $K_{d,d}$ and $H_{d,n}$ interchangeably.

Galvin and Tetali [44] gave a broad generalisation of Kahn's results to counting *homomorphisms* from a d-regular, bipartite G to any graph H (where this H may contain loops). A homomorphism from G to H is a map $\phi : V(G) \to V(H)$ such that $\phi(u)\phi(v) \in E(H)$ whenever $uv \in E(G)$. The case of H formed of two connected vertices, one with a self-loop, is that of counting independent sets. Via a modification of H and a limiting argument, they proved that for any $\lambda > 0$, the quantity $\frac{1}{|V(G)|}\log Z_G^{\mathrm{ind}}(\lambda)$ is maximised over d-regular bipartite graphs by $K_{d,d}$. Zhao [102] then removed the bipartite restriction in these results for independent sets by reducing the general case to the bipartite case, in particular proving that $H_{d,n}$ has the greatest number of independent sets of any d-regular graph on n vertices.

Theorem 1.1 (Kahn [57, 58], Galvin and Tetali [44], Zhao [102])**.** *For any d-regular graph G and $\lambda > 0$, $\frac{1}{|V(G)|} \log Z_G^{\mathrm{ind}}(\lambda) \le \frac{1}{2d} \log Z_{K_{d,d}}^{\mathrm{ind}}(\lambda)$.*

The method of Chapter 2 gives a strengthening of this theorem. We define the *occupancy fraction* $\alpha_G^{\mathrm{ind}}(\lambda)$ of the hard-core model on a graph G to be the expected fraction of vertices in a random independent set $\mathbf{I}$ drawn from the hard-core model on G. Then

$$\begin{aligned}
\alpha_G^{\mathrm{ind}}(\lambda) &= \frac{1}{|V(G)|}\,\mathbb{E}|\mathbf{I}| \\
&= \frac{1}{|V(G)|}\frac{\sum_{I\in\mathcal{I}(G)}|I|\lambda^{|I|}}{Z_G^{\mathrm{ind}}(\lambda)} \\
&= \frac{\lambda}{|V(G)|}\frac{1}{Z_G^{\mathrm{ind}}(\lambda)}\frac{\partial}{\partial\lambda}Z_G^{\mathrm{ind}}(\lambda) \\
&= \frac{\lambda}{|V(G)|}\frac{\partial}{\partial\lambda}\log Z_G^{\mathrm{ind}}(\lambda)\,,
\end{aligned}$$

and we note that, unlike the partition function, the occupancy fraction is an observable of the model. In Chapter 2 we prove that $K_{d,d}$ maximises the occupancy fraction over d-regular graphs, and show how this implies Theorem 1.1. By linearity of expectation, or simple manipulation of the partition functions, note that $\alpha_{H_{d,n}}^{\mathrm{ind}}(\lambda) = \alpha_{K_{d,d}}^{\mathrm{ind}}(\lambda)$ for any n divisible by $2d$, hence $H_{d,n}$ gives a family of graphs achieving the maximum. Our methods show (in a strong way) that these are the only optimal graphs, for more details see [26]. In Chapter 2 we also discuss the general features of our method for optimising observables of Gibbs distributions, which serve to unify the techniques presented in Chapters 2–4. We also discuss other applications to the occupancy fraction of the hard-core model, including upper and lower bounds for graphs of given girth, and lower bounds for triangle-free graphs and for d-regular, vertex-transitive, bipartite graphs.

1.1.2 The monomer-dimer model

The monomer-dimer model [51] on a graph G is a Gibbs distribution on *matchings* in G. A matching is a set of pairwise disjoint edges in a graph, and in the terminology of the model we say that edges of the matching are *dimers* and unmatched vertices are *monomers*. Matchings are related to independent sets by the following construction. Given a graph G, we form the *line graph* L of G by considering $E(G)$ to be the vertices of L, and stating that e, $f \in E(G)$ are adjacent as vertices in L if and only if e and f share a vertex in G. Then matchings in G correspond exactly to independent sets in L, and the monomer-dimer model on G induces the hard-core model on L.

In fact, the monomer-dimer model has its own rich history, arising in the context of adsorption of oxygen on a tungsten surface [83]; oxygen molecules consist of a pair of atoms that cover neighbouring tungsten atoms on the surface, much like a matching edge. There are some differences between the model here and that most relevant to this chemical problem, but not to first order [51]. The model we discuss arises in a similar chemical context in the work of Guggenheim [49] on mixtures of molecules on lattices.

Instead of using the same definitions as the hard-core model and having to change focus from G to its line graph, we use the following notation for dealing with matchings directly. The *monomer-dimer model* on a graph G at fugacity λ is the distribution on matchings $\mathcal{M}(G)$ in G such that

$$\mathbb{P}(M) = \frac{\lambda^{|M|}}{Z_G^{\mathrm{match}}(\lambda)}\,,$$

where $|M|$ is the number of edges in the matching M, and the partition function or *matching polynomial* is

$$Z_G^{\mathrm{match}}(\lambda) = \sum_{M \in \mathcal{M}(G)} \lambda^{|M|}\,.$$

The occupancy fraction of the monomer-dimer model (also known as the *dimer density*) is the expected fraction of edges in a random matching $\mathbf{M}$ drawn from the model,

$$\alpha_G^{\mathrm{match}}(\lambda) = \frac{1}{|E(G)|}\,\mathbb{E}|\mathbf{M}| = \frac{\lambda}{|E(G)|}\frac{\partial}{\partial\lambda}\log Z_G^{\mathrm{match}}(\lambda)\,.$$

Using the same general method as in Chapter 2, we show in Chapter 3 that $H_{d,n}$ also uniquely maximises the occupancy fraction of the monomer-dimer model in d-regular graphs. This result is rather similar to the the corresponding result for the hard-core model, but despite this similarity, the problem for matchings is somewhat harder. It seems that the entropy method does not give a result analogous to Theorem 1.1 on the level of the partition function, and using our method to optimise over the class of line graphs of d-regular graphs is more challenging than optimising over d-regular graphs.

In Chapter 3 we also discuss conjectures of Kahn [57] and Friedland, Krop, and Markström [38] which state respectively that over d-regular, n-vertex graphs, $H_{d,n}$ maximises the number of independent sets and matchings of every fixed size. Improving on results of Carroll, Galvin, and Tetali [14], and of Ilinca and Kahn [52], we give upper bounds for these problems which are larger than the conjectured values by factors polynomial in n. This proves the *asymptotic upper matching conjecture* of Friedland, Krop, Lundow, and Markström [39]. See [26] for further work on the above conjectures, where we resolve them exactly for a wide range of parameters.

1.1.3 The Potts model

In Chapter 4 we turn to colourings of 3-regular graphs. We consider the q-state Potts model [80] (with no external field), which is a Gibbs distribution on colourings $\sigma : V(G) \to [q]$ of the vertices of G with at most q colours. We call these colourings *q-colourings* and note that we allow monochromatic edges. If a q-colouring does not contain any monochromatic edges it is called *proper*.

The Potts model is a generalisation of the Ising model which relates to ferromagnets and other phenomena of solid-state physics, though here we focus on a combinatorial application of the model. In particular we consider the antiferromagnetic Potts model given by

$$\mathbb{P}(\sigma) = \frac{e^{-\beta m(\sigma)}}{Z_G^q(\beta)}\,,$$

where $\beta > 0$ is an inverse temperature parameter, $m(\sigma)$ is the number of monochromatic edges under the colouring σ, and the partition function is

$$Z_G^q(\beta) = \sum_{\sigma:V(G)\to[q]} e^{-\beta m(\sigma)}\,.$$

An observable of the Potts model analogous to the occupancy fraction is the *internal energy per particle* $U_G^q(\beta)$, given by

$$\begin{aligned} U_G^q(\beta) &= \frac{1}{|V(G)|}\,\mathbb{E}[m(\boldsymbol{\sigma})] \\ &= \frac{1}{|V(G)|}\frac{\sum_\sigma m(\sigma)e^{-\beta m(\sigma)}}{Z_G^q(\beta)} \\ &= -\frac{1}{|V(G)|}\frac{\partial}{\partial\beta}\log Z_G^q(\beta)\,, \end{aligned}$$

where $\boldsymbol{\sigma}$ is a random q-colouring of G from the model. Using the method of Chapter 2 we prove that $K_{3,3}$ minimises and that K_4 maximises $U_G^q(\beta)$ over 3-regular graphs for all q and $\beta > 0$. As a corollary we resolve the case $d = 3$ of a conjecture of Galvin and Tetali [44], showing that $H_{3,n}$ maximises the number of proper colourings over 3-regular, n-vertex graphs.

The work of Chapters 2, 3, and 4 is joint with Matthew Jenssen, Will Perkins, and Barnaby Roberts, and appears in [23, 25].

1.2 Regularity and counting

In Chapter 5 (and very briefly in Chapter 6) we consider Szemerédi regularity, which is a form of pseudorandomness. The basic structure which we consider analogous to a random model is the *regular pair*. The usual definition states that a pair of vertex sets (X, Y) is (ε, d)-regular if for any $X' \subseteq X$ with $|X'| \geq \varepsilon|X|$ and any $Y' \subseteq Y$ with $|Y'| \geq \varepsilon|Y|$, we have

$$e(X', Y') = (d \pm \varepsilon)|X'||Y'|\,,$$

where we note that d must be close to the density of (X, Y), namely $e(X, Y)/|X||Y|$. This 'approximately uniform edge distribution property' is related to a random model in which each edge between X and Y is present independently at random with probability d; in this model the expected value of $e(X', Y')$ is exactly $d|X'||Y'|$. In Chapter 5 we will prefer a different definition of regularity because it generalises more usefully to hypergraphs, which is to insist that the number of copies of C_4 (the four-cycle also known as $K_{2,2}$, see Figure 1.1b) is close to minimal for the given density. In dense graphs this definition is (by an easy application of the Cauchy–Schwarz inequality) equivalent to the one given above up to a polynomial change in ε.

A *regular partition* of a graph is a partition of its vertex set into parts of similar size such that most pairs of parts induce ε-regular bipartite graphs. The celebrated regularity lemma of Szemerédi [95] states that for any $\varepsilon > 0$ and $r_0 \in \mathbb{N}$ there is $r_1 \in \mathbb{N}$ such that every graph G admits an ε-regular partition with between r_0 and r_1 parts. This is useful because it is easy to work with a regular partition. Knowing the densities between all pairs of parts allows us to estimate not only the number of edges but also the number of copies of any given small graph in G; this statement is usually called the *counting lemma*. In this introduction we sketch a method for proving the counting lemma in dense graphs which forms the basis of the more general method for sparse hypergraphs developed in Chapter 5.

Giving the simplest non-trivial example, if X, Y, and Z are disjoint vertex sets in a graph G, each of size n, and each pair of sets induces an (ε, d)-regular bipartite graph, where $\varepsilon < d/2$, then the number of triangles with one vertex in each set is $(d^3 \pm \xi)n^3$, with an error ξ polynomial in ε and εd^{-1}. In applications, one can generally choose ε very small compared to d, and we will never be interested in knowing the formula more precisely.

To sketch a proof of the above assertion we use standard properties of a regular pair. We first observe that, by regularity, for all but at most $4\varepsilon n$ vertices $x \in X$ we have both $\deg(x, Y)$ and $\deg(x, Z)$ in the range $(d \pm \varepsilon)n$. We also note that any vertex is in at most n^2 triangles. Another standard consequence of regularity is that for any typical $x \in X$ (that is, with $\deg(x, Y)$ and $\deg(x, Z)$ in the range $(d \pm \varepsilon)n$), the pair $\big(N(x) \cap Y, N(x) \cap Z\big)$ *inherits regularity* and is $\big(\frac{\varepsilon}{d-\varepsilon}, d\big)$-regular. We now consider $N(x) \cap Y$ and note that similarly, for all but at most

$$\tfrac{2\varepsilon}{d-\varepsilon}|N(x) \cap Y| \leq 2\varepsilon\tfrac{d+\varepsilon}{d-\varepsilon}n \leq 6\varepsilon n$$

vertices $y \in N(x) \cap Y$, the vertices x and y have

$$\big(d \pm \tfrac{\varepsilon}{d-\varepsilon}\big)|N(X) \cap Z| = \big(d \pm \tfrac{\varepsilon}{d-\varepsilon}\big)(d \pm \varepsilon)n$$

common neighbours in Z (and so are in that many triangles); and the atypical vertices in $N(x) \cap Y$ contribute at most $(d + \varepsilon)n \leq n$ triangles each.

Pulling together these bounds there are at least zero and at most $4\varepsilon n^3 + 6\varepsilon n^3$

triangles using an atypical $x \in X$, and using a typical x but an atypical $y \in Y$ respectively. We also have the lower bound

$$(1-4\varepsilon)n \cdot (1-\tfrac{2\varepsilon}{d-\varepsilon})(d-\varepsilon)n \cdot (d-\tfrac{\varepsilon}{d-\varepsilon})(d-\varepsilon)n\,,$$

and the upper bound

$$n \cdot (d+\varepsilon)n \cdot (d+\tfrac{\varepsilon}{d-\varepsilon})(d+\varepsilon)n\,,$$

on the number of triangles using typical x and y. Given $\varepsilon < d/2$ we can bound $\varepsilon/(d-\varepsilon) \leq 2\varepsilon/d$, and hence the above sketch indeed shows that there are $(d^3 \pm \xi)n^3$ triangles with ξ polynomial in ε and ε/d.

In general (counting copies of some small graph H) one considers embedding H into G one vertex at a time, keeping track at each step of the number of ways to extend the next embedding. We need to do two things: argue that most ways of continuing the embedding are 'typical', and that 'atypical' choices do not contribute much. Here 'typical' simply means that neighbourhoods (and common neighbourhoods) of embedded vertices are about the size one would expect from the densities of the regular pairs, and we prove that most vertices are typical using regularity; and the atypical choices do not contribute simply because they are so few.

This sketch is not the only way to prove the counting lemma, however the same approach is also used in a wide variety of applications to find embeddings of large graphs into regular partitions Most notably this is true for the blow-up lemma [63] (which has also been extended to sparse graphs [1]). That this approach works is one of the major reasons why the regularity lemma has been so widely used in extremal graph theory.

If, however, one works with sparse graphs, say with $o(n^2)$ edges in an n-vertex graph, this sketch no longer works. There are versions of the regularity lemma which are useful in sparse graphs [61, 62, 92], but one cannot follow the above approach to count triangles using them. First, in the above sketch neighbourhoods inherit regularity simply because they are large fractions of the original regular pair. In sparse graphs, vertex neighbourhoods will typically be very small fractions of a regular pair. Second, we estimated the contribution of atypical vertices as the worst case which one would get in a complete graph. In sparse graphs, such an error term will no longer be small.

In general, the reason for these problems is that a counting lemma is simply false. However, if one restricts attention to subgraphs G of a suitably well-behaved (typical random, or pseudorandom) *ambient graph* Γ, where $e(G)$ is a large fraction of $e(\Gamma)$, then both problems can be avoided. One can estimate the contribution of atypical vertices using their contribution in Γ, which is small; and while vertex neighbourhoods still do not necessarily inherit regularity, typically they will do so. This statement is made precise by an *inheritance lemma* (see [2, 16]). Given such, one can prove a counting lemma in sparse graphs, and more generally do vertex-by-vertex embedding, much as one can in dense graphs.

In Chapter 5 we develop the equivalent tool for hypergraphs. As is fairly well known (and explained later), even when one works with dense hypergraphs, the hypergraph regularity lemma partitions a hypergraph into sparse parts, so that even for dense hypergraphs it is not obvious that vertex-by-vertex embedding is possible. This is one of the main reasons why hypergraph regularity theory has not had many applications compared to graph regularity theory. In sparse hypergraphs, as with graphs, it is not generally possible to perform vertex-by-vertex embedding (or indeed any embedding) but we prove that this is possible when we are presented with a sparse hypergraph that is a relatively dense subgraph of a well-behaved ambient hypergraph.

The development of these ideas for hypergraphs is a continuation of the work in [1, 2] which concerns inheritance lemmas and blow-up lemmas for sparse graphs. We intend to use these methods to prove a blow-up lemma for sparse hypergraphs, generalising the existing dense hypergraph blow-up lemma of Keevash [59].

This work is joint with Peter Allen and Jozef Skokan.

1.3 Ramsey theory

In Chapter 6 we turn to Ramsey theory. The prototypical problem in Ramsey theory is to find the smallest N such that every graph on at least N vertices contains either a clique or an independent set of size t. Ramsey [82] proved that for each t such an N exists.

It can be natural to phrase these questions in terms of edge-coloured complete graphs; from the description above suppose that the edges of the graph on N vertices are coloured with one colour, and the non-edges are coloured in another. The object we seek is then a monochromatic clique. This formulation also suggests a generalisation to many colours, and we define the *multicolour Ramsey number* $R_k(G)$ of a graph G to be the smallest N such whenever the edges of the complete graph K_N are coloured with k colours, one finds a monochromatic copy of G. In the case $k \geq 3$ determining the value of $R_k(G)$ for a given graph G is often difficult; there are only a few graphs G for which we know $R_k(G)$ exactly and frequently one has to settle for bounds on this quantity.

By way of introduction, we give simple upper and lower bounds on $R_2(K_t)$; a well-known weakening of the argument of Erdős and Szekeres [30], and a probabilistic argument of Erdős [28]. Though the methods used here differ from the methods of Chapter 6, the upper bound is a good, easy introduction to working with edge-coloured complete graphs, and the lower bound is an excellent introduction to the use of probabilistic methods in graph theory. The methods of Chapters 2 to 4 are highly probabilistic but significantly more involved; this introductory result merely serves to prepare the reader for probabilistic calculations.

Theorem 1.2.
$$\frac{t}{\sqrt{2}e}2^{t/2} < R_2(K_t) \leq 4^t\,.$$

Proof. For the upper bound, suppose the edges of a complete graph on at least 4^t vertices are coloured 'red' and 'blue'. Let $N_r(v)$ be the set of vertices connected to v by a red edge, and $N_b(v)$ be those connected by a blue edge. Select an arbitrary vertex $v_1 \in V(G)$ and let V_2 be the larger of $N_r(v_1)$ and $N_b(v_1)$. Continue by picking, for $i = 2, \ldots, 2t-1$ any vertex $v_i \in V_i$ and setting V_{i+1} to be the larger of $V_i \cap N_r(v_i)$ and $V_i \cap N_b(v_i)$. By this construction $|V_i| \geq 2^{2t+1-i}$ for each $i = 2, \ldots, 2t-1$, and the sequence $v_1, \ldots, v_{2t-1}$ has the property that each v_i is connected to all 'forward' v_j with $j > i$ by edges of the same colour. If at least t of the v_i are connected to their forward v_j by red edges we have a red clique of size t, otherwise we obtain a blue clique similarly.

For the lower bound, consider a colouring of the edges of K_N where each edge is coloured red or blue independently at random with probability $1/2$.

The expected number of monochromatic cliques of size t is $2^{1-\binom{t}{2}}\binom{N}{t}$. Set $N = \frac{t}{\sqrt{2}e}2^{t/2}$, and note that the expected expected number of monochromatic cliques of size t is then

$$2^{1-\binom{t}{2}}\binom{N}{t} < 2^{1-\binom{t}{2}}\frac{N^t}{t!} \leq 2^{-\binom{t}{2}}\left(\frac{t2^{(t-1)/2}}{e}\right)^t\left(\frac{e}{t}\right)^t = 1\,,$$

where we use that $t! \geq 2\left(\frac{t}{e}\right)^t$. Since the expectation is less than one, there must exist a colouring with zero monochromatic t-cliques. $\square$

In Chapter 6 we focus on the case where G is the n-vertex path P_n, and the case where n is even and G is the n-vertex cycle C_n. Since P_n is a subgraph of C_n we have $R_k(P_n) \leq R_k(C_n)$, and it is tempting to believe that for fixed k and even n the Ramsey numbers $R_k(P_n)$ and $R_k(C_n)$ are asymptotically equal. This is due to an application of the regularity lemma and the notion of connected matchings pioneered by Łuczak [34, 35, 73], which shows that given a bound on $R_k(P_n)$ and some nontrivial extra conditions (see below), one can derive a bound on $R_k(C_n)$. In practice it is often possible to obtain these nontrivial conditions, and progress on problems such as these often occurs simultaneously.

The method of Łuczak reduces the problem of finding a monochromatic C_n (or P_n) in a k-edge-coloured complete graph to that of finding a monochromatic component containing a sufficiently large matching in a *reduced graph*, which is an almost-complete k-edge-coloured graph. Edges of the reduced graph correspond to sufficiently dense regular pairs (with necessary additional properties we do not elaborate on) in a regular partition of the original graph. One uses e.g. the blow-up lemma [63] to show that for each edge of a matching in the reduced graph, a path may be found spanning the corresponding regular pair in the original graph. If this matching lies in a single connected component these paths may be joined to form a cycle.

We describe an elementary technique for finding monochromatic P_n in large k-edge-coloured complete graphs, and a modification of this technique to suit finding connected matchings that is slightly better for large n. This result is extended to C_n (for even n) via the regularity method.

This work is joint with Matthew Jenssen and Barnaby Roberts, and appears in [22].

2

Independent sets

In this chapter we prove a tight upper bound on the occupancy fraction of the hard-core model in d-regular graphs.

Theorem 2.1. *For all d-regular graphs G and all $\lambda > 0$, we have*

$$\alpha_G^{\mathrm{ind}}(\lambda) \le \alpha_{K_{d,d}}^{\mathrm{ind}}(\lambda) = \frac{\lambda(1+\lambda)^{d-1}}{2(1+\lambda)^d - 1}\,.$$

The maximum is achieved only by disjoint unions of $K_{d,d}$. That is, the quantity $\frac{1}{|V(G)|}\frac{\partial}{\partial\lambda}\log Z_G^{\mathrm{ind}}(\lambda)$ is uniquely maximised by $H_{d,n}$.

The assertion on the derivative of $\frac{1}{|V(G)|}\log Z_G^{\mathrm{ind}}(\lambda)$ is equivalent to the result on the occupancy fraction by a fact from Section 1.1.1,

$$\alpha_G^{\mathrm{ind}}(\lambda) = \frac{\lambda}{|V(G)|}\frac{\partial}{\partial\lambda}\log Z_G^{\mathrm{ind}}(\lambda)\,.$$

We also have $Z_G^{\mathrm{ind}}(0) = 1$ for any graph G, so can derive Theorem 1.1 from the above result.

Proof of Theorem 1.1 using Theorem 2.1. Let G be a d-regular graph. Then

$$\begin{aligned}\frac{1}{|V(G)|}\log Z_G^{\mathrm{ind}}(\lambda) &= \int_0^\lambda \frac{\alpha_G^{\mathrm{ind}}(t)}{t}\,\mathrm{d}t \\ &\le \int_0^\lambda \frac{\alpha_{K_{d,d}}^{\mathrm{ind}}(t)}{t}\,\mathrm{d}t \\ &= \frac{1}{2d}\log Z_{K_{d,d}}^{\mathrm{ind}}(\lambda)\,. \end{aligned}$$ □

In fact, Theorem 2.1 shows that the ratio $Z_{K_{d,d}}^{\mathrm{ind}}(\lambda)^{1/2d}/Z_G^{\mathrm{ind}}(\lambda)^{1/|V(G)|}$ is strictly increasing in λ for any d-regular graph G that is not $H_{d,n}$.

Using a variant of the method used to establish Theorem 2.1, we also prove a lower bound on the occupancy fraction in any d-regular, vertex-transitive, bipartite graph G. Let T_d be the infinite d-regular tree, and let $\alpha_{T_d}^{\mathrm{ind}}(\lambda)$ denote the occupancy fraction of the unique translation invariant hard-core measure on T_d at fugacity λ. One can obtain the occupancy fraction of an infinite graph as a limit of the occupancy fractions of a sequence of finite graphs that converges locally to the infinite graph in the sense of Benjamini and Schramm [8], but for our purposes it suffices to note that $\alpha_{T_d}^{\mathrm{ind}}(\lambda)$ is the solution of the equation

$$\frac{\alpha}{\lambda(1-\alpha)} = \left(\frac{1-2\alpha}{1-\alpha}\right)^d ,$$

see, for example, [9].

Theorem 2.2. *For any d-regular, vertex-transitive, bipartite graph G,*

$$\alpha_G^{\mathrm{ind}}(\lambda) > \alpha_{T_d}^{\mathrm{ind}}(\lambda)\,.$$

The corresponding statement for the normalised log partition function (the integrated version of Theorem 2.2) holds without the condition of vertex transitivity [89]. Theorem 2.2 itself may not hold without vertex transitivity (see Section 5 of [19] for a related discussion about matchings). For values of λ up to some critical $\lambda_c(T_d) = \frac{(d-1)^{d-1}}{(d-2)^d}$, known as the *uniqueness threshold*, there is a unique translation-invariant hard-core measure on T_d. Provided $\lambda \le \lambda_c(T_d)$, it is straightforward to show that the bound in Theorem 2.2 is asymptotically tight for the relevant class of graphs. Indeed, from the results of Weitz [98] any sequence of graphs G_n that converges locally to T_d has occupancy fraction $\alpha_{T_d}(\lambda) + o(1)$ as $n \to \infty$; so for example we can take a sequence of bipartite Cayley graphs of large girth.

The method developed to prove Theorems 2.1 and 2.2 is easy to generalise to other Gibbs distributions and to questions about optimising observables over different classes of graphs. We summarise a very general form of the method in Section 2.5, where we also give the necessary background on Gibbs distributions.

2.1 Related work

The results of Kahn [57], Galvin and Tetali [44], and Zhao [102] (see Theorem 1.1) culminating in the fact that $\frac{1}{|V(G)|} \log Z_G^{\text{ind}}(\lambda)$ is maximised over d-regular graphs by $K_{d,d}$ are based on the entropy method, a powerful tool for the type of problems we address here. Apart from the results mentioned above, see [41] and [81] for surveys of the method. A direct application of the method requires the graph G to be bipartite. Zhao [103] showed that in some, but not all applications of the method, this restriction can be removed by using a 'bipartite swapping trick'. An entropy-free proof of Galvin and Tetali's general theorem on counting homomorphisms was recently given by Lubetzky and Zhao [72]. Our method also does not use entropy, but in contrast to the other proofs it works directly for all d-regular graphs, without a reduction to the bipartite case. The method deals directly with the hard-core model instead of counting homomorphisms and seems to require more problem-specific information than the entropy method.

The technique of writing the expected size of an independent set in two ways (as we do here) was used by Alon [4] in proving lower bounds on the size of an independent set in a graph in which all vertex neighbourhoods are r-colourable. The idea of bounding the occupancy fraction instead of the partition function comes in part from work of Perkins [79] in improving, at low densities, the bounds on matchings of a given size in Ilinca and Kahn [52] and independent sets of a given size in Carroll, Galvin, and Tetali [14]. We study problems of this nature in Chapter 3. The use of linear programming for counting graph homomorphisms appears in Kopparty and Rossman [64], where they use a combination of entropy and linear programming to compute a related quantity, the homomorphism domination exponent, in chordal and series-parallel graphs.

In statistical physics, the analogue of the occupancy fraction in a general spin system is called the *mean magnetisation*; on general graphs it is $\#P$-hard to compute the magnetisation in the ferromagnetic Ising model, the monomer-dimer model, and the hard-core model [91, 94].

2.2 A sketch of the method

To introduce our method, we start by proving Theorem 2.1 under the assumption that G is triangle-free. In what follows, $\mathbf{I}$ will denote the random independent set drawn according to the hard-core model with fugacity λ on a d-regular, n-vertex graph G .

Given some independent set I, we say a vertex v is *occupied* if $v \in I$ and *uncovered* if none of its neighbours are in I: $N(v) \cap I = \emptyset$. Let p_v be the probability v is occupied and q_v be the probability v is uncovered. Note that q is often used to represent $1 - p$ in probabilistic settings, but here we have a different definition. The idea of considering q_v as we do here appears in Kahn's paper [57].

We will show that for every $\lambda > 0$ and any triangle-free G, $\alpha_G^{\mathrm{ind}}(\lambda)$ is maximised by $K_{d,d}$. Recall that by linearity of expectation the occupancy fraction is the same for any number of disjoint $K_{d,d}$'s.

The sketch relies on two key properties of the hard-core model. Firstly, for any vertex v, given that v is uncovered, v is occupied with probability $\lambda/(1+\lambda)$. Let $\mathcal{I} = \{I \in \mathcal{I}(G) : N(v) \cap I = \emptyset\}$ be the set of independent sets in G for which v is uncovered, and let $\mathcal{I}' = \{I \in \mathcal{I}(G) : v \in I\}$ consist of independent sets that contain v. Note that for any independent set I, $v \in I$ implies $N(v) \cap I = \emptyset$ so that $\mathcal{I}' \subseteq \mathcal{I}$. The property holds because each $I \in \mathcal{I}'$ has a corresponding $I \setminus \{v\} \in \mathcal{I} \setminus \mathcal{I}'$, and removing v 'costs' weight λ. When we consider $\mathbf{I}$, the random independent set chosen according to the hard-core model on G, we calculate

$$\mathbb{P}(v \in \mathbf{I} \,|\, v \text{ is uncovered}) = \frac{\sum_{I \in \mathcal{I}'} \lambda^{|I|}}{\sum_{J \in \mathcal{I}} \lambda^{|J|}} = \frac{\sum_{I \in \mathcal{I}'} \lambda^{|I|}}{\sum_{J \in \mathcal{I}'} (\lambda^{|J|-1} + \lambda^{|J|})} = \frac{\lambda}{1+\lambda}\,.$$

The second key property is that for any set $U \subseteq V(G)$ of vertices, conditioned on the fact that every vertex in U is uncovered, the probability that no vertex in U is occupied is $1/Z_F^{\mathrm{ind}}(\lambda)$, where F is the subgraph of G induced by U. This is simply because, conditioned on U being uncovered, $I \cap U$ may be any independent set in U, each such J occurring with probability proportional to $\lambda^{|J|}$. Now $Z_F^{\mathrm{ind}}(\lambda)$ is the normalising constant for this to be a probability distribution, hence the event that no vertex in U is occupied occurs with probability $1/Z_F^{\mathrm{ind}}(\lambda)$.

Letting $\alpha = \alpha_G^{\mathrm{ind}}(\lambda)$ and $n = |V(G)|$, we write

$$\begin{aligned}
\alpha &= \frac{1}{n}\sum_{v\in G} p_v \\
&= \frac{1}{n}\sum_{v\in G} \frac{\lambda}{1+\lambda} q_v && (2.1)\\
&= \frac{\lambda}{1+\lambda}\frac{1}{n}\sum_{v\in G}\sum_{j=0}^{d} \mathbb{P}[j \text{ neighbours of } v \text{ are uncovered}](1+\lambda)^{-j} && (2.2)\\
&= \frac{\lambda}{1+\lambda}\cdot \mathbb{E}[(1+\lambda)^{-Y}]
\end{aligned}$$

where Y is the random variable that counts the number of uncovered neighbours of a uniformly chosen vertex from G, with respect to the random independent set $\mathbf{I}$. Y is an integer valued random variable bounded between 0 and d. Equation (2.1) follows by the first key property, and (2.2) follows from the second. Conditioned on $U = \{u_1, \dots, u_j\}$ all uncovered, where the u_i's are neighbours of v, the probability that none are occupied is $(1+\lambda)^{-j}$. This is where we use the triangle-free assumption: there are no edges in U so the relevant partition function is $(1+\lambda)^j$.

We also have

$$\mathbb{E}Y = \frac{1}{n}\sum_{v\in G}\sum_{u\in N(v)} q_u = d\cdot\frac{1+\lambda}{\lambda}\alpha\,,$$

since each u appears in the double sum exactly d times as G is d-regular. This gives the identity

$$\mathbb{E}Y = d\cdot\mathbb{E}[(1+\lambda)^{-Y}]\,.$$

Now let

$$\alpha^* = \frac{\lambda}{d(1+\lambda)} \cdot \sup_{0 \le Y \le d} \{\mathbb{E}Y : \mathbb{E}Y = d \cdot \mathbb{E}[(1+\lambda)^{-Y}]\}$$

where the supremum is over all distributions of integer-valued random variables Y bounded between 0 and d.

For any λ and d there is a unique distribution Y supported only on 0 and d that satisfies the constraint $\mathbb{E}Y = d \cdot \mathbb{E}[(1+\lambda)^{-Y}]$. We claim that the supremum is uniquely achieved by this distribution. The claim follows from convexity, but we defer details to the proof of a more general statement in Section 2.3. Since the distribution on Y associated to $H_{d,n}$ satisfies the constraint and is supported on 0 and d, it must maximise α. Since disjoint unions of $K_{d,d}$'s are the only graphs whose associated distribution is supported on 0 and d, they uniquely achieve the maximum.

To recap, the method is the following:

(i) Define a random variable Y using randomness in the hard-core model on G and in choosing a random vertex of G. In the proof above, Y was the number of uncovered neighbours of a random vertex.

(ii) Write α in terms of expectations of functions of Y.

(iii) Add constraints that the random variable Y must satisfy for any graph G in our class. In the case above, the constraint was simply that $0 \le Y \le d$.

(iv) Show that the unique maximiser of α over all distributions Y satisfying the constraints is the distribution associated to the extremal graph, and therefore α is maximised by the extremal graph over the subset of distributions Y associated to d-regular graphs.

In Section 2.3 we give the full proof of Theorem 2.1. We prove the lower bound, Theorem 2.2, in Section 2.4.

2.3 Proof of Theorem 2.1

Here we show that Theorem 2.1 holds for all d-regular graphs.

For a vertex $v \in G$ and an independent set I, we define the *local view* at v to be the subgraph of G induced by the neighbours of v which are not adjacent to any vertex in $I \setminus N(v)$. We use the convention $v \notin N(v)$. The vertices in the local view may be uncovered or covered, but if they are covered it must be from another vertex in the local view. In a triangle-free graph the local view is always a set (possibly empty) of isolated vertices. Note that if $v \in I$, then the local view at v is necessarily empty.

Let $\mathbf{L}$ be the random local view at v when we draw I according to the hard-core model and choose vertex v uniformly at random from G. For any graph F, let p_F be the probability that $\mathbf{L}$ is isomorphic to F. Also let $Z_F(\lambda) = Z_F^{\text{ind}}(\lambda)$ be the partition function of the hard-core model on F at fugacity λ. Then we can write $\alpha = \alpha_G^{\text{ind}}(\lambda)$ in two ways:

$$\alpha = \frac{\lambda}{1+\lambda}\,\mathbb{E}\left[\frac{1}{Z_{\mathbf{L}}(\lambda)}\right] \tag{2.3}$$

and

$$\alpha = \frac{\lambda}{d}\,\mathbb{E}\left[\frac{Z'_{\mathbf{L}}(\lambda)}{Z_{\mathbf{L}}(\lambda)}\right], \tag{2.4}$$

where in both equations the expectations are over the random local view $\mathbf{L}$. Equation (2.3) holds because v itself is uncovered if and only if all vertices in the local view at v are unoccupied. Given that the $\mathbf{L}$ is isomorphic to F, the probability that all vertices in the local view are unoccupied is $\frac{1}{Z_F(\lambda)}$. Equation (2.4) follows by counting the expected number of occupied neighbours of v and dividing by d: only vertices in the local view can be occupied, and, given L, the expected number of occupied vertices in the free neighbourhood is $\frac{\lambda Z'_{\mathbf{L}}(\lambda)}{Z_{\mathbf{L}}(\lambda)}$.

Now let

$$\alpha^* = \frac{\lambda}{1+\lambda}\cdot\sup\left\{\mathbb{E}\left[\frac{1}{Z_{\mathbf{L}}(\lambda)}\right] : \frac{d}{1+\lambda}\cdot\mathbb{E}\left[\frac{1}{Z_{\mathbf{L}}(\lambda)}\right] = \mathbb{E}\left[\frac{Z'_{\mathbf{L}}(\lambda)}{Z_{\mathbf{L}}(\lambda)}\right]\right\} \tag{2.5}$$

where the supremum is over all distributions of the random local view $\mathbf{L}$ supported on graphs with at most d vertices. From (2.3) and (2.4), the distribution on $\mathbf{L}$ obtained from G satisfies the constraint above.

We claim that for any $\lambda > 0$, α^* is achieved uniquely by a distribution supported only on the empty graph and the graph consisting of d isolated vertices, $\overline{K_d}$. Theorem 2.1 follows since disjoint unions of $K_{d,d}$'s are the only graphs for which the free neighbourhood can only be the empty set or $\overline{K_d}$. To prove this claim we use the language of linear programming, see for example [13]. Any maximisation problem stated as a linear program has a corresponding dual program which is a minimisation problem, and the *strong duality theorem* states that any feasible value in the dual is an upper bound on the optimal value of the primal. Hence it suffices for us to show that the value corresponding to $K_{d,d}$, which is trivially feasible in the primal, is also feasible in the dual. The duality theorem then implies that the optimal values in both the primal and the dual are given by $K_{d,d}$.

Write $\mathcal{L}_d$ for the set of all graphs on at most d vertices. Equation (2.5) defines a linear program with the decision variables $\{p_F\}_{F\in\mathcal{L}_d}$. We write the linear program in standard form as

$$\begin{aligned} \alpha^* = \max \frac{\lambda}{2(1+\lambda)} & \sum_{F\in\mathcal{L}_d} p_F(a_F+b_F) \text{ s.t.} \\ & \sum_{F\in\mathcal{L}_d} p_F = 1 \\ & \sum_{F\in\mathcal{L}_d} p_F(a_F-b_F) = 0 \\ & p_F \geq 0 \ \ \forall F\in\mathcal{L}_d \end{aligned}$$

where $a_F = \frac{1}{Z_F(\lambda)}$ and $b_F = \frac{(1+\lambda)Z_F'(\lambda)}{dZ_F(\lambda)}$. We can calculate $a_\emptyset = 1$, $b_\emptyset = 0$, $a_{\overline{K}_d} = (1+\lambda)^{-d}$, $b_{\overline{K}_d} = 1$. The solution $p_\emptyset = \frac{1-(1+\lambda)^{-d}}{2-(1+\lambda)^{-d}}$ and $p_{\overline{K}_d} = \frac{1}{2-(1+\lambda)^{-d}}$ is the unique feasible solution supported only on $\emptyset$ and $\overline{K}_d$, and gives the objective value $\frac{\lambda(1+\lambda)^{d-1}}{2(1+\lambda)^d-1}$. Our claim is that this is the unique maximum.

The dual linear program is

$$\begin{aligned} \alpha^* = & \min \frac{\lambda}{2(1+\lambda)}\Lambda_1 \text{ s.t.} \\ & \Lambda_1 + \Lambda_2(a_F-b_F) \geq a_F+b_F \ \ \forall F\in\mathcal{L}_d \end{aligned}$$

where Λ_1, Λ_2 are the dual variables.

Guided by the candidate solution above we set $\Lambda_1 = \frac{2}{2-(1+\lambda)^{-d}}$, and $\Lambda_2 = 1-\Lambda_1$. With these values, the dual constraints corresponding to $F=\emptyset, \overline{K}_d$ hold with equality, and the objective value is $\frac{\lambda}{2(1+\lambda)}\Lambda_1 = \frac{\lambda(1+\lambda)^{d-1}}{2(1+\lambda)^d-1}$. To finish the proof we claim that Λ_1, Λ_2 are feasible for the dual program; which means showing that

$$\Lambda_1 + \Lambda_2(a_F - b_F) > a_F + b_F$$

for all $F \in \mathcal{L}_d \setminus \{\emptyset, \overline{K}_d\}$. Substituting our values of Λ_1, Λ_2, this inequality reduces to

$$\frac{\lambda Z_F'(\lambda)}{Z_F(\lambda) - 1} < \frac{d\lambda(1+\lambda)^{d-1}}{(1+\lambda)^d - 1}\,. \tag{2.6}$$

The left-hand side of (2.6) is the expected size of a random independent set $\mathbf{J}$ drawn from the hard-core model on F conditioned on $\mathbf{J}$ being non-empty. The right-hand side is the same quantity for $\overline{K}_d$.

Inequality (2.6) follows directly from the observation that, over all $F \in \mathcal{L}_d$, the graph $\overline{K}_d$ maximises the ratio of subsequent terms in the polynomial Z_F. Let $r_i = \binom{d}{i}$ be the coefficient of λ^i in $Z_{\overline{K}_d}(\lambda)$ and write $Z_F(\lambda) = 1 + \sum_{i=1}^{d} s_i \lambda^i$. We have $(i+1)r_{i+1} = (d-i)r_i$ and $(i+1)s_{i+1} \le (d-i)s_i$ by counting independent sets of size $i+1$.

To verify (2.6) we show that for each $1 \le k \le d$ the coefficient t_k of λ^k in the polynomial $(\lambda Z_{\overline{K}_d}')(Z_F - 1) - (\lambda Z_F')(Z_{\overline{K}_d} - 1)$ is non-negative. We have

$$\begin{aligned} t_k &= \sum_{i=1}^{k-1} i r_i s_{k-i} + \sum_{i=1}^{k-1} i r_{k-i} s_i \\ &= \sum_{i=1}^{\lfloor k/2 \rfloor} (k-2i)(r_{k-i}s_i - r_i s_{k-i})\,. \end{aligned}$$

Observe that term-by-term the above sum giving t_k is non-negative by comparing the ratio of successive coefficients in $Z_{\overline{K}_d}$ and Z_F. Furthermore, if $Z_F \ne Z_{\overline{K}_d}$ then at least one t_k must be positive, which completes the claim.

To see the optimiser is unique note that there is a unique distribution supported on $\emptyset$ and $\overline{K}_d$ satisfying the primal constraints, and fixing $\Lambda_1 = \alpha^{\text{ind}}_{K_{d,d}}(\lambda)$ in the dual gives a unique feasible value for Λ_2, since its coefficient $a_F - b_F$ takes different signs on $F = \emptyset, \overline{K}_d$. Therefore this is the unique optimal solution in the dual, and since all other dual constraints hold with strict inequality, any primal optimal solution must be supported on $\emptyset$ and $\overline{K}_d$. Disjoint unions of $K_{d,d}$'s are the only graphs whose distributions have this support. This completes the proof of Theorem 2.1.

2.4 Proof of Theorem 2.2

To prove Theorem 2.2 we will use the fact that occupancies of vertices on the same side of a bipartite graph are positively correlated. Recall that we let $\mathbf{I}$ be a random independent set drawn from the hard-core model, and for a vertex v we write p_v for the probability that $v \in \mathbf{I}$.

Lemma 2.3. *Let G be a bipartite graph with bipartition $\mathcal{E} \cup \mathcal{O}$. For any $r \geq 2$, let $u_1, u_2, \ldots, u_r \in \mathcal{E}$. Then*

$$\mathbb{P}[\{u_1, \ldots, u_r\} \subseteq \mathbf{I}] \geq \prod_{i=1}^{r} p_{u_i}$$

in the hard-core model for any λ. Similarly, let $\mathbf{U}$ be the random set of uncovered vertices of G. Then

$$\mathbb{P}[\{u_1, \ldots, u_r\} \subseteq \mathbf{U}] \geq \prod_{i=1}^{r} q_{u_i}$$

Moreover, the inequalities are strict when $\lambda > 0$ and at least two of the u_i's are in the same connected component of G.

The first part of the lemma follows by induction on r from the fact that $\mathbb{P}[u_1, u_2 \in \mathbf{I}] > \mathbb{P}[u_1 \in \mathbf{I}] \cdot \mathbb{P}[u_2 \in \mathbf{I}]$ when u_1, u_2 are in the same connected component and in the same part of the bipartition of G. In [97] this is shown to be a consequence of the FKG inequality; see also [36] and Corollary 1.5 of [6]. An intuitive reason for this fact (which can be turned into a rigorous argument using Weitz's tree [98]), is that conditioning on the event that a vertex v is occupied forbids its neighbours from being in the independent set; conditioning on the event that v is not occupied increases the probability each of its neighbours are occupied, and these effects propagate through the bipartite graph.

To prove the second part of the lemma, note that $p_{u_i} = \frac{\lambda}{1+\lambda} q_{u_i}$, and for $u_1, \ldots, u_r \in \mathcal{E}$, $\mathbb{P}[\{u_1, \ldots, u_r\} \subseteq \mathbf{I}] = (\frac{\lambda}{1+\lambda})^r \mathbb{P}[\{u_1, \ldots, u_r\} \subseteq \mathbf{U}]$, since there are no edges between the u_i's. Then the desired inequality follows from the first part of the lemma.

Proof of Theorem 2.2. Write α for $\alpha_G^{\mathrm{ind}}(\lambda)$. By vertex transitivity, for all v, $p_v = \alpha$ and $q_v = \frac{1+\lambda}{\lambda}\alpha$. Fix a vertex v and let Y be the number of uncovered neighbours of v. For $u \in N(v)$ let Y_u be the indicator random variable that u is uncovered.

$$\begin{aligned}\alpha &= \frac{\lambda}{1+\lambda}\mathbb{E}[(1+\lambda)^{-Y}] \\ &= \frac{\lambda}{1+\lambda}\mathbb{E}[(1+\lambda)^{-\sum_{u\in N(v)} Y_u}] \\ &= \frac{\lambda}{1+\lambda}\Big(\alpha + (1-\alpha)\,\mathbb{E}\Big[(1+\lambda)^{-\sum_{u\in N(v)} Y_u}\Big| v \notin \mathbf{I}\Big]\Big)\,,\end{aligned}$$

hence

$$\frac{\alpha}{\lambda(1-\alpha)} = \mathbb{E}\Big[(1+\lambda)^{-\sum_{u\in N(v)} Y_u}\Big| v \notin \mathbf{I}\Big]\,.$$

Now for $u \in N(v)$, let $\tilde{Y}_u$ be the indicator that u is uncovered, conditioned on the event $\{v \notin I\}$. For each u, $\tilde{Y}_u$ has a Bernoulli(p) distribution, where $p = \frac{1+\lambda}{\lambda}\frac{\alpha}{1-\alpha}$, and by Lemma 2.3 applied to $G \setminus v$, the $\tilde{Y}_u$'s are positively correlated. This gives

$$\begin{aligned}\frac{\alpha}{\lambda(1-\alpha)} &= \mathbb{E}[(1+\lambda)^{-\sum_{u\sim v}\tilde{Y}_u}] \\ &> \prod_{u\sim v}\mathbb{E}[(1+\lambda)^{-\tilde{Y}_u}] \\ &= \left(1 - p + \frac{p}{1+\lambda}\right)^d = \left(\frac{1-2\alpha}{1-\alpha}\right)^d\,.\end{aligned}$$

The function $\frac{\alpha}{\lambda(1-\alpha)}$ is increasing in α, the function $\left(\frac{1-2\alpha}{1-\alpha}\right)^d$ is decreasing in α, and the two functions are equal at $\alpha = \alpha_{T_d}(\lambda)$, so we conclude that $\alpha > \alpha_{T_d}(\lambda)$. □

2.5 A general method for Gibbs distributions

The hard-core model is an example of a Gibbs distribution, and we give a general description of such distributions in this section. We then state how the method detailed above can be applied in this general setting. In Chapter 3 we show how the method can give enough structural information to optimise over line graphs of d-regular graphs; and in Chapter 4 we apply the method to the Potts model, a different Gibbs distribution.

For a graph G and a finite set of spins Ω, a Gibbs distribution on G is a probability distribution on assignments of spins $\sigma : V(G) \to \Omega$. Given parameters λ_i and $\beta_{i,j}$ for $i, j \in \Omega$, we associate to each assignment of spins an energy of the form

$$H(\sigma) = \sum_{v \in V(G)} \lambda_{\sigma(v)} + \sum_{uv \in E(G)} \beta_{\sigma(u),\sigma(v)} ,$$

and insist that each σ occurs with probability proportional to $e^{H(\sigma)}$. The normalising constant $Z_G = \sum_\sigma e^{H(\sigma)}$ is called the *partition function* of the model. To obtain the hard-core model from this definition is simple. Take $\Omega = \{0, 1\}$ so that each σ corresponds to some subset $I_\sigma \subseteq V(G)$, with $\sigma(v) \in \{0, 1\}$ indicating whether $v \in I_\sigma$. Then set $\lambda_0 = 0$, $\lambda_1 = \log \lambda$, $\beta_{1,1} = -\infty$ and $\beta_{i,j} = 0$ otherwise. Now if σ corresponds to a subset of $V(G)$ which contains an edge, $H(\sigma) = -\infty$ which we take to mean I_σ occurs with probability zero. Otherwise, I_σ is an independent set and $H(\sigma) = |I_\sigma| \log \lambda$.

We introduce a final definition from statistical physics. Let the *free energy per particle* F_G be $\frac{1}{|V(G)|} \log Z_G$, and note that Theorem 1.1 may be interpreted as an upper bound on the free energy per particle of the hard-core model in d-regular graphs.

One can define observables such as the occupancy fraction by taking expectations over the distribution, and the examples in this thesis correspond to (logarithmic) derivatives of the partition function with respect to some of the parameters λ_i and $\beta_{i,j}$.

The method in this general setting now has the following form.

(i) Choose a Gibbs distribution, an observable, and a class of bounded-degree graphs.

(ii) For fixed depth t, consider *local views* generated by the following two-part experiment on any graph G from the class. Firstly, choose an assignment of spins $\sigma : V(G) \to \Omega$ from the Gibbs distribution, and secondly choose a vertex $v \in V(G)$ uniformly at random. Then record the graph structure of G from v to the depth-t neighbourhood of v and the spins that σ assigns to the boundary of this neighbourhood. See Figure 2.1 for examples.

(iii) Express the desired observable as an expectation over the random local view and formulate constraints on the distribution of local views that must hold whenever the two part experiment is carried out on a graph from the given class.

(iv) Formulate and solve a constrained optimisation problem for the observable as a linear program.

Figure 2.1: Example local views of depth $t = 2$ from a 3-regular graph. All graph structure on the white vertices is recorded. The boundary is coloured black, and we record σ restricted to these vertices. Note that we discard any edges within the boundary.

The proof of Theorem 2.1 follows this outline for the hard-core model, the occupancy fraction, and d-regular graphs. We take depth $t = 2$ and make a slight adjustment to the definition by considering v itself to be part of the boundary. This is merely for convenience in the proof, and it is a straightforward modification to express the proof with the definition of local view given above. In the hard-core model spins correspond to membership of the random independent set, hence given spins on the boundary (and v) we can see which neighbours of v are covered by an occupied boundary vertex. This yields the definition of local view exactly as in Section 2.3.

For Theorem 2.1 we only required one constraint on the distribution of the random local view, which we obtained by writing $\alpha_G^{\mathrm{ind}}(\lambda)$ two different ways and equating them, see (2.3) and (2.4). Increasing the depth of local views mean that they contain more information, which may yield more constraints on the distribution of the local views. This comes at the cost of a larger set of possible local views and lengthier calculations, however.

It is worth drawing attention to the key property of Gibbs distributions that allows this method to function. In several places we use the fact that conditioned on the spins at some boundary, the spins either side of the boundary are independent. This important fact, known as the *spatial Markov* *property*, lies behind the two key properties of the hard-core model given in Section 2.2. The derivations given there are straightforward to generalise to any Gibbs distribution of the above form, and any sort of boundary.

2.6 Further work

In this section we briefly discuss further applications of the method to independent sets and the hard-core model which focus on adding girth conditions to the class of graphs considered. Firstly, in [24] we prove an asymptotically tight lower bound on the average size of independent sets in a triangle-free graph on n vertices with maximum degree d. As a corollary we give a lower bound on the total number of independent sets in a triangle-free graph with maximum degree d that is asymptotically tight in the exponent. In both cases, tightness is exhibited by a random d-regular graph. The lower bounds of [24] are naturally expressed in terms of the Lambert W function, $W(z)$. For $z > 0$, $W(z)$ denotes the unique positive real satisfying the relation $W(z)e^{W(z)} = z$. It may be useful to note that for $z \geq e$ we have $W(z) \geq \log z - \log\log z$.

Theorem 2.4. *Let G be a triangle-free graph with maximum degree d. Then for all $\lambda > 0$,*

$$\alpha_G^{\text{ind}}(\lambda) \geq \frac{\lambda}{1+\lambda}\frac{W(d\log(1+\lambda))}{d\log(1+\lambda)}\,.$$

In particular, for $\lambda \geq 1/\log d$ we have $\alpha_G^{\text{ind}}(\lambda) \geq (1+o_d(1))\frac{\log d}{d}$.

Theorem 2.5. *Let G be a triangle-free graph on n vertices with maximum degree d. Then for all $\lambda > 0$,*

$$Z_G^{\text{ind}}(\lambda) \geq \exp\left(\left[W(d\log(1+\lambda))^2 + 2W(d\log(1+\lambda))\right]\frac{n}{2d}\right).$$

In particular, taking $\lambda = 1$, we see that G has at least $e^{\left(\frac{1}{2}+o_d(1)\right)\frac{\log^2 d}{d}n}$ independent sets.

Theorem 2.4 implies a well-known result of Shearer [93] on the Ramsey number $R(3,k)$. We define $R(3,k)$ to be the least N such that every graph on N vertices contains either a triangle or an independent set of size k, and note the bounds

$$\left(\frac{1}{4}+o(1)\right)\frac{k^2}{\log k} \le R(3,k) \le (1+o(1))\frac{k^2}{\log k}\,.$$

The upper bound is by Shearer [93] and the lower bound by independent work of Bohman and Keevash [11] and Fiz Pontiveros, Griffiths, and Morris [37].

To see that Theorem 2.4 directly implies the above upper bound, suppose that G is triangle free with no independent set of size k. Then G must have maximum degree less than k. Applying Theorem 2.4 we see the independence number is at least $(1+o_k(1))\frac{\log k}{k}n$ but less than k, and so $n < (1+o_k(1))\frac{k^2}{\log k}$ as required. Whether the upper bound can be improved is a major open question in Ramsey theory, see [24].

Perarnau and Perkins [78] apply the method to 3-regular graphs with girth conditions. Here the optimising graphs are the Petersen graph P^*, and the Heawood graph H^*, see Figure 2.2.

(a) The Petersen graph P^* (b) The Heawood graph H^*

Figure 2.2: Extremal graphs from results of Perarnau and Perkins [78]

Theorem 2.6. *For any triangle-free, cubic graph G, and every $0 < \lambda \le 1$, $\alpha_G^{\text{ind}}(\lambda) \ge \alpha_{P^*}^{\text{ind}}(\lambda)$, with equality if and only if G is a disjoint union of P^*'s.*

Theorem 2.7. *For any cubic graph G of girth at least 5, and every $\lambda > 0$, $\alpha_G^{\text{ind}}(\lambda) \le \alpha_{H^*}^{\text{ind}}(\lambda)$, with equality if and only if G is a disjoint union of H^*'s.*

Finally, we also note that stability follows naturally from this method. That is, when G is somehow 'far' from (disjoint unions of) the optimising graph in Theorems 2.1, 2.7, 3.1, and 4.1, the occupancy we obtain that the occupancy fraction is correspondingly 'far' from optimal. While the proofs presented here contain the relevant information, the details are discussed in [26], where we use stability to give tight bounds on certain coefficients of the relevant partition functions.

3

Matchings in regular graphs

In this chapter we prove a tight upper bound on the occupancy fraction of the monomer-dimer model in regular graphs.

Theorem 3.1. *For all d-regular graphs G and all $\lambda > 0$, we have*

$$\alpha_G^{\text{match}}(\lambda) \le \alpha_{K_{d,d}}^{\text{match}}(\lambda)\,.$$

The maximum is achieved only by disjoint unions of $K_{d,d}$. That is, the quantity $\frac{1}{|E(G)|}\frac{\partial}{\partial\lambda}\log Z_G^{\text{match}}(\lambda)$ is uniquely maximised by $H_{d,n}$.

As with the case of independent sets, since α_G^{match} is the normalised logarithmic derivative of $Z_G^{\text{match}}(\lambda)$, integrating the above result implies that $K_{d,d}$ (and thus also $H_{d,n}$) maximises $\frac{1}{|V(G)|}\log Z_G^{\text{match}}(\lambda)$ for any $\lambda > 0$, and in particular, with $\lambda = 1$, this shows that $H_{d,n}$ has the greatest total number of matchings of any d-regular graph on n vertices.

Theorem 3.2. *Let G be a d-regular graph and $\lambda > 0$. Then*

$$\frac{1}{|V(G)|}\log Z_G^{\text{match}}(\lambda) \le \frac{1}{2d}\log Z_{K_{d,d}}^{\text{match}}(\lambda)\,.$$

In Section 3.3 we use Theorem 3.2 to give new upper bounds on the number of matchings of a given size in d-regular graphs, proving the 'asymptotic upper matching conjecture' of Friedland, Krop, Lundow, and Markström [39]. The argument also gives new results for independent sets.

Let $i_k(G)$ be the number of independent sets of size k in a graph G, and

$m_k(G)$ the number of matchings of size k. Kahn [57] conjectured that $i_k(G)$ is maximised over d-regular, n-vertex graphs by $H_{d,n}$ for all k, and Friedland, Krop, and Markström [38] conjectured the same for $m_k(G)$. Previous bounds towards these conjectures were given in [14, 52, 79]. Here we adapt the method of Carroll, Galvin, and Tetali and use Theorem 3.2 to give bounds for both problems that fall short of the conjectures by a multiplicative factor of $2\sqrt{n}$, for all d and all k. For more recent developments based on these methods which include a proof of the above conjectures for a wide range of parameters, see [26].

Theorem 3.3. *Let $2d$ divide n. Then for all d-regular graphs G on n vertices,*

$$i_k(G) \leq 2\sqrt{n} \cdot i_k(H_{d,n})$$

and

$$m_k(G) \leq 2\sqrt{n} \cdot m_k(H_{d,n})\,.$$

We prove Theorem 3.1 in Section 3.2 before giving the bounds on the number of independent sets and matchings of a given size in Section 3.3.

3.1 Related work

Carroll, Galvin, and Tetali [14] used the entropy method to give an upper bound of $(1 + d\lambda)^{1/2}$ on $Z_G^{\text{match}}(\lambda)^{1/|V(G)|}$. It was previously conjectured [38, 41] that $K_{d,d}$ maximises $Z_G^{\text{match}}(\lambda)^{1/|V(G)|}$ over all d-regular graphs G, Theorem 3.2 resolves this conjecture.

In [20], Csikvári proved the 'lower matching conjecture' of [38] and in [19] gave a new lower bound on the number of perfect matchings of d-regular, vertex-transitive, bipartite graphs, in both comparing an arbitrary graph with the infinite d-regular tree (see also the recent extension by Lelarge [69] to irregular graphs). Proposition 2.10 in [19] states that the occupancy fraction of the monomer-dimer model on any d-regular, vertex-transitive, bipartite graph is at least that of the infinite d-regular tree; in Theorem 2.2 we proved the same fact for independent sets. Csikvári's techniques are different to the methods used here, but similar in that he bounds the occupancy fraction instead of directly working with the partition function. His results rely on an elegant interplay between the Heilmann–Lieb theorem [51] and Benjamini–Schramm convergence of bounded-degree graphs.

3.2 Proof of Theorem 3.1

Recall that we write $Z_G^{\mathrm{match}}(\lambda)$ for the partition function of the monomer-dimer model on a graph G at fugacity λ, and let $\mathbf{M}$ be a matching drawn from the model.

The proof follows the general framework of Section 2.5 (with depth $t = 2$) if one translates the problem to statements about independent sets in the line graph. Here our terminology avoids this translation.

Given some matching M, we refer to an edge as covered if an incident edge is in M. Given the random $\mathbf{M}$ from the model, let e be an edge of G chosen uniformly at random, with an arbitrary left/right orientation chosen at random, and now define the *local view* (centred on e) to be the subgraph of G containing all the incident edges to e that are not covered by edges outside of both e and its incident edges. In terms of the more general framework, this is equivalent to recording the spins on edges at distance 2 from e. Note that (in contrast to the proof in Section 2.3) here we do not consider e part of the boundary. We consider orientations of edges in terms of left and right endpoints, hence in this chapter we use the letter C for local views, leaving L free to mean 'left'. Given e and a local view C centred on e, we use the term *externally uncovered neighbour* to refer to an edge of C incident to e.

The possible local views C are completely defined by three parameters: $L, R, K \in \{0, 1, \dots, d-1\}$, counting the number of left and right neighbouring edges in C which do not form a triangle with e, and the number of triangles containing e in C. An example is pictured in Figure 3.1.

We now consider probabilities according to the two-part experiment that yields a local view; choosing $\mathbf{M}$ from the monomer-dimer model on a d-regular graph at fugacity λ, and picking an edge e (and left/right orientation) uniformly at random. Let $q(i, j, k) = \mathbb{P}[L = i, R = j, K = k]$, and observe that the matching polynomial for such a local view is $Z_{i,j,k}^{\mathrm{match}}(\lambda) := 1 + (i + j + 2k)\lambda + (k^2 + k(i + j - 1) + ij)\lambda^2$. For brevity we drop the superscript and may omit λ, so that $Z_{i,j,k}^{\mathrm{match}}(\lambda) = Z_{i,j,k}(\lambda) = Z_{i,j,k}$.

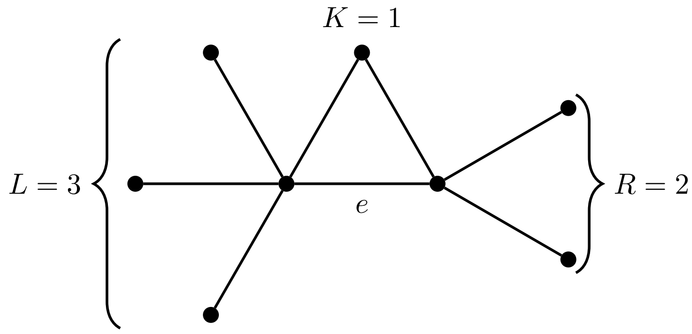


Figure 3.1: An example local view from the monomer-dimer model

We can write $\alpha_G^{\mathrm{match}}(\lambda)$ as the expected fraction of edges incident to e that are in the matching, as each edge in a d-regular graph is incident to exactly $2(d-1)$ other edges:

$$\begin{aligned}\alpha_G^{\mathrm{match}}(\lambda) &= \frac{1}{|E(G)|}\sum_e \sum_{f\sim e} \frac{\mathbb{P}[f \in \mathbf{M}]}{2(d-1)} \\ &= \mathbb{E}\left[\frac{\lambda Z'_{i,j,k}(\lambda)}{2(d-1)(\lambda + Z_{i,j,k}(\lambda))}\right] \\ &= \sum_{i,j,k} q(i,j,k)\frac{\lambda Z'_{i,j,k}(\lambda)}{2(d-1)(\lambda + Z_{i,j,k}(\lambda))}\,,\end{aligned}$$

where the expectation in the second line is over the random local view resulting from the two-part experiment described above. If we define

$$a(i,j,k) = \frac{\lambda Z'_{i,j,k}(\lambda)}{2(d-1)(\lambda + Z_{i,j,k}(\lambda))}$$

for the expected fraction of occupied neighbours of e in the local view given by i, j, k. Then the above expression can be written $\alpha_G^{\mathrm{match}}(\lambda) = \sum_{i,j,k} q(i,j,k)a(i,j,k)$.

We now need to introduce additional constraints before optimising $\alpha_G^{\mathrm{match}}(\lambda)$ over the $q(i,j,k)$. We could write multiple expressions for $\alpha_G^{\mathrm{match}}(\lambda)$, equate them, and solve the maximisation problem as we did for independent sets in Section 2.3. Using three expressions for $\alpha_G^{\mathrm{match}}(\lambda)$ we were able to prove Theorem 3.1 for the case $d = 3$, in which the optimiser is supported on only three values $q(0,0,0)$, $q(1,1,0)$, $q(2,2,0)$. In general we need at least $d-1$ constraints (in addition to the constraint that the $q(i,j,k)$ sum to one) as the distribution on local views induced by $K_{d,d}$ is supported on d values.

Instead, we write, for all t, two expressions for the marginal probability that the number of uncovered neighbours on a randomly chosen side of a random edge is equal to t. We find the two expressions by choosing uniformly: a random edge e, a random side left or right, and f, a random neighbouring edge of e from the given side. We first calculate the probability that e has t uncovered neighbours on the side containing f, then we calculate the probability that f has t uncovered neighbours on the side containing e.

Given a local view C with $L=i$, $R=j$, and $K=k$, e can have $0,1,i+k-1$, or $i+k$ uncovered left neighbours; an edge f to the left of e can have $0,1,i+k-2,i+k-1,i+k$, or $i+k+1$ uncovered right neighbours (depending on whether f itself is in the local view C).

For these quantities we make the definitions

$$\begin{aligned}\gamma^e_{i,j,k}(t) &= \mathbb{P}[e \text{ has } t \text{ uncovered left neighbours } | L=i, R=j, K=k]\,,\\ \gamma^f_{i,j,k}(t) &= \mathbb{P}[f \text{ has } t \text{ uncovered right neighbours } | L=i, R=j, K=k]\,,\end{aligned}$$

where f is a uniformly chosen left neighbour of the uniformly random oriented edge e.

Claim 3.4. *Let $\beta_t = 1+t\lambda$. Then we have*

$$\begin{aligned}\gamma^e_{i,j,k}(t) = \frac{1}{\lambda+Z_{i,j,k}}\Big(&\lambda\mathbb{1}_{t=0} + (i\lambda\beta_{j+k} + k\lambda\beta_{j+k-1})\mathbb{1}_{t=1}\\ &+ \beta_j\mathbb{1}_{t=i+k} + k\lambda\mathbb{1}_{t=i+k-1}\Big)\end{aligned} \tag{3.1}$$

$$\begin{aligned}\gamma^f_{i,j,k}(t) = \frac{1}{(d-1)(\lambda+Z_{i,j,k})}\Big(&(i\lambda\beta_{j+k} + k\lambda\beta_{j+k-1})\mathbb{1}_{t=0}\\ &+ \big[(d-1)\lambda + (d-2)(i\lambda\beta_{j+k} + k\lambda\beta_{j+k-1})\big]\mathbb{1}_{t=1}\\ &+ (i+k-1)k\lambda\mathbb{1}_{t=i+k-2}\\ &+ \big[(d-i-k)k\lambda + (i+k)j\lambda\big]\mathbb{1}_{t=i+k-1}\\ &+ \big[(d-1-i-k)j\lambda + (i+k)\big]\mathbb{1}_{t=i+k}\\ &+ (d-1-i-k)\mathbb{1}_{t=i+k+1}\Big)\,.\end{aligned} \tag{3.2}$$

Proof. To compute the functions $\gamma^e_{i,j,k}(t)$ we consider the following disjoint events: (i) no left edge and no right edge from a triangle is in the matching, (ii) e is in the matching, (iii) a left edge is in the matching, (iv) no left edge is in the matching, but a right edge from a triangle is in the matching. These events happen with probability $\frac{\beta_j}{\lambda+Z_{i,j,k}}$, $\frac{\lambda}{\lambda+Z_{i,j,k}}$, $\frac{i\lambda\beta_{j+k}+k\lambda\beta_{j+k-1}}{\lambda+Z_{i,j,k}}$, and $\frac{k\lambda}{\lambda+Z_{i,j,k}}$ respectively. Under these events the number of uncovered neighbours of e is $i+k$, 0, 1, and $i+k-1$ respectively. This gives (3.1).

To compute the functions $\gamma^f_{i,j,k}(t)$ we refine the above events to include the possible choices of f: f can be an edge outside the local view with probability $(d-1-i-k)/(d-1)$; an edge in the local view but not in a triangle with probability $i/(d-1)$; in the local view and in a triangle with probability $k/(d-1)$. If a left edge is in the matching we choose it as f with probability $1/(d-1)$, and if a right edge in a triangle is in the matching we choose f adjacent to it with probability $1/(d-1)$. Computing the number of uncovered neighbours of f in each case gives (3.2). □

We define a linear program with constraints imposing that the two different ways of writing these marginal probabilities are equal. This constraint for $t=d-1$ is redundant and we omit it.

$$\begin{aligned}\alpha^* = \max \sum_{i,j,k} &q(i,j,k)a(i,j,k) \quad \text{s.t.}\\ &q(i,j,k) \geq 0 \ \forall\ i,j,k\,,\\ &\sum_{i,j,k} q(i,j,k) = 1\,,\\ &\sum_{i,j,k} q(i,j,k)\frac{1}{2}\Big[\gamma^f_{i,j,k}(t)+\gamma^f_{j,i,k}(t)-\gamma^e_{i,j,k}(t)-\gamma^e_{j,i,k}(t)\Big]=0\\ &\qquad\qquad \forall\, t=0,\dots,d-2\,.\end{aligned}$$

Disjoint unions of copies of $K_{d,d}$ are the only graphs that induce a distribution $q(i,j,k)$ supported on triples with $i=j$ and $k=0$. This gives us a candidate solution to the linear program. To complete the proof Theorem 3.1 it suffices to show that this distribution is the unique optimiser of the above linear program.

The dual program is

$$\alpha^* = \min \Lambda_p \quad \text{s.t.}$$
$$\Lambda_p - a(i,j,k) + \sum_{t=0}^{d-2} \Lambda_t \frac{1}{2}\Big[\gamma^f_{i,j,k}(t) + \gamma^f_{j,i,k}(t) - \gamma^e_{i,j,k}(t) - \gamma^e_{j,i,k}(t)\Big] \geq 0\,,$$

so show that $K_{d,d}$ is optimal, we find values for the dual variables $\Lambda_0, \dots, \Lambda_{d-2}$ so that the dual constraints hold with $\Lambda_p = \alpha^{\text{match}}_{K_{d,d}}(\lambda)$. To find such values, we solve the system of equations generated by setting equality in the constraints corresponding to $i = j$ and $k = 0$ and solve for the variables Λ_t, with $t = 0, \dots, d-2$.

With this choice of values for the dual variables, we start by simplifying the form of the dual constraints with a substitution coming from equality in the $(i,j,k) = (0,0,0)$ constraint. The $(0,0,0)$ dual constraint has the simple form

$$\Lambda_0 - \Lambda_1 = \alpha^{\text{match}}_{K_{d,d}}(\lambda)\,.$$

Moreover, observe that from the $\mathbb{1}_{t=0}$ and $\mathbb{1}_{t=1}$ terms in $\gamma^e_{i,j,k}(t)$ and $\gamma^f_{i,j,k}(t)$, every dual constraint contains the term

$$\left[a(i,j,k) - \frac{\lambda}{(\lambda + Z_{i,j,k})}\right](\Lambda_0 - \Lambda_1) = \left[a(i,j,k) - \frac{\lambda}{(\lambda + Z_{i,j,k})}\right]\alpha^{\text{match}}_{K_{d,d}}(\lambda)\,.$$

With this simplification, we multiply through by $2(d-1)(\lambda + Z_{i,j,k})$ and expand $a(i,j,k)$ terms to obtain the following form of the dual constraints:

$$\begin{aligned}
\alpha^{\text{match}}_{K_{d,d}}(\lambda)\,&[\lambda Z'_{i,j,k} + 2(d-1)Z_{i,j,k}] - \lambda Z'_{i,j,k} \\
&+ \Lambda_{i+k-2} \cdot (i+k-1)k\lambda \\
&+ \Lambda_{i+k-1} \cdot [(d-i-k)k\lambda + (i+k)j\lambda - (d-1)k\lambda] \\
&+ \Lambda_{i+k} \cdot [(d-1-i-k)j\lambda + i + k - (d-1)\beta_j] \\
&+ \Lambda_{i+k+1} \cdot (d-1-i-k) \\
&+ \Lambda_{j+k-2} \cdot (j+k-1)k\lambda \\
&+ \Lambda_{j+k-1} \cdot [(d-j-k)k\lambda + (j+k)i\lambda - (d-1)k\lambda] \\
&+ \Lambda_{j+k} \cdot [(d-1-j-k)i\lambda + j + k - (d-1)\beta_i] \\
&+ \Lambda_{j+k+1} \cdot (d-1-j-k) \geq 0\,.
\end{aligned} \tag{3.3}$$

The $(i, i, 0)$ equality constraints now read

$$\begin{aligned}\alpha^{\text{match}}_{K_{d,d}}(\lambda)\beta_i(\beta_i + \tfrac{i\lambda}{d-1}) - \tfrac{i\lambda\beta_i}{d-1} + \Lambda_{i-1}\tfrac{i^2\lambda}{d-1}& \\ - \Lambda_i\tfrac{d-1-i+i^2\lambda}{d-1} + \Lambda_{i+1}\tfrac{d-1-i}{d-1} &= 0\,.\end{aligned} \tag{3.4}$$

With this we can write Λ_{i+k+1} in terms of Λ_{i+k} and Λ_{i+k-1}, and similarly for Λ_{j+k+1}. Substituting this into (3.3) and dividing by λ we derive the simplified form of the dual constraints:

$$\begin{aligned}&\lambda\big[(i-j)^2 + 2k\big](1 - d\alpha^{\text{match}}_{K_{d,d}}(\lambda)) \\ &\quad + \Lambda_{i+k-2}(i+k-1)k + \Lambda_{i+k-1}[k + (i+k)(j-i-2k)] \\ &\quad + \Lambda_{i+k}(i+k)(i+k-j) \\ &\quad + \Lambda_{j+k-2}(j+k-1)k + \Lambda_{j+k-1}[k + (j+k)(i-j-2k)] \\ &\quad + \Lambda_{j+k}(j+k)(j+k-i) \geq 0\,.\end{aligned} \tag{3.5}$$

Write $L(i, j, k)$ for the left-hand side of this inequality.

The marginal constraint for $t = d - 1$ was omitted, but we nonetheless introduce $\Lambda_{d-1} := 0$ in order to simplify the presentation of the argument. The $(d-1, d-1, 0)$ equality constraint gives Λ_{d-2} directly:

$$\Lambda_{d-2} = \frac{1}{(d-1)\lambda}\left[\lambda + (d-1)\lambda^2 - \alpha^{\text{match}}_{K_{d,d}}(\lambda)\beta_{d-1}\beta_d\right]\,.$$

With Λ_{d-1}, Λ_{d-2}, and the recurrence relation (3.4) the dual variables are fully determined. We do not give a closed-form expression for Λ_t as the values are used in an induction below. Using Λ_{d-1}, Λ_{d-2}, and (3.4) suffices for the proof.

We now reduce the problem of showing that the dual constraints (3.5) corresponding to triples (i, j, k) with $k > 0$ or $i \neq j$ hold with strict inequality to showing that a particular function is increasing. We go on to prove this fact in Claims 3.5 and 3.6.

