Friedlander–Keller Ray Expansions and Scalar Wave Reflection at Canonically–Perturbed Boundaries

R H Tew
School of Mathematical Sciences
University of Nottingham
University Park
Nottingham
NG7 2RD UK
E-mail: richard.tew@nottingham.ac.uk

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Abstract

This paper concerns the reflection of high–frequency, monochromatic linear waves of wavenumber $k\,(\gg 1)$ from smooth boundaries which are $O\left(k^{-1/2}\right)$ perturbations away from either a specified near–planar boundary or else from a given smooth, two–dimensional curve of general $O(1)$ curvature. For each class of perturbed boundary, we will consider separately plane and cylindrical wave incidence, with general amplitude profiles of each type of incident field.

This interfacial perturbation scaling is canonical in the sense that a ray approach requires a modification to the standard WKBJ 'ray ansatz' which, in turn, leads to a leading–order amplitude (or 'transport') equation which includes an extra term absent in a standard application of the geometrical theory of diffraction ('GTD'). This extra term is unique to this scaling, and the afore–mentioned modification that is required is an application of a generalised type of ray expansion first posed by F G Friedlander and J B Keller [1].

1 Introduction and Motivation

The interaction between monochromatic, propagating, linear waves (such as acoustic, elastic or electromagnetic waves) with obstructing boundaries is key to many topical scattering problems arising in modern applications of wave physics. Indeed, entire theories such as the geometrical theory of diffraction (GTD), relevant in the high–frequency domain, have been developed and applied very successfully for exactly such reasons.

One aim of this paper is to analyse examples from a relatively new class of time–harmonic, high–frequency scattering problems, in which standard methodologies have to be modified to account for reflection from obstructing boundaries having smooth perturbations (on a specified length scale) away from either a perfectly flat boundary or else one with a general $O(1)$ curvature profile. These modifications are to the WKBJ–type expansion applied in order to use ray methods, necessitating a modified 'ray' expansion of the type first considered by Friedlander and Keller [1].

This application of Friedlander and Keller's theory is a second aim of this paper, and might even be the first such application for any specific value of a key parameter α that they introduce, apart from two other applicable values ($\alpha = 0$ and $\alpha = 1/3$) which those authors identified as having already arisen in examples pre–existing their generalised work. Indeed, Keller and Lewis [2] make this point explicitly following equation (1.225) of their paper, and so our current work will add $\alpha = 1/2$ to the list of these only other known cases of practical relevance ($\alpha = 0$ amounts to the standard ray expansion and $\alpha = 1/3$ is relevant to, for example, creeping and whispering gallery wave propagation). We mention the case $\alpha = 1$ later, commenting that this case is not really special since it can be encapsulated within *complex* ray theory without any further modifications being necessary.

The first problem that we consider concerns reflection at near–planar interfaces, an example of which is the problem analysed by Engineer *et al* [3]. There, scattering by exterior plane wave incidence upon a two–dimensional 'canonically slender' obstruction was analysed in the high–frequency (*i.e.* $k \to \infty$) limit. In that context, 'canonically slender' referred to a body of finite length whose transverse profile away from its 'tips' was of a width commensurate with the requirements that the inner diffraction problem within a length scale $O\left(k^{-1}\right)$ of either tip region was the *full* Helmholtz equation $\left(\hat{\nabla}^2 + 1\right)\hat{\phi} = 0$ (in terms of scaled inner coordinates $x = k^{-1}\hat{x}$, $y = k^{-1}\hat{y}$ and with $\phi(x, y) = \hat{\phi}\left(\hat{x}, \hat{y}\right)$), rather than some 'parabolic wave equation' reduction of it, subject to data prescribed on a locally *parabolic* boundary, rather than an exactly flat half–plane, so that the full effects of the 'inner' boundary curvature were accounted for.

That argument supposes that the left–hand tip (for example) has a locally parabolic profile given by the equation

$$\hat{y}^2 = 2\beta^2\hat{x} \tag{1}$$

in which $|\hat{x}|$, $|\hat{y}|$ and the constant β are all $O(1)$ quantities. This inner form of the boundary must match smoothly with its equation $y = F\left(x; k\right)$ valid elsewhere for $|x| = O(1)$, and supposing $F\left(x; k\right) = k^{\gamma} f(x)$ for some γ to be determined, this leads to the 'asymptotic matching conditions'

$$\hat{y} = \beta k^{1/2}\sqrt{2k^{-1}\hat{x}} \sim k^{\gamma+1} f\left(k^{-1}\hat{x}\right), \tag{2}$$

from which $\gamma = -1/2$ and the condition $f(x) \sim \beta\sqrt{2x}$ as $x \to 0$, a condition which supplies the value of β since $f(x)$ is assumed known, both follow.

From this, we therefore take $y = k^{-1/2} f(x)$ to be the generic profile of the upper surface away from the tip region of what we now term a 'canonically near–planar' interface and we now consider reflection phenomena associated with both plane and cylindrical wave incidence upon it. In both cases, we consider incident waves with arbitrary amplitude profiles represented by amplitude pre–factors carried straightforwardly through the ray calculations to follow, though we do emphasise that our analysis pertains to perturbations from general, fully–infinite interfaces and not just from a semi–infinite half–plane as arises in the limiting case studied in [3].

Hence, the first part of this paper concerns the reflection of plane and cylindrical waves incident upon a Cartesian boundary profile $y = k^{-1/2} f(x)$, and we retain this notation here so that direct comparison can be made to the work of Engineer *et al* [3].

Generalising this, we then pose similar problems when the reflecting interface is a perturbation of the same length scale but from an underlying boundary with general curvature (rather than simply $y = 0$). If $\boldsymbol{x} = \boldsymbol{x_0}(s) = ((x_0(s), y_0(s)))$ is the original, unperturbed boundary ∂D, parametrised by arc–length s, then we consider reflection from a generalised profile given by $\partial \hat{D} : \boldsymbol{x} = \boldsymbol{x_0}(s) + k^{-1/2}\hat{\boldsymbol{x}}_{\boldsymbol{0}}(s)$ and in which the hatted coordinate variables are prescribed but general; notice that s is no longer arc–length along the perturbed boundary $\partial \hat{D}$, and we note the following relationship between s and $\hat{s}$, arc–length along the *perturbed* boundary, correct to $O\left(k^{-1}\right)$:

$$s \sim \hat{s} - k^{-1/2} \int_0^{\hat{s}} \boldsymbol{x}'_{\boldsymbol{0}}(u) \cdot \hat{\boldsymbol{x}}'_{\boldsymbol{0}}(u) du - \frac{1}{2} k^{-1} \int_0^{\hat{s}} \left(\hat{\boldsymbol{x}}'_{\boldsymbol{0}}(u) \cdot \hat{\boldsymbol{x}}'_{\boldsymbol{0}}(u) - 3 \left(\boldsymbol{x}'_{\boldsymbol{0}}(u) \cdot \hat{\boldsymbol{x}}'_{\boldsymbol{0}}(u) \right)^2 \right) du. \quad (3)$$

Restricting attention temporarily to *plane* wave incidence upon the first class $\left(y = k^{-1/2} f(x)\right)$ of reflecting boundaries (an exactly similar argument applies to cylindrical waves, and then to generally–curved boundaries for both types of incident wave), for which we will see that the incident field is of the general form $A_P^I(x, y) e^{ik(x \cos\theta - y \sin\theta)}$ in the high–frequency limit, then exponentials of the general form $A_P^I(x, y) e^{ikx \cos\theta \pm ik^{1/2} f(x) \sin\theta}$ are clearly inevitable when satisfying the imposed Neumann boundary condition. This guides us naturally towards seeking reflected fields with corresponding exponential factors $e^{iku(x,y) + ik^{1/2} v(x,y)}$ and straightaway we see the onset of a Friedlander–Keller – type ray expansion (here with $\alpha = 1/2$), which we describe further in the next section.

