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Abstract

Holographic algorithms were first introduced by Valiant as a new methodology to derive polynomial time algorithms. The algorithms introduced by Valiant are based on matchgates, which are intrinsically for problems over planar structures. In this paper we introduce two new families of holographic algorithms. These algorithms work over general, i.e., not necessarily planar, graphs. Instead of matchgates, the two underlying families of constraint functions are of the *affine* type and of the *product* type. These play the role of Kasteleyn's algorithm for counting planar perfect matchings. The new algorithms are obtained by transforming a problem to one of these two families by holographic reductions.

The tractability of affine and product-type constraint functions is known. The real challenge is to determine when some concrete problem, expressed by its constraint functions, has such a holographic reduction. We present a polynomial time algorithm to decide if a given counting problem has a holographic algorithm using the affine or product-type constraint functions. Our algorithm also finds a holographic transformation when one exists. We exhibit concrete problems that can be solved by the new holographic algorithms. When the constraint functions are symmetric, we further present a polynomial time algorithm for the same decision and search problems, where the complexity is measured in terms of the (exponentially more) succinct presentation of symmetric constraint functions. The algorithm for the symmetric case also shows that the recent dichotomy theorem for Holant problems with symmetric constraints is efficiently decidable. Our proof techniques are mainly algebraic, e.g., stabilizers and orbits of group actions.

1 Introduction

Recently a number of complexity dichotomy theorems have been obtained for counting problems. Typically, such dichotomy theorems assert that a vast majority of problems expressible within the framework are #P-hard, however an intricate subset manages to escape this fate. They exhibit a great deal of mathematical structure, which leads to a polynomial time algorithm. In recent dichotomy theorems, a pattern has emerged [14, 19, 21, 15, 35, 23, 12, 33]. Some of the tractable cases are expressible as "those problems for which there exists a *holographic algorithm*." However, this understanding has been largely restricted to problems where the local constraint functions are symmetric over the Boolean domain. In order to gain a better understanding, we must determine the full extent of holographic algorithms, beyond the symmetric constraints.

Holographic algorithms were first introduced by Valiant [46, 45]. They are applicable for any problem that can be expressed as the contraction of a tensor network. Valiant's algorithms have two main ingredients. The first ingredient is to encode computation in planar graphs using matchgates [44, 43, 9, 17, 10]. The result of the computation is then obtained by counting the number of

perfect matchings in a related planar graph, which can be done in polynomial time by Kasteleyn's (a.k.a. the FKT) algorithm [37, 42, 38]. The second ingredient is a holographic reduction, which is achieved by a choice of linear basis vectors. The computation can be carried out in any basis since the output of the computation is independent of the basis.

In this paper, we introduce two new families of holographic algorithms. These algorithms holographically reduce to problems expressible by either the *affine* type or the *product* type of constraint functions. Both types of problems are tractable over general (i.e. not necessarily planar) graphs [25], so the holographic algorithms are all polynomial time algorithms and work over general graphs. We present a polynomial time algorithm to decide if a given counting problem has a holographic algorithm over general graphs using the affine or product-type constraint functions. Our algorithm also finds a holographic algorithm when one exists. To formally state this result, we briefly introduce some notation.

The counting problems we consider are those expressible as a Holant problem [24, 22, 20, 25]. A Holant problem is defined by a set $\mathcal{F}$ of constraint functions, which we call signatures, and is denoted by Holant($\mathcal{F}$). An instance to Holant($\mathcal{F}$) is a tuple $\Omega = (G, \mathcal{F}, \pi)$ called a signature grid, where $G = (V, E)$ is a graph and π labels each vertex $v \in V$ and its incident edges with some $f_v \in \mathcal{F}$ and its input variables. Here f_v maps $\{0,1\}^{\deg(v)}$ to $\mathbb{C}$. We consider all possible 0-1 edge assignments. An assignment σ to the edges E gives an evaluation $\prod_{v \in V} f_v(\sigma|_{E(v)})$, where $E(v)$ denotes the incident edges of v and $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute

$$\text{Holant}_\Omega = \sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} f_v\left(\sigma|_{E(v)}\right).$$

For example, consider the problem of counting Perfect Matching on G. This problem corresponds to attaching the Exact-One function at every vertex of G. The Exact-One function is an example of a symmetric signature, which are functions that only depend on the Hamming weight of the input. We denote a symmetric signature by $f = [f_0, f_1, \ldots, f_n]$ where f_w is the value of f on inputs of Hamming weight w. For example, $[0, 1, 0, 0]$ is the Exact-One function on three bits. The output is 1 if and only if the input is 001, 010, or 100, and the output is 0 otherwise.

Holant problems contain both counting constraint satisfaction problems and counting graph homomorphisms as special cases. All three classes of problems have received considerable attention, which has resulted in a number of dichotomy theorems (see [40, 34, 29, 2, 28, 5, 31, 8] and [4, 3, 27, 1, 25, 7, 13, 30, 32, 14, 6]). Despite this success with #CSP and graph homomorphisms, the case with Holant problems is more difficult. A recent dichotomy theorem for Holant problems with symmetric signatures was obtained in [12]. But the general (i.e. not necessarily symmetric) case has a richer and more intricate structure. The same dichotomy for general signatures remains open. Our first main result makes a solid step forward in understanding holographic algorithms based on affine and product-type signatures in this more difficult setting.

Theorem 1.1. *There is a polynomial time algorithm to decide, given a finite set of signatures $\mathcal{F}$, whether* Holant($\mathcal{F}$) *admits a holographic algorithm based on affine or product-type signatures.*

These holographic algorithms for Holant($\mathcal{F}$) are all polynomial time in the size of the problem input Ω. The polynomial time decision algorithm of Theorem 1.1 is on another level; it decides based on any specific set of signatures $\mathcal{F}$ whether the counting problem Holant($\mathcal{F}$) defined by $\mathcal{F}$ has such a holographic algorithm.

However, symmetric signatures are an important special case. Because symmetric signatures can be presented exponentially more succinctly, we would like the decision algorithm to be efficient when measured in terms of this succinct presentation. An algorithm for this case needs to be exponentially faster than the one in Theorem 1.1. In Theorem 1.2, we present a polynomial time algorithm for the case of symmetric signatures. The increased efficiency is based on several signature invariants under orthogonal transformations.

Theorem 1.2. *There is a polynomial time algorithm to decide, given a finite set of symmetric signatures $\mathcal{F}$ expressed in the succinct notation, whether* Holant($\mathcal{F}$) *admits a holographic algorithm based on affine or product-type signatures.*

A dichotomy theorem classifies every set of signatures as defining either a tractable problem or an intractable problem (e.g. #P-hard). Yet it would be more useful if given a specific set of signatures, one could decide to which case it belongs. This is the decidability problem of a dichotomy theorem. In [12], a dichotomy regarding symmetric complex-weighted signatures for Holant problem was proved. However, the decidability problem was left open. Of the five tractable cases in this dichotomy theorem, three of them are easy to decide, but the remaining two cases are more challenging, which are (1) holographic algorithms using affine signatures and (2) holographic algorithms using product-type signatures. As a consequence of Theorem 1.2, this decidability is now proved.

Corollary 1.3. *The dichotomy theorem for symmetric complex-weighted Holant problems in [12] is decidable in polynomial time.*

Previous work on holographic algorithms focused almost exclusively on those with matchgates [46, 45, 16, 19, 17, 18, 33]. (This has led to a misconception in the community that holographic algorithms are always based on matchgates.) The first example of a holographic algorithm using something other than matchgates came in [24]. These holographic algorithms use generalized Fibonacci gates. A symmetric signature $f = [f_0, f_1, \ldots, f_n]$ is a generalized Fibonacci gate of type $\lambda \in \mathbb{C}$ if $f_{k+2} = \lambda f_{k+1} + f_k$ holds for all $k \in \{0, 1, \ldots, n-2\}$. The standard Fibonacci gates are of type $\lambda = 1$, in which case, the entries of the signature satisfy the recurrence relation of the Fibonacci numbers. The generalized Fibonacci gates were immediately put to use in a dichotomy theorem [22]. As it turned out, for nearly all values of λ, the generalized Fibonacci gates are holographically equivalent to product-type signatures. However, generalized Fibonacci gates are symmetric by definition. A main contribution of this paper is to extend the reach of holographic algorithms, other than those based on matchgates, beyond the symmetric case.