Putting $k = 0$ into (3.5) gives:

$$\begin{aligned}\frac{L(i,j,0)}{(j-i)} &= \lambda(j-i)(1 - d\alpha^{\text{match}}_{K_{d,d}}(\lambda)) + i\Lambda_{i-1} - i\Lambda_i - j\Lambda_{j-1} + j\Lambda_j \\ &= F_d(j) - F_d(i)\,,\end{aligned}$$

where

$$F_d(t) := t\Big[\lambda(1 - d\alpha^{\mathrm{match}}_{K_{d,d}}(\lambda)) + \Lambda_t - \Lambda_{t-1}\Big]\,. \tag{3.6}$$

From (3.5) we obtain

$$\begin{aligned} L(i-1, j-1, k+1) - L(i,j,k) = F_d(i+k) - F_d(i+k-1) \\ + F_d(j+k) - F_d(j+k-1). \end{aligned}$$

Therefore if $F_d(t)$ is strictly increasing, we have $L(i,j,0) > 0$ for $i \neq j$, and

$$L(i-1, j-1, k+1) > L(i,j,k) > \cdots > L(i+k, j+k, 0) \geq 0\,.$$

We first find an explicit expression for $F_d(t)$, where we write $Z_{K_{t,t}}$ for the matching polynomial of the graph $K_{t,t}$.

Claim 3.5. *For all $d \geq 2$ and $1 \leq t \leq d-1$,*

$$F_d(t) = \frac{t(d-1)}{Z_{K_{d,d}}} \sum_{\ell=t-1}^{d-2} \frac{(d-1-t)!}{(\ell+1-t)!} \lambda^{d-\ell} Z_{K_{\ell,\ell}}\,. \tag{3.7}$$

Proof. We will use the following two facts:

$$Z_{K_{d,d}} - \beta_{2d-1} Z_{K_{d-1,d-1}} + (d-1)^2\lambda^2 Z_{K_{d-2,d-2}} = 0 \tag{3.8}$$

$$\alpha^{\mathrm{match}}_{K_{d,d}}(\lambda) = \frac{\lambda Z_{K_{d-1,d-1}}}{Z_{K_{d,d}}}\,. \tag{3.9}$$

The first is a Laguerre polynomial identity, verifiable by hand; the second is a short calculation. The equality dual constraint (3.4) implies:

$$(d-1-t)F_d(t+1) = (t+1)[t\lambda F_d(t) + (d-1)\lambda - (d-1)\alpha^{\mathrm{match}}_{K_{d,d}}(\lambda)\beta_{d+t}]\,.$$

We first show that the right hand side of (3.7) satisfies the above recurrence relation. Using (3.9) this amounts to showing that the following expression is equal to zero for all $d \geq 2$ and $1 \leq t \leq d-1$:

$$\begin{aligned} \Phi_d(t) := (d-1-t)!\left(\sum_{\ell=t}^{d-2} \frac{\lambda^{d-\ell} Z_{K_{\ell,\ell}}}{(\ell-t)!} - t^2 \sum_{\ell=t-1}^{d-2} \frac{\lambda^{d+1-\ell} Z_{K_{\ell,\ell}}}{(\ell+1-t)!}\right) \\ - \lambda(Z_{K_{d,d}} - \beta_{d+t} Z_{K_{d-1,d-1}})\,. \end{aligned}$$

We proceed by induction on d. Note that when $d = 2$, $\Phi_2(1)$ is easily verified to be zero. Note that

$$\Phi_{d+1}(t) = \lambda\Big((d-t)\Phi_d(t) - Z_{K_{d+1,d+1}} + \beta_{2d+1}Z_{K_{d,d}} - d^2\lambda^2 Z_{K_{d-1,d-1}}\Big).$$

By the induction hypothesis and (3.8) the result follows. To complete the proof of the claim it suffices to show that (3.7) holds for $t = d-1$. Recalling that

$$\begin{aligned}\Lambda_{d-1} &= 0\,,\\ \Lambda_{d-2} &= \frac{1}{d-1} + \lambda - \frac{\alpha^{\text{match}}_{K_{d,d}}(\lambda)}{(d-1)\lambda}\beta_d\beta_{d-1}\,,\end{aligned}$$

substituting into (3.6), and using (3.8) and (3.9) we have

$$\begin{aligned}F_d(d-1) &= (d-1)\left[\lambda(1 - d\alpha^{\text{match}}_{K_{d,d}}(\lambda)) - \frac{1}{d-1} - \lambda + \frac{\alpha^{\text{match}}_{K_{d,d}}(\lambda)}{(d-1)\lambda}\beta_d\beta_{d-1}\right]\\ &= \frac{\alpha^{\text{match}}_{K_{d,d}}(\lambda)}{\lambda}\beta_{2d-1} - 1\\ &= \frac{1}{Z_{K_{d,d}}}\left[\beta_{2d-1}Z_{K_{d-1,d-1}} - Z_{K_{d,d}}\right]\\ &= \frac{(d-1)^2\lambda^2 Z_{K_{d-2,d-2}}}{Z_{K_{d,d}}}\,,\end{aligned}$$

verifying (3.7) for $t = d-1$. □

Using Claim 3.5 we prove the following.

Claim 3.6. *$F_d(t)$ is strictly increasing as a function of t.*

Proof. To prove that $F_d(t)$ is increasing, we show that

$$\begin{aligned}R_d(t) &:= \frac{Z_{K_{d,d}}}{(d-1)}\cdot\frac{F_d(t+1) - F_d(t)}{(d-2-t)!}\\ &= (t+1)\sum_{\ell=t}^{d-2}\frac{\lambda^{d-\ell}}{(\ell-t)!}Z_{K_{\ell,\ell}} - t(d-1-t)\sum_{\ell=t-1}^{d-2}\frac{\lambda^{d-\ell}}{(\ell+1-t)!}Z_{K_{\ell,\ell}}\end{aligned}$$

is positive for each t with $1 \le t \le d-2$. We do this by fixing t and inducting on d from $t+2$ upwards. A useful inequality will be $Z_{K_{t,t}} > t\lambda Z_{K_{t-1,t-1}}$ which comes from only counting matchings of $K_{t,t}$ that use a specific vertex.

Iterating this inequality we obtain

$$Z_{K_{t,t}} > \frac{t!}{\ell!}\lambda^{t-\ell} Z_{K_{\ell,\ell}} \text{ for } 0 \le \ell \le t-1\,. \tag{3.10}$$

For the base case of our induction, $d = t+2$, we have $R_d(d-2) = \lambda^2\big[Z_{K_{d-2,d-2}} - (d-2)\lambda Z_{K_{d-3,d-3}}\big]$ which by (3.10) is positive.

For the inductive step we have

$$R_{d+1}(t) = \lambda\Bigg[R_d(t) + \frac{\lambda}{(d-1-t)!} Z_{K_{d-1,d-1}} - \sum_{\ell=t-1}^{d-2} \frac{t\lambda^{d-\ell}}{(\ell-t+1)!} Z_{K_{\ell,\ell}}\Bigg]\,,$$

and so it is sufficient to show

$$\sum_{\ell=t-1}^{d-2} \frac{t\lambda^{d-\ell}}{(\ell+1-t)!} Z_{K_{\ell,\ell}} < \frac{\lambda}{(d-1-t)!} Z_{K_{d-1,d-1}}\,. \tag{3.11}$$

We use the inequality (3.10) in each term of the sum to see that the left-hand side of (3.11) is less than

$$\sum_{\ell=t-1}^{d-2} \frac{t\ell!\lambda}{(\ell+1-t)!(d-1)!} Z_{K_{d-1,d-1}}\,,$$

and so

$$\begin{aligned}
\sum_{\ell=t-1}^{d-2} \frac{t\lambda^{d-\ell}}{(\ell+1-t)!} Z_{K_{\ell,\ell}} &< \sum_{\ell=t-1}^{d-2} \frac{t\ell!\lambda}{(\ell+1-t)!(d-1)!} Z_{K_{d-1,d-1}} \\
&= \frac{\lambda Z_{K_{d-1,d-1}}}{(d-1-t)!} \cdot \sum_{\ell=t-1}^{d-2} \frac{t\ell!(d-1-t)!}{(\ell+1-t)!(d-1)!} \\
&= \frac{\lambda Z_{K_{d-1,d-1}}}{(d-1-t)!} \cdot \binom{d-1}{t}^{-1} \cdot \sum_{\ell=t-1}^{d-2} \binom{\ell}{t-1} \\
&= \frac{\lambda Z_{K_{d-1,d-1}}}{(d-1-t)!}\,,
\end{aligned}$$

hence (3.11) holds as required. □

This completes the proof of dual feasibility and shows our candidate solution to the primal program is optimal. The uniqueness of the solution follows from two facts. First, strict inequality in the dual constraints outside of the $(i, i, 0)$ constraints implies, by complementary slackness (see [13]), that the support of any optimal solution in the primal is contained in the set of $(i, i, 0)$ configurations. Second, the distribution induced by $K_{d,d}$ is the unique distribution satisfying the constraints with such a support. This follows from the fact that Λ_i is uniquely determined by (3.4) where we have set the $(i, i, 0)$ dual constraints to hold with equality, which in turn shows that the relevant $d \times d$ submatrix of the constraint matrix is full rank. This proves Theorem 3.1.

3.3 Matchings and independent sets of given size

To prove Theorem 3.3 we start with a fact about the independence and matching polynomials of $H_{d,n}$. In the argument we sometimes give $\mathbb{P}$ a subscript corresponding to the graph on which we consider the relevant probabilistic model.

Lemma 3.7. *For all $1 \leq k \leq n/2$, there exists a λ so that*

$$\frac{i_k(H_{d,n})\lambda^k}{Z^{\mathrm{ind}}_{H_{d,n}}(\lambda)} = \mathbb{P}_{H_{d,n}}[|\mathbf{I}| = k] > \frac{1}{2\sqrt{n}}\,,$$

and a λ so that

$$\frac{m_k(H_{d,n})\lambda^k}{Z^{\mathrm{match}}_{H_{d,n}}(\lambda)} = \mathbb{P}_{H_{d,n}}[|\mathbf{M}| = k] > \frac{1}{2\sqrt{n}}\,.$$

Proof. The distribution of the size of a random independent set I drawn from the hard-core model on $H_{d,n}$ is log-concave; that is,

$$\mathbb{P}_{H_{d,n}}[|\mathbf{I}| = j]^2 > \mathbb{P}_{H_{d,n}}[|\mathbf{I}| = j+1] \cdot \mathbb{P}_{H_{d,n}}[|\mathbf{I}| = j-1]$$

for all $1 < j < n/2$. This follows from two facts: the size distribution of the hard-core model on $K_{d,d}$ is log-concave, and the convolution of two log-concave distributions is again log-concave. The first fact is simply the calculation

$$\binom{d}{j}^2 > \binom{d}{j-1}\binom{d}{j+1}.$$

Now choose λ so that $\mathbb{P}_{H_{d,n}}[|\mathbf{I}| = k] = \mathbb{P}_{H_{d,n}}[|\mathbf{I}| = k+1]$. Log-concavity then implies that $\mathbb{P}_{H_{d,n}}[|\mathbf{I}| = k]$ is maximal. Some explicit computations for the variance for a single $K_{d,d}$ give that the variance of $|\mathbf{I}|$ is at most $n/8$; then via Chebyshev's inequality, with probability at least $2/3$ the size of $\mathbf{I}$ is one of at most $\frac{4}{3}\sqrt{n}$ values, and thus the largest probability of a single size is greater than $\frac{1}{2\sqrt{n}}$.

The proof for $m_k(H_{d,n})$ is the same: the variance of the size of a random matching is also at most $n/8$ (see [56]), and log-concavity of the size distribution on $K_{d,d}$ is verified via the inequality

$$\binom{d}{j}^4 j!^2 > \binom{d}{j-1}^2 (j-1)! \binom{d}{j+1}^2 (j+1)! \qquad \square$$

Proof of Theorem 3.3. Assume for sake of contradiction that $m_k(G) > 2\sqrt{n} \cdot m_k(H_{d,n})$. Choose λ according to Lemma 3.7. We have:

$$Z_G^{\text{match}}(\lambda) \geq m_k(G)\lambda^k > 2\sqrt{n} \cdot m_k(H_{d,n})\lambda^k > Z_{H_{d,n}}^{\text{match}}(\lambda)\,,$$

but this contradicts Theorem 3.1. The case of independent sets is identical. $\square$

The above proof is essentially the same as the proofs in Carroll, Galvin, and Tetali [14] with the small observation that λ can be chosen so that k is the most likely size of a matching (or independent set) drawn from $H_{d,n}$. The factor $2\sqrt{n}$ in both cases can surely be improved by using some regularity of the independent set and matchings sequence of a general d-regular graph; we leave this for future work.

As a consequence, we prove the asymptotic upper matching conjecture of Friedland, Krop, Lundow, and Markström [39]. Fix d and consider an infinite sequence of d-regular graphs $\mathcal{G}_d = G_1, G_2, \ldots$ such that G_n has n vertices. For any $\rho \in [0, 1/2]$, the *ρ-monomer entropy* is defined to be

$$h_{\mathcal{G}_d}(\rho) = \sup_{(k_n)} \limsup_{n\to\infty} \frac{\log m_{k_n}(G_n)}{n}\,,$$

where the supremum is taken over all integer sequences (k_n) with $k_n/n \to \rho$. Let $h_d(\rho) = \lim_{n\to\infty} \frac{\log m_{\lfloor \rho n \rfloor}(H_{d,n})}{n}$, where the limit is take over the sequence of integers divisible by $2d$. Then the conjecture states that for all $\mathcal{G}_d$ and all

$\rho \in [0, 1/2]$, we have $h_{\mathcal{G}_d}(\rho) \le h_d(\rho)$.

To prove this, first assume $\rho > 0$ since for $\rho = 0$ the result is trivially true. Assume for the sake of contradiction that $\limsup \frac{\log m_{k_n}(G_n)}{n} > h_d(\rho) + \varepsilon$ for some $\varepsilon > 0$. Take N_0 large enough that for all $n_1 \ge N_0$, divisible by $2d$, $\frac{\log m_{\lfloor \rho n_1 \rfloor}(H_{d,n_1})}{n_1} < h_d(\rho) + \varepsilon/2$. Now take some $n \ge N_0$ with $\frac{\log m_{k_n}(G_n)}{n} > h_d(\rho) + \varepsilon$, and let $n_1 = 2d \cdot \lceil n/2d \rceil$. Choose λ so that $m_{\lfloor \rho n_1 \rfloor}(H_{d,n_1}) > \frac{1}{2\sqrt{n_1}} M_{H_{d,n_1}}(\lambda)$. Note that since $\rho > 0$, this λ is bounded away from 0 as $n_1 \to \infty$. Then we have

$$\begin{aligned}\frac{\log Z^{\text{match}}_{G_n}(\lambda)}{n} \ge \frac{\log m_{k_n}(G_n)\lambda^{k_n}}{n} &> \frac{k_n}{n}\log\lambda + h_d(\rho) + \varepsilon \\ &= \rho\log\lambda + h_d(\rho) + \varepsilon + o(1) \text{ as } n \to \infty\end{aligned}$$

and

$$\begin{aligned}\frac{\log Z^{\text{match}}_{K_{d,d}}(\lambda)}{2d} = \frac{\log M_{H_{d,n_1}}(\lambda)}{n_1} &< \frac{\log\left(2\sqrt{n_1}\cdot m_{\lfloor \rho n_1 \rfloor}(H_{d,n_1})\lambda^{\lfloor \rho n_1 \rfloor}\right)}{n_1} \\ &< \frac{\log(2\sqrt{n_1})}{n_1} + \frac{\lfloor \rho n_1 \rfloor}{n_1}\log\lambda + h_d(\rho) + \varepsilon/2 \\ &= \rho\log\lambda + h_d(\rho) + \varepsilon/2 + o(1)\,,\end{aligned}$$

but this contradicts Theorem 3.1. With the same proof, the analogous statement for independent set entropy holds.

3.4 Remarks

Theorem 2.1 and Theorem 3.1 show that $K_{d,d}$ maximises the occupancy fraction of the hard-core model and the monomer-dimer model respectively. In both cases cases our results are neither implied by nor imply conjectures that the numbers of independent sets [57] and matchings [38] of each given size are maximised by disjoint unions of $K_{d,d}$'s; while we improve the known bounds in both cases, these conjectures remain open. Here we give even stronger conjectures:

Conjecture 3.8. *Let G be a d-regular, n-vertex graph, where $2d$ divides n. Then for all k, the ratio $\frac{i_k(G)}{i_{k-1}(G)}$ is maximised by $H_{d,n}$.*

Conjecture 3.9. *Let G be a d-regular, n-vertex graph, where $2d$ divides n. Then for all k, the ratio $\frac{m_k(G)}{m_{k-1}(G)}$ is maximised by $H_{d,n}$.*

Conjecture 3.8 also appeared in a draft of [79], and we discuss both conjectures in more detail in [26]. These conjectures are stronger than Theorems 2.1 and 3.1 and imply the conjectures of [57] and [38]. The relation to the work here is that Conjecture 3.8 can be stated as follows: the expected number of neighbours of uniformly random independent set of size k is minimised by $H_{d,n}$; and the analogous form of Conjecture 3.9 is similar. Theorems 2.1 and 3.1 show that such a statement is true when the random independent set or matching is chosen according to the hard-core model or monomer-dimer model instead of uniformly over those of a given size.

4

The Potts model

As an example of a Gibbs distribution, the Potts model is a probabilistic model of interacting *spins* on a graph. Here we use the term *colour* instead of spin to highlight a connection to extremal combinatorics which we cover in Section 4.1. An assignment of spins in the Potts model, which is simply a function $\sigma : V(G) \to [q]$, corresponds to a colouring of the vertices of a graph with at most q colours. The model is parametrised by an *inverse temperature* parameter β, and to obtain the Potts model from the general setup of Section 2.5, we take $\lambda_i = 0$ and $\beta_{i,i} = -\beta$ for all i, and $\beta_{i,j} = 0$ whenever $i \neq j$. Recall that we let $m(\sigma)$ denote the number of monochromatic edges of G under σ. Then the *energy* of a colouring σ is simply $H(\sigma) = -\beta m(\sigma)$, and we see that the Potts model is indeed a Gibbs distribution.

The model is *antiferromagnetic* if $\beta > 0$ and *ferromagnetic* if $\beta < 0$. For general statistical physics terminology, we refer the reader to Chapter 2 of [75], for example. The Potts model generalises the *Ising model* [53], which is the special case $q = 2$. See [100] for a survey of the Potts model.

The Potts partition function also plays an important role in graph theory as it is an evaluation of the Tutte polynomial (see [99]), which contains many properties of the graph in question. When β is positive, the model prefers colourings with fewer monochromatic edges, and the effect is intensified as β increases. When β is negative (the ferromagnetic Potts model), the distribution is biased towards colourings with more monochromatic edges.

The negative of the logarithmic derivative of $Z_G^q(\beta)$ with respect to β gives the expected number of monochromatic edges, or the *internal energy* of the model. If we scale by the number of vertices, this gives the *internal energy per particle*, $U_G^q(\beta)$. We recall from Section 1.1 that when $\boldsymbol{\sigma}$ is a random q-colouring of G chosen from the model,

$$\begin{aligned} U_G^q(\beta) &= \frac{1}{|V(G)|}\,\mathbb{E}[m(\boldsymbol{\sigma})] \\ &= \frac{1}{|V(G)|}\frac{\sum_\sigma m(\sigma)e^{-\beta m(\sigma)}}{Z_G^q(\beta)} \\ &= -\frac{1}{|V(G)|}\frac{\partial}{\partial\beta}\log Z_G^q(\beta)\,. \end{aligned}$$

When $\beta = 0$ there are no interactions in the model, and $Z_G^q(0) = q^{|V(G)|}$ for all G. Starting from here, we can integrate the internal energy per particle to obtain the scaled logarithm of the partition function, or the *free energy per particle*,

$$F_G^q(\beta) = \frac{1}{|V(G)|}\log Z_G^q(\beta) = \log q - \int_0^\beta U_G^q(t)\,\mathrm{d}t\,. \tag{4.1}$$

In this chapter we derive tight upper and lower bounds on $U_G^q(\beta)$ for cubic (3-regular) graphs in the anti-ferromagnetic ($\beta > 0$) regime. From (4.1) these bounds immediately imply corresponding tight bounds on the free energy per particle of the Ising and Potts models, and hence the respective partition functions. We determine, for every q, the maximum and minimum of both the internal energy and the free energy per particle as well as the family of graphs that achieve these bounds.

Theorem 4.1. *For any cubic graph G, any $q \geq 2$, and any $\beta > 0$,*

$$U_{K_{3,3}}^q(\beta) \leq U_G^q(\beta) \leq U_{K_4}^q(\beta)\,.$$

Furthermore, the respective equalities hold if and only if G is a union of $K_{3,3}$'s or a union of K_4's. As a corollary via (4.1)*, we have*

$$F_{K_4}^q(\beta) \leq F_G^q(\beta) \leq F_{K_{3,3}}^q(\beta)\,.$$

We conjecture that these bounds extend to higher regularity d and note that the case $d = 2$ is simply a calculation, see Section 4.4.

Conjecture 4.2. *For any d-regular graph G, any $q \geq 2$, and any $\beta > 0$,*

$$U^q_{K_{d,d}}(\beta) \leq U^q_G(\beta) \leq U^q_{K_{d+1}}(\beta)\,,$$

and in particular,

$$F^q_{K_{d+1}}(\beta) \leq F^q_G(\beta) \leq F^q_{K_{d,d}}(\beta)\,.$$

If we restrict ourselves to bipartite regular graphs, then Galvin (building on [44, 57]) proved that the maximiser of the free energy per particle is $K_{d,d}$.

Theorem 4.3 (Galvin [43])**.** *For any d-regular bipartite graph G, any β and any $q \geq 2$, $F^q_G(\beta) \leq F^q_{K_{d,d}}(\beta)$.*

Such a bound was known without the bipartite restriction in one case previously: in the anti-ferromagnetic Ising model. An extension of Galvin's result by Zhao [103] using the 'bipartite swapping trick' gives the following.

Theorem 4.4 (Zhao [103])**.** *For any d-regular graph G, $\beta > 0$, and $q = 2$ (the Ising model), $F^q_G(\beta) \leq F^q_{K_{d,d}}(\beta)$.*

Zhao's method does not work for $q \geq 3$, and in the ferromagnetic phase Galvin's result cannot be extended to all G; $K_{d,d}$ is not the maximiser. The clique K_{d+1} has a higher free energy for any d when $\beta < 0$. It is natural to conjecture that K_{d+1} is in fact extremal in this case, and also that Galvin's result can be extended to triangle-free graphs.

Conjecture 4.5. *For any d-regular G, any $q \geq 2$, and any $\beta < 0$,*

$$U^q_G(\beta) \leq U^q_{K_{d+1}}(\beta)\,,$$

and in particular,

$$F^q_G(\beta) \leq F^q_{K_{d+1}}(\beta)\,.$$

Moreover, if in addition G is triangle-free, then for any β,

$$U^q_G(\beta) \leq U^q_{K_{d,d}}(\beta)\,,$$

and in particular,

$$F^q_G(\beta) \leq F^q_{K_{d,d}}(\beta)\,.$$

The main work in this chapter is a proof of Theorem 4.1. The proof follows the same method as given in Chapters 2 and 3, outlined in general in Section 2.5.

We consider the following experiment. Fix q, β, and a d-regular graph G. Choose a vertex v uniformly from $V(G)$ and independently sample a colouring $\boldsymbol{\sigma} : V(G) \to [q]$ from the Potts model. Now for each neighbour u of v, record the number of its 'external' neighbours (neighbours outside $v \cup N(v)$) receiving each colour; and record any edges within $N(v)$. This is the sampling of a local view of depth 2 as described in Section 2.5, see Figure 4.1 for examples relevant to the Potts model. Note that although we have sampled a colouring $\boldsymbol{\sigma}$ of the whole graph, the colours of v and its neighbours do not form part of the local view. In fact it is best to think of these colours as having not been revealed. An important part of the method is the spatial Markov property; conditioned on the local view, the distribution of colourings of v and its neighbours can be determined. For fixed d and q there are only a finite number of possible local views; call this set of local views $\mathcal{L}_{d,q}$. Each d-regular graph G and inverse temperature β induces a probability distribution on $\mathcal{L}_{d,q}$.

Not all probability distributions on $\mathcal{L}_{d,q}$ can arise from a graph; there are certain consistency conditions that must hold. For example, the expected number of monochromatic edges incident to v must equal the expected number of monochromatic edges incident to a uniformly chosen neighbour of v. Moreover, we can compute both of these expectations given a probability distribution on $\mathcal{L}_{d,q}$; in fact they are both linear functions of the probabilities. For $d = 3$ this constraint is sufficient, but for larger d more are required. Another family of consistency conditions are that for every multiset S of size d from q colours, the probability $N(v)$ is coloured by S must be the same as the probability $N(u)$ is coloured by S for a uniformly chosen neighbour u of v. Finally, the quantity we wish to optimise, $U_G^q(\beta)$ is also a linear function of the probabilities in the distribution on $\mathcal{L}_{d,q}$.

So instead of maximising or minimising $U_G^q(\beta)$ over all d-regular graphs, we relax the problem and instead optimise over all probability distributions on $\mathcal{L}_{d,q}$ that satisfy the above consistency conditions. This is simply a linear program over $|\mathcal{L}_{d,q}|$ variables. For some values of d, q, and β we know this linear program is not tight although we conjecture it to be tight whenever $q \geq d + 1 \geq 3$ and $\beta > 0$.

This method builds on previous work on independent sets and matchings (see Chapters 2 and 3, and [24, 25, 78]) and the Widom–Rowlinson model [15], but here we generalise the previous approach in two ways. Firstly, we deal with q-spin models instead of 2-spin models; and secondly we deal with soft and hard constraints instead of just hard constraints. The family of linear programs in 3 for matchings was an infinite family indexed by two parameters (the vertex degree d and a fugacity parameter $\lambda > 0$), and the entire family could be solved analytically with a single proof via the duality theorem. Here the situation is worse: we have an infinite family of linear programs indexed by d, q, and β. Moreover, while the number of constraints for the matching program grew linearly in d, here the number of constraints needed can grow like the integer partition number of d. Here, we solve the program for $d = 3$ where there are 35 variables.

In Section 4.2 we solve the linear program (both the maximisation and minimisation problem) for $d = 3$, $q \geq 2$, and all $\beta > 0$, and we solve it in a somewhat mechanical way that does not reveal much about generalisations to higher d.

Nevertheless, we suspect that our method yields a program which is tight for a much wider set of parameters, including for $q \geq d + 1$ and $\beta > 0$. For some other parameter values the constraints described above are not enough to solve the internal energy minimisation problem for all d, q, and $\beta > 0$. It is easy to find values of β so that if $d \geq 4$ and $q \leq d$, the minimiser of the linear program is smaller than $U^q_{K_{d,d}}(\beta)$. Two challenges for future work are:

(i) Solve the infinite family of linear programs with $d \geq 4$, $q \geq d+1$, $\beta > 0$ that we conjecture is tight.

(ii) Find additional consistency conditions (constraints) that can further tighten the program for $q \leq d$; or prove that this cannot be done.

4.1 Maximising the number of q-colourings of d-regular graphs

If we take take $\beta \to \infty$ in the Potts model, we bias more and more against monochromatic edges, and thus if a *proper* q-colouring of G exists, the 'zero-temperature' anti-ferromagnetic Potts model is simply the uniform distribution over proper q-colourings of G. The limit of the partition function is $\lim_{\beta\to\infty} Z^q_G(\beta) = C_q(G)$, the number of proper q-colourings of G.

Maximising $C_q(G)$ over different families of graphs has been the study of much work in extremal combinatorics. Linial [70] asked which graph on n vertices with m edges maximises $C_q(G)$. After a series of bounds by Labeznik and coauthors [66, 67, 68], Loh, Pikhurko, and Sudakov [71] gave a complete answer to this question for a wide range of parameters q, n, m, using the regularity lemma to reduce the maximisation problem over graphs to a quadratic program in $2^q - 1$ variables.

A similar question in a very different setting is to ask which d-regular, n-vertex graph maximises the number of q-colourings; or, given that C_q is multiplicative when taking disjoint unions of graphs, which d-regular graph maximises $\frac{1}{|V(G)|}\log C_q(G)$? Although neither question specifies the sparsity of the graph, one can think of Linial's question as a question about dense graphs and this question as one about sparse graphs (and the techniques of [71] and this chapter reflect this: the regularity lemma primarily concerns dense graphs, while statistical mechanics is primarily concerned with sparse, regular graphs).

For regular graphs, Galvin and Tetali [44] conjectured that $K_{d,d}$ maximises the normalised number of q-colourings over all d-regular graphs.

Conjecture 4.6 (Galvin and Tetali [44])**.** *For any $q \geq 2$, $d \geq 1$, and all d-regular graphs G,*

$$\frac{1}{|V(G)|}\log C_q(G) \leq \frac{1}{2d}\log C_q(K_{d,d})\,. \tag{4.2}$$

In the same paper they prove that (4.2) holds for all d-regular, *bipartite* G. In the language of graph homomorphisms, $C_q(G)$ counts the number of homomorphisms from G into K_q, and their results holds for the number of homomorphisms of a d-regular bipartite G in to *any* target graph H.

Before this work, Conjecture 4.6 was not known for any pair (q, d) apart from the trivial cases $d = 1, d = 2$, and $q = 2$ (see Section 4.4). However, significant partial progress was made in addition to the bipartite case. Employing the bipartite swapping trick, Zhao [103] showed that for $q \geq (2n)^{2n-2}$, the bipartite restriction could be removed for graphs on n vertices. Galvin [40] then reduced the lower bound on q, showing that $q > 2\binom{nd/2}{4}$ suffices. Dependence on n in the number of colours is of course not ideal, as it does not prove Conjecture 4.6 for any pair (q, d), but it does restrict the class of possible counterexamples. In another direction of partial progress on Conjecture 4.6, Galvin [42] gave an upper bound on $C_q(G)$ for all d-regular G, that is tight, asymptotically in d, on a logarithmic scale.

The following lemma relates bounds on internal energy per particle to bounds on the number of q-colourings.

Lemma 4.7. *Fix d and q. If for all d-regular G, and all $\beta > 0$, we have $U_G^q(\beta) \geq U_{K_{d,d}}^q(\beta)$, then (4.2) holds.*

Proof. Let G be any d-regular graph. If $C_q(G) = 0$ then (4.2) clearly holds. Otherwise, we take logarithms and write

$$\begin{aligned}\frac{1}{|V(G)|}\log C_q(G) &= \lim_{\beta\to+\infty}\frac{1}{|V(G)|}\log Z_G^q(\beta)\\ &= \log q - \int_0^\infty U_G^q(\beta)\,\mathrm{d}\beta\\ &\leq \log q - \int_0^\infty U_{K_{d,d}}^q(\beta)\,\mathrm{d}\beta\\ &= \frac{1}{2d}\log C_q(K_{d,d})\,.\end{aligned}$$

□

As a corollary of Theorem 4.1 and Lemma 4.7, we prove Conjecture 4.6 for $d = 3$ and all q.

Corollary 4.8. *For any 3-regular graph G, and any $q \geq 2$,*

$$\frac{1}{|V(G)|}\log C_q(G) \leq \frac{1}{6}\log C_q(K_{3,3})\,,$$

with equality if and only if G is a union of $K_{3,3}$'s.

We remark that in a similar fashion, Theorem 4.1 gives that

$$\frac{1}{|V(G)|}\log C_q(G) \geq \frac{1}{4}\log C_q(K_4)$$

for all cubic graphs G, but this result was recently proved for all d by Csikvári (see [104]).

Theorem 4.9 (Csikvári)**.** *For all d, all $q \geq 2$, and all d-regular G,*

$$\frac{1}{|V(G)|}\log C_q(G) \geq \frac{1}{d+1}\log C_q(K_{d+1})\,.$$

Csikvári and Lin [21] also proved that for any d-regular, bipartite G,

$$C_q(G)^{1/|V(G)|} \geq q \left(\frac{q-1}{q} \right)^{d/2} ,$$

a result that, for q large enough as function of d, is tight asymptotically for a sequence of bipartite graphs of diverging girth.

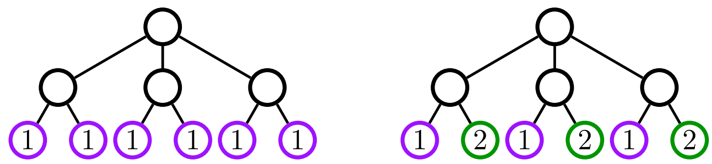
4.2 Proof of Theorem 4.1

In this section we prove Theorem 4.1 by formulating and solving the linear program described in the introduction. For brevity we drop the superscripts in notation for partition functions and internal energy of a graph G, writing Z_G and U_G for these quantities.

Recall the experiment which defines the local view: sample a colouring $\boldsymbol{\sigma} : V(G) \to [q]$ according to the q-colour Potts model with inverse temperature β and independently, uniformly at random sample a vertex $v \in V(G)$. The local view consists of the induced subgraph of G on $v \cup N(v)$, together with, for each $u \in N(v)$, the multiset of colours that appears in $N(u) \setminus (\{v\} \cup N(v))$. Four examples are pictured in Figure 4.1. As noted in the introduction, our calculations depend only on the *number* of 'external neighbours' of vertices $u \in N(v)$ which receive each colour, and not the graph structure between these external neighbours. For clarity we draw the vertices themselves. Let $\mathcal{C}_q$ denote all possible local views for the q-colour Potts model on cubic graphs.

As q grows larger the number of possible local views grows like $q^{d(d-1)}$. However, if we consider equivalence classes of the local views under permutations of the colours (as detailed below), the number of possible local views is bounded in terms of d. This makes the linear program finite for any fixed d. For a complete list of local views (up to equivalence) for $d = 3$ see Appendix A.

Suppose that the local view C arises from selecting the colouring σ and the vertex v. Recall that we refer to the coloured vertices at distance two from v as the *boundary*, and write V_C for the set of uncoloured vertices in C, so that the set of q-colourings of these vertices is $[q]^{V_C}$. The colouring $\boldsymbol{\sigma}$ induces a random *local colouring* $\boldsymbol{\chi} : V_C \to [q]$ that, by the spatial Markov property



(a) Local view C_1 (b) Local view C_2

(c) Local view with 1 triangle (d) Local view with 2 triangles

Figure 4.1: Example local views. Figures (a) and (b) show, up to permutations of the colours, the only local views which can arise in $K_{3,3}$. The coloured, numbered vertices are representations of the multiset of colours that appear in the boundary $N(u) \setminus (\{v\} \cup N(v))$ for $u \in N(v)$.

of the Potts model, is distributed according to the Potts model on C. For $\chi : V(C) \to [q]$, write $m(\chi)$ for the total number of monochromatic edges in C (including any monochromatic edges between V_C and the boundary), and given a vertex $u \in V_C$ write $m_u(\chi)$ for the number of monochromatic edges in C incident to u. Then, with the local partition function defined as

$$Z_C^q(\beta) = \sum_{\chi : V_C \to [q]} e^{-\beta m(\chi)} \,,$$

a local colouring χ is distributed according to

$$\mathbb{P}[\chi | C] = \frac{e^{-\beta m(\chi)}}{Z_C^q(\beta)} \,.$$

This fact means that we can interpret the internal energy per particle as an expectation over the random local view C and local colouring $\boldsymbol{\chi}$. Each edge of G is incident to exactly two vertices, hence

$$\begin{aligned}U_G(\beta) &= \frac{1}{|V(G)|}\mathbb{E}_{\boldsymbol{\sigma}}[m(\boldsymbol{\sigma})]\\ &= \frac{1}{2|V(G)|}\sum_{v\in V(G)}\sum_{u\in N(v)}\mathbb{P}(uv \text{ monochromatic})\\ &= \frac{1}{2}\mathbb{E}_C\Big[\sum_{u\in N(v)}\mathbb{P}(uv \text{ monochromatic}|C)\Big]\\ &= \frac{1}{2}\mathbb{E}_{C,\boldsymbol{\chi}}[m_v(\boldsymbol{\chi})]\,.\end{aligned}$$

Moreover, since G is regular, a neighbour of v chosen uniformly at random is distributed uniformly over $V(G)$, giving

$$U_G(\beta) = \frac{1}{2}\mathbb{E}_{C,\boldsymbol{\chi}}\Big[\frac{1}{3}\sum_{u\in N(v)} m_u(\boldsymbol{\chi})\Big]\,.$$

Given these observations, for a local view C we define

$$\begin{aligned}U_C^v &= \frac{1}{2Z_C}\sum_{\chi\in[q]^{V_C}} m_v(\chi)e^{-\beta m(\chi)}\,,\\ U_C^N &= \frac{1}{6Z_C}\sum_{\chi\in[q]^{V_C}}\sum_{u\in N(v)} m_u(\chi)e^{-\beta m(\chi)}\,,\end{aligned}$$

so that

$$U_G(\beta) = \mathbb{E}_C[U_C^v] = \mathbb{E}_C[U_C^N]\,,$$

giving us a constraint on probability distributions on local views that holds for all distributions arising from graphs.

We can now define the two linear programs for the q-colour Potts model,

$$\begin{aligned}\{U^{\min}, U^{\max}\} = \{\min,\max\}\sum_{C\in\mathcal{C}_q} p_C U_C^v \qquad &\text{s.t.}\\ p_C \geq 0\ \forall C\in\mathcal{C}_q\,,&\\ \sum_{C\in\mathcal{C}_q} p_C = 1\,,&\\ \sum_{C\in\mathcal{C}_q} p_C(U_C^v - U_C^N) = 0\,.&\end{aligned}$$

4.2.1 Minimising

For the minimisation problem, the dual program (with variables Λ, Δ) is

$$\begin{aligned} U^{\min} = {} & \max \Lambda \qquad \text{subject to} \\ & \Lambda + \Delta(U_C^v - U_C^N) \le U_C^v \qquad \forall C \in \mathcal{C}_q \,. \end{aligned}$$

For a given $\beta > 0$ and $q \ge 2$, to show that $U^{\min} = U_{K_{3,3}}$ via linear programming duality, we must find Δ^* so that the assignment $\Lambda = U_{K_{3,3}}$, $\Delta = \Delta^*$ is *feasible* for the dual. That is,

$$U_{K_{3,3}} + \Delta^*(U_C^v - U_C^N) \le U_C^v \tag{4.3}$$

for all $C \in \mathcal{C}_q$.

In fact it suffices to show (4.3) on a subset of $\mathcal{C}_q$. We say $C, C' \in \mathcal{C}_q$ are equivalent if

$$U_C^v = U_{C'}^v \quad \text{and} \quad U_C^N = U_{C'}^N$$

as functions of q and β. For instance if C is obtained from C' by a permutation of the colours, then C and C' are equivalent by symmetry. This equivalence relation partitions $\mathcal{C}_q$ into equivalence classes. Call this set of equivalence classes $\mathcal{C}'_q$. We always choose a representative member of the equivalence class that has an initial segment of the colours $[q]$ on its boundary, and write q_C for the total number of colours on the boundary of a local view C. For $d = 3$ and arbitrary q there are 35 non-isomorphic equivalence classes which we list in Appendix A.

We give the local views that arise in the optimising graphs names, writing C_1 and C_2 (see Figure 4.1) for representatives of the only two equivalence classes of local views are that can appear with positive probability when $G = K_{3,3}$. In K_4, the only local view that can arise is isomorphic to K_4 itself.

To find a value of Δ^* we solve the dual constraint (4.3) for the local view C_1 (see Figure 4.1a) to hold with equality when $\Lambda = U_{K_{3,3}}$:

$$U_{K_{3,3}} + \Delta^*(U_{C_1}^v - U_{C_1}^N) = U_{C_1}^v \,.$$

Writing $\lambda = e^{-\beta}$ (so that $0 < \lambda < 1$), we find

$$\begin{aligned}
Z_{C_1} &= (\lambda^3 + q - 1)^3 + (q-1)(\lambda^2 + \lambda + q - 2)^3 \,, \\
U^v_{C_1} &= \frac{3}{2Z_{C_1}} \left(\lambda^3(1+\lambda^3)^2 + (q-1)\lambda^2(\lambda^2+\lambda+q-2)^2\right) \\
U^N_{C_1} &= \frac{1}{2Z_{C_1}} \left(3\lambda^3(1+\lambda^3)^2 + (q-1)(\lambda+2\lambda^2)(\lambda^2+\lambda+q-2)^2\right)
\end{aligned}$$

and hence

$$U^N_{C_1} - U^v_{C_1} = \frac{1}{2Z_{C_1}} \lambda(1-\lambda)(q-1)(\lambda^2+\lambda+q-2)^2 \,.$$

Then we calculate

$$\begin{aligned}
Z_{K_{3,3}} &= q(\lambda^3+q-1)^3 + 3q(q-1)(\lambda^2+\lambda+q-2)^3 + \\
&\qquad\qquad\qquad q(q-1)(q-2)(3\lambda+q-3)^3 \,, \\
U_{K_{3,3}} &= \frac{3q}{2Z_{K_{3,3}}} \Bigg(\lambda^3(\lambda^3+q-1)^2 + \lambda(q-1)(q-2)(3\lambda+q-3)^2 + \\
&\qquad\qquad (q-1)(2\lambda^2+\lambda)(\lambda^2+\lambda+q-2)^2 \Bigg) \,,
\end{aligned}$$

and hence

$$\begin{aligned}
\Delta^* = -\frac{3q(1-\lambda)^2}{2(\lambda^2+\lambda+q-2)^2 Z_{K_{3,3}}} \Big[& 2\left(\lambda^2+2\lambda-1\right) q^5 \\
& + 2\left(\lambda^4+9\lambda^3+22\lambda^2+27\lambda+13\right)(\lambda-1)^6 \\
& + \left(\lambda^5+3\lambda^4+20\lambda^3-41\lambda+17\right) q^4 \\
& + \left(8\lambda^4+31\lambda^3+91\lambda^2+47\lambda-57\right)(\lambda-1)^2 q^3 \\
& + \left(25\lambda^4+82\lambda^3+146\lambda^2+22\lambda-95\right)(\lambda-1)^3 q^2 \\
& + \left(2\lambda^5+36\lambda^4+87\lambda^3+93\lambda^2-31\lambda-79\right)(\lambda-1)^4 q \Big] \,.
\end{aligned} \tag{4.4}$$

The term *slack* is used to describe the size of the left-hand side in the (slightly rearranged) dual constraint

$$U^v_C + \Delta^*(U^N_C - U^v_C) - U_{K_{3,3}} \geq 0 \,.$$

Considering the slack as a function S_C of C (and d, q, β), dual feasibility reduces to the following claim.

Claim 4.10. *For all $q \geq 2$ and all $\beta > 0$, the function*

$$\mathrm{S}_C = U_C^v + \Delta^*(U_C^N - U_C^v) - U_{K_{3,3}}$$

with Δ^* given by (4.4) *is identically* 0 *for $C \in \{C_1, C_2\}$ and strictly positive for all other $C \in \mathcal{C}'_q$.*

Claim 4.10 immediately proves that $U_G^q(\beta) \geq U_{K_{3,3},}^q(\beta)$. To show uniqueness, observe that strict positivity of the slack function implies via complementary slackness (see [13]) that the support of any distribution achieving the optimum must be contained in $\{C_1, C_2\}$; $K_{3,3}$ is the only connected graph whose distribution satisfies this. To see this note that, for any other connected cubic graph, there exists a vertex v with two neighbours u_1, u_2 such that the external neighbourhoods of u_1 and u_2 are distinct. Then there exists a colouring such that the external neighbours of u_1 are monochromatic, whilst those of u_2 are not. This means a local view not isomorphic to C_1 or C_2 appears with positive probability.

In order to prove Claim 4.10, we change variables and multiply the slack by a positive scaling factor, carefully chosen to result in a polynomial with positive coefficients. Write $r = q - 3$ and $t = e^\beta - 1 = 1/\lambda - 1$, so that for any $q \geq 3$ and $\beta > 0$ we have $r \geq 0$ and $t > 0$. It then suffices to show that the following scaling of the slack is non-negative:

$$\tilde{S}_C = \frac{4(1+t)^{17}(r(1+t)^2 + t^2 + 3t + 3)^2}{(3+r)t^2} Z_{K_{3,3}} Z_C \cdot \mathrm{S}_C \ . \tag{4.5}$$

In fact something stronger is true:

Claim 4.11. *For all $C \in \mathcal{C}'_q$, $\tilde{S}_C$ is a bivariate polynomial in r and t with all coefficients positive. The polynomial is identically* 0 *if and only if $C \in \{C_1, C_2\}$.*

For the case $q = 2$ we do something slightly different.

Claim 4.12. *For all $C \in \mathcal{C}'_2$ (which necessarily use at most* 2 *colours on the boundary), evaluating $\tilde{S}_C$ at $r = -1$ yields a polynomial in t with positive coefficients. The polynomial is identically* 0 *if and only if $C \in \{C_1, C_2\}$.*

Claims 4.11 and 4.12 are proved by simply computing the functions $\tilde{S}_C$ for each of the 35 equivalence classes in $\mathcal{C}'_q$, simplifying and collecting coefficients.

We include in Appendix B a computer program for the Sage mathematical software used to compute $\tilde{S}_C$ for each local view C, but we emphasise that we use a computer just to multiply polynomials and collect coefficients. Each of the steps the program performs are readily achievable by hand, though the number of steps and the size of the polynomials involved make this unappealing.

The output shows a list of $\tilde{S}_C$ for all 35 non-isomorphic local views, demonstrating that it is zero for C_1 and C_2 and a non-zero polynomial in r and t with non-negative coefficients for all other C. It also shows $\tilde{S}_C$ evaluated at $r = -1$ for local views C which use at most 2 colours on the boundary, yielding a non-zero polynomial in t with non-negative coefficients for all such C except C_1 and C_2, as desired.

4.2.2 Maximising

To show that K_4 is the unique maximiser of the linear program is somewhat more straightforward than the minimisation problem, largely because only one local view can arise in K_4; K_4 itself (with no boundary vertices). Since the distribution yielding K_4 as a local view with probability one is feasible in the linear program, it suffices to show that $U^v_{K_4} > U^v_C$ for all $C \neq K_4$.

Claim 4.13. *Let*

$$D^v_C = 2(1+t)^{14} Z_{K_4} Z_C t^{-2} (U^v_{K_4} - U^v_C)\,. \tag{4.6}$$

Then for all $C \in \mathcal{C}_q$, D^v_C is a polynomial $t = e^\beta - 1 = 1/\lambda - 1$ and $s = q - \max\{3, q_C\}$ with all positive coefficients, and identically 0 if and only if $C = K_4$.

Since local views with $q_C > q$ cannot occur, for $q \geq 3$ and $\beta > 0$ we have $s \geq 0$ and $t \geq 0$ and hence Claim 4.13 implies $U^v_{K_4} > U^v_C$ for all $C \neq K_4$. The quantity D^v_C is listed for all 35 non-isomorphic local views in Appendix A. Again, for $q = 2$ we must do more; for $C \in \mathcal{C}'_2$ we list D^v_C evaluated at $q = 2$, observing that it is a polynomial in t with non-negative coefficients, except for K_4 where it is identically zero.

As with the computations for $\tilde{S}_C$, we use a computer to multiply polynomials and obtain D^v_C for each local view, see Appendix B.

4.3 Extensions to $d \geq 4$

How might we extend Theorem 4.1 to graphs of larger degree? The minimisation program defined above in Section 4.2 is not tight in general: we can in fact see that it is insufficiently constrained by comparing the number of constraints, 2, to the number of equivalence classes of local views in the support of the distribution induced by $K_{d,d}$, which is the partition number of $d-1$ when $q \geq d-1$, and always greater than 2 if $d \geq 4$ and $q \geq 2$.

There is a large family of constraints that we can add to the program. Let $\mathcal{S}_{q,d}$ be the set of all *q-partitions* of size d; that is, partitions of d into at most q parts which we represent by vectors of length q with non-negative integer entries that sum to d, written in non-decreasing order. Any q-colouring χ of d vertices induces a q-partition; for instance if χ assigns the colours $\{1,4,2,2,1,2\}$, then the q-partition $H(\chi) = \{3,2,1,0\} \in \mathcal{S}_{4,6}$. Our family of constraints will be that for every $S \in \mathcal{S}_{q,d}$, the probability that the neighbours of v receive a colouring with q-partition S equals the average probability of the same for a neighbour of v.

Both of these probabilities can be computed as expectations over the random local view. For a local view C and a q-partition $S \in \mathcal{S}_{q,d}$ we define

$$\gamma_C^{v,S} := \frac{1}{Z_C} \sum_{\chi \in [q]^{V_C}} \mathbb{1}_{\{H(\chi(N(v)))=S\}} \cdot e^{-\beta m(\chi)} ,$$
$$\gamma_C^{N,S} := \frac{1}{d} \frac{1}{Z_C} \sum_{\chi \in [q]^{V_C}} \sum_{u \in N(v)} \mathbb{1}_{\{H(\chi(N(u)))=S\}} \cdot e^{-\beta m(\chi)} .$$

Observe that for any graph and any q-partition S, we must have

$$\mathbb{E}_C[\gamma_C^{v,S}] = \mathbb{E}_C[\gamma_C^{N,S}] .$$

Our minimisation program becomes

$$
\begin{aligned}
U^{\min} = \min \sum_C p_C U_C^v \qquad & \text{s.t.} \\
& p_C \geq 0 \ \forall C\,, \\
& \sum_C p_C = 1\,, \\
& \sum_C p_C (\gamma_C^{v,S} - \gamma_C^{N,S}) = 0 \ \forall S \in \mathcal{S}_{q,d}.
\end{aligned}
$$

This program is stronger than the one used in Section 4.2: the q-partition constraints together imply the constraint $\mathbb{E}_C[U_C^v] = \mathbb{E}_C[U_C^N]$.

We can solve this program for small values of d and fixed β, which leads us to the following conjecture.

Conjecture 4.14. *The above minimisation program is tight for $d \geq 3, q \geq d+1$ and all $\beta > 0$, and shows that*

$$
U_{K_{d,d}}^q(\beta) \leq U_G^q(\beta)
$$

for all d-regular G.

However we can also find values of β for $d \geq 4, q \leq d$ so that $U^{\min} < U_{K_{d,d}}^q$, and so we believe that this program is not tight in these cases.

4.4 2-regular graphs and other easy cases

Theorem 4.1 shows that $K_{3,3}$ is optimal on the level of internal energy per particle in the Potts model, and by Corollary 4.8 it maximises $\frac{1}{|V(G)|}\log C_q(G)$ over cubic graphs G. For arbitrary d, in the case $q = 2$, the fact that $K_{d,d}$ maximises $\frac{1}{|V(G)|}\log C_q(G)$ over d-regular G follows simply from the observation that $K_{d,d}$ is the smallest bipartite d-regular graph. Indeed for $q = 2$, if G is not bipartite then $C_q(G) = 0$ and if G is bipartite $C_q(G) = 2^{c(G)}$, where $c(G)$ is the number of connected components of G.

For $d = 2$, the only d-regular connected graphs are cycles, and there is an explicit formula for the q-colour Potts partition function of the n-cycle. In the language of statistical physics the 1-dimensional Potts model (including the 0-temperature Potts model) is *exactly solvable*:

$$Z^q_{C_n}(\beta) = (q-1)(e^{-\beta}-1)^n + (e^{-\beta}+q-1)^n.$$

One way to obtain this formula is to use the mapping of the Tutte polynomial $T(x,y)$ to the Potts partition function, given in [99], and then using the formula $T_{C_n}(x,y) = \frac{x^n - x}{x-1} + y$.

Taking the logarithmic derivative gives:

$$U^q_{C_n}(\beta) = \frac{e^{-\beta}}{e^{-\beta}-1} \cdot \frac{\left(1+\frac{q}{e^{-\beta}-1}\right)^{n-1} + q - 1}{\left(1+\frac{q}{e^{-\beta}-1}\right)^{n} + q - 1} \tag{4.7}$$

Proposition 4.15. *If $\beta > 0$ then*

$$U^q_{C_n}(\beta) > U^q_{C_{n+2}}(\beta) \text{ for } n \geq 3 \text{ odd},$$
$$U^q_{C_n}(\beta) < U^q_{C_{n+2}}(\beta) \text{ for } n \geq 4 \text{ even}.$$

If $\beta < 0$ then

$$U^q_{C_n}(\beta) > U^q_{C_{n+1}}(\beta) \text{ for all } n \geq 3.$$

Proof. Let $\beta > 0$ and suppose that $n \geq 3$ is odd. Let $x := 1 + \frac{q}{e^{-\beta}-1}$. By (4.7), we then have that $U^q_{C_n}(\beta) > U^q_{C_{n+2}}$ if and only if

$$x^n + x^{n+1} > x^{n-1} + x^{n+2}. \tag{4.8}$$

Since n is odd, (4.8) holds if and only if $x + x^2 > 1 + x^3$ which holds since $x < -1$. For even $n \geq 4$, the proof is the same.

Suppose now that $\beta < 0$ and $n \geq 3$. Defining x as before, we have that $x > 0$. In this case, the inequality $U^q_{C_n}(\beta) > U^q_{C_{n+1}}(\beta)$ simply reduces to the inequality $x^{n-1}(1-x)^2 > 0$. □

Letting $\mathbb{Z}_1$ denote a doubly-infinite path we have that

$$U^q_{\mathbb{Z}_1}(\beta) = \frac{e^{-\beta}}{e^{-\beta}+q-1} = \lim_{n\to\infty} U^q_{C_n}(\beta). \tag{4.9}$$

Taking the limit here is justified as the Potts model on $\mathbb{Z}_1$ is in the Gibbs uniqueness regime for all $q, \beta > 0$.

Corollary 4.16. *Conjectures 4.2, 4.5, and 4.6 hold for $d = 2$. Moreover, If $\beta > 0$ then*

$$U^q_{C_n}(\beta) > U^q_{\mathbb{Z}_1}(\beta) \text{ for } n \geq 3 \text{ odd},$$
$$U^q_{C_n}(\beta) < U^q_{\mathbb{Z}_1}(\beta) \text{ for } n \geq 4 \text{ even}.$$

If $\beta < 0$ then

$$U^q_{C_n}(\beta) > U^q_{\mathbb{Z}_1}(\beta) \text{ for all } n \geq 3.$$

Proof. This follows from Proposition 4.15 and (4.9). □

5

Hypergraph embedding

In this chapter we have two main results. The first is an inheritance lemma for uniform hypergraphs, which works for relatively dense subgraphs of sufficiently well-behaved ambient hypergraphs. The second main result is to define the concept of a *good partial embedding*. This concept characterises a setting where we are trying to find a homomorphism from some hypergraph H to a hypergraph G, and we have decided on images for some vertices of H, so we have a *partial embedding*. Such a setting is *good* if certain conditions are satisfied which imply, first, that we can choose an image for one further vertex in approximately the expected number of ways, second, that most of these choices again give a good partial embedding, and third, that the atypical choices do not cause too great an error for a counting lemma. Given the correct definition, the proof that it has these properties becomes quite straightforward. Using these two results, we prove a counting lemma. Slightly weaker counting lemmas have been proved in the literature (e.g. [76] for dense hypergraphs) or follow fairly easily from known results (e.g. [17] for sparse hypergraphs); we prove this version partly as a demonstration that our method allows for a direct generalisation of the approach used in dense graphs, and partly because it is needed as part of an inductive proof that a good partial embedding has the property we claim.

Finally, we use the good partial embedding approach to obtain lower bounds on the number of homomorphisms from H to G in the setting where H may have a large (growing with $v(G)$) number of vertices. Such lower bounds are not available from the older approaches. Of course, what one would really like is an analogue of the blow-up lemma [63] that gives the existence not just of many homomorphisms from H to G, but also the existence of an injective homomorphism (with positive weight), that is, of an embedding of H in G, under suitable conditions. We avoid such details here as we intend in future work to prove a blow-up lemma of this form.

The rest of this chapter is organised as follows. In Section 5.1 we state the main results, including the inheritance lemma, counting and embedding results, and the definition of a good partial embedding and other necessary concepts. In Section 5.2 we use the inheritance lemma to prove embedding and counting results for good partial embeddings. Section 5.3 contains a discussion of hypergraph regularity lemmas and the usual embedding and counting results in this setting. We prove slightly stronger versions of standard embedding and counting results for hypergraphs by constructing a good partial embedding and applying the results of Section 5.2. The technical work of the chapter is largely contained in the last three sections: in Section 5.4 we develop auxiliary results based on the Cauchy–Schwarz inequality which are used throughout the chapter, in Section 5.5 we prove the inheritance lemma, and finally in Section 5.6 we prove that a suitable random hypergraph has the pseudorandomness property necessary for our counting and inheritance lemmas.

5.1 Main results

Before we can state our main results, we need some definitions. Before we give these, we explain the way we will view hypergraphs in this chapter. The usual definition of a *hypergraph* consists of a vertex set V and edge set E containing subsets of V. Normally one is interested in *k-uniform hypergraphs* which are hypergraphs with the additional condition that E only contains subsets of V of size exactly k. When considering hypergraph regularity, one is often forced to consider *k-complexes* which correspond to a union of ℓ-uniform hypergraphs for $\ell \in [k]$ on the same vertex set with the additional property that the edge set E of a complex is down-closed: if $f \in E$ and $e \subseteq f$ then $e \in E$. We prefer to give alternative definitions to better separate the roles of complexes and hypergraphs in our methods.

We are primarily interested in finding homomorphisms from a complex H (as above) to a complex G that in applications is usually in some way 'inspired by' a uniform hypergraph. For this reason we exclusively use *complex* to refer to the object H whose vertices form the domain of the homomorphism, and *hypergraph* to refer to the 'host graph' G whose vertices form the image of the homomorphism. In particular, contrasting with usual 'uniform' usage, our definition of hypergraph allows for edges of each size from 0 upwards. Later we will generalise the hypergraphs to *weighted hypergraphs*, but we are not interested in weights on H, and so do not refer to 'weighted complexes'.

If for some $k \geq 1$, the complex H contains no edges of size greater than k, we say H is a k-complex (we do not insist that H contains edges of size exactly k). We will, however, not insist that our hypergraphs G are necessarily down-closed. As we will see when we come to the definition of a homomorphism, an edge of G whose subsets are not all contained in G cannot play a role in any homomorphisms from H to G, but it will nevertheless be convenient in the proof to allow such edges.

More importantly, we will work throughout with weighted hypergraphs. Given a vertex set V, a *weighted hypergraph* is a function from the power set of V to the non-negative reals. We think of a normal unweighted hypergraph as being equivalent to its characteristic function. This extra generality turns out not to complicate the proofs, and to rather simplify the notation. It is not essential to our approach; if one starts with unweighted hypergraphs, the functions appearing throughout will take only values $\{0,1\}$; that is, they are unweighted hypergraphs.

We use the letter Γ and calligraphic letters $\mathcal{G}$, $\mathcal{H}$ for weighted hypergraphs, and the corresponding lower case letters γ, g, and h for the weight functions.

A *homomorphism* ϕ from a complex H to a weighted hypergraph $\mathcal{G}$ is a map $\phi : V(H) \to V(\mathcal{G})$ such that $\big|\phi(e)\big| = |e|$ for each $e \in H$, and the *weight* of ϕ is

$$\mathcal{G}(\phi) := \prod_{e\in H} g\big(\phi(e)\big)\,.$$

Note that this product does run over $e = \emptyset$ and edges of size 1 in H. If $\mathcal{G}$ is an unweighted hypergraph, then the weight of ϕ is either 0 or 1, taking the latter value if and only if $\phi(e)$ is an edge of $\mathcal{H}$ (in the usual unweighted sense) for each $e \in F$, including edges of size one (vertices). In other words, this is if and only if ϕ is a homomorphism according to the usual unweighted definition from H to $\mathcal{G}$. We will be interested in summing the weights of homomorphisms, which is thus equivalent for unweighted hypergraphs to counting homomorphisms by the usual definition. Slightly abusing terminology for the sake of avoiding unwieldy phrases, we will talk about 'counting homomorphisms' or 'the number of homomorphisms' when what we really mean is the sum of weights of homomorphisms.

Bearing in mind that our weighted hypergraphs are 'inspired by' k-uniform hypergraphs, we wish to consider weighted hypergraphs which contain edges of size $0, 1, \ldots, k$, but not of size $k+1$. If the weight function is to generalise the indicator function for edges in the unweighted setting then we should say that $\mathcal{G}$ is a k-graph to mean that $g(e) = 0$ for any edge e of size at least $k+1$. We prefer an alternative definition for convenience of notation. If one is interested in weights in $\mathcal{G}$ of edges up to size k, one can ask for a homomorphism from a k-complex into $\mathcal{G}$, which naturally excludes any edges of size at least $k+1$.

It is more convenient for our purposes to say that $\mathcal{G}$ is a *k-graph* to mean that $g(e) = 1$ for all edges e of size at least $k+1$, so that such edges do not affect the weight of any homomorphisms into $\mathcal{G}$. This affords a certain amount of flexibility in the homomorphism counting methods we develop. For example, let H be a $(k+1)$-simplex (the down-closure of a single edge of size $k+1$), and H' be obtained from H by removing the edge of size $k+1$. If $\mathcal{G}$ is a k-graph then homomorphisms from H and H' to $\mathcal{G}$ receive the same weight and we do not need to distinguish between them.

We are usually not interested in counting general homomorphisms from H to $\mathcal{G}$; for simplicity we reduce to a *partite setting* where we have identified special image sets in $V(\mathcal{G})$ for each vertex of H. More formally, for the partite setting we will have a complex H on vertex set X, a k-graph $\mathcal{G}$ on vertex set V, a partition of X into disjoint sets $\{X_j\}_{j\in J}$ indexed by J, and a partition of V into disjoint sets $\{V_j\}_{j\in J}$ indexed by J. The sets X_j and V_j are called *parts*. We say a set of vertices (e.g. in X) is *crossing*, or *partite* if it contains at most one vertex from each part. As a shorthand, we say that H, $\mathcal{G}$ are J-partite to mean we have this setting; partitions of $V(H)$ and $V(\mathcal{G})$ indexed by J. If only the number of indices matters, we sometimes write e.g. k-partite to mean J-partite for some set J of size k.

Given this partite setting, a *partite homomorphism* from H to $\mathcal{G}$ is a homomorphism from H to $\mathcal{G}$ that maps each X_j into V_j. That is, given an index set J and partitions of X and V indexed by J, we consider special homomorphisms from X to V that 'respect' the partition. Given $x \in X_j$ we sometimes write V_x for the part V_j into which we intend to embed x; and for a subset e of X we write $V_e = \prod_{x\in e} V_x$ for the collection of partite $|e|$-sets with vertices in $\bigcup_{x\in e} V_x$.

We will always be interested in counting partite homomorphisms, Turning to counting weighted homomorphisms, and given the partite setup above we write $\mathcal{G}(H)$ for the expected weight of a uniformly random partite homomorphism from H to $\mathcal{G}$, that is, the normalised sum over all partite homomorphisms ϕ from H to $\mathcal{G}$ of the weight of ϕ,

$$\mathcal{G}(H) := \mathbb{E}\Big[\prod_{e\in H} g(\phi(e))\Big] = \Big(\prod_{j\in J}|V_j|^{-|X_j|}\Big)\sum_{\phi}\prod_{e\in H} g(\phi(e))\,.$$

If $\mathcal{G}$ is constant on the sets V_f for crossing $f \subseteq V(H)$, then we obtain $\mathcal{G}(H) = \mathcal{G}(\phi)$ for any partite homomorphism $\phi : H \to \mathcal{G}$, and a counting lemma states that $\mathcal{G}(H)$ is close to this 'expected value'.

In this chapter we will primarily work with partite homomorphisms which map exactly one vertex of H into each part of $\mathcal{G}$. We reduce the general setting to this one-vertex-per-part setting by the following somewhat standard 'copying process'.

Definition 5.1 (Standard construction)**.** Given an index set J, a k-complex H with vertex set X partitioned into $\{X_j\}_{j\in J}$ and a k-graph $\mathcal{G}$ with vertex set V partitioned into $\{V_j\}_{j\in J}$, the *standard construction* is as follows. Let $\mathcal{G}'$ be an X-partite k-graph with vertex sets $\{V'_x\}_{x\in X}$ where for each $x \in X$, the set V'_x is a copy of V_x, and where for each set $f \subseteq V(H)$ and each $e \in V'_f$ we define

$$g'(e) := \begin{cases} 1 & \text{if } f \notin H \\ g(e') & \text{if } f \in H\,, \end{cases}$$

where e' is the natural projection of e to $V(\mathcal{G})$.

This construction defines a new k-graph $\mathcal{G}'$ (together with a partition of its vertices indexed by X) whose vertices are all copies of vertices in $\mathcal{G}$, with weights given precisely so that J-partite homomorphism counts from H to $\mathcal{G}$ correspond to X-partite homomorphism counts from H to $\mathcal{G}'$. One is forced to consider H as J-partite for the former counts, and X-partite (with parts of size 1) for the latter.

That is, for $f \notin H$ the edges V'_f all have weight one in $\mathcal{G}'$, so for each $\phi : V(H) \to V(\mathcal{G}')$ we have

$$\mathcal{G}(\phi) = \prod_{f \in H} g(\phi(f)) = \prod_{f \subseteq V(H)} g'(\phi(f)) = \mathcal{G}'(\phi)\,,$$

where we abuse notation by identifying ϕ with its natural projection onto $V(\mathcal{G})$. Since it simplifies notation, in what follows we will always assume that if we are embedding some k-complex H to some $V(H)$-partite k-graph $\mathcal{G}$, and for $f \notin H$ we suppose $g(e) = 1$ for all $e \in V_f$.

We now define the *link graph* of a vertex v in a weighted hypergraph. Let J be an index set containing a 'special value' e.g. 0, and let $\mathcal{G}$ be a graph with vertex sets $\{V_j\}_{j \in J}$. For a vertex $v \in V_0$, let $\mathcal{G}_v$ be the graph on $\{V_j\}_{j \in J \setminus \{0\}}$ with weight function g_v defined as follows. For $f \subseteq J \setminus \{0\}$ and $e \in V_f$, we set

$$g_v(e) := g(e) \cdot g(v, e)\,.$$

Note that we write $g(v, e)$ for the more cumbersome $g(\{v\} \cup e)$ and we do allow $e = \emptyset$ in this definition.

We can now explain part of our embedding scheme. Given a complex H and a $V(H)$-partite setup where we want to find a partite homomorphism from H to a weighted graph $\mathcal{G}$, we start with a trivial partial embedding ϕ_0 from $H_0 := H$ to $\mathcal{G}_0 := \mathcal{G}$ in which no vertices are embedded. Now for each $t = 1, \ldots, v(H)$ in succession, we choose a vertex x_t of H_{t-1} and a vertex v_t of V_{x_t}. We set $\phi_t := \phi_{t-1} \cup \{x_t \to v_t\}$, we set $H_t := H_{t-1} \setminus \{x_t\}$, and set $\mathcal{G}_t := (\mathcal{G}_{t-1})_{v_t}$, that is we take the link graph. The graph $\mathcal{G}_{v(H)}$ is an empty weighted graph with weight function $g_{v(H)}$: the only edge it contains is the empty set, and its weight is

$$g_{v(H)}(\emptyset) = \prod_{e \subseteq V(H)} g(\phi(e)) = \mathcal{G}(\phi)\,.$$

This is the vertex-by-vertex embedding mentioned in the introduction. Obviously, in general the final value $\mathcal{G}(\phi)$ depends on the choices of the v_t made along the way. An important special case to bear in mind, however, is when for each $f \subseteq V(H)$ the function g is constant, say equal to $d(f)$, on V_f. In this case whatever choice we make at each step, we obtain the same answer. Furthermore, (trivially) at each step t, when we are to choose v_t the average weight in $\mathcal{G}_{t-1}$ of vertices in V_{x_t} depends only on the values $d(f)$ and not on the choices made; and a similar statement is true for the edges in each V_f. We will refer to these average weights in a constant $\mathcal{G}_{t-1}$ as the *expected values* (for given values of $d(f)$); we will want to compare the values obtained in this process when $\mathcal{G}$ is not necessarily constant on each V_f, but the average weight of edges $e \in V_f$ is roughly equal to $d(f)$, with these expected values obtained from the constant weighted graph.

In the case where a partition of $V(\mathcal{G})$ is indexed by a set J, and we have constants $d(f)$ which represent the average weight $\mathcal{G}$ gives to edges in V_f, we can reuse the notation for k-graphs to represent the product of densities that form the expected value of $\mathcal{G}(\phi)$. More formally, we have a k-graph $\mathcal{D}$ on vertex set J whose weight function is $f \mapsto d(f)$. If $\mathcal{G}$ is constant on each V_f, then (trivially) we have $\mathcal{G}(F)/g(\emptyset) = \mathcal{D}(F)/d(\emptyset)$. More generally, if $\mathcal{G}$ is not constant, but the edges are well-distributed (in a sense we will make precise later) and the density on each V_f is about $d(f)$, we will say $\mathcal{D}$ is *a density graph* for $\mathcal{G}$. Note that we *do not* insist that densities are given exactly by $\mathcal{D}$ (we allow a small error which we will specify later) and hence $\mathcal{D}$ is not given uniquely by $\mathcal{G}$. This turns out to be convenient for notation. Furthermore, at this point the reader already sees that our definition of weighted hypergraph, in which the empty set is given a weight, is not always quite convenient. We cannot keep control of the weight of the empty set, and as a result we have to scale explicitly by it in many formulae. But this piece of 'formal nonsense' does serve a purpose (keeping track of the embedded weight in a partial embedding) for which we would otherwise have to invent further notation, and avoids our having to explicitly exclude it throughout the argument.

Observe that if $\mathcal{D}$ is a density graph for the k-graph $\mathcal{G}$ (in our usual partite setting), and we have $j \in J$ and $v \in V_j$, then we would expect that the link graph $\mathcal{D}_j$ is a density graph for $\mathcal{G}_v$. Certainly this occurs in the model situation that $\mathcal{G}$ is constant on each V_f; much of the point of this chapter is that it also occurs when $\mathcal{G}$ is not constant but sufficiently well-behaved. However we should stress that when $\mathcal{G}$ is not constant, even if we have $g(\emptyset) = d(\emptyset)$, typically the weight of the empty set will be different in $\mathcal{D}_j$ and

$\mathcal{G}_v$, simply because although the average weight of vertices in V_j is $d(j)$, it may well be that no $v \in V_j$ actually has weight close to $d(j)$. This is what we mean by the above 'we cannot keep control of the weight of the empty set'.

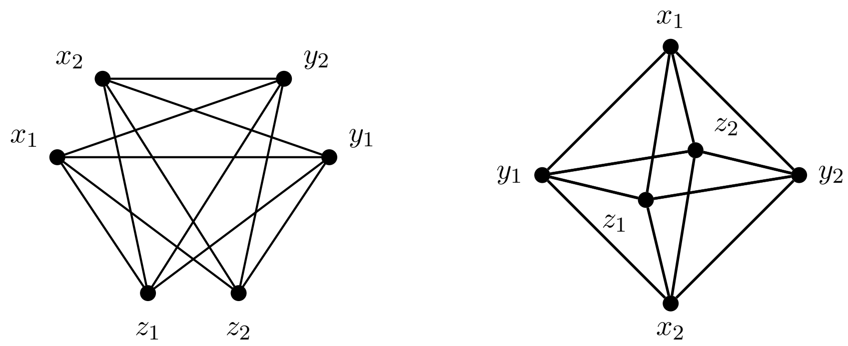
When $\mathcal{G}$ is not a constant hypergraph the value $\mathcal{G}(\phi)$ will depend substantially on the choices made; but a counting lemma states that $\mathcal{G}(H)$, the average (taken over all choices of partite ϕ) of $\mathcal{H}(\phi)$, is, up to a small error, the expected value $\frac{g(\emptyset)}{d(\emptyset)}\mathcal{D}(H)$. Roughly, we will prove it by following the above vertex-by-vertex embedding and showing that for most choices of ϕ_t the average weights of vertices (and larger sets) in $\mathcal{G}_t$ are, up to small errors, the expected values, and that the atypical choices do not contribute much to the average. Even more roughly, the underlying idea is that if the weights of edges in $\mathcal{G}$ in each V_f are on average $d(f)$ and furthermore they are evenly distributed then we will obtain this property. It is not too hard to see that even when $\mathcal{G}_{t-1}$ does have the desired good distribution, there can be choices of vertex $v_t = \phi_t(x_t)$ for which $\mathcal{G}_t$ does not have the expected average weights or the desired good distribution. This motivates the need for a *regularity inheritance lemma* which tells us that these bad choices are few.

Before we define precisely what we mean by 'good distribution', namely regularity, we give a word of warning. Taken on its own, the above sketch leads to a proof of a result called the *dense counting lemma*. This is a counting lemma valid only when all of the densities $d(f)$ are much larger than any of the initial error parameters in the regularity of $\mathcal{G}$. In practice, one does not obtain this ideal situation, and we will have to work harder to prove a more general counting lemma. Nothing in the previous paragraph is false, but there is an omission; namely that we will have to keep track of several additional weighted graphs, each of which must exhibit good behaviour and this leads to extra (but still few) atypical vertices. We come to this in Section 5.1.3.

5.1.1 Regularity of weighted hypergraphs

As mentioned in the introduction, a bipartite graph of density d is ε-regular if and only if the number of copies of C_4 it contains is close to minimal for that density. To generalise this to hypergraphs, we need to define the *octahedron graph.* We will need several related graphs later, so we give the general definition.

Given a vector $\mathbf{a}$ with k non-negative integer entries, we define $O^k(\mathbf{a})$ to be the k-partite complex whose jth part has $\mathbf{a}_j$ vertices, and which contains all crossing i-edges for each $1 \le i \le k$. Let $\mathbf{1}^k$ and $\mathbf{2}^k$ denote the k-vectors all of whose entries are respectively 1 and 2. Then $O^k(\mathbf{1}^k)$ is the complex generated by down-closure of a single k-uniform edge, while $O^k(\mathbf{2}^k)$ is 'the octahedron'. Note that $O^2(\mathbf{2}^2)$ is the down-closure of the usual graph C_4, and Figure 5.1 contains drawings of the octahedron $O^3(\mathbf{2}^3)$ that show it viewed as a complete 3-partite 3-complex and as a Platonic solid.



(a) Tripartite representation (b) Octahedron representation

Figure 5.1: Drawings of $O^3(\mathbf{2}^3)$ where every black line is an edge of size two and every triangle is an edge of size three.