2 The Friedlander–Keller Ray Expansion

The underlying field equation throughout our discussion is the two–dimensional Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) \phi = 0, \quad (4)$$

in which k is the large wavenumber and $\phi(x, y)$ is the field variable, which might be the potential function for a linear acoustic or elastic wave, or a component of an electromagnetic disturbance, for example. The constant k is given by ω / c, where we assume but suppress a multiplicative harmonic time dependence $e^{-i\omega t}$ throughout; c is the wavespeed of the medium.

We solve (4) in the domain $\hat{D}$ adjacent to some given boundary $\partial\hat{D}$, an $O\left(k^{-1/2}\right)$ perturbation away from another prescribed curve ∂D, on which a Neumann boundary condition for the total field is imposed (although we comment on the adjustments needed to account for Dirichlet data at appropriate points in the paper). Presented here are situations in which $\hat{D}$ is the two–dimensional region given by (i) $y > k^{-1/2}f(x)$ with $\partial\hat{D}$ being the boundary profile $y = k^{-1/2}f(x)$ and ∂D being the x–axis, $y = 0$, and (ii) $n > 0$ with $\partial\hat{D} : x = x_0(s) + k^{-1/2}\hat{x}_0(s), \quad y = y_0(s) + k^{-1/2}\hat{y}_0(s)$, where (s, n) are the standard curvilinear coordinates associated with the unperturbed boundary $\partial D : x = x_0(s), y = y_0(s)$, s being arc–length and n the normal distance along and from ∂D, respectively. The hatted perturbation coordinates are all taken as given, as is $f(x)$.

In the presence of the *unperturbed* boundary ∂D, we would ordinarily seek solutions to (4) in the singularly–perturbed, high–frequency limit $k \to \infty$ in the WKBJ–form

$$\phi(x, y) \sim e^{iku(x,y)} \sum_{n=0}^{\infty} \frac{A_n(x, y)}{(ik)^n}; \tag{5}$$

substitution into (4) and systematic comparison of terms at the various orders in k that arise then quickly yield the eikonal equation

$$\nabla u \cdot \nabla u = 1 \tag{6}$$

for u and the recursive family of transport equations

$$A_n \nabla^2 u + 2\nabla A_n \cdot \nabla u + \nabla^2 A_{n-1} = 0 \tag{7}$$

for the amplitude functions A_n ($n = 0, 1, 2, \ldots; A_{-1}(x, y) \equiv 0$). Such an expansion (5) is the basis of Keller's geometrical theory of diffraction as described in [4], applied in (for example) [5], [6], [7] and reviewed in [2].

Notice that the eikonal equation (6) for the phase decouples entirely from the amplitude equations (7), and is typically solved first (subject to appropriate boundary data) using Charpit's method (see, for example, [8] or [9]) by first introducing new dependent variables $p = \dfrac{\partial u}{\partial x}, q = \dfrac{\partial u}{\partial y}$ and showing that these quantities are conserved (*i.e.* $p = p_0(s), q = q_0(s)$, where the subscript zero once more denotes evaluation on ∂D) along the rays $\Gamma(s)$ given parametrically in terms of arc–length τ along them by

$$\Gamma(s) : \quad \frac{dx}{d\tau} = p_0(s), \frac{dy}{d\tau} = q_0(s). \tag{8}$$

The eikonal equation (6) now becomes

$$p_0^2(s) + q_0^2(s) = 1, \tag{9}$$

whilst differentiation of the boundary data $u = u_0(s)$ on ∂D gives

$$u_0'(s) = x_0'(s)p_0(s) + y_0'(s)q_0(s). \tag{10}$$

With $u_0(s)$ given, or calculable, equations (9) and (10), together with a radiation condition at infinity to fix the sign (or choice of branch) of $q_0(s) = \left(1 - p_0^2(s)\right)^{1/2}$, now provide $p_0(s)$ and $q_0(s)$, and hence the ray equations

$$\Gamma(s): \quad x = x_0(s) + \tau p_0(s), \quad y = y_0(s) + \tau q_0(s) \tag{11}$$

emanating from the boundary curve $\tau = 0 : x = x_0(s), y = y_0(s)$ and along which $\frac{du}{d\tau} = 1$ also now follow, implying

$$u(s,\tau) = u_0(s) + \tau \tag{12}$$

along them.

The leading–order transport equation, corresponding to $n = 0$ in (7), reduces to a homogeneous first–order ordinary differential equation along each ray, with general solution

$$A_0(s,\tau) = A_0(s,0)\left[\frac{a(s)}{\tau + a(s)}\right]^{1/2}, \tag{13}$$

in which

$$a(s) = \frac{q_0(s)x_0'(s) - p_0(s)y_0'(s)}{q_0(s)p_0'(s) - p_0(s)q_0'(s)}, \tag{14}$$

and this fully determines the leading–order solution; in principle, higher–order amplitude terms A_1,, A_2, ... can then be calculated from (7) in a recursive fashion.

Modifications to this *ansatz* are required whenever, for example, the rays associated with the incoming field are tangent to an obstructing boundary or else form a caustic; in the former case, for example, the exponent and denominator occurring in (5) are replaced by $iku(x,y) + ik^{1/3}v(x,y)$ and a factor proportional to $k^{n/3}$, respectively (see [10], [11], [12] and [13] and references cited therein for reviews of this and other related work.) The one–third power law in the exponent and amplitude expansion arises in that analysis because of the variable coefficients of the Helmholtz equation when written in local curvilinear coordinates near the point of tangential ray incidence.

This motivated Friedlander and Keller [1] to consider a generalised asymptotic *ansatz* for solutions ϕ of the Helmholtz equation in the modified WKBJ format

$$\phi(x,y,z) \sim e^{iku(x,y,z)+ik^{\alpha}v(x,y,z)} \sum_{n=0}^{\infty} \frac{A_n(x,y,z)}{k^{\lambda_n}}, \tag{15}$$

in which we have changed their original notation to one consistent with ours in the work presented here; in order to reproduce Friedlander and Keller's original *ansatz*, we simply replace ϕ, u, v and A_n in (15) by u, ϕ, χ and v_n, respectively. Also, their term proportional to k^{α} in the exponent of their exponential pre–factor had a real and negative co–efficient whereas we have replaced by one that is pure imaginary in order to make this *ansatz* directly relevant to the wave scattering applications we consider later in the paper.

In [1], the field equations satisfied by u, v and the A_n's are listed for all possible ranges of α and for appropriate corresponding values of λ_n, together with a commentary on special cases having important applications elsewhere in wave physics. For example, they observe that α can be restricted to the range $0 < \alpha \leq 1/2$, with the case $\alpha = 0$ coinciding with standard ray theory for the Helmholtz equation, and those cases for $\alpha > 1/2$ essentially result in v being constant so that this exponent can be absorbed into the multiplicative amplitude coefficients. It is important to remark that the constancy of v in those situations is a direct consequence of there only being *one* extra term (additional to the standard term u) within the exponent of the *ansatz* (15). Had we included more such terms, which would amount to a generalisation of (15), then we would not be able to draw this conclusion and we return to this point in the discussion at the end of the paper.

Another special case in this range deserving special attention is $\alpha = 1$ but this coincides with a Luneberg–Kline expansion, also cited in [1]. In this case the functions u and v can be combined to form a single exponent (whereby u is replaced by $u + iv$), and then a theory of *complex* rays applied [14]. Friedlander and Keller also cite the particular case of $\alpha = 1/3$ which, as we have already noted, is relevant for creeping and whispering gallery mode propagation around convex bodies of $O(1)$ curvature.

The results presented here complement this work of Friedlander and Keller in that it provides a concrete example of an application of their expansions for the case $\alpha = 1/2$ (which they do not consider), and also we go on and solve the resulting equations for the two exponent functions and the leading–order amplitude; Friedlander and Keller list the equations they satisfy but never actually solve them, either in general terms or via specific application.