The constraint functions we call signatures are essentially tensors. Our central object of study can be rephrased as the orbits of affine and product-type tensors when acted upon by the orthogonal group (and related groups). We show that one can efficiently decide if any such orbit of a given tensor intersects the set of affine or product-type tensors. This result also generalizes to a set of tensors as stated in Theorems 1.1 and 1.2. In contrast, this orbit problem with the general linear group acting on two arbitrary tensors is NP-hard [39]. The so-called orbit closure problem has a fundamental importance in the foundation of geometric complexity theory [41].

Our techniques are mainly algebraic. A particularly important insight is that an orthogonal transformation in the standard basis is equivalent to a diagonal transformation in the $\left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ basis, a type of correspondence as in Fourier transform. Since diagonal transformations are much easier

to understand, this gives us a great advantage in understanding orbits under orthogonal transformations. Also, the groups of transformations that stabilize the affine and product-type signatures play an important role in our proofs.

In Section 2, we review basic notation and state previous results, many of which come from [11], the full version of [12]. In Section 3, we present some example problems that are tractable by holographic algorithms using affine or product-type signatures. The proof of Theorem 1.1 spans two sections. The affine case is handled in Section 4 and the product-type case is handled in Section 5. The proof of Theorem 1.2 also spans two sections. Once again, the affine case is handled in Section 6 and the product-type case is handled in Section 7.

2 Preliminaries

2.1 Problems and Definitions

The framework of Holant problems is defined for functions mapping any $[q]^k \to \mathbb{F}$ for a finite q and some field $\mathbb{F}$. In this paper, we investigate some of the tractable complex-weighted Boolean Holant problems, that is, all functions are $[2]^k \to \mathbb{C}$. Strictly speaking, for consideration of models of computation, functions take complex algebraic numbers.

A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$, where π labels each vertex $v \in V$ and its incident edges with some $f_v \in \mathcal{F}$ and its input variables. The Holant problem on instance Ω is to evaluate $\text{Holant}_\Omega = \sum_\sigma \prod_{v \in V} f_v(\sigma \mid_{E(v)})$, a sum over all edge assignments $\sigma : E \to \{0, 1\}$.

A function f_v can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{2^{\deg(v)}}$, or as a tensor in $(\mathbb{C}^2)^{\otimes \deg(v)}$. We also use $f_{\mathbf{x}}$ to denote the value $f(\mathbf{x})$, where $\mathbf{x}$ is a binary string. A function $f \in \mathcal{F}$ is also called a *signature*. A symmetric signature f on k Boolean variables can be expressed as $[f_0, f_1, \ldots, f_k]$, where f_w is the value of f on inputs of Hamming weight w.

A Holant problem is parametrized by a set of signatures.

Definition 2.1. *Given a set of signatures $\mathcal{F}$, we define the counting problem* Holant($\mathcal{F}$) *as:*
Input: A signature grid $\Omega = (G, \mathcal{F}, \pi)$*;*
Output: Holant_Ω*.*

A signature f of arity n is *degenerate* if there exist unary signatures $u_j \in \mathbb{C}^2$ $(1 \le j \le n)$ such that $f = u_1 \otimes \cdots \otimes u_n$. A symmetric degenerate signature has the from $u^{\otimes n}$. For such signatures, it is equivalent to replace it by n copies of the corresponding unary signature. Replacing a signature $f \in \mathcal{F}$ by a constant multiple cf, where $c \neq 0$, does not change the complexity of Holant($\mathcal{F}$). It introduces a global factor to Holant_Ω.

We say a signature set $\mathcal{F}$ is tractable (resp. #P-hard) if the corresponding counting problem Holant($\mathcal{F}$) is tractable (resp. #P-hard). Similarly for a signature f, we say f is tractable (resp. #P-hard) if $\{f\}$ is.

2.2 Holographic Reduction

To introduce the idea of holographic reductions, it is convenient to consider bipartite graphs. For a general graph, we can always transform it into a bipartite graph while preserving the Holant value, as follows. For each edge in the graph, we replace it by a path of length two. (This operation is

called the *2-stretch* of the graph and yields the edge-vertex incidence graph.) Each new vertex is assigned the binary EQUALITY signature $(=_2) = [1, 0, 1]$.

We use Holant $(\mathcal{F} \mid \mathcal{G})$ to denote the Holant problem on bipartite graphs $H = (U, V, E)$, where each vertex in U or V is assigned a signature in $\mathcal{F}$ or $\mathcal{G}$, respectively. An input instance for this bipartite Holant problem is a bipartite signature grid and is denoted by $\Omega = (H;\ \mathcal{F} \mid \mathcal{G};\ \pi)$. Signatures in $\mathcal{F}$ are considered as row vectors (or covariant tensors); signatures in $\mathcal{G}$ are considered as column vectors (or contravariant tensors) [26].

For a 2-by-2 matrix T and a signature set $\mathcal{F}$, define $T\mathcal{F} = \{g \mid \exists f \in \mathcal{F} \text{ of arity } n,\ g = T^{\otimes n} f\}$, similarly for $\mathcal{F}T$. Whenever we write $T^{\otimes n} f$ or $T\mathcal{F}$, we view the signatures as column vectors; similarly for $fT^{\otimes n}$ or $\mathcal{F}T$ as row vectors.

Let T be an element of $\mathbf{GL}_2(\mathbb{C})$, the group of invertible 2-by-2 complex matrices. The holographic transformation defined by T is the following operation: given a signature grid $\Omega = (H;\ \mathcal{F} \mid \mathcal{G};\ \pi)$, for the same graph H, we get a new grid $\Omega' = (H;\ \mathcal{F}T \mid T^{-1}\mathcal{G};\ \pi')$ by replacing each signature in $\mathcal{F}$ or $\mathcal{G}$ with the corresponding signature in $\mathcal{F}T$ or $T^{-1}\mathcal{G}$.

Theorem 2.2 (Valiant's Holant Theorem [46])**.** *If there is a holographic transformation mapping signature grid Ω to Ω', then* $\text{Holant}_{\Omega} = \text{Holant}_{\Omega'}$.

Therefore, an invertible holographic transformation does not change the complexity of the Holant problem in the bipartite setting. Furthermore, there is a particular kind of holographic transformation, the orthogonal transformation, that preserves the binary equality and thus can be used freely in the standard setting. Let $\mathbf{O}_2(\mathbb{C})$ be the group of 2-by-2 complex matrices that are orthogonal. Recall that a matrix T is orthogonal if $TT^{\mathtt{T}} = I$.

Theorem 2.3 (Theorem 2.6 in [20])**.** *Suppose $T \in \mathbf{O}_2(\mathbb{C})$ and let $\Omega = (H, \mathcal{F}, \pi)$ be a signature grid. Under a holographic transformation by T, we get a new grid $\Omega' = (H, T\mathcal{F}, \pi')$ and* $\text{Holant}_{\Omega} = \text{Holant}_{\Omega'}$.

We also use $\mathbf{SO}_2(\mathbb{C})$ to denote the group of special orthogonal matrices, i.e. the subgroup of $\mathbf{O}_2(\mathbb{C})$ with determinant 1.

2.3 Tractable Signature Sets without a Holographic Transformation

The following two signature sets are tractable without a holographic transformation [25].

Definition 2.4. *A k-ary function $f(x_1, \ldots, x_k)$ is* affine *if it has the form*

$$\lambda \chi_{Ax=0} \cdot i^{\sum_{j=1}^{n} \langle \mathbf{v}_j, x \rangle},$$

where $\lambda \in \mathbb{C}$, $x = (x_1, x_2, \ldots, x_k, 1)^{\mathtt{T}}$, A is a matrix over $\mathbb{F}_2$, $\mathbf{v}_j$ is a vector over $\mathbb{F}_2$, and χ is a 0-1 indicator function such that $\chi_{Ax=0}$ is 1 iff $Ax = 0$. Note that the dot product $\langle \mathbf{v}_j, x \rangle$ is calculated over $\mathbb{F}_2$, while the summation $\sum_{j=1}^{n}$ on the exponent of $i = \sqrt{-1}$ is evaluated as a sum mod 4 of 0-1 terms. We use $\mathscr{A}$ to denote the set of all affine functions.