We are now in a position to define regularity for hypergraphs. Even when we are working in the 'dense case', that is, we are thinking of $\mathcal{G}$ as a relatively dense subgraph of the complete hypergraph (as opposed to some much sparser 'ambient hypergraph'), we will often need to introduce a graph Γ which is not complete and of which $\mathcal{G}$ is a relatively dense subgraph. The reader should always think of Γ as being a hypergraph whose good behaviour we have already established (and we are trying to show that $\mathcal{G}$ is also well-behaved).

Definition 5.2 (Regularity of hypergraphs)**.** Given $k \ge 1$ and non-negative real numbers ε, d, let $\mathcal{G}$ and Γ be k-partite hypergraphs on the same vertex parts. Suppose that for each e with $|e| < k$ we have $g(e) = \gamma(e)$, and suppose that for each e with $|e| = k$ we have $g(e) \le \gamma(e)$. Then we say that $\mathcal{G}$ is

(ε, d)*-regular* with respect to Γ if

$$\mathcal{G}(O^k(\mathbf{1}^k)) = (d \pm \varepsilon)\Gamma(O^k(\mathbf{1}^k)) \quad \text{and} \quad \mathcal{G}(O^k(\mathbf{2}^k)) \leq (d^{2^k} + \varepsilon)\Gamma(O^k(\mathbf{2}^k))\,.$$

We say that $\mathcal{G}$ is ε*-regular* with respect to Γ to mean that the corresponding (ε, d)-regularity statement holds with $d = \mathcal{G}(O^k(\mathbf{1}^k))/\Gamma(O^k(\mathbf{1}^k))$.

Note that in this definition we do not specify the octahedron density of $\mathcal{G}$ but only give an upper bound. The definition is only useful for graphs Γ such that a matching lower bound holds for all $\mathcal{G}$, which we will see (Corollary 5.20) is the case when Γ is sufficiently well behaved.

Regularity for k-graphs is not usually discussed for $k = 1$, but we use the notion as a shorthand for relative density in this chapter. The definition makes sense when $k = 1$, but only the first part of the assertion, that $\mathcal{G}$ has density close to d with respect to Γ, is important. For any 1-graph $\mathcal{G}$ on a vertex set V, we have

$$\mathcal{G}(O^1(\mathbf{2}^1)) = \mathbb{E}[g(u)g(v)|u, v \in V] = \mathbb{E}[g(v)|v \in V]^2 = \mathcal{G}(O^1(\mathbf{1}^1))^2\,,$$

and so imposing the upper bound on octahedron count is superfluous, as essentially the same upper bound (the change in ε being unimportant) follows from the density.

5.1.2 Inheritance of regularity

In this section we state our regularity inheritance lemma. Recall that for 2-graphs, a regularity inheritance lemma states that, given vertex sets X, Y and Z, neighbourhoods of vertices $z \in Z$ on one or two sides of the regular pair (X, Y) usually induce regular subpairs (provided that we are working within a sufficiently well-behaved ambient graph). For 2-graphs (see [2, 16]) the cases 'one side' and 'two sides' are usually stated as separate lemmas, and the quantitative requirement for 'well-behaved' are a little different. In this chapter, we will not try to optimise this quantitative requirement and so state one lemma which covers all cases.

In the 2-graph case, in addition to a regularity inheritance lemma one usually needs to make use of the (trivial) observation that given a regular pair (X, Y) in G, if Y' is a subset of Y which is not too small then most vertices in X

have about the expected neighbourhood in Y'. Another way of phrasing this is to define a partite weighted graph $\mathcal{G}$ on $X \cup Y$, with weights on the crossing 2-edges corresponding to edges of G and weights on the vertices of Y being the characteristic function of Y'; then for most $v \in X$ the link 1-graph $\mathcal{G}_v$ has about the expected density (recall that regularity is trivial for 1-graphs). We will need a generalisation of this observation to graphs of higher uniformity, where we will need not only that the link graph typically has the right density but also that it is typically regular. It is convenient to state this too as part of our general regularity inheritance lemma.

Informally, the idea is the following. If $\mathcal{G} \subseteq \Gamma$ are $\{0, \dots, k\}$-partite weighted graphs, which are equal on all edges except those in $V_{[k]}$ and $V_{\{0,\dots,k\}}$, and we have that $\mathcal{G}[V_{[k]}]$ and $\mathcal{G}[V_{\{0,\dots,k\}}]$ are respectively (ε, d)-regular and (ε, d')-regular with respect to Γ, and Γ is sufficiently well-behaved, then for most $v \in V_0$ the graph $\mathcal{G}_v$ is (ε', dd')-regular with respect to Γ_v, where ε' is not too much larger than ε.

To state formally what 'well-behaved' means, we require the following notation for blowups of a k-complex R with vertex set J. Let $\mathbf{a}$ be a vector with non-negative integer entries indexed by J, and write $R(\mathbf{a})$ for the blow-up of R where the jth vertex is blown up into $\mathbf{a}_j$ copies. That is, $R(\mathbf{a})$ has vertex set $\bigcup_{j\in J}\{j_1, \dots, j_{\mathbf{a}_j}\}$ and $e \in R(\mathbf{a})$ if and only if, under the projection $\rho : V(R(\mathbf{a})) \to J$ given by identifying j_i with j, we have $\rho(e) \in R$. This means that if $\mathbf{a}_j = 0$ we remove the jth vertex of R and any edges that contain it. To represent two copies of $R(1, \mathbf{a})$ which share the same first vertex but are otherwise disjoint we write $+2R(0, \mathbf{a})$. In the statement of the lemma we also use notation $\mathcal{H}^{(\ell)}$ to mean the k-graph which gives the same weight as $\mathcal{H}$ to edges of size ℓ but weight 1 to all other crossing edges, a natural extension of this to $\mathcal{H}^{(\leq \ell)}$ to mean the weight has been set to 1 for all crossing edges of size greater than ℓ, and $\mathcal{H} \cdot \mathcal{H}'$ to mean the k-graph whose weight function is the pointwise product $h \cdot h'$. Given two weighted graphs $\mathcal{H}$, $\mathcal{H}'$ on the same vertex set, we say $\mathcal{H} \leq \mathcal{H}'$ to mean that $h \leq h'$ pointwise. Finally, for a single part V_j of a partite k-graph $\mathcal{H}$, and $U \subseteq V_j$, we write $\|U\|_{\mathcal{H}} := \mathbb{E}[\mathbb{1}_{v\in U} h(v)]$, where the expectation is over a uniform choice of $v \in V_j$, so that $\|V_j\|_{\mathcal{H}}$ is the average weight of a vertex in V_j.

Lemma 5.3. *For all $k \geq 1$ and ε', $d_0 > 0$, provided ε, $\eta > 0$ are small* *enough that*

$$\min\{\varepsilon', 2^{-k}\} \geq 2^{2^{k+6}} k^3 (\varepsilon^{1/16} + \eta^{1/32}) d_0^{-2^{k+1}},$$

the following holds for all $\mathcal{P}$ *and all* d, $d' \geq d_0$.

Let $\{V_j\}_{0 \leq j \leq k}$ *be vertex sets, and* $\mathcal{P}$ *be a density graph on* $\{0, \ldots, k\}$. *Let* $\mathcal{G} \leq \Gamma$ *be* $(k+1)$*-partite* $(k+1)$*-graphs on* $V_0, \ldots, V_k$ *such that*

(INH1) *for all complexes* R *of the form* $+2O^k(\mathbf{a})$ *or* $O^{k+1}(\mathbf{b})$, *where* $\mathbf{a} \in \{0,1,2\}^k$ *and* $\mathbf{b} \in \{0,1,2\}^{k+1}$, *we have*

$$\Gamma(R) = (1 \pm \eta) \tfrac{\gamma(\emptyset)}{p(\emptyset)} \mathcal{P}(R),$$

(INH2) $\mathcal{G}$ *gives the same weight as* Γ *to every edge except those of size* $k+1$ *and those in* $V_{[k]}$,

(INH3) $\mathcal{G}^{(k+1)} \cdot \Gamma^{(\leq k)}$ *is* (ε, d')*-regular with respect to* Γ,

(INH4) $\mathcal{G}[V_1, \ldots, V_k]$ *is* (ε, d)*-regular with respect to* $\Gamma[V_1, \ldots, V_k]$.

Then there exists a set $V_0' \subseteq V_0$ *with* $\|V_0'\|_\Gamma \geq (1-\varepsilon')\|V_0\|_\Gamma$ *such that for every* $v \in V_0'$ *the graph* $\mathcal{G}_v$ *is* (ε', dd')*-regular with respect to* Γ_v.

This is the promised regularity inheritance lemma. The quantification of the constants is crucial for the definition of a good partial embedding in the following Section 5.1.3; in order for a useful counting lemma to follow from our approach one needs to be able to control the regularity error parameters at every step of a vertex-by-vertex embedding, and work with underlying densities much smaller than these errors. Observe that in the statement above, the quantities d and d' are relative densities of parts of $\mathcal{G}$ with respect to Γ; they need to be large compared to ε' (and ε) in order for the statement to be interesting, and η also needs to be small compared to ε'. But the densities $p(e)$ from $\mathcal{P}$, which by (INH1) are approximately the absolute densities in Γ, can be (and in applications usually will be) very small compared to all other quantities. In typical applications d, d', ε, ε', η will be constants fixed in a proof and independent of $v(\mathcal{G})$, while the $p(e)$ may well tend to zero as $v(\mathcal{G})$ grows.

5.1.3 Good partial embeddings and counting

At last, we come to the second main result of this chapter, namely the definitions that allow us to work in a hypergraph regularity setting, with a potentially sparse but sufficiently well-behaved ambient hypergraph Γ, and perform embedding and counting vertex by vertex.

To begin with, we need to define what 'sufficiently well-behaved' for the ambient graph Γ means. Roughly, it means that we can count accurately copies of small complexes (and the count corresponds to what we would have in a random hypergraph of the same density) and that this property is typically hereditary in the sense that for most vertices v we can count in the link Γ_v, and we can count in typical links of Γ_v, and so on. The important point separating this definition from simply 'we can count all small subgraphs accurately' is that we may take links a large (depending on the number of vertices of Γ) number of times.

Definition 5.4 (Typically hereditary counting)**.** Given $k \geq 1$, a vertex set J endowed with a linear order, and a density k-graph $\mathcal{D}$ on J, we say the J-partite k-graph $\mathcal{H}$ is an (η, c^*)-*THC graph* if the following two properties hold.

(THC1) For each J-partite k-complex R with at most 4 vertices in each part and at most c^* vertices in total, we have

$$\mathcal{H}(R) = \big(1 \pm v(R)\eta\big)\tfrac{h(\emptyset)}{d(\emptyset)}\mathcal{D}(F)\,.$$

(THC2) If $|J| \geq 2$ and x is the first vertex of J, there is a set $V'_x \subseteq V_x$ with $\|V'_x\|_{\mathcal{H}} \geq (1-\eta)\|V_x\|_{\mathcal{H}}$ such that for each $v \in V'_x$ the graph $\mathcal{H}_v$ is an (η, c^*)-THC graph on $J \setminus \{x\}$ with density graph $\mathcal{D}_x$.

It is trivial that when $\mathcal{H}$ is the complete J-partite k-graph (that is, it assigns weight 1 to all J-partite edges) then for any c^* it is a $(0, c^*)$-THC graph, with density graph $\mathcal{D}$ being the complete k-graph on J (and the ordering on J is irrelevant). This is the setting we obtain (from the standard construction) when we are interested in embedding a k-complex F on J into a dense partite k-graph $\mathcal{G}$, which we think of as a relatively dense subgraph of the complete k-graph on $V(\mathcal{G})$.

More interestingly, if Γ is a sparse k-graph with vertex partition $\{V_j\}_{j\in J}$, and F is a J-partite k-complex which we want to embed into some (relatively dense) subgraph $\mathcal{G}$ of Γ, then applying the standard construction we obtain $X = V(F)$, vertex sets $\{V'_x\}_{x\in X}$, and X-partite k-graphs $\mathcal{G}' \subseteq \Gamma'$ such that we want to find embeddings of F where each vertex $x \in X$ is embedded to V'_x. Note that since X could be comparable in size to $V(\Gamma)$, the graph Γ' could be much bigger than Γ. But if F has small maximum degree, and small degeneracy (see below) then most edges of $\mathcal{G}'$ and Γ' receive weight one. It is easy to check that in this setup, if Γ is a typical random k-uniform hypergraph $G^{(k)}(n,p)$, then the resulting Γ' will be an (η, c^*)-THC graph, provided that p is large enough depending mainly on c^* and two properties of F: the maximum degree and the vertex-degeneracy, which we explain below. See Section 5.6 for a proof the relevant THC property in a random hypergraph.

We write $\Delta(F)$ for the maximum degree of F, which is the largest number of edges in which a single vertex is contained. Since we deal with down-closed F, given any vertex x of F, there are at most $\Delta(F)$ vertices which share an edge of F with x, which we exploit when finding embeddings of F. Given a fixed linear order of $V(F)$, we write $\mathrm{vdeg}(F)$ for the *vertex-degeneracy* of F, which we define as

$$\mathrm{vdeg}(F) := \max_{e\in F}\Big|\{x \in V(F) : x \le y \text{ for all } y \in e, \text{ and } \\ \{x\} \cup e' \in F \text{ for some } \emptyset \ne e' \subseteq e\}\Big| .$$

Then given a fixed ordering of $V(F)$ and a partial embedding ϕ for an initial segment of $V(F)$, for any unembedded edge $e \in F$ there can be at most $\mathrm{vdeg}(F)$ vertices x for which there is $y \in e$ with $\{x, y\} \in F$.

We can now state our counting and embedding lemmas. In the dense case, that is, when Γ is a complete J-partite k-graph, our counting lemma, Theorem 5.5, is more or less the same as that given in [76]. The notion of regularity used there is that obtained by the regularity lemma of [86], which is slightly stronger than the octahedron minimality we use. The embedding lemma, Theorem 5.6, is (as far as we know) not found in this form in the literature, but it does follow fairly easily from [76]; a related but rather harder statement is found in [18]. In the sparse case, our Theorem 5.5 essentially follows from the results of [76] and of [17], though again it is not explicitly stated. However we would like to stress that the main novelty here is that our proofs go by vertex-by-vertex embedding. As is standard in this context, we write e.g. $0 < \eta_0 \ll d_1, \dots, d_k$ to mean that there is an increasing function f such that the argument is valid for $0 < \eta_0 \le f(d_1, \dots, d_k)$.

Theorem 5.5 (Counting lemma for sparse hypergraphs)**.** *For all $k \ge 2$, finite sets J, and J-partite k-complexes F, given parameters η_k, η_0 and ε_ℓ, d_ℓ for $1 \le \ell \le k$ such that $0 < \eta_0 \ll d_1, \dots, d_k$, η_k, and for all ℓ we have $0 < \varepsilon_\ell \ll d_\ell, \dots, d_k$, η_k, the following holds.*

Let $c^* = \max\{2v(F) - 1, 4k^2 + k\}$. For any J-partite weighted k-graphs $\mathcal{G} \subseteq \Gamma$ and density graphs $\mathcal{D}$, $\mathcal{P}$, where Γ is an (η_0, c^*)-THC graph with density graph $\mathcal{P}$, and where for each $e \subseteq J$ of size $1 \le \ell \le k$, the graph $\mathcal{G}[V_e]$ is ε_ℓ-regular with relative density $d(e) \ge d_\ell$ with respect to the graph obtained from $\mathcal{G}[V_e]$ by replacing layer ℓ with Γ, we have

$$\mathcal{G}(F) = \big(1 \pm v(F)\eta_k\big) \tfrac{g(\emptyset)}{d(\emptyset)p(\emptyset)} \mathcal{D}(F)\mathcal{P}(F)\,.$$

Theorem 5.6 (Embedding lemma for sparse hypergraphs)**.** *For all $k \ge 2$ and $\Delta \ge 1$, given parameters η_k, η_0, and ε_ℓ, d_ℓ for $1 \le \ell \le k$ such that $0 < \eta_0 \ll d_1, \dots, d_k$, η_k, Δ, and for all ℓ we have $0 < \varepsilon_\ell \ll d_\ell, \dots, d_k$, η_k, Δ, the following holds.*

Let $\mathcal{G} \subseteq \Gamma$ be J-partite weighted k-graphs with associated density graphs $\mathcal{D}$, $\mathcal{P}$, where Γ is an $(\eta_0, 4k^2 + k)$-THC graph with density graph $\mathcal{P}$, and where for each $e \subseteq J$ of size $1 \le \ell \le k$, the graph $\mathcal{G}[V_e]$ is ε_ℓ-regular with relative density $d(e) \ge d_\ell$ with respect to the graph obtained from $G[V_e]$ by replacing layer ℓ with Γ. Then we have

$$\mathcal{G}(F) \ge (1 - \eta_k)^{v(F)} \tfrac{g(\emptyset)}{d(\emptyset)p(\emptyset)} \mathcal{D}(F)\mathcal{P}(F)\,.$$

for all J-partite k-complexes F of maximum degree Δ.

In Section 5.3 we prove these results and discuss how they relate to the structure one can obtain by existing regularity lemmas, but give a motivating sketch of this structure here.

When the (strong) hypergraph regularity lemma is applied to a k-uniform subgraph of Γ, one ends up working with a subgraph $\mathcal{G}$ of Γ which has the following properties. First, there is a vertex partition $\{V_j\}_{j\in J}$ indexed by J of $V(\mathcal{G}) = V(\Gamma)$. Second, for each $f \subseteq J$ with $2 \le |f| \le k$, the graph $\mathcal{G}[V_f]$ is $(\varepsilon_{|f|}, d(f))$-regular with respect to the graph whose weight function is equal to that of $\mathcal{G}$ on edges of size at most $|f| - 1$ and to Γ on edges of size $|f|$. Here one should think of the edges of $\mathcal{G}$ of size $k-1$ and less as being output by the regularity lemma, and the k-edges as being the subgraph of Γ which we are regularising. The difficulty is that, while we always have $\varepsilon_{|f|} \ll d(f)$, and indeed $\varepsilon_\ell \ll d(f)$ for any f with $|f| \ge \ell$, it may be the case that ε_ℓ is large compared to the $d(f)$ with $|f| < \ell$. Note that in this setting $\mathcal{G}$ always gives weight one to vertices; for the purpose of the following sketch we will assume that $\mathcal{G}$ has this property, though we will eventually get rid of that assumption.

The solution to this is to separate counting and embedding into several steps. To begin with, we can count any small hypergraph to high precision in the ambient Γ by assumption. We define a hypergraph whose edges are given weight equal to Γ on edges of size 3 and above, but equal to $\mathcal{G}$ on edges of size two (and one). We can think of this hypergraph as being very regular and dense relative to Γ: the relative density parameters are $d(e)$ for $|e| = 2$ which are much larger than the regularity parameter ε_2. Using our regularity inheritance lemma, we show that we can count any small hypergraph to high precision in this new hypergraph. This means we can now think of our new hypergraph as a well-behaved ambient hypergraph, and consider the hypergraph whose edges have weight equal to Γ on edges of size 4 and above, but equal to $\mathcal{G}$ on edges of size 3 and below. The same argument shows we can count small hypergraphs to high precision in this hypergraph too, and so on. Our approach thus keeps track of a *stack* of hypergraphs, where we assume that we can count in the bottom *level* Γ and inductively bootstrap our way to counting in the top layer $\mathcal{G}$ by using the fact that each level is relatively dense and very regular with respect to the level below.

In general, we may have a more complicated setup because we have embedded some vertices. We begin by defining abstractly the structure we consider, and will then move on to giving the conditions it must satisfy in order that we can work with it. It is convenient to introduce a complex F and a partial embedding of that complex in order to define the update rule; we do not need to specify the graph into which F is partially embedded.

Definition 5.7 (Stack of candidate graphs, update rule)**.** Let $k \geq 2$, and suppose that a k-complex F, a partial embedding ϕ of F, and disjoint vertex sets V_x for each $x \in V(F)$ which is unembedded (that is, $x \notin \operatorname{dom}\phi$) are given. Suppose that for each $0 \leq \ell \leq k$ and each $e \subseteq V(F) \setminus \operatorname{dom}\phi$ we are given a subgraph $\mathcal{C}^{(\ell)}(e)$ of V_e. We write $\mathcal{C}^{(\ell)}$ for the union of the $\mathcal{C}^{(\ell)}(e)$; that is, the graph with parts $\{V_x\}_{x\in V(F)\setminus\operatorname{dom}\phi}$ whose weight function is equal to that of $\mathcal{C}^{(\ell)}(e)$ on V_e for each $e \subseteq V(F) \setminus \operatorname{dom}\phi$. If $\mathcal{C}^{(0)} \geq \mathcal{C}^{(1)} \geq \cdots \geq \mathcal{C}^{(k)}$ then we call the collection of $k+1$ graphs a *stack of candidate graphs*, and $\mathcal{C}^{(\ell)}$ is the *layer ℓ candidate graph*.

Given $x \in V(F) \setminus \operatorname{dom}\phi$ and $v \in V_x$, we form a stack of candidate graphs corresponding to the partial embedding $\phi\cup\{x \mapsto v\}$ according to the following *update rule*. For each $0 \leq \ell \leq k$, we let $\mathcal{C}^{(\ell)}_{x\mapsto v} := \mathcal{C}^{(\ell)}_v$ be the link graph of v in $\mathcal{C}^{(\ell)}$. Note that trivially since $\mathcal{C}^{(\ell)} \leq \mathcal{C}^{(\ell-1)}$ we have $\mathcal{C}^{(\ell)}_{x\mapsto v} \leq \mathcal{C}^{(\ell-1)}_{x\mapsto v}$ for each $1 \leq \ell \leq k$, so that this indeed gives a stack of candidate graphs.

It will be important in what follows that we think of each $\mathcal{C}^{(\ell)}$ both as specifying weights for an ongoing embedding of F, and also as a partite graph into which we expect to know the number of embeddings of some (small, not necessarily related to F) complex R.

We are now in a position to define a *good partial embedding* (GPE). Informally, this is a partial embedding of F together with a stack of candidate graphs, such that for each $1 \leq \ell \leq k$ the graph $\mathcal{C}^{(\ell)}$ is relatively dense and regular with respect to $\mathcal{C}^{(\ell-1)}$. We specify the relative density of each $\mathcal{C}^{(\ell)}(e)$ explicitly in terms of numbers a density k-graph $\mathcal{D}^{(\ell)}$ with densities $d^{(\ell)}(f) \in [0,1]$ for each $1 \leq \ell \leq k$ and $f \subseteq V(F)$, which we think of as being the relative densities in the trivial GPE. We denote by $\mathcal{D}^{(\ell)}_\phi$ the density k-graph obtained from $\mathcal{D}^{(\ell)}$ by repeatedly taking the neighbourhood of vertices $x \in \operatorname{dom}\phi$, so that $\mathcal{D}^{(\ell)}_\phi$ gives the 'current' relative densities of $\mathcal{C}^{(\ell)}$.

We will need a collection of parameters which describe, respectively, the minimum relative densities in each layer of the stack (with respect to the layer below) at any step of the embedding (denoted δ_ℓ), the required accuracy of counting in each layer (denoted η_ℓ), and the regularity required in each layer. The regularity parameters are somewhat complicated. In general, one should focus on the best- and worst-case regularity; it is necessary to have the other parameters, but one only needs the extra granularity they offer in certain parts of the argument. Briefly, when we say $\mathcal{C}^{(\ell)}(e)$ is $\varepsilon_{\ell,r,h}$-regular, the ℓ indicates the layer in the stack, $r = |e|$ gives the uniformity, and h is the number of *hits*, that is, how many times in creating ϕ we previously degraded the regularity of $\mathcal{C}^{(\ell)}(e)$. This will turn out to be equal to

$$\pi_\phi(e) := \left|\{x \in \operatorname{dom}\phi : \{x\} \cup e' \in F \text{ for some } \emptyset \neq e' \subseteq e\}\right| .$$

Our definition of vertex-degeneracy was chosen precisely to make $\pi_\phi(e) \leq \operatorname{vdeg}(F)$ hold for all unembedded $e \in F$ whenever ϕ is a partial embedding of F with $\operatorname{dom}\phi$ an initial segment of $V(F)$.

Definition 5.8 (Ensemble of parameters, valid ensemble)**.** For integers k, c^*, h^*, and Δ, an *ensemble of parameters* is a collection $\delta_1, \dots, \delta_k$ of *minimum relative densities*, $\eta_0, \dots, \eta_k$ of *counting accuracy parameters*, and $(\varepsilon_{\ell,r,h})_{\ell,r\in[k],h\in\{0,\dots,h^*\}}$ of *regularity parameters*. For each $\ell \in [k]$ we define the *best-case regularity* $\varepsilon_\ell := \min_{r\in[k],h\in\{0,\dots,h^*\}} \varepsilon_{\ell,r,h}$ and the *worst-case regularity* $\varepsilon'_\ell := \max_{r\in[k],h\in\{0,\dots,h^*\}} \varepsilon_{\ell,r,h}$.

An ensemble of parameters is *valid* if the following statements hold for each $1 \leq \ell \leq k$.

(VE1) We have

$$\eta_0 \ll \delta_1, \dots, \delta_\ell, \eta_\ell, k, c^* ,$$

and for all $\ell' \in [\ell]$, we have

$$\varepsilon'_{\ell'} \ll \delta_{\ell'}, \dots, \delta_\ell, \eta_\ell, k, c^*, \Delta ,$$

such that

$$\eta_0 \leq \frac{\eta_\ell}{72(k+1)c^*} \prod_{0<\ell''\leq\ell} \delta_{\ell''}^{c^*} ,$$

$$\varepsilon'_{\ell'} \leq \frac{\eta_\ell \delta_{\ell'}}{72k(k+1)\Delta^2} \prod_{\ell'<\ell''\leq\ell} \delta_{\ell''}^{c^*} .$$

(VE2) For each $r \in [k]$ and $0 \leq h \leq h^* - 1$, we have $\varepsilon_{\ell,r,h} \ll \varepsilon_{\ell,r,h+1}, \delta_\ell$ small enough for Lemma 5.3 (inheritance and link regularity) with input δ_ℓ and $\varepsilon_{\ell,r,h+1}$. In particular $\varepsilon_{\ell,r,h}$ increases with h.

(VE3) For each $r \in [k-1]$ we have $\varepsilon_{\ell,r+1,h^*} \leq \varepsilon_{\ell,r,0}$.

(VE4) The counting accuracy $(4k+1)\eta_{\ell-1}$ is good enough for each application of Lemma 5.3 as above. That is, with inputs δ_ℓ and

$\varepsilon_{\ell,r,h}$ for $1 \le h \le h^*$ we have $(4k+1)\eta_{\ell-1}$ small enough to apply Lemma 5.3.

By this definition, we always have $\varepsilon_\ell = \varepsilon_{\ell,k,1}$ and $\varepsilon'_\ell = \varepsilon_{\ell,1,h^*}$. It is important to observe that we can obtain a valid ensemble of parameters by starting with δ_k and η_k, choosing $\varepsilon_{k,1,h^*} = \varepsilon'_k$ to satisfy

$$\varepsilon_{k,1,h^*} \le \frac{\eta_k \delta_k}{72k(k+1)\Delta^2}\,,$$

then choosing in order

$$\varepsilon_{k,1,h^*-1} \gg \cdots \gg \varepsilon_{k,1,0} \gg \varepsilon_{k,2,h^*} \gg \cdots \gg \varepsilon_{k,2,1} \gg \cdots \gg \varepsilon_{k,k,0} = \varepsilon_k\,,$$

at which point we can calculate the required accuracy of counting η_{k-1} and given δ_{k-1} choose ε'_{k-1} to match it, and repeat this process down the stack. In particular, this order of choosing constants is compatible with the strong hypergraph regularity lemma (see Section 5.3), to which we would first input ε_k, be given a d_{k-1} which means we can specify δ_{k-1}, then choose ε_{k-1}, and be able to calculate δ_{k-2}, and so on.

Given a partial embedding, a stack of candidate graphs, $1 \le \ell \le k$ and $e \subseteq V(F) \setminus \operatorname{dom}\phi$ with $|e| \ge 1$, we let $\overline{\mathcal{C}}^{(\ell-1)}(e)$ denote the subgraph of $\mathcal{C}^{(\ell-1)}$ induced by $\bigcup_{x\in e} V_x$. We let $\widetilde{\mathcal{C}}^{(\ell)}(e)$ denote the graph obtained from $\overline{\mathcal{C}}^{(\ell-1)}(e)$ by replacing the weights of edges in V_e with the weights of $\mathcal{C}^{\ell}(e)$. We will always consider regularity of $\widetilde{\mathcal{C}}^{(\ell)}(e)$ with respect to $\overline{\mathcal{C}}^{(\ell-1)}(e)$. This may seem strange—if we are working with unweighted graphs, presumably there are edges at all levels of the complex $\overline{\mathcal{C}}^{(\ell-1)}(e)$ which are not in $\overline{\mathcal{C}}^{(\ell)}(e)$, and so we are insisting on a regularity involving some edges of $\mathcal{C}^{(\ell)}(e)$ which do not contribute to the count of embeddings into $\mathcal{C}^{(\ell)}$. But it turns out to be necessary.

Definition 5.9 (Good partial embedding)**.** Given $k \ge 2$, a k-complex F of maximum degree Δ, integers c^*, h^*, and for each $0 \le \ell \le k$ a density k-graph $\mathcal{D}^{(\ell)}$ on $V(F)$, let $\delta_1, \dots, \delta_k$, $\eta_0, \dots, \eta_k$, and $(\varepsilon_{\ell,r,h})_{\ell,r\in[k],h\in[h^*]}$ be a valid ensemble of parameters. Given $1 \le \ell \le k$, we say that a partial embedding ϕ of $V(F)$ together with a stack of candidate graphs $\mathcal{C}^{(0)}, \dots, \mathcal{C}^{(\ell)}$ is an *ℓ-good partial embedding* (ℓ-GPE) if the following hold.

(GPE1) The graph $\mathcal{C}^{(0)}$ is an (η_0, c^*)-THC graph with density graph $\mathcal{D}^{(0)}_\phi$.

(GPE2) For each $1 \le \ell' \le \ell$ and $\emptyset \neq e \subseteq V(F) \setminus \operatorname{dom} \phi$, the graph $\widetilde{\mathcal{C}}^{(\ell')}(e)$ is (ε, d)-regular with respect to $\overline{\mathcal{C}}^{(\ell'-1)}(e)$, where

$$\varepsilon = \varepsilon_{\ell', |e|, \pi_\phi(e)}\,, \qquad \text{and} \qquad d = d_\phi^{(\ell')}(e) = \prod_{\substack{f \subseteq V(F),\\ e \subseteq f,\, f \setminus e \subseteq \operatorname{dom} \phi}} d^{(\ell')}(f)\,.$$

(GPE3) The parameters $\delta_1, \ldots, \delta_\ell$ are 'global' lower bounds on the relative density terms in the sense that for each $1 \le \ell' \le \ell$ and $\emptyset \neq e \subseteq V(F) \setminus \operatorname{dom} \phi$, we have

$$\delta_{\ell'} \le \prod_{\substack{f \subseteq V(F),\\ e \subseteq f}} d^{(\ell')}(f)\,.$$

When we have a k-good partial embedding, we will usually simply say *good partial embedding* (GPE).

If we were told that the trivial partial embedding was good, and that for every x and $v \in V_x$, extending a good partial embedding ϕ of F to $\phi \cup \{x \mapsto v\}$ and using the update rule to obtain a new stack of candidate graphs would result in a good partial embedding, then we would rather trivially conclude the desired counting lemma. We would simply count the number of ways to complete the embedding: when we come to embed some x to $\mathcal{C}^{(k)}(x)$ (with respect to the current GPE ϕ) the density of $\mathcal{C}^{(k)}(x)$ would be

$$\prod_{\ell=0}^{k} d_\phi^{(\ell)}(x) = \prod_{\ell=0}^{k} \prod_{\substack{f \subseteq V(F),\, x \in f,\\ f \setminus \{x\} \subseteq \operatorname{dom} \phi}} d^{(\ell)}(f)\,,$$

up to a relative error which is small provided that for each ℓ, all the $\varepsilon_{\ell,r,h}$ are small enough compared to the $d^{(\ell)}(f)$. Since this formula does not depend on a specific ϕ but only on $\operatorname{dom} \phi$ (so, on the order we embed the vertices) we conclude that the total weight of embeddings of F is

$$\frac{c^{(k)}(\emptyset)}{\prod_{0 \le \ell \le k} d^{(\ell)}(\emptyset)} \prod_{\ell=0}^{k} \mathcal{D}^{(\ell)}(F) = \frac{c^{(k)}(\emptyset)}{\prod_{0 \le \ell \le k} d^{(\ell)}(\emptyset)} \prod_{\ell=0}^{k} \prod_{f \subseteq V(F)} d^{(\ell)}(f)\,,$$

up to a relative error which is small provided that for each ℓ, all the $\varepsilon_{\ell,r,h}$ are small enough given the $d^{(\ell)}(f)$ and $v(F)^{-1}$. This is the statement we would like to prove. Of course, it is unrealistic to expect that we always get a good partial embedding when we extend a good partial embedding. However, it is enough if we usually get a good partial embedding, and the next lemma states that this is the case.

Lemma 5.10 (One-step lemma)**.** *Given $k \geq 2$, a k-complex F of maximum degree Δ and vertex-degeneracy* $\mathrm{vdeg}(F) \leq \Delta'$*, positive integers c^* and h^*, a valid ensemble of parameters, a partial embedding ϕ and stack of candidate graphs $\mathcal{C}^{(0)}, \dots, \mathcal{C}^{(k)}$ giving a GPE, let $B_0(x)$ denote the set of vertices $v \in V_x$ such that condition* (GPE1) *does not hold for the extension $\phi \cup \{x \mapsto v\}$ and the updated candidate graph $\mathcal{C}^{(0)}_{x \mapsto v}$. For $1 \leq \ell \leq k$, let $B_\ell(x)$ denote the set of vertices $v \in V_x$ such that $\phi \cup \{x \mapsto v\}$ and the updated candidate graphs do not form an ℓ-GPE.*

Then for every $1 \leq \ell \leq k$ such that $\ell(4k+1) \leq c^*$ and $\ell(4k+1) + k\Delta' \leq h^*$, we have

$$\|B_\ell(x) \setminus B_{\ell-1}(x)\|_{\mathcal{C}^{(\ell-1)}(x)} \leq k\Delta^2 \varepsilon'_\ell \|V_x\|_{\mathcal{C}^{(\ell-1)}(x)} .$$

The point of this collection of bounds on atypical vertices is that if a vertex v is in $B_\ell(x) \setminus B_{\ell-1}(x)$ for some ℓ, then we will be able to upper bound the count of F-copies extending $\phi \cup \{x \mapsto v\}$ in terms of the count of those F-copies in $\mathcal{C}^{(\ell-1)}$ (which we show we can estimate accurately). This upper bound is bigger than the number we would like to get (the count in $\mathcal{C}^{(k)}$) by the reciprocal of a product of some $d^{(\ell')}(f)$ terms, for various f but only for $\ell' \geq \ell$. In particular, if $v(F) - |\mathrm{dom}\,\phi|$ is not too large then this product is much larger than ε'_ℓ, so that the vertices of $B_\ell(x) \setminus B_{\ell-1}(x)$ in total do not contribute much to the overall count.

The corresponding counting lemma is then the following.

Lemma 5.11 (Counting lemma for GPEs)**.** *Given $k \geq 2$, positive integers Δ, c^*, h^*, and a valid ensemble of parameters, let ϕ be a partial embedding of a k-complex F of maximum degree Δ, and suppose that for some $1 \leq \ell \leq k$, the stack of candidate graphs $\mathcal{C}^{(0)}, \dots, \mathcal{C}^{(\ell)}$ gives an ℓ-GPE. Write $r = v(F) - |\mathrm{dom}\,\phi|$ and suppose that we have $c^* \geq \max\{2r-1, \ell(4k+1)\}$, $h^* \geq \ell(4k+1) + \mathrm{vdeg}(F)$, and $r\eta_\ell \leq 1/2$. Then*

$$\mathcal{C}^{(\ell)}(F - \mathrm{dom}\,\phi) = (1 \pm r\eta_\ell) \frac{c^{(\ell)}(\emptyset)}{\prod_{0 \leq \ell' \leq \ell} d^{(\ell')}_\phi(\emptyset)} \prod_{0 \leq \ell' \leq \ell} \mathcal{D}^{(\ell')}_\phi(F - \mathrm{dom}\,\phi) .$$

The right-hand side consists of a relative error term and a product of densities, where the $\emptyset$ terms correspond to edges of F which are fully embedded by ϕ, and the remaining terms correspond to the expected weight of edges not yet fully embedded by ϕ.

The proofs of Lemmas 5.10 and 5.11 are an intertwined induction, which we give in the following Section 5.2. Specifically, to prove Lemma 5.10 for some $\ell \geq 1$ we assume Lemma 5.11 for $\ell' < \ell$, and to prove Lemma 5.11 for $\ell \geq 1$ we assume Lemma 5.10 for $\ell' \leq \ell$. The base case is provided by the observation that the counting conditions we require to prove Lemma 5.10 for $\ell = 1$, in $\mathcal{C}^{(0)}$, hold because (GPE1) states that $\mathcal{C}^{(0)}$ is a THC graph.

If one is only interested in a lower bound for the purpose of embedding, our methods are significantly simpler because we trivially have zero as a lower bound for the total weight of embeddings using bad vertices, and one can afford the luxury of ignoring levels below k of the stack. Controlling this error is what requires $c^* \geq 2r - 1$ in Lemma 5.11, but we would like to depend less on the global structure of F in an embedding result, stated as Lemma 5.12 below.

Lemma 5.12 (Embedding lemma for GPEs). *For $k \geq 2$, positive integers Δ, Δ', $c^* \geq k(4k+1)$, $h^* \geq k(4k+1) + k\Delta'$, and a valid ensemble of parameters, let ϕ be a partial embedding of a k-complex F of maximum degree Δ and vertex-degeneracy at most Δ', and suppose that the stack of candidate graphs $\mathcal{C}^{(0)}, \dots, \mathcal{C}^{(k)}$ gives a k-GPE. Write $r = v(F) - |\mathrm{dom}\, \phi|$.*

Then we have

$$\mathcal{C}^{(k)}(F - \mathrm{dom}\, \phi) \geq (1 - \eta_k)^r \frac{c^{(k)}(\emptyset)}{\prod_{0 \leq \ell \leq k} d_\phi^{(\ell)}(\emptyset)} \prod_{0 \leq \ell \leq k} \mathcal{D}_\phi^{(\ell)}(F - \mathrm{dom}\, \phi) \,.$$

Note that although Lemmas 5.11 and 5.12 only explicitly allow for counting embeddings in a partite graph where one vertex is embedded to each part, it is easy to deduce versions where multiple vertices may be embedded into each part by applying the standard construction at each layer of the stack. It is trivial to check that for layers 1 to k the required regularity is carried over, and the homomorphism counts imposed on the bottom layer are similarly preserved by duplication.

5.2 Embedding and counting

In this section we prove Lemmas 5.10, 5.11, and 5.12. As mentioned above, we prove the first two lemmas together, by induction on ℓ in each lemma. We begin by assuming Lemma 5.11 for $\ell' < \ell$ in order to prove the bound on $B_\ell(x)$ claimed in Lemma 5.10. We will use Lemma 5.11 to show that the various counting conditions for Lemma 5.3 are met; the rest is simply bookkeeping.

Proof of Lemma 5.10 for $\ell \geq 1$. If a vertex $v \in V_x$ is in $B_\ell(x) \setminus B_{\ell-1}(x)$, then by definition there is a failure of regularity in the graph $\mathcal{C}^{(\ell)}_{x\mapsto v}$ (obtained by applying the update rule to $\mathcal{C}^{(\ell)}$). Specifically, there is some $e \subseteq F \setminus \big(\mathrm{dom}(\phi) \cup \{x\}\big)$ such that, although $\widetilde{\mathcal{C}}^{(\ell)}(e)$ is $(\varepsilon_{\ell,|e|,\pi_\phi(e)}, d)$-regular (with d as given in (GPE2)) with respect to $\overline{\mathcal{C}}^{(\ell-1)}(e)$, the graph $\widetilde{\mathcal{C}}^{(\ell)}_{x\mapsto v}(e)$ is not $\big(\varepsilon_{\ell,|e|,\pi_{\phi\cup\{x\to v\}}(e)}, d_x\big)$-regular (with d_x as given in (GPE2)) with respect to $\overline{\mathcal{C}}^{(\ell-1)}_{x\mapsto v}(e)$.

First, observe that if $\pi_\phi(e) = \pi_{\phi\cup\{x\mapsto v\}}(e)$, then this failure of regularity is impossible: we have $\widetilde{\mathcal{C}}^{(\ell)}(e) = \widetilde{\mathcal{C}}^{(\ell)}_{x\mapsto v}(e)$ and $\overline{\mathcal{C}}^{(\ell-1)}(e) = \overline{\mathcal{C}}^{(\ell-1)}_{x\mapsto v}(e)$. Thus there is an edge of F which both contains x and some non-empty subset of e; since there are at most Δ edges of F containing x, each of whose at most $k-1$ other vertices are in at most $\Delta - 1$ different edges of F, there are in total at most $\Delta + (k-1)\Delta(\Delta-1) \leq k\Delta^2$ choices of e.

Thus, in order to prove the ℓ case of Lemma 5.10, it suffices to show that for any given non-empty $e \subseteq F \setminus \big(\mathrm{dom}(\phi) \cup \{x\}\big)$, the total weight of vertices v in $\mathcal{C}^{(\ell)}(x)$ such that $\widetilde{\mathcal{C}}^{(\ell)}_{x\mapsto v}(e)$ is not $\big(\varepsilon_{\ell,|e|,\pi_\phi(e)+1}, d_x\big)$-regular, with respect to $\overline{\mathcal{C}}^{(\ell-1)}_{x\mapsto v}(e)$, is at most $\varepsilon'_\ell \|V_x\|_{\mathcal{C}^{(\ell-1)}(x)}$. The idea is that Lemma 5.3 should provide this desired bound.

To that end, let $V = V_x \cup \bigcup_{y\in e} V_y$, let $\Gamma := \mathcal{C}^{(\ell-1)}[V]$ with the inherited vertex partition, and let $\mathcal{G}$ be obtained from Γ by replacing the edges in V_e and $V_{\{x\}\cup e}$ with those from $\mathcal{C}^{(\ell)}$. Now by (GPE2) the graphs $\mathcal{G} \cap V_e$ and $\mathcal{G} \cap V_{e\cup\{x\}}$ are both $\varepsilon := \varepsilon_{\ell,|e|,\pi_\phi(e)}$-regular with densities $d, d' \geq \delta_\ell$ with respect to Γ. With $\varepsilon' := \varepsilon_{\ell,|e|,\pi_\phi(e)+1}$, $d_x = dd'$, and the update rule; by definition failure of regularity in the sense of an ℓ-GPE coincides with failure to inherit regularity in Lemma 5.3. The conditions (VE2) and (VE3) state that the constants above are compatible with Lemma 5.3 in this case, and the conclusion of Lemma 5.3 is the desired bound. It only remains to show that all the conditions of Lemma 5.3 are met. By construction, we have (INH2), while (INH3) and (INH4) are given by (GPE2). Thus to complete the proof of Lemma 5.10 we only need to show that the counting condition (INH1) holds.

Write $s = |e|$ and $e' = \{x\} \cup e$. Note that we have $0 < s \le k$. We now justify that for any given k-complex R of the form $O^{s+1}(\mathbf{a})$ or $+2O^s(0, \mathbf{b})$ with $\mathbf{a} \in \{0,1,2\}^{e'}$ and $\mathbf{b} \in \{0,1,2\}^e$, we can accurately count R in Γ. This verifies (INH1).

We separate two cases. First, if $\ell = 1$ then Γ is an induced subgraph of $\mathcal{C}^{(0)}$. By (GPE1), $\mathcal{C}^{(0)}$ is an (η_0, c^*)-THC graph, and thus by (THC1), $c^* \ge 4k+1$, and (VE4), we obtain the required count immediately.

The second, slightly more difficult case is $\ell > 1$. Here we aim to deduce the required count of R from the $\ell - 1$ case of Lemma 5.11 (which is valid by induction). We obtain a stack of candidate graphs by applying the standard construction (with the e'-partite k-complex R) to the graphs $\mathcal{C}^{(i)}[V]$ for $i = 0, \dots, k$. Now the required count follows immediately from Lemma 5.11 and condition (VE4) on $\eta_{\ell-1}$, provided that we can justify that the trivial partial embedding of R (in which no vertices are embedded) together with this stack of candidate graphs forms an $(\ell-1)$-GPE. To do this we need to specify the valid ensemble of parameters we use. These are identical to the valid ensemble we are provided with, *except* that we shift the indices for hits in the regularity parameters, that is, we use $\varepsilon_{\ell',r,h}$ with $h_0 \le h \le h^*$ where $h_0 = \max\{\pi_\phi(f) : \emptyset \ne f \subseteq e'\}$. Recall that we have $h_0 \le \mathrm{vdeg}(F)$.

By construction the property (GPE1) for the trivial embedding of R is implied by (GPE1) for ϕ and F. For (GPE2), we use the assumption that ϕ is an $(\ell-1)$-GPE, so for each edge f and each $1 \le \ell' \le k-1$, we have that $\widetilde{\mathcal{C}}^{(\ell')}(f)$ is $(\varepsilon_{\ell',|f|,\pi_\phi(f)}, d_f)$-regular (with d_f as given in (GPE2)) with respect to $\overline{\mathcal{C}}^{(\ell'-1)}(f)$. Since $\pi_\phi(f) \le h_0$ we have $\varepsilon_{\ell',|f|,\pi_\phi(f)} \le \varepsilon_{\ell',|f|,h_0}$, and so indeed the trivial partial embedding of R with the given stack of candidate graphs satisfies the conditions (GPE2) for $1 \le \ell' \le \ell - 1$ and the shifted regularity parameters.

Finally we must verify that the shifted ensemble is valid and suitable for use in Lemma 5.11. The 'length' of the sequences of shifted regularity parameters is $h_0^* := h^* - h_0 \le h^*$, hence (VE1) and (VE4) are implied by the same conditions for the unshifted ensemble. The property (VE2) is unchanged by shifting, and (VE3) holds because we have $\varepsilon_{\ell,r+1,h_0^*} \le \varepsilon_{\ell,r+1,h_0} \le \varepsilon_{\ell,r,0} \le \varepsilon_{\ell,r,h_0}$. For counting R with the height $\ell-1$ case of Lemma 5.11 we need $c^* \ge \max\{8k+1, (\ell-1)(4k+1)\}$,

$$h_0^* \ge h^* - \Delta' \ge \ell(4k+1) \ge (\ell-1)(4k+1) + \mathrm{vdeg}(R)\,,$$

and $(4k+1)\eta_\ell \le 1/2$, which hold for this case by the assumptions of Lemma 5.10 because $\ell \ge 2$. $\square$

The second part of the intertwined induction is a proof of Lemma 5.11. We first give a proof of Lemma 5.12 which assumes Lemma 5.10 (for $\ell \le k$), because it serves as a good introduction to aspects of the method without the notation necessary for the induction on ℓ, or calculations involving bad vertices.

Proof of Lemma 5.12. We prove Lemma 5.12 by induction on $r = v(F) - |\mathrm{dom}\,\phi|$, assuming the $\ell \le k$ cases of Lemma 5.10.

The statement for $r = 0$ is a tautology, since then $F - \mathrm{dom}\,\phi = \emptyset$ and the empty set appears identically on both sides of the required count.

For $r = 1$, the statement follows directly from the definition of a GPE, without the need to apply Lemma 5.10. The empty set is dealt with explicitly, so here we consider consider the weights $\mathcal{C}^{(\ell)}(x)$ as functions on V_x, and by the *density* of $\mathcal{C}^{(\ell)}(x)$ we mean $\|V_x\|_{\mathcal{C}^{(\ell)}(x)}$. Let $V(F) = \{x\}$, and note that by (GPE1) we know that $\mathcal{C}^{(0)}(x)$ has density

$$\|V_x\|_{\mathcal{C}^{(0)}(x)} = (1 \pm \eta_0) d_\phi^{(0)}(x)\,,$$

and by (GPE2), for each $1 \le \ell' \le \ell$, the graph $\mathcal{C}^{(\ell')}(x)$ is a subgraph (in the sense of a weighted 1-graph) of $\mathcal{C}^{(\ell'-1)}(x)$ of relative density

$$d_\phi^{(\ell')}(x) \pm \varepsilon'_{\ell'}\,.$$

Thus $\mathcal{C}^{(k)}$ has density

$$(1 \pm \eta_0) d_\phi^{(0)}(x) \prod_{\ell \in [k]} \left(1 \pm \frac{\varepsilon'_\ell}{\delta_\ell}\right) d_\phi^{(\ell)}(x) = (1 \pm \eta_k) \prod_{0 \leq \ell \leq k} d_\phi^{(\ell)}(x)\,, \tag{5.1}$$

because we have a valid ensemble of parameters ensuring for $\ell \in [k]$ that η_0, $\varepsilon'_\ell \ll \delta_\ell$, η_k, k by (VE1). Multiplied by the weight $c^{(k)}(\emptyset)$, this is the desired expression for $\mathcal{C}^{(\ell)}(F - \operatorname{dom}\phi)$ in the case $r = 1$.

For $r \geq 2$, fix any $x \in V(F)$. We will consider embedding x to some $v \in V_x$ and use induction on r to count the contribution from good choices of v. The key observation is that the update rule implies

$$\mathcal{C}^{(\ell)}(F - \operatorname{dom}\phi) = \mathbb{E}\big[\mathcal{C}^{(\ell)}_{x \mapsto v}(F - \operatorname{dom}\phi - \{x\})\big]\,,$$

where the expectation is over a uniformly random choice of $v \in V_x$. We separate three types of density term in the desired counting statement: $d^{(\ell)}(\emptyset)$ terms, $d^{(\ell)}(x)$ terms, and the remaining terms for which we write

$$\xi(\ell) := \frac{\mathcal{D}_\phi^{(\ell)}(F - \operatorname{dom}\phi)}{d_\phi^{(\ell)}(\emptyset) d_\phi^{(\ell)}(x)} = \frac{\mathcal{D}^{(\ell)}_{\phi \cup \{x \mapsto v\}}(F - \operatorname{dom}\phi - x)}{d^{(\ell)}_{\phi \cup \{x \mapsto v\}}(\emptyset)}\,, \tag{5.2}$$

where the second expression for $\xi(\ell)$ comes from the update rule. Note that despite the appearance of v in the notation on the right-hand side, as an expected density ξ does not depend on the choice of v. The weight of the empty set is dealt with explicitly, analysing the choice of $v \in V_x$ gives the $d_\phi^{(\ell)}(x)$ terms, and the $\xi(\ell)$ terms are found by induction on r. We can afford to ignore bad vertices for a lower bound, we merely need to estimate $\|V_x \setminus B_k(x)\|_{\mathcal{C}^{(k)}(x)}$ with Lemma 5.10. For brevity we write B_ℓ for $B_\ell(x)$ in the following calculations.

For B_0 we have

$$\|B_0\|_{\mathcal{C}^{(k)}(x)} \leq \|B_0\|_{\mathcal{C}^{(0)}(x)} \leq \eta_0 \|V_x\|_{\mathcal{C}^{(0)}(x)} \leq 2\eta_0 d_\phi^{(0)}(x)$$

by (GPE1) and the condition (THC2) of a THC-graph. The same argument as for (5.1) gives that the density of $\mathcal{C}^{(\ell')}(x)$ satisfies

$$\begin{aligned}\|V_x\|_{\mathcal{C}^{(\ell')}(x)} &= (1\pm\eta_0)d_\phi^{(0)}(x)\prod_{\ell''\in[\ell']}\Big(1\pm\frac{\varepsilon'_{\ell''}}{\delta_{\ell''}}\Big)d_\phi^{(\ell'')}(x)\\ &= (1\pm\eta_{\ell'})\prod_{0\le\ell''\le\ell}d_\phi^{(\ell'')}(x)\,.\end{aligned}\tag{5.3}$$

Then by Lemma 5.10 and (5.3), we calculate for $1\le\ell\le k$ the bound

$$\begin{aligned}\|B_\ell\setminus B_{\ell-1}\|_{\mathcal{C}^{(k)}(x)} &\le \|B_\ell\setminus B_{\ell-1}\|_{\mathcal{C}^{(\ell-1)}(x)}\\ &\le k\Delta^2\varepsilon'_\ell\cdot\|V_x\|_{\mathcal{C}^{(\ell-1)}(x)}\\ &\le 2k\Delta^2\varepsilon'_\ell\prod_{0\le\ell'<\ell}d_\phi^{(\ell')}(x)\,.\end{aligned}$$

We next give a short calculation which shows that

$$\|V_x\setminus B_k\|_{\mathcal{C}^{(k)}(x)}\ge(1-\eta_k)\prod_{0\le\ell\le k}d_\phi^{(\ell)}(x)\,,\tag{5.4}$$

by a careful collection of density terms and 'compensating' error terms from lower levels of the stack. We have

$$\begin{aligned}\|V_x\setminus B_k\|_{\mathcal{C}^{(k)}(x)} &\ge \|V_x\|_{\mathcal{C}^{(k)}(x)}-\|B_0\|_{\mathcal{C}^{(0)}(x)}-\sum_{\ell\in[k]}\|B_\ell\setminus B_{\ell-1}\|_{\mathcal{C}^{(k)}(x)}\\ &\ge \left(\begin{aligned}&(1-\eta_0)\prod_{\ell'\in[k]}\Big(1-\frac{\varepsilon'_{\ell'}}{\delta_{\ell'}}\Big)\\ &\qquad-\frac{2\eta_0}{\prod_{\ell'\in[k]}\delta_{\ell'}}-\sum_{\ell\in[k]}\frac{2k\Delta^2\varepsilon'_\ell}{\prod_{\ell'=\ell}^k\delta_{\ell'}}\end{aligned}\right)\prod_{0\le\ell\le k}d_\phi^{(\ell)}(x)\\ &\ge(1-\eta_k)\prod_{0\le\ell\le k}d_\phi^{(\ell)}(x)\,.\end{aligned}$$

The $\delta_{\ell'}$ terms in the denominators of the second line correspond to 'missing densities' lost because we can only account for failure of a regularity condition in level ℓ' of the stack with the regularity properties of that level. We can afford to write $\delta_{\ell'}$ terms instead of $d_\phi^{(\ell')}(x)$ because we have $\delta_{\ell'}\le d_\phi^{(\ell')}(x)$ by (GPE3). For B_0 the missing densities are for levels $\ell'\in[k]$ but there is a very small η_0 to compensate, and for $B_\ell\setminus B_{\ell-1}$ we have a product of missing densities from levels ℓ to k of the stack, but a comparatively small ε'_ℓ to compensate.

With (5.4) in hand, we finish the proof with the induction on r. For any $v \in V_x \setminus B_k$, note that applying the induction hypothesis is valid as the required lower bounds on c^*, h^* still hold, and we have

$$\begin{aligned}
&\mathcal{C}^{(k)}_{x\mapsto v}(F - \operatorname{dom}\phi - x) \\
&\qquad \geq (1-\eta_k)^{r-1} \frac{c^{(k)}_{x\mapsto v}(\emptyset)}{\prod_{0\leq \ell\leq k} d^{(\ell)}_{\phi\cup\{x\mapsto v\}}(\emptyset)} \prod_{0\leq \ell\leq k} \mathcal{D}^{(\ell)}_{\phi\cup\{x\mapsto v\}}(F - \operatorname{dom}\phi - x) \\
&\qquad = (1-\eta_k)^{r-1} c^{(k)}_{x\mapsto v}(\emptyset) \prod_{0\leq \ell\leq k} \xi(\ell)\,,
\end{aligned}$$

where we have separated out the only term $c^{(k)}_{x\mapsto v}(\emptyset)$ which depends on v, so that the remaining product over ℓ is independent of v. By the update rule we have $c^{(k)}_{x\mapsto v}(\emptyset) = c^{(k)}(\emptyset)c^{(k)}(v)$, which gives

$$\begin{aligned}
\mathcal{C}^{(k)}(F - \operatorname{dom}\phi) &= \mathbb{E}\Big[\mathcal{C}^{(k)}_{x\mapsto v}(F - \operatorname{dom}\phi - x)\Big| v \in V_x\Big] \\
&\geq (1-\eta_k)^{r-1} c^{(k)}(\emptyset) \|V_x \setminus B_k\|_{\mathcal{C}^{(k)}(x)} \prod_{0\leq \ell\leq k} \xi(\ell) \\
&= (1-\eta_k)^{r} \frac{c^{(k)}(\emptyset)}{\prod_{0\leq \ell\leq k} d^{(\ell)}_{\phi}(\emptyset)} \prod_{0\leq \ell\leq k} \mathcal{D}^{(\ell)}_{\phi}(F - \operatorname{dom}\phi)\,,
\end{aligned}$$

where for the last line we observe that density terms involving x are taken care of by $\|V_x \setminus B_k\|_{\mathcal{C}^{(k)}(x)}$ via (5.4), and the other terms are given by ξ via (5.2). □

The proof for Lemma 5.11 is similar, but we must proceed by induction on the height ℓ of the GPE and handle bad vertices more carefully. For the latter consideration, we use the following consequence of the Cauchy–Schwarz inequality, which we prove along with several related tools in Section 5.4.

Lemma 5.13. *Let W, X, and Y be discrete random variables such that W takes values in $[0,1]$, X takes values in the non-negative reals, and Y is real-valued. Suppose also that for $0 \leq \varepsilon \leq 1$ and $d \geq 0$ we have*

$$\mathbb{E}[XY] = (1\pm\varepsilon)d\,\mathbb{E}\,X \qquad \textit{and} \qquad \mathbb{E}[XY^2] \leq (1+\varepsilon)d^2\,\mathbb{E}\,X\,.$$

Then

$$\mathbb{E}[WXY] = \left(1 - \varepsilon \pm 2\sqrt{\frac{\varepsilon\,\mathbb{E}\,X}{\mathbb{E}[WX]}}\right) d\,\mathbb{E}[WX]\,,$$

and

$$\mathbb{E}[WXY^2] = \Big(1 - 2\varepsilon \pm 7\sqrt{\varepsilon}\frac{\mathbb{E}\,X}{\mathbb{E}[WX]}\Big)\, d^2\,\mathbb{E}[WX]\,.$$

Proof of Lemma 5.11 for $\ell \geq 1$. Given ℓ, we prove Lemma 5.11 for height ℓ by induction on $r = v(F) - |\mathrm{dom}\,\phi|$, assuming the $\ell' \leq \ell$ cases of Lemma 5.10 and $\ell' < \ell$ cases of Lemma 5.11. Below we write B_ℓ as a shorthand for $B_\ell(x)$.

As in the previous proof, the case $r = 0$ is a tautology, and the statement for $r = 1$ follows directly from the definition of ℓ-GPE. The same applications of properties (GPE1), (GPE2), and (VE1) as for (5.1) and (5.3) give that for $\ell' \leq \ell$ we have

$$\begin{aligned} \|V_x\|_{\mathcal{C}^{(\ell')}(x)} &= (1 \pm \eta_0) d_\phi^{(0)}(x) \prod_{\ell'' \in [\ell']} \Big(1 \pm \frac{\varepsilon'_{\ell''}}{\delta_{\ell''}}\Big) d_\phi^{(\ell'')}(x) \\ &= (1 \pm \eta_{\ell'}) \prod_{0 \leq \ell'' \leq \ell'} d_\phi^{(\ell'')}(x)\,. \end{aligned} \tag{5.5}$$

When $r = 1$, with $\ell' = \ell$, and multiplied by the factor $c^{(\ell)}(\emptyset)$, this is the desired statement.

Now given $r \geq 2$, fix $x \in V(F)$. We use the statement of Lemma 5.11 for heights $\ell' < \ell$ and with the complex $F - x$, and (the induction assumption in this proof) for height ℓ.

We have a partition of V_x into the bad vertices B_0, and $B_{\ell'} \setminus B_{\ell'-1}$ for $\ell' \in [\ell]$, and the good vertices $V_x \setminus B_\ell$. As in the previous proof, we separately consider density terms for $\emptyset$, x, and the ones of the form $\xi(\ell')$ obtained via the induction on r. From (5.2) recall that ξ is independent of v. The desired counting statement is then

$$\mathcal{C}^{(\ell)}(F - \mathrm{dom}\,\phi) = (1 \pm r\eta_\ell) c^{(\ell)}(\emptyset) \prod_{0 \leq \ell' \leq \ell} d_\phi^{(\ell')}(x)\xi(\ell')\,.$$

As before, by Lemma 5.10, (5.5), and the fact that in any valid ensemble we have $\eta_{\ell'-1} < 1$ for all ℓ', we calculate for $1 \leq \ell' \leq \ell$ the bound

$$\begin{aligned} \|B_{\ell'} \setminus B_{\ell'-1}\|_{\mathcal{C}^{(\ell')}} &\leq \|B_{\ell'} \setminus B_{\ell'-1}(x)\|_{\mathcal{C}^{(\ell'-1)}(x)} \\ &\leq k\Delta^2 \varepsilon'_{\ell'} \cdot \|V_x\|_{\mathcal{C}^{(\ell'-1)}(x)} \\ &\leq 2k\Delta^2 \varepsilon'_{\ell'} \prod_{0 \leq \ell'' < \ell'} d_\phi^{(\ell'')}(x)\,. \end{aligned} \tag{5.6}$$

By definition, for each $v \in V_x \setminus B_{\ell'}(x)$, the partial embedding $\phi \cup \{x \to v\}$ together with the stack of candidate graphs $\mathcal{C}^{(0)}_{x\mapsto v}, \dots, \mathcal{C}^{(\ell')}_{x\mapsto v}$ obtained by the update rule is an ℓ'-GPE. Applying for each $1 \le \ell' \le \ell$ the ℓ' case of Lemma 5.11 with the partial embedding $\phi \cup \{x \mapsto v\}$ and updated candidate graphs (where we note that c^* and h^* are large enough and $\eta_{\ell'}$ small enough for this to be valid), it follows that for each such choice of v we have

$$\begin{aligned}
&\mathcal{C}^{(\ell')}_{x\mapsto v}(F - \operatorname{dom}\phi - x) \\
&\quad = (1 \pm (r-1)\eta_{\ell'}) \frac{c^{(\ell')}_{x\mapsto v}(\emptyset)}{\prod_{0\le \ell'' \le \ell'} d^{(\ell'')}_{\phi\cup\{x\mapsto v\}}(\emptyset)} \prod_{0\le \ell'' \le \ell'} \mathcal{D}^{(\ell'')}_{\phi\cup\{x\mapsto v\}}(F - \operatorname{dom}\phi - x) \\
&\quad = (1 \pm (r-1)\eta_{\ell'}) c^{(\ell')}(\emptyset) c^{(\ell')}(v) \prod_{0\le \ell'' \le \ell'} \xi(\ell'')\,,
\end{aligned}$$

where the second line follows from the update rule and definition of ξ. We will carefully account for the empty set in level ℓ and not below. Then by the fact that $\mathcal{C}^{(\ell')} \le \mathcal{C}^{(\ell)}$, for each $1 \le \ell' \le \ell$ and $v \in V_x \setminus B_{\ell'}$ we have

$$\begin{aligned}
\mathcal{C}^{(\ell)}_{x\mapsto v}(F - \operatorname{dom}\phi - x) &\le \tfrac{c^{(\ell)}(\emptyset)}{c^{(\ell')}(\emptyset)} \mathcal{C}^{(\ell')}_{x\mapsto v}(F - \operatorname{dom}\phi - x) \\
&= (1 \pm (r-1)\eta_{\ell'}) c^{(\ell)}(\emptyset) c^{(\ell')}(v) \prod_{0\le \ell'' \le \ell'} \xi(\ell'')\,.
\end{aligned} \tag{5.7}$$

Putting (5.5), (5.6), and (5.7) together will give us the required lower bound on $\mathcal{C}^{(\ell)}(F)$, but for the upper bound we still need to show that the contribution made by $v \in B_0$ is small. Letting F' be the k-complex on $r' := 2r - 1 \le c^*$ vertices obtained by taking two disjoint copies of F and identifying each vertex in $\operatorname{dom}\phi \cup \{x\}$ with the corresponding vertex in the other copy, we have the following counts in the bottom layer of the stack by (GPE1),

$$\begin{aligned}
\mathcal{C}^{(0)}(F - \operatorname{dom}\phi) &= (1 \pm r\eta_0) \tfrac{c^{(0)}(\emptyset)}{d^{(0)}_{\phi}(\emptyset)} \mathcal{D}^{(0)}_{\phi}(F - \operatorname{dom}\phi)\,, \\
&= (1 \pm r\eta_0) c^{(0)}(\emptyset) d^{(0)}_{\phi}(x) \xi(0)\,, & (5.8) \\
\mathcal{C}^{(0)}(F' - \operatorname{dom}\phi) &= (1 \pm r'\eta_0) \tfrac{c^{(0)}(\emptyset)}{d^{(0)}_{\phi}(\emptyset)} \mathcal{D}^{(0)}_{\phi}(F' - \operatorname{dom}\phi) \\
&= (1 \pm r'\eta_0) c^{(0)}(\emptyset) d^{(0)}_{\phi}(x) \xi(0)^2\,. & (5.9)
\end{aligned}$$

From this, apply Lemma 5.13 to the experiment of choosing $v \in V_x$, with

$$\begin{aligned} X &:= c^{(0)}_{x\mapsto v}(\emptyset)\,,\\ Y &:= \mathcal{C}^{(0)}_{x\mapsto v}(F - \operatorname{dom}\phi - x)/c^{(0)}_{x\mapsto v}(\emptyset)\,,\\ W &:= \mathbb{1}_{v\in B_0(x)}\,. \end{aligned}$$

Property (GPE1) gives $\mathbb{E}\,X = (1 \pm \eta_0)c^{(0)}(\emptyset)d^{(0)}_\phi(x)$, and statements (5.8) and (5.9) give bounds on $\mathbb{E}[XY]$ and $\mathbb{E}[XY^2]$. We also have $\mathbb{E}[WX] \le \eta_0\,\mathbb{E}\,X$ by (GPE1) and condition (THC2). Hence we conclude

$$\mathbb{E}[WXY] \le 5r'\eta_0 \cdot c^{(0)}(\emptyset)d^{(0)}_\phi(x)\xi(0) \le 10\eta_0 c^* \cdot c^{(0)}(\emptyset)d^{(0)}_\phi(x)\xi(0)\,.$$

Again, taking care to deal with the empty set in level ℓ, we deduce the upper bound bound

$$10\eta_0 c^* \cdot c^{(\ell)}(\emptyset)d^{(0)}_\phi(x)\xi(0) \tag{5.10}$$

on the contribution to $\mathcal{C}^{(\ell)}(F - \operatorname{dom}\phi)$ from vertices $v \in B_0$.

To complete the proof we substitute these bounds into the expression

$$\begin{aligned} \mathcal{C}^{(\ell)}(F - \operatorname{dom}\phi) = \mathbb{E}\Big[&\mathbb{1}_{v\notin B_\ell}\mathcal{C}^{(\ell)}_{x\mapsto v}(F - \operatorname{dom}\phi - x)\Big]\\ &\pm \sum_{\ell'\in[\ell]} \mathbb{E}\Big[\mathbb{1}_{v\in B_{\ell'}\setminus B_{\ell'-1}}\mathcal{C}^{(\ell)}_{x\mapsto v}(F - \operatorname{dom}\phi - x)\Big]\\ &\pm \mathbb{E}\Big[\mathbb{1}_{v\in B_0}\mathcal{C}^{(0)}_{x\mapsto v}(F - \operatorname{dom}\phi - x)\Big]\,. \end{aligned}$$

Using (5.5), (5.6), (5.7), and (5.10), we obtain

$$
\begin{aligned}
&\mathcal{C}^{(\ell)}(F - \operatorname{dom}\phi) \\
&\quad = (1 \pm (r-1)\eta_\ell) c^{(\ell)}(\emptyset) \|V_x \setminus B_\ell\|_{\mathcal{C}^{(\ell)}(x)} \cdot \prod_{0 \le \ell'' \le \ell} \xi(\ell'') \\
&\qquad \pm \sum_{\ell' \in [\ell]} (1 + (r-1)\eta_{\ell'}) c^{(\ell)}(\emptyset) \|B_{\ell'} \setminus B_{\ell'-1}\|_{\mathcal{C}^{(\ell')}(x)} \cdot \prod_{0 \le \ell'' \le \ell'} \xi(\ell') \\
&\qquad \pm 10\eta_0 c^* \cdot c^{(\ell)}(\emptyset) d_\phi^{(0)}(x) \xi(0) \\
&\quad = (1 \pm (r-1)\eta_\ell) c^{(\ell)}(\emptyset) \Big(1 \pm \frac{\|B_\ell\|_{\mathcal{C}^{(\ell)}(x)}}{\|V_x\|_{\mathcal{C}^{(\ell)}(x)}}\Big)(1 \pm \eta_0) \\
&\qquad\qquad \cdot \Big(\prod_{\ell'' \in [\ell]} \Big(1 + \frac{\varepsilon'_{\ell''}}{\delta_{\ell''}}\Big)\Big) \cdot \prod_{0 \le \ell'' \le \ell} d_\phi^{(\ell'')}(x) \xi(\ell'') \\
&\qquad \pm \sum_{\ell' \in [\ell]} (1 + (r-1)\eta_{\ell'}) c^{(\ell)}(\emptyset) \cdot 2k\Delta^2 \frac{\varepsilon'_{\ell'}}{d_\phi^{(\ell')}(x)} \cdot \prod_{0 \le \ell'' \le \ell'} d_\phi^{(\ell'')}(x) \xi(\ell'') \\
&\qquad \pm 10\eta_0 c^* \cdot c^{(\ell)}(\emptyset) d_\phi^{(0)}(x) \xi(0) \,.
\end{aligned}
$$

This is almost the desired statement. By collecting terms we have

$$
\mathcal{C}^{(\ell)}(F - \operatorname{dom}\phi) = (1 \pm r\eta_\ell) c^{(\ell)}(\emptyset) \prod_{0 \le \ell'' \le \ell} d_\phi^{(\ell'')}(x) \xi(\ell'') \,,
$$

with a relative error given by $r\eta_\ell$, provided the following holds:

$$
\begin{aligned}
1 + r\eta_\ell \ge (1 + (r-1)\eta_\ell) &\Big(1 + \frac{\|B_\ell(x)\|_{\mathcal{C}^{(\ell)}(x)}}{\|V_x\|_{\mathcal{C}^{(\ell)}(x)}}\Big)(1 + \eta_0) \prod_{\ell'' \in [\ell]} \Big(1 + \frac{\varepsilon'_{\ell''}}{\delta_{\ell''}}\Big) \\
&+ \sum_{\ell' \in [\ell]} (1 + (r-1)\eta_{\ell'}) 2k\Delta^2 \cdot \frac{\varepsilon'_{\ell'}}{\delta_{\ell'} \prod_{\ell' < \ell'' \le \ell} \delta_{\ell''} \xi(\ell'')} \\
&+ \frac{10\eta_0 c^*}{\prod_{0 < \ell'' \le \ell} \delta_{\ell''} \xi(\ell'')} \,.
\end{aligned}
$$

The definition of a valid ensemble is chosen to make this inequality hold. Considering the right-hand side, the first line can be made at most $1 + (r - 2/3)\eta_\ell$, and each of the two remaining terms can be made at most $\eta_\ell/3$. Essentially the point is that where we have products of 'missing' minimum densities in the denominator of error terms, there is an $\varepsilon'_{\ell'}$ or η_0 to compensate in the numerator. The $\varepsilon'_{\ell'}$ parameters are chosen to be small enough to compensate for any product of minimum densities from the same level or higher, and η_0 is small enough to compensate for any densities in layers above 0.

Here we require the upper bound on r, since it implies the $\xi(\ell')$ terms corresponding to edges remaining after x is embedded cannot be too small. We give the required calculations below, relying on the facts that for all $\ell' \geq 1$, we have

$$\delta_{\ell'} \leq d_\phi^{(\ell')}(x)\,, \qquad \text{and} \qquad \delta_{\ell'}^{c^*-1} \leq \xi(\ell')\,. \tag{5.11}$$

The first bound states the contribution to the final count at level ℓ' from embedding x is at least $\delta_{\ell'}$, which holds by assumption: $\delta_{\ell'}$ is a minimum density. Then with $2r-1 \leq c^*$ the first inequality implies the second because $\xi(\ell')$ is a product over the remaining $r-1$ vertices of their contributions. The next claim deals with the smaller two error terms, and a subsequent claim deals with the main term.