We have already presented arguments which motivate an *ansatz* for the reflected fields ϕ^R for either of the two classes of problems we consider of the form

$$\phi^R(x,y) = A\,(x,y;k)\,e^{iku(x,y)+ik^{1/2}v(x,y)}. \tag{16}$$

Substitution into the Helmholtz equation (4) shows that u still satisfies the eikonal equation (6), and subsequently that v is coupled to u via the equation

$$\nabla u \cdot \nabla v = 0, \tag{17}$$

in agreement with Friedlander and Keller [1]. The amplitude $A\,(x,y;k)$, which still carries k as a parameter, now satisfies

$$\nabla^2 A + ik^{1/2}\left(A\nabla^2 v + 2\nabla v\cdot\nabla A\right) + ik\left(2\nabla u\cdot\nabla A + A\nabla^2 u + iA\nabla v\cdot\nabla v\right) = 0 \tag{18}$$

which, in turn, prompts a modified ‘ray’, or Friedlander–Keller type, expansion

$$A(x,y;k) = \sum_{n=0}^{\infty}\frac{A_n(x,y)}{k^{n/2}}. \tag{19}$$

Substitution of (19) into (18) and extracting like powers of k, we obtain the recursive system of transport equations

$$-i\nabla^2 A_n + A_{n+1}\nabla^2 v + 2\nabla v\cdot\nabla A_{n+1} + 2\nabla u\cdot\nabla A_{n+2} + A_{n+2}\nabla^2 u + iA_{n+2}\nabla v\cdot\nabla v = 0, \tag{20}$$

which is valid for $n = -2, -1, 0, 1, \ldots$ provided we take $A_{-2}(x,y) \equiv 0$, $A_{-1}(x,y) \equiv 0$; the case $n = -2$ yields the leading-order amplitude equation for A_0 in the form

$$2\nabla u \cdot \nabla A_0 + A_0 \left(\nabla^2 u + i\nabla v \cdot \nabla v\right) = 0. \tag{21}$$

(At this point, we respectfully note a typographical error in the paper of Friedlander and Keller ([1]) at the corresponding point in their more general analysis. In their equation (10), the term $\nabla^2\chi$ should, in fact, be $(\nabla\chi)^2$. This is an isolated error and all subsequent formulae arising from this result are correct). Notice that this transport equation differs from the standard form ((7) with $n = 0$ and $A_{-1}(x,y) \equiv 0$) because of the inclusion of the final term involving $\nabla v \cdot \nabla v$; this is solely due to the $O\left(k^{1/2}\right)$ term in the exponent in (16); only for precisely this power does this happen – any other power will leave the original leading–order transport equation unaffected, further strengthening the claim that the $O\left(k^{-1/2}\right)$ scale of the boundary perturbations considered here are indeed canonical.

We end this section by noting that since $\nabla u = (p_0(s), q_0(s)) = \left(\dfrac{dx}{d\tau}, \dfrac{dy}{d\tau}\right)$ along a ray (along which s is constant and τ is arc–length measured from the boundary) we see that $\nabla u \cdot \nabla v = 0$ implies $\dfrac{dv}{d\tau} = 0$, so that v is invariant along the rays *i.e.* $v = v_0(s)$ where $v_0(s)$ is the value of v on ∂D. Also, a straightforward calculation using the ray equations yields the useful derivative relationships

$$\frac{\partial F}{\partial x} = \frac{q_0 \frac{\partial F}{\partial s} - (y_0' + \tau q_0')\frac{\partial F}{\partial \tau}}{q_0 x_0' - p_0 y_0' + \tau\left(p_0' q_0 - q_0' p_0\right)}, \qquad \frac{\partial F}{\partial y} = -\frac{p_0 \frac{\partial F}{\partial s} - (x_0' + \tau p_0')\frac{\partial F}{\partial \tau}}{q_0 x_0' - p_0 y_0' + \tau\left(p_0' q_0 - q_0' p_0\right)} \tag{22}$$

for general F, allowing $\nabla^2 u = \dfrac{\partial p_0(s)}{\partial x} + \dfrac{\partial q_0(s)}{\partial y}$ and $\nabla v = v_0'(s)\nabla s$ to be evaluated. The leading–order transport equation (21) for $A_0(s,\tau)$ along each ray then follows as

$$2\frac{dA_0}{d\tau} + A_0\left\{\frac{1}{\tau + a(s)} + \frac{ib(s)}{(\tau + a(s))^2}\right\} = 0, \tag{23}$$

where $a(s)$ is given by (14) and

$$b(s) = \frac{\left(v_0'(s)\right)^2}{\left(q_0(s)p_0'(s) - p_0(s)q_0'(s)\right)^2}. \tag{24}$$

This can be solved to give the general formula for the leading–order amplitude in our interpretation of a Friedlander–Keller ray expansion with $\alpha = 1/2$ in the form

$$A_0(s,\tau) = A_0(s,0)\left(\frac{a(s)}{\tau + a(s)}\right)^{1/2} \exp\left(-\frac{ib(s)\tau}{2a(s)\left(\tau + a(s)\right)}\right). \tag{25}$$

This generic theory plays a role when considering reflection at perturbed boundaries of arbitrary underlying curvature; for those which are near–planar we nonetheless deliberately adopt a partial differential equation approach to illustrate an additional, alternative means of solution available in that case.

3 Formulation of the Boundary Value Problems

Writing the total potential ϕ_W^T as the superposition of an incident field ϕ_W^I and reflected field ϕ_W^R ($W = P, C$ for plane and cylindrical wave incidence, respectively)

$$\phi_W^T(x,y) = \phi_W^I(x,y) + \phi_W^R(x,y), \tag{26}$$

the reflected field ϕ_W^R satisfies the Helmholtz equation (4) subject to Neumann data

$$\frac{\partial \phi_W^R}{\partial n} = -\frac{\partial \phi_W^I}{\partial n} \quad \text{on } \partial\hat{D} \tag{27}$$

(with

$$\phi_W^R = -\phi_W^I \quad \text{on } \partial\hat{D} \tag{28}$$

being the analogous relation for Dirichlet data) on $\partial\hat{D}$ and a 'radiation condition'

$$\phi_W^R \quad \text{'outgoing' in } \hat{D}, \tag{29}$$

where $\dfrac{\partial}{\partial n}$ denotes the normal derivative at the rigid boundary $\partial\hat{D}$. The condition (29) will be dealt with in each of the two cases (P and C) separately; it essentially means that the field represented by ϕ_W^R must propagate *away* from, and not *towards*, the boundary $\partial\hat{D}$.

Regarding the functional forms of ϕ_W^I ($\alpha = P, C$), we have already noted that the general form of an incoming *plane* wave propagating towards the x–axis of a Cartesian coordinate system has a phase which gives a leading–order form

$$\phi_P^I(x,y) \sim A_P^I(x,y)e^{ik(x\cos\theta - y\sin\theta)}, \tag{30}$$

where θ is the angle of incidence measured from the x–axis. Of course, the incident phase $u_P^I(x,y) = x\cos\theta - y\sin\theta$ satisfies the eikonal equation (6) and this, together with (7) setting $n = 0$, yields

$$\cos\theta\frac{\partial A_P^I}{\partial x} - \sin\theta\frac{\partial A_P^I}{\partial y} = 0, \tag{31}$$

so that $A_P^I(x,y) = F_P^I(x + y\cot\theta)$ for some function F_P^I of a single variable. We assume that F_P^I is given, so that the incident plane wave has the general, variable–amplitude form

$$\phi_P^I(x,y) \sim F_P^I(x + y\cot\theta)\, e^{ik(x\cos\theta - y\sin\theta)} \tag{32}$$

at leading order.