Notice that there is no restriction on the number of rows in the matrix A. It is permissible that A is the zero matrix so that $\chi_{Ax=0} = 1$ holds for all x. An equivalent way to express the exponent of i is as a quadratic polynomial where all cross terms have an even coefficient.

Definition 2.5. *A function is of* product type *if it can be expressed as a product of unary functions, binary equality functions* ($[1,0,1]$)*, and binary disequality functions* ($[0,1,0]$)*. We use $\mathscr{P}$ to denote the set of product-type functions.*

It can be shown (cf. Lemma A.1 in [36], the full version of [35]) that if f is a symmetric signature in $\mathscr{P}$, then f is either degenerate, binary disequality, or of the form $[a,0,\ldots,0,b]$ for some $a,b\in\mathbb{C}$. It is known that the set of non-degenerate symmetric signatures in $\mathscr{A}$ is precisely the nonzero signatures ($\lambda\neq 0$) in $\mathscr{F}_1\cup\mathscr{F}_2\cup\mathscr{F}_3$ with arity at least two, where $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ are three families of signatures defined as

$$\mathscr{F}_1=\left\{\lambda\left([1,0]^{\otimes k}+i^r[0,1]^{\otimes k}\right)\mid\lambda\in\mathbb{C},k=1,2,\ldots,r=0,1,2,3\right\},$$
$$\mathscr{F}_2=\left\{\lambda\left([1,1]^{\otimes k}+i^r[1,-1]^{\otimes k}\right)\mid\lambda\in\mathbb{C},k=1,2,\ldots,r=0,1,2,3\right\},\text{ and}$$
$$\mathscr{F}_3=\left\{\lambda\left([1,i]^{\otimes k}+i^r[1,-i]^{\otimes k}\right)\mid\lambda\in\mathbb{C},k=1,2,\ldots,r=0,1,2,3\right\}.$$

Let $\mathscr{F}_{123}=\mathscr{F}_1\cup\mathscr{F}_2\cup\mathscr{F}_3$ be the union of these three sets of signatures. We explicitly list all the signatures in $\mathscr{F}_{123}$ up to an arbitrary constant multiple from $\mathbb{C}$:

1. $[1,0,\ldots,0,\pm 1]$; $(\mathscr{F}_1, r=0,2)$
2. $[1,0,\ldots,0,\pm i]$; $(\mathscr{F}_1, r=1,3)$
3. $[1,0,1,0,\ldots,0 \text{ or } 1]$; $(\mathscr{F}_2, r=0)$
4. $[1,-i,1,-i,\ldots,(-i) \text{ or } 1]$; $(\mathscr{F}_2, r=1)$
5. $[0,1,0,1,\ldots,0 \text{ or } 1]$; $(\mathscr{F}_2, r=2)$
6. $[1,i,1,i,\ldots,i \text{ or } 1]$; $(\mathscr{F}_2, r=3)$
7. $[1,0,-1,0,1,0,-1,0,\ldots,0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r=0)$
8. $[1,1,-1,-1,1,1,-1,-1,\ldots,1 \text{ or } (-1)]$; $(\mathscr{F}_3, r=1)$
9. $[0,1,0,-1,0,1,0,-1,\ldots,0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r=2)$
10. $[1,-1,-1,1,1,-1,-1,1,\ldots,1 \text{ or } (-1)]$. $(\mathscr{F}_3, r=3)$

2.4 $\mathscr{A}$-transformable and $\mathscr{P}$-transformable Signatures

The tractable sets $\mathscr{A}$ and $\mathscr{P}$ are still tractable under a suitable holographic transformation. This is captured by the following definition.

Definition 2.6. *A set $\mathcal{F}$ of signatures is $\mathscr{A}$-transformable (resp. $\mathscr{P}$-transformable) if there exists a holographic transformation T such that $\mathcal{F}\subseteq T\mathscr{A}$ (resp. $\mathcal{F}\subseteq T\mathscr{P}$) and $[1,0,1]T^{\otimes 2}\in\mathscr{A}$ (resp. $[1,0,1]T^{\otimes 2}\in\mathscr{P}$).*

To refine the above definition, we consider the stabilizer group of $\mathscr{A}$,

$$\mathrm{Stab}(\mathscr{A})=\{T\in\mathbf{GL}_2(\mathbb{C})\mid T\mathscr{A}\subseteq\mathscr{A}\}.$$

Technically this set is the left stabilizer group of $\mathscr{A}$, but it turns out that the left and right stabilizer groups of $\mathscr{A}$ coincide (Lemma 8.3 in [11]).

The following matrices are useful. Let $D=\left[\begin{smallmatrix}1&0\\0&i\end{smallmatrix}\right]$ and $H_2=\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1&1\\1&-1\end{smallmatrix}\right]$. Also let $X=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ and $Z=\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1&1\\i&-i\end{smallmatrix}\right]$. Note that $Z=DH_2$ and that $D^2Z=\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1&1\\-i&i\end{smallmatrix}\right]=ZX$, hence $X=Z^{-1}D^2Z$. It is easy to verify that $D,H_2,X,Z\in\mathrm{Stab}(\mathscr{A})$. In fact, $\mathrm{Stab}(\mathscr{A})=\mathbb{C}^*\cdot\langle D,H_2\rangle$, all scalar multiples of the group generated by D and H_2 (Lemma 8.3 in [11]).

The next lemma is the first step toward understanding the $\mathscr{A}$-transformable signatures. In this lemma and throughout the paper, we use α to denote $\frac{1+i}{\sqrt{2}}=\sqrt{i}=e^{\frac{\pi i}{4}}$.

Lemma 2.7 (Lemma 8.5 in [11]). *Let $\mathcal{F}$ be a set of signatures. Then $\mathcal{F}$ is $\mathscr{A}$-transformable iff there exists an $H \in \mathbf{O}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{A}$ or $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$.*

The three sets $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$ capture all symmetric $\mathscr{A}$-transformable signatures.

Definition 2.8. *A symmetric signature f of arity n is in, respectively, $\mathscr{A}_1$, or $\mathscr{A}_2$, or $\mathscr{A}_3$ if there exist an $H \in \mathbf{O}_2(\mathbb{C})$ and nonzero constant $c \in \mathbb{C}$ such that f has the form, respectively, $cH^{\otimes n}\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n}\right)$, or $cH^{\otimes n}\left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n}\right)$, or $cH^{\otimes n}\left(\left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n}\right)$, where $\beta = \alpha^{tn+2r}$, $r \in \{0,1,2,3\}$, and $t \in \{0,1\}$.*

For $i \in \{1,2,3\}$, when such an orthogonal H exists, we say that $f \in \mathscr{A}_i$ with transformation H. If $f \in \mathscr{A}_i$ with I_2, then we say f is in the canonical form of $\mathscr{A}_i$.

Lemma 2.9 (Lemma 8.10 in [11]). *Let f be a non-degenerate symmetric signature. Then f is $\mathscr{A}$-transformable iff $f \in \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$.*

We also have a similar characterization for $\mathscr{P}$-transformable signatures using the stabilizer group of $\mathscr{P}$,

$$\mathrm{Stab}(\mathscr{P}) = \{T \in \mathbf{GL}_2(\mathbb{C}) \mid T\mathscr{P} \subseteq \mathscr{P}\}.$$

The group $\mathrm{Stab}(\mathscr{P})$ is generated by matrices of the form $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \nu \end{smallmatrix}\right]$ for any $\nu \in \mathbb{C}$ and $X = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Lemma 2.10 (Lemma 8.11 in [11]). *Let $\mathcal{F}$ be a set of signatures. Then $\mathcal{F}$ is $\mathscr{P}$-transformable iff there exists an $H \in \mathbf{O}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{P}$ or $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P}$.*

Definition 2.11. *A symmetric signature f of arity n is in $\mathscr{P}_1$ if there exist an $H \in \mathbf{O}_2(\mathbb{C})$ and a nonzero $c \in \mathbb{C}$ such that $f = cH^{\otimes n}\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n}\right)$, where $\beta \neq 0$.*

It is easy to check that $\mathscr{A}_1 \subset \mathscr{P}_1$. We define $\mathscr{P}_2 = \mathscr{A}_2$. For $i \in \{1,2\}$, when such an H exists, we say that $f \in \mathscr{P}_i$ with transformation H. If $f \in \mathscr{P}_i$ with I_2, then we say f is in the canonical form of $\mathscr{P}_i$.