Claim 5.14. (VE1) *implies both*

$$\sum_{\ell' \in [\ell]} (1+(r-1)\eta_{\ell'})2k\Delta^2 \cdot \frac{\varepsilon'_{\ell'}}{\delta_{\ell'} \prod_{\ell'<\ell''\leq \ell} \delta_{\ell''}\xi(\ell'')} \leq \frac{\eta_\ell}{3}\,,$$

and

$$\frac{10\eta_0 c^*}{\prod_{0<\ell''\leq\ell} \delta_{\ell''}\xi(\ell'')} \leq \frac{10\eta_0 c^*}{\prod_{0<\ell''\leq\ell} \delta_{\ell''}^{c^*}} \leq \frac{\eta_\ell}{3}\,.$$

Proof. For the first statement, since we have $(r-1)\eta_{\ell'} \leq 1/2$ and (5.11) it suffices to ensure that

$$\varepsilon'_{\ell'} \leq \frac{\eta_\ell}{9k^2\Delta^2}\delta_{\ell'} \prod_{\ell'<\ell''\leq\ell} \delta_{\ell''}^{c^*}$$

for each $\ell' \in [\ell]$, which holds by (VE1). In the second statement the first inequality holds by (5.11), and the second holds by (VE1). □

Claim 5.15. (VE1) *implies*

$$(1+(r-1)\eta_\ell)\Big(1+\frac{\|B_\ell(x)\|_{\mathcal{C}^{(\ell)}(x)}}{\|V_x\|_{\mathcal{C}^{(\ell)}(x)}}\Big)(1+\eta_0)\prod_{\ell''\in[\ell]}\Big(1+\frac{\varepsilon'_{\ell''}}{\delta_{\ell''}}\Big) \leq 1+\Big(r-\frac{2}{3}\Big)\eta_\ell$$

Proof. First we bound $\|B_\ell(x)\|_{\mathcal{C}^{(\ell)}(x)}$. By (5.5), (5.6), and $\|B_0(x)\|_{\mathcal{C}^{(0)}(x)} \leq$

$\eta_0\|V_x\|_{\mathcal{C}^{(0)}(x)}$, we have

$$\begin{aligned}\|B_\ell(x)\|_{\mathcal{C}^{(\ell)}(x)} &\leq \|B_0(x)\|_{\mathcal{C}^{(0)}(x)} + \sum_{\ell'\in[\ell]} \|B_{\ell'}(x)\setminus B_{\ell'-1}(x)\|_{\mathcal{C}^{(\ell')}(x)} \\ &\leq 2\eta_0 d_\phi^{(0)}(x) + \sum_{\ell'\in[\ell]} 2k\Delta^2\varepsilon'_{\ell'} \prod_{0\leq\ell''<\ell'} d_\phi^{(\ell'')}(x)\,.\end{aligned}$$

Hence (using that $\eta_\ell < 1/2$), we have

$$\begin{aligned}\frac{\|B_\ell(x)\|_{\mathcal{C}^{(\ell)}(x)}}{\|V_x\|_{\mathcal{C}^{(\ell)}(x)}} &\leq \frac{4\eta_0}{\prod_{\ell'\in[\ell]} d^{(\ell')}(x)} + \sum_{\ell'\in[\ell]} 4k\Delta^2 \frac{\varepsilon'_{\ell'}}{\prod_{\ell'\leq\ell''\leq\ell} d^{(\ell'')}(x)} \\ &\leq \frac{4\eta_0}{\prod_{\ell'\in[\ell]}\delta_{\ell'}} + \sum_{\ell'\in[\ell]} 4k\Delta^2\frac{\varepsilon'_{\ell'}}{\prod_{\ell'\leq\ell''\leq\ell}\delta_{\ell''}}\,.\end{aligned}$$

For the claim, by $r\eta_\ell < 1/2$ it now suffices to show

$$\begin{aligned}\left(1+\frac{4\eta_0}{\prod_{\ell'\in[\ell]}\delta_{\ell'}} + \sum_{\ell'\in[\ell]} 4k\Delta^2\frac{\varepsilon'_{\ell'}}{\prod_{\ell'\leq\ell''\leq\ell}\delta_{\ell''}}\right)(1+\eta_0)\prod_{\ell''\in[\ell]}\left(1+\frac{\varepsilon'_{\ell''}}{\delta_{\ell''}}\right) \\ \leq 1+\frac{\eta_\ell}{9} \leq \frac{1+(r-2/3)\eta_\ell}{1+(r-1)\eta_\ell}\,.\end{aligned}$$

We use that $\eta_\ell \ll k, 1$. The first bracketed term and $(1+\eta_0)$ are each at most $1+\eta_\ell/36$ by (VE1), and similarly we have

$$1+\frac{\varepsilon'_{\ell''}}{\delta_{\ell''}} \leq 1+\frac{\eta_\ell}{72k} \leq \left(1+\frac{\eta_\ell}{36}\right)^{1/k},$$

which shows the product over $[\ell]$ is also at most $1+\eta_\ell/36$. It follows that the expression is at most $(1+\eta_\ell/36)^3 \leq 1+\eta_\ell/9$ as required. □

This completes the proof of Lemma 5.11. □

5.3 Relating GPEs and regularity lemmas

There are several approaches to generalising Szemerédi's regularity lemma to hypergraphs [47, 77, 85, 86]. Recall that main idea is to partition a hypergraph into a bounded number of pieces, almost all of which are pseudorandom. Difficulties arise in giving a precise formulation of pseudorandomness that is both weak enough to be found by a regularity lemma and strong enough to support a counting lemma. We use a notion of octahedron minimality as our pseudorandomness condition (Definition 5.2), and in this section we describe how existing results imply that we can partition arbitrary hypergraphs into pieces which have the necessary structure.

In dense hypergraphs, the combined use of (strong) regularity lemmas with compatible counting lemmas constitute the standard hypergraph regularity method [84, 87]. Our Theorem 5.5 is essentially a version of the counting lemma of [76] for use with our definition of regularity. In the following subsection we show how to derive the setup of Theorem 5.5 from the strong regularity lemma of [86], allowing our Theorem 5.5 to be used in many applications of the standard hypergraph regularity method.

Versions of these tools for sparse graphs are less well-developed, but notably the weak regularity lemma and accompanying counting lemma of Conlon, Fox, and Zhao [17], give a general technique for transferring results for dense hypergraphs to a sparse setting. We show how combined use of the regularity methods of [17] and [76, 86] can yield the setup of Theorems 5.5 and 5.6, giving a powerful vertex-by-vertex method for counting and embedding in sparse hypergraphs. At the end of the section we prove Theorems 5.5 and 5.6.

5.3.1 Dense hypergraphs

We use results of Rödl and Skokan [86], and Nagle, Rödl, and Schacht [76], which allow us to partition a k-uniform hypergraph $\mathcal{H}$ into 'pieces' $\mathcal{G}$ which satisfy the hypotheses of Theorems 5.5 and 5.6 when Γ is taken to be the constant function 1.

We must use *both* the strong regularity lemma of [86] and the corresponding counting lemma [76] because our definition of regularity is related to octahedron counts which one obtains from the Rödl–Skokan version of regularity by a counting lemma. The precise structure of the partition into pieces which results from the strong regularity lemma is rather technical, but for our purposes it suffices to observe that, applied to $\mathcal{H}$, we obtain a number of k-complexes whose k-layers form a partition of $E(\mathcal{H})$. We choose to represented these k-complexes as weighted k-graphs with weights in $\{0,1\}$, and call them *slices* of $\mathcal{H}$. The technical part of the definition relates to how the levels below k of different slices overlap, which we do not discuss.

In general one can show that all but a constant proportion of the edges of $\mathcal{H}$ lie in slices that have the regularity structure we seek, but in this sketch we make the simplifying assumption that *all* slices have the required regularity. In applications one deals with this proportion separately without use of a counting lemma, either making the constant proportion so small that the edges in irregular slices have a negligible effect, or assuming strong properties of $\mathcal{H}$ which imply it can be fully partitioned into regular slices.

The slices we obtain from combined use of the strong regularity lemma and the counting lemma are as follows. Firstly $V(\mathcal{H})$ is partitioned into $\{V_j\}_{j\in J}$, such that each slice $\mathcal{G}$ of $\mathcal{H}$ is a k-partite k-graph on $\{V_j\}_{j\in f}$ for some $f \in \binom{J}{k}$. For any constant $\varepsilon_k > 0$ and family of functions $\varepsilon_2, \dots, \varepsilon_{k-1}$ where $\varepsilon_\ell : [0,1]^{k-\ell} \to \mathbb{R}_{>0}$, we obtain lower bounds $d_1, \dots, d_{k-1}$ such that every slice $\mathcal{G}$ on vertex sets $\{V_j\}_{j\in f}$ has the following properties.

(i) The vertex partition is equitable such that for all j, $|V_j| \geq d_1|V(\mathcal{H})|$.

(ii) For each $e \subsetneq f$ with $|e| = \ell \geq 2$, $\mathcal{G}[V_e]$ is $\varepsilon_\ell(d_\ell, \dots, d_{k-1})$-regular with relative density $d(e) \geq d_\ell$ with respect to the ℓ-graph obtained from $\mathcal{G}[V_e]$ by replacing layer ℓ with the constant 1.

(iii) $\mathcal{G}$ is ε_k-regular with respect to the k-graph obtained from $\mathcal{G}$ by replacing layer k with the constant 1.

Note that we do not control the k-level relative density, $d(f)$ say, of $\mathcal{G}$ above, but in applications for a given constant d_k we can make ε_k small enough that any slice where $d(f) < d_k$ may be neglected. This formulation permits $\varepsilon_\ell \ll d_\ell, \dots, d_k$, and hence for suitable ε_ℓ functions these slices have the required properties for the use of Theorems 5.5 and 5.6 in the case $\Gamma = 1$.

To prove Theorems 5.5 and 5.6 for dense hypergraphs it is enough to construct a stack of candidate graphs from a slice of the above form such that the empty partial embedding is a GPE, and apply Lemmas 5.10 and 5.11 as required. There are additional complications for sparse hypergraphs which we deal with first.

5.3.2 Sparse hypergraphs

We can represent a sparse, pseudorandom k-uniform hypergraph Γ as a weighted k-graph Γ which is an $(\eta, 2c^*)$-THC graph with density graph $\mathcal{P}$

which is equal to some $0 < p \leq 1$ at the k-layer and equal to 1 elsewhere. In fact, for this sketch suppose that Γ has a slightly stronger version of the THC properties which allow us to count complexes with at most 8 vertices in each part. For an arbitrary $\mathcal{H} \subseteq \Gamma$, we wish to partition $\mathcal{H}$ and Γ into slices of the form $\mathcal{G} \subseteq \Gamma'$ which satisfy the hypotheses of Theorems 5.5 and 5.6.

As above, when considering an application of Theorem 5.5 or 5.6, we will have $\varepsilon_k > 0$ and functions $\varepsilon_\ell : [0,1]^{k-\ell} \to \mathbb{R}_{>0}$ to ensure that any errors in level ℓ are small enough compared to relative densities from levels ℓ and above. In addition we will have some maximum permitted error η_0 for the THC properties of the eventual slices Γ'.

In order to apply the methods of Conlon, Fox, and Zhao [17], we require $\mathcal{H}$ to have a property known as *upper regularity*, but this holds for any $\mathcal{H}$ because of our assumptions on Γ, see [17]. We restrict our attention to k-partite pieces of Γ' which can be obtained from the general case by the standard construction. Then let $|f| = k$ and Γ be k-partite on vertex sets $\{U_j\}_{j \in f}$.

With the weak regularity lemma of [17], given $\nu > 0$ (to be determined later), we find a dense approximation $\tilde{\mathcal{H}}$ of $\mathcal{H}$ with a prescribed structure on $s \ll 1/\nu$ 'pieces'. V_f is partitioned into k-uniform hypergraphs $\mathcal{Q}_1, \ldots, \mathcal{Q}_s$, (with functions q_i to indicate their edges) such that each $\mathcal{Q}_i$ is the set of k-cliques in some $(k-1)$-uniform hypergraph, and we have densities $\alpha_1, \ldots, \alpha_s \in [0,1]$ such that with k-level weight given by

$$\tilde{h}(x_f) = \sum_{i \in [s]} \alpha_i q_i(x_f) \in [0,1]\,,$$

we can view $\tilde{\mathcal{H}}$ as a 'density-weighted' union of the dense, k-uniform hypergraphs $\mathcal{Q}_1, \ldots, \mathcal{Q}_s$.

The sense in which $\tilde{\mathcal{H}}$ approximates $\mathcal{H}$ is that the functions $p^{-1}h$ and $\tilde{h}$ from V_f to $\mathbb{R}_{\geq 0}$ are a *ν-discrepancy pair.* This means that for any functions $u_e : U_e \to [0,1]$ where $e \subsetneq f$ we have

$$\left|\mathbb{E}\Big[(p^{-1}h(x_f) - \tilde{h}(x_f)) \prod_{e \subsetneq f} u_e(x_e) \Big| x_f \in U_f\Big]\right| \leq \nu\,. \tag{5.12}$$

In contrast to the strong hypergraph regularity lemma, this is a global statement: we have not refined the partition $\{U_j\}_{j\in f}$. We will have to take ν small enough that this global statement gives usable bounds even when the u_e functions are used to restrict our attention more locally.

To obtain the required structure for our results, we first use the methods for dense hypergraphs of the previous subsection, and then use the counting lemma of [17] which relates subgraph counts in graphs whose weight functions are a discrepancy pair. We require a standard, slight generalisation of the methods described in the previous subsection, where we start the regularity lemma with a prescribed partition which must be refined, and simultaneously regularise some finite number s of k-uniform hypergraphs. Applied to the $\mathcal{Q}_i$, these methods partition $\tilde{\mathcal{H}}$ into a number of slices (depending on s and the ε_ℓ) with the property that each slice has its k-layer contained within exactly one $\mathcal{Q}_i$. In particular, given s and the ε_ℓ, we obtain a bounded number of slices $\tilde{\mathcal{G}}$ with minimum relative densities $d_1, \dots, d_{k-1}$ and vertex sets $V_j \subseteq U_j$ of size at least $d_1|U_j|$ such that (i) to (iii) hold. In this setting the k-layer of each $\tilde{\mathcal{G}}$ is complete with respect to the layer below in the sense that $\tilde{g}(x_f) = \prod_{e \subsetneq f} \tilde{g}(x_e) \in \{0, 1\}$. This means we may use the u_e functions for $e \subsetneq f$ in (5.12) to restrict attention to k-edges supported by the slice $\tilde{\mathcal{G}}$. The fact that we know $d_1, \dots, d_{k-1}$ in terms of s and the ε_ℓ means that we can choose ν small enough that (5.12) is still accurate when rescaled to be over V_f instead of U_f.

The graphs $\mathcal{G}$ and Γ' that we show satisfy the hypotheses of Theorems 5.5 and 5.6 are $\mathcal{H}$ restricted to $\tilde{\mathcal{G}}$, and $\Gamma' := \Gamma[V_f]$ respectively, where we consider both $\mathcal{G}$ and Γ' to have vertex sets V_j for $j \in f$.

Firstly, we show that Γ' is an (η', c^*)-THC graph with density k-graph $\mathcal{P}$, which we recall is p at layer k and 1 on $e \subsetneq f$. The main idea is that accurate counting of small subgraphs is a 'linear forms condition', and powerful combinatorial techniques exist to manage graphs which satisfy these conditions. A standard argument using repeated applications of the Cauchy–Schwarz inequality shows that the $(\eta, 2c^*)$-THC condition (strengthened to allow complexes with up to 8 vertices in each part) on Γ implies that, up to errors controlled by η and k, the function $p^{-1}\gamma$ on U_f may be replaced by 1 in counting expressions which involve subgraphs of Γ. The argument is related to the Gowers–Cauchy–Schwarz inequality [46], proofs of generalised von Neumann theorems [48, 96], and the rather simpler methods in Section 5.4. Conlon, Fox, and Zhao [17, Section 6.2] give a full exposition of this fact, showing (in more generality than we need here) that when we restrict attention to slices Γ' on V_f instead of U_f we can still count any k-complex F on at most c^* vertices. The loss in the accuracy of these statements is controlled by polynomials in η of degree bounded in k, and in d_1^{-k} which gives a bound on how small V_f can be. Then provided $\eta d_1^{-k} \ll \eta_0$, k we have $\Gamma'(F) = (1 \pm \eta_0)p^m$ for any F with $v(F) \leq r$ and exactly m edges of size k, which shows Γ' is an (η_0, c^*)-THC graph.

Now we turn to $\mathcal{G}$, which is $\mathcal{H}$ restricted to $\tilde{\mathcal{G}}$, hence we write $g(x_f) = h(x_f)\tilde{g}(x_f)$. Multiplying by $\tilde{g}$ and rescaling (5.12) so the expectation is over V_f instead of U_f, we have for any functions $u_e : V_e \to [0,1]$,

$$\left|\mathbb{E}\Big[\Big(p^{-1}g(x_f) - \alpha_i\tilde{g}(x_f)\Big)\prod_{e \subsetneq f} u_e(x_e)\Big| x_f \in V_f\Big]\right| \leq \nu d_1^{-k}\,.$$

This means $p^{-1}g$ and $\alpha_i\tilde{g}$ are a νd_1^{-k}-discrepancy pair (as functions $V_f \to \mathbb{R}_{\geq 0}$), and by the THC condition in Γ' which gives us the necessary linear forms conditions, the counting lemma of [17] implies that small subgraph counts agree to accuracy ν' provided $\nu d_1^{-k} \ll \nu'$, k.

Since $\mathcal{G}$ and Γ' agree with $\tilde{\mathcal{G}}$ on edges of size up to $k-1$, the methods for dense hypergraphs in the previous subsection give that for all $e \subsetneq f$ of size $2 \leq \ell \leq k-1$, the graph $G[V_e]$ is ε_ℓ-regular with relative density $d(e) \geq d_\ell$ with respect to the graph obtained from $G[V_e]$ by replacing layer ℓ with Γ', where we use that $\Gamma' = 1$ at layers below k. For layer k, we need to count $O^k(\mathbf{1}^k)$ and $O^k(\mathbf{2}^k)$ in $\mathcal{G}$ and Γ', and the counting lemma of [17] implies

$$\begin{aligned}
\mathcal{G}(O^k(\mathbf{1}^k)) &= \Big(\alpha_i\tilde{\mathcal{G}}(O^k(\mathbf{1}^k)) \pm \nu'\Big)\,p \\
&= (\alpha_i \pm \varepsilon_k)\Gamma'(O^k(\mathbf{1}^k))\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(O^k(\mathbf{2}^k)) &= \Big(\alpha_i^{2^k}\tilde{\mathcal{G}}(O^k(\mathbf{2}^k)) \pm \nu'\Big)\,p^{2^k} \\
&\leq (\alpha_i^{2^k} + \varepsilon_k)\Gamma'(O^k(\mathbf{2}^k))\,,
\end{aligned}$$

provided $\eta_0 \ll \varepsilon_k$ and ν', $\varepsilon_{k-1} \ll \varepsilon_k$, $d_2, \ldots, d_{k-1}$, k. For the second count we need to lower bound $O^k(\mathbf{2}^k)$ in $\tilde{\mathcal{G}}$, which can be done with accuracy that tends to zero with ε_{k-1} by Theorem 5.5 for $(k-1)$-graphs since at layer k we know that $\tilde{\mathcal{G}}$ is complete with respect to layer $k-1$. A careful analysis of the constants shows that, given any functions ε_ℓ and an η_0 as we would have for either Theorem 5.5 or 5.6, we can choose ν and η small enough (and r

large enough) that this sketch yields slices of the desired form.

Again, we do not control the densities α_i, and we should allow for irregular slices in the dense regularity methods, but in applications ε_k can be chosen small enough in terms of some lower bound d_k that irregular slices and slices with $\alpha_i \leq d_k$ can be ignored.

5.3.3 Proofs of Theorems 5.5 and 5.6

Having sketched constructions which show how to obtain the hypotheses of Theorems 5.5 and 5.6, we now turn to their proofs.

Proof of Theorem 5.5. We construct a stack of candidate graphs which show that the trivial partial embedding of F is a GPE, and apply Lemma 5.11.

First, let $\mathcal{C}^{(0)} = \Gamma$, and note that this is an (η_0, c^*)-THC graph with density graph $\mathcal{D}^{(0)} = \mathcal{P}$ by assumption. For higher levels in the stack let $\mathcal{C}^{(\ell)}$ be the graph obtained from $\mathcal{C}^{(\ell-1)}$ by replacing weights on edges of size ℓ by those in $\mathcal{G}$.

For (GPE2) recall that we must consider the regularity of $\widetilde{\mathcal{C}}^{(\ell)}(e)$ with respect to $\overline{\mathcal{C}}^{(\ell-1)}(e)$ for all $e \subseteq J$, and that $\overline{\mathcal{C}}^{(\ell-1)}(e)$ is simply the subgraph of $\mathcal{C}^{(\ell)}$ induced by V_x for $x \in e$. By the definitions of the $\mathcal{C}^{(\ell)}$ in terms of $\mathcal{G}$ and Γ, $\widetilde{\mathcal{C}}^{(\ell)}(e)$ is identical to $\overline{\mathcal{C}}^{(\ell-1)}(e)$ except when $|e| = \ell$ in which case we obtain $\widetilde{\mathcal{C}}^{(\ell)}(e)$ from $\overline{\mathcal{C}}^{(\ell-1)}(e)$ by replacing the weights in V_e with those from $\mathcal{G}$. Then the required regularity holds trivially with $d^{(\ell)}(e) = 1$ for $|e| \neq \ell$ because the graphs are equal, and when $|e| = \ell$ the required regularity holds with $d^{(\ell)}(e) = d(e)$ by the assumptions on $\mathcal{G} \subseteq \Gamma$.

In order to apply Lemma 5.11 with $\Delta = \Delta(F)$, c^* is as required by assumption, but we must give a suitable h^* and valid ensemble of parameters. Let $h^* = 4k^2 + k + \mathrm{vdeg}(F)$, and choose an ensemble of parameters as described following Definition 5.8. We must ensure that η_0 and the ε_ℓ are small enough that this ensemble is indeed valid, but this is allowed by the dependencies among the constants. Then the conclusion of Lemma 5.11 gives the desired bounds because by construction $\prod_{0 \leq \ell \leq k} d^{(\ell)}(e) = d(e)p(e)$, giving

$$\mathcal{G}(F) = \mathcal{C}^{(k)}(F) = \big(1 \pm v(F)\eta_k\big) \tfrac{g(\emptyset)}{d(\emptyset)p(\emptyset)} \mathcal{D}(F)\mathcal{P}(F)\,. \qquad \square$$

Proof of Theorem 5.6. The proof is almost identical to the above, but we use

Lemma 5.12. The construction of the stack of candidate graphs is identical, and again we use $h^* = 4k^2 + k + \mathrm{vdeg}(F) \leq 4k^2 + k + k\Delta$ and ensure that η_0 and the ε_ℓ are small enough that the same construction of an ensemble is valid. Lemma 5.12 gives the required lower bound. $\square$

5.4 Applying the Cauchy–Schwarz inequality

We can always consider a k-complex on t vertices as t-partite with parts of size one, and in this case we represent the sum giving $\mathcal{H}(F)$ by an expectation as follows. A partite homomorphism $\phi : F \to \mathcal{H}$ must map vertex j of F to a vertex $x_j \in V_j$ of $\mathcal{H}$, so for $|J| = t$ indexing the vertex sets, a partite homomorphism ψ from F to $\mathcal{H}$ is equivalent to a vector of vertices $x_J \in V_J$, and we have

$$\mathcal{H}(F) = \mathbb{E}\Big[\prod_{e\in F} h(x_e)\Big| x_J \in V_J\Big] \,,$$

where the expectation is over the uniform distribution on vectors $x_J \in V_J$, and we write x_e for the natural projection of x_e onto V_e.

Let $\mathbf{a} \in \{0,1,2\}^J$. We use the following notation for the count of octahedra such as $O^k(\mathbf{a})$ in $\mathcal{H}$. Suppose that for $j \in J$ and $i \in [\mathbf{a}_j]$, vertices $x_j^{(i)}$ are chosen uniformly at random (with replacement) from V_j. For $e \subseteq J$ and $\omega \in \prod_{j\in J}[\mathbf{a}_j]$ we write $x_e^{(\omega)}$ for the vector indexed by $j \in e$ of vertices $x_j^{(\omega_j)}$. Then we have the notation

$$\mathcal{H}(F(\mathbf{a})) = \mathbb{E}\Big[\prod_{\substack{e\in F\\ \omega:\omega_i\in[\mathbf{a}_i]}} h(x_e^{(\omega)})\Big| x_j^{(i)} \in V_j \text{ for each } j \in J \text{ and } i \in [\mathbf{a}_j]\Big] \,,$$

for the expected weight of a uniformly random partite homomorphism from $O^k(\mathbf{a})$ to $\mathcal{H}$. With this notation in place, we turn to the main tool of the chapter.

5.4.1 Basic applications

We make extensive use of the Cauchy–Schwarz inequality in the form $\mathbb{E}[XY]^2 \leq \mathbb{E}[X]\,\mathbb{E}[XY^2]$, where we take care to ensure that X takes non-negative values throughout. First, we restate Lemma 5.13 and give a proof.

Lemma 5.13. *Let W, X, and Y be discrete random variables such that W takes values in $[0,1]$, X takes values in the non-negative reals, and Y is real-valued. Suppose also that for $0 \le \varepsilon \le 1$ and $d \ge 0$ we have*

$$\mathbb{E}[XY] = (1 \pm \varepsilon) d \,\mathbb{E}\, X \qquad \textit{and} \qquad \mathbb{E}[XY^2] \le (1+\varepsilon) d^2 \,\mathbb{E}\, X\,.$$

Then

$$\mathbb{E}[WXY] = \left(1 - \varepsilon \pm 2\sqrt{\frac{\varepsilon \,\mathbb{E}\, X}{\mathbb{E}[WX]}}\right) d\,\mathbb{E}[WX]\,,$$

and

$$\mathbb{E}[WXY^2] = \left(1 - 2\varepsilon \pm 7\sqrt{\varepsilon}\frac{\mathbb{E}\, X}{\mathbb{E}[WX]}\right) d^2\,\mathbb{E}[WX]\,.$$

Proof. For the first statement, observe that X, W, and $1-W$ are all non-negative random variables. Then we have

$$\begin{aligned}(1+\varepsilon)d^2\,\mathbb{E}\, X \ge \mathbb{E}[XY^2] &= \mathbb{E}[WXY^2] + \mathbb{E}[(1-W)XY^2] \\ &\ge \frac{\mathbb{E}[WXY]^2}{\mathbb{E}[WX]} + \frac{\mathbb{E}[(1-W)XY]^2}{\mathbb{E}[(1-W)X]}\,,\end{aligned} \tag{5.13}$$

where the second inequality is by two applications of the Cauchy–Schwarz inequality. Given fixed $\mathbb{E}[WXY]$ the right hand side is minimised when $\mathbb{E}[XY] = (1-\varepsilon)d\,\mathbb{E}\, X$, so we may assume

$$\mathbb{E}[(1-W)XY] = (1-\varepsilon)d\,\mathbb{E}\, X - \mathbb{E}[WXY]\,.$$

Let $\mathbb{E}[WXY] = (1-\varepsilon+c)d\,\mathbb{E}[WX]$. Then from (5.13) we have

$$\begin{aligned}(1+\varepsilon)\,\mathbb{E}\, X &\ge (1-\varepsilon+c)^2\,\mathbb{E}[WX] + \frac{\big((1-\varepsilon)\,\mathbb{E}\, X - (1-\varepsilon+c)\,\mathbb{E}[WX]\big)^2}{\mathbb{E}[(1-W)X]} \\ &= (1-\varepsilon+c)^2\,\mathbb{E}[WX] + \frac{\big((1-\varepsilon)\,\mathbb{E}[(1-W)X] - c\,\mathbb{E}[WX]\big)^2}{\mathbb{E}[(1-W)X]} \\ &= (1-\varepsilon)^2\,\mathbb{E}\, X + \frac{c^2\,\mathbb{E}[X]\,\mathbb{E}[WX]}{\mathbb{E}[(1-W)X]}\,,\end{aligned}$$

and so

$$3\varepsilon - \varepsilon^2 \ge c^2\frac{\mathbb{E}[WX]}{\mathbb{E}[(1-W)X]} \ge c^2\frac{\mathbb{E}[WX]}{\mathbb{E}\, X}\,,$$

which is a contradiction if $c^2 \ge 4\varepsilon\,\mathbb{E}[X]/\,\mathbb{E}[WX]$, as required.

For the second statement, we have by the Cauchy–Schwarz inequality and the first part,

$$\begin{aligned}\mathbb{E}[WXY^2] \geq \frac{\mathbb{E}[WXY]^2}{\mathbb{E}[WX]} &\geq \left(1-\varepsilon-2\sqrt{\frac{\varepsilon\,\mathbb{E}\,X}{\mathbb{E}[WX]}}\right)^2 d^2\,\mathbb{E}[WX] \\ &\geq \left(1-2\varepsilon-4\sqrt{\frac{\varepsilon\,\mathbb{E}\,X}{\mathbb{E}[WX]}}\right) d^2\,\mathbb{E}[WX]\,,\end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{E}[(1-W)XY^2] &\geq \frac{\mathbb{E}[(1-W)XY]^2}{\mathbb{E}[(1-W)X]} \\ &\geq \frac{\Big((1-\varepsilon)\,\mathbb{E}[(1-W)X]-2\sqrt{\varepsilon\,\mathbb{E}[X]\,\mathbb{E}[WX]}\Big)^2}{\mathbb{E}[(1-W)X]}d^2 \\ &\geq \Big((1-2\varepsilon)\,\mathbb{E}[(1-W)X]-4\sqrt{\varepsilon\,\mathbb{E}[X]\,\mathbb{E}[WX]}\Big)d^2\,,\end{aligned}$$

so that

$$\begin{aligned}\mathbb{E}[WXY^2] &\leq (1+\varepsilon)d^2\,\mathbb{E}\,X \\ &\qquad -\Big((1-2\varepsilon)\,\mathbb{E}[(1-W)X]-4\sqrt{\varepsilon\,\mathbb{E}[X]\,\mathbb{E}[WX]}\Big)d^2 \\ &\leq \left(1-2\varepsilon+7\sqrt{\varepsilon}\frac{\mathbb{E}\,X}{\mathbb{E}[WX]}\right)d^2\,\mathbb{E}[WX]\,.\end{aligned}$$

□

Corollary 5.16. *Let X and Y be random variables such that X takes values in the non-negative reals and Y is real-valued. Suppose also that for $0 \leq \varepsilon \leq 1$ and $d \geq 0$ we have*

$$\mathbb{E}[XY] = (1\pm\varepsilon)d\,\mathbb{E}\,X \qquad \textit{and} \qquad \mathbb{E}[XY^2] \leq (1+\varepsilon)d^2\,\mathbb{E}\,X\,.$$

Let W be the indicator of the event that $Y = (1\pm 2\varepsilon^{1/4})d$. Then

$$\mathbb{E}[WX] \geq (1-4\varepsilon^{1/4})\,\mathbb{E}\,X\,.$$

Proof. Write $\varepsilon' = 2\varepsilon^{1/4}$ and let Z indicate the event that $Y > (1+\varepsilon')d$.

Then using Lemma 5.13 with weight $1 - Z$ we have

$$\begin{aligned}(1+\varepsilon)d\,\mathbb{E}\,X \geq \mathbb{E}[XY] &= \mathbb{E}[(1-Z)XY] + \mathbb{E}[ZXY] \\ &\geq \Big(1-\varepsilon-2\sqrt{\tfrac{\varepsilon\,\mathbb{E}\,X}{\mathbb{E}[(1-Z)X]}}\Big)\,d\,\mathbb{E}[(1-Z)X] + (1+\varepsilon')d\,\mathbb{E}[ZX] \\ &\geq \Big(1-\varepsilon-2\sqrt{\varepsilon}+\varepsilon'\tfrac{\mathbb{E}[ZX]}{\mathbb{E}\,X}\Big)\,d\,\mathbb{E}\,X\,,\end{aligned}$$

which implies that $\mathbb{E}[ZX] \leq 2\varepsilon^{1/4}\,\mathbb{E}\,X$.

With a similar argument we deal with the event that $Y < (1-\varepsilon')d$, now using the letter Z for the indicator of this event we calculate

$$\begin{aligned}&(1-\varepsilon)d\,\mathbb{E}\,X \\ &\quad \leq \mathbb{E}[XY] = \mathbb{E}[(1-Z)XY] + \mathbb{E}[ZXY] \\ &\quad \leq \left(1-\varepsilon+2\sqrt{\frac{\varepsilon\,\mathbb{E}\,X}{\mathbb{E}[(1-Z)X]}}\right)d\,\mathbb{E}[(1-Z)X] + (1-\varepsilon')d\,\mathbb{E}[ZX] \\ &\quad \leq \left(1+2\sqrt{\varepsilon}-\varepsilon'\frac{\mathbb{E}[ZX]}{\mathbb{E}\,X}\right)d\,\mathbb{E}\,X\,,\end{aligned}$$

which implies that $\mathbb{E}[ZX] \leq 2\varepsilon^{1/4}\,\mathbb{E}\,X$. Together, the two arguments prove that $\mathbb{E}[WX] \geq (1-4\varepsilon^{1/4})\,\mathbb{E}\,X$ as required. □

Corollary 5.17. *Let X and Y be random variables such that X takes values in the non-negative reals and Y is real-valued. Suppose also that for a natural number $t \geq 2$, and reals $0 \leq \varepsilon < 2^{2-2t}$ and $d \geq 0$ we have*

$$\mathbb{E}[XY] = (1\pm\varepsilon)d\,\mathbb{E}\,X \qquad \textit{and} \qquad \mathbb{E}[XY^{2^t}] \leq (1+\varepsilon)d^{2^t}\,\mathbb{E}\,X\,.$$

Let W be the indicator of the event that $Y = (1\pm 2\varepsilon^{1/8})d$. Then

$$\mathbb{E}[WX] \geq (1-4\varepsilon^{1/8})\,\mathbb{E}\,X\,.$$

Proof. Let $Z := Y^{2^{t-1}}$ and $\tilde{d} := d^{2^{t-1}}$. Then by the Cauchy–Schwarz inequality we have

$$\mathbb{E}[XZ]^2 \leq \mathbb{E}[X]\,\mathbb{E}[XZ^2] = \mathbb{E}[X]\,\mathbb{E}[XY^{2^t}] \leq (1+\varepsilon)\tilde{d}^2\,\mathbb{E}[X]^2\,.$$

By $t-1$ further applications of the Cauchy–Schwarz inequality we also have

$$\mathbb{E}[XZ] \geq \mathbb{E}[X]^{1-2^{t-1}}\,\mathbb{E}[XY]^{2^{t-1}} \geq (1-\varepsilon)^{2^{t-1}}\tilde{d}\,\mathbb{E}\,X \geq (1-2^{t-1}\varepsilon)\tilde{d}\,\mathbb{E}\,X\,.$$

With $\tilde{\varepsilon} := \varepsilon^{1/2} \geq 2^{t-1}\varepsilon$ this implies

$$\mathbb{E}[XZ] = (1 \pm \tilde{\varepsilon})\tilde{d}\,\mathbb{E}\,X\,, \qquad\qquad \mathbb{E}[XZ^2] \leq (1+\tilde{\varepsilon})\tilde{d}^2\,\mathbb{E}\,X\,.$$

The result now follows from Corollary 5.16. Note that $Z = (1 \pm 2\tilde{\varepsilon}^{1/4})\tilde{d}$ implies the event $Y = (1 \pm 2\varepsilon^{1/8})d$ which is indicated by W, hence by Corollary 5.16 we obtain $\mathbb{E}[WX] \geq (1 - 4\varepsilon^{1/8})\,\mathbb{E}\,X$. $\square$

5.4.2 Lower bounds on octahedra

The common theme in the following results is an application of the Cauchy–Schwarz inequality to the expectation in a normalised homomorphism count.

Lemma 5.18. *For every natural number $k \geq 2$, vertex set J of size k, index $i \in J$ and vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \{1,2\}^J$ which satisfy $\mathbf{a}_j = \mathbf{b}_j = \mathbf{c}_j$ for all $j \in J \setminus \{i\}$ and $\mathbf{a}_i = 0$, $\mathbf{b}_i = 1$, $\mathbf{c}_i = 2$ the following holds. Let $\mathcal{H}$ be a k-partite k-graph on vertex set $\{V_j\}_{j\in J}$. Then*

$$\mathcal{H}(O^k(\mathbf{c})) \geq \frac{\mathcal{H}(O^k(\mathbf{b}))^2}{\mathcal{H}(O^{k-1}(\mathbf{a}))}\,.$$

Proof. We prove the case $J = \{0,1,\ldots,k-1\}$ and $i = 0$, writing $f = [k-1]$ for the indices on which $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ agree. The other cases follow by relabelling indices.

Observe that a copy of $O^k(\mathbf{c})$ simply consists of two copies of $O^k(\mathbf{b})$ agreeing on a copy of $O^{k-1}(\mathbf{a})$. Let X be the random variable giving the weight of a uniform random copy of $O^{k-1}(\mathbf{a})$, and Y be the random variable which, given a uniform random copy of $O^{k-1}(\mathbf{a})$, returns the total weight of the ways to extend it to a copy of $O^k(\mathbf{b})$. More concretely, we choose uniformly at random (with replacement) vertices $x_j^{(i)} \in V_j$ for each $i \in [\mathbf{a}_j]$, and let

$$X := \prod_{\substack{e\subseteq f,\\ \omega:\omega_i\in[\mathbf{a}_i]}} g(x_e^{(\omega)})\,, \qquad \text{and} \qquad Y := \mathbb{E}\Big[\prod_{\substack{e\subseteq f,\\ \omega:\omega_i\in[\mathbf{a}_i]}} g(x_0, x_e^{(\omega)})\Big| x_0 \in V_0\Big]\,.$$

Thus we have $\mathbb{E}\,X = \mathcal{H}(O^{k-1}(\mathbf{a}))$, $\mathbb{E}[XY] = \mathcal{H}(O^k(\mathbf{b}))$, and $\mathbb{E}[XY^2] = \mathcal{H}(O^k(\mathbf{c}))$. Since X is a non-negative random variable, the Cauchy-Schwarz inequality $\mathbb{E}[XY]^2 \leq \mathbb{E}[X]\,\mathbb{E}[XY^2]$ gives the required statement. $\square$

Lemma 5.18 justifies the term 'minimal' used in the following definition.

Definition 5.19. Let $\mathcal{H}$ be a k-partite k-graph and $\eta \geq 0$. Then we say that $\mathcal{H}$ is *η-minimal* if, for every $i \in [k]$ and for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1,2\}^k$ which satisfy $\mathbf{a}_j = \mathbf{b}_j = \mathbf{c}_j$ for all $j \in [k] \setminus \{i\}$ and $\mathbf{a}_i = 0$, $\mathbf{b}_i = 1$, $\mathbf{c}_i = 2$, we have

$$\mathcal{H}(O^k(\mathbf{c})) \leq (1+\eta)\frac{\mathcal{H}(O^k(\mathbf{b}))^2}{\mathcal{H}(O^{k-1}(\mathbf{a}))}\,.$$

Suppose Γ and $\mathcal{G}$ are k-partite k-graphs, and $\mathcal{G}$ agrees with Γ on edges of size $k-1$ and less. Suppose furthermore that the density of $\mathcal{G}$ relative to Γ is d. If Γ is a complete graph, then it is well known that $\mathcal{G}$ has at least d^{2^k} times as many octahedra as Γ. For general Γ this statement is false, but we will now show that if Γ is η-minimal it is approximately true (and generalise it).

Corollary 5.20. *For all natural numbers $k \geq 2$ and vectors $\mathbf{s}$, $\mathbf{s}' \in \{1,2\}^k$ with $\mathbf{s} \geq \mathbf{s}'$ pointwise and such that $\mathbf{s}$ has t more 2 entries than $\mathbf{s}'$, the following holds. Suppose that $\mathcal{G}$ and Γ are k-partite k-graphs on the same partite vertex set, with $g(e) = \gamma(e)$ for all e with $|e| < k$, and suppose $\mathcal{G}(O^k(\mathbf{s}')) = d \cdot \Gamma(O^k(\mathbf{s}'))$. Moreover suppose that Γ is η-minimal. Then*

$$\mathcal{G}(O^k(\mathbf{s})) \geq \frac{d^{2^t}}{(1+\eta)^{2^t-1}}\Gamma(O^k(\mathbf{s}))\,.$$

Proof. We prove the case $\mathbf{s} = (\mathbf{2}^{t+a}, \mathbf{1}^{k-t-a})$ where $a \geq 0$ is an integer; the other cases follow by relabelling indices. Letting $\mathbf{s}^{(i)} := (\mathbf{2}^{t+a-i}, \mathbf{1}^{k+i-t-a})$ and $\mathbf{r}^{(i)} := (\mathbf{2}^{t+a-i-1}, 0, \mathbf{1}^{k+i-t-a})$, we have $\mathbf{s}^{(0)} = \mathbf{s}$ and $\mathbf{s}^{(t)} = \mathbf{s}'$. By Lemma 5.18 and η-minimality of Γ respectively, for each $0 \leq i \leq t-1$ we have

$$\mathcal{G}(O^k(\mathbf{s}^{(i)})) \geq \frac{\mathcal{G}(O^k(\mathbf{s}^{(i+1)}))^2}{\mathcal{G}(O^{k-1}(\mathbf{r}^{(i+1)}))}$$
$$\Gamma(O^k(\mathbf{s}^{(i)})) \leq (1+\eta)\frac{\Gamma(O^k(\mathbf{s}^{(i+1)}))^2}{\Gamma(O^{k-1}(\mathbf{r}^{(i+1)}))}\,.$$

Note that since $\mathcal{G}$ and Γ agree on edges of size less than k, the denominators in both fractions are equal, so for each $0 \leq i \leq t-1$ we have

$$\frac{\mathcal{G}(O^k(\mathbf{s}^{(i)}))}{\Gamma(O^k(\mathbf{s}^{(i)}))} \geq \frac{1}{1+\eta}\left(\frac{\mathcal{G}(O^k(\mathbf{s}^{(i+1)}))}{\Gamma(O^k(\mathbf{s}^{(i+1)}))}\right)^2,$$

and thus

$$\frac{\mathcal{G}\big(O^k(\mathbf{s}^{(0)})\big)}{\Gamma\big(O^k(\mathbf{s}^{(0)})\big)} \geq \frac{1}{(1+\eta)^{2^t-1}}\left(\frac{\mathcal{G}\big(O^k(\mathbf{s}^{(t)})\big)}{\Gamma\big(O^k(\mathbf{s}^{(t)})\big)}\right)^{2^t} = \frac{d^{2^t}}{(1+\eta)^{2^t-1}}\,,$$

as desired. □

In particular, it follows that if $\mathcal{G}$ is (ε, d)-regular with respect to the η-minimal Γ, then $\mathcal{G}$ is itself ε'-minimal, where ε' is small provided η is sufficiently small and ε is small enough compared to d.

Corollary 5.21. *Given ε', $d > 0$, then for ε, η small enough that*

$$\varepsilon' \geq \max\left\{1 - \tfrac{(1-\varepsilon/d)^{2^k}}{(1+\eta)^{2^k-1}},\ (1+\varepsilon d^{-2^k})(1+\eta)^{2^k-1} - 1\right\},$$

the following holds. Let Γ and $\mathcal{G}$ be k-partite k-graphs on the same partite vertex set, such that $\gamma(e) = g(e)$ whenever $|e| < k$. Suppose that $\mathcal{G}$ is (ε, d)-regular with respect to Γ, and that Γ is η-minimal. Then $\mathcal{G}$ is ε'-minimal, and for each $\mathbf{c} \in \{1,2\}^k$ we have $\mathcal{G}\big(O^k(\mathbf{c})\big) = (1 \pm \varepsilon')d^r\Gamma\big(O^k(\mathbf{c})\big)$ with $r = \prod_{i\in[k]} \mathbf{c}_i$.

Moreover we note that if the above inequality for ε' is tight, we have

$$\varepsilon' \leq 2^{2^k}\big(\varepsilon d^{-2^k} + \eta\big)\,.$$

Proof. We begin with the second statement, comparing $\mathcal{G}\big(O^k(\mathbf{c})\big)$ to $\Gamma\big(O^k(\mathbf{c})\big)$. Corollary 5.20 with $\mathbf{s} = \mathbf{c}$ and $\mathbf{s}' = \mathbf{1}^k$ and the regularity bound on $\mathcal{G}\big(O^k(\mathbf{1}^k)\big)$ give the required lower bound, since for any $1 \leq r \leq 2^k$ we have by choice of ε and η,

$$\frac{(1-\varepsilon/d)^r}{(1+\eta)^{r-1}} \geq \frac{(1-\varepsilon/d)^{2^k}}{(1+\eta)^{2^k-1}} \geq 1 - \varepsilon'\,.$$

To obtain the upper bound, suppose for contradiction that $\mathcal{G}\big(O^k(\mathbf{c})\big) > (1+\varepsilon')d^r\Gamma\big(O^k(\mathbf{c})\big)$, where $r := \prod_{i\in[k]} \mathbf{c}_i$. Then applying Corollary 5.20 with $\mathbf{s} = \mathbf{2}^k$ and $\mathbf{s}' = \mathbf{c}$, we have

$$\begin{aligned}\mathcal{G}\big(O^k(\mathbf{2}^k)\big) &> \frac{\big((1+\varepsilon')d^r\big)^{2^k/r}}{(1+\eta)^{2^k/r-1}}\Gamma\big(O^k(\mathbf{2}^k)\big) \geq \frac{(1+\varepsilon')^{2^k/r}}{(1+\eta)^{2^k-1}}d^{2^k}\Gamma\big(O^k(\mathbf{2}^k)\big)\\ &\geq (d^{2^k} + \varepsilon)\Gamma\big(O^k(\mathbf{2}^k)\big)\,,\end{aligned}$$

by $1 \le r \le 2^k$ and choice of ε, η. This contradicts the (ε, d)-regularity of $\mathcal{G}$ with respect to Γ.

The minimality argument is essentially identical. Suppose for contradiction that $\mathcal{G}$ is not ε'-minimal, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \{0,1,2\}^k$ be vectors witnessing this. That is, these vectors agree on $[k] \setminus \{j\}$ for some $j \in [k]$ and we have $\mathbf{a}_j = 0$, $\mathbf{b}_j = 1$, $\mathbf{c}_j = 2$, and

$$\mathcal{G}\big(O^k(\mathbf{c})\big) > (1+\varepsilon')\frac{\mathcal{G}\big(O^k(\mathbf{b})\big)^2}{\mathcal{G}\big(O^{k-1}(\mathbf{a})\big)}\,. \tag{5.14}$$

Observe that $\mathbf{b}$ cannot contain any zero entries, since otherwise the three octahedron counts are the same as in Γ, and since $\varepsilon' \ge \eta$ the three vectors then witness that Γ is not η-minimal. Let t be the number of 2 entries in $\mathbf{b}$. By Corollary 5.20, we have $\mathcal{G}\big(O^k(\mathbf{b})\big) \ge \frac{d^{2^t}}{(1+\eta)^{2^t-1}}\Gamma\big(O^k(\mathbf{b})\big)$, so since $\mathcal{G}$ and Γ agree on edges of size at most $k-1$, we have

$$\mathcal{G}\big(O^k(\mathbf{c})\big) \overset{(5.14)}{>} (1+\varepsilon')\frac{d^{2^{t+1}}}{(1+\eta)^{2^{t+1}-2}}\frac{\Gamma\big(O^k(\mathbf{b})\big)^2}{\Gamma\big(O^{k-1}(\mathbf{a})\big)} \ge \frac{(1+\varepsilon')d^{2^{t+1}}}{(1+\eta)^{2^{t+1}-1}}\Gamma\big(O^k(\mathbf{c})\big)\,,$$

where the second inequality uses the η-minimality of Γ. Applying Corollary 5.20 with $\mathbf{s} = \mathbf{2}^k$ and $\mathbf{s}' = \mathbf{c}$, we obtain

$$\begin{aligned}\mathcal{G}\big(O^k(\mathbf{2}^k)\big) &> (1+\eta)^{2^{k-t-1}-1}\left(\frac{(1+\varepsilon')d^{2^{t+1}}}{(1+\eta)^{2^{t+1}-1}}\right)^{2^{k-t-1}}\Gamma\big(O^k(\mathbf{2}^k)\big)\\ &= \frac{(1+\varepsilon')^{2^{k-t-1}}}{(1+\eta)^{2^k-1}}d^{2^k}\Gamma\big(O^k(\mathbf{2}^k)\big)\,.\end{aligned}$$

Since $t \le k-1$, and since $(1+\varepsilon')(1+\eta)^{1-2^k}d^{2^k} \ge d^{2^k}+\varepsilon$, this is the desired contradiction to the (ε, d)-regularity of $\mathcal{G}$ with respect to Γ. $\square$

5.4.3 Regular subgraphs of regular graphs

In this section we show that, given an η-minimal graph Γ, if we replace the ℓ-edges $V_{[\ell]}$ by a subgraph which is relatively dense and regular with respect to $\Gamma[V_1, \dots, V_\ell]$ then the result is still η'-minimal. This is a generalisation of the *slicing lemma* for 2-graphs, which says that large subsets of a regular pair induce a regular pair; in other words, replacing 1-edges with a relatively dense subgraph preserves regularity of the 2-edges. Note that regularity is a trivial condition for 1-graphs.

Lemma 5.22. *Given ε', $d > 0$, then for ε, η small enough that*

$$\begin{aligned}\min\{\varepsilon', 1/2\} \geq \eta + \max\Big\{&2^7k^3\Big(1 - \tfrac{(1-\varepsilon/d)^{2^{k-1}}}{(1+\eta)^{2^{k-1}-1}}\Big),\\ &2^7k^3\left((1+\varepsilon d^{-2^{k-1}})(1+\eta)^{2^{k-1}-1} - 1\right),\\ &2^9k^3\left((1+100\sqrt{\eta}d^{-2^{k-1}})(1+2\eta) - 1\right)\Big\},\end{aligned}$$

the following holds. Let Γ be an η-minimal k-partite k-graph with parts $V_1, \dots, V_k$, and let $\mathcal{G}$ be a subgraph on the same vertex set, which agrees with Γ except on $V_{[\ell]}$ for some $\ell < k$, and which has the property that $\mathcal{G}[V_1, \dots, V_\ell]$ is (ε, d)-regular with respect to $\Gamma[V_1, \dots, V_\ell]$. Then $\mathcal{G}$ is ε'-minimal, and for each $\mathbf{s} \in \{0,1,2\}^k$ we have $\mathcal{G}(O^k(\mathbf{s})) = (1 \pm \varepsilon')d^r\Gamma(O^k(\mathbf{s}))$ with $r := \prod_{i\in[\ell]} \mathbf{s}_i$. Moreover, we note that when $\varepsilon' < 1/2$ and the above inequality for ε' is tight, we have

$$\varepsilon' \leq 2^{2^{k-1}+18}k^3(\varepsilon + \sqrt{\eta})d^{-2^{k-1}}.$$

Proof. Let ξ be maximal such that $\big(\frac{1+2k\xi}{1-2k\xi}\big)^2(1+\eta) \leq 1 + \varepsilon'$, noting that this gives $2^{-7}k^{-3}(\varepsilon' - \eta) \leq \xi \leq (4k)^{-1}\varepsilon'$. The choice of ε, η ensure that when Corollary 5.21 is applied (e.g. to $\mathcal{G}[V_1, \dots, V_\ell]$ and $\Gamma[V_1, \dots, V_\ell]$) with $k_{\text{C5.21}} = \ell$ and ε, η as in this lemma, the resulting $\varepsilon'_{\text{C5.21}}$ is at most ξ, and that

$$(1+100\sqrt{\eta}d^{-2^{k-1}})(1+2\eta) \leq 1 + \tfrac{1}{4}\xi\,. \tag{5.15}$$

The following claim, and choice of ξ, gives the desired counting in $\mathcal{G}$.

Claim 5.23. *Given $\mathbf{s} \in \{0,1,2\}^k$, let $q := \sum_{i\in[k]} \mathbf{s}_i$ and $r := \prod_{i\in[\ell]} s_i$. Then we have*

$$\mathcal{G}(O^k(\mathbf{s})) = (1 \pm q\xi)d^r\Gamma(O^k(\mathbf{s}))\,.$$

The required counting statements in $\mathcal{G}$ follow because $q \leq 2k$ and $\xi \leq (4k)^{-1}\varepsilon'$. The desired ε'-minimality follows directly from this claim and η-minimality of Γ. Indeed, let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors in $\{0,1,2\}^k$ agreeing at all indices except j, and with $\mathbf{a}_j = 0$, $\mathbf{b}_j = 1$, $\mathbf{c}_j = 2$. Let $t_{\mathbf{a}} = \prod_{i\in[\ell]} a_i$, and define similarly $t_{\mathbf{b}}$ and $t_{\mathbf{c}}$. Note that either $j \in [\ell]$ and we have $t_{\mathbf{a}} = 0$ and $t_{\mathbf{c}} = 2t_{\mathbf{b}}$, or $j \notin [\ell]$ and all three are equal. Since $\sum_{i\in[k]} \mathbf{c}_i \le 2k$, we have by the claim,

$$\begin{aligned}\mathcal{G}(O^k(\mathbf{c})) &\le (1+2k\xi)d^{t_\mathbf{c}}\Gamma(O^k(\mathbf{c}))\\ &\le (1+2k\xi)(1+\eta)d^{t_\mathbf{c}}\frac{\Gamma(O^k(\mathbf{b}))^2}{\Gamma(O^k(\mathbf{a}))}\\ &\le (1+2k\xi)(1+\eta)d^{t_\mathbf{c}}\frac{(1-2k\xi)^{-2}d^{-2t_\mathbf{b}}\mathcal{G}(O^k(\mathbf{b}))^2}{(1+2k\xi)^{-1}d^{-t_\mathbf{a}}\mathcal{G}(O^k(\mathbf{a}))}\\ &= \left(\frac{1+2k\xi}{1-2k\xi}\right)^2(1+\eta)\frac{\mathcal{G}(O^k(\mathbf{b}))^2}{\mathcal{G}(O^k(\mathbf{a}))}\,,\end{aligned}$$

as desired. It remains only to prove the claim, which we now do by induction on the number of zeroes in $\mathbf{s}$ outside $[\ell]$.

Proof of Claim 5.23. The base case is that all entries of $\mathbf{s}$ outside $[\ell]$ are equal to zero. Note that if $\mathbf{s} = \mathbf{0}^k$ then the claim is trivial, so we assume this is not the case, and hence $q = \sum_{i\in[k]} \mathbf{s}_i \ge 1$. Write $\mathbf{s}' \in \{0,1,2\}^\ell$ for the first ℓ entries of $\mathbf{s}$, which for the base case are the only entries which may be non-zero, giving $\mathcal{G}(O^k(\mathbf{s})) = \mathcal{G}[V_1,\ldots,V_\ell](O^\ell(\mathbf{s}'))$. Since $\mathcal{G}[V_1,\ldots,V_\ell]$ is (ε,d)-regular with respect to $\Gamma[V_1,\ldots,V_\ell]$, which is η-minimal, by Corollary 5.21 and choice of ε, η, the claim statement follows.

For the induction step, suppose that $j \notin [\ell]$ is such that $s_j \ne 0$. For $i = 0,1,2$, let $\mathbf{s}^{(i)}$ be the vector equal to $\mathbf{s}$ at all entries except the jth, and with $\mathbf{s}^{(i)}_j = i$. By induction, the claim statement holds for $\mathbf{s}^{(0)}$. Again, write $\mathbf{s}' \in \{0,1,2\}^\ell$ for the first ℓ entries of $\mathbf{s}$. We define random variables W, X, Y as follows. The random experiment we perform is to choose, for each $i \in [\ell]$, uniformly at random (with replacement) $\mathbf{s}^{(0)}_i$ vertices in V_i. We let X be the weight of the copy of $O^{k-1}(\mathbf{s}^{(0)})$ in Γ on these vertices, WX be the weight of the copy of $O^{k-1}(\mathbf{s}^{(0)})$ in $\mathcal{G}$ on these vertices, and XY be the expected weight, over a uniformly random choice of vertex in V_j, of the copy of $O^k(\mathbf{s}^{(1)})$ in Γ. Note that since $\mathcal{G}$ is a subgraph of Γ, we always have $0 \le W \le 1$. More formally (and dealing with the trivial exceptional case $X = 0$), let $x^{(m)}_i \in V_i$ be chosen independently, uniformly at random for each $i \in [k] \setminus \{j\}$ and

$m \in [\mathbf{s}_i^{(0)}]$. Write $\Omega = \prod_{i\in[k]\setminus\{j\}}[\mathbf{s}_i]$ and $\Omega' = \prod_{i\in[k]}[\mathbf{s}_i^{(1)}]$, and define

$$\begin{aligned} X &:= \prod_{e\subseteq[k]\setminus\{j\}} \prod_{\omega\in\Omega} \gamma\big(x_e^{(\omega)}\big)\,, \\ Y &:= \mathbb{E}\Big[\prod_{e\subseteq[k],\, j\in e} \prod_{\omega\in\Omega'} \gamma\big(x_e^{(\omega)}\big)\Big| x_j^{(1)} \in V_j\Big]\,, \quad \text{and} \\ W &:= \begin{cases} \frac{1}{X}\prod_{e\subseteq[k]\setminus\{j\}}\prod_{\omega\in\Omega} g\big(x_e^{(\omega)}\big) & \text{if } X>0 \\ 1 & \text{if } X=0\,. \end{cases} \end{aligned}$$

The key feature of these definitions is that $\mathbb{E}[XY^i] = \Gamma\big(O^k(\mathbf{s}^{(i)})\big)$ for each $i = 0, 1, 2$, and similarly $\mathbb{E}[WXY^i] = \mathcal{G}\big(O^k(\mathbf{s}^{(i)})\big)$. If $\mathbb{E}\,X = 0$ then trivially the claim holds, since $\Gamma\big(O^k(\mathbf{s})\big) = 0$. So we may assume $\mathbb{E}\,X > 0$, and let d' be such that $\mathbb{E}[XY] = d'\,\mathbb{E}\,X$. By Lemma 5.18 and the η-minimality of Γ, we have $\mathbb{E}[XY^2] = (1\pm\eta)\,\mathbb{E}[XY]^2/\,\mathbb{E}\,X = (1\pm\eta)(d')^2\,\mathbb{E}\,X$. We are thus in a position to apply Lemma 5.13, with $\varepsilon_{\mathrm{L5.13}} = \eta$. We obtain

$$\begin{aligned} \mathbb{E}[WXY] &= \Big(1-\eta\pm 2\sqrt{\tfrac{\eta\,\mathbb{E}\,X}{\mathbb{E}[WX]}}\Big)d'\cdot\mathbb{E}[WX] \quad \text{and} \\ \mathbb{E}[WXY^2] &= \Big(1-2\eta\pm 7\sqrt{\eta}\frac{\mathbb{E}\,X}{\mathbb{E}[WX]}\Big)(d')^2\,\mathbb{E}[WX]\,. \end{aligned}$$

Recall that from the induction hypothesis with $q := \sum_{i\in[k]}\mathbf{s}_i^{(0)}$ and $r := \prod_{i\in[\ell]}\mathbf{s}_i^{(0)}$ we have

$$\mathbb{E}[WX] = \mathcal{G}\big(O^{k-1}(\mathbf{s}^{(0)})\big) = (1\pm q\xi)d^r\Gamma\big(O^k(\mathbf{s}^{(0)})\big) = (1\pm q\xi)d^r\,\mathbb{E}\,X\,.$$

This gives

$$\begin{aligned} \mathcal{G}\big(O^k(\mathbf{s}^{(1)})\big) &= \Big(1-\eta\pm 2\sqrt{\eta(1+2q\xi)d^{-r}}\Big)(1\pm q\xi)d^r\Gamma\big(O^k(\mathbf{s}^{(1)})\big)\,, \quad \text{and} \\ \mathcal{G}\big(O^k(\mathbf{s}^{(2)})\big) &= \Big(1-2\eta\pm 7\sqrt{\eta}(1+2q\xi)d^{-r}\Big)(1\pm q\xi)(1\pm 2\eta)d^r\Gamma\big(O^k(\mathbf{s}^{(2)})\big)\,, \end{aligned}$$

where we use the η-minimality of Γ in obtaining the second statement. By choice of ξ and (5.15), this proves the claim for $\mathbf{s}^{(i)}$ with $i = 1, 2$, and in particular for $\mathbf{s}$, as desired. □

The proof of Claim 5.23 completes the proof of Lemma 5.22. □

5.5 Inheritance of regularity

Our goal in this section is to prove Lemma 5.3. Note that in proving the counting and embedding lemmas (see Section 5.2) for k-graphs, we must apply Lemma 5.3 for $k_{\mathrm{L5.3}}$-graphs where $k_{\mathrm{L5.3}}$ takes values up to k, which means we must mention $(k+1)$-partite $(k+1)$-graphs in the proof below. Our definitions mean that in Section 5.2, whenever we are applying Lemma 5.3 to a $(k+1)$-partite $(k+1)$-graph, the graphs are trivial and equal to 1 on edges of size $k+1$. This feature is visible in the graph case: when $k=2$ the inheritance lemmas of [2, 16] involve a 3-partite graphs, and to deduce similar results from our inheritance lemma one must form a 3-graph from this 3-partite graph by giving edges of size 3 weight 1.

We with a brief outline of the method for proving Lemma 5.3. First, let $\mathcal{H}$ be the k-graph on $V_0, \ldots, V_k$ with edge weights

$$h(e) := \begin{cases} \gamma(e) & e \notin V_{[k]} \\ g(e) & e \in V_{[k]} \,. \end{cases}$$

By Lemma 5.22 and (INH3), $\mathcal{G}$ is regular with respect to $\mathcal{H}$, and by (INH4), $\mathcal{H}[V_1, \ldots, V_k]$ is regular with respect to $\Gamma[V_1, \ldots, V_k]$. This, together with Corollary 5.21, in particular allows us to estimate $\mathcal{G}\big(O^{k+1}(\mathbf{1}^{k+1})\big)$ and $\mathcal{G}\big(O^{k+1}(1, \mathbf{2}^k)\big)$ accurately.

These two quantities are by definition equal to the averages, over $v \in V_0$, of $\mathcal{G}_v\big(O^k(\mathbf{1}^k)\big)$ and $\mathcal{G}_v\big(O^k(\mathbf{2}^k)\big)$ respectively. Using (INH1), we conclude that on average the relative density of $\mathcal{G}_v$ with respect to Γ_v is about dd', and the number of octahedra it contains is about $(dd')^{2^k}$ times the number of octahedra in Γ_v.

However, we can also give a lower bound on the average number of octahedra in $\mathcal{G}_v$ using its density relative to Γ_v and Corollary 5.20, whenever Γ_v satisfies the counting conditions of that lemma. The assumption (INH1) implies that these counting conditions are typically satisfied, and the few atypical vertices do not much affect the argument. Using the defect Cauchy–Schwarz inequality and the fact that we know the average density of $\mathcal{G}_v$ relative to Γ_v, we conclude that the only way this lower bound does not contradict the previous estimate is if typically $\mathcal{G}_v$ has density about dd' relative to Γ_v and number of octahedra about $(dd')^{2^k}$ times the number in Γ_v. In other words, $\mathcal{G}_v$ is typically (ε', dd')-regular with respect to Γ_v, as desired.

Proof of Lemma 5.3. We use the letter v for a vertex in V_0 to draw attention to the special role of the set V_0, but use x_j for a vertex in V_j when $j \in [k]$. As in the proof of Lemmas 5.11 and 5.12, we use the correspondence between copies of $O^{k+1}(1, \mathbf{a})$ in $\mathcal{G}$ or Γ, and the average of the counts of $O^k(\mathbf{a})$ in the graphs $\mathcal{G}_v$ or Γ_v over $v \in V_0$. More precisely, we have for any $\mathbf{a} \in \{0, 1, 2\}^k$,

$$\mathcal{G}(O^{k+1}(1, \mathbf{a})) = \mathbb{E}[\mathcal{G}_v(O^k(\mathbf{a}))|v \in V_0] , \tag{5.16}$$

$$\Gamma(O^{k+1}(1, \mathbf{a})) = \mathbb{E}[\Gamma_v(O^k(\mathbf{a}))|v \in V_0] . \tag{5.17}$$

When Γ_v is well-behaved (in a way we make precise below) we are able to count carefully in $\mathcal{G}_v$ but when Γ_v is not well-behaved, we can bound weights in $\mathcal{G}_v$ from above by those in Γ_v.

Though in general it is difficult to control the weight of the empty set, in this proof we only embed a single vertex into V_0, hence there is not much to control. Instead of the usual sprinkling of weights involving the empty set, for this proof we can assume without loss of generality that $\gamma(\emptyset) = g(\emptyset) = p(\emptyset) = 1$ and avoid most of these factors. We will have similar correcting factors when counting in $\mathcal{G}_v$ and Γ_v, however.

The first step of the proof is to use the counting conditions in Γ to establish the existence of $U \subseteq V_0$ such that for each vertex $v \in U$, the link Γ_v is well-behaved. We also give additional properties of Γ and U that are useful later. Property (INH1) specifies a kind of pseudorandomness for Γ, and the natural definition of a well-behaved vertex $v \in V_0$ is that its link Γ_v is similarly pseudorandom, so our definition of U will involve control of the counts of $O^k(\mathbf{a})$ in links. As ever, we must deal carefully with the weight of the empty set in these links, but for edges of size greater than one, we will see that (INH1) implies concentration of the edge weights by the Cauchy–Schwarz inequality. We state the definition in terms of $\mathcal{P}$ rather than $\mathcal{P}_v$ for more convenient use later.

Write $\eta' := 2^{3/2}\eta^{1/4}$ (and note we have $\eta' < 1/2$), and let $U \subseteq V_0$ be those vertices $v \in V_0$ such that for any $\mathbf{a} \in \{0,1,2\}^k \setminus \{\mathbf{0}^k\}$ we have

$$\Gamma_v(O^k(\mathbf{a})) = (1 \pm \eta')\frac{\gamma(v)}{p(0)}\mathcal{P}(O^{k+1}(1,\mathbf{a}))\,. \tag{5.18}$$

The counting assumptions (INH1) are a form of pseudorandomness which suggests that U will be a large subset of V_0, which we prove in the necessary weighted setting below.

Claim 5.24.

(i) Γ *is* 16η*-minimal.*

(ii) *For* $v \in U$*,* Γ_v *is* $16\eta'$*-minimal.*

(iii) *The contribution to* $\Gamma(O^{k+1}(\mathbf{1}^{k+1}))$ *from homomorphisms which use a vertex in* $V_0 \setminus U$ *is at most* $3^{k+3}\eta'\mathcal{P}(O^{k+1}(\mathbf{1}^{k+1}))$*.*

Proof. To see (i), we use (INH1). Let $j \in \{0\} \cup [k]$, and vectors $\mathbf{a},\mathbf{b},\mathbf{c} \in \{0,1,2\}^{k+1}$ be equal on $\{0\} \cup [k] \setminus \{j\}$ and satisfy $\mathbf{a}_j = 0$, $\mathbf{b}_j = 1$, $\mathbf{c}_j = 2$. Then by (INH1) we have

$$\begin{aligned}\Gamma(O^{k+1}(\mathbf{a}))\Gamma(O^{k+1}(\mathbf{c})) &\le (1+\eta)^2 \cdot \mathcal{P}(O^{k+1}(\mathbf{a}))\mathcal{P}(O^{k+1}(\mathbf{c}))\\ &= (1+\eta)^2 \cdot \mathcal{P}(O^{k+1}(\mathbf{b}))^2\\ &\le \frac{(1+\eta)^2}{(1-\eta)^2} \cdot \Gamma(O^{k+1}(\mathbf{b}))^2\,,\end{aligned}$$

which shows Γ is minimal with parameter $(1+\eta)^2(1-\eta)^{-2} - 1 \le 16\eta$.

The proof of (ii) is similar but we use the definition of U. Let $j \in [k]$, and $\mathbf{a},\mathbf{b},\mathbf{c} \in \{0,1,2\}^k$ be equal on $[k] \setminus \{j\}$ and satisfy $\mathbf{a}_j = 0$, $\mathbf{b}_j = 1$, $\mathbf{c}_j = 2$. If $\mathbf{a} = \mathbf{0}^k$ then the required bound is trivial, otherwise by the fact that $v \in U$ we have

$$\begin{aligned}\Gamma_v(O^k(\mathbf{a}))\Gamma_v(O^k(\mathbf{c})) &\le (1+\eta')^2 \cdot \frac{\gamma(v)^2}{p(0)^2}\mathcal{P}(O^{k+1}(1,\mathbf{a}))\mathcal{P}(O^{k+1}(1,\mathbf{c}))\\ &= (1+\eta')^2 \cdot \frac{\gamma(v)^2}{p(0)^2}\mathcal{P}(O^{k+1}(1,\mathbf{b}))^2\\ &\le \frac{(1+\eta')^2}{(1-\eta')^2} \cdot \Gamma_v(O^k(\mathbf{b}))^2\,,\end{aligned}$$

which shows that when $v \in U$, Γ_v is minimal with parameter $(1+\eta')^2(1-\eta')^{-2}-1 \leq 16\eta'$.

Part (iii) resembles a step in the proof of Lemma 5.11 involving $\mathcal{C}^{(0)}$. We first establish a lower bound on $\|U\|_\Gamma$. Fix $\mathbf{a} \in \{0,1,2\}^k$ and recall that $+2O^{k+1}(0,\mathbf{a})$ is the $(k+1)$-complex obtained by taking two vertex-disjoint copies of $O^{k+1}(1,\mathbf{a})$ and identifying their first vertices. Consider the experiment where $v \in V_0$ is chosen uniformly at random, and let

$$X := \gamma(v)\,, \qquad\qquad \text{and} \qquad\qquad Y := \frac{\Gamma_v(O^k(\mathbf{a}))}{\gamma(v)}\,.$$

By (5.17) and (INH1) we have

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[\Gamma_v(O^k(\mathbf{a}))] = \Gamma(O^{k+1}(1,\mathbf{a})) = (1\pm\eta)\mathcal{P}(O^{k+1}(1,\mathbf{a})) \\
&= (1\pm\eta)p(0)\cdot\frac{\mathcal{P}(O^{k+1}(1,\mathbf{a}))}{p(0)}\,, \\
\mathbb{E}[XY^2] &= \mathbb{E}\left[\frac{\Gamma_v(O^k(\mathbf{a}))^2}{\gamma(v)}\right] = \Gamma(+2O^{k+1}(0,\mathbf{a})) = (1\pm\eta)\mathcal{P}(+2O^{k+1}(0,\mathbf{a})) \\
&= (1\pm\eta)p(0)\cdot\left(\frac{\mathcal{P}(O^{k+1}(1,\mathbf{a}))}{p(0)}\right)^2\,.
\end{aligned}$$

Noting that $\mathbb{E}\,X = (1\pm\eta)p(0)$ by (INH1), we can apply Lemma 5.13 and Corollary 5.16 in the arguments below with an appropriate $\mathbf{a}$, $\varepsilon_{\mathrm{L5.13}} = \varepsilon_{\mathrm{C5.16}} := 4\eta$, and

$$d_{\mathrm{L5.13}} = d_{\mathrm{C5.16}} := \frac{\mathcal{P}(O^{k+1}(1,\mathbf{a}))}{p(0)}\,.$$

To bound $\|U\|_\Gamma$, for $\mathbf{a} \in \{0,1,2\}^k \setminus \{\mathbf{0}^k\}$, let $U_\mathbf{a} \subseteq V_0$ be those vertices v which satisfy

$$\Gamma_v(O^k(\mathbf{a})) = (1\pm\eta')\frac{\gamma(v)}{p(0)}\mathcal{P}(O^{k+1}(1,\mathbf{a}))\,,$$

so that U is the intersection of the 3^k-1 different $U_\mathbf{a}$. By Corollary 5.16 we have $\|U_\mathbf{a}\|_\Gamma \geq (1-2\eta')\|V_0\|_\Gamma$, and hence $\|U\|_\Gamma \geq (1-3^{k+1}\eta')\|V_0\|_\Gamma$, so that

$$\|V_0 \setminus U\|_\Gamma \leq 3^{k+1}\eta'\|V_0\|_\Gamma\,.$$

The contribution to $\Gamma\big(O^{k+1}(\mathbf{1}^{k+1})\big)$ from homomorphisms that use a vertex in $V_0 \setminus U$ can be written as

$$\mathbb{E}\big[\Gamma_v\big(O^k(\mathbf{1}^k)\big)\mathbb{1}_{v\in V_0\setminus U}\big]\,,$$

which is a weighting of $\mathbb{E}\big[\Gamma_v\big(O^k(\mathbf{1}^k)\big)\big]$ by the weight $W := \mathbb{1}_{v\in V_0\setminus U}$. We apply Lemma 5.13 with this weight W and X, Y as above with $\mathbf{a} = \mathbf{1}^k$ to obtain

$$\begin{aligned}\mathbb{E}\big[\Gamma_v\big(O^k(\mathbf{1}^k)\big)\mathbb{1}_{v\in V_0\setminus U}\big] &\le \left(1 - 4\eta + 4\sqrt{\tfrac{\eta\|V_0\|_\Gamma}{\|V_0\setminus U\|_\Gamma}}\right)\frac{\mathcal{P}\big(O^{k+1}(\mathbf{1}^{k+1})\big)}{p(0)}\|V_0 \setminus U\|_\Gamma \\ &\le (1+\eta)\big(3^{k+1}\eta' + 4\sqrt{3^{k+1}\eta\eta'}\big)\mathcal{P}\big(O^{k+1}(\mathbf{1}^{k+1})\big)\,,\end{aligned}$$

and note that the coefficient of $\mathcal{P}\big(O^{k+1}(\mathbf{1}^{k+1})\big)$ here is at most $3^{k+3}\eta'$. □

With the set U understood, we proceed by counting $O^{k+1}(1, \mathbf{2}^k)$ in $\mathcal{G}$ two different ways. Firstly, we estimate counts of $O^{k+1}(\mathbf{1}^{k+1})$ and $O^{k+1}(1, \mathbf{2}^k)$ in $\mathcal{G}$ with Corollary 5.21 and Lemma 5.22. We give crude values of the constants that work in the argument, but make no effort to optimise them.