For the case of an incident *cylindrical* wave ϕ_C^I, we suppose that its source is located at the remote point $(0, h)$, and so has an associated phase function

$$u_C^I(x,y) = \left[x^2 + (y-h)^2\right]^{1/2}, \tag{33}$$

which of course also satisfies (6). Writing the incident field in this case as

$$\phi_C^I(x,y) \sim A_C^I(x,y)e^{ik\left[x^2+(y-h)^2\right]^{1/2}}, \tag{34}$$

(7) now implies the leading–order transport equation

$$x\frac{\partial A_C^I}{\partial x} + (y-h)\frac{\partial A_C^I}{\partial y} + \frac{1}{2}A_C^I = 0, \tag{35}$$

which has general solution

$$A_C^I(x,y) = F_C^I\left(\frac{y-h}{x}\right)\left[x^2+(y-h)^2\right]^{-1/4} \tag{36}$$

for any arbitrary function F_C^I of the single variable $\dfrac{y-h}{x}$.
As with F_P^I previously, we take F_C^I to be given and so we maintain as general as possible leading–order amplitude variation in both cases. Notice that, in this second case, $\dfrac{y-h}{x} = \tan\theta$, where now θ is defined to be the polar angle measured from the source location and is *not* related to the angle of incidence defined in ϕ_P^I. This allows us to write

$$F_C^I\left(\frac{y-h}{x}\right) = D_C^I(\theta), \tag{37}$$

where the (known) function $D_C^I(\theta)$ is the leading–order directivity of ϕ_C^I which, in turn, can now be expressed in the more recognisable form

$$\phi_C^I \sim \frac{D_C^I(\theta)}{R^{1/2}}e^{ikR}, \tag{38}$$

where $R = \left[x^2+(y-h)^2\right]^{1/2}$ is radial distance measured from the source.

Our final comments on the problem formulation concern the satisfaction of the Neumann boundary condition (27).

First, in doing so we must simultaneously balance (i) the $O(k)$ and (ii) the $O\left(k^{1/2}\right)$ terms within the various exponents that arise, as well as (iii) the $O(1)$ terms within the multiplicative amplitude, and this contains some subtle features. For example, if we consider the near–planar boundary $y = k^{-1/2}f(x)$, then a typical exponent (whether in ϕ_W^I or ϕ_W^R) evaluated on this perturbed boundary is of the form

$$\begin{aligned} &iku\left(x, k^{-1/2}f(x)\right) + ik^{1/2}v\left(x, k^{-1/2}f(x)\right) \sim \\ &iku(x,0) + ik^{1/2}\left(f(x)\frac{\partial u}{\partial y}(x,0) + v(x,0)\right) + i\frac{1}{2}f^2(x)\frac{\partial^2 u}{\partial y^2}(x,0) + if(x)\frac{\partial v}{\partial y}(x,0) \end{aligned} \tag{39}$$

(with an exactly similar argument available for the other more general class of boundaries considered). The point is that we 'linearise' the exponent onto the unperturbed boundary ($y=0$ in this case) *before* proceeding with the ray treatment, and we see that derivatives of u *alone* appear within the $O\left(k^{1/2}\right)$ terms whilst derivatives of *both* u and v arise in the final $O(1)$ terms. That is to say, the solution for u, which is found independently of that for either v or the leading–order amplitude, feeds into that for v via the boundary condition, whilst the solutions for *both* u and v play a role in providing part of the boundary data (via the final two $O(1)$ terms, or their analogues, on the right–hand–side of (39)) for the leading–order amplitude.

Our second point regarding (27) concerns its mathematical implementation. Of course, $\dfrac{\partial}{\partial n} = \boldsymbol{n} \cdot \nabla$ where $\boldsymbol{n}$ is the outward–pointing unit normal to the interface, and in the two–dimensional problems that we consider here we have either $\boldsymbol{n} \propto \left(-k^{-1/2} f'(x), 1\right)$ for the near–planar case or $\boldsymbol{n} \propto \left(-\left(y_0'(s) + k^{-1/2}\hat{y}_0'(s)\right), x_0'(s) + k^{-1/2}\hat{x}_0'(s)\right)$ for the generally–curved examples, allowing us to replace (27) by either

$$\frac{\partial \phi_W^T}{\partial y} = k^{-1/2} f'(x) \frac{\partial \phi_W^T}{\partial x} \tag{40}$$

on $\partial\hat{D}$, or else

$$\left(y_0'(s) + k^{-1/2}\hat{y}_0'(s)\right) \frac{\partial \phi_W^T}{\partial x} = \left(x_0'(s) + k^{-1/2}\hat{x}_0'(s)\right) \frac{\partial \phi_W^T}{\partial y} \tag{41}$$

on $\partial\hat{D}$, respectively. The analogue of both (40) and (41) for Dirichlet data is encapsulated in the single condition (28). Notice also that (41) appears to mix the Cartesian (x, y) and curvilinear (s, n) coordinate systems; this is justified since x and y derivatives of ϕ_W^T necessarily involve $\dfrac{\partial u_W}{\partial x}$, $\dfrac{\partial u_W}{\partial y}$ and $\dfrac{\partial v_W}{\partial x}$, $\dfrac{\partial v_W}{\partial y}$ when we consider Friedlander–Keller exponentials $e^{iku_W(x,y)+ik^{1/2}v_W(x,y)}$, and these derivatives arise naturally (via $\nabla u, \nabla v$) in the subsequent ray analysis.

Strictly speaking, both sides of (41) should be multiplied by $\dfrac{ds}{d\hat{s}}$, where $\hat{s}$ is arc–length along the *perturbed* boundary $\partial\hat{D}$; the equation relating s to $\hat{s}$ is (3).

We now have all of the mathematical machinery that we need, and the solution strategy is clear: having specified the incident field, we pose an *ansatz* and then use the boundary condition (27) to establish boundary data for the various u, v and A_0 that arise, and then use it to solve (6), (17) and (21), in that order, for these quantities to be determined explicitly. When all three of these quantities have been found, we will say that the leading–order solution for the reflected field has been constructed.

4 Near–Planar Boundaries

4.1 Plane Wave Incidence

Guided by the previous discussions leading to (32), we write down the leading–order expression for the *total* field in the form

$$\phi_P^T(x, y) \sim F_P^I\left(x + y\cot\theta\right) e^{ik(x\cos\theta - y\sin\theta)} + A_P^R(x, y) e^{iku_P(x,y)+ik^{1/2}v_P(x,y)}, \tag{42}$$

in which $A_P^R(x.y)$ is the amplitude of the reflected field to leading order. Of course, the overall potential ϕ_P^T must satisfy (40), and we have already noted that this will require that the exponents arising in each of the two terms on the right–hand–side of (42) must agree on $\partial\hat{D}: \quad y = k^{-1/2} f(x)$; this in turn gives two further equations (one at $O(k)$, the other at $O\left(k^{1/2}\right)$) providing boundary data for u_P and, subsequently, v_P, respectively.

Hence, the problem for $u_P(x, y)$ is to solve the eikonal equation (6) subject to data

$$u_P(x, 0) = x \cos\theta, \tag{43}$$

together with the 'radiation condition' that these reflected rays propagate *away* from the boundary and into the region $y > 0$. Equation (12) and the ray analysis around it then give

$$u_P(x, y) = x \cos\theta + y \sin\theta, \tag{44}$$

so that (17) now in turn gives

$$\cos\theta \frac{\partial v_P}{\partial x} + \sin\theta \frac{\partial v_P}{\partial y} = 0, \tag{45}$$

so that $v_P(x, y) = v_P(\xi)$ is a function of the single variable $\xi = x - y \cot\theta$. The argument surrounding (39) now provides the data for v_P in the form

$$-f(x) \sin\theta = f(x) \frac{\partial u_P}{\partial y} + v_P(x, 0), \tag{46}$$

from all of which we conclude that

$$v_P(x, y) = -2 \sin\theta f(x - y \cot\theta). \tag{47}$$

Hence, the solutions for u_P and v_P are now both fully determined, with only the latter depending explicitly upon the boundary profile $f(x)$ and the former essentially seeing no difference from the totally rigid *flat* boundary case.