Lemma 2.12 (Lemma 8.13 in [11]). *Let f be a non-degenerate symmetric signature. Then f is $\mathscr{P}$-transformable iff $f \in \mathscr{P}_1 \cup \mathscr{P}_2$.*

3 Some Example Problems

3.1 A Fibonacci-like Problem

Fibonacci gates were introduced in [24]. These lead to tractable counting problems, and holographic algorithms based on Fibonacci gates work over general (i.e. not necessarily planar) graphs. However, Fibonacci gates are symmetric by definition. An example of a Fibonacci gate is the signature $f = [f_0, f_1, f_2, f_3] = [1, 0, 1, 1]$. Its entries satisfy the recurrence relation of the Fibonacci numbers, i.e. $f_2 = f_1 + f_0$ and $f_3 = f_2 + f_1$. For Holant(f), the input is a 3-regular graph, and the problem is to count spanning subgraphs such that no vertex has degree 1.

A symmetric signature $g = [g_0, g_1, \ldots, g_n]$ is a generalized Fibonacci gate of type $\lambda \in \mathbb{C}$ if $g_{k+2} = \lambda g_{k+1} + g_k$ holds for all $k \in \{0, 1, \ldots, n-2\}$. The standard Fibonacci gates are of type $\lambda = 1$. An example of a generalized Fibonacci gate is $g = [3, 1, 3, 1]$, which has type $\lambda = 0$. In contrast to Holant(f), the problem Holant(g) permits all possible spanning subgraphs. The output

is the sum of the weights of each spanning subgraph. The weight of a spanning subgraph S is $3^{k(S)}$, where $k(S)$ is the number of vertices of even degree in S. Since $g = [3, 1, 3, 1]$ is Fibonacci, the problem Holant(g) is computable in polynomial time. One new family of holographic algorithms in this paper extends Fibonacci gates to asymmetric signatures.

In full notation, $g = (3, 1, 1, 3, 1, 3, 3, 1)^{\mathtt{T}} \in \{0, 1\}^3$. Consider the asymmetric signature $h = (3, 1, -1, -3, -1, -3, 3, 1)^{\mathtt{T}}$. This signature h differs from g by a negative sign in four entries. Although h is not a generalized Fibonacci gate or even a symmetric signature, it still defines a tractable Holant problem. Under a holographic transformation by Z^{-1}, where $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$,

$$\text{Holant}(h) = \text{Holant}\left(=_2 \mid h\right) = \text{Holant}\left(=_2 (Z^{-1})^{\otimes 2} \mid Z^{\otimes 4} h\right) = \text{Holant}\left([1, 0, -1] \mid \hat{h}\right),$$

where $\hat{h} = 2i\sqrt{2}(0, 1, 0, 0, 0, 0, 2i, 0)$. Both $[1, 0, -1](x_1, x_2) = \text{EQUALITY}(x_1, x_2) \cdot [1, -1](x_1)$ and $\hat{h}(x_1, x_2, x_3) = 2i\sqrt{2} \cdot \text{EQUALITY}(x_1, x_2) \cdot \text{DISEQUALITY}(x_2, x_3) \cdot [1, 2i](x_1)$ are product-type signatures.

It turns out that for all values of $\lambda \neq \pm 2i$, the generalized Fibonacci gates of type λ are $\mathscr{P}$-transformable. The value of λ indicates under which holographic transformation the signatures become product type. For $\lambda = \pm 2i$, the generalized Fibonacci gates of type λ are vanishing, which means the output is always zero for every possible input.

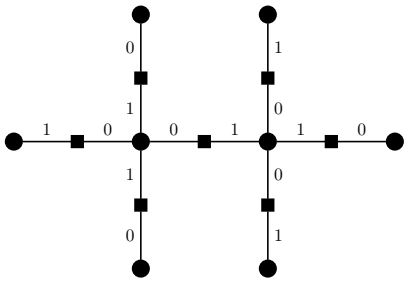
3.2 Some Cycle Cover Problems and Orientation Problems

To express some problems involving asymmetric signatures of arity 4, it is convenient to arrange the 16 outputs into a 4-by-4 matrix.

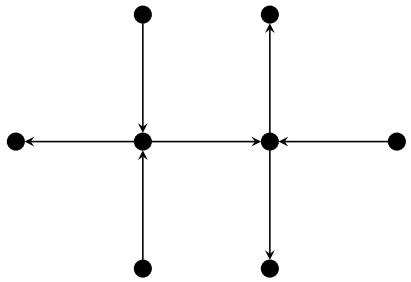
Definition 3.1 (Definition 6.2 in [11]). *The* signature matrix *of a signature $f(x_1, x_2, x_3, x_4)$ is $M_f = \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} \\ f_{1100} & f_{1110} & f_{1101} & f_{1111} \end{bmatrix}$, where the row is indexed by two bits (x_1, x_2) and the column is indexed by two bits (x_4, x_3) in reverse order. This ordering is for convenience in some proofs.*

Consider the problem of counting the number of cycle covers in a given graph. This problem is #P-hard even when restricted to planar 4-regular graphs [33]. As a Holant problem, its expression is Holant(f), where $f(x_1, x_2, x_3, x_4)$ is the symmetric signature $[0, 0, 1, 0, 0]$. The signature matrix of f is $M_f = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$. The six entries in the support of f, which are all of Hamming weight two (indicating that a cycle cover passes through each vertex exactly twice), can be divided into two parts, namely $\{0011, 0110, 1100, 1001\}$ and $\{0101, 1010\}$. In the planar setting, this corresponds to a pairing of adjacent or non-adjacent incident edges. Both sets are invariant under cyclic permutations.

Suppose we removed the inputs 0101 and 1010 from the support of f, which are the two 1's on the anti-diagonal in the middle of M_f. Call the resulting signature g, which has signature matrix $M_g = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$. These new 0's impose a constraint on the types of cycle covers allowed. We call a cycle cover *valid* if it satisfies this new constraint. A valid cycle cover must not pass through a vertex in a "crossing" way. Counting the number of such cycle covers over 4-regular graphs can be done in polynomial time, even without the planarity restriction, e.g., for a graph embedded on a surface of arbitrary genus. The signature $g(x_1, x_2, x_3, x_4) = \text{DIS-EQUALITY}(x_1, x_3) \cdot \text{DIS-EQUALITY}(x_2, x_4)$ is of the product type $\mathscr{P}$, therefore Holant(g) is tractable.



(a) An admissible assignment to this graph fragment. The circle vertices are assigned $\hat{g}$ and the square vertices are assigned $\neq_2$.



(b) The orientation induced by the assignment in (a).

Figure 1: A fragment of an instance to Holant $(\neq_2 \mid \hat{g})$, which must be a $(2, 4)$-regular bipartite graph. Note the saddle orientation of the edges incident to the two vertices with all four edges depicted.

Under a holographic transformation by $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$, we obtain the problem

$$\text{Holant}(g) = \text{Holant}\left(=_2 \mid g\right) = \text{Holant}\left(=_2 Z^{\otimes 2} \mid (Z^{-1})^{\otimes 4} g\right) = \text{Holant}\left(\neq_2 \mid \hat{g}\right),$$

where $M_{\hat{g}} = \left[\begin{smallmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix}\right]$. This problem has the following interpretation. It is a Holant problem on bipartite graphs. On the right side of the bipartite graph, the vertices must all have degree 4 and are assigned the signature $\hat{g}$. On the left side, the vertices must all have degree 2 and are assigned the binary disequality constraint $\neq_2$. The disequality constraints suggest an orientation between their two neighboring vertices of degree 4 (see Figure 1). By convention, we view the edge as having its tail assigned 0 and its head assigned 1. Then every valid assignment in this bipartite graph naturally corresponds to an orientation in the original 4-regular graph.