Let $\mathcal{H}$ have layer $k+1$ given by $\mathcal{G}$, and lower layers given by Γ. Then by assumption (INH3) $\mathcal{H}$ is (ε, d')-regular with respect to Γ, and we obtain $\mathcal{G}$ from $\mathcal{H}$ by replacing weights on $V_{[k]}$ with those from $\mathcal{G}$. By Claim 5.24(i), and Corollary 5.21 for $(k+1)$-graphs, $\mathcal{H}$ is ε_m-minimal where

$$\begin{aligned}\varepsilon_m &:= 2^{2^{k+1}}\Big(\varepsilon(d')^{-2^{k+1}} + \eta\Big) \\ &> \max\Big\{1 - \tfrac{(1-\varepsilon/d')^{2^{k+1}}}{(1+\eta)^{2^{k+1}-1}},\ \big(1+\varepsilon(d')^{-2^{k+1}}\big)(1+\eta)^{2^{k+1}-1} - 1\Big\}\,.\end{aligned}$$

We can now apply Lemma 5.22 for $(k+1)$-graphs to $\mathcal{G}$ and $\mathcal{H}$ to obtain the required counts in $\mathcal{G}$. With $\varepsilon_{\mathrm{L5.22}} = \varepsilon$, $\eta_{\mathrm{L5.22}} = \varepsilon_m$ as above, and $d_{\mathrm{L5.22}} = d$, we obtain that for

$$\varepsilon'_m := 2^{2^k+22}k^3\big(\varepsilon^{1/2}(d')^{-2^k} + \eta^{1/2}\big)d^{2^{-k}}\,,$$

the $(k+1)$-graph $\mathcal{G}$ is ε'_m-minimal, and the remaining assertions of Corollary 5.21 and Lemma 5.22 give

$$\begin{aligned}\mathcal{G}(O^{k+1}(\mathbf{1}^{k+1})) &= (1\pm\varepsilon'_m)d\cdot\mathcal{H}(O^{k+1}(\mathbf{1}^{k+1}))\\ &= (1\pm\varepsilon_m)(1\pm\varepsilon'_m)dd'\Gamma(O^{k+1}(\mathbf{1}^{k+1}))\\ &= (1\pm\varepsilon''_m)dd'\cdot\mathcal{P}(O^{k+1}(\mathbf{1}^{k+1}))\,,\end{aligned}\tag{5.19}$$

$$\begin{aligned}\mathcal{G}(O^{k+1}(1,\mathbf{2}^k)) &\le (1+\varepsilon'_m)d^{2^k}\mathcal{H}(O^{k+1}(1,\mathbf{2}^k))\\ &\le (1+\varepsilon_m)(1+\varepsilon'_m)(dd')^{2^k}\Gamma(O^{k+1}(1,\mathbf{2}^k))\\ &\le (1+\varepsilon''_m)(dd')^{2^k}\cdot\mathcal{P}(O^{k+1}(1,\mathbf{2}^k))\,,\end{aligned}\tag{5.20}$$

where

$$\varepsilon''_m = 2^{2^k+25}k^3(\varepsilon^{1/2}(d')^{-2^{k+1}}+\eta^{1/2})d^{2^{-k}}\,.$$

The second method for counting $O^{k+1}(1,\mathbf{2}^k)$ involves counting $O^k(\mathbf{2}^k)$ in the links of vertices $v\in V_0$. We have $\mathcal{G}_v\le\Gamma_v$ and since we do not try to control $\mathcal{G}_v$ directly when $v\notin U$, we define

$$d_v = \begin{cases}\frac{\mathcal{G}_v(O^k(\mathbf{1}^k))}{\Gamma_v(O^k(\mathbf{1}^k))} & \text{if } v\in U\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Claim 5.25. *Writing*

$$\zeta := \max\left\{\varepsilon''_m+\eta'+\tfrac{3^{k+3}\eta'}{dd'},\ \varepsilon''_m+2\eta'+2\eta'\varepsilon''_m,\ \tfrac{(1+16\eta')^{2^k-1}}{1-16\eta'}(1+\varepsilon''_m)-1\right\},$$

we have

$$\begin{aligned}\mathbb{E}[\gamma(v)d_v] &= (1\pm\zeta)dd'\cdot p(0)\,, \qquad \textit{and}\\ \mathbb{E}\big[\gamma(v)d_v^{2^k-1}\big] &\le (1+\zeta)(dd')^{2^k-1}\cdot p(0)\,.\end{aligned}$$

Moreover, we note that a crude calculation gives

$$\zeta\le 2^{2^{k+1}+50}k^3\big(\varepsilon^{1/2}+\eta^{1/4}\big)(dd')^{-2^{k+1}}\,.$$

Proof. First we bound $\mathbb{E}[\gamma(v)d_v]$. By (5.16) we have

$$\begin{aligned}\mathcal{G}(O^{k+1}(\mathbf{1}^{k+1})) &= \mathbb{E}[\mathcal{G}_v(O^k(\mathbf{1}^k))]\\ &\le \mathbb{E}\big[d_v\Gamma_v(O^k(\mathbf{1}^k))\big]+\mathbb{E}\big[\Gamma_v(O^k(\mathbf{1}^k))\mathbb{1}_{v\in V_0\setminus U}\big]\,.\end{aligned}$$

By the definition (5.18) of U, for the first expectation we have an upper bound on $\Gamma_v(O^k(\mathbf{1}^k))$ which depends only on $\gamma(v)$, and by Claim 5.24(iii) we have a bound on the final expectation which represents copies of $O^{k+1}(\mathbf{1}^{k+1})$ using a vertex in $V_0 \setminus U$. We combine these facts with (5.19) to obtain a lower bound on $\mathbb{E}[\gamma_0(v)d_v]$. That is,

$$
\begin{aligned}
&(1-\varepsilon''_m)dd' \cdot \mathcal{P}\big(O^{k+1}(\mathbf{1}^{k+1})\big) \\
&\qquad \le \mathcal{G}\big(O^{k+1}(\mathbf{1}^{k+1})\big) \\
&\qquad \le (1+\eta')\frac{\mathcal{P}\big(O^{k+1}(\mathbf{1}^{k+1})\big)}{p(0)}\,\mathbb{E}[\gamma(v)d_v] + 3^{k+3}\eta'\mathcal{P}\big(O^{k+1}(\mathbf{1}^{k+1})\big)\,,
\end{aligned}
$$

which yields the lower bound

$$
\mathbb{E}[\gamma(v)d_v] \ge \left(1-\varepsilon''_m-\eta'-\frac{3^{k+3}\eta'}{dd'}\right)dd'\cdot p(0)\,.
$$

For a corresponding upper bound we have

$$
\begin{aligned}
(1+\varepsilon''_m)dd' \cdot \mathcal{P}\big(O^{k+1}(\mathbf{1}^{k+1})\big) \ge \mathcal{G}\big(O^{k+1}(\mathbf{1}^{k+1})\big) &\ge \mathbb{E}\big[d_v\Gamma_v\big(O^k(\mathbf{1}^k)\big)\big] \\
&\ge (1-\eta')\frac{\mathcal{P}\big(O^{k+1}(\mathbf{1}^{k+1})\big)}{p(0)}\,\mathbb{E}[\gamma(v)d_v]\,,
\end{aligned}
$$

by the definition of U and (5.19). We conclude

$$
\mathbb{E}[\gamma(v)d_v] \le \big(1+\varepsilon''_m+2\eta'+2\eta'\varepsilon''_m\big)\,dd'\cdot p(0)\,.
$$

For the second statement, Claim 5.24(ii) means that when $v \in U$ we can apply Corollary 5.20 to $\mathcal{G}_v \le \Gamma_v$ and obtain a lower bound on $\mathcal{G}_v(O^k(\mathbf{2}^k))$,

$$
\begin{aligned}
\mathcal{G}_v\big(O^k(\mathbf{2}^k)\big) &\ge \frac{d_v^{2^k}}{(1+16\eta')^{2^k-1}}\Gamma_v\big(O^k(\mathbf{2}^k)\big) \\
&\ge \frac{1-\eta'}{(1+16\eta')^{2^k-1}}\,\frac{\mathcal{P}\big(O^{k+1}(1,\mathbf{2}^k)\big)}{p(0)}\cdot\gamma(v)d_v^{2^k}\,.
\end{aligned}
\tag{5.21}
$$

Then by (5.16) again,

$$
p(0)\mathcal{G}\big(O^{k+1}(1,\mathbf{2}^k)\big) \ge \frac{1-\eta'}{(1+16\eta')^{2^k-1}}\mathcal{P}\big(O^{k+1}(1,\mathbf{2}^k)\big)\,\mathbb{E}\big[\gamma(v)d_v^{2^k}\big]\,,
$$

which together with (5.20) implies the required upper bound

$$\mathbb{E}\big[\gamma(v)d_v^{2^k}\big] \le \frac{(1+16\eta')^{2^k-1}}{1-16\eta'}(1+\varepsilon''_m)(dd')^{2^k}\cdot p(0)\,. \qquad \square$$

Claim 5.25 means that we have concentration of d_v by Corollary 5.17 with $X := \gamma(v)$ and $Y := d_v$. Writing $U_{\mathrm{conc}} \subseteq U$ for the vertices v with $d_v = (1\pm 2\zeta^{1/8})dd'$, we have

$$\|U_{\mathrm{conc}}\|_\Gamma \ge (1-4\zeta^{1/8})p(0)\,. \tag{5.22}$$

It remains to show that for almost all of the weight in U_{conc}, $\mathcal{G}_v$ is regular in the sense that the weight of $O^k(\mathbf{2}^k)$ is close to minimal. Let $U_{\mathrm{reg}} \subseteq U$ be the vertices $v \in U$ with

$$\mathcal{G}_v\big(O^k(\mathbf{2}^k)\big) \le (d_v^{2^k}+\varepsilon')\Gamma_v\big(O^k(\mathbf{2}^k)\big)\,.$$

For all vertices $v \in U$ we have the lower bound (5.21) on $\mathcal{G}_v(O^k(\mathbf{2}^k))$, hence we are supposing that for $v \in U\setminus U_{\mathrm{reg}}$ we have an additive improvement on (5.21) of at least $\varepsilon'\Gamma_v(O^k(\mathbf{2}^k))$. Then we have

$$\begin{aligned}
\mathcal{G}\big(O^{k+1}(1,\mathbf{2}^k)\big) &= \mathbb{E}\big[\mathcal{G}_v\big(O^k(\mathbf{2}^k)\big)\big] \\
&\ge \frac{1}{(1+16\eta')^{2^k-1}}\,\mathbb{E}\Big[\big(d_v^{2^k}+\varepsilon'\mathbb{1}_{v\in U\setminus U_{\mathrm{reg}}}\big)\Gamma_v\big(O^k(\mathbf{2}^k)\big)\Big] \\
&\ge \frac{1-\eta'}{(1+16\eta')^{2^k-1}}\Big(\mathbb{E}\big[\gamma(v)d_v^{2^k}\big]+\varepsilon'\|U\setminus U_{\mathrm{reg}}\|_\Gamma\Big)\frac{\mathcal{P}\big(O^{k+1}(1,\mathbf{2}^k)\big)}{p(0)} \\
&\ge \frac{1-\eta'}{(1+16\eta')^{2^k-1}}\left(\frac{\mathbb{E}[\gamma_0(v)d_v]^{2^k}}{\|U\|_\Gamma^{2^k-1}}+\varepsilon'\|U\setminus U_{\mathrm{reg}}\|_\Gamma\right)\frac{\mathcal{P}\big(O^{k+1}(1,\mathbf{2}^k)\big)}{p(0)} \\
&\ge \frac{(1-\eta')(1-\zeta)^{2^k}}{\big((1+16\eta')(1+\eta)\big)^{2^k-1}}\left((dd')^{2^k}+\frac{\varepsilon'\|U\setminus U_{\mathrm{reg}}\|_\Gamma}{p(0)}\right)\mathcal{P}\big(O^{k+1}(1,\mathbf{2}^k)\big)\,,
\end{aligned}$$

where the fourth line is by the Cauchy–Schwarz inequality, and the fifth is by Claim 5.25, and the fact that $\|U\|_\Gamma \le \|V_0\|_\Gamma \le (1+\eta)p(0)$. With (5.20) we have

$$\begin{aligned}\|U\setminus U_{\mathrm{reg}}\|_\Gamma &\le \frac{1}{\varepsilon'}\bigg(\frac{\big((1+16\eta')(1+\eta)\big)^{2^k-1}(1+\varepsilon''_m)}{(1-\eta')(1-\zeta)^{2^k}}-1\bigg)(dd')^{2^k}p(0) \qquad (5.23)\\ &\le \frac{1}{\varepsilon'}\cdot 2^{2^{k+2}+54}k^3\big(\varepsilon^{1/2}+\eta^{1/4}\big)(dd')^{-2^k}p(0)\\ &\le 2^{2^{k+2}+54}k^3\big(\varepsilon^{1/4}+\eta^{1/8}\big)(dd')^{-2^k}p(0)\,,\end{aligned}$$

where for the last line we use that $\varepsilon' \ge \max\{\varepsilon^{1/4},\, \eta^{1/8}\}$.

Now, for Lemma 5.3 we may take $V_0' = U_{\mathrm{conc}} \cap U_{\mathrm{reg}}$, since then for $v \in V_0'$ the link $\mathcal{G}_v$ inherits both the desired relative density and regularity from $\mathcal{G}$. Moreover, by (5.22) and (5.23) we have

$$\begin{aligned}\|V_0\|_\Gamma &\ge \|U_{\mathrm{conc}}\|_\Gamma - \|U\setminus U_{\mathrm{reg}}\|_\Gamma\\ &\ge \Big(1-4\zeta^{1/8}-2^{2^{k+2}+54}k^3\big(\varepsilon^{1/4}+\eta^{1/8}\big)(dd')^{-2^k}\Big)\,p(0)\\ &\ge \Big(1-2^{2^{k+6}}k^3\big(\varepsilon^{1/16}+\eta^{1/32}\big)(dd')^{-2^k}\Big)\,\|V_0\|_\Gamma\,,\end{aligned}$$

where we again use (INH1) for the last line. To complete the proof, observe that we choose ε, η in terms of ε', d, d', and k to satisfy

$$\min\{\varepsilon', 2^{-k}\} \ge 2^{2^{k+6}}k^3\big(\varepsilon^{1/16}+\eta^{1/32}\big)(dd')^{-2^k}\,. \qquad \square$$

5.6 THC in random hypergraphs

In this section we consider a random k-uniform hypergraph on n vertices, which we view as a k-graph that is complete (i.e. weight 1) on edges of size at most $k-1$, and for which weights of k-edges are independent Bernoulli random variables (taking values in $\{0,1\}$) with probability p. Let $\Gamma = G^{(k)}(n,p)$ be this k-graph.

For a finite set J, let F be a J-partite k-complex on a vertex set X, and let $V(\Gamma)$ be partitioned into $\{V_j\}_{j\in J}$. Suppose that X comes with a linear order. By the standard construction we obtain vertex sets $\{V_x'\}_{x\in X}$, and an X-partite k-graph Γ'. Note that, as observed after Definition 5.1, partite homomorphism counts in Γ and in Γ' are in correspondence. In particular, the counting property (THC1) is equivalent to asking for the same bounds on homomorphism counts in Γ. Furthermore, if we embed an initial segment of

X and update Γ' by taking links of all the embedded vertices, then (THC1) in the link graph is the same as asking for a count of rooted homomorphisms in Γ. There is a slight subtlety here, namely that if we embed two vertices of X to (automatically) different vertices of Γ' which correspond to one vertex of Γ, then the complex we count in a link of Γ' and that which we count rooted in Γ are not quite the same.

We would like to know that in this setup Γ' is well-behaved enough to apply the main results of this chapter, which we might expect to be true provided p is not too small. We prove that for $c^* \in \mathbb{N}$ and $\eta > 0$, provided p and the parts V_j are large enough, with high probability Γ' is an (η, c^*)-THC graph.

To state the requirements on p formally we use another definition of degeneracy more suited to the random k-graph. Suppose that X is equipped with a fixed ordering, and let

$$\deg_k(F) := \max_{e \in F} \left|\{f \in F^{(k)} : e \subseteq f,\ f \setminus e \text{ precedes } e\}\right|,$$

where $f \setminus e$ precedes e if and only if each vertex of $f \setminus e$ comes before every vertex of e in the order on X. Then, when embedding vertices in order, part-way through the process an edge e can be the set of unembedded vertices for at most $\deg_k(F)$ edges of size k in F. We make no attempt to optimise the dependence of p on the relevant parameters.

Lemma 5.26. *Let $\eta > 0$ be a real number, c^*, Δ, $d \in \mathbb{N}$, and J be a finite set. Suppose that F is a J-partite k-complex of maximum degree Δ and degeneracy $\deg_k(F) \le d$ with vertex set X equipped with some fixed ordering. For some fixed $0 < \varepsilon < 1$, let $\Gamma = G^{(k)}(n,p)$ be a random k-graph where $\min\{p^{4^k c^* d}, p^{4^k \Delta + d}\} \ge (2\log n) n^{\varepsilon - 1}$. Suppose also that $(1-\eta)^\Delta \ge 1/2$ and $|X| \le n$. Then with probability at least $1 - o(1)$ the following holds.*

For any partition $\{V_j\}_{j \in J}$ of $V(\Gamma)$ into parts of size at least $n_0 = n/\log n$, writing Γ' for the X-partite graph obtained by the standard construction to F, Γ, and $\{V_j\}_{j\in J}$, we have that Γ' in an (η, c^*)-THC graph with density graph $\mathcal{Q}$ that gives weight p to edges of $F^{(k)}$ and weight 1 elsewhere.

The proof will involve showing that counts of complexes R in Γ' and in related k-graphs obtained by taking links are close to their expectation, and such counts will correspond to counts of weighted homomorphism-like objects in Γ. In order to avoid trying to deal with Γ and Γ' simultaneously, we first state and prove the required property of Γ, which is rather technical.

Let Y be an initial segment of X and $\phi : Y \to V(\Gamma)$ be a partite map. Let Z be a vertex set disjoint from Y, equipped with a map ρ that associates each $z \in Z$ to some $\rho(z) \in X \setminus Y$. For convenience we extend ρ to be the identity map on Y. Let R be a J-partite k-complex on Z, and let R_ϕ be the hypergraph with vertex set $\operatorname{im}\phi \cup Z$, and edge set

$$E(R_\phi) := E(R) \cup \{f \subseteq \operatorname{dom}\phi \cup Z : f \cap Z \neq \emptyset,\, \rho(f) \in F\}\,.$$

We view R_ϕ as J-partite in the following way. Each vertex in $\operatorname{im}\phi$ is in V_j for some $j \in J$, which naturally gives an association to the index j, and vertices in Z are related to indices j through the map $\rho : Z \to X$ and the partition of X into parts indexed by J. We write V_z for the V_j to which $z \in Z$ is associated in this way.

Then the homomorphism-like objects we consider in Γ are partite maps ψ from R_ϕ to Γ, where we insist that ψ extends the identity map on $\operatorname{im}\phi$. This definition is rather difficult to parse, but a certain amount of complexity is necessary to deal with the case that ϕ is not injective. In any case, the idea is that ψ signifies a copy of R_ϕ in Γ 'rooted' at some fixed vertices specified by $\operatorname{im}\phi$. We are interested in weighting such ψ according to the subset $R_\phi^{(\geq 2)} \subseteq R_\phi$ of edges of size at least 2, preferring to deal separately with the empty set (which has weight 1 in this setup) and vertex weights. For $z \in Z$, let $U_z \subseteq V_z$ be a set of exactly $n_1 := np^d/(2\log n)$ vertices. We define

$$N(\phi, R, U_Z) := \sum_{\psi} \prod_{e \in R_\phi^{(\geq 2)}} \gamma(\psi(e))\,, \tag{5.24}$$

where the sum is over all maps $\psi : \operatorname{im}\phi \cup Z \to V(\Gamma)$ such that $\psi(w) = w$ for any $w \in \operatorname{im}\phi$ and $\psi(z) \in U_z$ for all $z \in Z$. Note that with $Y = \phi = \emptyset$ we have $R_\phi = R$, and since Γ is complete on edges of size at most 1, $n_1^{-|Z|} N(\phi, R)$ is then the partite count of copies of R in Γ that lie on U_Z.

The main probabilistic tool we require for counting in Γ' is a statement that for any suitable R, the count $N(\phi, R, U_Z)$ is close to its expectation with very high probability. It turns out that we are interested in R of the following form. Given Y, let F^4 be the complex F with each vertex blown up into 4 copies. A *suitable* R is any subcomplex of F^4 on at most c^* vertices which uses no copies of vertices in Y. Considering suitable R is what requires us to work with $4^k\Delta$ and $4^k d$ in what follows.

Claim 5.27. *Consider the setup of Lemma 5.26, the above definitions, and a suitable R. Then with probability at least $1 - \exp\big(-O(n^{1+\varepsilon})\big)$ we have*

$$N(\phi, R, U_Z) = (1 \pm \tfrac{1}{\log n}) n_1^{|Z|} \prod_{e \in R_\phi^{(\geq 2)}} q(e)\,. \tag{5.25}$$

Proof. Formally, we proceed by induction on $|Z|$. The claim is trivial if $|Z| \leq 1$, as the product over $E_2(R_\phi)$ is empty.

If $|Z| \geq 2$, note that it suffices to consider injective maps ψ in (5.24). Any non-injective partite map $\psi' : \operatorname{im}\phi \cup Z \to V(\Gamma)$ of the form considered in (5.24) is an injective partite map into $V(\Gamma)$ from the complex R' on a vertex set Z' formed from R by identifying any vertices of z with the same images under ψ'. Applying the claim to R' (which is on fewer vertices), we see that with probability at least $1 - \exp\big(-O(n^{1+\varepsilon})\big)$, these non-injective maps contribute an amount at most twice expectation of $N(\psi, R', U_{Z'})$. Comparing the expectations of $N(\psi, R, U_Z)$ and $N(\psi, R', U_{Z'})$, identifying a pair z, z' of vertices in Z 'costs' a factor n_1 but can gain a factor up to $p^{-4^k\Delta}$ since the edges involving z, of which there are at most $4^k\Delta$, are now coupled in Γ with those containing z'. Then the assumptions on n and p imply that the contribution to $N(\phi, R, U_Z)$ from non-injective homomorphisms is at most a factor $O(n^{-\varepsilon})$ times the expected contribution from injective homomorphisms. Write N^* for the contribution to $N(\phi, R, U_Z)$ from injective ψ, noting that the above argument shows that $N(\phi, R, U_Z) = \big(1 \pm O(n^{-\varepsilon})\big)N^*$.

For each injective ψ, the term $\mathbf{X}^*_\psi := \prod_{e \in R_\phi^{(\geq 2)}} \gamma(\psi(e))$ appearing in N^* is a product of independent Bernoulli random variables with probabilities given by $q(e)$. The $\mathbf{X}^*_\psi$ themselves are therefore 'partly dependent' Bernoulli random variables, each with the same probability

$$p^* := \prod_{e \in R_\phi^{(\geq 2)}} q(e)\,.$$

Since we consider only the edges $R_\phi^{(\geq 2)}$, if $\mathbf{X}^*_\psi$ and $\mathbf{X}^*_{\psi'}$ are dependent it must be because they agree on at least two vertices of Z. Then each $\mathbf{X}^*_\psi$ can be dependent on at most $\binom{|Z|}{2} n_1^{|Z|-2}$ other variables $\mathbf{X}^*_{\psi'}$. We apply a theorem of Janson [54, Corollary 2.6] which bounds the probability of large deviations in sums of partly dependent random variables.

Theorem 5.28 (Janson [54])**.** *Let Ψ be an index set, and $N^* = \sum_{\psi\in\Psi}\mathbf{X}^*_\psi$, such that each $\mathbf{X}^*_\psi$ is a Bernoulli random variable with probability $p^*\in(0,1)$. Let Δ^*_1 be one more than the maximum degree of the graph on vertex set Ψ such that ψ and ψ' are adjacent if and only if $\mathbf{X}^*_\psi$ and $\mathbf{X}^*_{\psi'}$ are dependent. Then for any $\delta>0$,*

$$\mathbb{P}\big[N^* = (1\pm\delta)\,\mathbb{E}\,N^*\big] \geq 1 - 2\exp\left(-\frac{3\delta^2|\Psi|p^*\big(1-\Delta^*_1/|\Psi|\big)}{8\Delta^*_1}\right). \qquad (5.26)$$

In the setup above, we have $n_1^{|Z|}(1-|Z|/n_1)^{|Z|} \leq |\Psi| \leq n_1^{|Z|}$ and $\Delta^*_1 \leq |Z|^2 n_1^{|Z|-2}$. Since $|Z|\leq c^*$ is bounded by a constant, this means $|\Psi| = (1\pm O(n_1^{-1}))n_1^{|Z|}$,

$$\frac{|\Psi|}{\Delta^*_1} = \Omega(n_1^2)\,, \qquad\qquad \text{and} \qquad\qquad \frac{\Delta^*_1}{|\Psi|} = O(n_1^{-2})\,.$$

Moreover, we know that $p^* \leq p^{(|Z|-1)4^k d}$ since embedding the first vertex of Z is 'free', and each remaining vertex can be the last vertex of at most $4^k d$ edges which occur with probability p each. Then for $\delta := 1/(2\log n)$, the exponent on the right-hand side of (5.26) is

$$-\Omega\big(p^{4^k d|Z|}n^2\big) = -\Omega\big(n^{1+\varepsilon}\big)\,,$$

by the assumptions on n and p. The claim follows since the event that (5.25) that we wish to control occurs with probability 1 provided $N^* = (1\pm\delta)\,\mathbb{E}\,N^*$ and n is large enough. We have

$$\mathbb{E}\,N^* = |\Psi|\prod_{e\in R_\phi^{(\geq 2)}} q(e) = \big(1\pm O(n_1^{-1})\big)n_1^{|Z|}\prod_{e\in R_\phi^{(\geq 2)}} q(e)\,,$$

and hence for large enough n, with probability at least $1 - \exp\big(-O(n^{1+\varepsilon})\big)$,

$$\begin{aligned} N(\phi, R) &= \big(1 \pm O(n^{-\varepsilon})\big)N^* \\ &= \big(1 \pm O(n^{-\varepsilon})\big)\Big(1 \pm \tfrac{2}{\log n}\Big)\,\mathbb{E}\,N^* \\ &= (1 \pm \tfrac{1}{\log n})n_1^{|Z|} \prod_{e \in R_\phi^{(\geq 2)}} q(e)\,. \end{aligned} \qquad \square$$

With the main probabilistic argument complete, we can now apply Claim 5.27 to the problem of showing Γ' is an (η, c^*)-THC graph.

Proof of Lemma 5.26. We start with a sketch of the proof. Given a fixed partition $\{V_j\}_{j \in J}$, to verify Γ' is an (η, c^*)-THC graph we must count suitable X-partite complexes R in graphs obtained from Γ' by embedding vertices of F. We are not required to consider arbitrary embeddings, at each step we are permitted by (THC2) to avoid a 'bad set' of potential images, which we will exploit in due course.

At first, no vertex of F has been embedded and we count R in Γ', which by the standard construction is the same as counting R in Γ. By Claim 5.27 with $Y = \phi = \emptyset$ and a union bound over suitable R, with high probability we have the required accurate counts of R in Γ. In fact, these counts are accurate enough to imply deterministically that there is a small 'bad set' which, if avoided, allows us to embed the next vertex x and continue the argument with 'well-behaved' vertex weights in Γ'_x.

When some initial segment Y of X has been embedded, say by a map $\phi' : Y \to V(\Gamma')$, we always have an associated map $\phi : Y \to V(\Gamma)$ obtained by identifying the copies of parts V_j made in the standard construction. Write $\Gamma'_{\phi'}$ for the k-graph obtained from Γ' by taking the link of vertices in $\operatorname{im}\phi'$ By construction, the required counts of complexes R in $\Gamma'_{\phi'}$ correspond to counts of R_ϕ in Γ, which we can control with Claim 5.27. We handle vertex weights separately, and apply Claim 5.27 with subsets $U_z \subseteq V_z$ of vertices that receive weight 1 in $\Gamma'_{\phi'}$. We then take a union bound over choices of partition to complete the lemma.

The notion of 'well behaved' for vertex weights in $\Gamma'_{\phi'}$ that we maintain is as follows. Recall that given a partition $\{V_j\}_{j \in J}$, we have $V(\Gamma')$ partitioned into $\{V'_x\}_{x \in X}$ where V'_x is a copy of the V_j into which x will be embedded. Since we view vertices in Z as copies of vertices in X, we also write V'_z for the part of Γ' into which z should be embedded. Given $Y \subseteq X$ and ϕ, ϕ' as above, let $\mathcal{Q}_\phi$ be the density k-graph obtained from $\mathcal{Q}$ by taking links of vertices in $\operatorname{im}\phi$. For a fixed suitable R on vertex set Z, and $z \in Z$, let $\mathcal{A}_{Y,z}$ be the event that $\|V'_z\|_{\Gamma_W} = (1 \pm \eta)^{\pi(z)} q_Y(z)$, where $\pi(z) := \big|\{y \in Y : \{y,z\} \in F\}\big| \le \Delta$, and let $\mathcal{A}_Y$ be the intersection of $\mathcal{A}_{Y,z}$ for all $z \in Z$. The event $\mathcal{A}_\emptyset$ holds with probability 1 because Γ gives weight 1 to all vertices, and by avoiding bad vertices we will maintain $\mathcal{A}_Y$ as we embed.

We are now ready to give the main proof. Supposing that the initial segment $Y \subseteq X$ has been embedded, we have the associated partite maps ψ and ψ' from Y to $V(\Gamma)$ and $V(\Gamma')$ respectively, we count copies of suitable R in Γ'.

Given $\mathcal{A}_Y$, since we have by assumption $(1-\eta)^\Delta \ge 1/2$, $|V_z| \ge n/\log n$, and $q_Y(z) \ge p^d$, we can apply Claim 5.27 for every collection of U_z such that $U_z \subseteq V_z$ is of size exactly n_1. There are $\prod_{z\in Z}\binom{|V_z|}{n_1} = e^{O(n)}$ choices of collection, hence by Claim 5.27 and a union bound over collections, conditioned on $\mathcal{A}_Y$ for all $z \in Z$, with probability at least $1-\exp\big(-O(n^{1+\varepsilon})\big)$, the $N(\phi, R, U_Z)$ counts are close to their expectation for all such U_Z. In particular, the $N(\phi, R, U_Z)$ are 'correct' for the collections where every $u \in U_z$ receives weight 1 as a vertex in $\Gamma'_{\phi'}$. The count $N(\phi, R, U_Z)$ deals with edges of R_ϕ of size at least 2, hence by the above argument and averaging over sets U_z of vertices that receive weight 1 in $\Gamma'_{\phi'}$, we obtain that with high probability the count $\Gamma'_{\phi'}(R)$ is close to its expectation.

More precisely, by a union bound over the constant number of complexes R to consider, and by averaging over the choice of collection $\{U_z\}_{z\in Z}$, we have, conditioned on $\mathcal{A}_Y$, with probability at least $1-\exp\big(-O(n^{1+\varepsilon})\big)$,

$$\Gamma'_{\phi'}(R) = \big(1 \pm \tfrac{1}{\log n}\big)\Big(\prod_{e \in R_\phi^{(\ge 2)}} q(e)\Big)\prod_{z\in Z}\|V'_z\|_{\Gamma'_{\phi'}}\,, \tag{5.27}$$

for all suitable k-complexes R.

Let x be the next vertex to embed. To prove the lemma it now suffices to show that there is a subset $\tilde{V}'_x \subseteq V'_x$ with $\|\tilde{V}'_x\|_{\Gamma'_{\phi'}} \ge (1-\eta)\|V'_x\|_{\Gamma'_{\phi'}}$ such that $\mathcal{A}_{Y\cup\{x\}}$ holds. Then the above argument after x has been embedded, and a union bound over the total number of vertices to embed (at most n) gives the result.

Suppose that we embed x to $w \in V'_x$. Since F has maximum degree Δ, there are at most Δ vertices $z \in \rho(Z)$ with $\Gamma'_{\phi' \cup \{x \mapsto w\}}[V'_z] \neq \Gamma'_{\phi'}[V'_z]$. Let Z' be the set of these vertices. The counts (5.27) imply that for $z \in Z'$,

$$|V'_x|^{-1} \sum_{u \in V_x} \gamma'_{\phi'}(u) = (1 \pm \tfrac{1}{\log n}) \|V'_x\|_{\Gamma'_{\phi'}} ,$$

$$|V'_x|^{-1}|V'_z|^{-1} \sum_{uv \in V'_{xz}} \gamma'_{\phi'}(u)\gamma'_{\phi'}(v)\gamma'_{\phi'}(u,v)$$
$$= (1 \pm \tfrac{1}{\log n}) q_Y(x,z) \|V_z\|_{\Gamma'_{\phi'}} \cdot \|V'_x\|_{\Gamma'_{\phi'}} ,$$

$$|V'_x|^{-1} \sum_{u \in V'_x} \gamma'_{\phi'}(u) \Big(|V'_z|^{-1} \sum_{v \in V'_z} \gamma'_{\phi'}(v)\gamma'_{\phi'}(u,v) \Big)^2$$
$$= (1 \pm \tfrac{1}{\log n}) q_Y(x,z)^2 \|V'_z\|^2_{\Gamma'_{\phi'}} \cdot \|V'_x\|_{\Gamma'_{\phi'}}$$

hence we may apply Corollary 5.16 with $\varepsilon_{\mathrm{C5.16}} := 4/\log n$ and $d_{\mathrm{C5.16}} := q_Y(x,z)\|V_z\|_{\Gamma'_{\phi'}}$ to obtain the following.

For each $z \in Z'$ there is a set $B_z \subseteq V'_x$ with $\|B_z\|_{\Gamma'_{\phi'}} \leq 8(\log n)^{-1/4}\|V'_z\|_{\Gamma'_{\phi'}}$ such that for all $w \in V'_x \setminus B_z$, if x is embedded to w we have

$$\|V'_z\|_{\Gamma'_{\phi' \cup \{x \mapsto w\}}} = \Big(1 \pm \frac{4}{(\log n)^{1/4}}\Big) q_Y(x,z) \|V'_z\|_{\Gamma'_{\phi'}} .$$

Set $\tilde{V}'_x = V'_x \setminus \bigcup_{z \in Z'} B_z$, so that

$$\|\tilde{V}'_x\|_{\Gamma'_{\phi'}} \geq \Big(1 - \frac{8\Delta}{(\log n)^{1/4}}\Big) \|V'_x\|_{\Gamma'_W} ,$$

which is at least $(1-\eta)\|V'_x\|_{\Gamma'_{\phi'}}$ for large enough n. Then given $\mathcal{A}_Y$ and the counts (5.27), we have a small 'bad set' which, if avoided when embedding x, implies $\mathcal{A}_{Y \cup \{x\}}$ holds deterministically. So we can maintain well-behaved vertex weights throughout the embedding, and we may repeat the probabilistic argument above to control the counting properties (5.27) after each vertex is embedded. There are at most n embeddings, and hence with probability at least $1 - \exp\big(-O(n^{1+\varepsilon})\big)$ the partition $\{V_j\}_{j \in J}$ yields a Γ' with the required properties. To complete the proof we take a union bound over the $e^{O(n)}$ possible partitions. $\square$

6

Multicolour Ramsey numbers of paths and even cycles

In this chapter we prove upper bounds on the multicolour Ramsey numbers of the n-vertex path P_n, and for even n, the n-vertex cycle C_n.

The 2-colour Ramsey number of a path was completely determined by Gerencsér and Gyárfás [45] who showed that for $n \geq 2$,

$$R_2(P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor .$$

For 3 colours, Faudree and Schelp [33] conjectured that

$$R_3(P_n) = \begin{cases} 2n-2 & \text{for } n \text{ even}, \\ 2n-1 & \text{for } n \text{ odd}. \end{cases}$$

This conjecture was resolved for large n by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [50], but for $k \geq 4$ much less is known. A well-known upper bound $R_k(P_n) \leq kn$ follows easily by observing that any k-edge-colouring of the edges of the complete graph on kn vertices contains a colour class with at least $(kn-1)\frac{n}{2}$ edges by the pigeonhole principle. A result of Erdős and Gallai [29] (see Lemma 6.5) then implies that any graph on kn vertices with this many edges contains a copy of P_n. Despite the simplicity of this observation, the bound was only recently improved upon by Sárközy [90] who proved a stability version of Lemma 6.5 and showed that for $k \geq 4$ and n sufficiently large,

$$R_k(P_n) \leq \left(k - \frac{k}{16k^3 + 1}\right) n \, .$$

We improve on the above result for all $k \geq 4$, reducing the upper bound on $R_k(P_n)$ by an amount that does not deteriorate as k grows. Our method is similar to that of [90] in that we also use results of Erdős and Gallai [29], and Kopylov [65] to bound the number of edges in the densest two colours. Our improvement comes from using more information about the densest colour in order to obtain stronger bounds on the number of edges in the second densest. For paths we prove the following.

Theorem 6.1. *For $k \geq 4$ and all $n \geq 64k$,*

$$R_k(P_n) \leq \left(k - \frac{1}{4} + \frac{1}{2k}\right) n \, .$$

If n is much larger we can in fact slightly improve on this bound and extend it to even cycles, see Theorem 6.2 below.

The fact that P_n is a subgraph of C_n and the regularity method of Łuczak in [73] mean that we can apply similar methods to the problem of determining $R_k(C_n)$ for even n. In the case of two colours Faudree and Schelp [32], and independently Rosta [88], showed that $R_2(C_n) = \frac{3n}{2} + 1$ for even $n \geq 6$. For three colours, Benevides and Skokan [7] proved that $R_3(C_n) = 2n$ for sufficiently large even n. For $k \geq 4$ colours, again very little is known. Łuczak, Simonovits, and Skokan [74] showed that for n even, $R_k(C_n) \leq kn + o(n)$; and recently Sárközy [90] improved this upper bound to $\left(k - \frac{k}{16k^3+1}\right) n + o(n)$. Here we obtain a slight strengthening of Theorem 6.1 for large n.

Theorem 6.2. *For $k \geq 4$ and n even*

$$R_k(C_n) \leq \left(k - \frac{1}{4}\right) n + o(n) \, .$$

It is interesting to note that odd cycles behave very differently in this context. Recently Jenssen and Skokan [55] showed, via analytic methods, that for $k \geq 4$ and n odd and sufficiently large, $R_k(C_n) = 2^{k-1}(n-1) + 1$. This resolved a conjecture of Bondy and Erdős [12] for large n.

6.1 Lower bounds

Before proving Theorems 6.1 and 6.2 we briefly discuss lower bounds. Constructions based on finite affine planes (see [10]) show that $R_k(P_n) \geq (k-1)(n-1)$, when $k-1$ is a prime power and this lower bound is thought to be closer to the truth than our upper bound. Yongqi, Yuansheng, Feng, and Bingxi [101] provide a construction which shows that $R_k(C_n) \geq (k-1)(n-2)+2$ for any k and for even n. This construction can easily be modified to give a lower bound on $R_k(P_n)$ for any k and any n. We sketch this construction below.

To see that $R_k(P_n) \geq 2(k-1)\left(\lfloor \frac{n}{2} \rfloor - 1\right)+1$, consider a complete graph G on vertices $\{0,1,\ldots,2k-3\}$ and for $1 \leq i \leq k-1$ colour the edges from vertex i to vertices $i+1,\ldots,i+k-2$ and the edges from vertex $i+k-1$ to vertices $i+k,\ldots,i+2k-3$ (taken modulo $2k-2$) with colour c_i. Then each colour $c_1,\ldots,c_{k-1}$ consists of two vertex-disjoint stars, each on $k-1$ vertices. The remaining edges are those of the form $\{j, j+k-1\}$ for $j=0,\ldots,k-2$ which are coloured with the final colour c_k. The final colour forms a matching on $k-1$ edges. Construct G' by 'blowing up' each vertex i of G into a set V_i of $\lfloor \frac{n}{2} \rfloor - 1$ vertices and colour the edges within V_i with colour c_k. Edges between sets V_i and V_j in G' are coloured with the same colour as the edge $\{i,j\}$ in G.

There is no monochromatic P_n in G' because in colours $c_1,\ldots,c_{k-1}$, components are bipartite with smallest part size $\lfloor \frac{n}{2} \rfloor - 1$, hence cannot contain a P_n. The components in colour c_k have less than n vertices and so cannot contain a P_n. Again, this lower bound is generally considered to be closer to the truth than our upper bound.

6.2 Methods

We omit floor and ceiling signs whenever they are not crucial. To prove Theorem 6.1 we will proceed by contradiction. We take a complete graph on $N=(k-\frac{1}{4}+\frac{1}{2k})n$ vertices whose edges have been coloured with k colours and suppose it contains no monochromatic P_n. First we show that the densest colour has only a few components and these are not too large. For the other colours we consider the edges between these components and use the multipartite structure to bound the number of such edges. This gives a bound on the total number of edges which is less than $\binom{N}{2}$; yielding the desired contradiction. The proof is given in Section 6.3.

Recall that we use the regularity method of Łuczak [34, 35, 73] which reduces the problem to that of finding a monochromatic connected matching in the reduced graph. We use the term *connected matching of* t *edges* to mean a connected graph which contains a matching of t edges. We deduce Theorem 6.2 from the following result, which we prove in Section 6.4.

Theorem 6.3. *Let $k \geq 4$ be a positive integer, and let $0 \leq \delta < \frac{1}{64k^2}$. Then for even $n \geq 32k$ and $N = (k - \frac{1}{4})n$ the following holds. Suppose that G is a k-edge-coloured, N-vertex graph with at least $(1-\delta)\binom{N}{2}$ edges, then we may find a monochromatic connected matching of $\frac{n}{2}$ edges in G.*

The statement we use to deduce Theorem 6.2 from Theorem 6.3 is from a paper of Figaj and Łuczak [34, Lemma 3].

Lemma 6.4 (Figaj and Łuczak [34])**.** *Let $t > 0$ be a real number. If for every $\varepsilon > 0$ there exists $\delta > 0$ and an n_1 such that for every even $n > n_1$ and any k-edge-coloured graph G with $v(G) > (1+\varepsilon)tn$ and $e(G) \geq (1-\delta)\binom{v(G)}{2}$ has a monochromatic connected matching of $\frac{n}{2}$ edges, then asymptotically as $n \to \infty$ we have for even n that $R_k(C_n) \leq (t + o(1))n$.*

Theorem 6.2 follows from Theorem 6.3 by applying Lemma 6.4 with $t = k - \frac{1}{4}$ and for any positive ε choosing $\delta < \frac{1}{64k^2}$ and $n_1 \geq 32k$. Note that Lemma 6.4 hides the details of the regularity method, which is convenient for our purposes, but it obscures one detail the knowledgeable reader may wish to note. Let m be an even natural number. In using the regularity method to prove an upper bound on $R_k(C_m)$, one is interested in finding a connected matching in the reduced graph such that in the original graph we find a C_m. A single edge in the reduced graph yields a path spanning the corresponding parts in the original graph, which means that to find C_m we will apply Theorem 6.3 with a value of n that is *smaller* than m. Though this may suggest it is preferable to state Theorem 6.3 with the parameters renamed, keeping n allows us to use the same notation in the technical proofs of this chapter, hiding the necessary change in n inside the 'black box' of Lemma 6.4.

The remainder of this chapter is devoted to proving Theorems 6.1 and 6.3. We will need the following extremal results for graphs not containing an n-vertex path.

Lemma 6.5 (Erdős and Gallai [29])**.** *Let H be a graph which does not contain an n-vertex path. Then*

$$e(H) \leq \frac{n-2}{2} v(H) .$$

The following simplified version of a result due to Kopylov [65] improves on the above result for connected graphs.

Lemma 6.6. *Let H be a connected graph which does not contain an n-vertex path. Then*

$$e(H) \leq \frac{n}{2} \max \left\{ n, v(H) - \frac{n}{4} \right\} .$$

Our next result gives a slight improvement of Lemma 6.6 under the additional assumption that H is c-partite. In this case the bound on $e(H)$ can be improved when $v(H)$ is small.

Lemma 6.7. *Let H be a c-partite connected graph which does not contain an n-vertex path. Then*

$$e(H) \leq \begin{cases} \left(1 - \frac{1}{c}\right) \frac{v(H)^2}{2} & \text{for } v(H) \leq n\sqrt{\frac{c}{c-1}} \,, \\ \frac{n^2}{2} & \text{for } n\sqrt{\frac{c}{c-1}} < v(H) \leq \frac{5n}{4} \,, \\ \frac{n}{2}\left(v(H) - \frac{n}{4}\right) & \text{for } \frac{5n}{4} < v(H) \,. \end{cases}$$

Proof. In the 'small' case $v(H) \leq n\sqrt{\frac{c}{c-1}}$ we simply use that a c-partite graph has at most as many edges as the complete balanced c-partite graph on the same number of vertices. Therefore we conclude that $e(H) \leq \left(1 - \frac{1}{c}\right) \frac{v(H)^2}{2}$ without the assumption that H contains no copy of P_n.

The 'medium' case where $n\sqrt{\frac{c}{c-1}} < v(H) \leq \frac{5n}{4}$ and the remaining 'large' case follow directly from Lemma 6.6 and make no use of the c-partite assumption on H. $\square$

Lemma 6.7 is already strong enough for us to prove Theorem 6.1, however we require another modification to prove Theorem 6.2. We will defer its proof to Section 6.4.

Lemma 6.8. *Let H be a c-partite connected graph which does not contain a matching of $\frac{n}{2}$ edges. Suppose further that there is a c-partition of H such that the sum of the sizes of any two parts is at least n. Then*

$$e(H) \leq \begin{cases} \left(1-\frac{1}{c}\right)\frac{v(H)^2}{2} & \text{for } v(H) \leq n\sqrt{\frac{c}{c-1}}\,, \\ \frac{n^2}{2} & \text{for } n\sqrt{\frac{c}{c-1}} < v(H) \leq \frac{5n}{4}\,, \\ \frac{n}{2}\left(v(H)-\frac{n}{4}\right) & \text{for } \frac{5n}{4} < v(H) < \frac{31n}{16}\,, \\ \frac{n}{2}\left(v(H)-\frac{7n}{16}\right) & \text{for } \frac{31n}{16} \leq v(H)\,. \end{cases}$$

6.3 Paths

Proof of Theorem 6.1. Let $\alpha = \frac{1}{4} - \frac{1}{2k}$ and let G be a k-edge-coloured complete graph on $N = (k-\alpha)n$ vertices. Let 'blue' be one of these colours. We proceed by contradiction, supposing that G contains no monochromatic n-vertex path. Over all such G consider the one in which blue has the most edges. In particular G has at least as many blue edges as any other colour. The main idea of our argument is to use bounds on the sizes and the number of blue components to bound the number of edges of G which lie inside blue components, and then to bound the number of edges in each other colour that lie between different blue components.

Let B denote the blue subgraph of G and let $B_1, \ldots, B_c$ be the connected components of B. Let 'red' be the colour that has the most edges lying between blue connected components and let R' denote the c-partite graph of those red edges. We will prove the following two bounds. Firstly the number of edges (of any colour) within blue components satisfies

$$\sum_{i=1}^{c} \binom{v(B_i)}{2} \leq \left(k - 2\alpha + 5\alpha^2\right)\frac{n^2}{2}\,, \tag{6.1}$$

and secondly the number of red edges between blue components satisfies

$$e(R') \leq \left(k - \alpha - \frac{1}{4}\right)\frac{n^2}{2}\,. \tag{6.2}$$

It follows that

$$e(G) \leq (k-1)e(R')+\sum_{i=1}^{c}\binom{v(B_i)}{2} \leq \Big((k-1)(k-\alpha-\tfrac{1}{4})+(k-2\alpha+5\alpha^2)\Big)\frac{n^2}{2}\,.$$

Since $e(G) = \binom{N}{2} = (k-\alpha)(k-\alpha-\frac{1}{n})\frac{n^2}{2}$ it is easy to verify that this fails for $\alpha = \frac{1}{4} - \frac{1}{2k}$ and $n \geq 64k$, reaching the desired contradiction.

We now proceed with proving inequalities (6.1) and (6.2). As a first step toward proving (6.1), we establish bounds on the size of blue components. We first argue that there cannot be large blue components.

Claim 6.9. *There is no blue component in G on more than $\frac{5n}{4}$ vertices.*

Proof. For contradiction, suppose there is a blue component B_1 on βn vertices with $\beta > \frac{5}{4}$. In this case, by Lemma 6.6 we have $e(B_1) \leq (\beta - \frac{1}{4})\frac{n^2}{2}$. Using Lemma 6.5 on the rest of the blue graph B, we obtain

$$e(B) \leq \left(\beta - \frac{1}{4}\right)\frac{n^2}{2} + (k-\alpha-\beta)\,\frac{n^2}{2} = \left(k-\alpha-\frac{1}{4}\right)\frac{n^2}{2}\,.$$

Since blue is the densest colour we have $e(G) \leq k \cdot e(B)$ and hence

$$\binom{N}{2} = (k-\alpha)\left(k-\alpha-\frac{1}{n}\right)\frac{n^2}{2} \leq k\left(k-\alpha-\frac{1}{4}\right)\frac{n^2}{2}$$
$$\alpha^2 - k\alpha + \frac{k}{4} - \frac{k-\alpha}{n} \leq 0\,.$$

This fails when $\alpha = \frac{1}{4} - \frac{1}{2k}$ and $n \geq 64k$. □

The main application of Claim 6.9 is that now when applying Lemma 6.6 to a blue component B_i we obtain the bound $e(B_i) \leq \frac{n^2}{2}$. Using this fact we get a tighter bound on the size of blue components. Let x be defined by the equation

$$xn = \sum_{i=1}^{c}\max\{v(B_i)-n,0\}\,.$$

We refer to x as the *excess* size of blue components. The motivation for this definition is that we expect blue components to be of size approximately n.

Claim 6.10. *We have $x < \alpha$.*

Proof. Let $B_1, \ldots, B_\ell$ be the blue components with more than n vertices. By Lemma 6.6 and Claim 6.9 we have that there are at most $\frac{n^2}{2}$ edges in each of $B_1, \ldots, B_\ell$. Using Lemma 6.5 on the rest of the blue graph we have

$$e(B) \leq \ell\frac{n^2}{2} + (k-\alpha-\ell-x)\frac{n^2}{2} = (k-\alpha-x)\frac{n^2}{2}\,.$$

Since blue is the densest colour we have $e(G) \leq k \cdot e(B)$, and so

$$\binom{N}{2} = (k-\alpha)\left(k-\alpha-\frac{1}{n}\right)\frac{n^2}{2} \leq k(k-\alpha-x)\frac{n^2}{2}$$

therefore

$$x \leq \alpha - \frac{\alpha^2}{k} + \frac{k-\alpha}{kn}\,,$$

and in particular $x < \alpha$ for $\alpha = \frac{1}{4} - \frac{1}{2k}$ and $n \geq 64k$. □

With this bound on the excess, we can prove (6.1), completing the first part of the proof.

Proof of inequality (6.1). By convexity, $\sum_{i=1}^{c}\binom{v(B_i)}{2}$ is maximised when there is one blue component of size $(1+x)n$ which has all the excess, $(k-2)$ components of size n and one component of size $(1-\alpha-x)n$. Note that this is at least $n/2$ as $x, \alpha < 1/4$. It follows that

$$\begin{aligned}\sum_{i=1}^{c}\binom{v(B_i)}{2} &\leq \left((1+x)^2 + (k-2) + (1-\alpha-x)^2\right)\frac{n^2}{2}\\ &= \left(k - 2\alpha + \alpha^2 + 2\alpha x + 2x^2\right)\frac{n^2}{2}\,.\end{aligned}$$

Using the bound $x < \alpha$ from Claim 6.10 completes the argument. □

The second step of the proof of Theorem 6.1 is to bound the number of red edges which lie between different blue components, establishing (6.2). We begin with the following claim.

Claim 6.11. *The number, c, of blue components of G is at most $\frac{4}{3}(k-\alpha)+1$.*

Proof. It suffices to show that all but at most one blue component contain more than $\frac{3n}{4}$ vertices. Suppose for contradiction that B_1 and B_2 each have at most $\frac{3n}{4}$ vertices, and let b satisfy $bn = v(B_1 \cup B_2)$. Note that, by the maximality assumption on blue, $B_1 \cup B_2$ must contain at least $n-1$ vertices. If not, putting a blue clique on $V(B_1 \cup B_2)$ would increase $e(B)$ without creating a blue P_n in G. We therefore have $\frac{n-1}{n} \le b \le \frac{3}{2}$. $e(B_1 \cup B_2)$ is maximal when both components are cliques and by convexity is maximised when $v(B_1) = \frac{3n}{4}$, $v(B_2) = \left(b - \frac{3}{4}\right) n$. Using Lemma 6.5 on the rest of the blue graph we have

$$\begin{aligned} e(B) &\le \frac{9}{16}\frac{n^2}{2} + \left(b - \frac{3}{4}\right)^2 \frac{n^2}{2} + (k - \alpha - b)\frac{n^2}{2} \\ &\le \left(b^2 - \frac{5}{2}b + k - \alpha + \frac{9}{8}\right)\frac{n^2}{2}\,. \end{aligned}$$

Under the constraint $\frac{n-1}{n} \le b \le \frac{3}{2}$, the quadratic function $b^2 - \frac{5b}{2}$ is maximised at $b = \frac{n-1}{n}$, hence

$$e(B) \le \left(k - \alpha - \frac{3}{8} + \frac{1}{2n} + \frac{1}{n^2}\right)\frac{n^2}{2}\,.$$

Since blue is the densest colour we have $e(G) \le k \cdot e(B)$ which gives

$$\binom{N}{2} = (k-\alpha)\left(k - \alpha - \frac{1}{n}\right)\frac{n^2}{2} \le k\left(k - \alpha - \frac{3}{8} + \frac{1}{2n} + \frac{1}{n^2}\right)\frac{n^2}{2}$$

hence

$$\alpha^2 - \alpha k + \frac{3k}{8} - \frac{3k - 2\alpha}{2n} - \frac{k}{n^2} \le 0\,.$$

However this fails for $\alpha = \frac{1}{4} - \frac{1}{2k}$ and $n \ge 64k$; the desired contradiction. $\square$

Using the above bound on c, the next claim uses Lemma 6.7 to bound the number of edges of R'.

Claim 6.12. *Let H be a c-partite connected graph on at most $(k-\alpha)n$ vertices which does not contain an n-vertex path. Then*

$$\frac{e(H)}{v(H)} \le \frac{n}{2}\left(1 - \frac{1}{4(k-\alpha)}\right)\,.$$

Proof. We use Lemma 6.7 to break the proof into three cases depending on the size of H. Firstly in the case where $v(H) \le n\sqrt{\frac{c}{c-1}}$ we have $\frac{e(H)}{v(H)} \le (1 - \frac{1}{c})\frac{v(H)}{2}$. Since $v(H) \le n\sqrt{\frac{c}{c-1}}$ this is at most $\frac{n}{2}\sqrt{\frac{c-1}{c}} \le \frac{n}{2}\left(1 - \frac{1}{2c}\right)$. By

Claim 6.11 we know that $c \leq \frac{4}{3}(k-\alpha)+1$. This gives a bound of

$$\frac{e(H)}{v(H)} \leq \frac{n}{2}\left(1-\frac{3}{8(k-\alpha)+6}\right) \leq \frac{n}{2}\left(1-\frac{1}{4(k-\alpha)}\right).$$

Next suppose $n\sqrt{\frac{c}{c-1}} < v(H) \leq \frac{5n}{4}$. Then, by Lemma 6.7, we have $\frac{e(H)}{v(H)} \leq \frac{n}{2}\sqrt{\frac{c-1}{c}}$. As shown in the previous case $\frac{n}{2}\sqrt{\frac{c-1}{c}}$ is at most $\frac{n}{2}\left(1-\frac{1}{4(k-\alpha)}\right)$. Finally suppose $v(H) > \frac{5n}{4}$. Then $\frac{e(H)}{v(H)} \leq \frac{n}{2}\left(1-\frac{n}{4v(H)}\right)$. This is maximised when $v(H)$ is as large as possible giving $\frac{e(H)}{v(H)} \leq \frac{n}{2}\left(1-\frac{1}{4(k-\alpha)}\right)$. □

We can now deduce (6.2) from Claim 6.12. There will be a connected component H of R' with at least as high a ratio $e(H)/v(H)$ as the overall ratio $e(R')/v(R')$. Therefore if R' had more than $\left(k-\alpha-\frac{1}{4}\right)\frac{n^2}{2}$ edges there would be a connected component H satisfying

$$\frac{e(H)}{v(H)} > \frac{1}{N}\left(k-\alpha-\frac{1}{4}\right)\frac{n^2}{2} = \frac{n}{2}\left(1-\frac{1}{4(k-\alpha)}\right).$$

This contradicts Claim 6.12, completing the proof. □

6.4 Even Cycles

The proof of Theorem 6.3 closely resembles the arguments of the previous section. We make three changes, the first two of which are only minor adjustments. We must work with the value $\alpha = \frac{1}{4}$ instead of the value $\frac{1}{4}-\frac{1}{2k}$, and we must permit the host graph G to have as few as $(1-\delta)\binom{N}{2}$ edges for some small $\delta > 0$ which we choose. The more significant change is that we apply Lemma 6.8 instead of Lemma 6.7 to bound the number of edges between blue components.

The reason we are able to improve upon the result of Theorem 6.1 is that when looking only for a connected matching (rather than a path) we can better deal with large components of the graph R' consisting of red edges between blue components. In particular, the tight case of Claim 6.12 is when $v(H) > \frac{5n}{4}$ where we can do no better than assume R' consists of one large connected component. The improvement in this case is given by Lemma 6.8, where we get a better bound on $e(H)$ when H is a component of R' with at least $\frac{31n}{16}$ vertices.

Proof of Lemma 6.8. First note that the three bounds for the range $v(H) < \frac{31n}{16}$ follow directly from Lemma 6.7 since, for even n, a copy of P_n contains a matching of $\frac{n}{2}$ edges. We therefore assume H is a c-partite, connected graph on at least $\frac{31n}{16}$ vertices, in which the sizes of any two parts sum to at least n and which contains no matching of $\frac{n}{2}$ edges. We will show that $e(H) \leq \frac{n}{2}(v(H) - \frac{7n}{16})$.

Let $A = \{v \in H : d(v) \geq n\}$ and let M denote a maximal matching in $H' := H \backslash A$. We may assume $v(M) < n$ as H does not contain any matching with $n/2$ edges. We will bound $e(H)$ by first bounding $e(H')$ and then bounding the number of edges incident to A. First note that

$$|A| \leq \frac{n}{2} - \frac{v(M)}{2}\,, \tag{6.3}$$

otherwise we could greedily extend M to a matching of size $\frac{n}{2}$ in H, contradicting the assumption of the lemma. For $v \in H'$ let $d^*(v)$ denote the number of neighbours of v in $H' \backslash M$. Now let $\{u, v\}$ be an edge of M. Note that either $d^*(u) \leq 1$ or $d^*(v) \leq 1$ else we could replace the edge $\{u, v\}$ with a pair of edges $\{u, x\}$, $\{v, y\}$ to get a larger matching in H'. Let us denote the edges of M by $\{u_i, v_i\}$ for $i = 1, \ldots, \frac{v(M)}{2}$ and assume without loss of generality that $d^*(u_i) \leq 1$ for all i. Since each edge of H' is incident to an edge of M by maximality it follows that

$$\begin{aligned} e(H') &\leq \sum_{i=1}^{\frac{1}{2}v(M)} (d(v_i) + d^*(u_i)) + \binom{\frac{1}{2}v(M)}{2} \\ &\leq \frac{v(M)}{2}\left(n + \frac{v(M)}{4}\right) \\ &\leq \frac{5}{8}n^2 - \frac{3}{2}|A|n + \frac{|A|^2}{2}\,, \end{aligned} \tag{6.4}$$

where for the second inequality we used that $d(v) < n$ for all $v \in H'$ by the definition of A, and $d^*(u_i) \leq 1$ for all i by assumption. For the last inequality we used (6.3). We now turn our attention to bounding the number of edges incident to A. Recall that H is c-partite and let t denote the size of its smallest part. First let us suppose that $t \leq |A|$. Since we assume the sum of any two parts of H is at least n, it follows that the second smallest part of H has size at least $n-t$ (note that $t \leq \frac{n}{2}$ by (6.3)). It follows that at most t vertices of A have degree $v(H)-t$ and the rest have degree at most $v(H)-n+t$ so that

$$\sum_{v\in A} d(v) \leq t(v(H)-t) + (|A|-t)(v(H)-n+t)\,.$$

Considering the right hand side as a quadratic function in t we see that it is maximised when $t = \frac{n+|A|}{4}$ and so

$$\sum_{v\in A} d(v) \leq \frac{|A|^2}{8} + \left(v(H) - \frac{3}{4}n\right)|A| + \frac{n^2}{8}\,. \tag{6.5}$$

Since $e(H) \leq e(H') + \sum_{v\in A} d(v)$ it follows by (6.4) and (6.5) that

$$e(H) \leq \frac{5}{8}|A|^2 + \left(v(H) - \frac{9}{4}n\right)|A| + \frac{3}{4}n^2\,.$$

We consider the right hand side as a quadratic function in $|A|$ and optimise under the constraint $0 \leq |A| \leq \frac{n}{2}$. The maximum must occur at either $|A| = 0$ or $|A| = \frac{n}{2}$ and it is simple to check that the latter is the maximiser under the assumption that $v(H) \geq \frac{31}{16}n$. It follows that $e(H) \leq \frac{n}{2}v(H) - \frac{7}{32}n^2$ as claimed. It remains to consider the case where $t \geq |A|$. Recall that the maximum degree of H is at most $v(H)-t$ and so

$$\sum_{v\in A} d(v) \leq |A|(v(H)-t) \leq |A|(v(H)-|A|) \leq \frac{n}{2}v(H) - \frac{n^2}{4}\,,$$

where for the last inequality we again use the bound $|A| \leq \frac{n}{2}$. The result follows. $\square$

Proof of Theorem 6.3. Let $\alpha = \frac{1}{4}$, $0 < \delta < \frac{1}{64k^2}$ and let G be a k-edge-coloured graph on $N = (k-\alpha)n$ vertices with at least $(1-\delta)\binom{N}{2}$ edges. We proceed by contradiction, supposing that G contains no monochromatic connected matching of $\frac{n}{2}$ edges. Over all such G consider the one in which blue has the most edges.

Let $B_1, \ldots, B_c$ be the blue connected components of G, suppose that red has the most edges between blue connected components and let R' denote the c-partite graph of red edges which lie between blue components. The method is the same as the previous section. We establish the following two bounds.

$$\sum_{i=1}^{c}\binom{v(B_i)}{2} \le \left(k-2\alpha+5\alpha^2\right)\frac{n^2}{2} = \left(k-\frac{3}{16}\right)\frac{n^2}{2}\,, \tag{6.6}$$

$$e(R') \le \left(k-\alpha-\frac{7}{16}\right)\frac{n^2}{2} = \left(k-\frac{11}{16}\right)\frac{n^2}{2}\,. \tag{6.7}$$

We then deduce

$$e(G) \le (k-1)e(R') + \sum_{i=1}^{c}\binom{v(B_i)}{2} \le \left(k^2-\frac{11}{16}k+\frac{1}{2}\right)\frac{n^2}{2}\,.$$

Since $e(G) \ge (1-\delta)\binom{N}{2} = (1-\delta)(k-\frac{1}{4})(k-\frac{1}{4}-\frac{1}{n})\frac{n^2}{2}$ it is easy to verify that with $\delta < \frac{1}{64k^2}$ and $n \ge 32k$ we reach the desired contradiction.

It remains to prove the inequalities (6.6) and (6.7). We start by showing Claims 6.9 and 6.10 have direct analogues here.

Claim 6.13. *There is no blue component on more than $\frac{5n}{4}$ vertices.*

Proof. For contradiction, suppose there is a blue component B_1 on βn vertices with $\beta > \frac{5n}{4}$. In this case, by Lemma 6.6 we have $e(B_1) \le (\beta - \frac{1}{4})\frac{n^2}{2}$. Using Lemma 6.5 on the rest of the blue graph, B, we obtain

$$e(B) \le \left(\beta-\frac{1}{4}\right)\frac{n^2}{2} + (k-\alpha-\beta)\frac{n^2}{2} = \left(k-\alpha-\frac{1}{4}\right)\frac{n^2}{2}\,.$$

Since blue is the densest colour we have $e(G) \le k\cdot e(B)$ and hence

$$(1-\delta)\binom{N}{2} = (1-\delta)(k-\alpha)\left(k-\alpha-\frac{1}{n}\right)\frac{n^2}{2} \le k\left(k-\alpha-\frac{1}{4}\right)\frac{n^2}{2}$$
$$(1-\delta)\alpha^2-(1-2\delta)k\alpha+\frac{k}{4}-(1-\delta)\frac{k-\alpha}{n}-\delta k^2 \le 0\,.$$

This fails with $\alpha = \frac{1}{4}$, $n \ge 32k$ and $\delta < \frac{1}{64k^2}$. $\square$

Using the above claim we get a tighter bound on the size of blue components.

Let x be the excess size of blue components,

$$xn = \sum_{i=1}^{c} \max\{v(B_i) - n, 0\}$$

as before.

Claim 6.14. *We have $x < \alpha$.*

Proof. Let $B_1, \ldots, B_\ell$ be the blue components with more than n vertices. By Lemma 6.6 and Claim 6.13 we have that there are at most $\frac{n^2}{2}$ edges in each of $B_1, \ldots, B_\ell$. Using Lemma 6.5 on the rest of the blue graph we have

$$e(B) \le \ell \frac{n^2}{2} + (k - \alpha - \ell - x)\frac{n^2}{2} = (k - \alpha - x)\frac{n^2}{2}\,.$$

Since blue is the densest colour we have $e(G) \le k \cdot e(B)$, and so

$$(1-\delta)\binom{N}{2} = (1-\delta)(k-\alpha)\left(k - \alpha - \frac{1}{n}\right)\frac{n^2}{2} \le k(k-\alpha-x)\frac{n^2}{2}\,.$$

Therefore

$$x \le (1-2\delta)\alpha - (1-\delta)\frac{\alpha^2}{k} + \delta k + (1-\delta)\frac{k-\alpha}{kn}\,,$$

and in particular $x < \alpha$ for $\alpha = \frac{1}{4}$, $n \ge 32k$ and $\delta < \frac{1}{64k^2}$. □

Inequality (6.6) follows from Claim 6.14 in the exact same way as inequality (6.1) follows from Claim 6.10.

We require the same bound as before on the number of blue components, now with $\alpha = \frac{1}{4}$.

Claim 6.15. *The number, c, of blue components of G is at most $\frac{4}{3}(k-\alpha)+1$.*

Proof. The proof follows that of Claim 6.11 replacing $\binom{N}{2}$ with $(1-\delta)\binom{N}{2}$. With $\alpha = \frac{1}{4}$, $n \ge 32k$, and $\delta < \frac{1}{64k^2}$ the required contradiction holds. □

The final claim is the improved version of Claim 6.12 which makes use of Lemma 6.8 and the above claim bounding c.