In order to calculate the leading–order amplitude $A_P^R(x, y)$, we substitute the results so far into (21), to obtain the boundary value problem

$$\cos\theta \frac{\partial A_P^R}{\partial x}(x, y) + \sin\theta \frac{\partial A_P^R}{\partial y}(x, y) = -2i \left[f'(x - y \cot\theta)\right]^2 A_P^R(x, y), \tag{48}$$

$$A_P^R(x, 0) = F_P^I(x) e^{-2i \cos\theta f(x) f'(x)}, \tag{49}$$

where the boundary condition (49) follows from the leading–order contribution to (40). Introducing a second independent variable $\zeta = x + y \tan\theta$, the partial differential equation for A_P^R becomes

$$\frac{\partial A_P^R}{\partial \zeta} = -2i \cos\theta \left[f'(\xi)\right]^2 A_P^R. \tag{50}$$

This equation is easily solved by separating variables and doing so, applying the boundary condition and then reverting back to the original Cartesian coordinates yields the solution for A_P^R in the final form

$$\begin{aligned} A_P^R(x,y) &= F_P^I(x - y\cot\theta) \\ &\times \exp\left[-\frac{2iy}{\sin\theta}\left(f'(x-y\cot\theta)\right)^2 - 2i\cos\theta f(x-y\cot\theta)\, f'(x-y\cot\theta)\right] \end{aligned} \tag{51}$$

and this completes the solution for the reflected field for the case of variable–amplitude plane wave incidence subject to Neumann data. Dirichlet data is accommodated instead by simply negating the expressions on the right–hand–sides of (49) and (51).

4.2 Cylindrical Wave Incidence

With the total field

$$\begin{aligned} \phi_C^T(x,y) &\sim F_C^I\left(\frac{y-h}{x}\right)\left[x^2+(y-h)^2\right]^{-1/4} e^{ik\left[x^2+(y-h)^2\right]^{1/2}} \\ &+ A_C^R(x,y)e^{iku_C(x,y)+ik^{1/2}v_C(x,y)}, \end{aligned} \tag{52}$$

the methodology is as before and as far as the 'phase functions' u_C, v_C (in an obvious notation) are concerned it is straightforward to obtain the two boundary relations arising at orders k and $k^{1/2}$, respectively, within the exponents as

$$u_C(x,0) = \left(x^2+h^2\right)^{1/2} \tag{53}$$

and

$$f(x)\frac{\partial u_C}{\partial y}(x,0) + v_C(x,0) = -\frac{h}{(x^2+h^2)^{1/2}} f(x). \tag{54}$$

The eikonal problem for u_C is straightforward to solve, the 'outgoing' solution being

$$u_C(x,y) = \left[x^2+(y+h)^2\right]^{1/2}, \tag{55}$$

corresponding to an outgoing (in $y > 0$) cylindrical wave emanating from the image source $(0,-h)$. One way of seeing this from a partial differential equations perspective and without the need to recourse to ray methods is to observe that $u_C(x,0) = \left(x^2+h^2\right)^{1/2} = R(x,0)$, where $R(x,y) = [x^2+(y-h)^2]^{1/2}$ is planar distance measured from the source point $(0,h)$. Since $R(x,\pm y)$ both satisfy the eikonal equation and the boundary condition, u_C must be one or the other. Taking the positive sign yields a field incoming towards the boundary and is therefore rejected since it violates the radiation condition. Hence, $u_C(x,y) = R(x,-y)$ and (55) emerges. Given this solution, we find from (17) that v_C satisfies

$$x\frac{\partial v_C}{\partial x} + (y+h)\frac{\partial v_C}{\partial y} = 0, \tag{56}$$

so that v_C has the self–similar form $v_C(x,y) = g(\eta)$, $\eta = \dfrac{y+h}{x} = \tan\Theta$, for some function to be determined g. Notice that v_C depends only upon the polar angle Θ measured from the *image* source and so this term within the exponent can be regarded as a k–dependent modification to the final directivity associated with the reflected field.

Combining (54) and (55) provides the refined boundary condition

$$v_C(x,0) = -\frac{2h}{(x^2+h^2)^{1/2}} f(x), \tag{57}$$

from which

$$g(\eta) = -\frac{2\eta}{(1+\eta^2)^{1/2}} f\left(\frac{h}{\eta}\right) \tag{58}$$

is immediate, and $v_C(x,y) = g\left(\dfrac{y+h}{x}\right)$ is determined; all that remains to be done is to compute the leading–order amplitude A_C^R. We begin by substituting our solutions for u_C and v_C into (21), which reveals that

$$x\frac{\partial A_C^R}{\partial x} + (y+h)\frac{\partial A_C^R}{\partial y} + \frac{1}{2}A_C^R = -\frac{iA_C^R}{2x}\left[1+\left(\frac{y+h}{x}\right)^2\right]^{3/2}\left(g'\left(\frac{y+h}{x}\right)\right)^2 \tag{59}$$

and, guided by our construction of the incident field (which is the main reason for including that discussion), we now set

$$A_C^R(x,y) = \left[x^2+(y+h)^2\right]^{-1/4}\hat{A}_C^R(x,y) \tag{60}$$

so that

$$x\frac{\partial \hat{A}_C^R}{\partial x} + (y+h)\frac{\partial \hat{A}_C^R}{\partial y} = \frac{\hat{A}_C^R}{x}\Lambda\left(\frac{y+h}{x}\right) \tag{61}$$

where

$$\Lambda(\eta) = -\frac{i}{2}\left(1+\eta^2\right)^{3/2}\left[g'(\eta)\right]^2 \tag{62}$$

is a known function of the single variable η.

Equation (61) admits an exact solution

$$\hat{A}_C^R(x,y) = F_C^R\left(\frac{y+h}{x}\right)\exp\left[-\frac{1}{x}\Lambda\left(\frac{y+h}{x}\right)\right] \tag{63}$$

for a function $F_C^R(\eta)$ to be determined, and piecing all of this information together we deduce that

$$A_C^R(x,y) = F_C^R\left(\frac{y+h}{x}\right)\left[x^2+(y+h)^2\right]^{-1/4}\exp\left[-\frac{1}{x}\Lambda\left(\frac{y+h}{x}\right)\right] \tag{64}$$

and we can identify F_C^R as another function dependent solely on the polar angle Θ measured from the image source, and therefore (like v_C) closely allied to the directivity of the reflected field. We shall be more precise about this shortly, linking it directly to the prescribed incoming directivity F_C^I as prescribed in the given incident field. We also note that Λ is pure imaginary, of $O(1)$ modulus and vanishes if g, and hence f, does. It therefore describes an $O(1)$ shift in the phase due to the presence of the boundary undulations described by $f(x)$.

The determination of the reflected field in this case is complete once the 'directivity' F_C^R has been found, and this is done by appealing to the one remaining condition to be satisfied, namely (40). Feeding into that condition all that has been found so far then gives, after considerable algebra, the closed–form expression

$$F_C^R(\eta) = F_C^I(-\eta)\exp\left[\frac{2i}{\eta\left(1+\eta^2\right)^{1/2}} f'\left(\frac{h}{\eta}\right)\left(\eta f\left(\frac{h}{\eta}\right) - h\left(1+\eta^2\right) f'\left(\frac{h}{\eta}\right)\right)\right]; \qquad (65)$$

this now determines the leading–order amplitude A_C^R of the reflected field explicitly and the reflection problem is solved, noting that the equivalent relation to (65) appropriate to Dirichlet data is obtained by negating this result.

5 Perturbed Boundaries of General Curvature

In this section we replace u_W and v_W ($W = P, C$) by μ_C and ν_C, respectively, to avoid possible confusion with the near–planar boundary analysis just considered. This is purely notational; all underlying field equations and boundary conditions remain unchanged.