If the four inputs 0011, 0110, 1100, and 1001 were in the support of $\hat{g}$, then the Holant sum would be over all possible orientations with an even number of incoming edges at each vertex. As it is, the sum is over all possible orientations with an even number of incoming edges at each vertex that also forbid those four types of orientations at each vertex, as specified by $\hat{g}$. The following orientations are admissible by $\hat{g}$: The orientation of the edges are such that at each vertex all edges are oriented out (source vertex), or all edges are oriented in (sink vertex), or the edges are cyclically oriented in, out, in, out (saddle vertex).

Thus, the output of Holant $(\neq_2 \mid \hat{g})$ is a weighted sum over of these admissible orientations. Each admissible orientation O contributes a weight $(-1)^{s(O)}$ to the sum, where $s(O)$ is the number of source and sink vertices in an orientation O. We can express this as $\sum_{O \in \mathcal{O}(G)} (-1)^{s(O)}$, where $\mathcal{O}(G)$ is the set of admissible orientations for G, which are those orientations that only contain source, sink, and saddle vertices. In words, the value is the number of admissible orientations with an even number of sources and sinks minus the number of admissible orientations with an odd number of sources and sinks. This orientation problem may seem quite different from the restricted cycle cover problem we started with, but they are, in fact, the same problem. Since Holant(g) is tractable, so is Holant $(\neq_2 \mid \hat{g})$.

Now, consider a slight generalization of this orientation problem.

Problem: #λ-SourceSinkSaddleOrientations

Input: An undirected 4-regular graph G.
Output: $\sum_{O \in \mathcal{O}(G)} \lambda^{s(O)}$.

For $\lambda = -1$, we recover the orientation problem from above. For $\lambda = 1$, the problem is also tractable since, when viewed as a bipartite Holant problem on the $(2,4)$-regular bipartite vertex-edge incidence graph, the disequality constraint on the vertices of degree 2 and the constraint on the vertices of degree 4 are both product-type functions. As a function of x_1, x_2, x_3, x_4, the constraint on the degree 4 vertices is $\textsc{Equality}(x_1, x_3) \cdot \textsc{Equality}(x_2, x_4)$. Let $s_{k,m}(G)$ be the number of $O \in \mathcal{O}(G)$ such that $s(O) \equiv k \pmod{m}$. Then the output of this problem with $\lambda = 1$ is $s_{0,2}(G) + s_{1,2}(G)$ and the output of this problem with $\lambda = -1$ is $s_{0,2}(G) - s_{1,2}(G)$. Therefore, we can compute both $s_{0,2}(G)$ and $s_{1,2}(G)$. However, more is possible.

For $\lambda = i$, the problem is tractable using affine constraints. In the $(2,4)$-regular bipartite vertex-edge incidence graph, the disequality constraint assigned to the vertices of degree 2 is affine. On the vertices of degree 4, the assigned constraint function is an affine signature since the affine support is defined by the affine linear system $x_1 = x_3$ and $x_2 = x_4$ while the quadratic polynomial in the exponent of i is $3x_1^2 + 3x_2^2 + 2x_1x_2 + 1$. Although the output is a complex number, the real and imaginary parts encode separate information. The real part is $s_{0,4}(G) - s_{2,4}(G)$ and the imaginary part is $s_{1,4}(G) - s_{3,4}(G)$. Since $s_{0,2}(G) = s_{0,4}(G) + s_{2,4}(G)$ and $s_{1,2}(G) = s_{1,4}(G) + s_{3,4}(G)$, we can actually compute all four quantities $s_{0,4}(G)$, $s_{1,4}(G)$, $s_{2,4}(G)$, and $s_{3,4}(G)$ in polynomial time.

3.3 An Enigmatic Problem

Some problems may be a challenge for the human intelligence to grasp. But in a platonic view of computational complexity, they are no less valid problems. For example, consider the problem $\text{Holant}((1+c^2)^{-1}[1,0,-i] \mid f)$ where f has the signature matrix

$$\begin{bmatrix} 0 & (4+4i)\left(28+20\sqrt{2}+\sqrt{2\left(799+565\sqrt{2}\right)}\right) & (4+4i)\left(28+20\sqrt{2}+\sqrt{2\left(799+565\sqrt{2}\right)}\right) & -8i\left(13+9\sqrt{2}+2\sqrt{82+58\sqrt{2}}\right) \\ (4+4i)\left(28+20\sqrt{2}+\sqrt{2\left(799+565\sqrt{2}\right)}\right) & -8i\left(13+9\sqrt{2}+2\sqrt{82+58\sqrt{2}}\right) & 8i\left(18+13\sqrt{2}+4\sqrt{41+29\sqrt{2}}\right) & (-4+4i)\left(12+8\sqrt{2}+\sqrt{274+194\sqrt{2}}\right) \\ (4+4i)\left(28+20\sqrt{2}+\sqrt{2\left(799+565\sqrt{2}\right)}\right) & 8i\left(18+13\sqrt{2}+4\sqrt{41+29\sqrt{2}}\right) & -8i\left(13+9\sqrt{2}+2\sqrt{82+58\sqrt{2}}\right) & (-4+4i)\left(12+8\sqrt{2}+\sqrt{274+194\sqrt{2}}\right) \\ -8i\left(13+9\sqrt{2}+2\sqrt{82+58\sqrt{2}}\right) & (-4+4i)\left(12+8\sqrt{2}+\sqrt{274+194\sqrt{2}}\right) & (-4+4i)\left(12+8\sqrt{2}+\sqrt{274+194\sqrt{2}}\right) & -16\left(13+9\sqrt{2}+2\sqrt{82+58\sqrt{2}}\right) \end{bmatrix}$$

and $c = 1 + \sqrt{2} + \sqrt{2(1+\sqrt{2})}$. Most likely no one has ever considered this problem before. Yet this nameless problem is $\mathscr{A}$-transformable under $T = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & c \\ -c & 1 \end{smallmatrix}\right]$, and hence it is really the *same* problem as a more comprehensible problem defined by $\hat{f}$. Namely,

$$\text{Holant}((1+c^2)^{-1}[1,0,-i] \mid f) = \text{Holant}((1+c^2)^{-1}[1,0,-i]T^{\otimes 2} \mid (T^{-1})^{\otimes 4} f) = \text{Holant}([1,0,1] \mid \hat{f}) = \text{Holant}(\hat{f}),$$

where $M_{\hat{f}} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$. We can express $\hat{f}$ as $\hat{f}(x_1, x_2, x_3, x_4) = i^{Q(x)}$, where $Q(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1)$. Therefore, $\hat{f}$ is affine, which means that $\text{Holant}(\hat{f})$ as well as $\text{Holant}((1+c^2)^{-1}[1,0,-i] \mid f)$ are tractable. Furthermore, notice that $\hat{f}$ only contains integers even though $(1+c^2)^{-1}[1,0,-i]$ and f contain many complex numbers with irrational real and imaginary parts. Thus, $\text{Holant}((1+c^2)^{-1}[1,0,-i] \mid f)$ is not only tractable, but it always outputs an integer. Apparent anomalies like $\text{Holant}((1+c^2)^{-1}[1,0,-i] \mid f)$, however contrived they may seem to be to the human eye, behoove the creation of a systematic theory to understand and characterize the tractable cases.

4 General $\mathscr{A}$-transformable Signatures

Let f be a signature of arity n. It is given as a column vector in $\mathbb{C}^{2^n}$ with bit length N, which is on the order of 2^n. We denote its entries by $f_{\mathbf{x}} = f(\mathbf{x})$ indexed by $\mathbf{x} \in \{0,1\}^n$. The entries are from a fixed degree algebraic extension of $\mathbb{Q}$ and we may assume basic bit operations in the field take unit time.

Notice that the number of general affine signatures of arity n are on the order of 2^{n^2}. Hence a naive check of the membership of affine signatures would result in a super-polynomial running time in N. Instead, we present a polynomial time algorithm.

Lemma 4.1. *There is an algorithm to decide whether a given signature f of arity n belongs to $\mathscr{A}$ with running time polynomial in N, the bit length of f.*

Proof. If f is identically zero, then f is trivially in $\mathscr{A}$, so assume that f is not identically zero. We first normalize f so that the first nonzero entry of f is 1. If there exists a nonzero entry of f after normalization that is not a power of i, then $f \notin \mathscr{A}$, so assume that all entries are now a power of i.