Claim 6.16. *Let H be a c-partite connected graph which does not contain a matching of $\frac{n}{2}$ edges. Suppose further that there is a c-partition of H such that the sum of the sizes of any two parts is at least n. Then*

$$\frac{e(H)}{v(H)} \leq \frac{n}{2}\left(1 - \frac{7}{16(k-\alpha)}\right).$$

Proof. We prove this using Lemma 6.8 in the same way that we proved Claim 6.12 using Lemma 6.7; breaking into cases depending on the size of H. Firstly if $v(H) \leq n\sqrt{\frac{c}{c-1}}$, or if $n\sqrt{\frac{c}{c-1}} < v(H) \leq \frac{5n}{4}$, the argument is identical to that of Claim 6.12 giving in both cases

$$\frac{e(H)}{v(H)} \leq \frac{n}{2}\left(1 - \frac{3}{4(k-\alpha)+6}\right) \leq \frac{n}{2}\left(1 - \frac{7}{16(k-\alpha)}\right).$$

If $\frac{5n}{4} < v(H) < \frac{31n}{16}$ then

$$\frac{e(H)}{v(H)} \leq \frac{n}{2}\left(1 - \frac{n}{4v(H)}\right) \leq \frac{n}{2}\left(1 - \frac{4}{31}\right) \leq \frac{n}{2}\left(1 - \frac{7}{16(k-\alpha)}\right).$$

Finally if $\frac{31n}{16} \leq v(H)$ we have $\frac{e(H)}{v(H)} \leq \frac{n}{2}\left(1 - \frac{7n}{16v(H)}\right)$. This is maximised when $v(H)$ is as large as possible and so we have

$$\frac{e(H)}{v(H)} \leq \frac{n}{2}\left(1 - \frac{7}{16(k-\alpha)}\right). \qquad \square$$

We can now deduce inequality (6.7) from Claim 6.16. Suppose for contradiction that $e(R') > \left(k - \alpha - \frac{7}{16}\right)\frac{n^2}{2}$. Then, by the pigeonhole principle, there is a connected component of H with

$$\frac{e(H)}{v(H)} > \frac{1}{N}\left(k - \alpha - \frac{7}{16}\right)\frac{n^2}{2} = \frac{n}{2}\left(1 - \frac{7}{16(k-\alpha)}\right).$$

This contradicts Claim 6.16, proving (6.7) and completing the proof of Theorem 6.3. $\square$

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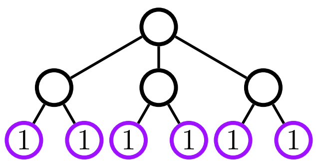
A

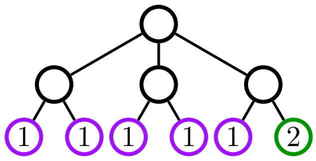
List of local views in the Potts model on cubic graphs

Local view 1 of 35 (named C_1)



$$
\begin{aligned}
Z_C &= \lambda^9 + 4\lambda^6(q-1) + 3\lambda^5(q-1) + 3\lambda^4(q-1)(q-2) + 3\lambda^4(q-1) + 9\lambda^3(q-1)(q-2) + \\
&\quad 3\lambda^2(q-1)(q-2)(q-3) + 4\lambda^3(q-1) + 6\lambda^2(q-1)(q-2) + 3\lambda(q-1)(q-2)(q-3) + \\
&\quad (q-1)(q-2)(q-3)(q-4) + 3\lambda(q-1)(q-2) + 4(q-1)(q-2)(q-3) + \\
&\quad 4(q-1)(q-2) + q - 1. \\
2Z_C U_C^v &= 3\lambda^9 + 6\lambda^6(q-1) + 3\lambda^5(q-1) + 6\lambda^4(q-1) + 9\lambda^3(q-1)(q-2) + 6\lambda^3(q-1) + \\
&\quad 6\lambda^2(q-1)(q-2) + 3\lambda(q-1)(q-2)(q-3) + 3\lambda(q-1)(q-2). \\
6Z_C U_C^N &= 9\lambda^9 + 24\lambda^6(q-1) + 15\lambda^5(q-1) + 12\lambda^4(q-1)(q-2) + 12\lambda^4(q-1) + \\
&\quad 27\lambda^3(q-1)(q-2) + 6\lambda^2(q-1)(q-2)(q-3) + 12\lambda^3(q-1) + 12\lambda^2(q-1)(q-2) + \\
&\quad 3\lambda(q-1)(q-2)(q-3) + 3\lambda(q-1)(q-2). \\
\tilde{S}_C &= 0. \\
D_C^U &= 3(2s^4t^7 + 14s^4t^6 + 12s^3t^7 + 42s^4t^5 + 91s^3t^6 + 27s^2t^7 + 70s^4t^4 + 300s^3t^5 + 221s^2t^6 + \\
&\quad 27st^7 + 70s^4t^3 + 554s^3t^4 + 796s^2t^5 + 237st^6 + 10t^7 + 42s^4t^2 + 616s^3t^3 + 1619s^2t^4 + \\
&\quad 928st^5 + 94t^6 + 14s^4t + 411s^3t^2 + 1994s^2t^3 + 2071st^4 + 400t^5 + 2s^4 + 152s^3t + \\
&\quad 1481s^2t^2 + 2816st^3 + 978t^4 + 24s^3 + 612s^2t + 2325st^2 + 1464t^3 + 108s^2 + 1080st + \\
&\quad 1341t^2 + 216s + 702t + 162)(s+3)(s+2)(t+1)^4. \\
D_C^U|_{q=2} &= 6(t^2+2t+2)(t+2)^4(t+1)^4.
\end{aligned}
$$

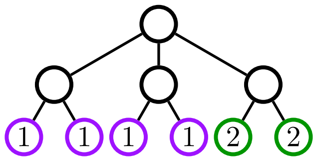
Local view 2 of 35



$$\begin{aligned}
Z_C &= \lambda^8 + \lambda^7 + \lambda^6(q-2) + \lambda^6 + 5\lambda^5(q-2) + \lambda^4(q-2)(q-3) + 5\lambda^5 + 11\lambda^4(q-2) + \\
&\quad 10\lambda^3(q-2)(q-3) + 2\lambda^2(q-2)(q-3)(q-4) + 5\lambda^4 + 19\lambda^3(q-2) + 15\lambda^2(q-2)(q-3) + \\
&\quad 5\lambda(q-2)(q-3)(q-4) + (q-2)(q-3)(q-4)(q-5) + \lambda^3 + 15\lambda^2(q-2) + \\
&\quad 17\lambda(q-2)(q-3) + 7(q-2)(q-3)(q-4) + \lambda^2 + 9\lambda(q-2) + 12(q-2)(q-3) + \lambda + 5q - 10. \\
2Z_C U_C^v &= 3\lambda^8 + 2\lambda^7 + 2\lambda^6(q-2) + \lambda^6 + 5\lambda^5(q-2) + 8\lambda^5 + 12\lambda^4(q-2) + 6\lambda^3(q-2)(q-3) + \\
&\quad 7\lambda^4 + 19\lambda^3(q-2) + 12\lambda^2(q-2)(q-3) + 3\lambda(q-2)(q-3)(q-4) + 2\lambda^3 + \\
&\quad 16\lambda^2(q-2) + 9\lambda(q-2)(q-3) + \lambda^2 + 3\lambda(q-2). \\
6Z_C U_C^N &= 8\lambda^8 + 7\lambda^7 + 6\lambda^6(q-2) + 6\lambda^6 + 25\lambda^5(q-2) + 4\lambda^4(q-2)(q-3) + 25\lambda^5 + 44\lambda^4(q-2) + \\
&\quad 30\lambda^3(q-2)(q-3) + 4\lambda^2(q-2)(q-3)(q-4) + 20\lambda^4 + 57\lambda^3(q-2) + 30\lambda^2(q-2)(q-3) + \\
&\quad 5\lambda(q-2)(q-3)(q-4) + 3\lambda^3 + 30\lambda^2(q-2) + 17\lambda(q-2)(q-3) + 2\lambda^2 + 9\lambda(q-2) + \lambda. \\
\tilde{S}_C &= (4r^8t^{15} + 60r^8t^{14} + 42r^7t^{15} + 424r^8t^{13} + 662r^7t^{14} + 194r^6t^{15} + 1872r^8t^{12} + 4930r^7t^{13} + \\
&\quad 3202r^6t^{14} + 514r^5t^{15} + 5772r^8t^{11} + 22994r^7t^{12} + 25050r^6t^{13} + 8862r^5t^{14} + 851r^4t^{15} + \\
&\quad 13156r^8t^{10} + 75032r^7t^{11} + 123078r^6t^{12} + 72648r^5t^{13} + 15324r^4t^{14} + 895r^3t^{15} + \\
&\quad 22880r^8t^9 + 181196r^7t^{10} + 424020r^6t^{11} + 375096r^5t^{12} + 131468r^4t^{13} + 16896r^3t^{14} + \\
&\quad 576r^2t^{15} + 30888r^8t^8 + 334044r^7t^9 + 1082836r^6t^{10} + 1361314r^5t^{11} + 712016r^4t^{12} + \\
&\quad 151828r^3t^{13} + 11526r^2t^{14} + 202rt^{15} + 32604r^8t^7 + 478020r^7t^8 + 2113148r^6t^9 + \\
&\quad 3669346r^5t^{10} + 2716791r^4t^{11} + 861812r^3t^{12} + 108975r^2t^{13} + 4396rt^{14} + 28t^{15} + \\
&\quad 26884r^8t^6 + 534618r^7t^7 + 3202540r^6t^8 + 7568730r^5t^9 + 7715100r^4t^{10} + 3451813r^3t^{11} + \\
&\quad 648892r^2t^{12} + 44221rt^{13} + 704t^{14} + 17160r^8t^5 + 466734r^7t^6 + 3793562r^6t^7 + \\
&\quad 12135518r^5t^8 + 16794876r^4t^9 + 10309598r^3t^{10} + 2725907r^2t^{11} + 277292rt^{12} + \\
&\quad 7710t^{13} + 8320r^8t^4 + 315146r^7t^5 + 3507018r^6t^6 + 15216656r^5t^7 + 28456722r^4t^8 + \\
&\quad 23648956r^3t^9 + 8550366r^2t^{10} + 1222221rt^{11} + 51336t^{12} + 2964r^8t^3 + 161482r^7t^4 + \\
&\quad 2506578r^6t^5 + 14895064r^5t^6 + 37743412r^4t^7 + 42293290r^3t^8 + 20637616r^2t^9 + \\
&\quad 4022318rt^{10} + 237894t^{11} + 732r^8t^2 + 60740r^7t^3 + 1358926r^6t^4 + 11273998r^5t^5 + \\
&\quad 39109440r^4t^6 + 59288290r^3t^7 + 38909474r^2t^8 + 10202454rt^9 + 821076t^{10} + 112r^8t + \\
&\quad 15824r^7t^2 + 540560r^6t^3 + 6473218r^5t^4 + 31355250r^4t^5 + 65005470r^3t^6 + \\
&\quad 57600846r^2t^7 + 20256420rt^8 + 2186316t^9 + 8r^8 + 2552r^7t + 148864r^6t^2 + \\
&\quad 2727300r^5t^3 + 19081926r^4t^4 + 55206558r^3t^5 + 66794490r^2t^6 + 31642596rt^7 + \\
&\quad 4566960t^8 + 192r^7 + 25368r^6t + 795600r^5t^2 + 8527320r^4t^3 + 35630622r^3t^4 + \\
&\quad 60082830r^2t^5 + 38788956rt^6 + 7523064t^7 + 2016r^6 + 143640r^5t + 2640600r^4t^2 + \\
&\quad 16909452r^3t^3 + 41140170r^2t^4 + 36949770rt^5 + 9745272t^6 + 12096r^5 + 506520r^4t + \\
&\quad 5569776r^3t^2 + 20755440r^2t^3 + 26847126rt^4 + 9829350t^5 + 45360r^4 + 1138536r^3t + \\
&\quad 7286112r^2t^2 + 14409900rt^3 + 7579170t^4 + 108864r^3 + 1592136r^2t + 5400432rt^2 + \\
&\quad 4330260t^3 + 163296r^2 + 1265544rt + 1735020t^2 + 139968r + 437400t + 52488)(t+1)^2t. \\
\tilde{S}_C|_{q=2} &= 4(t^2+t+1)(t+2)^9(t+1)^2t.
\end{aligned}$$

$$\begin{aligned}
D_C^U &= (6s^5t^{10}+60s^5t^9+42s^4t^{10}+270s^5t^8+449s^4t^9+117s^3t^{10}+720s^5t^7+2163s^4t^8+1336s^3t^9+\\
&\quad 162s^2t^{10}+1260s^5t^6+6181s^4t^7+6875s^3t^8+1972s^2t^9+111st^{10}+1512s^5t^5+11599s^4t^6+\\
&\quad 20997s^3t^7+10815s^2t^8+1439st^9+30t^{10}+1260s^5t^4+14931s^4t^5+42146s^3t^6+35222s^2t^7+\\
&\quad 8398st^8+414t^9+720s^5t^3+13349s^4t^4+58090s^3t^5+75472s^2t^6+29118st^7+2568t^8+\\
&\quad 270s^5t^2+8183s^4t^3+55671s^3t^4+111207s^2t^5+66507st^6+9468t^7+60s^5t+3291s^4t^2+\\
&\quad 36625s^3t^3+114140s^2t^4+104641st^5+23030t^6+6s^5+784s^4t+15827s^3t^2+80595s^2t^3+\\
&\quad 114932st^4+38665t^5+84s^4+4056s^3t+37479s^2t^2+87087st^3+45420t^4+468s^3+10368s^2t+\\
&\quad 43623st^2+36918t^3+1296s^2+13068st+19926t^2+1782s+6480t+972)(s+3)(t+1).\\
D_C^U|_{q=2} &= 2(3t^4+12t^3+20t^2+16t+6)(t+2)^4(t+1).
\end{aligned}$$

Local view 3 of 35



$$\begin{aligned}
Z_C &= \lambda^8+2\lambda^7+2\lambda^6(q-2)+2\lambda^6+7\lambda^5(q-2)+3\lambda^4(q-2)(q-3)+3\lambda^5+11\lambda^4(q-2)+\\
&\quad 9\lambda^3(q-2)(q-3)+3\lambda^2(q-2)(q-3)(q-4)+3\lambda^4+14\lambda^3(q-2)+\\
&\quad 15\lambda^2(q-2)(q-3)+3\lambda(q-2)(q-3)(q-4)+(q-2)(q-3)(q-4)(q-5)+2\lambda^3+\\
&\quad 14\lambda^2(q-2)+12\lambda(q-2)(q-3)+8(q-2)(q-3)(q-4)+2\lambda^2+10\lambda(q-2)+\\
&\quad 16(q-2)(q-3)+\lambda+7q-14.\\
2Z_CU_C^v &= 2\lambda^8+4\lambda^7+2\lambda^6(q-2)+4\lambda^6+7\lambda^5(q-2)+5\lambda^5+14\lambda^4(q-2)+9\lambda^3(q-2)(q-3)+\\
&\quad 4\lambda^4+16\lambda^3(q-2)+6\lambda^2(q-2)(q-3)+3\lambda(q-2)(q-3)(q-4)+2\lambda^3+8\lambda^2(q-2)+\\
&\quad 12\lambda(q-2)(q-3)+2\lambda^2+10\lambda(q-2)+\lambda.\\
6Z_CU_C^N &= 8\lambda^8+14\lambda^7+12\lambda^6(q-2)+12\lambda^6+35\lambda^5(q-2)+12\lambda^4(q-2)(q-3)+15\lambda^5+44\lambda^4(q-2)+\\
&\quad 27\lambda^3(q-2)(q-3)+6\lambda^2(q-2)(q-3)(q-4)+12\lambda^4+42\lambda^3(q-2)+30\lambda^2(q-2)(q-3)+\\
&\quad 3\lambda(q-2)(q-3)(q-4)+6\lambda^3+28\lambda^2(q-2)+12\lambda(q-2)(q-3)+4\lambda^2+10\lambda(q-2)+\lambda.\\
\tilde{S}_C &= 2(4r^4t^9+36r^4t^8+27r^3t^9+144r^4t^7+252r^3t^8+69r^2t^9+336r^4t^6+1053r^3t^7+\\
&\quad 666r^2t^8+79rt^9+504r^4t^5+2586r^3t^6+2895r^2t^7+786rt^8+34t^9+504r^4t^4+\\
&\quad 4113r^3t^5+7446r^2t^6+3540rt^7+348t^8+336r^4t^3+4392r^3t^4+12495r^2t^5+9496rt^6+\\
&\quad 1620t^7+144r^4t^2+3147r^3t^3+14190r^2t^4+16746rt^5+4520t^6+36r^4t+1458r^3t^2+\\
&\quad 10905r^2t^3+20154rt^4+8352t^5+4r^4+396r^3t+5466r^2t^2+16569rt^3+10620t^4+48r^3+\\
&\quad 1620r^2t+8982rt^2+9315t^3+216r^2+2916rt+5454t^2+432r+1944t+324)(r^4t^7+\\
&\quad 8r^4t^6+4r^3t^7+27r^4t^5+40r^3t^6+6r^2t^7+50r^4t^4+164r^3t^5+76r^2t^6+5rt^7+55r^4t^3+\\
&\quad 360r^3t^4+370r^2t^5+68rt^6+2t^7+36r^4t^2+460r^3t^3+948r^2t^4+373rt^5+24t^6+13r^4t+\\
&\quad 344r^3t^2+1402r^2t^3+1090rt^4+141t^5+2r^4+140r^3t+1204r^2t^2+1848rt^3+462t^4+\\
&\quad 24r^3+558r^2t+1824rt^2+891t^3+108r^2+972rt+1008t^2+216r+621t+162)(t+2)t.\\
\tilde{S}_C|_{q=2} &= 2(t^2+2t+2)^3(t+2)^9t.\\
D_C^U &= (6s^5t^{11}+66s^5t^{10}+44s^4t^{11}+330s^5t^9+501s^4t^{10}+130s^3t^{11}+990s^5t^8+2612s^4t^9+\\
&\quad 1524s^3t^{10}+193s^2t^{11}+1980s^5t^7+8224s^4t^8+8238s^3t^9+2316s^2t^{10}+143st^{11}+2772s^5t^6+\\
&\quad 17360s^4t^7+27073s^3t^8+12904s^2t^9+1749st^{10}+42t^{11}+2772s^5t^5+25774s^4t^6+60024s^3t^7+\\
&\quad 44023s^2t^8+9996st^9+522t^{10}+1980s^5t^4+27440s^4t^5+94143s^3t^6+102037s^2t^7+\\
&\quad 35230st^8+3048t^9+990s^5t^3+20932s^4t^4+106442s^3t^5+168439s^2t^6+84987st^7+11048t^8+\\
&\quad 330s^5t^2+11204s^4t^3+86643s^3t^4+201738s^2t^5+147118st^6+27618t^7+66s^5t+4005s^4t^2+\\
&\quad 49698s^3t^3+175033s^2t^4+186160st^5+49947t^6+6s^5+860s^4t+19109s^3t^2+107664s^2t^3+\\
&\quad 171928st^4+66586t^5+84s^4+4428s^3t+44661s^2t^2+113448st^3+65352t^4+468s^3+\\
&\quad 11232s^2t+50913st^2+46260t^3+1296s^2+13986st+22518t^2+1782s+6804t+972)(s+3).
\end{aligned}$$

$$D_C^U|_{q=2} = 2(2t^2+4t+3)(t^2+2t+2)(t+2)^4.$$

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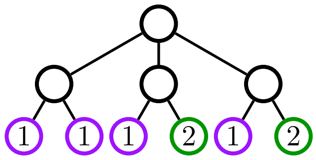

$$\begin{aligned}
Z_C &= \lambda^7+3\lambda^6+2\lambda^5(q-2)+4\lambda^5+15\lambda^4(q-2)+7\lambda^3(q-2)(q-3)+\\
&\quad \lambda^2(q-2)(q-3)(q-4)+4\lambda^4+18\lambda^3(q-2)+21\lambda^2(q-2)(q-3)+\\
&\quad 7\lambda(q-2)(q-3)(q-4)+(q-2)(q-3)(q-4)(q-5)+3\lambda^3+20\lambda^2(q-2)+\\
&\quad 18\lambda(q-2)(q-3)+6(q-2)(q-3)(q-4)+\lambda^2+7\lambda(q-2)+9(q-2)(q-3)+3q-6.\\
2Z_CU_C^v &= 3\lambda^7+6\lambda^6+4\lambda^5(q-2)+6\lambda^5+14\lambda^4(q-2)+3\lambda^3(q-2)(q-3)+6\lambda^4+24\lambda^3(q-2)+\\
&\quad 18\lambda^2(q-2)(q-3)+3\lambda(q-2)(q-3)(q-4)+3\lambda^3+14\lambda^2(q-2)+6\lambda(q-2)(q-3)+\\
&\quad \lambda(q-2).\\
6Z_CU_C^N &= 7\lambda^7+18\lambda^6+10\lambda^5(q-2)+20\lambda^5+60\lambda^4(q-2)+21\lambda^3(q-2)(q-3)+\\
&\quad 2\lambda^2(q-2)(q-3)(q-4)+16\lambda^4+54\lambda^3(q-2)+42\lambda^2(q-2)(q-3)+\\
&\quad 7\lambda(q-2)(q-3)(q-4)+9\lambda^3+40\lambda^2(q-2)+18\lambda(q-2)(q-3)+2\lambda^2+7\lambda(q-2).\\
\tilde{S}_C &= 2(2r^8t^{15}+30r^8t^{14}+20r^7t^{15}+212r^8t^{13}+318r^7t^{14}+87r^6t^{15}+936r^8t^{12}+2385r^7t^{13}+\\
&\quad 1466r^6t^{14}+214r^5t^{15}+2886r^8t^{11}+11188r^7t^{12}+11658r^6t^{13}+3831r^5t^{14}+322r^4t^{15}+\\
&\quad 6578r^8t^{10}+36681r^7t^{11}+58024r^6t^{12}+32319r^5t^{13}+6177r^4t^{14}+298r^3t^{15}+\\
&\quad 11440r^8t^9+88936r^7t^{10}+201983r^6t^{11}+170565r^5t^{12}+55484r^4t^{13}+6244r^3t^{14}+\\
&\quad 161r^2t^{15}+15444r^8t^8+164526r^7t^9+520248r^6t^{10}+629660r^5t^{11}+310756r^4t^{12}+\\
&\quad 60227r^3t^{13}+3822r^2t^{14}+44rt^{15}+16302r^8t^7+236160r^7t^8+1022736r^6t^9+\\
&\quad 1720691r^5t^{10}+1215887r^4t^{11}+359076r^3t^{12}+40197r^2t^{13}+1274rt^{14}+4t^{15}+\\
&\quad 13442r^8t^6+264846r^7t^7+1560072r^6t^8+3590661r^5t^9+3521231r^4t^{10}+1489548r^3t^{11}+\\
&\quad 256633r^2t^{12}+14992rt^{13}+172t^{14}+8580r^8t^5+231782r^7t^6+1858797r^6t^7+\\
&\quad 5816311r^5t^8+7790635r^4t^9+4567498r^3t^{10}+1130413r^2t^{11}+103508rt^{12}+2372t^{13}+\\
&\quad 4160r^8t^4+156837r^7t^5+1727442r^6t^6+7360789r^5t^7+13388834r^4t^8+10701410r^3t^9+\\
&\quad 3668451r^2t^{10}+485621rt^{11}+17988t^{12}+1482r^8t^3+80508r^7t^4+1240394r^6t^5+\\
&\quad 7266070r^5t^6+17988696r^4t^7+19491608r^3t^8+9091388r^2t^9+1668073rt^{10}+90342t^{11}+\\
&\quad 366r^8t^2+30325r^7t^3+675120r^6t^4+5541140r^5t^5+18862222r^4t^6+27783569r^3t^7+\\
&\quad 17529532r^2t^8+4368276rt^9+328710t^{10}+56r^8t+7908r^7t^2+269389r^6t^3+3202160r^5t^4+\\
&\quad 15286059r^4t^5+30939606r^3t^6+26486622r^2t^7+8905641rt^8+908856t^9+4r^8+1276r^7t+\\
&\quad 74348r^6t^2+1356117r^5t^3+9390348r^4t^4+26655057r^3t^5+31312386r^2t^6+14250276rt^7+\\
&\quad 1956834t^8+96r^7+12684r^6t+397044r^5t^2+4228425r^4t^3+17423676r^3t^4+\\
&\quad 28681668r^2t^5+17874702rt^6+3312711t^7+1008r^6+71820r^5t+1316520r^4t^2+\\
&\quad 8356311r^3t^3+19965096r^2t^4+17407062rt^5+4406076t^6+6048r^5+253260r^4t+\\
&\quad 2773548r^3t^2+10213695r^2t^3+12910104rt^4+4560867t^5+22680r^4+569268r^3t+\\
&\quad 3622644r^2t^2+7054047rt^3+3605148t^4+54432r^3+796068r^2t+2679804rt^2+\\
&\quad 2106081t^3+81648r^2+632772rt+858762t^2+69984r+218700t+26244)(t+1)^2t.\\
\tilde{S}_C|_{q=2} &= 2(t^2+2t+2)(t+2)^9(t+1)^2t.
\end{aligned}$$

$$
\begin{aligned}
D_C^U \;&= (6s^5t^9 + 54s^5t^8 + 38s^4t^9 + 216s^5t^7 + 377s^4t^8 + 95s^3t^9 + 504s^5t^6 + 1656s^4t^7 + \\
&1037s^3t^8 + 118s^2t^9 + 756s^5t^5 + 4231s^4t^6 + 4986s^3t^7 + 1405s^2t^8 + 73st^9 + 756s^5t^4 + \\
&6934s^4t^5 + 13903s^3t^6 + 7356s^2t^7 + 937st^8 + 18t^9 + 504s^5t^3 + 7563s^4t^4 + 24835s^3t^5 + \\
&22309s^2t^6 + 5310st^7 + 246t^8 + 216s^5t^2 + 5492s^4t^3 + 29523s^3t^4 + 43336s^2t^5 + \\
&17450st^6 + 1500t^7 + 54s^5t + 2561s^4t^2 + 23384s^3t^3 + 56085s^2t^4 + 36767st^5 + 5320t^6 + \\
&6s^5 + 696s^4t + 11909s^3t^2 + 48486s^2t^3 + 51720st^4 + 12118t^5 + 84s^4 + 3540s^3t + \\
&27057s^2t^2 + 48792st^3 + 18477t^4 + 468s^3 + 8856s^2t + 29907st^2 + 18990t^3 + 1296s^2 + \\
&10854st + 12798t^2 + 1782s + 5184t + 972)(s + 3)(t + 1)^2.
\end{aligned}
$$

$$
D_C^U|_{q=2} = 6(t^2 + 2t + 2)(t + 2)^4(t + 1)^2.
$$

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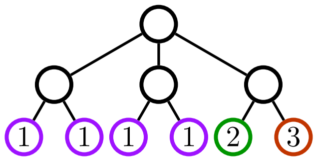

$$
\begin{aligned}
Z_C \;&= 3\lambda^7 + \lambda^6(q-3) + 2\lambda^6 + 3\lambda^5(q-3) + \lambda^4(q-3)(q-4) + 6\lambda^5 + 19\lambda^4(q-3) + \\
&10\lambda^3(q-3)(q-4) + 2\lambda^2(q-3)(q-4)(q-5) + 24\lambda^4 + 35\lambda^3(q-3) + \\
&21\lambda^2(q-3)(q-4) + 5\lambda(q-3)(q-4)(q-5) + (q-3)(q-4)(q-5)(q-6) + 14\lambda^3 + \\
&41\lambda^2(q-3) + 32\lambda(q-3)(q-4) + 11(q-3)(q-4)(q-5) + 12\lambda^2 + 49\lambda(q-3) + \\
&33(q-3)(q-4) + 18\lambda + 27q - 79.
\end{aligned}
$$

$$
\begin{aligned}
2Z_CU_C^v \;&= 7\lambda^7 + 2\lambda^6(q-3) + 2\lambda^6 + \lambda^5(q-3) + 8\lambda^5 + 20\lambda^4(q-3) + 6\lambda^3(q-3)(q-4) + 30\lambda^4 + \\
&31\lambda^3(q-3) + 12\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + 16\lambda^3 + 32\lambda^2(q-3) + \\
&18\lambda(q-3)(q-4) + 10\lambda^2 + 25\lambda(q-3) + 8\lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_CU_C^N \;&= 21\lambda^7 + 6\lambda^6(q-3) + 12\lambda^6 + 15\lambda^5(q-3) + 4\lambda^4(q-3)(q-4) + 30\lambda^5 + 76\lambda^4(q-3) + \\
&30\lambda^3(q-3)(q-4) + 4\lambda^2(q-3)(q-4)(q-5) + 96\lambda^4 + 105\lambda^3(q-3) + \\
&42\lambda^2(q-3)(q-4) + 5\lambda(q-3)(q-4)(q-5) + 42\lambda^3 + 82\lambda^2(q-3) + \\
&32\lambda(q-3)(q-4) + 24\lambda^2 + 49\lambda(q-3) + 18\lambda.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C \;&= 2(6r^7t^{14} + 88r^7t^{13} + 58r^6t^{14} + 600r^7t^{12} + 902r^6t^{13} + 242r^5t^{14} + 2520r^7t^{11} + 6522r^6t^{12} + \\
&3986r^5t^{13} + 568r^4t^{14} + 7282r^7t^{10} + 29044r^6t^{11} + 30502r^5t^{12} + 9882r^4t^{13} + 817r^3t^{14} + \\
&15312r^7t^9 + 88950r^6t^{10} + 143666r^5t^{11} + 79780r^4t^{12} + 14927r^3t^{13} + 729r^2t^{14} + \\
&24156r^7t^8 + 198108r^6t^9 + 465130r^5t^{10} + 396172r^4t^{11} + 126492r^3t^{12} + 13834r^2t^{13} + \\
&378rt^{14} + 29040r^7t^7 + 330792r^6t^8 + 1094660r^5t^9 + 1351836r^4t^{10} + 659334r^3t^{11} + \\
&122062r^2t^{12} + 7328rt^{13} + 88t^{14} + 26730r^7t^6 + 420576r^6t^7 + 1930768r^5t^8 + 3352802r^4t^9 + \\
&2362335r^3t^{10} + 663832r^2t^{11} + 66585rt^{12} + 1712t^{13} + 18744r^7t^5 + 409086r^6t^6 + \\
&2592356r^5t^7 + 6232646r^4t^8 + 6155251r^3t^9 + 2485939r^2t^{10} + 374998rt^{11} + 15834t^{12} + \\
&9856r^7t^4 + 302902r^6t^5 + 2662222r^5t^6 + 8821590r^4t^7 + 12028550r^3t^8 + 6780484r^2t^9 + \\
&1460034rt^{10} + 91644t^{11} + 3768r^7t^3 + 168050r^6t^4 + 2080858r^5t^5 + 9553402r^4t^6 + \\
&17911252r^3t^7 + 13890298r^2t^8 + 4152645rt^9 + 369072t^{10} + 990r^7t^2 + 67740r^6t^3 + \\
&1218558r^5t^4 + 7878116r^4t^5 + 20425384r^3t^6 + 21712632r^2t^7 + 8891904rt^8 + 1090674t^9 + \\
&160r^7t + 18754r^6t^2 + 518442r^5t^3 + 4870194r^4t^4 + 17755560r^3t^5 + 26030328r^2t^6 + \\
&14558202rt^7 + 2434536t^8 + 12r^7 + 3192r^6t + 151494r^5t^2 + 2188854r^4t^3 + 11585304r^3t^4 + \\
&23827338r^2t^5 + 18316728rt^6 + 4166100t^7 + 252r^6 + 27216r^5t + 676170r^4t^2 + 5503788r^3t^3 + \\
&16402338r^2t^4 + 17633430rt^5 + 5491800t^6 + 2268r^5 + 128520r^4t + 1800090r^3t^2 + \\
&8239320r^2t^3 + 12797514rt^4 + 5553036t^5 + 11340r^4 + 362880r^3t + 2856870r^2t^2 + \\
&6797682rt^3 + 4245210t^4 + 34020r^3 + 612360r^2t + 2501442rt^2 + 2383830t^3 + 61236r^2 + \\
&571536rt + 931662t^2 + 61236r + 227448t + 26244)(rt^2 + 2rt + t^2 + r + 3t + 3)(t + 1)t.
\end{aligned}
$$

$$D_C^U = (6s^5t^{10}+60s^5t^9+38s^4t^{10}+270s^5t^8+409s^4t^9+94s^3t^{10}+720s^5t^7+1987s^4t^8+1085s^3t^9+112s^2t^{10}+1260s^5t^6+5733s^4t^7+5673s^3t^8+1379s^2t^9+62st^{10}+1512s^5t^5+10871s^4t^6+17675s^3t^7+7739s^2t^8+814st^9+12t^{10}+1260s^5t^4+14147s^4t^5+36295s^3t^6+26036s^2t^7+4908st^8+168t^9+720s^5t^3+12789s^4t^4+51267s^3t^5+58010s^2t^6+17904st^7+1092t^8+270s^5t^2+7927s^4t^3+50395s^3t^4+89222s^2t^5+43578st^6+4376t^7+60s^5t+3223s^4t^2+34013s^3t^3+95752s^2t^4+73556st^5+11860t^6+6s^5+776s^4t+15075s^3t^2+70719s^2t^3+86848st^4+22398t^5+84s^4+3960s^3t+34383s^2t^2+70707st^3+29568t^4+468s^3+9936s^2t+38007st^2+26874t^3+1296s^2+12204st+16146t^2+1782s+5832t+972)(s+3)(t+1).$$

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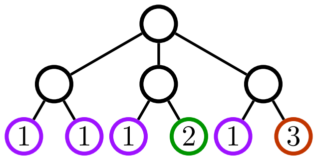


$$Z_C = \lambda^7+2\lambda^6+2\lambda^5(q-3)+9\lambda^5+14\lambda^4(q-3)+7\lambda^3(q-3)(q-4)+\lambda^2(q-3)(q-4)(q-5)+16\lambda^4+36\lambda^3(q-3)+24\lambda^2(q-3)(q-4)+7\lambda(q-3)(q-4)(q-5)+(q-3)(q-4)(q-5)(q-6)+23\lambda^3+56\lambda^2(q-3)+39\lambda(q-3)(q-4)+10(q-3)(q-4)(q-5)+18\lambda^2+47\lambda(q-3)+27(q-3)(q-4)+10\lambda+20q-58.$$

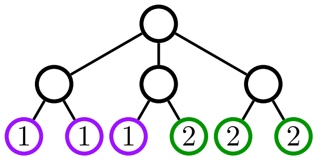
$$2Z_CU_C^v = 3\lambda^7+4\lambda^6+4\lambda^5(q-3)+13\lambda^5+12\lambda^4(q-3)+3\lambda^3(q-3)(q-4)+22\lambda^4+36\lambda^3(q-3)+18\lambda^2(q-3)(q-4)+3\lambda(q-3)(q-4)(q-5)+23\lambda^3+44\lambda^2(q-3)+15\lambda(q-3)(q-4)+14\lambda^2+15\lambda(q-3)+2\lambda.$$

$$6Z_CU_C^N = 7\lambda^7+12\lambda^6+10\lambda^5(q-3)+45\lambda^5+56\lambda^4(q-3)+21\lambda^3(q-3)(q-4)+2\lambda^2(q-3)(q-4)(q-5)+64\lambda^4+108\lambda^3(q-3)+48\lambda^2(q-3)(q-4)+7\lambda(q-3)(q-4)(q-5)+69\lambda^3+112\lambda^2(q-3)+39\lambda(q-3)(q-4)+36\lambda^2+47\lambda(q-3)+10\lambda.$$

$$\tilde{S}_C = 2(4r^8t^{14}+56r^8t^{13}+40r^7t^{14}+366r^8t^{12}+600r^7t^{13}+176r^6t^{14}+1480r^8t^{11}+4199r^7t^{12}+2816r^6t^{13}+445r^5t^{14}+4136r^8t^{10}+18169r^7t^{11}+21018r^6t^{12}+7564r^5t^{13}+706r^4t^{14}+8448r^8t^9+54288r^7t^{10}+96982r^6t^{11}+59987r^5t^{12}+12712r^4t^{13}+718r^3t^{14}+13002r^8t^8+118446r^7t^9+308972r^6t^{10}+294200r^5t^{11}+106786r^4t^{12}+13669r^3t^{13}+455r^2t^{14}+15312r^8t^7+194514r^7t^8+718618r^6t^9+996614r^5t^{10}+554989r^4t^{11}+121337r^3t^{12}+9155r^2t^{13}+162rt^{14}+13860r^8t^6+244134r^7t^7+1257620r^6t^8+2465659r^5t^9+1993742r^4t^{10}+666580r^3t^{11}+85773r^2t^{12}+3466rt^{13}+24t^{14}+9592r^8t^5+235212r^7t^6+1681324r^6t^7+4591516r^5t^8+5235414r^4t^9+2533375r^3t^{10}+497251r^2t^{11}+34343rt^{12}+560t^{13}+4994r^8t^4+173032r^7t^5+1724480r^6t^6+6533287r^5t^7+10357086r^4t^8+7046204r^3t^9+1995702r^2t^{10}+210118rt^{11}+5922t^{12}+1896r^8t^3+95631r^7t^4+1349604r^6t^5+7132877r^5t^6+15669127r^4t^7+14783937r^3t^8+5869103r^2t^9+889848rt^{10}+38352t^{11}+496r^8t^2+38489r^7t^3+792922r^6t^4+5942151r^5t^5+18203048r^4t^6+23753970r^3t^7+13040976r^2t^8+2764029rt^9+171504t^{10}+80r^8t+10660r^7t^2+338974r^6t^3+3715869r^5t^4+16146918r^4t^5+29346600r^3t^6+22228443r^2t^7+6497001rt^8+562590t^9+6r^8+1818r^7t+99636r^6t^2+1690497r^5t^3+10758312r^4t^4+27720468r^3t^5+29185326r^2t^6+11737197rt^7+1398330t^8+144r^7+18018r^6t+528660r^5t^2+5217750r^4t^3+19693881r^3t^4+29352942r^2t^5+16367319rt^6+2676078t^7+1512r^6+101682r^5t+1740420r^4t^2+10198683r^3t^3+22247946r^2t^4+17522973rt^5+3961710t^6+9072r^5+357210r^4t+3637548r^3t^2+12318642r^2t^3+14173947rt^4+4513968t^5+34020r^4+799470r^3t+4709340r^2t^2+8400267rt^3+3897234t^4+81648r^3+1112454r^2t+3449628rt^2+2474226t^3+122472r^2+879174rt+1093500t^2+104976r+301806t+39366)(t+1)^3t.$$

$$
\begin{aligned}
D_C^U &= (6s^5t^9 + 54s^5t^8 + 36s^4t^9 + 216s^5t^7 + 361s^4t^8 + 85s^3t^9 + 504s^5t^6 + 1600s^4t^7 + \\
&945s^3t^8 + 99s^2t^9 + 756s^5t^5 + 4119s^4t^6 + 4620s^3t^7 + 1208s^2t^8 + 56st^9 + 756s^5t^4 + \\
&6794s^4t^5 + 13079s^3t^6 + 6476s^2t^7 + 750st^8 + 12t^9 + 504s^5t^3 + 7451s^4t^4 + 23685s^3t^5 + \\
&20091s^2t^6 + 4392st^7 + 180t^8 + 216s^5t^2 + 5436s^4t^3 + 28503s^3t^4 + 39881s^2t^5 + \\
&14873st^6 + 1152t^7 + 54s^5t + 2545s^4t^2 + 22822s^3t^3 + 52676s^2t^4 + 32282st^5 + 4238t^6 + \\
&6s^5 + 694s^4t + 11733s^3t^2 + 46404s^2t^3 + 46778st^4 + 10008t^5 + 84s^4 + 3516s^3t + \\
&26337s^2t^2 + 45426st^3 + 15864t^4 + 468s^3 + 8748s^2t + 28611st^2 + 16992t^3 + 1296s^2 + \\
&10638st + 11934t^2 + 1782s + 5022t + 972)(s + 3)(t + 1)^2.
\end{aligned}
$$

Local view 7 of 35

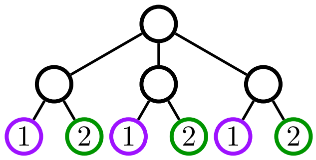


$$
\begin{aligned}
Z_C &= 2\lambda^7 + 4\lambda^6 + 7\lambda^5(q-2) + \lambda^4(q-2)(q-3) + 2\lambda^5 + 12\lambda^4(q-2) + 10\lambda^3(q-2)(q-3) + \\
&2\lambda^2(q-2)(q-3)(q-4) + 2\lambda^4 + 15\lambda^3(q-2) + 15\lambda^2(q-2)(q-3) + \\
&5\lambda(q-2)(q-3)(q-4) + (q-2)(q-3)(q-4)(q-5) + 4\lambda^3 + 16\lambda^2(q-2) + \\
&17\lambda(q-2)(q-3) + 7(q-2)(q-3)(q-4) + 2\lambda^2 + 11\lambda(q-2) + 12(q-2)(q-3) + 4q - 8. \\
2Z_CU_C^v &= 4\lambda^7 + 8\lambda^6 + 7\lambda^5(q-2) + 4\lambda^5 + 16\lambda^4(q-2) + 6\lambda^3(q-2)(q-3) + 2\lambda^4 + 15\lambda^3(q-2) + \\
&12\lambda^2(q-2)(q-3) + 3\lambda(q-2)(q-3)(q-4) + 4\lambda^3 + 14\lambda^2(q-2) + 9\lambda(q-2)(q-3) + \\
&2\lambda^2 + 5\lambda(q-2). \\
6Z_CU_C^N &= 14\lambda^7 + 24\lambda^6 + 35\lambda^5(q-2) + 4\lambda^4(q-2)(q-3) + 10\lambda^5 + 48\lambda^4(q-2) + \\
&30\lambda^3(q-2)(q-3) + 4\lambda^2(q-2)(q-3)(q-4) + 8\lambda^4 + 45\lambda^3(q-2) + 30\lambda^2(q-2)(q-3) + \\
&5\lambda(q-2)(q-3)(q-4) + 12\lambda^3 + 32\lambda^2(q-2) + 17\lambda(q-2)(q-3) + 4\lambda^2 + 11\lambda(q-2). \\
\tilde{S}_C &= 2(4r^7t^{14} + 60r^7t^{13} + 38r^6t^{14} + 420r^7t^{12} + 602r^6t^{13} + 154r^5t^{14} + 1816r^7t^{11} + 4460r^6t^{12} + \\
&2582r^5t^{13} + 347r^4t^{14} + 5412r^7t^{10} + 20440r^6t^{11} + 20256r^5t^{12} + 6160r^4t^{13} + 472r^3t^{14} + \\
&11748r^7t^9 + 64620r^6t^{10} + 98330r^5t^{11} + 51139r^4t^{12} + 8865r^3t^{13} + 390r^2t^{14} + 19140r^7t^8 + \\
&148848r^6t^9 + 329320r^5t^{10} + 262580r^4t^{11} + 77743r^3t^{12} + 7733r^2t^{13} + 183rt^{14} + \\
&23760r^7t^7 + 257292r^6t^8 + 803624r^5t^9 + 929933r^4t^{10} + 421286r^3t^{11} + 71408r^2t^{12} + \\
&3808rt^{13} + 38t^{14} + 22572r^7t^6 + 338712r^6t^7 + 1471540r^5t^8 + 2399420r^4t^9 + 1573944r^3t^{10} + \\
&407025r^2t^{11} + 36814rt^{12} + 820t^{13} + 16324r^7t^5 + 340998r^6t^6 + 2051924r^5t^7 + \\
&4645999r^4t^8 + 4284089r^3t^9 + 1599239r^2t^{10} + 219669rt^{11} + 8228t^{12} + 8844r^7t^4 + \\
&261122r^6t^5 + 2187730r^5t^6 + 6851974r^4t^7 + 8753443r^3t^8 + 4579412r^2t^9 + 904252rt^{10} + \\
&51094t^{11} + 3480r^7t^3 + 149660r^6t^4 + 1773798r^5t^5 + 7729240r^4t^6 + 13630488r^3t^7 + \\
&9850467r^2t^8 + 2715769rt^9 + 219468t^{10} + 940r^7t^2 + 62240r^6t^3 + 1076160r^5t^4 + \\
&6633028r^4t^5 + 16247086r^3t^6 + 16163688r^2t^7 + 6134631rt^8 + 689322t^9 + 156r^7t + \\
&17752r^6t^2 + 473618r^5t^3 + 4261338r^4t^4 + 14747784r^3t^5 + 20328300r^2t^6 + 10586070rt^7 + \\
&1631466t^8 + 12r^7 + 3108r^6t + 142908r^5t^2 + 1986834r^4t^3 + 10033272r^3t^4 + 19499346r^2t^5 + \\
&14024340rt^6 + 2954988t^7 + 252r^6 + 26460r^5t + 635400r^4t^2 + 4960188r^3t^3 + \\
&14044428r^2t^4 + 14199246rt^5 + 4117014t^6 + 2268r^5 + 124740r^4t + 1684260r^3t^2 + \\
&7366356r^2t^3 + 10821276rt^4 + 4394088t^5 + 11340r^4 + 351540r^3t + 2660040r^2t^2 + \\
&6023322rt^3 + 3540510t^4 + 34020r^3 + 591948r^2t + 2316276rt^2 + 2091258t^3 + 61236r^2 + \\
&551124rt + 857304t^2 + 61236r + 218700t + 26244)(rt^2 + 2rt + t^2 + r + 3t + 3)(t + 1)t. \\
\tilde{S}_C|_{q=2} &= 4(t^4 + 3t^3 + 6t^2 + 6t + 3)(t + 2)^9(t + 1)t.
\end{aligned}
$$

$$
\begin{aligned}
D_C^U \quad &= (6s^4t^8 + 48s^4t^7 + 34s^3t^8 + 168s^4t^6 + 293s^3t^7 + 72s^2t^8 + 336s^4t^5 + 1107s^3t^6 + \\
&666s^2t^7 + 68st^8 + 420s^4t^4 + 2396s^3t^5 + 2697s^2t^6 + 667st^7 + 24t^8 + 336s^4t^3 + \\
&3250s^3t^4 + 6267s^2t^5 + 2862st^6 + 246t^7 + 168s^4t^2 + 2829s^3t^3 + 9165s^2t^4 + 7073st^5 + \\
&1104t^6 + 48s^4t + 1543s^3t^2 + 8655s^2t^3 + 11077st^4 + 2870t^5 + 6s^4 + 482s^3t + 5160s^2t^2 + \\
&11310st^3 + 4774t^4 + 66s^3 + 1776s^2t + 7383st^2 + 5250t^3 + 270s^2 + 2826st + 3762t^2 + \\
&486s + 1620t + 324)(st^2 + 2st + t^2 + s + 3t + 3)(s + 3)(t + 1).
\end{aligned}
$$

$$D_C^U|_{q=2} = 4(2t^2 + 4t + 3)(t + 2)^4(t + 1).$$

Local view 8 of 35 (named C_2)



$$
\begin{aligned}
Z_C \quad &= 2\lambda^6 + 6\lambda^5 + 6\lambda^4(q-2) + 6\lambda^4 + 39\lambda^3(q-2) + 33\lambda^2(q-2)(q-3) + \\
&9\lambda(q-2)(q-3)(q-4) + (q-2)(q-3)(q-4)(q-5) + 2\lambda^3 + 12\lambda^2(q-2) + \\
&15\lambda(q-2)(q-3) + 5(q-2)(q-3)(q-4) + 6\lambda(q-2) + 7(q-2)(q-3) + 2q - 4.
\end{aligned}
$$

$$
\begin{aligned}
2Z_CU_C^v &= 6\lambda^6 + 12\lambda^5 + 12\lambda^4(q-2) + 6\lambda^4 + 39\lambda^3(q-2) + 24\lambda^2(q-2)(q-3) + \\
&3\lambda(q-2)(q-3)(q-4) + 6\lambda^2(q-2) + 3\lambda(q-2)(q-3).
\end{aligned}
$$

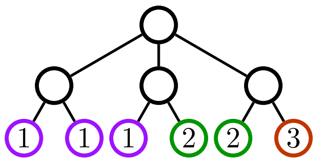
$$
\begin{aligned}
6Z_CU_C^N &= 12\lambda^6 + 30\lambda^5 + 24\lambda^4(q-2) + 24\lambda^4 + 117\lambda^3(q-2) + 66\lambda^2(q-2)(q-3) + \\
&9\lambda(q-2)(q-3)(q-4) + 6\lambda^3 + 24\lambda^2(q-2) + 15\lambda(q-2)(q-3) + 6\lambda(q-2).
\end{aligned}
$$

$$\tilde{S}_C = 0.$$

$$
\begin{aligned}
D_C^U \quad &= 3(2s^5t^8 + 16s^5t^7 + 12s^4t^8 + 56s^5t^6 + 109s^4t^7 + 28s^3t^8 + 112s^5t^5 + 429s^4t^6 + 289s^3t^7 + \\
&32s^2t^8 + 140s^5t^4 + 958s^4t^5 + 1277s^3t^6 + 372s^2t^7 + 18st^8 + 112s^5t^3 + 1330s^4t^4 + 3178s^3t^5 + \\
&1840s^2t^6 + 232st^7 + 4t^8 + 56s^5t^2 + 1177s^4t^3 + 4898s^3t^4 + 5097s^2t^5 + 1280st^6 + 56t^7 + \\
&16s^5t + 649s^4t^2 + 4805s^3t^3 + 8715s^2t^4 + 3943st^5 + 344t^6 + 2s^5 + 204s^4t + 2937s^3t^2 + \\
&9487s^2t^3 + 7471st^4 + 1178t^5 + 28s^4 + 1024s^3t + 6457s^2t^2 + 9019st^3 + 2466t^4 + 156s^3 + \\
&2520s^2t + 6855st^2 + 3288t^3 + 432s^2 + 3024st + 2790t^2 + 594s + 1404t + 324)(s+3)(t+1)^3.
\end{aligned}
$$

$$D_C^U|_{q=2} = 12(t + 2)^4(t + 1)^3.$$

Local view 9 of 35

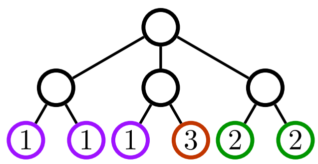


$$
\begin{aligned}
Z_C \quad &= 4\lambda^6 + \lambda^5(q-3) + 9\lambda^5 + 17\lambda^4(q-3) + 7\lambda^3(q-3)(q-4) + \lambda^2(q-3)(q-4)(q-5) + \\
&15\lambda^4 + 34\lambda^3(q-3) + 24\lambda^2(q-3)(q-4) + 7\lambda(q-3)(q-4)(q-5) + \\
&(q-3)(q-4)(q-5)(q-6) + 22\lambda^3 + 54\lambda^2(q-3) + 39\lambda(q-3)(q-4) + \\
&10(q-3)(q-4)(q-5) + 18\lambda^2 + 50\lambda(q-3) + 27(q-3)(q-4) + 12\lambda + 19q - 56.
\end{aligned}
$$

$$
\begin{aligned}
2Z_CU_C^v &= 9\lambda^6 + 2\lambda^5(q-3) + 13\lambda^5 + 16\lambda^4(q-3) + 3\lambda^3(q-3)(q-4) + 20\lambda^4 + 36\lambda^3(q-3) + \\
&18\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + 24\lambda^3 + 40\lambda^2(q-3) + \\
&15\lambda(q-3)(q-4) + 11\lambda^2 + 17\lambda(q-3) + 4\lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_CU_C^N &= 24\lambda^6+5\lambda^5(q-3)+45\lambda^5+68\lambda^4(q-3)+21\lambda^3(q-3)(q-4)+2\lambda^2(q-3)(q-4)(q-5)+\\
&\quad 60\lambda^4+102\lambda^3(q-3)+48\lambda^2(q-3)(q-4)+7\lambda(q-3)(q-4)(q-5)+66\lambda^3+\\
&\quad 108\lambda^2(q-3)+39\lambda(q-3)(q-4)+36\lambda^2+50\lambda(q-3)+12\lambda.\\
\tilde{S}_C &= (12r^8t^{15}+184r^8t^{14}+122r^7t^{15}+1320r^8t^{13}+1992r^7t^{14}+544r^6t^{15}+5876r^8t^{12}+\\
&\quad 15210r^7t^{13}+9440r^6t^{14}+1393r^5t^{15}+18148r^8t^{11}+72024r^7t^{12}+76538r^6t^{13}+\\
&\quad 25621r^5t^{14}+2245r^4t^{15}+41184r^8t^{10}+236472r^7t^{11}+384604r^6t^{12}+219969r^5t^{13}+\\
&\quad 43624r^4t^{14}+2338r^3t^{15}+70928r^8t^9+570044r^7t^{10}+1339354r^6t^{11}+1169849r^5t^{12}+\\
&\quad 395367r^4t^{13}+47785r^3t^{14}+1542r^2t^{15}+94380r^8t^8+1041984r^7t^9+3423180r^6t^{10}+\\
&\quad 4310637r^5t^{11}+2218972r^4t^{12}+455499r^3t^{13}+32926r^2t^{14}+590rt^{15}+97812r^8t^7+\\
&\quad 1470240r^7t^8+6631728r^6t^9+11657023r^5t^{10}+8629373r^4t^{11}+2689500r^3t^{12}+\\
&\quad 328617r^2t^{13}+13048rt^{14}+100t^{15}+78936r^8t^6+1614138r^7t^7+9913608r^6t^8+\\
&\quad 23896081r^5t^9+24636774r^4t^{10}+11009206r^3t^{11}+2034488r^2t^{12}+135646rt^{13}+\\
&\quad 2272t^{14}+49192r^8t^5+1378544r^7t^6+11526036r^6t^7+37802919r^5t^8+53345249r^4t^9+\\
&\quad 33108193r^3t^{10}+8742594r^2t^{11}+877585rt^{12}+24480t^{13}+23244r^8t^4+908202r^7t^5+\\
&\quad 10419640r^6t^6+46517906r^5t^7+89191552r^4t^8+75581141r^3t^9+27633528r^2t^{10}+\\
&\quad 3949192rt^{11}+164994t^{12}+8060r^8t^3+453192r^7t^4+7262426r^6t^5+44512308r^5t^6+\\
&\quad 116070972r^4t^7+133368090r^3t^8+66387341r^2t^9+13093898rt^{10}+775944t^{11}+\\
&\quad 1936r^8t^2+165780r^7t^3+3831788r^6t^4+32840950r^5t^5+117535058r^4t^6+\\
&\quad 183375982r^3t^7+123452570r^2t^8+33050346rt^9+2694588t^{10}+288r^8t+41964r^7t^2+\\
&\quad 1481170r^6t^3+18341500r^5t^4+91824204r^4t^5+196417116r^3t^6+179152290r^2t^7+\\
&\quad 64680591rt^8+7136856t^9+20r^8+6572r^7t+395932r^6t^2+7504140r^5t^3+\\
&\quad 54335520r^4t^4+162509274r^3t^5+202857912r^2t^6+98961318rt^7+14682726t^8+\\
&\quad 480r^7+65436r^6t+2122812r^5t^2+23566050r^4t^3+101961792r^3t^4+177733062r^2t^5+\\
&\quad 118376208rt^6+23662314t^7+5040r^6+371196r^5t+7070220r^4t^2+46945980r^3t^3+\\
&\quad 118302876r^2t^4+109802790rt^5+29879766t^6+30240r^5+1311660r^4t+14969988r^3t^2+\\
&\quad 57897990r^2t^3+77562684rt^4+29332044t^5+113400r^4+2955204r^3t+19664532r^2t^2+\\
&\quad 40391460rt^3+21991500t^4+272160r^3+4143636r^2t+14641236rt^2+12196170t^3+\\
&\quad 408240r^2+3303828rt+4726836t^2+349920r+1145988t+131220)(t+1)^2t.\\
D_C^U &= (6s^5t^9+54s^5t^8+34s^4t^9+216s^5t^7+343s^4t^8+74s^3t^9+504s^5t^6+1530s^4t^7+\\
&\quad 836s^3t^8+77s^2t^9+756s^5t^5+3965s^4t^6+4152s^3t^7+965s^2t^8+37st^9+756s^5t^4+\\
&\quad 6584s^4t^5+11941s^3t^6+5322s^2t^7+514st^8+6t^9+504s^5t^3+7269s^4t^4+21970s^3t^5+\\
&\quad 16997s^2t^6+3153st^7+96t^8+216s^5t^2+5338s^4t^3+26862s^3t^4+34739s^2t^5+\\
&\quad 11217st^6+666t^7+54s^5t+2515s^4t^2+21848s^3t^3+47241s^2t^4+25587st^5+2662t^6+\\
&\quad 6s^5+690s^4t+11405s^3t^2+42834s^2t^3+38955st^4+6826t^5+84s^4+3468s^3t+\\
&\quad 25005s^2t^2+39720st^3+11745t^4+468s^3+8532s^2t+26235st^2+13644t^3+1296s^2+\\
&\quad 10206st+10368t^2+1782s+4698t+972)(s+3)(t+1)^2.
\end{aligned}
$$

Local view 10 of 35



$$
\begin{aligned}
Z_C &= \lambda^7+6\lambda^6+6\lambda^5(q-3)+\lambda^4(q-3)(q-4)+9\lambda^5+17\lambda^4(q-3)+10\lambda^3(q-3)(q-4)+\\
&\quad 2\lambda^2(q-3)(q-4)(q-5)+13\lambda^4+33\lambda^3(q-3)+21\lambda^2(q-3)(q-4)+\\
&\quad 5\lambda(q-3)(q-4)(q-5)+(q-3)(q-4)(q-5)(q-6)+18\lambda^3+44\lambda^2(q-3)+\\
&\quad 32\lambda(q-3)(q-4)+11(q-3)(q-4)(q-5)+18\lambda^2+48\lambda(q-3)+33(q-3)(q-4)+\\
&\quad 13\lambda+27q-78.
\end{aligned}
$$

$$
\begin{aligned}
2Z_C U_C^v &= 2\lambda^7 + 9\lambda^6 + 5\lambda^5(q-3) + 16\lambda^5 + 20\lambda^4(q-3) + 6\lambda^3(q-3)(q-4) + 16\lambda^4 + \\
&\quad 27\lambda^3(q-3) + 12\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + 15\lambda^3 + 34\lambda^2(q-3) + \\
&\quad 18\lambda(q-3)(q-4) + 17\lambda^2 + 25\lambda(q-3) + 6\lambda. \\
6Z_C U_C^N &= 7\lambda^7 + 36\lambda^6 + 30\lambda^5(q-3) + 4\lambda^4(q-3)(q-4) + 45\lambda^5 + 68\lambda^4(q-3) + \\
&\quad 30\lambda^3(q-3)(q-4) + 4\lambda^2(q-3)(q-4)(q-5) + 52\lambda^4 + 99\lambda^3(q-3) + \\
&\quad 42\lambda^2(q-3)(q-4) + 5\lambda(q-3)(q-4)(q-5) + 54\lambda^3 + 88\lambda^2(q-3) + \\
&\quad 32\lambda(q-3)(q-4) + 36\lambda^2 + 48\lambda(q-3) + 13\lambda. \\
\tilde{S}_C &= (12r^8t^{16} + 204r^8t^{15} + 124r^7t^{16} + 1628r^8t^{14} + 2224r^7t^{15} + 560r^6t^{16} + 8092r^8t^{13} + \\
&\quad 18744r^7t^{14} + 10598r^6t^{15} + 1449r^5t^{16} + 28028r^8t^{12} + 98452r^7t^{13} + 94258r^6t^{14} + \\
&\quad 28902r^5t^{15} + 2360r^4t^{16} + 71708r^8t^{11} + 360432r^7t^{12} + 522470r^6t^{13} + 270832r^5t^{14} + \\
&\quad 49475r^4t^{15} + 2491r^3t^{16} + 140140r^8t^{10} + 974620r^7t^{11} + 2018518r^6t^{12} + 1581392r^5t^{13} + \\
&\quad 487190r^4t^{14} + 54609r^3t^{15} + 1672r^2t^{16} + 213356r^8t^9 + 2012516r^7t^{10} + 5759548r^6t^{11} + \\
&\quad 6435446r^5t^{12} + 2989397r^4t^{13} + 562841r^3t^{14} + 38050r^2t^{15} + 652rt^{16} + 255684r^8t^8 + \\
&\quad 3235848r^7t^9 + 12548376r^6t^{10} + 19342576r^5t^{11} + 12786028r^4t^{12} + 3617707r^3t^{13} + \\
&\quad 408294r^2t^{14} + 15296rt^{15} + 112t^{16} + 241956r^8t^7 + 4092924r^7t^8 + 21284460r^6t^9 + \\
&\quad 44395064r^5t^{10} + 40401943r^4t^{11} + 16220843r^3t^{12} + 2738395r^2t^{13} + 169957rt^{14} + \\
&\quad 2704t^{15} + 180180r^8t^6 + 4085224r^7t^7 + 28395276r^6t^8 + 79339936r^5t^9 + 97524496r^4t^{10} + \\
&\quad 53770313r^3t^{11} + 12834156r^2t^{12} + 1184677rt^{13} + 31002t^{14} + 104468r^8t^5 + \\
&\quad 3206368r^7t^6 + 29885390r^6t^7 + 111539609r^5t^8 + 183382229r^4t^9 + 136262671r^3t^{10} + \\
&\quad 44532146r^2t^{11} + 5786935rt^{12} + 223846t^{13} + 46228r^8t^4 + 1957860r^7t^5 + \\
&\quad 24726482r^6t^6 + 123728950r^5t^7 + 271397042r^4t^8 + 269204933r^3t^9 + 118267914r^2t^{10} + \\
&\quad 20973866rt^{11} + 1136790t^{12} + 15092r^8t^3 + 911704r^7t^4 + 15910942r^6t^5 + \\
&\quad 107913884r^5t^6 + 317106478r^4t^7 + 418949958r^3t^8 + 245144867r^2t^9 + 58282811rt^{10} + \\
&\quad 4296036t^{11} + 3428r^8t^2 + 312972r^7t^3 + 7805102r^6t^4 + 73211456r^5t^5 + 291496218r^4t^6 + \\
&\quad 515224702r^3t^7 + 400729520r^2t^8 + 126590649rt^9 + 12475590t^{10} + 484r^8t + 74692r^7t^2 + \\
&\quad 2821456r^6t^3 + 37868944r^5t^4 + 208559766r^4t^5 + 498997944r^3t^6 + 518288724r^2t^7 + \\
&\quad 217141005rt^8 + 28367550t^9 + 32r^8 + 11072r^7t + 708772r^6t^2 + 14435652r^5t^3 + \\
&\quad 113843214r^4t^4 + 376569828r^3t^5 + 528629382r^2t^6 + 295127820rt^7 + 51019974t^8 + \\
&\quad 768r^7 + 110544r^6t + 3824244r^5t^2 + 45825120r^4t^3 + 217058472r^3t^4 + 420758442r^2t^5 + \\
&\quad 316844460rt^6 + 72824292t^7 + 8064r^6 + 628992r^5t + 12826620r^4t^2 + 92375316r^3t^3 + \\
&\quad 256228218r^2t^4 + 265946328rt^5 + 82254042t^6 + 48384r^5 + 2230200r^4t + 27370764r^3t^2 + \\
&\quad 115416576r^2t^3 + 171153648rt^4 + 72789678t^5 + 181440r^4 + 5044032r^3t + 36268236r^2t^2 + \\
&\quad 81676188rt^3 + 49514166t^4 + 435456r^3 + 7103376r^2t + 27267516rt^2 + 25051356t^3 + \\
&\quad 653184r^2 + 5692032rt + 8899632t^2 + 559872r + 1985796t + 209952)(t+1)t. \\
D_C^U &= (6s^5t^{10} + 60s^5t^9 + 38s^4t^{10} + 270s^5t^8 + 407s^4t^9 + 96s^3t^{10} + 720s^5t^7 + 1969s^4t^8 + 1093s^3t^9 + \\
&\quad 121s^2t^{10} + 1260s^5t^6 + 5663s^4t^7 + 5639s^3t^8 + 1450s^2t^9 + 75st^{10} + 1512s^5t^5 + 10717s^4t^6 + \\
&\quad 17367s^3t^7 + 7901s^2t^8 + 944st^9 + 18t^{10} + 1260s^5t^4 + 13937s^4t^5 + 35357s^3t^6 + 25842s^2t^7 + \\
&\quad 5395st^8 + 240t^9 + 720s^5t^3 + 12607s^4t^4 + 49699s^3t^5 + 56251s^2t^6 + 18571st^7 + 1434t^8 + \\
&\quad 270s^5t^2 + 7829s^4t^3 + 48813s^3t^4 + 85169s^2t^5 + 42861st^6 + 5142t^7 + 60s^5t + 3193s^4t^2 + \\
&\quad 33049s^3t^3 + 90800s^2t^4 + 69516st^5 + 12422t^6 + 6s^5 + 772s^4t + 14747s^3t^2 + 67239s^2t^3 + \\
&\quad 80330st^4 + 21349t^5 + 84s^4 + 3912s^3t + 33051s^2t^2 + 65271st^3 + 26610t^4 + 468s^3 + 9720s^2t + \\
&\quad 35631st^2 + 23796t^3 + 1296s^2 + 11772st + 14580t^2 + 1782s + 5508t + 972)(s+3)(t+1).
\end{aligned}
$$

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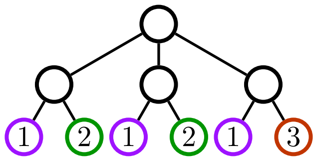

$$
\begin{aligned}
Z_C &= \lambda^6 + 6\lambda^5 + 4\lambda^4(q-3) + 21\lambda^4 + 47\lambda^3(q-3) + 33\lambda^2(q-3)(q-4) + \\
&\quad 9\lambda(q-3)(q-4)(q-5) + (q-3)(q-4)(q-5)(q-6) + 25\lambda^3 + 66\lambda^2(q-3) + \\
&\quad 42\lambda(q-3)(q-4) + 9(q-3)(q-4)(q-5) + 21\lambda^2 + 44\lambda(q-3) + 22(q-3)(q-4) + \\
&\quad 6\lambda + 14q - 41.
\end{aligned}
$$

$$
\begin{aligned}
2Z_C U_C^v &= 3\lambda^6 + 13\lambda^5 + 8\lambda^4(q-3) + 32\lambda^4 + 51\lambda^3(q-3) + 24\lambda^2(q-3)(q-4) + \\
&\quad 3\lambda(q-3)(q-4)(q-5) + 21\lambda^3 + 42\lambda^2(q-3) + 12\lambda(q-3)(q-4) + 11\lambda^2 + 10\lambda(q-3) + \lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_C U_C^N &= 6\lambda^6 + 30\lambda^5 + 16\lambda^4(q-3) + 84\lambda^4 + 141\lambda^3(q-3) + 66\lambda^2(q-3)(q-4) + \\
&\quad 9\lambda(q-3)(q-4)(q-5) + 75\lambda^3 + 132\lambda^2(q-3) + 42\lambda(q-3)(q-4) + 42\lambda^2 + \\
&\quad 44\lambda(q-3) + 6\lambda.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C &= (2r^4t^8 + 16r^4t^7 + 12r^3t^8 + 56r^4t^6 + 104r^3t^7 + 28r^2t^8 + 112r^4t^5 + 396r^3t^6 + 260r^2t^7 + \\
&\quad 31rt^8 + 140r^4t^4 + 864r^3t^5 + 1064r^2t^6 + 301rt^7 + 14t^8 + 112r^4t^3 + 1180r^3t^4 + \\
&\quad 2504r^2t^5 + 1298rt^6 + 138t^7 + 56r^4t^2 + 1032r^3t^3 + 3704r^2t^4 + 3246rt^5 + 612t^6 + \\
&\quad 16r^4t + 564r^3t^2 + 3524r^2t^3 + 5146rt^4 + 1596t^5 + 2r^4 + 176r^3t + 2104r^2t^2 + 5292rt^3 + \\
&\quad 2676t^4 + 24r^3 + 720r^2t + 3444rt^2 + 2952t^3 + 108r^2 + 1296rt + 2088t^2 + 216r + 864t + \\
&\quad 162)(4r^4t^6 + 24r^4t^5 + 15r^3t^6 + 60r^4t^4 + 120r^3t^5 + 21r^2t^6 + 80r^4t^3 + 378r^3t^4 + \\
&\quad 210r^2t^5 + 12rt^6 + 60r^4t^2 + 612r^3t^3 + 828r^2t^4 + 150rt^5 + 2t^6 + 24r^4t + 543r^3t^2 + \\
&\quad 1650r^2t^3 + 741rt^4 + 36t^5 + 4r^4 + 252r^3t + 1767r^2t^2 + 1836rt^3 + 225t^4 + 48r^3 + \\
&\quad 972r^2t + 2421rt^2 + 702t^3 + 216r^2 + 1620rt + 1161t^2 + 432r + 972t + 324)(t+1)^3t.
\end{aligned}
$$

$$
\begin{aligned}
D_C^U &= (6s^5t^8 + 48s^5t^7 + 32s^4t^8 + 168s^5t^6 + 299s^4t^7 + 65s^3t^8 + 336s^5t^5 + 1203s^4t^6 + 708s^3t^7 + \\
&\quad 64s^2t^8 + 420s^5t^4 + 2734s^4t^5 + 3273s^3t^6 + 793s^2t^7 + 31st^8 + 336s^5t^3 + 3850s^4t^4 + 8464s^3t^5 + \\
&\quad 4188s^2t^6 + 423st^7 + 6t^8 + 168s^5t^2 + 3447s^4t^3 + 13479s^3t^4 + 12337s^2t^5 + 2508st^6 + 90t^7 + \\
&\quad 48s^5t + 1919s^4t^2 + 13596s^3t^3 + 22309s^2t^4 + 8369st^5 + 576t^6 + 6s^5 + 608s^4t + 8507s^3t^2 + \\
&\quad 25530s^2t^3 + 17218st^4 + 2110t^5 + 84s^4 + 3024s^3t + 18147s^2t^2 + 22512st^3 + 4872t^4 + 468s^3 + \\
&\quad 7344s^2t + 18405st^2 + 7299t^3 + 1296s^2 + 8640st + 6966t^2 + 1782s + 3888t + 972)(s+3)(t+1)^3.
\end{aligned}
$$

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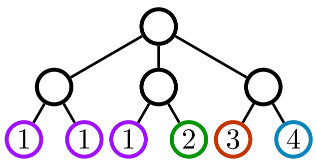

$$
\begin{aligned}
Z_C &= 3\lambda^6 + \lambda^5(q-4) + 13\lambda^5 + 16\lambda^4(q-4) + 7\lambda^3(q-4)(q-5) + \lambda^2(q-4)(q-5)(q-6) + \\
&\quad 28\lambda^4 + 52\lambda^3(q-4) + 27\lambda^2(q-4)(q-5) + 7\lambda(q-4)(q-5)(q-6) + \\
&\quad (q-4)(q-5)(q-6)(q-7) + 62\lambda^3 + 96\lambda^2(q-4) + 60\lambda(q-4)(q-5) + \\
&\quad 14(q-4)(q-5)(q-6) + 63\lambda^2 + 132\lambda(q-4) + 57(q-4)(q-5) + 69\lambda + 72q - 270.
\end{aligned}
$$

$$
\begin{aligned}
2Z_C U_C^v &= 7\lambda^6 + 2\lambda^5(q-4) + 18\lambda^5 + 14\lambda^4(q-4) + 3\lambda^3(q-4)(q-5) + 36\lambda^4 + 48\lambda^3(q-4) + \\
&\quad 18\lambda^2(q-4)(q-5) + 3\lambda(q-4)(q-5)(q-6) + 62\lambda^3 + 70\lambda^2(q-4) + \\
&\quad 24\lambda(q-4)(q-5) + 45\lambda^2 + 49\lambda(q-4) + 24\lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_CU_C^N &= 18\lambda^6+5\lambda^5(q-4)+65\lambda^5+64\lambda^4(q-4)+21\lambda^3(q-4)(q-5)+2\lambda^2(q-4)(q-5)(q-6)+\\
&\quad 112\lambda^4+156\lambda^3(q-4)+54\lambda^2(q-4)(q-5)+7\lambda(q-4)(q-5)(q-6)+186\lambda^3+\\
&\quad 192\lambda^2(q-4)+60\lambda(q-4)(q-5)+126\lambda^2+132\lambda(q-4)+69\lambda.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C &= (16r^8t^{15}+244r^8t^{14}+162r^7t^{15}+1740r^8t^{13}+2636r^7t^{14}+722r^6t^{15}+7696r^8t^{12}+\\
&\quad 20038r^7t^{13}+12492r^6t^{14}+1855r^5t^{15}+23608r^8t^{11}+94384r^7t^{12}+100890r^6t^{13}+\\
&\quad 33977r^5t^{14}+3013r^4t^{15}+53196r^8t^{10}+308024r^7t^{11}+504556r^6t^{12}+290433r^5t^{13}+\\
&\quad 58106r^4t^{14}+3178r^3t^{15}+90948r^8t^9+737640r^7t^{10}+1747296r^6t^{11}+1537093r^5t^{12}+\\
&\quad 523395r^4t^{13}+64071r^3t^{14}+2130r^2t^{15}+120120r^8t^8+1338852r^7t^9+4437864r^6t^{10}+\\
&\quad 5632945r^5t^{11}+2921010r^4t^{12}+605057r^3t^{13}+44502r^2t^{14}+826rt^{15}+123552r^8t^7+\\
&\quad 1875216r^7t^8+8538732r^6t^9+15140187r^5t^{10}+11295061r^4t^{11}+3547182r^3t^{12}+\\
&\quad 438079r^2t^{13}+17756rt^{14}+140t^{15}+98956r^8t^6+2043138r^7t^7+12671352r^6t^8+\\
&\quad 30829109r^5t^9+32052624r^4t^{10}+14430020r^3t^{11}+2687270r^2t^{12}+181280rt^{13}+\\
&\quad 3096t^{14}+61204r^8t^5+1731468r^7t^6+14620050r^6t^7+48419903r^5t^8+68948979r^4t^9+\\
&\quad 43132355r^3t^{10}+11467674r^2t^{11}+1159491rt^{12}+32700t^{13}+28704r^8t^4+\\
&\quad 1131854r^7t^5+13112924r^6t^6+59128656r^5t^7+114466310r^4t^8+97838603r^3t^9+\\
&\quad 36026236r^2t^{10}+5177882rt^{11}+217566t^{12}+9880r^8t^3+560416r^7t^4+9066690r^6t^5+\\
&\quad 56130224r^5t^6+147837930r^4t^7+171460688r^3t^8+86024723r^2t^9+17065506rt^{10}+\\
&\quad 1014972t^{11}+2356r^8t^2+203428r^7t^3+4745340r^6t^4+41074710r^5t^5+148510546r^4t^6+\\
&\quad 234009984r^3t^7+158932344r^2t^8+42835854rt^9+3505356t^{10}+348r^8t+51104r^7t^2+\\
&\quad 1819612r^6t^3+22749912r^5t^4+115062546r^4t^5+248672040r^3t^6+229006584r^2t^7+\\
&\quad 83337705rt^8+9240984t^9+24r^8+7944r^7t+482544r^6t^2+9230220r^5t^3+\\
&\quad 67506948r^4t^4+204027858r^3t^5+257309676r^2t^6+126669798rt^7+18917874t^8+\\
&\quad 576r^7+79128r^6t+2589408r^5t^2+29025540r^4t^3+126899568r^3t^4+223571502r^2t^5+\\
&\quad 150407388rt^6+30312468t^7+6048r^6+449064r^5t+8632440r^4t^2+57905820r^3t^3+\\
&\quad 147505860r^2t^4+138382506rt^5+38018970t^6+36288r^5+1587600r^4t+18296928r^3t^2+\\
&\quad 71526564r^2t^3+96890904rt^4+37032714t^5+136080r^4+3578904r^3t+24062832r^2t^2+\\
&\quad 49983156rt^3+27524124t^4+326592r^3+5021352r^2t+17939232rt^2+15119460t^3+\\
&\quad 489888r^2+4006584rt+5799924t^2+419904r+1390932t+157464)(t+1)^2t.
\end{aligned}
$$

$$
\begin{aligned}
D_C^U &= (6s^5t^9+54s^5t^8+62s^4t^9+216s^5t^7+597s^4t^8+252s^3t^9+504s^5t^6+2554s^4t^7+\\
&\quad 2592s^3t^8+502s^2t^9+756s^5t^5+6373s^4t^6+11842s^3t^7+5502s^2t^8+486st^9+756s^5t^4+\\
&\quad 10224s^4t^5+31569s^3t^6+26804s^2t^7+5673st^8+180t^9+504s^5t^3+10937s^4t^4+\\
&\quad 54156s^3t^5+76288s^2t^6+29453st^7+2250t^8+216s^5t^2+7802s^4t^3+62030s^3t^4+\\
&\quad 139968s^2t^5+89481st^6+12471t^7+54s^5t+3579s^4t^2+47454s^3t^3+171860s^2t^4+\\
&\quad 175686st^5+40473t^6+6s^5+958s^4t+23385s^3t^2+141342s^2t^3+231611st^4+85122t^5+\\
&\quad 114s^4+6736s^3t+75126s^2t^2+205364st^3+120732t^4+864s^3+23424s^2t+118272st^2+\\
&\quad 115824t^3+3264s^2+40192st+72672t^2+6144s+27136t+4608)(s+4)(t+1)^2.
\end{aligned}
$$

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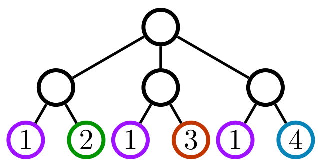

$$
\begin{aligned}
Z_C &= \lambda^6+3\lambda^5+3\lambda^4(q-4)+36\lambda^4+51\lambda^3(q-4)+33\lambda^2(q-4)(q-5)+\\
&\quad 9\lambda(q-4)(q-5)(q-6)+(q-4)(q-5)(q-6)(q-7)+58\lambda^3+126\lambda^2(q-4)+\\
&\quad 69\lambda(q-4)(q-5)+13(q-4)(q-5)(q-6)+93\lambda^2+132\lambda(q-4)+49(q-4)(q-5)+\\
&\quad 51\lambda+57q-214.
\end{aligned}
$$

$$2Z_CU_C^v = 3\lambda^6 + 6\lambda^5 + 6\lambda^4(q-4) + 60\lambda^4 + 57\lambda^3(q-4) + 24\lambda^2(q-4)(q-5) + 3\lambda(q-4)(q-5)(q-6) + 54\lambda^3 + 84\lambda^2(q-4) + 21\lambda(q-4)(q-5) + 57\lambda^2 + 36\lambda(q-4) + 12\lambda.$$

$$6Z_CU_C^N = 6\lambda^6 + 15\lambda^5 + 12\lambda^4(q-4) + 144\lambda^4 + 153\lambda^3(q-4) + 66\lambda^2(q-4)(q-5) + 9\lambda(q-4)(q-5)(q-6) + 174\lambda^3 + 252\lambda^2(q-4) + 69\lambda(q-4)(q-5) + 186\lambda^2 + 132\lambda(q-4) + 51\lambda.$$

$$\tilde{S}_C = 3(2r^4t^8 + 16r^4t^7 + 12r^3t^8 + 56r^4t^6 + 104r^3t^7 + 28r^2t^8 + 112r^4t^5 + 396r^3t^6 + 260r^2t^7 + 31rt^8 + 140r^4t^4 + 864r^3t^5 + 1064r^2t^6 + 301rt^7 + 14t^8 + 112r^4t^3 + 1180r^3t^4 + 2504r^2t^5 + 1298rt^6 + 138t^7 + 56r^4t^2 + 1032r^3t^3 + 3704r^2t^4 + 3246rt^5 + 612t^6 + 16r^4t + 564r^3t^2 + 3524r^2t^3 + 5146rt^4 + 1596t^5 + 2r^4 + 176r^3t + 2104r^2t^2 + 5292rt^3 + 2676t^4 + 24r^3 + 720r^2t + 3444rt^2 + 2952t^3 + 108r^2 + 1296rt + 2088t^2 + 216r + 864t + 162)(2r^3t^4 + 8r^3t^3 + 7r^2t^4 + 12r^3t^2 + 38r^2t^3 + 9rt^4 + 8r^3t + 73r^2t^2 + 57rt^3 + 4t^4 + 2r^3 + 60r^2t + 138rt^2 + 27t^3 + 18r^2 + 144rt + 81t^2 + 54r + 108t + 54)(rt + r + 3)(t+1)^4t.$$

$$D_C^U = 3(2s^5t^8 + 16s^5t^7 + 20s^4t^8 + 56s^5t^6 + 175s^4t^7 + 79s^3t^8 + 112s^5t^5 + 667s^4t^6 + 752s^3t^7 + 155s^2t^8 + 140s^5t^4 + 1448s^4t^5 + 3111s^3t^6 + 1589s^2t^7 + 152st^8 + 112s^5t^3 + 1960s^4t^4 + 7320s^3t^5 + 7095s^2t^6 + 1658st^7 + 60t^8 + 56s^5t^2 + 1695s^4t^3 + 10733s^3t^4 + 18047s^2t^5 + 7918st^6 + 690t^7 + 16s^5t + 915s^4t^2 + 10056s^3t^3 + 28656s^2t^4 + 21659st^5 + 3477t^6 + 2s^5 + 282s^4t + 5885s^3t^2 + 29144s^2t^3 + 37179st^4 + 10122t^5 + 38s^4 + 1968s^3t + 18570s^2t^2 + 41100st^3 + 18684t^4 + 288s^3 + 6784s^2t + 28640st^2 + 22448t^3 + 1088s^2 + 11520st + 17184t^2 + 2048s + 7680t + 1536)(s+4)(t+1)^3.$$