5.1 Plane Wave Incidence

As before, the phase of the incoming plane wave provides the eikonal boundary data for μ_P, this time in the form

$$\mu_P(s,0) = x_0(s)\cos\theta - y_0(s)\sin\theta, \qquad (66)$$

so that

$$\mu_P(s,\tau) = x_0(s)\cos\theta - y_0(s)\sin\theta + \tau \qquad (67)$$

follows along the as yet unknown rays from (12). Also, in this case (10) yields

$$x_0'(s)p_0(s) + y_0'(s)q_0(s) = x_0'(s)\cos\theta - y_0'(s)\sin\theta, \qquad (68)$$

so that eliminating q_0 between (9) and (10) in this case now gives a quadratic equation for $p_0(s)$ in the form

$$p_0^2(s) + 2x_0'(s)p_0(s)\left(y_0'(s)\sin\theta - x_0'(s)\cos\theta\right) + \left(y_0'(s)\sin\theta - x_0'(s)\cos\theta\right)^2 - \left(y_0'(s)\right)^2 = 0; \quad (69)$$

in deriving (69) the relation $\left(x_0'(s)\right)^2 + \left(y_0'(s)\right)^2 = 1$, a result which crucially requires the parameter s to be *specifically* arc–length, has been used repeatedly.
We note from (68) that there is always a trivial solution

$$p_0(s) = \cos\theta, \quad q_0(s) = -\sin\theta \qquad (70)$$

satisfying both (68) and (69), but this reproduces the incident field and is rejected in favour of the other solution; once found, (68) then provides an unambiguous solution for $q_0(s)$. Knowing that $p_0(s) - \cos\theta$ divides the left–hand–side of (69) allows us to find its second factor, leading us to the desired solution

$$p_0(s) = \Big((x_0'(s))^2 - (y_0'(s))^2\Big)\cos\theta - 2x_0'(s)y_0'(s)\sin\theta; \tag{71}$$

the corresponding solution for $q_0(s)$ follows from (68) as

$$q_0(s) = 2x_0'(s)y_0'(s)\cos\theta + \Big((x_0'(s))^2 - (y_0'(s))^2\Big)\sin\theta, \tag{72}$$

with $p_0^2(s) + q_0^2(s) = 1$ being an easy check. This now completely fixes the ray directions along which the solutions for μ_P and ν_P (to be determined) are valid, but equally importantly supply the boundary data (and therefore simultaneously the full solution) for ν_P. This is because

$$\nu_p(s) = \hat{x}_0(s)\cos\theta - \hat{y}_0(s)\sin\theta - \nabla\mu_P \cdot \hat{\boldsymbol{x}}_{\mathbf{0}}\ (s) \tag{73}$$

where $\nabla\mu_P = (p_0(s), q_0(s))$ and, after considerable labour, this eventually leads to the final expression

$$\nu_P(s) = 2\left(\hat{x}_0(s)y_0'(s) - \hat{y}_0(s)x_0'(s)\right)\left(x_0'(s)\sin\theta + y_0'(s)\cos\theta\right). \tag{74}$$

leaving just the leading–order amplitude to be determined.

Making use of the facts that, in this case,

$$q_0(s)x_0'(s) - p_0(s)y_0'(s) = x_0'(s)\sin\theta + y_0'(s)\cos\theta, \tag{75}$$

and

$$q_0(s)p_0'(s) - p_0(s)q_0'(s) = \frac{p_0'(s)}{q_0(s)} = -2\kappa_0(s), \tag{76}$$

where $\kappa_0(s)$ is the curvature of the unperturbed boundary, detailed and laborious algebra reveals that the reflected field boundary amplitude $\tilde{A}_P^R(s,0)$ arising in (25) is given by

$$\begin{aligned}\tilde{A}_P^R(s,0) &= F_P^I\left(x_0(s) + y_0(s)\cot\theta\right)\exp\Big[-\frac{i\kappa_0(s)\left(q_0(s)\hat{x}_0(s) - p_0(s)\hat{y}_0(s)\right)^2}{\left(x_0'(s)\sin\theta + y_0'(s)\cos\theta\right)} \\ &- 2i\left(q_0(s)\hat{x}_0(s) - p_0(s)\hat{y}_0(s)\right)\left(\hat{x}_0'(s)y_0'(s) - \hat{y}_0'(s)x_0'(s)\right)\Big],\end{aligned} \tag{77}$$

and the Dirichlet data case is covered by negating this result. Tilde's have been put on the reflected leading–order amplitudes solely to avoid confusion with those arising in the previous section. The leading–order amplitude of the reflected field is now given by (77) and (25), with

$$a(s) = -\frac{1}{2}\rho_0(s)\left(x_0'(s)\sin\theta + y_0'(s)\cos\theta\right) \tag{78}$$

and

$$\begin{aligned}b(s) &= \rho_0^2(s)\big[\left(\hat{x}_0'(s)y_0'(s) - \hat{y}_0'(s)x_0'(s)\right)\left(x_0'(s)\sin\theta + y_0'(s)\cos\theta\right) \\ &+ \kappa_0(s)\left(q_0(s)\hat{x}_0(s) - p_0(s)\hat{y}_0(s)\right)\big]^2,\end{aligned} \tag{79}$$

where $\rho_0(s) = \kappa_0^{-1}(s)$ is the radius of curvature of the unperturbed boundary, and the reflected field is now fully determined correct to leading order.

5.2 Cylindrical Wave Incidence

With cylindrically–spreading wavefronts emanating from the source location $(0, h)$, the eikonal boundary data for μ_C is now

$$\mu_C(s, 0) = R_0(s) = \left[(x_0(s))^2 + (y_0(s) - h)^2\right]^{1/2}, \tag{80}$$

and if we define $c_0(s)$, $s_0(s)$ and $\psi_0(s)$ through the relations

$$c_0(s) = \cos\psi_0(s) = \frac{x_0(s)}{R_0(s)}, \quad s_0(s) = \sin\psi_0(s) = \frac{y_0(s) - h}{R_0(s)}, \tag{81}$$

so that $(c_0(s))^2 + (s_0(s))^2 = 1$, then (10) becomes

$$p_0(s)x_0'(s) + q_0(s)y_0'(s) = c_0(s)x_0'(s) + s_0(s)y_0'(s) \tag{82}$$

and $p_0(s)$ then satisfies the quadratic equation

$$p_0^2(s) - 2p_0(s)x_0'(s)\left(c_0(s)x_0'(s) + s_0(s)y_0'(s)\right) + \left(c_0(s)x_0'(s) + s_0(s)y_0'(s)\right)^2 - \left(y_0'(s)\right)^2 = 0. \tag{83}$$

The relevant solutions for p_o and q_0 are found to be

$$p_0(s) = c_0(s)\left[(x_0'(s))^2 - (y_0'(s))^2\right] + 2s_0(s)x_0'(s)y_0'(s) \tag{84}$$

$$q_0(s) = -s_0(s)\left[(x_0'(s))^2 - (y_0'(s))^2\right] + 2c_0(s)x_0'(s)y_0'(s). \tag{85}$$

It is useful to note that in this case, (76) must be replaced by

$$\frac{p_0'(s)}{q_0(s)} = \psi_0'(s) - 2\kappa_0(s). \tag{86}$$

With these, we of course now have the ray solution

$$\mu_C(s, \tau) = R_0(s) + \tau \tag{87}$$

valid along the rays given by (11). The same general method as before is used to compute ν_C and the upshot is the compactly factorised expression

$$\nu_C(s) = \frac{2}{R_0(s)}\left[x_0(s)y_0'(s) - (y_0(s) - h)\,x_0'(s)\right]\left[y_0'(s)\hat{x}_0(s) - x_0'(s)\hat{y}_0(s)\right]. \tag{88}$$

This now determines completely the k–dependent terms within the exponent of the reflected field, and we are also able to write down expressions for $a(s)$ and $b(s)$ relevant to this case in the forms

$$a(s) = \frac{R_0(s)\psi_0'(s)}{\psi_0'(s) - 2\kappa_0(s)} \tag{89}$$

and

$$b(s) = \frac{4}{(\psi_0'(s) - 2\kappa_0(s))^2}\Big[- R_0'(s)\psi_0'(s)\left(y_0'(s)\hat{x}_0(s) - x_0'(s)\hat{y}_0(s)\right) + \kappa_0(s)\left(q_0(s)\hat{x}_0(s) - p_0(s)\hat{y}_0(s)\right) + R_0(s)\psi_0'(s)\left(y_0'(s)\hat{x}_0'(s) - x_0'(s)\hat{y}_0'(s)\right)\Big]^2 \quad (90)$$

where, for completeness, we also note that $\psi_0(s)$ defined through (81) satisfies

$$\psi_0'(s) = \frac{1}{R_0^2(s)}\left[x_0(s)y_0'(s) - (y_0(s) - h)\, x_0'(s)\right]. \quad (91)$$