The next step is to decide if the support $\mathcal{S} \neq \emptyset$ of f forms an affine linear subspace. We try to build a basis for $\mathcal{S}$ inductively. It may end successfully or find an inconsistency. We choose the index of the first nonzero entry $\mathbf{b}_0 \in \mathcal{S}$ as our first basis element. Assume we have a set of basis elements $\mathcal{B} = \{\mathbf{b}_0, \ldots, \mathbf{b}_k\} \subseteq \mathcal{S}$. Consider the affine linear span $\mathrm{Span}(\mathcal{B})$. We check if $\mathrm{Span}(\mathcal{B}) \subseteq \mathcal{S}$. If not, then $\mathcal{S}$ is not affine and $f \notin \mathscr{A}$, so suppose that this is the case. If $\mathrm{Span}(\mathcal{B}) = \mathcal{S}$, then we are done. Lastly, if $\mathcal{S} - \mathrm{Span}(\mathcal{B}) \neq \emptyset$, then pick the next element $\mathbf{b}_{k+1} \in \mathcal{S} - \mathrm{Span}(\mathcal{B})$. Let $\mathcal{B}' = \mathcal{B} \cup \{\mathbf{b}_{k+1}\}$ and repeat with the new basis set $\mathcal{B}'$.

Now assume that $\mathcal{S}$ is an affine subspace, that we have a linear system defining it, and that every nonzero entry of f is a power of i. If $\mathcal{S}$ has dimension 0, then $\mathcal{S}$ is a single point, and $f \in \mathscr{A}$. Otherwise, $\dim(\mathcal{S}) = r \geq 1$, and (after reordering) $x_1, \ldots, x_r$ are free variables of the linear system defining $\mathcal{S}$. For each $\mathbf{x} \in \{0,1\}^r$, let $\mathbf{y} \in \{0,1\}^{n-r}$ be the unique extension such that $\mathbf{xy} \in \mathcal{S}$. Define $p_{\mathbf{x}} \in \mathbb{Z}_4$ such that $f_{\mathbf{xy}} = i^{p_{\mathbf{x}}} \neq 0$. We want to decide if there exists a quadratic polynomial

$$Q(\mathbf{x}) = \sum_{j=1}^{r} c_j x_j^2 + 2 \sum_{1 \leq k < \ell \leq r} c_{k\ell} x_k x_\ell + c,$$

where $c, c_j, c_{k\ell} \in \mathbb{Z}_4$, for $1 \leq j \leq r$ and $1 \leq k < \ell \leq r$, such that $Q(\mathbf{x}) \equiv p_{\mathbf{x}} \pmod 4$ for all $\mathbf{x} \in \{0,1\}^r$. Setting $\mathbf{x} = \mathbf{0} \in \{0,1\}^r$ determines c. Setting exactly one $x_j = 1$ and the rest to 0 determines c_j. Setting exactly two $x_k = x_\ell = 1$ and the rest to 0 determines $c_{k\ell}$. Then we verify if $Q(\mathbf{x})$ is consistent with f, and $f \in \mathscr{A}$ iff it is so. □

For later use, we note the following corollary.

Corollary 4.2. *There is an algorithm to decide whether a given signature f of arity n belongs to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$ with running time polynomial in N, the bit length of f.*

Proof. For $\mathrm{arity}(f) = n$, just check if $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right]^{\otimes n} f \in \mathscr{A}$ by Lemma 4.1. □

We can strengthen Lemma 2.7 by restricting to orthogonal transformations within $\mathbf{SO}_2(\mathbb{C})$.

Lemma 4.3. *Let $\mathcal{F}$ be a set of signatures. Then $\mathcal{F}$ is $\mathscr{A}$-transformable iff there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{A}$ or $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$.*

Proof. Sufficiency is obvious by Lemma 2.7.

Assume that $\mathcal{F}$ is $\mathscr{A}$-transformable. By Lemma 2.7, there exists an $H \in \mathbf{O}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{A}$ or $\mathcal{F} \subseteq H\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$. If $H \in \mathbf{SO}_2(\mathbb{C})$, we are done, so assume that $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. We want to find an $H' \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H'\mathscr{A}$ or $\mathcal{F} \subseteq H'\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$. Let $H' = H\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right] \in \mathbf{SO}_2(\mathbb{C})$. There are two cases to consider.

1. Suppose $\mathcal{F} \subseteq H\mathscr{A}$. Then since $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{A})$,
$$\begin{aligned}\mathcal{F} &\subseteq H\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]\mathscr{A}\\ &= H'\mathscr{A}.\end{aligned}$$
2. Suppose $\mathcal{F} \subseteq H\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$. Then since $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{A})$, which commutes with $\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]$,
$$\begin{aligned}\mathcal{F} &\subseteq H\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]\mathscr{A}\\ &= H\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}\\ &= H'\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}.\end{aligned}$$

□

We now observe some properties of a signature under transformations in $\mathbf{SO}_2(\mathbb{C})$. Let f be a signature and $H = \left[\begin{smallmatrix}a & b\\ -b & a\end{smallmatrix}\right] \in \mathbf{SO}_2(\mathbb{C})$ where $a^2 + b^2 = 1$. Notice that $v_0 = (1, i)$ and $v_1 = (1, -i)$ are row eigenvectors of H with eigenvalues $a - bi$ and $a + bi$ respectively. Let $Z' = \left[\begin{smallmatrix}1 & i\\ 1 & -i\end{smallmatrix}\right]$. Then $Z'H = TZ'$, where $T = \left[\begin{smallmatrix}a-bi & 0\\ 0 & a+bi\end{smallmatrix}\right]$.

For a vector $\mathbf{u} = (u_1, \ldots, u_n) \in \{0, 1\}^n$ of length n, let

$$v_{\mathbf{u}} = v_{u_1} \otimes v_{u_2} \otimes \ldots \otimes v_{u_n},$$

and let $w(\mathbf{u})$ be the Hamming weight of $\mathbf{u}$. Then $v_{\mathbf{u}}$ is a row eigenvector of the 2^n-by-2^n matrix $H^{\otimes n}$ with eigenvalue $(a-bi)^{n-w(\mathbf{u})}(a+bi)^{w(\mathbf{u})} = (a-bi)^{n-2w(\mathbf{u})} = (a+bi)^{2w(\mathbf{u})-n}$ since $(a+bi)(a-bi) = a^2 + b^2 = 1$. In this paper, the following Z'-transformation plays an important role. For any function f on $\{0, 1\}^n$, we define

$$\hat{f} = Z'^{\otimes n} f.$$

Then $\hat{f}_{\mathbf{u}} = \langle v_{\mathbf{u}}, f\rangle$, as a dot product.

Lemma 4.4. *Suppose f and g are signatures of arity n and let $H = \left[\begin{smallmatrix}a & b\\ -b & a\end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix}a-bi & 0\\ 0 & a+bi\end{smallmatrix}\right]$. Then $g = H^{\otimes n} f$ iff $\hat{g} = T^{\otimes n}\hat{f}$.*

Proof. Since $Z'H = TZ'$,

$$\begin{aligned}g = H^{\otimes n} f &\iff Z'^{\otimes n} g = Z'^{\otimes n} H^{\otimes n} f\\ &\iff Z'^{\otimes n} g = T^{\otimes n} Z'^{\otimes n} f\\ &\iff \hat{g} = T^{\otimes n}\hat{f}.\end{aligned}$$

□

We note that $v_{\mathbf{u}}^{\mathtt{T}}$ is also a column eigenvector of $H^{\otimes n}$ with eigenvalue $(a - bi)^{2w(\mathbf{u})-n}$. Now we characterize the signatures that are invariant under transformations in $\mathbf{SO}_2(\mathbb{C})$.