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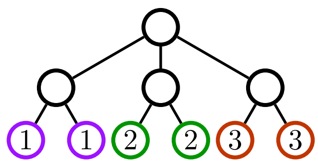

$$Z_C = 3\lambda^7 + \lambda^6(q-3) + 6\lambda^6 + 9\lambda^5(q-3) + 3\lambda^4(q-3)(q-4) + 9\lambda^5 + 18\lambda^4(q-3) + 9\lambda^3(q-3)(q-4) + 3\lambda^2(q-3)(q-4)(q-5) + 12\lambda^4 + 28\lambda^3(q-3) + 24\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + (q-3)(q-4)(q-5)(q-6) + 15\lambda^3 + 45\lambda^2(q-3) + 21\lambda(q-3)(q-4) + 12(q-3)(q-4)(q-5) + 18\lambda^2 + 36\lambda(q-3) + 40(q-3)(q-4) + 12\lambda + 38q - 108.$$

$$2Z_CU_C^v = 3\lambda^7 + 12\lambda^6 + 9\lambda^5(q-3) + 15\lambda^5 + 18\lambda^4(q-3) + 9\lambda^3(q-3)(q-4) + 12\lambda^4 + 30\lambda^3(q-3) + 6\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + 15\lambda^3 + 18\lambda^2(q-3) + 21\lambda(q-3)(q-4) + 12\lambda^2 + 36\lambda(q-3) + 12\lambda.$$

$$6Z_CU_C^N = 21\lambda^7 + 6\lambda^6(q-3) + 36\lambda^6 + 45\lambda^5(q-3) + 12\lambda^4(q-3)(q-4) + 45\lambda^5 + 72\lambda^4(q-3) + 27\lambda^3(q-3)(q-4) + 6\lambda^2(q-3)(q-4)(q-5) + 48\lambda^4 + 84\lambda^3(q-3) + 48\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + 45\lambda^3 + 90\lambda^2(q-3) + 21\lambda(q-3)(q-4) + 36\lambda^2 + 36\lambda(q-3) + 12\lambda.$$

$$\tilde{S}_C = 6(r^4t^7 + 8r^4t^6 + 4r^3t^7 + 27r^4t^5 + 40r^3t^6 + 6r^2t^7 + 50r^4t^4 + 164r^3t^5 + 76r^2t^6 + 5rt^7 + 55r^4t^3 + 360r^3t^4 + 370r^2t^5 + 68rt^6 + 2t^7 + 36r^4t^2 + 460r^3t^3 + 948r^2t^4 + 373rt^5 + 24t^6 + 13r^4t + 344r^3t^2 + 1402r^2t^3 + 1090rt^4 + 141t^5 + 2r^4 + 140r^3t + 1204r^2t^2 + 1848rt^3 + 462t^4 + 24r^3 + 558r^2t + 1824rt^2 + 891t^3 + 108r^2 + 972rt + 1008t^2 + 216r + 621t + 162)(2r^3t^6 + 12r^3t^5 + 11r^2t^6 + 30r^3t^4 + 69r^2t^5 + 21rt^6 + 40r^3t^3 + 183r^2t^4 + 136rt^5 + 14t^6 + 30r^3t^2 + 263r^2t^3 + 376rt^4 + 92t^5 + 12r^3t + 216r^2t^2 + 573rt^3 + 260t^4 + 2r^3 + 96r^2t + 510rt^2 + 414t^3 + 18r^2 + 252rt + 396t^2 + 54r + 216t + 54)(rt^2 + 2rt + t^2 + r + 3t + 3)(t+2)(t+1)t.$$

$$\begin{aligned}
D_C^U \quad &= 3(2s^4t^8+16s^4t^7+12s^3t^8+56s^4t^6+99s^3t^7+28s^2t^8+112s^4t^5+363s^3t^6+233s^2t^7+30st^8+\\
&140s^4t^4+772s^3t^5+874s^2t^6+246st^7+12t^8+112s^4t^3+1040s^3t^4+1936s^2t^5+918st^6+\\
&96t^7+56s^4t^2+907s^3t^3+2772s^2t^4+2063st^5+348t^6+16s^4t+499s^3t^2+2623s^2t^3+\\
&3082st^4+766t^5+2s^4+158s^3t+1596s^2t^2+3144st^3+1156t^4+22s^3+568s^2t+2131st^2+\\
&1258t^3+90s^2+870st+966t^2+162s+468t+108)(st^2+2st+t^2+s+3t+3)(s+3)(t+1).
\end{aligned}$$

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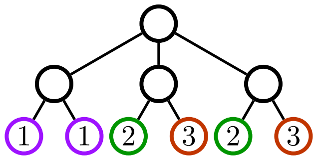

$$\begin{aligned}
Z_C \quad &= 2\lambda^6+15\lambda^5+21\lambda^4(q-3)+7\lambda^3(q-3)(q-4)+\lambda^2(q-3)(q-4)(q-5)+12\lambda^4+\\
&28\lambda^3(q-3)+24\lambda^2(q-3)(q-4)+7\lambda(q-3)(q-4)(q-5)+\\
&(q-3)(q-4)(q-5)(q-6)+14\lambda^3+58\lambda^2(q-3)+39\lambda(q-3)(q-4)+\\
&10(q-3)(q-4)(q-5)+30\lambda^2+49\lambda(q-3)+27(q-3)(q-4)+6\lambda+19q-55.
\end{aligned}$$

$$\begin{aligned}
2Z_CU_C^v &= 4\lambda^6+25\lambda^5+22\lambda^4(q-3)+3\lambda^3(q-3)(q-4)+16\lambda^4+30\lambda^3(q-3)+18\lambda^2(q-3)(q-4)+\\
&3\lambda(q-3)(q-4)(q-5)+14\lambda^3+42\lambda^2(q-3)+15\lambda(q-3)(q-4)+20\lambda^2+17\lambda(q-3)+2\lambda.
\end{aligned}$$

$$\begin{aligned}
6Z_CU_C^N &= 12\lambda^6+75\lambda^5+84\lambda^4(q-3)+21\lambda^3(q-3)(q-4)+2\lambda^2(q-3)(q-4)(q-5)+48\lambda^4+\\
&84\lambda^3(q-3)+48\lambda^2(q-3)(q-4)+7\lambda(q-3)(q-4)(q-5)+42\lambda^3+116\lambda^2(q-3)+\\
&39\lambda(q-3)(q-4)+60\lambda^2+49\lambda(q-3)+6\lambda.
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_C \quad &= 2(6r^8t^{15}+94r^8t^{14}+59r^7t^{15}+688r^8t^{13}+988r^7t^{14}+252r^6t^{15}+3120r^8t^{12}+7727r^7t^{13}+\\
&4514r^6t^{14}+610r^5t^{15}+9802r^8t^{11}+37420r^7t^{12}+37702r^6t^{13}+11702r^5t^{14}+911r^4t^{15}+\\
&22594r^8t^{10}+125434r^7t^{11}+194742r^6t^{12}+104357r^5t^{13}+18780r^4t^{14}+852r^3t^{15}+\\
&39468r^8t^9+308166r^7t^{10}+695544r^6t^{11}+574402r^5t^{12}+179005r^4t^{13}+19014r^3t^{14}+\\
&480r^2t^{15}+53196r^8t^8+573066r^7t^9+1819074r^6t^{10}+2183288r^5t^{11}+1049694r^4t^{12}+\\
&194327r^3t^{13}+11760r^2t^{14}+144rt^{15}+55770r^8t^7+821184r^7t^8+3597912r^6t^9+\\
&6071290r^5t^{10}+4242621r^4t^{11}+1215280r^3t^{12}+129735r^2t^{13}+3996rt^{14}+16t^{15}+\\
&45474r^8t^6+914055r^7t^7+5478900r^6t^8+12759962r^5t^9+12531878r^4t^{10}+5222615r^3t^{11}+\\
&867936r^2t^{12}+48258rt^{13}+552t^{14}+28600r^8t^5+790208r^7t^6+6475188r^6t^7+\\
&20636932r^5t^8+27961299r^4t^9+16376750r^3t^{10}+3970620r^2t^{11}+347760rt^{12}+\\
&7536t^{13}+13624r^8t^4+526195r^7t^5+5938254r^6t^6+25892270r^5t^7+47997990r^4t^8+\\
&38764066r^3t^9+13220244r^2t^{10}+1699353rt^{11}+59328t^{12}+4758r^8t^3+265028r^7t^4+\\
&4190846r^6t^5+25197866r^5t^6+63913684r^4t^7+70582724r^3t^8+33187851r^2t^9+\\
&6016896rt^{10}+311850t^{11}+1150r^8t^2+97732r^7t^3+2234998r^6t^4+18863029r^5t^5+\\
&66016256r^4t^6+99716067r^3t^7+64072452r^2t^8+16027956rt^9+1178496t^{10}+172r^8t+\\
&24910r^7t^2+871848r^6t^3+10665930r^5t^4+52456332r^4t^5+109320894r^3t^6+\\
&96005988r^2t^7+32809878rt^8+3337308t^9+12r^8+3924r^7t+234846r^6t^2+\\
&4409280r^5t^3+31486464r^4t^4+92251521r^3t^5+111712392r^2t^6+52135812rt^7+\\
&7249500t^8+288r^7+39060r^6t+1258038r^5t^2+13818600r^4t^3+58843800r^3t^4+\\
&100151586r^2t^5+64386144rt^6+12219822t^7+3024r^6+221508r^5t+4185810r^4t^2+\\
&27462564r^3t^3+67948470r^2t^4+61341543rt^5+16018560t^6+18144r^5+782460r^4t+\\
&8852490r^3t^2+33775056r^2t^3+44299386rt^4+16221708t^5+68040r^4+1762236r^3t+\\
&11612970r^2t^2+23485464rt^3+12477564t^4+163296r^3+2469852r^2t+8632818rt^2+\\
&7064010t^3+244944r^2+1968300rt+2781864t^2+209952r+682344t+78732)(t+1)^2t.
\end{aligned}$$

$$
\begin{aligned}
D_C^U &= (6s^5t^9 + 54s^5t^8 + 34s^4t^9 + 216s^5t^7 + 341s^4t^8 + 76s^3t^9 + 504s^5t^6 + 1516s^4t^7 + \\
&840s^3t^8 + 86s^2t^9 + 756s^5t^5 + 3923s^4t^6 + 4110s^3t^7 + 1018s^2t^8 + 50st^9 + 756s^5t^4 + \\
&6514s^4t^5 + 11717s^3t^6 + 5378s^2t^7 + 618st^8 + 12t^9 + 504s^5t^3 + 7199s^4t^4 + 21480s^3t^5 + \\
&16691s^2t^6 + 3432st^7 + 156t^8 + 216s^5t^2 + 5296s^4t^3 + 26274s^3t^4 + 33592s^2t^5 + \\
&11326st^6 + 888t^7 + 54s^5t + 2501s^4t^2 + 21442s^3t^3 + 45482s^2t^4 + 24652st^5 + 2984t^6 + \\
&6s^5 + 688s^4t + 11253s^3t^2 + 41400s^2t^3 + 36785st^4 + 6744t^5 + 84s^4 + 3444s^3t + \\
&24393s^2t^2 + 37542st^3 + 10860t^4 + 468s^3 + 8424s^2t + 25155st^2 + 12456t^3 + 1296s^2 + \\
&9990st + 9666t^2 + 1782s + 4536t + 972)(s+3)(t+1)^2.
\end{aligned}
$$

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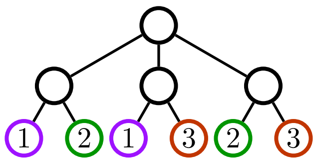

$$
\begin{aligned}
Z_C &= 9\lambda^5 + 3\lambda^4(q-3) + 18\lambda^4 + 51\lambda^3(q-3) + 33\lambda^2(q-3)(q-4) + 9\lambda(q-3)(q-4)(q-5) + \\
&(q-3)(q-4)(q-5)(q-6) + 27\lambda^3 + 60\lambda^2(q-3) + 42\lambda(q-3)(q-4) + \\
&9(q-3)(q-4)(q-5) + 18\lambda^2 + 48\lambda(q-3) + 22(q-3)(q-4) + 9\lambda + 13q - 39.
\end{aligned}
$$

$$
\begin{aligned}
2Z_CU_C^v &= 21\lambda^5 + 6\lambda^4(q-3) + 24\lambda^4 + 57\lambda^3(q-3) + 24\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + \\
&27\lambda^3 + 36\lambda^2(q-3) + 12\lambda(q-3)(q-4) + 6\lambda^2 + 12\lambda(q-3) + 3\lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_CU_C^N &= 45\lambda^5 + 12\lambda^4(q-3) + 72\lambda^4 + 153\lambda^3(q-3) + 66\lambda^2(q-3)(q-4) + 9\lambda(q-3)(q-4)(q-5) + \\
&81\lambda^3 + 120\lambda^2(q-3) + 42\lambda(q-3)(q-4) + 36\lambda^2 + 48\lambda(q-3) + 9\lambda.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C &= 6(2r^4t^8 + 16r^4t^7 + 12r^3t^8 + 56r^4t^6 + 104r^3t^7 + 28r^2t^8 + 112r^4t^5 + 396r^3t^6 + 260r^2t^7 + \\
&31rt^8 + 140r^4t^4 + 864r^3t^5 + 1064r^2t^6 + 301rt^7 + 14t^8 + 112r^4t^3 + 1180r^3t^4 + \\
&2504r^2t^5 + 1298rt^6 + 138t^7 + 56r^4t^2 + 1032r^3t^3 + 3704r^2t^4 + 3246rt^5 + 612t^6 + 16r^4t + \\
&564r^3t^2 + 3524r^2t^3 + 5146rt^4 + 1596t^5 + 2r^4 + 176r^3t + 2104r^2t^2 + 5292rt^3 + 2676t^4 + \\
&24r^3 + 720r^2t + 3444rt^2 + 2952t^3 + 108r^2 + 1296rt + 2088t^2 + 216r + 864t + 162)(r^3t^4 + \\
&4r^3t^3 + 3r^2t^4 + 6r^3t^2 + 18r^2t^3 + 3rt^4 + 4r^3t + 36r^2t^2 + 24rt^3 + t^4 + r^3 + 30r^2t + \\
&66rt^2 + 9t^3 + 9r^2 + 72rt + 36t^2 + 27r + 54t + 27)(rt^2 + 2rt + t^2 + r + 3t + 3)(t+1)^3t.
\end{aligned}
$$

$$
\begin{aligned}
D_C^U &= 3(2s^5t^8 + 16s^5t^7 + 10s^4t^8 + 56s^5t^6 + 95s^4t^7 + 18s^3t^8 + 112s^5t^5 + 387s^4t^6 + 207s^3t^7 + \\
&14s^2t^8 + 140s^5t^4 + 888s^4t^5 + 993s^3t^6 + 198s^2t^7 + 4st^8 + 112s^5t^3 + 1260s^4t^4 + 2638s^3t^5 + \\
&1144s^2t^6 + 75st^7 + 56s^5t^2 + 1135s^4t^3 + 4288s^3t^4 + 3585s^2t^5 + 555st^6 + 6t^7 + 16s^5t + \\
&635s^4t^2 + 4395s^3t^3 + 6777s^2t^4 + 2133st^5 + 78t^6 + 2s^5 + 202s^4t + 2785s^3t^2 + 8017s^2t^3 + \\
&4821st^4 + 406t^5 + 28s^4 + 1000s^3t + 5845s^2t^2 + 6733st^3 + 1158t^4 + 156s^3 + 2412s^2t + \\
&5775st^2 + 1992t^3 + 432s^2 + 2808st + 2088t^2 + 594s + 1242t + 324)(s+3)(t+1)^3.
\end{aligned}
$$

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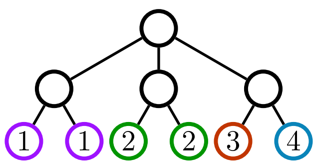

$$
\begin{aligned}
Z_C ={}& 8\lambda^6 + 5\lambda^5(q-4) + \lambda^4(q-4)(q-5) + 14\lambda^5 + 22\lambda^4(q-4) + 10\lambda^3(q-4)(q-5) + \\
& 2\lambda^2(q-4)(q-5)(q-6) + 32\lambda^4 + 51\lambda^3(q-4) + 27\lambda^2(q-4)(q-5) + \\
& 5\lambda(q-4)(q-5)(q-6) + (q-4)(q-5)(q-6)(q-7) + 48\lambda^3 + 84\lambda^2(q-4) + \\
& 47\lambda(q-4)(q-5) + 15(q-4)(q-5)(q-6) + 60\lambda^2 + 115\lambda(q-4) + 66(q-4)(q-5) + \\
& 66\lambda + 92q - 340.
\end{aligned}
$$

$$
\begin{aligned}
2Z_C U_C^v ={}& 10\lambda^6 + 3\lambda^5(q-4) + 24\lambda^5 + 24\lambda^4(q-4) + 6\lambda^3(q-4)(q-5) + 38\lambda^4 + 39\lambda^3(q-4) + \\
& 12\lambda^2(q-4)(q-5) + 3\lambda(q-4)(q-5)(q-6) + 38\lambda^3 + 54\lambda^2(q-4) + \\
& 27\lambda(q-4)(q-5) + 48\lambda^2 + 63\lambda(q-4) + 34\lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_C U_C^N ={}& 48\lambda^6 + 25\lambda^5(q-4) + 4\lambda^4(q-4)(q-5) + 70\lambda^5 + 88\lambda^4(q-4) + 30\lambda^3(q-4)(q-5) + \\
& 4\lambda^2(q-4)(q-5)(q-6) + 128\lambda^4 + 153\lambda^3(q-4) + 54\lambda^2(q-4)(q-5) + \\
& 5\lambda(q-4)(q-5)(q-6) + 144\lambda^3 + 168\lambda^2(q-4) + 47\lambda(q-4)(q-5) + 120\lambda^2 + \\
& 115\lambda(q-4) + 66\lambda.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C ={}& 2(8r^8t^{15}+128r^8t^{14}+82r^7t^{15}+956r^8t^{13}+1392r^7t^{14}+368r^6t^{15}+4420r^8t^{12}+11038r^7t^{13}+ \\
& 6624r^6t^{14}+948r^5t^{15}+14144r^8t^{11}+54196r^7t^{12}+55680r^6t^{13}+18056r^5t^{14}+1541r^4t^{15}+ \\
& 33176r^8t^{10} + 184156r^7t^{11} + 289740r^6t^{12} + 160506r^5t^{13} + 30924r^4t^{14} + 1629r^3t^{15} + \\
& 58916r^8t^9 + 458504r^7t^{10} + 1043222r^6t^{11} + 883024r^5t^{12} + 289661r^4t^{13} + 34186r^3t^{14} + \\
& 1099r^2t^{15} + 80652r^8t^8 + 863784r^7t^9 + 2751704r^6t^{10} + 3361112r^5t^{11} + 1679622r^4t^{12} + \\
& 335637r^3t^{13} + 23874r^2t^{14} + 431rt^{15} + 85800r^8t^7 + 1253496r^7t^8 + 5490868r^6t^9 + \\
& 9373024r^5t^{10} + 6741057r^4t^{11} + 2043386r^3t^{12} + 244030r^2t^{13} + 9612rt^{14} + 74t^{15} + \\
& 70928r^8t^6+1412442r^7t^7+8437832r^6t^8+19776896r^5t^9+19830664r^4t^{10}+8621729r^3t^{11}+ \\
& 1552812r^2t^{12} + 101652rt^{13} + 1696t^{14} + 45188r^8t^5 + 1235632r^7t^6 + 10065316r^6t^7 + \\
& 32142232r^5t^8 + 44164209r^4t^9 + 26693890r^3t^{10} + 6866402r^2t^{11} + 673120rt^{12} + \\
& 18504t^{13} + 21788r^8t^4 + 832302r^7t^5 + 9318640r^6t^6 + 40560068r^5t^7 + 75808326r^4t^8 + \\
& 62632709r^3t^9 + 22323598r^2t^{10} + 3110230rt^{11} + 127104t^{12} + 7696r^8t^3 + 423892r^7t^4 + \\
& 6640336r^6t^5 + 39732640r^5t^6 + 101104052r^4t^7 + 113385106r^3t^8 + 55087475r^2t^9 + \\
& 10596752rt^{10} + 612252t^{11} + 1880r^8t^2 + 158008r^7t^3 + 3576284r^6t^4 + 29963866r^5t^5 + \\
& 104753312r^4t^6 + 159659830r^3t^7 + 105032966r^2t^8 + 27461787rt^9 + 2182776t^{10} + \\
& 284r^8t + 40696r^7t^2 + 1409062r^6t^3 + 17081560r^5t^4 + 83618838r^4t^5 + 174864732r^3t^6 + \\
& 155994678r^2t^7 + 55084350rt^8 + 5934978t^9 + 20r^8 + 6476r^7t + 383416r^6t^2 + \\
& 7124736r^5t^3 + 50496480r^4t^4 + 147743910r^3t^5 + 180484740r^2t^6 + 86211756rt^7 + \\
& 12515220t^8 + 480r^7 + 64428r^6t + 2052504r^5t^2 + 22328910r^4t^3 + 94568004r^3t^4 + \\
& 161373168r^2t^5 + 105302484rt^6 + 20628378t^7 + 5040r^6 + 365148r^5t + 6824520r^4t^2 + \\
& 44388648r^3t^3 + 109522476r^2t^4 + 99604242rt^5 + 26585820t^6 + 30240r^5 + 1288980r^4t + \\
& 14423400r^3t^2 + 54629154r^2t^3 + 71690832rt^4 + 26594406t^5 + 113400r^4 + 2900772r^3t + \\
& 18909288r^2t^2 + 38032416rt^3 + 20300220t^4 + 272160r^3 + 4061988r^2t + 14049288rt^2 + \\
& 11461338t^3+408240r^2+3233844rt+4525632t^2+349920r+1119744t+131220)(t+1)^2t.
\end{aligned}
$$

$$
\begin{aligned}
D_C^U ={}& (6s^5t^9 + 54s^5t^8 + 66s^4t^9 + 216s^5t^7 + 621s^4t^8 + 290s^3t^9 + 504s^5t^6 + 2610s^4t^7 + \\
& 2839s^3t^8 + 636s^2t^9 + 756s^5t^5 + 6429s^4t^6 + 12474s^3t^7 + 6444s^2t^8 + 694st^9 + 756s^5t^4 + \\
& 10224s^4t^5 + 32293s^3t^6 + 29436s^2t^7 + 7250st^8 + 300t^9 + 504s^5t^3 + 10881s^4t^4 + \\
& 54278s^3t^5 + 79695s^2t^6 + 34248st^7 + 3228t^8 + 216s^5t^2 + 7746s^4t^3 + 61401s^3t^4 + \\
& 141060s^2t^5 + 96427st^6 + 15696t^7 + 54s^5t + 3555s^4t^2 + 46718s^3t^3 + 169317s^2t^4 + \\
& 178904st^5 + 45660t^6 + 6s^5 + 954s^4t + 23035s^3t^2 + 137778s^2t^3 + 227277st^4 + 88242t^5 + \\
& 114s^4 + 6672s^3t + 73230s^2t^2 + 197844st^3 + 118164t^4 + 864s^3 + 23040s^2t + 113760st^2 + \\
& 110000t^3 + 3264s^2 + 39168st + 68704t^2 + 6144s + 26112t + 4608)(s+4)(t+1)^2.
\end{aligned}
$$

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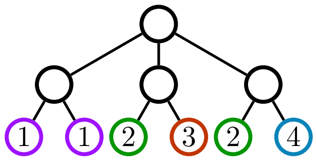

$$
\begin{aligned}
Z_C ={}& \lambda^6 + 18\lambda^5 + 20\lambda^4(q-4) + 7\lambda^3(q-4)(q-5) + \lambda^2(q-4)(q-5)(q-6) + 29\lambda^4 + \\
& 46\lambda^3(q-4) + 27\lambda^2(q-4)(q-5) + 7\lambda(q-4)(q-5)(q-6) + \\
& (q-4)(q-5)(q-6)(q-7) + 48\lambda^3 + 100\lambda^2(q-4) + 60\lambda(q-4)(q-5) + \\
& 14(q-4)(q-5)(q-6) + 79\lambda^2 + 131\lambda(q-4) + 57(q-4)(q-5) + 62\lambda + 72q - 269.
\end{aligned}
$$

$$
\begin{aligned}
2Z_C U_C^v ={}& 2\lambda^6 + 28\lambda^5 + 20\lambda^4(q-4) + 3\lambda^3(q-4)(q-5) + 38\lambda^4 + 42\lambda^3(q-4) + \\
& 18\lambda^2(q-4)(q-5) + 3\lambda(q-4)(q-5)(q-6) + 46\lambda^3 + 72\lambda^2(q-4) + \\
& 24\lambda(q-4)(q-5) + 56\lambda^2 + 49\lambda(q-4) + 22\lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_C U_C^N ={}& 6\lambda^6 + 90\lambda^5 + 80\lambda^4(q-4) + 21\lambda^3(q-4)(q-5) + 2\lambda^2(q-4)(q-5)(q-6) + 116\lambda^4 + \\
& 138\lambda^3(q-4) + 54\lambda^2(q-4)(q-5) + 7\lambda(q-4)(q-5)(q-6) + 144\lambda^3 + 200\lambda^2(q-4) + \\
& 60\lambda(q-4)(q-5) + 158\lambda^2 + 131\lambda(q-4) + 62\lambda.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C ={}& 2(8r^8t^{15} + 124r^8t^{14} + 79r^7t^{15} + 898r^8t^{13} + 1310r^7t^{14} + 341r^6t^{15} + 4030r^8t^{12} + 10141r^7t^{13} + \\
& 6040r^6t^{14} + 841r^5t^{15} + 12532r^8t^{11} + 48600r^7t^{12} + 49878r^6t^{13} + 15880r^5t^{14} + 1295r^4t^{15} + \\
& 28600r^8t^{10} + 161210r^7t^{11} + 254718r^6t^{12} + 139589r^5t^{13} + 26021r^4t^{14} + 1272r^3t^{15} + \\
& 49478r^8t^9 + 391964r^7t^{10} + 899515r^6t^{11} + 758024r^5t^{12} + 243019r^4t^{13} + 27157r^3t^{14} + \\
& 774r^2t^{15} + 66066r^8t^8 + 721500r^7t^9 + 2326416r^6t^{10} + 2844442r^5t^{11} + 1400713r^4t^{12} + \\
& 269106r^3t^{13} + 17548r^2t^{14} + 262rt^{15} + 68640r^8t^7 + 1023672r^7t^8 + 4551414r^6t^9 + \\
& 7812872r^5t^{10} + 5575465r^4t^{11} + 1644121r^3t^{12} + 184466r^2t^{13} + 6350rt^{14} + 36t^{15} + \\
& 55484r^8t^6 + 1128555r^7t^7 + 6857772r^6t^8 + 16226476r^5t^9 + 16239803r^4t^{10} + 6933022r^3t^{11} + \\
& 1194327r^2t^{12} + 71075rt^{13} + 964t^{14} + 34606r^8t^5 + 966670r^7t^6 + 8022195r^6t^7 + \\
& 25945424r^5t^8 + 35763164r^4t^9 + 21388831r^3t^{10} + 5333160r^2t^{11} + 488713rt^{12} + \\
& 11646t^{13} + 16354r^8t^4 + 638021r^7t^5 + 7284896r^6t^6 + 32197645r^5t^7 + 60635369r^4t^8 + \\
& 49892797r^3t^9 + 17416598r^2t^{10} + 2313698rt^{11} + 85614t^{12} + 5668r^8t^3 + 318640r^7t^4 + \\
& 5092978r^6t^5 + 31006824r^5t^6 + 79797163r^4t^7 + 89629023r^3t^8 + 43006542r^2t^9 + \\
& 8002700rt^{10} + 431364t^{11} + 1360r^8t^2 + 116556r^7t^3 + 2691774r^6t^4 + 22979909r^5t^5 + \\
& 81504000r^4t^6 + 125033068r^3t^7 + 81812339r^2t^8 + 20920710rt^9 + 1583880t^{10} + 202r^8t + \\
& 29480r^7t^2 + 1041069r^6t^3 + 12870136r^5t^4 + 64075503r^4t^5 + 135448356r^3t^6 + \\
& 120933135r^2t^7 + 42138435rt^8 + 4389372t^9 + 14r^8 + 4610r^7t + 278152r^6t^2 + \\
& 5272320r^5t^3 + 38072178r^4t^4 + 113010813r^3t^5 + 138938274r^2t^6 + 65990052rt^7 + \\
& 9367074t^8 + 336r^7 + 45906r^6t + 1491336r^5t^2 + 16548345r^4t^3 + 71312688r^3t^4 + \\
& 123070806r^2t^5 + 80401734rt^6 + 15544899t^7 + 3528r^6 + 260442r^5t + 4966920r^4t^2 + \\
& 32942484r^3t^3 + 82549962r^2t^4 + 75631401rt^5 + 20088162t^6 + 21168r^5 + 920430r^4t + \\
& 10515960r^3t^2 + 40589343r^2t^3 + 53963496rt^4 + 20072043t^5 + 79380r^4 + 2074086r^3t + \\
& 13812120r^2t^2 + 28281312rt^3 + 15243876t^4 + 190512r^3 + 2908710r^2t + 10281816rt^2 + \\
& 8525655t^3 + 285768r^2 + 2319678rt + 3318408t^2 + 244944r + 804816t + 91854)(t+1)^2t.
\end{aligned}
$$

$$
\begin{aligned}
D_C^U ={}& (6s^5t^9 + 54s^5t^8 + 62s^4t^9 + 216s^5t^7 + 595s^4t^8 + 254s^3t^9 + 504s^5t^6 + 2540s^4t^7 + \\
& 2588s^3t^8 + 517s^2t^9 + 756s^5t^5 + 6331s^4t^6 + 11744s^3t^7 + 5555s^2t^8 + 523st^9 + 756s^5t^4 + \\
& 10154s^4t^5 + 31177s^3t^6 + 26650s^2t^7 + 5887st^8 + 210t^9 + 504s^5t^3 + 10867s^4t^4 + \\
& 53386s^3t^5 + 75058s^2t^6 + 29662st^7 + 2469t^8 + 216s^5t^2 + 7760s^4t^3 + 61162s^3t^4 + \\
& 136931s^2t^5 + 88138st^6 + 12972t^7 + 54s^5t + 3565s^4t^2 + 46880s^3t^3 + 167917s^2t^4 + \\
& 170707st^5 + 40332t^6 + 6s^5 + 956s^4t + 23177s^3t^2 + 138438s^2t^3 + 223879st^4 + 82398t^5 + \\
& 114s^4 + 6704s^3t + 73974s^2t^2 + 198932st^3 + 115260t^4 + 864s^3 + 23232s^2t + 115456st^2 + \\
& 110576t^3 + 3264s^2 + 39680st + 70112t^2 + 6144s + 26624t + 4608)(s+4)(t+1)^2.
\end{aligned}
$$

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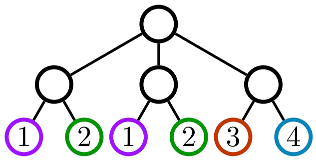

$$
\begin{aligned}
Z_C &= 6\lambda^5+2\lambda^4(q-4)+32\lambda^4+55\lambda^3(q-4)+33\lambda^2(q-4)(q-5)+9\lambda(q-4)(q-5)(q-6)+\\
&\quad (q-4)(q-5)(q-6)(q-7)+64\lambda^3+120\lambda^2(q-4)+69\lambda(q-4)(q-5)+\\
&\quad 13(q-4)(q-5)(q-6)+84\lambda^2+136\lambda(q-4)+49(q-4)(q-5)+58\lambda+56q-212.\\
2Z_CU_C^v &= 14\lambda^5+4\lambda^4(q-4)+50\lambda^4+63\lambda^3(q-4)+24\lambda^2(q-4)(q-5)+3\lambda(q-4)(q-5)(q-6)+\\
&\quad 66\lambda^3+78\lambda^2(q-4)+21\lambda(q-4)(q-5)+46\lambda^2+38\lambda(q-4)+16\lambda.\\
6Z_CU_C^N &= 30\lambda^5+8\lambda^4(q-4)+128\lambda^4+165\lambda^3(q-4)+66\lambda^2(q-4)(q-5)+9\lambda(q-4)(q-5)(q-6)+\\
&\quad 192\lambda^3+240\lambda^2(q-4)+69\lambda(q-4)(q-5)+168\lambda^2+136\lambda(q-4)+58\lambda.\\
\tilde{S}_C &= 2(2r^4t^8+16r^4t^7+12r^3t^8+56r^4t^6+104r^3t^7+28r^2t^8+112r^4t^5+396r^3t^6+260r^2t^7+\\
&\quad 31rt^8+140r^4t^4+864r^3t^5+1064r^2t^6+301rt^7+14t^8+112r^4t^3+1180r^3t^4+\\
&\quad 2504r^2t^5+1298rt^6+138t^7+56r^4t^2+1032r^3t^3+3704r^2t^4+3246rt^5+612t^6+\\
&\quad 16r^4t+564r^3t^2+3524r^2t^3+5146rt^4+1596t^5+2r^4+176r^3t+2104r^2t^2+5292rt^3+\\
&\quad 2676t^4+24r^3+720r^2t+3444rt^2+2952t^3+108r^2+1296rt+2088t^2+216r+864t+\\
&\quad 162)(4r^4t^6+24r^4t^5+15r^3t^6+60r^4t^4+120r^3t^5+21r^2t^6+80r^4t^3+378r^3t^4+\\
&\quad 210r^2t^5+12rt^6+60r^4t^2+612r^3t^3+828r^2t^4+150rt^5+2t^6+24r^4t+543r^3t^2+\\
&\quad 1650r^2t^3+741rt^4+36t^5+4r^4+252r^3t+1767r^2t^2+1836rt^3+225t^4+48r^3+\\
&\quad 972r^2t+2421rt^2+702t^3+216r^2+1620rt+1161t^2+432r+972t+324)(t+1)^3t.\\
D_C^U &= (6s^5t^8+48s^5t^7+58s^4t^8+168s^5t^6+511s^4t^7+218s^3t^8+336s^5t^5+1959s^4t^6+\\
&\quad 2113s^3t^7+398s^2t^8+420s^5t^4+4274s^4t^5+8871s^3t^6+4223s^2t^7+352st^8+336s^5t^3+\\
&\quad 5810s^4t^4+21130s^3t^5+19395s^2t^6+4061st^7+120t^8+168s^5t^2+5043s^4t^3+31304s^3t^4+\\
&\quad 50489s^2t^5+20349st^6+1500t^7+48s^5t+2731s^4t^2+29589s^3t^3+81725s^2t^4+\\
&\quad 57913st^5+8154t^6+6s^5+844s^4t+17447s^3t^2+84468s^2t^3+102701st^4+25302t^5+\\
&\quad 114s^4+5872s^3t+54558s^2t^2+116628st^3+49236t^4+864s^3+20160s^2t+83104st^2+\\
&\quad 61776t^3+3264s^2+34048st+48992t^2+6144s+22528t+4608)(s+4)(t+1)^3.
\end{aligned}
$$

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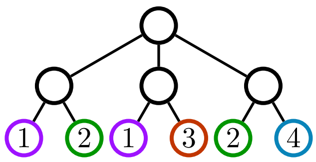

$$
\begin{aligned}
Z_C &= 6\lambda^5+2\lambda^4(q-4)+32\lambda^4+55\lambda^3(q-4)+33\lambda^2(q-4)(q-5)+9\lambda(q-4)(q-5)(q-6)+\\
&\quad (q-4)(q-5)(q-6)(q-7)+64\lambda^3+120\lambda^2(q-4)+69\lambda(q-4)(q-5)+\\
&\quad 13(q-4)(q-5)(q-6)+84\lambda^2+136\lambda(q-4)+49(q-4)(q-5)+58\lambda+56q-212.\\
2Z_CU_C^v &= 14\lambda^5+4\lambda^4(q-4)+50\lambda^4+63\lambda^3(q-4)+24\lambda^2(q-4)(q-5)+3\lambda(q-4)(q-5)(q-6)+\\
&\quad 66\lambda^3+78\lambda^2(q-4)+21\lambda(q-4)(q-5)+46\lambda^2+38\lambda(q-4)+16\lambda.\\
6Z_CU_C^N &= 30\lambda^5+8\lambda^4(q-4)+128\lambda^4+165\lambda^3(q-4)+66\lambda^2(q-4)(q-5)+9\lambda(q-4)(q-5)(q-6)+\\
&\quad 192\lambda^3+240\lambda^2(q-4)+69\lambda(q-4)(q-5)+168\lambda^2+136\lambda(q-4)+58\lambda.
\end{aligned}
$$

$$\begin{aligned}
\tilde{S}_C &= 2(2r^4t^8 + 16r^4t^7 + 12r^3t^8 + 56r^4t^6 + 104r^3t^7 + 28r^2t^8 + 112r^4t^5 + 396r^3t^6 + 260r^2t^7 + \\
&\quad 31rt^8 + 140r^4t^4 + 864r^3t^5 + 1064r^2t^6 + 301rt^7 + 14t^8 + 112r^4t^3 + 1180r^3t^4 + \\
&\quad 2504r^2t^5 + 1298rt^6 + 138t^7 + 56r^4t^2 + 1032r^3t^3 + 3704r^2t^4 + 3246rt^5 + 612t^6 + \\
&\quad 16r^4t + 564r^3t^2 + 3524r^2t^3 + 5146rt^4 + 1596t^5 + 2r^4 + 176r^3t + 2104r^2t^2 + 5292rt^3 + \\
&\quad 2676t^4 + 24r^3 + 720r^2t + 3444rt^2 + 2952t^3 + 108r^2 + 1296rt + 2088t^2 + 216r + 864t + \\
&\quad 162)(4r^4t^6 + 24r^4t^5 + 15r^3t^6 + 60r^4t^4 + 120r^3t^5 + 21r^2t^6 + 80r^4t^3 + 378r^3t^4 + \\
&\quad 210r^2t^5 + 12rt^6 + 60r^4t^2 + 612r^3t^3 + 828r^2t^4 + 150rt^5 + 2t^6 + 24r^4t + 543r^3t^2 + \\
&\quad 1650r^2t^3 + 741rt^4 + 36t^5 + 4r^4 + 252r^3t + 1767r^2t^2 + 1836rt^3 + 225t^4 + 48r^3 + \\
&\quad 972r^2t + 2421rt^2 + 702t^3 + 216r^2 + 1620rt + 1161t^2 + 432r + 972t + 324)(t+1)^3t. \\
D_C^U &= (6s^5t^8 + 48s^5t^7 + 58s^4t^8 + 168s^5t^6 + 511s^4t^7 + 218s^3t^8 + 336s^5t^5 + 1959s^4t^6 + \\
&\quad 2113s^3t^7 + 398s^2t^8 + 420s^5t^4 + 4274s^4t^5 + 8871s^3t^6 + 4223s^2t^7 + 352st^8 + 336s^5t^3 + \\
&\quad 5810s^4t^4 + 21130s^3t^5 + 19395s^2t^6 + 4061st^7 + 120t^8 + 168s^5t^2 + 5043s^4t^3 + 31304s^3t^4 + \\
&\quad 50489s^2t^5 + 20349st^6 + 1500t^7 + 48s^5t + 2731s^4t^2 + 29589s^3t^3 + 81725s^2t^4 + \\
&\quad 57913st^5 + 8154t^6 + 6s^5 + 844s^4t + 17447s^3t^2 + 84468s^2t^3 + 102701st^4 + 25302t^5 + \\
&\quad 114s^4 + 5872s^3t + 54558s^2t^2 + 116628st^3 + 49236t^4 + 864s^3 + 20160s^2t + 83104st^2 + \\
&\quad 61776t^3 + 3264s^2 + 34048st + 48992t^2 + 6144s + 22528t + 4608)(s+4)(t+1)^3.
\end{aligned}$$

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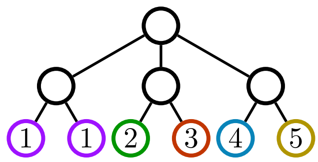

$$\begin{aligned}
Z_C &= 21\lambda^5 + 19\lambda^4(q-5) + 7\lambda^3(q-5)(q-6) + \lambda^2(q-5)(q-6)(q-7) + 44\lambda^4 + \\
&\quad 64\lambda^3(q-5) + 30\lambda^2(q-5)(q-6) + 7\lambda(q-5)(q-6)(q-7) + \\
&\quad (q-5)(q-6)(q-7)(q-8) + 104\lambda^3 + 148\lambda^2(q-5) + 81\lambda(q-5)(q-6) + \\
&\quad 18(q-5)(q-6)(q-7) + 164\lambda^2 + 255\lambda(q-5) + 99(q-5)(q-6) + 204\lambda + 185q - 837. \\
2Z_CU_C^v &= 31\lambda^5 + 18\lambda^4(q-5) + 3\lambda^3(q-5)(q-6) + 56\lambda^4 + 54\lambda^3(q-5) + 18\lambda^2(q-5)(q-6) + \\
&\quad 3\lambda(q-5)(q-6)(q-7) + 96\lambda^3 + 102\lambda^2(q-5) + 33\lambda(q-5)(q-6) + 116\lambda^2 + \\
&\quad 99\lambda(q-5) + 76\lambda. \\
6Z_CU_C^N &= 105\lambda^5 + 76\lambda^4(q-5) + 21\lambda^3(q-5)(q-6) + 2\lambda^2(q-5)(q-6)(q-7) + 176\lambda^4 + \\
&\quad 192\lambda^3(q-5) + 60\lambda^2(q-5)(q-6) + 7\lambda(q-5)(q-6)(q-7) + 312\lambda^3 + 296\lambda^2(q-5) + \\
&\quad 81\lambda(q-5)(q-6) + 328\lambda^2 + 255\lambda(q-5) + 204\lambda.
\end{aligned}$$

$$
\begin{aligned}
\tilde{S}_C \;=\;& 2(10r^8t^{14} + 144r^8t^{13} + 99r^7t^{14} + 964r^8t^{12} + 1533r^7t^{13} + 430r^6t^{14} + 3976r^8t^{11} + \\
& 11022r^7t^{12} + 7136r^6t^{13} + 1072r^5t^{14} + 11286r^8t^{10} + 48758r^7t^{11} + 54918r^6t^{12} + \\
& 18986r^5t^{13} + 1679r^4t^{14} + 23320r^8t^9 + 148228r^7t^{10} + 259776r^6t^{11} + 155835r^5t^{12} + \\
& 31583r^4t^{13} + 1692r^3t^{14} + 36168r^8t^8 + 327534r^7t^9 + 843710r^6t^{10} + 785811r^5t^{11} + \\
& 275450r^4t^{12} + 33608r^3t^{13} + 1068r^2t^{14} + 42768r^8t^7 + 542400r^7t^8 + 1990048r^6t^9 + \\
& 2719785r^5t^{10} + 1476282r^4t^{11} + 310277r^3t^{12} + 22268r^2t^{13} + 380rt^{14} + 38742r^8t^6 + \\
& 683760r^7t^7 + 3514868r^6t^8 + 6834669r^5t^9 + 5432027r^4t^{10} + 1762685r^3t^{11} + \\
& 216929r^2t^{12} + 8324rt^{13} + 56t^{14} + 26752r^8t^5 + 659295r^7t^6 + 4721776r^6t^7 + \\
& 12858321r^5t^8 + 14515701r^4t^9 + 6880744r^3t^{10} + 1303789r^2t^{11} + 85568rt^{12} + 1320t^{13} + \\
& 13860r^8t^4 + 483837r^7t^5 + 4847426r^6t^6 + 18395595r^5t^7 + 29049328r^4t^8 + \\
& 19520168r^3t^9 + 5391911r^2t^{10} + 544098rt^{11} + 14436t^{12} + 5224r^8t^3 + 266010r^7t^4 + \\
& 3784112r^6t^5 + 20107425r^5t^6 + 44223420r^4t^7 + 41501360r^3t^8 + 16221041r^2t^9 + \\
& 2383945rt^{10} + 97464t^{11} + 1354r^8t^2 + 106242r^7t^3 + 2210998r^6t^4 + 16708357r^5t^5 + \\
& 51457222r^4t^6 + 67173962r^3t^7 + 36604192r^2t^8 + 7604559rt^9 + 453414t^{10} + 216r^8t + \\
& 29138r^7t^2 + 937552r^6t^3 + 10388432r^5t^4 + 45534522r^4t^5 + 83176107r^3t^6 + \\
& 62948034r^2t^7 + 18208905rt^8 + 1535850t^9 + 16r^8 + 4912r^7t + 272738r^6t^2 + \\
& 4685910r^5t^3 + 30160152r^4t^4 + 78399711r^3t^5 + 82912248r^2t^6 + 33258087rt^7 + \\
& 3905586t^8 + 384r^7 + 48720r^6t + 1449450r^5t^2 + 14497740r^4t^3 + 55370394r^3t^4 + \\
& 83251908r^2t^5 + 46586205rt^6 + 7579062t^7 + 4032r^6 + 275184r^5t + 4780350r^4t^2 + \\
& 28411182r^3t^3 + 62738118r^2t^4 + 49831119rt^5 + 11290914t^6 + 24192r^5 + 967680r^4t + \\
& 10011222r^3t^2 + 34413336r^2t^3 + 40090140rt^4 + 12866850t^5 + 90720r^4 + 2168208r^3t + \\
& 12990294r^2t^2 + 23537466rt^3 + 11055528t^4 + 217728r^3 + 3020976r^2t + 9539694rt^2 + \\
& 6954660t^3 + 326592r^2 + 2391120rt + 3032640t^2 + 279936r + 822312t + 104976)(t+1)^3t.
\end{aligned}
$$

$$
\begin{aligned}
D_C^U \;=\;& (6s^5t^8 + 48s^5t^7 + 90s^4t^8 + 168s^5t^6 + 759s^4t^7 + 536s^3t^8 + 336s^5t^5 + 2805s^4t^6 + 4752s^3t^7 + \\
& 1584s^2t^8 + 420s^5t^4 + 5934s^4t^5 + 18498s^3t^6 + 14712s^2t^7 + 2320st^8 + 336s^5t^3 + 7860s^4t^4 + \\
& 41323s^3t^5 + 60150s^2t^6 + 22500st^7 + 1344t^8 + 168s^5t^2 + 6675s^4t^3 + 57969s^3t^4 + \\
& 141615s^2t^5 + 96312st^6 + 13584t^7 + 48s^5t + 3549s^4t^2 + 52305s^3t^3 + 210243s^2t^4 + \\
& 238396st^5 + 60672t^6 + 6s^5 + 1080s^4t + 29645s^3t^2 + 201705s^2t^3 + 374142st^4 + 157376t^5 + \\
& 144s^4 + 9648s^3t + 122175s^2t^2 + 381903st^3 + 260616t^4 + 1380s^3 + 42720s^2t + 247875st^2 + \\
& 283140t^3 + 6600s^2 + 93600st + 197500t^2 + 15750s + 81000t + 15000)(s+5)(t+1)^3.
\end{aligned}
$$

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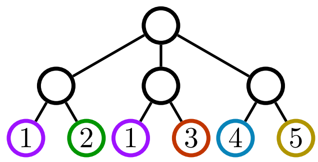

$$
\begin{aligned}
Z_C \;=\;& 3\lambda^5 + \lambda^4(q-5) + 44\lambda^4 + 59\lambda^3(q-5) + 33\lambda^2(q-5)(q-6) + 9\lambda(q-5)(q-6)(q-7) + \\
& (q-5)(q-6)(q-7)(q-8) + 109\lambda^3 + 180\lambda^2(q-5) + 96\lambda(q-5)(q-6) + \\
& 17(q-5)(q-6)(q-7) + 204\lambda^2 + 278\lambda(q-5) + 88(q-5)(q-6) + 199\lambda + 153q - 699.
\end{aligned}
$$

$$
\begin{aligned}
2Z_CU_C^v \;=\;& 7\lambda^5 + 2\lambda^4(q-5) + 72\lambda^4 + 69\lambda^3(q-5) + 24\lambda^2(q-5)(q-6) + 3\lambda(q-5)(q-6)(q-7) + \\
& 117\lambda^3 + 120\lambda^2(q-5) + 30\lambda(q-5)(q-6) + 122\lambda^2 + 82\lambda(q-5) + 57\lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_CU_C^N \;=\;& 15\lambda^5 + 4\lambda^4(q-5) + 176\lambda^4 + 177\lambda^3(q-5) + 66\lambda^2(q-5)(q-6) + 9\lambda(q-5)(q-6)(q-7) + \\
& 327\lambda^3 + 360\lambda^2(q-5) + 96\lambda(q-5)(q-6) + 408\lambda^2 + 278\lambda(q-5) + 199\lambda.
\end{aligned}
$$

$$\tilde{S}_C = 2(2r^4t^8 + 16r^4t^7 + 12r^3t^8 + 56r^4t^6 + 104r^3t^7 + 28r^2t^8 + 112r^4t^5 + 396r^3t^6 + 260r^2t^7 + 31rt^8 + 140r^4t^4 + 864r^3t^5 + 1064r^2t^6 + 301rt^7 + 14t^8 + 112r^4t^3 + 1180r^3t^4 + 2504r^2t^5 + 1298rt^6 + 138t^7 + 56r^4t^2 + 1032r^3t^3 + 3704r^2t^4 + 3246rt^5 + 612t^6 + 16r^4t + 564r^3t^2 + 3524r^2t^3 + 5146rt^4 + 1596t^5 + 2r^4 + 176r^3t + 2104r^2t^2 + 5292rt^3 + 2676t^4 + 24r^3 + 720r^2t + 3444rt^2 + 2952t^3 + 108r^2 + 1296rt + 2088t^2 + 216r + 864t + 162)(5r^4t^6 + 30r^4t^5 + 18r^3t^6 + 75r^4t^4 + 147r^3t^5 + 24r^2t^6 + 100r^4t^3 + 468r^3t^4 + 249r^2t^5 + 12rt^6 + 75r^4t^2 + 762r^3t^3 + 1008r^2t^4 + 168rt^5 + t^6 + 30r^4t + 678r^3t^2 + 2040r^2t^3 + 876rt^4 + 36t^5 + 5r^4 + 315r^3t + 2202r^2t^2 + 2241rt^3 + 252t^4 + 60r^3 + 1215r^2t + 3006rt^2 + 837t^3 + 270r^2 + 2025rt + 1431t^2 + 540r + 1215t + 405)(t+1)^3t.$$

$$D_C^U = (6s^5t^8 + 48s^5t^7 + 86s^4t^8 + 168s^5t^6 + 737s^4t^7 + 486s^3t^8 + 336s^5t^5 + 2757s^4t^6 + 4453s^3t^7 + 1354s^2t^8 + 420s^5t^4 + 5884s^4t^5 + 17787s^3t^6 + 13216s^2t^7 + 1860st^8 + 336s^5t^3 + 7840s^4t^4 + 40506s^3t^5 + 56274s^2t^6 + 19247st^7 + 1008t^8 + 168s^5t^2 + 6681s^4t^3 + 57584s^3t^4 + 136711s^2t^5 + 87123st^6 + 11004t^7 + 48s^5t + 3557s^4t^2 + 52377s^3t^3 + 207559s^2t^4 + 225599st^5 + 52716t^6 + 6s^5 + 1082s^4t + 29787s^3t^2 + 201945s^2t^3 + 366091st^4 + 145164t^5 + 144s^4 + 9688s^3t + 123105s^2t^2 + 381903st^3 + 251836t^4 + 1380s^3 + 43020s^2t + 250525st^2 + 282390t^3 + 6600s^2 + 94600st + 200250t^2 + 15750s + 82250t + 15000)(s+5)(t+1)^3.$$

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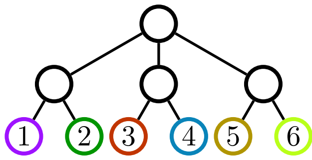

$$Z_C = 54\lambda^4 + 63\lambda^3(q-6) + 33\lambda^2(q-6)(q-7) + 9\lambda(q-6)(q-7)(q-8) + (q-6)(q-7)(q-8)(q-9) + 162\lambda^3 + 240\lambda^2(q-6) + 123\lambda(q-6)(q-7) + 21(q-6)(q-7)(q-8) + 378\lambda^2 + 474\lambda(q-6) + 139(q-6)(q-7) + 486\lambda + 328q - 1752.$$

$$2Z_CU_C^v = 90\lambda^4 + 75\lambda^3(q-6) + 24\lambda^2(q-6)(q-7) + 3\lambda(q-6)(q-7)(q-8) + 180\lambda^3 + 162\lambda^2(q-6) + 39\lambda(q-6)(q-7) + 234\lambda^2 + 144\lambda(q-6) + 144\lambda.$$

$$6Z_CU_C^N = 216\lambda^4 + 189\lambda^3(q-6) + 66\lambda^2(q-6)(q-7) + 9\lambda(q-6)(q-7)(q-8) + 486\lambda^3 + 480\lambda^2(q-6) + 123\lambda(q-6)(q-7) + 756\lambda^2 + 474\lambda(q-6) + 486\lambda.$$

$$\tilde{S}_C = 6(2r^4t^8 + 16r^4t^7 + 12r^3t^8 + 56r^4t^6 + 104r^3t^7 + 28r^2t^8 + 112r^4t^5 + 396r^3t^6 + 260r^2t^7 + 31rt^8 + 140r^4t^4 + 864r^3t^5 + 1064r^2t^6 + 301rt^7 + 14t^8 + 112r^4t^3 + 1180r^3t^4 + 2504r^2t^5 + 1298rt^6 + 138t^7 + 56r^4t^2 + 1032r^3t^3 + 3704r^2t^4 + 3246rt^5 + 612t^6 + 16r^4t + 564r^3t^2 + 3524r^2t^3 + 5146rt^4 + 1596t^5 + 2r^4 + 176r^3t + 2104r^2t^2 + 5292rt^3 + 2676t^4 + 24r^3 + 720r^2t + 3444rt^2 + 2952t^3 + 108r^2 + 1296rt + 2088t^2 + 216r + 864t + 162)(2r^3t^4 + 8r^3t^3 + 7r^2t^4 + 12r^3t^2 + 38r^2t^3 + 9rt^4 + 8r^3t + 73r^2t^2 + 57rt^3 + 4t^4 + 2r^3 + 60r^2t + 138rt^2 + 27t^3 + 18r^2 + 144rt + 81t^2 + 54r + 108t + 54)(rt + r + 3)(t+1)^4t.$$

$$D_C^U = 3(2s^4t^6 + 12s^4t^5 + 32s^3t^6 + 30s^4t^4 + 203s^3t^5 + 190s^2t^6 + 40s^4t^3 + 537s^3t^4 + 1270s^2t^5 + 496st^6 + 30s^4t^2 + 759s^3t^3 + 3549s^2t^4 + 3471st^5 + 480t^6 + 12s^4t + 605s^3t^2 + 5318s^2t^3 + 10224st^4 + 3480t^5 + 2s^4 + 258s^3t + 4515s^2t^2 + 16253st^3 + 10770t^4 + 46s^3 + 2062s^2t + 14740st^2 + 18190t^3 + 396s^2 + 7248st + 17700t^2 + 1512s + 9432t + 2160)(st + s + 3t + 6)(s+6)(t+1)^4.$$

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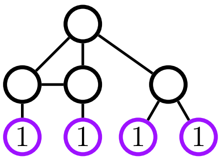

$$
\begin{aligned}
Z_C &= \lambda^8 + 5\lambda^5(q-1) + 5\lambda^4(q-1) + 8\lambda^3(q-1)(q-2) + \lambda^2(q-1)(q-2)(q-3) + \\
&\quad 2\lambda^3(q-1) + 9\lambda^2(q-1)(q-2) + 6\lambda(q-1)(q-2)(q-3) + (q-1)(q-2)(q-3)(q-4) + \\
&\quad 2\lambda^2(q-1) + 6\lambda(q-1)(q-2) + 3(q-1)(q-2)(q-3) + \lambda(q-1) + 2(q-1)(q-2). \\
2Z_C U_C^v &= 3\lambda^8 + 8\lambda^5(q-1) + 7\lambda^4(q-1) + 5\lambda^3(q-1)(q-2) + 4\lambda^3(q-1) + 11\lambda^2(q-1)(q-2) + \\
&\quad 3\lambda(q-1)(q-2)(q-3) + 2\lambda^2(q-1) + 2\lambda(q-1)(q-2). \\
6Z_C U_C^N &= 9\lambda^8 + 28\lambda^5(q-1) + 23\lambda^4(q-1) + 27\lambda^3(q-1)(q-2) + 2\lambda^2(q-1)(q-2)(q-3) + \\
&\quad 6\lambda^3(q-1) + 19\lambda^2(q-1)(q-2) + 7\lambda(q-1)(q-2)(q-3) + 4\lambda^2(q-1) + \\
&\quad 8\lambda(q-1)(q-2) + 2\lambda(q-1). \\
\tilde{S}_C &= 2(2r^8t^{15} + 30r^8t^{14} + 22r^7t^{15} + 210r^8t^{13} + 348r^7t^{14} + 107r^6t^{15} + 910r^8t^{12} + 2572r^7t^{13} + \\
&\quad 1772r^6t^{14} + 300r^5t^{15} + 2730r^8t^{11} + 11776r^7t^{12} + 13752r^6t^{13} + 5168r^5t^{14} + 529r^4t^{15} + \\
&\quad 6006r^8t^{10} + 37338r^7t^{11} + 66265r^6t^{12} + 41910r^5t^{13} + 9431r^4t^{14} + 597r^3t^{15} + 10010r^8t^9 + \\
&\quad 86812r^7t^{10} + 221472r^6t^{11} + 211780r^5t^{12} + 79589r^4t^{13} + 10995r^3t^{14} + 415r^2t^{15} + \\
&\quad 12870r^8t^8 + 152856r^7t^9 + 543324r^6t^{10} + 744278r^5t^{11} + 420433r^4t^{12} + 96326r^3t^{13} + \\
&\quad 7942r^2t^{14} + 158rt^{15} + 12870r^8t^7 + 207504r^7t^8 + 1009908r^6t^9 + 1923580r^5t^{10} + \\
&\quad 1550155r^4t^{11} + 530694r^3t^{12} + 72342r^2t^{13} + 3208rt^{14} + 24t^{15} + 10010r^8t^6 + 218922r^7t^7 + \\
&\quad 1447350r^6t^8 + 3771308r^5t^9 + 4214356r^4t^{10} + 2048859r^3t^{11} + 415447r^2t^{12} + 30651rt^{13} + \\
&\quad 544t^{14} + 6006r^8t^5 + 179476r^7t^6 + 1611639r^6t^7 + 5704600r^5t^8 + 8707559r^4t^9 + \\
&\quad 5851225r^3t^{10} + 1677481r^2t^{11} + 183991rt^{12} + 5566t^{13} + 2730r^8t^4 + 113388r^7t^5 + \\
&\quad 1393760r^6t^6 + 6705848r^5t^7 + 13897154r^4t^8 + 12729936r^3t^9 + 5026737r^2t^{10} + 777101rt^{11} + \\
&\quad 35186t^{12} + 910r^8t^3 + 54208r^7t^4 + 928196r^6t^5 + 6121480r^5t^6 + 17247938r^4t^7 + \\
&\quad 21428316r^3t^8 + 11506629r^2t^9 + 2441885rt^{10} + 155742t^{11} + 210r^8t^2 + 18982r^7t^3 + \\
&\quad 467361r^6t^4 + 4301614r^5t^5 + 16628045r^4t^6 + 28079804r^3t^7 + 20421375r^2t^8 + 5877780rt^9 + \\
&\quad 513150t^{10} + 30r^8t + 4596r^7t^2 + 172198r^6t^3 + 2284180r^5t^4 + 12340032r^4t^5 + 28600134r^3t^6 + \\
&\quad 28254375r^2t^7 + 10995174rt^8 + 1298340t^9 + 2r^8 + 688r^7t + 43824r^6t^2 + 886950r^5t^3 + \\
&\quad 6918813r^4t^4 + 22431804r^3t^5 + 30413628r^2t^6 + 16062570rt^7 + 2559519t^8 + 48r^7 + 6888r^6t + \\
&\quad 237708r^5t^2 + 2835810r^4t^3 + 13294656r^3t^4 + 25226640r^2t^5 + 18278622rt^6 + 3948642t^7 + \\
&\quad 504r^6 + 39312r^5t + 801900r^4t^2 + 5760450r^3t^3 + 15819651r^2t^4 + 16042698rt^5 + 4752189t^6 + \\
&\quad 3024r^5 + 139860r^4t + 1722060r^3t^2 + 7256466r^2t^3 + 10654092rt^4 + 4415958t^5 + 11340r^4 + \\
&\quad 317520r^3t + 2297808r^2t^2 + 5180274rt^3 + 3108213t^4 + 27216r^3 + 449064r^2t + 1740852rt^2 + \\
&\quad 1603800t^3 + 40824r^2 + 361584rt + 572994t^2 + 34992r + 126846t + 13122)(r+2)(t+1)^3. \\
\tilde{S}_C|_{q=2} &= 2(4t^3 + 7t^2 + 6t + 2)(t+2)^8(t+1)^3. \\
D_C^U &= (4s^4t^8 + 32s^4t^7 + 22s^3t^8 + 112s^4t^6 + 195s^3t^7 + 44s^2t^8 + 224s^4t^5 + 753s^3t^6 + 436s^2t^7 + \\
&\quad 38st^8 + 280s^4t^4 + 1658s^3t^5 + 1859s^2t^6 + 423st^7 + 12t^8 + 224s^4t^3 + 2280s^3t^4 + 4496s^2t^5 + \\
&\quad 1989st^6 + 150t^7 + 112s^4t^2 + 2007s^3t^3 + 6790s^2t^4 + 5272st^5 + 774t^6 + 32s^4t + 1105s^3t^2 + \\
&\quad 6584s^2t^3 + 8734st^4 + 2240t^5 + 4s^4 + 348s^3t + 4011s^2t^2 + 9341st^3 + 4068t^4 + 48s^3 + 1404s^2t + \\
&\quad 6327st^2 + 4812t^3 + 216s^2 + 2484st + 3645t^2 + 432s + 1620t + 324)(s+3)(s+2)(t+1)^3. \\
D_C^U|_{q=2} &= 2(3t^2 + 6t + 4)(t+2)^4(t+1)^3.
\end{aligned}
$$

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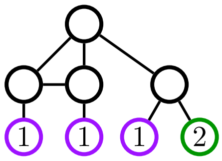

$$
\begin{aligned}
Z_C \;&= \lambda^7 + \lambda^6 + \lambda^5(q-2) + 2\lambda^5 + 8\lambda^4(q-2) + 2\lambda^3(q-2)(q-3) + 6\lambda^4 + 16\lambda^3(q-2) + \\
&21\lambda^2(q-2)(q-3) + 8\lambda(q-2)(q-3)(q-4) + (q-2)(q-3)(q-4)(q-5) + 5\lambda^3 + \\
&29\lambda^2(q-2) + 24\lambda(q-2)(q-3) + 6(q-2)(q-3)(q-4) + \lambda^2 + 9\lambda(q-2) + \\
&8(q-2)(q-3) + 2q - 4.
\end{aligned}
$$

$$
\begin{aligned}
2Z_C U_C^v \;&= 3\lambda^7 + 2\lambda^6 + 2\lambda^5(q-2) + 4\lambda^5 + 12\lambda^4(q-2) + 2\lambda^3(q-2)(q-3) + 10\lambda^4 + \\
&20\lambda^3(q-2) + 17\lambda^2(q-2)(q-3) + 3\lambda(q-2)(q-3)(q-4) + 5\lambda^3 + 21\lambda^2(q-2) + \\
&8\lambda(q-2)(q-3) + 2\lambda(q-2).
\end{aligned}
$$

$$
\begin{aligned}
6Z_C U_C^N \;&= 8\lambda^7 + 7\lambda^6 + 6\lambda^5(q-2) + 12\lambda^5 + 38\lambda^4(q-2) + 8\lambda^3(q-2)(q-3) + 26\lambda^4 + \\
&52\lambda^3(q-2) + 45\lambda^2(q-2)(q-3) + 9\lambda(q-2)(q-3)(q-4) + 16\lambda^3 + 63\lambda^2(q-2) + \\
&28\lambda(q-2)(q-3) + 3\lambda^2 + 12\lambda(q-2).
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C \;&= (4r^8t^{14} + 56r^8t^{13} + 44r^7t^{14} + 364r^8t^{12} + 652r^7t^{13} + 208r^6t^{14} + 1456r^8t^{11} + 4484r^7t^{12} + \\
&3266r^6t^{13} + 554r^5t^{14} + 4004r^8t^{10} + 18968r^7t^{11} + 23784r^6t^{12} + 9220r^5t^{13} + 910r^4t^{14} + \\
&8008r^8t^9 + 55132r^7t^{10} + 106466r^6t^{11} + 71092r^5t^{12} + 16066r^4t^{13} + 940r^3t^{14} + 12012r^8t^8 + \\
&116468r^7t^9 + 327264r^6t^{10} + 336710r^5t^{11} + 131148r^4t^{12} + 17675r^3t^{13} + 590r^2t^{14} + \\
&13728r^8t^7 + 184404r^7t^8 + 730716r^6t^9 + 1094526r^5t^{10} + 656896r^4t^{11} + 152915r^3t^{12} + \\
&11937r^2t^{13} + 202rt^{14} + 12012r^8t^6 + 222288r^7t^7 + 1222092r^6t^8 + 2583346r^5t^9 + \\
&2257058r^4t^{10} + 809875r^3t^{11} + 109880r^2t^{12} + 4490rt^{13} + 28t^{14} + 8008r^8t^5 + 204996r^7t^6 + \\
&1555188r^6t^7 + 4565542r^5t^8 + 5629710r^4t^9 + 2939665r^3t^{10} + 615979r^2t^{11} + 44347rt^{12} + \\
&712t^{13} + 4004r^8t^4 + 143924r^7t^5 + 1513152r^6t^6 + 6137370r^5t^7 + 10513862r^4t^8 + \\
&7744626r^3t^9 + 2361556r^2t^{10} + 263944rt^{11} + 7662t^{12} + 1456r^8t^3 + 75724r^7t^4 + \\
&1120138r^6t^5 + 6305888r^5t^6 + 14935738r^4t^7 + 15280200r^3t^8 + 6567844r^2t^9 + 1069469rt^{10} + \\
&48700t^{11} + 364r^8t^2 + 28952r^7t^3 + 621008r^6t^4 + 4927678r^5t^5 + 16217196r^4t^6 + \\
&22941188r^3t^7 + 13683768r^2t^8 + 3138850rt^9 + 208914t^{10} + 56r^8t + 7604r^7t^2 + 250026r^6t^3 + \\
&2882722r^5t^4 + 13392248r^4t^5 + 26337976r^3t^6 + 21709866r^2t^7 + 6902040rt^8 + 647076t^9 + \\
&4r^8 + 1228r^7t + 69104r^6t^2 + 1224152r^5t^3 + 8278770r^4t^4 + 23007300r^3t^5 + 26361756r^2t^6 + \\
&11567754rt^7 + 1500912t^8 + 92r^7 + 11736r^6t + 356688r^5t^2 + 3714366r^4t^3 + 15050172r^3t^4 + \\
&24378930r^2t^5 + 14857560rt^6 + 2656044t^7 + 924r^6 + 63828r^5t + 1143180r^4t^2 + \\
&7147080r^3t^3 + 16898112r^2t^4 + 14554674rt^5 + 3607848t^6 + 5292r^5 + 216000r^4t + \\
&2328372r^3t^2 + 8509590r^2t^3 + 10702854rt^4 + 3745116t^5 + 18900r^4 + 465588r^3t + \\
&2941272r^2t^2 + 5726376rt^3 + 2924586t^4 + 43092r^3 + 624024r^2t + 2105352rt^2 + 1665522t^3 + \\
&61236r^2 + 475308rt + 653184t^2 + 49572r + 157464t + 17496)(rt + r + 2t + 3)(t+1)^3.
\end{aligned}
$$

$$
\tilde{S}_C|_{q=2} = 4(t^3 + 2t^2 + 3t + 1)(t+2)^8(t+1)^3.
$$

$$
\begin{aligned}
D_C^U \;&= (4s^4t^8 + 32s^4t^7 + 22s^3t^8 + 112s^4t^6 + 195s^3t^7 + 44s^2t^8 + 224s^4t^5 + 753s^3t^6 + 436s^2t^7 + \\
&38st^8 + 280s^4t^4 + 1658s^3t^5 + 1859s^2t^6 + 423st^7 + 12t^8 + 224s^4t^3 + 2280s^3t^4 + 4496s^2t^5 + \\
&1989st^6 + 150t^7 + 112s^4t^2 + 2007s^3t^3 + 6790s^2t^4 + 5272st^5 + 774t^6 + 32s^4t + 1105s^3t^2 + \\
&6584s^2t^3 + 8734st^4 + 2240t^5 + 4s^4 + 348s^3t + 4011s^2t^2 + 9341st^3 + 4068t^4 + 48s^3 + 1404s^2t + \\
&6327st^2 + 4812t^3 + 216s^2 + 2484st + 3645t^2 + 432s + 1620t + 324)(st + s + t + 2)(s+3)(t+1)^2.
\end{aligned}
$$

$$
D_C^U|_{q=2} = 2(3t^2 + 6t + 4)(t+2)^4(t+1)^2.
$$

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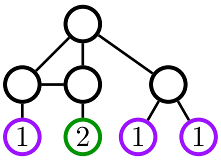

$$
\begin{aligned}
Z_C \;&= \lambda^7+2\lambda^6+2\lambda^5(q-2)+3\lambda^5+12\lambda^4(q-2)+7\lambda^3(q-2)(q-3)+\lambda^2(q-2)(q-3)(q-4)+\\
&4\lambda^4+15\lambda^3(q-2)+15\lambda^2(q-2)(q-3)+6\lambda(q-2)(q-3)(q-4)+\\
&(q-2)(q-3)(q-4)(q-5)+3\lambda^3+17\lambda^2(q-2)+21\lambda(q-2)(q-3)+\\
&7(q-2)(q-3)(q-4)+2\lambda^2+15\lambda(q-2)+12(q-2)(q-3)+\lambda+4q-8.\\
2Z_CU_C^v &= 3\lambda^7+4\lambda^6+4\lambda^5(q-2)+5\lambda^5+14\lambda^4(q-2)+5\lambda^3(q-2)(q-3)+6\lambda^4+17\lambda^3(q-2)+\\
&11\lambda^2(q-2)(q-3)+3\lambda(q-2)(q-3)(q-4)+3\lambda^3+13\lambda^2(q-2)+\\
&11\lambda(q-2)(q-3)+2\lambda^2+9\lambda(q-2)+\lambda.\\
6Z_CU_C^N &= 8\lambda^7+13\lambda^6+11\lambda^5(q-2)+17\lambda^5+54\lambda^4(q-2)+23\lambda^3(q-2)(q-3)+\\
&2\lambda^2(q-2)(q-3)(q-4)+18\lambda^4+49\lambda^3(q-2)+33\lambda^2(q-2)(q-3)+\\
&7\lambda(q-2)(q-3)(q-4)+10\lambda^3+39\lambda^2(q-2)+25\lambda(q-2)(q-3)+5\lambda^2+18\lambda(q-2)+\lambda.\\
\tilde{S}_C \;&= (4r^9t^{16}+64r^9t^{15}+56r^8t^{16}+480r^9t^{14}+936r^8t^{15}+344r^7t^{16}+2240r^9t^{13}+7332r^8t^{14}+\\
&6008r^7t^{15}+1218r^6t^{16}+7280r^9t^{12}+35724r^8t^{13}+49178r^7t^{14}+22248r^6t^{15}+\\
&2747r^5t^{16}+17472r^9t^{11}+121160r^8t^{12}+250336r^7t^{13}+190418r^6t^{14}+52515r^5t^{15}+\\
&4112r^4t^{16}+32032r^9t^{10}+303264r^8t^{11}+886706r^7t^{12}+1013216r^6t^{13}+470127r^5t^{14}+\\
&82245r^4t^{15}+4115r^3t^{16}+45760r^9t^9+579436r^8t^{10}+2316724r^7t^{11}+3749964r^6t^{12}+\\
&2615189r^5t^{13}+769711r^4t^{14}+85886r^3t^{15}+2682r^2t^{16}+51480r^9t^8+862004r^8t^9+\\
&4617556r^7t^{10}+10232436r^6t^{11}+10113999r^5t^{12}+4473676r^4t^{13}+838579r^3t^{14}+\\
&58030r^2t^{15}+1046rt^{16}+45760r^9t^7+1009008r^8t^8+7160496r^7t^9+21287484r^6t^{10}+\\
&28825339r^5t^{11}+18070074r^4t^{12}+5084426r^3t^{13}+588544r^2t^{14}+23182rt^{15}+188t^{16}+\\
&32032r^9t^6+932360r^8t^7+8729532r^7t^8+34432992r^6t^9+62605753r^5t^{10}+\\
&53770851r^4t^{11}+21423163r^3t^{12}+3711467r^2t^{13}+242417rt^{14}+4196t^{15}+17472r^9t^5+\\
&677820r^8t^6+8393552r^7t^7+43753866r^6t^8+105665559r^5t^9+121896961r^4t^{10}+\\
&66498786r^3t^{11}+16279291r^2t^{12}+1582660rt^{13}+44778t^{14}+7280r^9t^4+383604r^8t^5+\\
&6343346r^7t^6+43812864r^6t^7+140019440r^5t^8+214675450r^4t^9+157253303r^3t^{10}+\\
&52635547r^2t^{11}+7205113rt^{12}+300812t^{13}+2240r^9t^3+165672r^8t^4+3728048r^7t^5+\\
&34451082r^6t^6+146114290r^5t^7+296722376r^4t^8+288879206r^3t^9+129709697r^2t^{10}+\\
&24218883rt^{11}+1416222t^{12}+480r^9t^2+52784r^8t^3+1670218r^7t^4+21045200r^6t^5+\\
&119638620r^5t^6+322830314r^4t^7+416463980r^3t^8+248394900r^2t^9+62114124rt^{10}+\\
&4937946t^{11}+64r^9t+11700r^8t^2+551396r^7t^3+9789224r^6t^4+76032544r^5t^5+\\
&275439646r^4t^6+472527116r^3t^7+373394412r^2t^8+123892263rt^9+13161564t^{10}+4r^9+\\
&1612r^8t+126496r^7t^2+3351300r^6t^3+36753882r^5t^4+182267148r^4t^5+420322494r^3t^6+\\
&441829224r^2t^7+194096880rt^8+27317358t^9+104r^8+18016r^7t+796200r^6t^2+\\
&13059396r^5t^3+91653024r^4t^4+289844232r^3t^5+409893066r^2t^6+239481954rt^7+\\
&44576406t^8+1200r^7+117264r^6t+3215376r^5t^2+33833700r^4t^3+151766334r^3t^4+\\
&294750576r^2t^5+231755796rt^6+57332502t^7+8064r^6+489888r^5t+8640000r^4t^2+\\
&58269132r^3t^3+160869240r^2t^4+173884968rt^5+57878226t^6+34776r^5+1362312r^4t+\\
&15448320r^3t^2+64316268r^2t^3+99015534rt^4+45328896t^5+99792r^4+2522016r^3t+\\
&17723448r^2t^2+41275980rt^3+26953560t^4+190512r^3+2997648r^2t+11838960rt^2+\\
&11731068t^3+233280r^2+2076192rt+3507948t^2+166212r+638604t+52488)(t+1)^2.\\
\tilde{S}_C|_{q=2} &= 2(t^4+4t^3+7t^2+6t+1)(t^2+2t+2)(t+2)^8(t+1)^2.
\end{aligned}
$$

$$
\begin{aligned}
D_C^U &= (4s^5t^9+36s^5t^8+28s^4t^9+144s^5t^7+269s^4t^8+79s^3t^9+336s^5t^6+1152s^4t^7+802s^3t^8+\\
&113s^2t^9+504s^5t^5+2887s^4t^6+3645s^3t^7+1195s^2t^8+82st^9+504s^5t^4+4666s^4t^5+\\
&9743s^3t^6+5700s^2t^7+890st^8+24t^9+336s^5t^3+5043s^4t^4+16889s^3t^5+16121s^2t^6+\\
&4401st^7+264t^8+144s^5t^2+3644s^4t^3+19692s^3t^4+29821s^2t^5+13053st^6+1338t^7+\\
&36s^5t+1697s^4t^2+15439s^3t^3+37425s^2t^4+25626st^5+4126t^6+4s^5+462s^4t+7843s^3t^2+\\
&31854s^2t^3+34530st^4+8552t^5+56s^4+2340s^3t+17715s^2t^2+31909st^3+12342t^4+312s^3+\\
&5832s^2t+19485st^2+12378t^3+864s^2+7128st+8316t^2+1188s+3402t+648)(s+3)(t+1)^2.\\
D_C^U|_{q=2} &= 4(t^2+2t+2)(t+2)^4(t+1)^2.
\end{aligned}
$$