Referring once more to (25), we now need to find $\tilde{A}_C^R(s, 0)$ to finalise the solution and, as before, this utilises (41) and so must take account of the $O(1)$ contributions in all phase terms in reflected and incident fields. After some particularly involved algebra, the final result turns out to be

$$\tilde{A}_C^R(s, 0) = F_C^I\left(\frac{y_0(s) - h}{x_0(s)}\right)(R_0(s))^{-1/2}\exp\left[\frac{i\left((y_0(s) - h)\,\hat{x}_0(s) - x_0(s)\hat{y}_0(s)\right)^2}{2\left[x_0^2(s) + (y_0(s) - h)^2\right]^{3/2}} - iF_1(s)\right] \quad (92)$$

where

$$F_1(s) = \frac{1}{2}\left(q_0(s)\hat{x}_0(s) - p_0(s)\hat{y}_0(s)\right)\Bigg[\left(\frac{1}{R_0(s)} + 2\kappa_0(s)\right)\left(q_0(s)\hat{x}_0(s) - p_0(s)\hat{y}_0(s)\right) - \frac{4\left(x_0(s)x_0'(s) + (y_0(s) - h)\, y_0'(s)\right)}{\left(x_0^2(s) + (y_0(s) - h)^2\right)}\left(y_0'(s)\hat{x}_0(s) - x_0'(s)\hat{y}_0(s)\right) + 4\left(y_0'(s)\hat{x}_0'(s) - x_0'(s)\hat{y}_0'(s)\right)\Bigg] \quad (93)$$

and so the solution is now complete for the Neumann data case; simply negating the final overall result covers Dirichlet data.

We end this section by commenting that for either type of incoming wave, the latter class of perturbed boundaries of general curvature can, in principle at least, encapsulate the former, of a near–planar nature, by setting $(x_0(s), y_0(s)) = (s, 0)$ and $(\hat{x}_0(s), \hat{y}_0(s)) = (0, f(s))$ in the notation of the analysis presented.
If we do so then lengthy calculation does indeed reveal that the results of Section 4 are reproduced in every detail from those in Section 5. We do not present the details, but remark instead that this is to be interpreted in the current context as an independent check on the accuracy of the results presented, rather than as an alternative means of arriving at the 'near–planar' results without the need to perform that analysis separately.

6 Scattering by Perturbed Circles

To illustrate the application of our results, we consider the case when the obstructing boundary is a perturbed circle with plane polar representation

$$r = r_0 + k^{-1/2} r_0^{1/2} f(\theta), \tag{94}$$

where $f(\theta)$ specifies the undulations superimposed onto the circle $r = r_0$.
It is easy to see from this that the Cartesian representation is encapsulated in terms of arc–length s measured counter–clockwise from the Cartesian point $(r_0, 0)$ (based on a natural origin located at the centre of the circle $r = r_0$) by

$$x_0(s) = r_0 \cos\left(\frac{s}{r_0}\right), \quad y_0(s) = r_0 \sin\left(\frac{s}{r_0}\right) \tag{95}$$

and

$$\hat{x}_0(s) = r_0^{1/2} f\left(\frac{s}{r_0}\right) \cos\frac{s}{r_0}, \quad \hat{y}_0(s) = r_0^{1/2} f\left(\frac{s}{r_0}\right) \sin\left(\frac{s}{r_0}\right). \tag{96}$$

Hence, in this case we also have

$$R_0(s) = \left[r_0^2 \cos^2\left(\frac{s}{r_0}\right) + \left(r_0 \sin\left(\frac{s}{r_0}\right) - h\right)^2\right]^{1/2} \tag{97}$$

and

$$\psi_0(s) = \cos^{-1}\left[\frac{r_0 \cos\left(\frac{s}{r_0}\right)}{R_0(s)}\right] = \sin^{-1}\left[\frac{r_0 \sin\left(\frac{s}{r_0}\right) - h}{R_0(s)}\right]. \tag{98}$$

Owing to the high degree of symmetry in the unperturbed boundary, we are free to select without any loss of generality an incoming (a) plane wave propagating parallel to the y–axis in the sense of y–decreasing (corresponding to selecting $\theta = \pi/2$ in the notation of Section 5.1) and (b) cylindrical wave emanating from the source point $(0, h)$, where $h \gg r_0$.
In both cases, there are 'limiting' incident rays which are tangent to the *unperturbed* boundary, and which separates the region on the scatterer from which reflected rays radiate from that in which they do not (*i.e.* the 'illuminated or lit' and 'unilluminated or dark' portions of the boundary). These 'lit' regions correspond to

$$0 \le \frac{s}{r_0} \le \pi, \quad \sin^{-1}\left(\frac{r_0}{h}\right) \le s/a \le \pi - \sin^{-1}\left(\frac{r_0}{h}\right) \tag{99}$$

for our choices of plane, cylindrical wave incidence, respectively. We have seen already that the geometry of the scattered field is largely dictated by the ray directions $(p_0(s), q_0(s))$, and referring back to (70) and then (71), (72) for the plane wave case – an exactly similar thing occurs for cylindrical waves – we note that there is choice in their selection. In that instance, we rejected the former possibility since it generated a family of rays parallel to the incoming ones and thereby violated standard laws of ray reflection (the 'outgoing wave' condition for one thing). However, this is *precisely* the solution we choose for the radiated

rays leaving the unlit portion of the boundary since not only do they then exactly coincide with the incoming rays but also can be shown, after calculation, to have an associated field amplitude equal in magnitude but opposite in sign. Exact cancellation between the two fields then occurs and the total field is therefore zero to leading–order in the region of space that the obstructing scatterer 'blocks' the incoming field; in other words, the 'geometrical shadow' is easily constructed in both cases, though we omit the technical details here in this account to concentrate instead on the reflected fields present elsewhere.
Having gone through the general construction of the ray solution in the main body of the paper, we summarise the results relevant to the reflected fields in these examples by simply listing for completeness those functions arising in that analysis and which can be used to substitute directly into the relevant expressions that arose. Since we itemise them in separate sections for plane and cylindrical wave incidence, and therefore no confusion is likely, we drop any subscripts ('P' for plane and 'C' for cylindrical) in those original formulae. In all cases, the solution to the analogous problem involving Dirichlet data is obtained by multiplying the final result presented here by -1.

6.1 Plane Wave Incidence: $\theta = \pi/2$

$$p_0(s) = \sin\left(\frac{2s}{r_0}\right), \quad q_0(s) = -\cos\left(\frac{2s}{r_0}\right) \tag{100}$$

$$\tau(s,r) = \sqrt{r^2 - r_0^2\cos^2\left(\frac{s}{r_0}\right)} - r_0\sin\left(\frac{s}{r_0}\right) \tag{101}$$

$$\mu = -r_0\sin\left(\frac{s}{r_0}\right) + \tau \tag{102}$$

$$\nu(s) = -2r_0^{1/2}\sin\left(\frac{s}{r_0}\right) f\left(\frac{s}{r_0}\right) \tag{103}$$

$$a(s) = \frac{1}{2}r_0\sin\left(\frac{s}{r_0}\right) \tag{104}$$

$$b(s) = r_0^3\left[\frac{d}{ds}\left(f\left(\frac{s}{r_0}\right)\sin\left(\frac{s}{r_0}\right)\right)\right]^2 \tag{105}$$

$$\begin{aligned} A_0(s,0) &= F_P^I\left(r_0\cos\left(\frac{s}{r_0}\right)\right) \\ &\times \exp\left[if^2\left(\frac{s}{r_0}\right)\frac{\cos^2\left(\frac{s}{r_0}\right)}{\sin\left(\frac{s}{r_0}\right)} + 2if\left(\frac{s}{r_0}\right)f'\left(\frac{s}{r_0}\right)\cos\left(\frac{s}{r_0}\right)\right] \end{aligned} \tag{106}$$