Lemma 4.5. *Let f be a signature. Then f is invariant under transformations in $\mathbf{SO}_2(\mathbb{C})$ (up to a nonzero constant) iff the support of $\hat{f}$ contains at most one Hamming weight.*

Proof. This clearly holds when f is identically zero, so assume that f contains a nonzero entry and has arity n. Such an f is invariant under any H (up to a nonzero constant) iff f is a column eigenvector of $H^{\otimes n}$. Take $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n}$ has $n+1$ distinct eigenvalues $(a-bi)^{n-w}(a+bi)^w$, for $0 \leq w \leq n$. Then f is a column eigenvector of $H^{\otimes n}$ iff it is a nonzero linear combination of $v_{\mathbf{u}}^{\mathsf{T}}$ of the same Hamming weight $w(\mathbf{u})$. Hence f is invariant under H iff the support of $\hat{f}$ contains at most one Hamming weight. □

Using Lemma 4.5, we can efficiently decide if there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \mathscr{A}$.

Lemma 4.6. *There is an algorithm to decide, for any input signature f of arity n, whether there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \mathscr{A}$ with running time polynomial in N. If so, either $f \in \mathscr{A}$ and f is invariant under any transformation in $\mathbf{SO}_2(\mathbb{C})$, or there exist at most $8n$ many $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \mathscr{A}$, and they can all be computed in time polynomial in N.*

Proof. Compute $\hat{f} = Z'^{\otimes n} f$. If the support of $\hat{f}$ contains at most one Hamming weight, then by Lemma 4.5, f is invariant under any $H \in \mathbf{SO}_2(\mathbb{C})$. Therefore we only need to directly decide if $f \in \mathscr{A}$, which we do by Lemma 4.1.

Now assume there are at least two nonzero entries of $\hat{f}$ that are of distinct Hamming weight. Let $\mathbf{u}_1, \mathbf{u}_2 \in \{0,1\}^n$ be such that $\hat{f}_{\mathbf{u}_1}$ and $\hat{f}_{\mathbf{u}_2}$ are nonzero, and $0 < w(\mathbf{u}_2) - w(\mathbf{u}_1) \leq n$. Suppose there exists an $H = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \in \mathbf{SO}_2(\mathbb{C})$ such that $g = H^{\otimes n} f \in \mathscr{A}$. Then by Lemma 4.4, we have $\hat{g} = T^{\otimes n} \hat{f}$, where $T = \left[\begin{smallmatrix} a-bi & 0 \\ 0 & a+bi \end{smallmatrix}\right]$ is a diagonal transformation. Since $Z' = \sqrt{2} H_2 D \in \mathrm{Stab}(\mathscr{A})$, we have $\hat{g} = Z'^{\otimes n} g \in \mathscr{A}$. Also since T is diagonal, both $\hat{g}_{\mathbf{u_1}}$ and $\hat{g}_{\mathbf{u_2}}$ are nonzero. Therefore, there must exist an $r \in \{0,1,2,3\}$ such that

$$i^r = \frac{\hat{g}_{\mathbf{u_2}}}{\hat{g}_{\mathbf{u_1}}} = \frac{(a+bi)^{2w(\mathbf{u_2})-n} \hat{f}_{\mathbf{u_2}}}{(a+bi)^{2w(\mathbf{u_1})-n} \hat{f}_{\mathbf{u_1}}} = (a+bi)^{2w(\mathbf{u}_2)-2w(\mathbf{u}_1)} \frac{\hat{f}_{\mathbf{u_2}}}{\hat{f}_{\mathbf{u_1}}}. \tag{1}$$

Recall that $0 < w(\mathbf{u}_2) - w(\mathbf{u}_1) \leq n$. Then given $\hat{f}_{\mathbf{u_1}}$ and $\hat{f}_{\mathbf{u_2}}$, there are at most $8n$ many solutions for $a, b \in \mathbb{C}$ such that $a + bi$ satisfies (1) (with 4 possible values of r) and $a^2 + b^2 = 1$. Each (a, b) solution corresponds to a distinct $H \in \mathbf{SO}_2(\mathbb{C})$. □

We also want to efficiently decide if there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$.

Lemma 4.7. *There is an algorithm to decide, for any input signature f of arity n, whether there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$ with running time polynomial in N. If so, either $f \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$ and f is invariant under any transformation in $\mathbf{SO}_2(\mathbb{C})$, or there exist $O(nN^{16})$ many $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$, and they can all be computed in polynomial time in N.*

Proof. Compute $\hat{f} = Z'^{\otimes n} f$. If the support of $\hat{f}$ contains at most one Hamming weight, then by Lemma 4.5, f is invariant under any $H \in \mathbf{SO}_2(\mathbb{C})$. Therefore we only need to directly decide if $f \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$, which we do by Corollary 4.2.

Now assume there are at least two nonzero entries of $\hat{f}$ that are of distinct Hamming weight. Let $\mathbf{u}_1, \mathbf{u}_2 \in \{0,1\}^n$ be such that $\hat{f}_{\mathbf{u}_1}$ and $\hat{f}_{\mathbf{u}_2}$ are nonzero, and $0 < w(\mathbf{u}_2) - w(\mathbf{u}_1) \leq n$. We derive necessary conditions for the existence of $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$. Thus, assume such an $H = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$ exists, where $a^2 + b^2 = 1$.

Let $g = H^{\otimes n} f$. Then $\hat{g} = Z'^{\otimes n} g \in \left[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]\mathscr{A}$. By Lemma 4.4, we have $\hat{g} = T^{\otimes n}\hat{f}$, where $T = \left[\begin{smallmatrix} a-bi & 0 \\ 0 & a+bi \end{smallmatrix}\right]$. Thus $\hat{g}_{\mathbf{u}} = (a+bi)^{2w(\mathbf{u})-n}\hat{f}_{\mathbf{u}}$ for any $\mathbf{u} \in \{0,1\}^n$. Let $t = w(\mathbf{u}_1) - w(\mathbf{u}_2)$. Then

$$\frac{\hat{g}_{\mathbf{u}_1}}{\hat{g}_{\mathbf{u}_2}} = \frac{(a+bi)^{2w(\mathbf{u}_1)-n}\hat{f}_{\mathbf{u}_1}}{(a+bi)^{2w(\mathbf{u}_2)-n}\hat{f}_{\mathbf{u}_2}} = (a+bi)^{2t}\frac{\hat{f}_{\mathbf{u}_1}}{\hat{f}_{\mathbf{u}_2}}.$$

Hence

$$(a+bi)^{2t} = \frac{\hat{f}_{\mathbf{u}_2}}{\hat{f}_{\mathbf{u}_1}} \cdot \frac{\hat{g}_{\mathbf{u}_1}}{\hat{g}_{\mathbf{u}_2}}.$$

Notice that $\hat{g} \in \left[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]\mathscr{A}$. We claim that the value of each entry in $\hat{g}$ as well as the number of possible values is bounded by a polynomial in N, and hence so are the ratios between them. Recall that $\hat{g} = \left[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right]^{\otimes n}\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]^{\otimes n} h$, for some $h \in \mathscr{A}$. Then every nonzero entry of h is a power of i, up to a constant factor λ. This constant factor cancels when taking ratios of entries, so we omit it. Let $h' = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]^{\otimes n} h$. Then every entry of h' is a power of α or 0. Moreover, each entry of $\left[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right]^{\otimes n}$ is also a power of α. Therefore every entry of $\hat{g}$ is an exponential sum of 2^n terms, each a power of α or 0. Let c_0 denote the number of 0 and c_i (for $1 \le i \le 8$) denote the number of α^i in an entry $\hat{g}_{\mathbf{u}}$ of $\hat{g}$. Then we have

$$c_0 + \sum_{i=1}^{8} c_i = 2^n \qquad \text{and} \qquad \sum_{i=1}^{8} c_i \alpha^i = \hat{g}_{\mathbf{u}}.$$

Clearly the total number of possible values of entries in $\hat{g}_{\mathbf{u}}$ is at most the number of possible choices of $(c_0, \ldots, c_8)$. There are at most $\binom{2^n+8}{8} = O(N^8)$ many choices of $(c_0, \ldots, c_8)$. Thus the number of all possible ratios is at most $O(N^{16})$, and can all be enumerated in time polynomial in N.

For any possible value of the ratio $\frac{\hat{g}_{\mathbf{u}_1}}{\hat{g}_{\mathbf{u}_2}}$, each possible value of $\frac{\hat{f}_{\mathbf{u}_2}}{\hat{f}_{\mathbf{u}_1}}$ gives at most $2n$ many different transformations H. Therefore, the total number of transformations is bounded by $O(nN^{16})$, and we can find them in time polynomial in N. □

Now we give an algorithm that efficiently decides if a set of signatures is $\mathscr{A}$-transformable.