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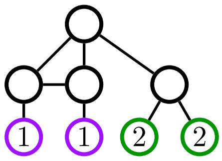

$$
\begin{aligned}
Z_C &= \lambda^7+3\lambda^6+3\lambda^5(q-2)+2\lambda^5+11\lambda^4(q-2)+8\lambda^3(q-2)(q-3)+\\
&\lambda^2(q-2)(q-3)(q-4)+2\lambda^4+14\lambda^3(q-2)+12\lambda^2(q-2)(q-3)+\\
&6\lambda(q-2)(q-3)(q-4)+(q-2)(q-3)(q-4)(q-5)+5\lambda^3+16\lambda^2(q-2)+\\
&24\lambda(q-2)(q-3)+7(q-2)(q-3)(q-4)+3\lambda^2+19\lambda(q-2)+11(q-2)(q-3)+2q-4.\\
2Z_CU_C^v &= 2\lambda^7+7\lambda^6+4\lambda^5(q-2)+4\lambda^5+15\lambda^4(q-2)+5\lambda^3(q-2)(q-3)+2\lambda^4+14\lambda^3(q-2)+\\
&11\lambda^2(q-2)(q-3)+3\lambda(q-2)(q-3)(q-4)+6\lambda^3+16\lambda^2(q-2)+\\
&11\lambda(q-2)(q-3)+3\lambda^2+8\lambda(q-2).\\
6Z_CU_C^N &= 8\lambda^7+21\lambda^6+18\lambda^5(q-2)+10\lambda^5+47\lambda^4(q-2)+27\lambda^3(q-2)(q-3)+\\
&2\lambda^2(q-2)(q-3)(q-4)+8\lambda^4+48\lambda^3(q-2)+25\lambda^2(q-2)(q-3)+\\
&7\lambda(q-2)(q-3)(q-4)+18\lambda^3+34\lambda^2(q-2)+29\lambda(q-2)(q-3)+7\lambda^2+24\lambda(q-2).
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C &= 2(2r^9t^{17} + 34r^9t^{16} + 30r^8t^{17} + 272r^9t^{15} + 528r^8t^{16} + 193r^7t^{17} + 1360r^9t^{14} + \\
&\quad 4374r^8t^{15} + 3523r^7t^{16} + 706r^6t^{17} + 4760r^9t^{13} + 22648r^8t^{14} + 30289r^7t^{15} + 13377r^6t^{16} + \\
&\quad 1632r^5t^{17} + 12376r^9t^{12} + 82082r^8t^{13} + 162831r^7t^{14} + 119478r^6t^{15} + 32080r^5t^{16} + \\
&\quad 2488r^4t^{17} + 24752r^9t^{11} + 220948r^8t^{12} + 612812r^7t^{13} + 667658r^6t^{14} + 297594r^5t^{15} + \\
&\quad 50655r^4t^{16} + 2511r^3t^{17} + 38896r^9t^{10} + 457366r^8t^{11} + 1712852r^7t^{12} + 2612872r^6t^{13} + \\
&\quad 1728785r^5t^{14} + 487512r^4t^{15} + 52850r^3t^{16} + 1620r^2t^{17} + 48620r^9t^9 + 743600r^8t^{10} + \\
&\quad 3680782r^7t^{11} + 7595415r^6t^{12} + 7037884r^5t^{13} + 2942040r^4t^{14} + 526887r^3t^{15} + \\
&\quad 35199r^2t^{16} + 606rt^{17} + 48620r^9t^8 + 961246r^8t^9 + 6209918r^7t^{10} + 16974252r^6t^{11} + \\
&\quad 21291234r^5t^{12} + 12454960r^4t^{13} + 3299365r^3t^{14} + 362982r^2t^{15} + 13588rt^{16} + 100t^{17} + \\
&\quad 38896r^9t^7 + 993564r^8t^8 + 8325537r^7t^9 + 29774808r^6t^{10} + 49529088r^5t^{11} + \\
&\quad 39212682r^4t^{12} + 14513697r^3t^{13} + 2355570r^2t^{14} + 144792rt^{15} + 2316t^{16} + 24752r^9t^6 + \\
&\quad 821106r^8t^7 + 8918899r^7t^8 + 41485986r^6t^9 + 90436566r^5t^{10} + 94982130r^4t^{11} + \\
&\quad 47533145r^3t^{12} + 10756318r^2t^{13} + 972620rt^{14} + 25488t^{15} + 12376r^9t^5 + 539448r^8t^6 + \\
&\quad 7633285r^7t^7 + 46159759r^6t^8 + 131141028r^5t^9 + 180641225r^4t^{10} + 119871384r^3t^{11} + \\
&\quad 36619498r^2t^{12} + 4605495rt^{13} + 177048t^{14} + 4760r^9t^4 + 278278r^8t^5 + 5188915r^7t^6 + \\
&\quad 41000622r^6t^7 + 151799532r^5t^8 + 272864112r^4t^9 + 237500090r^3t^{10} + 96108204r^2t^{11} + \\
&\quad 16285593rt^{12} + 868478t^{13} + 1360r^9t^3 + 110348r^8t^4 + 2766954r^7t^5 + 28898418r^6t^6 + \\
&\quad 140185306r^5t^7 + 328975930r^4t^8 + 373890368r^3t^9 + 198351034r^2t^{10} + 44457254rt^{11} + \\
&\quad 3187250t^{12} + 272r^9t^2 + 32482r^8t^3 + 1133006r^7t^4 + 15960212r^6t^5 + 102642729r^5t^6 + \\
&\quad 316324390r^4t^7 + 469894193r^3t^8 + 325496542r^2t^9 + 95546900rt^{10} + 9044796t^{11} + \\
&\quad 34r^9t + 6688r^8t^2 + 344012r^7t^3 + 6760153r^6t^4 + 58825220r^5t^5 + 241007787r^4t^6 + \\
&\quad 470973223r^3t^7 + 426633918r^2t^8 + 163435575rt^9 + 20235840t^{10} + 2r^9 + 860r^8t + \\
&\quad 72976r^7t^2 + 2120112r^6t^3 + 25820158r^5t^4 + 143594604r^4t^5 + 373934139r^3t^6 + \\
&\quad 446108022r^2t^7 + 223468308rt^8 + 36075132t^9 + 52r^8 + 9656r^7t + 463788r^6t^2 + \\
&\quad 8377308r^5t^3 + 65438517r^4t^4 + 232019454r^3t^5 + 369537102r^2t^6 + 243912546rt^7 + \\
&\quad 51460191t^8 + 600r^7 + 63168r^6t + 1891944r^5t^2 + 22003380r^4t^3 + 109990818r^3t^4 + \\
&\quad 239161248r^2t^5 + 210998169rt^6 + 58650588t^7 + 4032r^6 + 265356r^5t + 5137236r^4t^2 + \\
&\quad 38403396r^3t^3 + 118164771r^2t^4 + 142629660rt^5 + 53020413t^6 + 17388r^5 + 742392r^4t + \\
&\quad 9284544r^3t^2 + 42931296r^2t^3 + 73581210rt^4 + 37477242t^5 + 49896r^4 + 1383480r^3t + \\
&\quad 10768788r^2t^2 + 27879876rt^3 + 20221731t^4 + 95256r^3 + 1656288r^2t + 7272504rt^2 + \\
&\quad 8008794t^3 + 116640r^2 + 1156194rt + 2178252t^2 + 83106r + 358668t + 26244)(t+1). \\
\tilde{S}_C|_{q=2} &= 2(3t^6 + 14t^5 + 33t^4 + 48t^3 + 41t^2 + 18t + 2)(t+2)^8(t+1). \\
D_C^U &= \\
&\quad (4s^4t^8+32s^4t^7+22s^3t^8+112s^4t^6+195s^3t^7+44s^2t^8+224s^4t^5+753s^3t^6+436s^2t^7+38st^8+ \\
&\quad 280s^4t^4+1658s^3t^5+1859s^2t^6+423st^7+12t^8+224s^4t^3+2280s^3t^4+4496s^2t^5+1989st^6+ \\
&\quad 150t^7+112s^4t^2+2007s^3t^3+6790s^2t^4+5272st^5+774t^6+32s^4t+1105s^3t^2+6584s^2t^3+ \\
&\quad 8734st^4+2240t^5+4s^4+348s^3t+4011s^2t^2+9341st^3+4068t^4+48s^3+1404s^2t+6327st^2+ \\
&\quad 4812t^3+216s^2+2484st+3645t^2+432s+1620t+324)(st^2+2st+t^2+s+2t+2)(s+3)(t+1). \\
D_C^U|_{q=2} &= 2(3t^2 + 6t + 4)(t+2)^4(t+1).
\end{aligned}
$$

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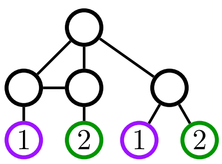

$$\begin{aligned}
Z_C = {} & 2\lambda^6+4\lambda^5+7\lambda^4(q-2)+\lambda^3(q-2)(q-3)+4\lambda^4+25\lambda^3(q-2)+24\lambda^2(q-2)(q-3)+\\
& 8\lambda(q-2)(q-3)(q-4)+(q-2)(q-3)(q-4)(q-5)+4\lambda^3+18\lambda^2(q-2)+\\
& 21\lambda(q-2)(q-3)+6(q-2)(q-3)(q-4)+2\lambda^2+13\lambda(q-2)+9(q-2)(q-3)+2q-4.
\end{aligned}$$

$$\begin{aligned}
2Z_CU_C^v = {} & 6\lambda^6+8\lambda^5+15\lambda^4(q-2)+2\lambda^3(q-2)(q-3)+4\lambda^4+23\lambda^3(q-2)+17\lambda^2(q-2)(q-3)+\\
& 3\lambda(q-2)(q-3)(q-4)+4\lambda^3+14\lambda^2(q-2)+8\lambda(q-2)(q-3)+2\lambda^2+5\lambda(q-2).
\end{aligned}$$

$$\begin{aligned}
6Z_CU_C^N = {} & 14\lambda^6+22\lambda^5+33\lambda^4(q-2)+4\lambda^3(q-2)(q-3)+18\lambda^4+83\lambda^3(q-2)+53\lambda^2(q-2)(q-3)+\\
& 9\lambda(q-2)(q-3)(q-4)+14\lambda^3+40\lambda^2(q-2)+24\lambda(q-2)(q-3)+4\lambda^2+15\lambda(q-2).
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_C = {} & 2(2r^9t^{14}+28r^9t^{13}+26r^8t^{14}+182r^9t^{12}+386r^8t^{13}+150r^7t^{14}+728r^9t^{11}+2656r^8t^{12}+\\
& 2356r^7t^{13}+503r^6t^{14}+2002r^9t^{10}+11228r^8t^{11}+17132r^7t^{12}+8349r^6t^{13}+1082r^5t^{14}+\\
& 4004r^9t^9+32582r^8t^{10}+76452r^7t^{11}+64091r^6t^{12}+18948r^5t^{13}+1555r^4t^{14}+\\
& 6006r^9t^8+68662r^8t^9+233930r^7t^{10}+301645r^6t^{11}+153338r^5t^{12}+28627r^4t^{13}+\\
& 1501r^3t^{14}+6864r^9t^7+108372r^8t^8+519240r^7t^9+972583r^6t^{10}+760298r^5t^{11}+\\
& 243619r^4t^{12}+28877r^3t^{13}+942r^2t^{14}+6006r^9t^6+130152r^8t^7+862272r^7t^8+\\
& 2272875r^6t^9+2581016r^5t^{10}+1270386r^4t^{11}+257436r^3t^{12}+18801r^2t^{13}+349rt^{14}+\\
& 4004r^9t^5+119526r^8t^6+1088424r^7t^7+3970673r^6t^8+6347086r^5t^9+4535361r^4t^{10}+\\
& 1408325r^3t^{11}+174724r^2t^{12}+7178rt^{13}+58t^{14}+2002r^9t^4+83534r^8t^5+1049514r^7t^6+\\
& 5268479r^6t^7+11661826r^5t^8+11727539r^4t^9+5278986r^3t^{10}+999565r^2t^{11}+\\
& 69174rt^{12}+1224t^{13}+728r^9t^3+43736r^8t^4+769380r^7t^5+5335860r^6t^6+16265434r^5t^7+\\
& 22653365r^4t^8+14339118r^3t^9+3926007r^2t^{10}+412337rt^{11}+12172t^{12}+182r^9t^2+\\
& 16636r^8t^3+422140r^7t^4+4105380r^6t^5+17307668r^5t^6+33210656r^4t^7+29103325r^3t^8+\\
& 11188194r^2t^9+1693081rt^{10}+75318t^{11}+28r^9t+4346r^8t^2+168116r^7t^3+2362312r^6t^4+\\
& 13983454r^5t^5+37136434r^4t^6+44838724r^3t^7+23843502r^2t^8+5054880rt^9+\\
& 322518t^{10}+2r^9+698r^8t+45942r^7t^2+985904r^6t^3+8444864r^5t^4+31522428r^4t^5+\\
& 52697454r^3t^6+38594097r^2t^7+11302191rt^8+1007352t^9+52r^8+7712r^7t+282138r^6t^2+\\
& 3696948r^5t^3+19994868r^4t^4+47016732r^3t^5+47675988r^2t^6+19212930rt^7+\\
& 2361150t^8+600r^7+49560r^6t+1109106r^5t^2+9190584r^4t^3+31347252r^3t^4+\\
& 44728416r^2t^5+24946542rt^6+4213728t^7+4032r^6+204120r^5t+2893806r^4t^2+\\
& 15143868r^3t^3+31369248r^2t^4+24618654rt^5+5750352t^6+17388r^5+558684r^4t+\\
& 5010498r^3t^2+15944688r^2t^3+18174942rt^4+5970834t^5+49896r^4+1016064r^3t+\\
& 5550606r^2t^2+9730692rt^3+4643244t^4+95256r^3+1183896r^2t+3569184rt^2+\\
& 2621484t^3+116640r^2+801900rt+1014768t^2+83106r+240570t+26244)(t+1)^4.
\end{aligned}$$

$$\tilde{S}_C|_{q=2} = 4(t^3+2t^2+3t+1)(t+2)^8(t+1)^4.$$

$$\begin{aligned}
D_C^U = {} & (4s^5t^8+32s^5t^7+26s^4t^8+112s^5t^6+229s^4t^7+66s^3t^8+224s^5t^5+881s^4t^6+640s^3t^7+\\
& 82s^2t^8+280s^5t^4+1936s^4t^5+2699s^3t^6+872s^2t^7+50st^8+224s^5t^3+2660s^4t^4+6496s^3t^5+\\
& 4015s^2t^6+579st^7+12t^8+112s^5t^2+2341s^4t^3+9796s^3t^4+10551s^2t^5+2895st^6+150t^7+\\
& 32s^5t+1289s^4t^2+9500s^3t^3+17430s^2t^4+8277st^5+810t^6+4s^5+406s^4t+5791s^3t^2+\\
& 18639s^2t^3+14942st^4+2510t^5+56s^4+2028s^3t+12639s^2t^2+17608st^3+4932t^4+312s^3+\\
& 4968s^2t+13329st^2+6384t^3+864s^2+5940st+5400t^2+1188s+2754t+648)(s+3)(t+1)^3.
\end{aligned}$$

$$D_C^U|_{q=2} = 8(t+2)^4(t+1)^3.$$

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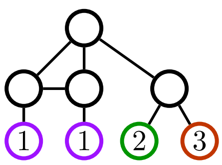

$$
\begin{aligned}
Z_C \; &= 3\lambda^6 + \lambda^5(q-3) + 4\lambda^5 + 6\lambda^4(q-3) + 2\lambda^3(q-3)(q-4) + 8\lambda^4 + 28\lambda^3(q-3) + \\
&\quad 21\lambda^2(q-3)(q-4) + 8\lambda(q-3)(q-4)(q-5) + (q-3)(q-4)(q-5)(q-6) + 30\lambda^3 + \\
&\quad 59\lambda^2(q-3) + 48\lambda(q-3)(q-4) + 10(q-3)(q-4)(q-5) + 20\lambda^2 + 65\lambda(q-3) + \\
&\quad 26(q-3)(q-4) + 16\lambda + 16q - 48. \\
2Z_C U_C^v &= 7\lambda^6 + 2\lambda^5(q-3) + 8\lambda^5 + 8\lambda^4(q-3) + 2\lambda^3(q-3)(q-4) + 12\lambda^4 + 36\lambda^3(q-3) + \\
&\quad 17\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + 38\lambda^3 + 43\lambda^2(q-3) + \\
&\quad 17\lambda(q-3)(q-4) + 10\lambda^2 + 22\lambda(q-3) + 6\lambda. \\
6Z_C U_C^N &= 21\lambda^6 + 6\lambda^5(q-3) + 24\lambda^5 + 30\lambda^4(q-3) + 8\lambda^3(q-3)(q-4) + 36\lambda^4 + 92\lambda^3(q-3) + \\
&\quad 45\lambda^2(q-3)(q-4) + 9\lambda(q-3)(q-4)(q-5) + 96\lambda^3 + 129\lambda^2(q-3) + \\
&\quad 55\lambda(q-3)(q-4) + 48\lambda^2 + 76\lambda(q-3) + 18\lambda. \\
\tilde{S}_C \; &= 2(2r^9t^{16} + 32r^9t^{15} + 30r^8t^{16} + 240r^9t^{14} + 496r^8t^{15} + 190r^7t^{16} + 1120r^9t^{13} + \\
&\quad 3846r^8t^{14} + 3272r^7t^{15} + 679r^6t^{16} + 3640r^9t^{12} + 18564r^8t^{13} + 26410r^7t^{14} + 12221r^6t^{15} + \\
&\quad 1527r^5t^{16} + 8736r^9t^{11} + 62426r^8t^{12} + 132624r^7t^{13} + 102994r^6t^{14} + 28740r^5t^{15} + \\
&\quad 2259r^4t^{16} + 16016r^9t^{10} + 155064r^8t^{11} + 463766r^7t^{12} + 539596r^6t^{13} + 253080r^5t^{14} + \\
&\quad 44405r^4t^{15} + 2210r^3t^{16} + 22880r^9t^9 + 294294r^8t^{10} + 1197400r^7t^{11} + 1967252r^6t^{12} + \\
&\quad 1384544r^5t^{13} + 408290r^4t^{14} + 45262r^3t^{15} + 1385r^2t^{16} + 25740r^9t^8 + 435292r^8t^9 + \\
&\quad 2361194r^7t^{10} + 5292570r^6t^{11} + 5268304r^5t^{12} + 2331773r^4t^{13} + 433889r^3t^{14} + \\
&\quad 29438r^2t^{15} + 508rt^{16} + 22880r^9t^7 + 507078r^8t^8 + 3627360r^7t^9 + 10869330r^6t^{10} + \\
&\quad 14786648r^5t^{11} + 9260658r^4t^{12} + 2584320r^3t^{13} + 293553r^2t^{14} + 11122rt^{15} + 84t^{16} + \\
&\quad 16016r^9t^6 + 466752r^8t^7 + 4387218r^7t^8 + 17381892r^6t^9 + 31669522r^5t^{10} + \\
&\quad 27125551r^4t^{11} + 10706419r^3t^{12} + 1820973r^2t^{13} + 114948rt^{14} + 1868t^{15} + 8736r^9t^5 + \\
&\quad 338338r^8t^6 + 4191352r^7t^7 + 21873963r^6t^8 + 52798808r^5t^9 + 60625090r^4t^{10} + \\
&\quad 32719790r^3t^{11} + 7862259r^2t^{12} + 741213rt^{13} + 19884t^{14} + 3640r^9t^4 + 191100r^8t^5 + \\
&\quad 3152270r^7t^6 + 21732973r^6t^7 + 69245503r^5t^8 + 105464877r^4t^9 + 76315475r^3t^{10} + \\
&\quad 25054353r^2t^{11} + 3331527rt^{12} + 132918t^{13} + 1120r^9t^3 + 82446r^8t^4 + 1846672r^7t^5 + \\
&\quad 16990394r^6t^6 + 71672540r^5t^7 + 144314633r^4t^8 + 138581496r^3t^9 + 60964129r^2t^{10} + \\
&\quad 11061474rt^{11} + 621114t^{12} + 240r^9t^2 + 26264r^8t^3 + 826050r^7t^4 + 10341016r^6t^5 + \\
&\quad 58346176r^5t^6 + 155827088r^4t^7 + 197993517r^3t^8 + 115556569r^2t^9 + 28064376rt^{10} + \\
&\quad 2146944t^{11} + 32r^9t + 5826r^8t^2 + 272744r^7t^3 + 4803298r^6t^4 + 36958760r^5t^5 + \\
&\quad 132301778r^4t^6 + 223259032r^3t^7 + 172433508r^2t^8 + 55509378rt^9 + 5675940t^{10} + 2r^9 + \\
&\quad 804r^8t + 62686r^7t^2 + 1645892r^6t^3 + 17855238r^5t^4 + 87369846r^4t^5 + 197965242r^3t^6 + \\
&\quad 203193063r^2t^7 + 86509998rt^8 + 11709072t^9 + 52r^8 + 8960r^7t + 392346r^6t^2 + \\
&\quad 6358824r^5t^3 + 43978902r^4t^4 + 136518204r^3t^5 + 188376498r^2t^6 + 106570620rt^7 + \\
&\quad 19053252t^8 + 600r^7 + 58128r^6t + 1574046r^5t^2 + 16305300r^4t^3 + 71730702r^3t^4 + \\
&\quad 135859680r^2t^5 + 103381704rt^6 + 24537924t^7 + 4032r^6 + 241920r^5t + 4197150r^4t^2 + \\
&\quad 27738072r^3t^3 + 74652678r^2t^4 + 78079788rt^5 + 24919488t^6 + 17388r^5 + 669816r^4t + \\
&\quad 7437258r^3t^2 + 30171852r^2t^3 + 44949330rt^4 + 19729494t^5 + 49896r^4 + 1233792r^3t + \\
&\quad 8443278r^2t^2 + 19029816rt^3 + 11919636t^4 + 95256r^3 + 1458000r^2t + 5571018rt^2 + \\
&\quad 5298372t^3 + 116640r^2 + 1003104rt + 1627128t^2 + 83106r + 306180t + 26244)(t+1)^2. \\
D_C^U \; &= (4s^4t^8 + 32s^4t^7 + 22s^3t^8 + 112s^4t^6 + 195s^3t^7 + 44s^2t^8 + 224s^4t^5 + 753s^3t^6 + 436s^2t^7 + \\
&\quad 38st^8 + 280s^4t^4 + 1658s^3t^5 + 1859s^2t^6 + 423st^7 + 12t^8 + 224s^4t^3 + 2280s^3t^4 + 4496s^2t^5 + \\
&\quad 1989st^6 + 150t^7 + 112s^4t^2 + 2007s^3t^3 + 6790s^2t^4 + 5272st^5 + 774t^6 + 32s^4t + 1105s^3t^2 + \\
&\quad 6584s^2t^3 + 8734st^4 + 2240t^5 + 4s^4 + 348s^3t + 4011s^2t^2 + 9341st^3 + 4068t^4 + 48s^3 + 1404s^2t + \\
&\quad 6327st^2 + 4812t^3 + 216s^2 + 2484st + 3645t^2 + 432s + 1620t + 324)(st+s+2)(s+3)(t+1)^2.
\end{aligned}
$$

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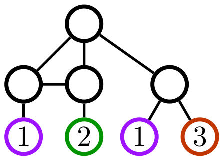

$$
\begin{aligned}
Z_C &= \lambda^6 + 6\lambda^5 + 6\lambda^4(q-3) + \lambda^3(q-3)(q-4) + 13\lambda^4 + 31\lambda^3(q-3) + 24\lambda^2(q-3)(q-4) + \\
&\quad 8\lambda(q-3)(q-4)(q-5) + (q-3)(q-4)(q-5)(q-6) + 21\lambda^3 + 60\lambda^2(q-3) + \\
&\quad 45\lambda(q-3)(q-4) + 10(q-3)(q-4)(q-5) + 27\lambda^2 + 59\lambda(q-3) + 27(q-3)(q-4) + \\
&\quad 11\lambda + 19q - 55. \\
2Z_C U_C^v &= 3\lambda^6+14\lambda^5+13\lambda^4(q-3)+2\lambda^3(q-3)(q-4)+19\lambda^4+33\lambda^3(q-3)+17\lambda^2(q-3)(q-4)+ \\
&\quad 3\lambda(q-3)(q-4)(q-5)+21\lambda^3+42\lambda^2(q-3)+17\lambda(q-3)(q-4)+19\lambda^2+23\lambda(q-3)+5\lambda. \\
6Z_C U_C^N &= 7\lambda^6 + 35\lambda^5 + 29\lambda^4(q-3) + 4\lambda^3(q-3)(q-4) + 57\lambda^4 + 103\lambda^3(q-3) + \\
&\quad 53\lambda^2(q-3)(q-4) + 9\lambda(q-3)(q-4)(q-5) + 71\lambda^3 + 134\lambda^2(q-3) + \\
&\quad 51\lambda(q-3)(q-4) + 61\lambda^2 + 67\lambda(q-3) + 12\lambda. \\
\tilde{S}_C &= (4r^8t^{14} + 56r^8t^{13} + 48r^7t^{14} + 364r^8t^{12} + 708r^7t^{13} + 242r^6t^{14} + 1456r^8t^{11} + 4844r^7t^{12} + \\
&\quad 3786r^6t^{13} + 678r^5t^{14} + 4004r^8t^{10} + 20376r^7t^{11} + 27422r^6t^{12} + 11284r^5t^{13} + 1166r^4t^{14} + \\
&\quad 8008r^8t^9+58872r^7t^{10}+121894r^6t^{11}+86746r^5t^{12}+20640r^4t^{13}+1267r^3t^{14}+12012r^8t^8+ \\
&\quad 123596r^7t^9 + 371544r^6t^{10} + 408452r^5t^{11} + 168452r^4t^{12} + 23817r^3t^{13} + 849r^2t^{14} + \\
&\quad 13728r^8t^7 + 194436r^7t^8 + 821604r^6t^9 + 1316462r^5t^{10} + 840728r^4t^{11} + 206198r^3t^{12} + \\
&\quad 16940r^2t^{13} + 320rt^{14} + 12012r^8t^6 + 232848r^7t^7 + 1359432r^6t^8 + 3073206r^5t^9 + \\
&\quad 2868048r^4t^{10}+1090508r^3t^{11}+155508r^2t^{12}+6779rt^{13}+52t^{14}+8008r^8t^5+213312r^7t^6+ \\
&\quad 1709964r^6t^7 + 5359846r^5t^8 + 7077040r^4t^9 + 3938135r^3t^{10} + 871224r^2t^{11} + 66006rt^{12} + \\
&\quad 1166t^{13} + 4004r^8t^4 + 148764r^7t^5 + 1643298r^6t^6 + 7096018r^5t^7 + 13029736r^4t^8 + \\
&\quad 10277361r^3t^9 + 3330281r^2t^{10} + 391838rt^{11} + 12044t^{12} + 1456r^8t^3 + 77748r^7t^4 + \\
&\quad 1200818r^6t^5+7167552r^5t^6+18187392r^4t^7+19994068r^3t^8+9193464r^2t^9+1585738rt^{10}+ \\
&\quad 75836t^{11} + 364r^8t^2 + 29528r^7t^3 + 656862r^6t^4 + 5497634r^5t^5 + 19343858r^4t^6 + \\
&\quad 29463014r^3t^7+18908970r^2t^8+4631567rt^9+325224t^{10}+56r^8t+7704r^7t^2+260846r^6t^3+ \\
&\quad 3152508r^5t^4+15602488r^4t^5+33049912r^3t^6+29443716r^2t^7+10074894rt^8+1005702t^9+ \\
&\quad 4r^8+1236r^7t+71092r^6t^2+1310722r^5t^3+9395664r^4t^4+28084716r^3t^5+34881678r^2t^6+ \\
&\quad 16588062rt^7+2314836t^8+92r^7+11904r^6t+373572r^5t^2+4096536r^4t^3+17795280r^3t^4+ \\
&\quad 31283154r^2t^5 + 20776716rt^6 + 4032126t^7 + 924r^6 + 65340r^5t + 1222560r^4t^2 + \\
&\quad 8151660r^3t^3 + 20899890r^2t^4 + 19696446rt^5 + 5343678t^6 + 5292r^5 + 223560r^4t + \\
&\quad 2551392r^3t^2 + 10080558r^2t^3 + 13903488rt^4 + 5361876t^5 + 18900r^4 + 488268r^3t + \\
&\quad 3315492r^2t^2+7078266rt^3+4006908t^4+43092r^3+664848r^2t+2452356rt^2+2158812t^3+ \\
&\quad 61236r^2 + 516132rt + 790236t^2 + 49572r + 174960t + 17496)(rt + r + 2t + 3)(t + 1)^3. \\
D_C^U &= (4s^5t^8 + 32s^5t^7 + 24s^4t^8 + 112s^5t^6 + 215s^4t^7 + 57s^3t^8 + 224s^5t^5 + 839s^4t^6 + 565s^3t^7 + \\
&\quad 69s^2t^8+280s^5t^4+1866s^4t^5+2435s^3t^6+736s^2t^7+44st^8+224s^5t^3+2590s^4t^4+5986s^3t^5+ \\
&\quad 3439s^2t^6+489st^7+12t^8+112s^5t^2+2299s^4t^3+9211s^3t^4+9239s^2t^5+2406st^6+138t^7+ \\
&\quad 32s^5t + 1275s^4t^2 + 9101s^3t^3 + 15677s^2t^4 + 6905st^5 + 696t^6 + 4s^5 + 404s^4t + 5641s^3t^2 + \\
&\quad 17259s^2t^3+12742st^4+2054t^5+56s^4+2004s^3t+12045s^2t^2+15565st^3+3984t^4+312s^3+ \\
&\quad 4860s^2t+12303st^2+5304t^3+864s^2+5724st+4752t^2+1188s+2592t+648)(s+3)(t+1)^3.
\end{aligned}
$$

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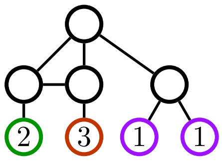

$$
\begin{aligned}
Z_C &= 3\lambda^6 + \lambda^5(q-3) + 9\lambda^5 + 15\lambda^4(q-3) + 7\lambda^3(q-3)(q-4) + \lambda^2(q-3)(q-4)(q-5) + \\
&\quad 10\lambda^4 + 27\lambda^3(q-3) + 18\lambda^2(q-3)(q-4) + 6\lambda(q-3)(q-4)(q-5) + \\
&\quad (q-3)(q-4)(q-5)(q-6) + 17\lambda^3 + 45\lambda^2(q-3) + 39\lambda(q-3)(q-4) + \\
&\quad 11(q-3)(q-4)(q-5) + 24\lambda^2 + 60\lambda(q-3) + 33(q-3)(q-4) + 14\lambda + 27q - 77.
\end{aligned}
$$

$$
\begin{aligned}
2Z_C U_C^v &= 7\lambda^6 + 2\lambda^5(q-3) + 15\lambda^5 + 18\lambda^4(q-3) + 5\lambda^3(q-3)(q-4) + 14\lambda^4 + 27\lambda^3(q-3) + \\
&\quad 11\lambda^2(q-3)(q-4) + 3\lambda(q-3)(q-4)(q-5) + 17\lambda^3 + 31\lambda^2(q-3) + \\
&\quad 20\lambda(q-3)(q-4) + 18\lambda^2 + 33\lambda(q-3) + 10\lambda.
\end{aligned}
$$

$$
\begin{aligned}
6Z_C U_C^N &= 21\lambda^6 + 6\lambda^5(q-3) + 49\lambda^5 + 66\lambda^4(q-3) + 23\lambda^3(q-3)(q-4) + 2\lambda^2(q-3)(q-4)(q-5) + \\
&\quad 44\lambda^4 + 89\lambda^3(q-3) + 39\lambda^2(q-3)(q-4) + 7\lambda(q-3)(q-4)(q-5) + 57\lambda^3 + \\
&\quad 101\lambda^2(q-3) + 46\lambda(q-3)(q-4) + 56\lambda^2 + 71\lambda(q-3) + 16\lambda.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C &= 2(2r^9t^{16} + 32r^9t^{15} + 30r^8t^{16} + 240r^9t^{14} + 500r^8t^{15} + 191r^7t^{16} + 1120r^9t^{13} + \\
&\quad 3904r^8t^{14} + 3330r^7t^{15} + 686r^6t^{16} + 3640r^9t^{12} + 18954r^8t^{13} + 27187r^7t^{14} + 12548r^6t^{15} + \\
&\quad 1546r^5t^{16} + 8736r^9t^{11} + 64038r^8t^{12} + 137928r^7t^{13} + 107398r^6t^{14} + 29719r^5t^{15} + \\
&\quad 2284r^4t^{16} + 16016r^9t^{10} + 159640r^8t^{11} + 486558r^7t^{12} + 570750r^6t^{13} + 267015r^5t^{14} + \\
&\quad 46164r^4t^{15} + 2228r^3t^{16} + 22880r^9t^9 + 303732r^8t^{10} + 1265216r^7t^{11} + 2107256r^6t^{12} + \\
&\quad 1488160r^5t^{13} + 435527r^4t^{14} + 47289r^3t^{15} + 1397r^2t^{16} + 25740r^9t^8 + 449878r^8t^9 + \\
&\quad 2508230r^7t^{10} + 5729730r^6t^{11} + 5757222r^5t^{12} + 2546224r^4t^{13} + 468068r^3t^{14} + \\
&\quad 30988r^2t^{15} + 518rt^{16} + 22880r^9t^7 + 524238r^8t^8 + 3866424r^7t^9 + 11865360r^6t^{10} + \\
&\quad 16389450r^5t^{11} + 10324641r^4t^{12} + 2868751r^3t^{13} + 320902r^2t^{14} + 11874rt^{15} + 88t^{16} + \\
&\quad 16016r^9t^6 + 482196r^8t^7 + 4683051r^7t^8 + 19084404r^6t^9 + 35504178r^5t^{10} + \\
&\quad 30785410r^4t^{11} + 12187312r^3t^{12} + 2058396r^2t^{13} + 127874rt^{14} + 2044t^{15} + 8736r^9t^5 + \\
&\quad 348920r^8t^6 + 4471226r^7t^7 + 24088782r^6t^8 + 59682900r^5t^9 + 69808403r^4t^{10} + \\
&\quad 38054727r^3t^{11} + 9152010r^2t^{12} + 855668rt^{13} + 22652t^{14} + 3640r^9t^4 + 196534r^8t^5 + \\
&\quad 3353679r^7t^6 + 23935160r^6t^7 + 78652998r^5t^8 + 122758494r^4t^9 + 90331359r^3t^{10} + \\
&\quad 29905312r^2t^{11} + 3974529rt^{12} + 157296t^{13} + 1120r^9t^3 + 84474r^8t^4 + 1955200r^7t^5 + \\
&\quad 18655334r^6t^6 + 81499663r^5t^7 + 169115669r^4t^8 + 166228537r^3t^9 + 74280460r^2t^{10} + \\
&\quad 13576423rt^{11} + 761354t^{12} + 240r^9t^2 + 26784r^8t^3 + 868512r^7t^4 + 11283094r^6t^5 + \\
&\quad 66152431r^5t^6 + 183029936r^4t^7 + 239556490r^3t^8 + 143028836r^2t^9 + 35262818rt^{10} + \\
&\quad 2715174t^{11} + 32r^9t + 5908r^8t^2 + 284148r^7t^3 + 5190156r^6t^4 + 41599812r^5t^5 + \\
&\quad 155004162r^4t^6 + 271093024r^3t^7 + 215677563r^2t^8 + 71020089rt^9 + 7369116t^{10} + 2r^9 + \\
&\quad 810r^8t + 64568r^7t^2 + 1754882r^6t^3 + 19857670r^5t^4 + 101558130r^4t^5 + 239896251r^3t^6 + \\
&\quad 255372768r^2t^7 + 112047975rt^8 + 15516234t^9 + 52r^8 + 9104r^7t + 411204r^6t^2 + \\
&\quad 6951528r^5t^3 + 50417382r^4t^4 + 164074044r^3t^5 + 236403936r^2t^6 + 138853683rt^7 + \\
&\quad 25605288t^8 + 600r^7 + 59640r^6t + 1681776r^5t^2 + 18310590r^4t^3 + 84894156r^3t^4 + \\
&\quad 169053750r^2t^5 + 134568351rt^6 + 33209784t^7 + 4032r^6 + 250992r^5t + 4580820r^4t^2 + \\
&\quad 32058612r^3t^3 + 91358064r^2t^4 + 100749744rt^5 + 33708312t^6 + 17388r^5 + 703836r^4t + \\
&\quad 8309304r^3t^2 + 35958654r^2t^3 + 56976534rt^4 + 26448444t^5 + 49896r^4 + 1315440r^3t + \\
&\quad 9678204r^2t^2 + 23432976rt^3 + 15679332t^4 + 95256r^3 + 1580472r^2t + 6566832rt^2 + \\
&\quad 6754914t^3 + 116640r^2 + 1108080rt + 1977048t^2 + 83106r + 345546t + 26244)(t+1)^2.
\end{aligned}
$$

$$D_C^U = (4s^5t^9+36s^5t^8+26s^4t^9+144s^5t^7+251s^4t^8+70s^3t^9+336s^5t^6+1082s^4t^7+709s^3t^8+100s^2t^9+504s^5t^5+2733s^4t^6+3231s^3t^7+1033s^2t^8+76st^9+504s^5t^4+4456s^4t^5+8705s^3t^6+4852s^2t^7+788st^8+24t^9+336s^5t^3+4861s^4t^4+15284s^3t^5+13657s^2t^6+3732st^7+252t^8+144s^5t^2+3546s^4t^3+18123s^3t^4+25444s^2t^5+10703st^6+1200t^7+36s^5t+1667s^4t^2+14491s^3t^3+32539s^2t^4+20682st^5+3442t^6+4s^5+458s^4t+7519s^3t^2+28500s^2t^3+28087st^4+6692t^5+56s^4+2292s^3t+16419s^2t^2+26797st^3+9366t^4+312s^3+5616s^2t+17217st^2+9570t^3+864s^2+6696st+6858t^2+1188s+3078t+648)(s+3)(t+1)^2.$$

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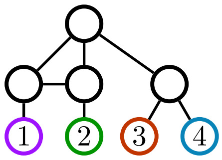

$$Z_C = 8\lambda^5+5\lambda^4(q-4)+\lambda^3(q-4)(q-5)+20\lambda^4+37\lambda^3(q-4)+24\lambda^2(q-4)(q-5)+8\lambda(q-4)(q-5)(q-6)+(q-4)(q-5)(q-6)(q-7)+48\lambda^3+102\lambda^2(q-4)+69\lambda(q-4)(q-5)+14(q-4)(q-5)(q-6)+88\lambda^2+153\lambda(q-4)+57(q-4)(q-5)+72\lambda+72q-268.$$

$$2Z_CU_C^v = 20\lambda^5+11\lambda^4(q-4)+2\lambda^3(q-4)(q-5)+30\lambda^4+43\lambda^3(q-4)+17\lambda^2(q-4)(q-5)+3\lambda(q-4)(q-5)(q-6)+54\lambda^3+70\lambda^2(q-4)+26\lambda(q-4)(q-5)+58\lambda^2+59\lambda(q-4)+30\lambda.$$

$$6Z_CU_C^N = 48\lambda^5+25\lambda^4(q-4)+4\lambda^3(q-4)(q-5)+88\lambda^4+123\lambda^3(q-4)+53\lambda^2(q-4)(q-5)+9\lambda(q-4)(q-5)(q-6)+160\lambda^3+228\lambda^2(q-4)+78\lambda(q-4)(q-5)+200\lambda^2+173\lambda(q-4)+80\lambda.$$

$$\begin{aligned}
\tilde{S}_C = {} & 2(2r^9t^{15}+30r^9t^{14}+30r^8t^{15}+210r^9t^{13}+468r^8t^{14}+188r^7t^{15}+910r^9t^{12}+3406r^8t^{13}+\\
& 3082r^7t^{14}+659r^6t^{15}+2730r^9t^{11}+15340r^8t^{12}+23532r^7t^{13}+11408r^6t^{14}+1440r^5t^{15}+\\
& 6006r^9t^{10}+47814r^8t^{11}+111016r^7t^{12}+91792r^6t^{13}+26378r^5t^{14}+2044r^4t^{15}+\\
& 10010r^9t^9+109252r^8t^{10}+361928r^7t^{11}+455516r^6t^{12}+224150r^5t^{13}+39680r^4t^{14}+\\
& 1882r^3t^{15}+12870r^9t^8+189046r^8t^9+863786r^7t^{10}+1559456r^6t^{11}+1172686r^5t^{12}+\\
& 356593r^4t^{13}+38846r^3t^{14}+1076r^2t^{15}+12870r^9t^7+252252r^8t^8+1559184r^7t^9+\\
& 3902050r^6t^{10}+4225742r^5t^{11}+1969513r^4t^{12}+369982r^3t^{13}+23783r^2t^{14}+343rt^{15}+\\
& 10010r^9t^6+261690r^8t^7+2167704r^7t^8+7373180r^6t^9+11112784r^5t^{10}+\\
& 7481176r^4t^{11}+2160581r^3t^{12}+241096r^2t^{13}+8209rt^{14}+46t^{15}+6006r^9t^5+\\
& 211068r^8t^6+2340492r^7t^7+10714936r^6t^8+22037174r^5t^9+20710820r^4t^{10}+\\
& 8661988r^3t^{11}+1492527r^2t^{12}+89207rt^{13}+1206t^{14}+2730r^9t^4+131274r^8t^5+\\
& 1962686r^7t^6+12075719r^6t^7+33561442r^5t^8+43181117r^4t^9+25274768r^3t^{10}+\\
& 6326238r^2t^{11}+588063rt^{12}+14190t^{13}+910r^9t^3+61828r^8t^4+1267948r^7t^5+\\
& 10555476r^6t^6+39581306r^5t^7+69057053r^4t^8+55480210r^3t^9+19480875r^2t^{10}+\\
& 2642632rt^{11}+100314t^{12}+210r^9t^2+21346r^8t^3+619752r^7t^4+7098628r^6t^5+\\
& 36153148r^5t^6+85415728r^4t^7+93318869r^3t^8+45073097r^2t^9+8603313rt^{10}+\\
& 480120t^{11}+30r^9t+5100r^8t^2+221872r^7t^3+3607292r^6t^4+25367752r^5t^5+\\
& 81712316r^4t^6+121278082r^3t^7+79839699r^2t^8+21007956rt^9+1657206t^{10}+2r^9+\\
& 754r^8t+54926r^7t^2+1340974r^6t^3+13429356r^5t^4+59961492r^4t^5+121769814r^3t^6+\\
& 109189197r^2t^7+39232647rt^8+4278276t^9+52r^8+8408r^7t+344298r^6t^2+\\
& 5192352r^5t^3+33141552r^4t^4+93663768r^3t^5+115280100r^2t^6+56533950rt^7+\\
& 8433882t^8+600r^7+54600r^6t+1384290r^5t^2+13353390r^4t^3+54178488r^3t^4+\\
& 93151512r^2t^5+62863398rt^6+12819654t^7+4032r^6+227556r^5t+3701970r^4t^2+\\
& 22799448r^3t^3+56528604r^2t^4+53471502rt^5+15033600t^6+17388r^5+631260r^4t+\\
& 6584490r^3t^2+24907986r^2t^3+34127082rt^4+13485366t^5+49896r^4+1165752r^3t+\\
& 7510158r^2t^2+15788682rt^3+9073620t^4+95256r^3+1382184r^2t+4983444rt^2+\\
& 4420656t^3+116640r^2+954990rt+1465290t^2+83106r+293058t+26244)(t+1)^3.
\end{aligned}$$

$$\begin{aligned}
D_C^U = {} & (4s^5t^8+32s^5t^7+42s^4t^8+112s^5t^6+361s^4t^7+176s^3t^8+224s^5t^5+1357s^4t^6+\\
& 1614s^3t^7+372s^2t^8+280s^5t^4+2916s^4t^5+6479s^3t^6+3596s^2t^7+402st^8+224s^5t^3+\\
& 3920s^4t^4+14900s^3t^5+15278s^2t^6+4033st^7+180t^8+112s^5t^2+3377s^4t^3+21506s^3t^4+\\
& 37371s^2t^5+17904st^6+1848t^7+32s^5t+1821s^4t^2+19970s^3t^3+57722s^2t^4+\\
& 46119st^5+8442t^6+4s^5+562s^4t+11655s^3t^2+57767s^2t^3+75748st^4+22554t^5+\\
& 76s^4+3908s^3t+36610s^2t^2+81534st^3+38928t^4+576s^3+13424s^2t+56256st^2+\\
& 44808t^3+2176s^2+22720st+33696t^2+4096s+15104t+3072)(s+4)(t+1)^3.
\end{aligned}$$

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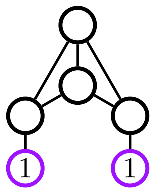

$$\begin{aligned}
Z_C = {} & \lambda^7+\lambda^5(q-1)+6\lambda^4(q-1)+2\lambda^3(q-1)(q-2)+5\lambda^3(q-1)+15\lambda^2(q-1)(q-2)+\\
& 7\lambda(q-1)(q-2)(q-3)+(q-1)(q-2)(q-3)(q-4)+2\lambda^2(q-1)+\\
& 6\lambda(q-1)(q-2)+3(q-1)(q-2)(q-3)+\lambda(q-1)+2(q-1)(q-2).
\end{aligned}$$

$$\begin{aligned}
2Z_CU_C^v = {} & 3\lambda^7+3\lambda^5(q-1)+10\lambda^4(q-1)+4\lambda^3(q-1)(q-2)+5\lambda^3(q-1)+10\lambda^2(q-1)(q-2)+\\
& 3\lambda(q-1)(q-2)(q-3)+2\lambda^2(q-1)+4\lambda(q-1)(q-2)+\lambda(q-1).
\end{aligned}$$

$$
\begin{aligned}
6Z_C U_C^N &= 9\lambda^7 + 7\lambda^5(q-1) + 30\lambda^4(q-1) + 8\lambda^3(q-1)(q-2) + 19\lambda^3(q-1) + \\
&\quad 38\lambda^2(q-1)(q-2) + 9\lambda(q-1)(q-2)(q-3) + 6\lambda^2(q-1) + 8\lambda(q-1)(q-2) + \lambda(q-1). \\
\tilde{S}_C &= 2(4r^8t^{13} + 52r^8t^{12} + 43r^7t^{13} + 312r^8t^{11} + 601r^7t^{12} + 202r^6t^{13} + 1144r^8t^{10} + 3869r^7t^{11} + \\
&\quad 3028r^6t^{12} + 540r^5t^{13} + 2860r^8t^9 + 15191r^7t^{10} + 20867r^6t^{11} + 8680r^5t^{12} + 897r^4t^{13} + \\
&\quad 5148r^8t^8 + 40590r^7t^9 + 87561r^6t^{10} + 63967r^5t^{11} + 15488r^4t^{12} + 951r^3t^{13} + 6864r^8t^7 + \\
&\quad 77946r^7t^8 + 249672r^6t^9 + 286459r^5t^{10} + 122037r^4t^{11} + 17660r^3t^{12} + 636r^2t^{13} + \\
&\quad 6864r^8t^6 + 110682r^7t^7 + 510972r^6t^8 + 870418r^5t^9 + 582615r^4t^{10} + 148697r^3t^{11} + \\
&\quad 12644r^2t^{12} + 251rt^{13} + 5148r^8t^5 + 117678r^7t^6 + 772338r^6t^7 + 1896176r^5t^8 + \\
&\quad 1883784r^4t^9 + 755755r^3t^{10} + 113399r^2t^{11} + 5249rt^{12} + 46t^{13} + 2860r^8t^4 + 93687r^7t^5 + \\
&\quad 873102r^6t^6 + 3048260r^5t^7 + 4362023r^4t^8 + 2595899r^3t^9 + 612010r^2t^{10} + 49715rt^{11} + \\
&\quad 978t^{12} + 1144r^8t^3 + 55165r^7t^4 + 738294r^6t^5 + 3662690r^5t^6 + 7449312r^4t^7 + 6378524r^3t^8 + \\
&\quad 2228185r^2t^9 + 283566rt^{10} + 9634t^{11} + 312r^8t^2 + 23353r^7t^3 + 461272r^6t^4 + 3290362r^5t^5 + \\
&\quad 9506758r^4t^6 + 11554675r^3t^7 + 5798142r^2t^8 + 1090936rt^9 + 57660t^{10} + 52r^8t + \\
&\quad 6731r^7t^2 + 206995r^6t^3 + 2183128r^5t^4 + 9071459r^4t^5 + 15644259r^3t^6 + 11121681r^2t^7 + \\
&\quad 2999361rt^8 + 233508t^9 + 4r^8 + 1184r^7t + 63185r^6t^2 + 1040021r^5t^3 + 6394875r^4t^4 + \\
&\quad 15846771r^3t^5 + 15952293r^2t^6 + 6080067rt^7 + 676602t^8 + 96r^7 + 11760r^6t + 336915r^5t^2 + \\
&\quad 3238095r^4t^3 + 11869965r^3t^4 + 17136468r^2t^5 + 9223605rt^6 + 1446633t^7 + 1008r^6 + \\
&\quad 66528r^5t + 1115505r^4t^2 + 6394347r^3t^3 + 13634190r^2t^4 + 10494252rt^5 + 2317167t^6 + \\
&\quad 6048r^5 + 234360r^4t + 2346921r^3t^2 + 7817769r^2t^3 + 8861886rt^4 + 2788263t^5 + 22680r^4 + \\
&\quad 526176r^3t + 3062043r^2t^2 + 5408775rt^3 + 2496339t^4 + 54432r^3 + 734832r^2t + 2263545rt^2 + \\
&\quad 1621053t^3 + 81648r^2 + 583200rt + 725355t^2 + 69984r + 201204t + 26244)(r+2)(t+1)^5. \\
\tilde{S}_C|_{q=2} &= 2(t^4 + 5t^3 + 12t^2 + 12t + 8)(t+2)^7(t+1)^5. \\
D_C^U &= (2s^3t^6 + 12s^3t^5 + 9s^2t^6 + 30s^3t^4 + 57s^2t^5 + 13st^6 + 40s^3t^3 + 153s^2t^4 + 87st^5 + 6t^6 + \\
&\quad 30s^3t^2 + 225s^2t^3 + 249st^4 + 42t^5 + 12s^3t + 192s^2t^2 + 397st^3 + 126t^4 + 2s^3 + 90s^2t + \\
&\quad 384st^2 + 214t^3 + 18s^2 + 216st + 234t^2 + 54s + 162t + 54)(st + s + 2t + 3)(s+3)(s+2)(t+1)^4. \\
D_C^U|_{q=2} &= 4(t+2)^4(t+1)^4.
\end{aligned}
$$

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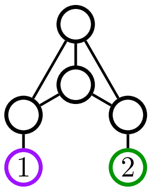
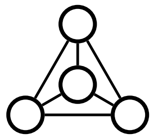


$$
\begin{aligned}
Z_C &= 2\lambda^6 + \lambda^5(q-2) + 2\lambda^5 + 4\lambda^4(q-2) + 2\lambda^3(q-2)(q-3) + 2\lambda^4 + 17\lambda^3(q-2) + \\
&\quad 15\lambda^2(q-2)(q-3) + 7\lambda(q-2)(q-3)(q-4) + (q-2)(q-3)(q-4)(q-5) + 6\lambda^3 + \\
&\quad 20\lambda^2(q-2) + 27\lambda(q-2)(q-3) + 7(q-2)(q-3)(q-4) + 4\lambda^2 + 21\lambda(q-2) + \\
&\quad 11(q-2)(q-3) + 2q - 4. \\
2Z_C U_C^v &= 6\lambda^6 + 3\lambda^5(q-2) + 4\lambda^5 + 8\lambda^4(q-2) + 4\lambda^3(q-2)(q-3) + 2\lambda^4 + 19\lambda^3(q-2) + \\
&\quad 10\lambda^2(q-2)(q-3) + 3\lambda(q-2)(q-3)(q-4) + 8\lambda^3 + 16\lambda^2(q-2) + \\
&\quad 13\lambda(q-2)(q-3) + 4\lambda^2 + 11\lambda(q-2). \\
6Z_C U_C^N &= 16\lambda^6 + 7\lambda^5(q-2) + 12\lambda^5 + 20\lambda^4(q-2) + 8\lambda^3(q-2)(q-3) + 10\lambda^4 + 65\lambda^3(q-2) + \\
&\quad 38\lambda^2(q-2)(q-3) + 9\lambda(q-2)(q-3)(q-4) + 24\lambda^3 + 52\lambda^2(q-2) + \\
&\quad 35\lambda(q-2)(q-3) + 10\lambda^2 + 27\lambda(q-2).
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C \quad &= 2(4r^9t^{14}+56r^9t^{13}+53r^8t^{14}+364r^9t^{12}+782r^8t^{13}+311r^7t^{14}+1456r^9t^{11}+5354r^8t^{12}+\\
&4832r^7t^{13}+1060r^6t^{14}+4004r^9t^{10}+22544r^8t^{11}+34816r^7t^{12}+17332r^6t^{13}+2316r^5t^{14}+\\
&8008r^9t^9+65219r^8t^{10}+154196r^7t^{11}+131337r^6t^{12}+39801r^5t^{13}+3375r^4t^{14}+\\
&12012r^9t^8+137126r^8t^9+468955r^7t^{10}+611406r^6t^{11}+316853r^5t^{12}+60787r^4t^{13}+\\
&3294r^3t^{14}+13728r^9t^7+216084r^8t^8+1036032r^7t^9+1953617r^6t^{10}+1549056r^5t^{11}+\\
&507416r^4t^{12}+61882r^3t^{13}+2084r^2t^{14}+12012r^9t^6+259248r^8t^7+1714584r^7t^8+\\
&4532826r^6t^9+5196588r^5t^{10}+2601672r^4t^{11}+539998r^3t^{12}+40572r^2t^{13}+777rt^{14}+\\
&8008r^9t^5+237963r^8t^6+2159352r^7t^7+7875952r^6t^8+12656061r^5t^9+9154266r^4t^{10}+\\
&2898053r^3t^{11}+368617r^2t^{12}+15562rt^{13}+130t^{14}+4004r^9t^4+166298r^8t^5+\\
&2079597r^7t^6+10411054r^6t^7+23079416r^5t^8+23386011r^4t^9+10681470r^3t^{10}+\\
&2065453r^2t^{11}+146527rt^{12}+2660t^{13}+1456r^9t^3+87098r^8t^4+1524080r^7t^5+\\
&10521252r^6t^6+32016814r^5t^7+44738506r^4t^8+28598227r^3t^9+7961479r^2t^{10}+\\
&854726rt^{11}+25834t^{12}+364r^9t^2+33152r^8t^3+836696r^7t^4+8089266r^6t^5+\\
&33954977r^5t^6+65118602r^4t^7+57362099r^3t^8+22316625r^2t^9+3439366rt^{10}+\\
&156416t^{11}+56r^9t+8669r^8t^2+333652r^7t^3+4657791r^6t^4+27397004r^5t^5+\\
&72476656r^4t^6+87578534r^3t^7+46902435r^2t^8+10082132rt^9+655872t^{10}+4r^9+\\
&1394r^8t+91361r^7t^2+1947660r^6t^3+16555585r^5t^4+61388212r^4t^5+102294306r^3t^6+\\
&75087492r^2t^7+22187058rt^8+2008428t^9+104r^8+15376r^7t+559087r^6t^2+\\
&7265440r^5t^3+38953572r^4t^4+90978576r^3t^5+92034801r^2t^6+37230984rt^7+\\
&4624848t^8+1200r^7+98616r^6t+2188881r^5t^2+17956608r^4t^3+60654834r^3t^4+\\
&85967946r^2t^5+47881692rt^6+8131536t^7+8064r^6+405216r^5t+5684337r^4t^2+\\
&29395476r^3t^3+60250743r^2t^4+46981404rt^5+10971288t^6+34776r^5+1106028r^4t+\\
&9789147r^3t^2+30725676r^2t^3+34631793rt^4+11309382t^5+99792r^4+2004912r^3t+\\
&10777293r^2t^2+18600840rt^3+8772300t^4+190512r^3+2326968r^2t+6881031rt^2+\\
&4966920t^3+233280r^2+1568808rt+1940598t^2+166212r+468018t+52488)(t+1)^4.\\
\tilde{S}_C|_{q=2} &= 4(2t^4+7t^3+12t^2+12t+4)(t+2)^7(t+1)^4.\\
D_C^U \quad &= (2s^3t^6+12s^3t^5+9s^2t^6+30s^3t^4+57s^2t^5+13st^6+40s^3t^3+153s^2t^4+87st^5+6t^6+\\
&30s^3t^2+225s^2t^3+249st^4+42t^5+12s^3t+192s^2t^2+397st^3+126t^4+2s^3+90s^2t+384st^2+\\
&214t^3+18s^2+216st+234t^2+54s+162t+54)(st+s+2t+3)(st+s+t+2)(s+3)(t+1)^3.\\
D_C^U|_{q=2} &= 4(t+2)^4(t+1)^3.
\end{aligned}
$$

Local view 35 of 35 (named K_4)



$$
\begin{aligned}
Z_C &= \lambda^6 q+4\lambda^3(q-1)q+3\lambda^2(q-1)q+6\lambda(q-1)(q-2)q+(q-1)(q-2)(q-3)q.\\
2Z_CU_C^v &= 3\lambda^6 q+6\lambda^3(q-1)q+3\lambda^2(q-1)q+3\lambda(q-1)(q-2)q.\\
6Z_CU_C^N &= 9\lambda^6 q+18\lambda^3(q-1)q+9\lambda^2(q-1)q+9\lambda(q-1)(q-2)q.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_C \quad &= 6(2r^5t^8 + 16r^5t^7 + 14r^4t^8 + 56r^5t^6 + 125r^4t^7 + 40r^3t^8 + 112r^5t^5 + 485r^4t^6 + 392r^3t^7 + \\
&\quad 59r^2t^8 + 140r^5t^4 + 1070r^4t^5 + 1668r^3t^6 + 620r^2t^7 + 45rt^8 + 112r^5t^3 + 1470r^4t^4 + \\
&\quad 4032r^3t^5 + 2857r^2t^6 + 496rt^7 + 14t^8 + 56r^5t^2 + 1289r^4t^3 + 6068r^3t^4 + 7512r^2t^5 + \\
&\quad 2445rt^6 + 160t^7 + 16r^5t + 705r^4t^2 + 5832r^3t^3 + 12328r^2t^4 + 6942rt^5 + 838t^6 + 2r^5 + \\
&\quad 220r^4t + 3500r^3t^2 + 12962r^2t^3 + 12358rt^4 + 2556t^5 + 30r^4 + 1200r^3t + 8550r^2t^2 + \\
&\quad 14160rt^3 + 4908t^4 + 180r^3 + 3240r^2t + 10260rt^2 + 6093t^3 + 540r^2 + 4320rt + 4833t^2 + \\
&\quad 810r + 2268t + 486)(rt^2 + 2rt + t^2 + r + 3t + 3)^2(r + 3)(r + 2)(t + 1)^6. \\
\tilde{S}_C|_{q=2} &= 12(t^2 + t + 2)(t + 2)^7(t + 1)^6. \\
D_C^U \quad &= 0.
\end{aligned}
$$

B

Potts model computations

The verification that, for each of the local views shown in Appendix A, the scaled slack (4.5) and the scaled difference (4.6) are non-negative (and zero where required) was done with the aid of a computer. Here we describe some additional considerations required to perform this verification for arbitrary q and β, and give a computer program which implements these ideas.

As noted in Section 4.2, the number of equivalence classes of local views we must consider is bounded independently of q, and we only consider representatives of each equivalence class that use an initial segment of colours from $\{1, \ldots, 6\}$ on the boundary. In order to compute the partition function and other properties of a local view C, one is required to consider the q^4 possible local colourings of V_C. This too can be done in a way that is bounded independently of q by considering equivalence classes of local colourings.

Let C be a local view (that uses an initial segment of $\{1, \ldots, 6\}$ on the boundary) and recall that q_C is the largest colour appearing on the boundary of C. Then given a local colouring χ of V_C, we can only see at most $q_C + 4$ colours. After permuting colours not used on the boundary, we may assume that χ consists only of colours in $[q_C]$ and initial segment of $\{q_C{+}1, \ldots, q_C{+}4\}$ (which may be empty). This means we are considering equivalence classes of local colourings and choosing a representative $\tilde{\chi}$ of each class such that, together with the colours on the boundary, we only ever colour C with an initial segment of $[q_C + 4]$.

Then for arbitrary q it suffices to consider at most $q_C + 4 \leq 10$ colours in the calculations for Z_C, U_C^v, and U_C^N. Given the set $Q_C \subseteq [q_C + 4]^{V_C}$ of *representative* local colourings $\tilde{\chi}$ such that $\tilde{\chi}$ uses an initial segment (which may be empty) of the colours $\{q_C + 1, \dots, q_C + 4\}$, and writing ℓ for the largest colour used in $\tilde{\chi}$, the Potts model on C induces the distribution

$$\tilde{\chi} \mapsto \begin{cases} \frac{e^{-\beta m(\tilde{\chi})}}{Z_C}\binom{q-q_C}{\ell-q_C} & \text{if } \ell > q_C \\ \frac{e^{-\beta m(\tilde{\chi})}}{Z_C} & \text{otherwise} \end{cases}$$

on $\tilde{\chi} \in Q_C$.

The consideration of these equivalence classes of local colourings means that for any β, q, and C, the quantities Z_C, U_C^v, and U_C^N may be computed by summing over Q_C whose size is bounded independently of q. Using this simplification, we used a *SageMath*[1] computer program to compute the scaled slack function $\tilde{S}_C$ and the scaled difference D_C^v for each of the 35 local views. The program can be used to generate the data for Appendix A so that the reader may verify the proof, and in addition the program can verify the required coefficients are non-negative and print these observations.

Program listing

```
# Define a generator for all possible local views in cubic
# graphs, represented as a pair of lists. The first list
# comprises the colors of the boundary: external neighbors of
# each u in N(v). The second list comprises the edges inside
# N(v) = [1, 2, 3]
import itertools as its
from sage.combinat.permutation import Permutations_mset

# helper functions to manupulate colorings of the boundary
def color_Nus(Nu_sizes, tc):
    f = lambda (a, s), l:(a + [range(s, s+l)], s+l)
    ix, _ = reduce(f, Nu_sizes, ([], 0))
    return tuple(map(lambda Nu: map(lambda i: tc[i], Nu), ix))
def tc_hash(tc):
    s = int(len(tc)-1).bit_length()
    return sum(map(lambda (i, c): c << (i*s),
                                        zip(its.count(0), tc)))
def permute_cNu(cNu, g_perm, c_perm):
    return map(lambda Nu: map(c_perm, Nu), [cNu[g_perm(i)]
                                   for i in range(len(cNu))])
is_minimal = lambda cNu: all(map(is_sorted, cNu))
```

[1] http://www.sagemath.org

```
is_sorted = lambda l: all(l[i] <= l[i+1]
                                    for i in xrange(len(l)-1))

def gen_LVs():
    # Iterate over possible graphs on N(v): empty, an edge, two
    # edges, and a triangle
    all_Nv_edges = [[], [[1,2]], [[1,2],[2,3]],
                    [[1,2],[1,3],[2,3]]]
    all_Nus      = [[[4,5],[6,7],[8,9]], [[4],[5],[6,7]],
                    [[4],[],[5]], [[],[],[]]]
    # Symmetry groups for the possible graphs (indices from 0)
    all_sym_gps  = map(PermutationGroup,
              [[(0,1),(1,2)], [(0,1)], [(0,2)], [(0,1),(1,2)]])

    for Nv_edges, Nus, Nv_sym_gp in its.izip(all_Nv_edges,
                                             all_Nus, all_sym_gps):
        Nu_sizes = map(len, Nus)

        # There are 6-2*len(Nv_edges) vertices on the boundary
        for parts in Partitions(6 - 2*len(Nv_edges)):

            color_set = range(1, 1+len(parts))
            color_freqs = zip(color_set, parts)
            color_multiset = reduce(lambda a, t: a+[t[0]]*t[1],
                                    color_freqs, [])

            # Create the symmetry group for the colors
            def accum_gens(a, kg):
                colors = [f[0] for f in kg[1]]
                # Tuples representing generators for symmetric
                # group on these colors
                new_gens = ([tuple(colors[0:2]), tuple(colors)]
                                    if len(colors) > 1 else [])
                return a + new_gens
            gens = reduce(accum_gens,
                  its.groupby(color_freqs, lambda t: t[1]), [])
            color_gp = PermutationGroup(gens, domain=color_set)

            # Iterate through the isomorphic colorings
            found_tcs = set()
            for coloring in Permutations_mset(color_multiset):
                tc = tuple(coloring)
                cNu = color_Nus(Nu_sizes, tc)
                if not is_minimal(cNu):
                    continue # we only need 'minimal' colorings
                if tc_hash(tc) not in found_tcs:
                # When a new coloring is found, yield an LV and
                # store all isomorphic colorings to check
                # against in future
                    yield [list(cNu), Nv_edges]
                    for g_pm, c_pm in its.product(Nv_sym_gp,
                                                  color_gp):
                        isocNu = map(sorted,
                                 permute_cNu(cNu, g_pm, c_pm))
```

```
                        found_tcs.add(tc_hash(flatten(isocNu)))

# Name variables for the partition functions
q, r, s, t, varDelta = var('q, r, s, t, varDelta')
lam = var('lam', latex_name='\\lam')
b = var('b', latex_name='\\beta')

# Z and U for K33 and K4
ZK33(q, b) = (q*(exp(-3*b)+q-1)^3 +
              3*q*(q-1)*(exp(-2*b)+exp(-b)+q-2)^3 +
              q*(q-1)*(q-2)*(3*exp(-b)+q-3)^3)
UK33(q, b) = -ZK33(q, b).derivative(b)/ZK33(q, b)/6

ZK4(q, b) = (q*exp(-6*b) + 4*q*(q-1)*exp(-3*b) +
             3*q*(q-1)*exp(-2*b) + 6*q*(q-1)*(q-2)*exp(-b) +
             q*(q-1)*(q-2)*(q-3))
UK4(q, b) = -ZK4(q, b).derivative(b)/ZK4(q, b)/4

# Put all the local views in a list
LVs = list(gen_LVs())

# Give names to LVs that occur in the optimizing graphs
C1 = [[[1, 1], [1, 1], [1, 1]], []]
C2 = [[[1, 2], [1, 2], [1, 2]], []]
K4 = [[[], [], []], [[1, 2], [1, 3], [2, 3]]]

# Helper functions for lists
first = lambda l: l[0]
rest = lambda l: l[1:]

# returns true for a list which is empty, a singleton,
# or consists of repetitions of one value
def constQ(l):
    if len(l) <= 1: return true
    else: return l[0] == l[1] and constQ(rest(l))

# returns true for nonempty lists of nonnegative numbers
def nonNegativeListQ(l):
    return (len(l) > 0 and
            reduce(lambda x,y: x and y,
                   map(lambda c: c >= 0,l),True))

# Find the largest color used in a local view
def findqC(C):
    cols = flatten(C[0])
    if cols == []: return 0
    else: return max(cols)

# Count the monochromatic edges in local view C with local
# coloring chi. There are three terms, monochromatic edges
# incident to v, those from u in N(v) to an external neighbor,
# and those inside N(v)
def mChi(C, chi):
    cv = first(chi)
```

```
    cN = rest(chi)
    return (cN.count(cv) +
        sum(map(lambda j: C[0][j].count(cN[j]), range(0,3))) +
        map(lambda l: constQ([cN[j-1] for j in l]),
            C[1]).count(true))

# Count the monochromatic edges incident to v
def mvChi(C, chi):
    cv = first(chi)
    cN = rest(chi)
    return cN.count(cv)

# Sum the number the monochromatic edges incident to u in N(v)
# In fact this simply counts all monochromatic edges in the
# local view, and double counts any inside N(v)
def mNChi(C, chi):
    cv = first(chi)
    cN = rest(chi)
    return (mChi(C,chi) +
        map(lambda l: constQ([cN[j-1] for j in l]),
            C[1]).count(true))

# returns True if chi uses an initial seqment of {qC+1,...,q}
def validChi(chi, qC):
    return (Set(chi).intersection(Set(range(qC+1,max(chi)+1)))
            == Set(range(qC+1,max(chi)+1)))

# returns the total weight of (a class of) local colourings
# divide by Z to obtain the probability
def zChi(C, q, b, chi):
    qC = findqC(C)
    if validChi(chi, qC):
        return (binomial(q-qC, max(max(chi)-qC, 0)) *
                exp(-b*mChi(C, chi)))
    else:
        return 0

# Partition function for a local view
def Z(C, q, b):
    chis = Tuples(range(1, findqC(C)+5),4).list()
    return sum(zChi(C, q, b, chi) for chi in chis)

# Internal energy per particle from the perspective of v
def Uv(C, q, b):
    chis = Tuples(range(1, findqC(C)+5),4).list()
    return 1/2/Z(C, q, b)*sum(mvChi(C, chi)*zChi(C, q, b, chi)
                              for chi in chis)

# Internal energy per particle from the perspective of N(v)
def UN(C, q, b):
    chis = Tuples(range(1, findqC(C)+5),4).list()
    return 1/6/Z(C, q, b)*sum(mNChi(C, chi)*zChi(C, q, b, chi)
                              for chi in chis)

```

```
# We solve DualConstraint == 0 on C1 (or C2) to obtain Delta
# Note the change of notation to lam = e^(-b)
def DualConstraint(C, q, lam, varDelta):
    b = -log(lam)
    return Uv(C,q,b)+varDelta*(UN(C,q,b)-Uv(C,q,b))-UK33(q,b)

Delta(q,lam) = solve(DualConstraint(C1, q, lam, varDelta) == 0,
                     varDelta)[0].rhs()

# Slack in the dual constraint
def Slack(C, q, lam):
    b = -log(lam)
    return (Uv(C,q,b)+Delta(q,lam)*(UN(C,q,b)-Uv(C,q,b))-
            UK33(q,b))

# Scaling and reparametrisation of the slack
def ScaledSlack(C, r, t):
    q = r+3
    lam = 1/(1+t)
    b = log(1+t)
    ee = (ZK33(q,b)*Z(C,q,b)*4*(1+t)^17*(3+3*t+t^2+r*(1+t)^2)^2
          /(3+r)/t^2*Slack(C, q, lam)).simplify_log()
    if ee == 0: return ee # sage can't factor '0'
    else: return ee.factor()

# Subtract Uv for C from Uv for K4
def UvComparison(C, q, b):
    return UK4(q,b) - Uv(C,q,b)

# Scaling and reparametrisation of the difference
def ScaledUvComparison(C, s, t):
    q = s+max(3, findqC(C))
    b = log(1+t)
    ee = (ZK4(q, b)*Z(C, q, b)*2*(1+t)^14
          /t^2*UvComparison(C, q, b)).simplify_log()
    if ee == 0: return ee # sage can't factor '0'
    else: return ee.factor()

# Row of useful observations for a local view
def ObsRow(C):
    b = -log(lam)
    print('Computing ObsRow for local view %s' % C)
    return [C, simplify(Z(C,q,b)),
            simplify(2*Z(C,q,b)*Uv(C,q,b)),
            simplify(6*Z(C, q, b)*UN(C, q, b)),
            ScaledSlack(C, r, t),
            ScaledSlack(C,-1,t) if findqC(C) <= 2 else false,
            ScaledUvComparison(C, s, t),
            (ScaledUvComparison(C, -1, t)
                if findqC(C) <= 2 else false)]

# Check the required observations, should all print 'True'
def RequiredObservations(C):
    ss0 = '  ScaledSlack is zero...                         %s'
```

```
    ss1 = '  ScaledSlack has nonneg coeffs...              %s'
    ss2 = '  ScaledSlack at q=2 has nonneg coeffs...       %s'
    dv0 = '  ScaledUvComparison is zero...                 %s'
    dv1 = '  ScaledUvComparison has nonneg coeffs...       %s'
    dv2 = '  ScaledUvComparison at q=2 has nonneg coeffs... %s'

    print('Testing local view %s' % C)

    qC = findqC(C)
    ss = ScaledSlack(C, r, t)
    if C == C1 or C == C2:
        print(ss0 % ss.is_zero())
    else:
        print(ss1 %
            nonNegativeListQ(ss.polynomial(QQ).coefficients()))
        if qC <= 2:
            print(ss2 % nonNegativeListQ(ss(r=-1)
                .polynomial(QQ).coefficients()))

    dv = ScaledUvComparison(C, s, t)
    if C == K4:
        print(dv0 % dv.is_zero())
    else:
        print(dv1 %
            nonNegativeListQ(dv.polynomial(QQ).coefficients()))
        if qC <= 2:
            print(dv2 % nonNegativeListQ(dv(s=-1)
                .polynomial(QQ).coefficients()))

# Check the required observations
map(RequiredObservations, LVs);
print('Done')

# Generate the observation table
ObsTable = map(ObsRow, LVs)
save(ObsTable, os.path.join(os.getcwd(), 'ObsTable'))
save(LVs, os.path.join(os.getcwd(), 'LVs'))
print('Done')
```