6.2 Cylindrical Wave Incidence

$$p_0(s) = -\cos\left(\frac{2s}{r_0} - \psi_0(s)\right), \quad q_0(s) = -\sin\left(\frac{2s}{r_0} - \psi_0(s)\right) \tag{107}$$

$$\tau = r_0 \cos\left(\frac{s}{r_0} - \psi_0(s)\right) - \sqrt{r^2 - r_0^2 \sin^2\left(\frac{s}{r_0} - \psi_0(s)\right)} \tag{108}$$

$$\begin{aligned} \mu &= \left[r_0^2 \cos^2\left(\frac{s}{r_0}\right) + \left(r_0 \sin\left(\frac{s}{r_0}\right) - h\right)^2\right]^{1/2} \\ &+ r_0 \cos\left(\frac{s}{r_0} - \psi_0(s)\right) - \sqrt{r^2 - r_0^2 \sin^2\left(\frac{s}{r_0} - \psi_0(s)\right)} \end{aligned} \tag{109}$$

$$\nu = 2\cos\left(\frac{s}{r_0} - \psi_0(s)\right) r_0^{1/2} f\left(\frac{s}{r_0}\right) \tag{110}$$

$$a(s) = \frac{R_0(s) r_0 \cos\left(\frac{s}{r_0} - \psi_0(s)\right)}{r_0 \cos\left(\frac{s}{r_0}\right) - 2R_0(s)} \tag{111}$$

$$\begin{aligned} b(s) &= 4r_0 \\ &\times \left[\frac{R_0(s)\cos\left(\frac{s}{r_0} - \psi_0(s)\right) f'\left(\frac{s}{r_0}\right) - \left(R_0(s) - r_0\cos\left(\frac{s}{r_0} - \psi_0(s)\right)\sin\left(\frac{s}{r_0} - \psi_0(s)\right) f\left(\frac{s}{r_0}\right)\right)}{r_0\cos\left(\frac{s}{r_0} - \psi_0(s)\right) - 2R_0(s)}\right]^2 \end{aligned} \tag{112}$$

$$\begin{aligned} A_0(s,0) &= F_C^I\left(\frac{y_0(s) - h}{x_0(s)}\right)(R_0(s))^{-1/2} \\ &\times \exp\left[\left(\frac{2r_0}{R_0(s)} - 1\right) f^2\left(\frac{s}{r_0}\right)\sin^2\left(\frac{s}{r_0} - \psi_0(s)\right) + 2if\left(\frac{s}{r_0}\right) f'\left(\frac{s}{r_0}\right)\sin\left(\frac{s}{r_0} - \psi_0(s)\right)\right]. \end{aligned} \tag{113}$$

7 Discussion and Concluding Remarks

Our first observation is that the presence of the boundary perturbation does not in itself induce amplifications in the *amplitudes* (by way of focusing or caustic formation, for example) of the reflected fields for either type of wave incidence. Any amplification that is present is primarily due to the geometry of the *unperturbed* boundary, and is located parametrically at focal points or caustic curves in the physical domain satisfying the equation $\tau + a(s) = 0$ (in the notation of, say, equation (25)). However, such instances *are* now singularities of the *phase* distortion in the perturbed cases because of the corrections to the phase via the exponential factor in (25) (noting that $a(s)$ and $b(s)$ are both real–valued) that the perturbation undulations induce.

So although no *new* caustic curves (or focal points) are induced within the reflected field because of these perturbing undulations – those that exist would have been present anyway even in their absence – it is the case that the field variation across those that do exist, classically described by Airy functions in the case of caustics ([15], [16]), will now need some adjustment to account for this modified singular phase variation across the caustic and this is currently under investigation.

A special class of problems that can arise, not discussed at all here, pertains to incoming plane waves for which the angle of incidence θ (as defined earlier within the main text of the paper) is small *i.e.* the incoming plane wave is at, or is close to, grazing incidence and when it would meet the boundary perturbation broadside on. More specifically, if $\theta = O\left(k^{-1/2}\right)$, then the incoming plane wave $\phi_P^I(x,y)$ can be approximated, having first set $\theta = k^{-1/2}\hat{\theta}$ for $|\hat{\theta}| = O(1)$ and fixed, by

$$\phi_P^I(x,y) \propto e^{ikx - ik^{1/2}\hat{\theta}y - \frac{1}{2}i\theta^2 x} \tag{114}$$

where the error in the exponent, for $O(1)$ values of x and y, is of $O\left(k^{-1/2}\right)$ and would constitute a higher–order correction to the amplitude, and not the phase (*e.g.* the omitted exponential is $e^{ik^{-1/2}\hat{\theta}^3 y/6} \sim 1 - \dfrac{1}{6}ik^{-1/2}\hat{\theta}^3 y$ if $|y| = O(1)$). We see from (114) that once more an ansatz of the form (16) is required, but for completely different reasons to before – these originally being motivated strongly by the geometry of the boundary $\partial\hat{D} : y = k^{-1/2}f(x)$ rather than the form of the incoming field itself. Indeed, to make matters even more interesting, when the modified incident field (114) valid away from the boundary is actually evaluated upon it, then propagating exponentials such as $e^{ikx - i\left(f(x)\hat{\theta} - \frac{1}{2}\hat{\theta}^2 x\right)}$ emerge, and rather than containing $O\left(k^{1/2}\right)$ terms within the exponent to justify (16), it does not actually contain *any*. A completely separate analysis is required for this class of problems, and this is also being pursued elsewhere.

Another extension currently under investigation include reflection of *elastic* waves at both near–planar and perturbed but otherwise generally curved free surfaces. A new feature within this class of problems is mode conversion, whereby an elastic wave of one type (such as longitudinal 'P' or vertically–polarised shear 'SV') can, and generally will, give rise to a reflected wave involving the other. This not only brings about the possibility of total internal reflection but also that of exciting a Rayleigh surface wave. In either case propagation *along* the perturbed boundary is a key characteristic, allowing these waves to accrue the cumulative effects of the boundary undulations; this is in stark contrast to the reflection–only problems considered here for which the interaction of rays – incoming or reflected – occurs in a pointwise fashion.

Finally, we remark that the Friedlander–Keller expansions that we have considered all involve exponentials of the form $e^{iku + ik^{1/2}v}$; considering the exponent as a sequence of terms with decreasing powers of $k^{1/2}$ we see immediately that this is the most general possible, it being impossible to 'fit' any more terms in between the 'u' and 'v' terms (and we note that we can disregard any terms in k^0, since these lead to terms that can be absorbed into amplitude considerations if we choose to).
Suppose instead that the relevant power of k was one–third, rather than one–half, as occurs in creeping and whispering gallery wave propagation (referred to earlier). In those examples, it is well–known that full descriptions can be obtained using Friedlander–Keller expansions

with exponents of the form $iku + ik^{1/3}v$. However, there appears to be no *a priori* reason why there shouldn't also be an intermediate term proportional to $ik^{2/3}$, prompting a more general Friedlander–Keller exponent of the form $iku(x,y) + ik^{2/3}v(x,y) + ik^{1/3}w(x,y)$ exhibiting all intermediate powers of $k^{1/3}$ up to and including k.
This strongly motivates a completely separate study of a yet further generalised Friedlander–Keller ray expansion of the form

$$\phi(x,y) \sim \exp\left[i\sum_{r=1}^{m} k^{r/m}u_r(x,y)\right]\sum_{n=0}^{\infty}\frac{A_n(x,y)}{k^{n/m}} \tag{115}$$

for any integer $m > 1$. A specific example is wave scattering by a perturbed circle of profile $r = r_0 + k^{-1/3}F(\theta)$, in terms of plane polar coordinates r and θ. In that case, a full *ansatz* of the form (115) with $m = 3$ is required, with all three terms in the exponent being needed. Similar issues arise in scattering by near–planar boundaries of the more general form $y = k^{-1/n}f_n(x)$, and all of these aspects are currently under investigation and review.

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