Theorem 4.8. *There is a polynomial time algorithm to decide, for any finite set of signatures $\mathcal{F}$, whether $\mathcal{F}$ is $\mathscr{A}$-transformable. If so, at least one transformation can be found.*

Proof. By Lemma 4.3, we only need to decide if there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{A}$ or $\mathcal{F} \subseteq H\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]\mathscr{A}$. To every signature in $\mathcal{F}$, we apply Lemma 4.6 or Lemma 4.7 to check each case, respectively. If no H exists for some signature, then $\mathcal{F}$ is not $\mathscr{A}$-transformable. Otherwise, every signature is $\mathscr{A}$-transformable for some $H \in \mathbf{SO}_2(\mathbb{C})$. If every signature in $\mathcal{F}$ is invariant under transformations in $\mathbf{SO}_2(\mathbb{C})$, then $\mathcal{F}$ is $\mathscr{A}$-transformable. Otherwise, there exists an $f \in \mathcal{F}$ that is not invariant under transformations in $\mathbf{SO}_2(\mathbb{C})$. The number of possible transformations that work for f is bounded by a polynomial in the size of the presentation of f. We simply try all such transformations on all other signatures in $\mathcal{F}$ that are not invariant under transformations in $\mathbf{SO}_2(\mathbb{C})$, respectively using Lemma 4.1 or Corollary 4.2 to check if the transformation works. □

5 General $\mathscr{P}$-transformable Signatures

We begin with the counterpart to Lemma 4.3, which strengthens Lemma 2.10 by restricting to either orthogonal transformations within $\mathbf{SO}_2(\mathbb{C})$ or no orthogonal transformation at all.

Lemma 5.1. *Let $\mathcal{F}$ be a set of signatures. Then $\mathcal{F}$ is $\mathscr{P}$-transformable iff $\mathcal{F} \subseteq \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P}$ or there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{P}$.*

Proof. Sufficiency is obvious by Lemma 2.10.

Assume that $\mathcal{F}$ is $\mathscr{P}$-transformable. By Lemma 2.10, there exists an $H \in \mathbf{O}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{P}$ or $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P}$. There are two cases to consider.

1. Suppose $\mathcal{F} \subseteq H\mathscr{P}$. If $H \in \mathbf{SO}_2(\mathbb{C})$, then we are done, so assume that $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. We want to find an $H' \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H'\mathscr{P}$. Let $H' = H \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \mathbf{SO}_2(\mathbb{C})$. Then
$$\begin{aligned} \mathcal{F} &\subseteq H \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \mathscr{P} \\ &= H'\mathscr{P} \end{aligned}$$
since $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{P})$.
2. Suppose $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P}$. If $H = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \in \mathbf{SO}_2(\mathbb{C})$, then
$$\begin{aligned} \mathcal{F} &\subseteq H \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P} \\ &\subseteq \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \left[\begin{smallmatrix} a+bi & 0 \\ 0 & a-bi \end{smallmatrix}\right] \mathscr{P} \\ &\subseteq \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P} \end{aligned}$$
since $H \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \left[\begin{smallmatrix} a+bi & 0 \\ 0 & a-bi \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a+bi & 0 \\ 0 & a-bi \end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{P})$. Otherwise, $H = \left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right] \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ and
$$\begin{aligned} \mathcal{F} &\subseteq H \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P} \\ &\subseteq \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & a-bi \\ a+bi & 0 \end{smallmatrix}\right] \mathscr{P} \\ &\subseteq \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P} \end{aligned}$$
since $H \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & a-bi \\ a+bi & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & a-bi \\ a+bi & 0 \end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{P})$. □

The "building blocks" of $\mathscr{P}$ are signatures whose support is contained in two entries with complement indices. Recall that two signatures are considered the same if one is a nonzero constant multiple of the other.

Definition 5.2. *A k-ary function f is a* generalized equality *if it is $[0,0], [1,0]$, $[0,1]$, or satisfies*
$$\exists \mathbf{x} \in \{0,1\}^k, \quad \forall \mathbf{y} \in \{0,1\}^k, \quad f_{\mathbf{y}} = 0 \iff \mathbf{y} \notin \{\mathbf{x}, \overline{\mathbf{x}}\}.$$
We use $\mathscr{E}$ to denote the set of all generalized equality functions.

For any set $\mathcal{F}$, we let $\langle \mathcal{F} \rangle$ denote the closure under function products without shared variables. If we view signatures as tensors, then $\langle \cdot \rangle$ is the closure under tensor products. That is, if $f(\mathbf{x}_1, \mathbf{x}_2) = f_1(\mathbf{x}_1) f_2(\mathbf{x}_2)$, then $f = f_1 \otimes f_2$ with a correct ordering of indices. By definition, one can show that $\mathscr{P} = \langle \mathscr{E} \rangle$.

Definition 5.3. *We call a function f of arity n on variable set $\mathbf{x}$* reducible *if there exist f_1 and f_2 of arities n_1 and n_2 on variable sets $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, such that $1 \leq n_1, n_2 \leq n-1$, $\mathbf{x}_1 \cup \mathbf{x}_2 = \mathbf{x}$, $\mathbf{x}_1 \cap \mathbf{x}_2 = \emptyset$, and $f(\mathbf{x}) = f_1(\mathbf{x}_1) f_2(\mathbf{x}_2)$. Otherwise we call f* irreducible.

Note that all unary functions, including $[0, 0]$, are irreducible. However, the identically zero function of arity greater than one is reducible. Definition 5.2 is a slight modification of a similar definition for $\mathscr{E}$ that appeared in Section 2 of [21]. For both definitions of $\mathscr{E}$, it follows that $\mathscr{P} = \langle \mathscr{E} \rangle$. The motivation for our slight change in the definition is so that every signature in $\mathscr{E}$ is irreducible.

If a function f is reducible, then we can factor it into functions of smaller arity. This procedure can be applied recursively and terminates when all components are irreducible. Therefore any function has at least one irreducible factorization. We show that such a factorization is unique for functions that are not identically zero.

Lemma 5.4. *Let f be a function of arity n on variables $\mathbf{x}$ that is not identically zero. Assume there exist irreducible functions f_i and g_j, and two partitions $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_j\}$ of $\mathbf{x}$ for $1 \leq i \leq k$ and $1 \leq j \leq k'$, such that*

$$f(\mathbf{x}) = \prod_{i=1}^{k} f_i(\mathbf{x}_i) = \prod_{j=1}^{k'} g_j(\mathbf{y}_j).$$

Then $k = k'$, the partitions are the same, and $\{f_i\}$ and $\{g_j\}$ are the same up to a permutation.

Proof. Since f is not identically zero, none of f_i or g_j is identically zero. Fix an assignment $\mathbf{u}_2, \ldots, \mathbf{u}_k$ such that $c = \prod_{i=2}^{k} f_i(\mathbf{u}_i) \neq 0$. Let $\mathbf{z}_j = \mathbf{y}_j \cap \mathbf{x}_1$, and $\mathbf{v}_j = \mathbf{y}_j \cap (\cup_{i=2}^{k} \mathbf{x}_i)$ for $1 \leq j \leq k'$. Let the assignments $\mathbf{u}_2, \ldots, \mathbf{u}_k$ restricted to $\mathbf{v}_j$ be $\mathbf{w}_j$. Then we have

$$cf_1(\mathbf{x}_1) = f_1(\mathbf{x}_1) \prod_{i=2}^{k} f_i(\mathbf{u}_i) = \prod_{j=1}^{k'} g_j(\mathbf{z}_j, \mathbf{w}_j).$$

Define new functions $h_j(\mathbf{z}_j) = g_j(\mathbf{z}_j, \mathbf{w}_j)$ for $1 \leq j \leq k'$. Then

$$f_1(\mathbf{x}_1) = \frac{1}{c} \prod_{j=1}^{k'} h_j(\mathbf{z}_j).$$

Since f_1 is irreducible, there cannot be two $\mathbf{z}_j$ that are nonempty. And yet, $\mathbf{x}_1 = \cup_{j=1}^{k'} \mathbf{z}_j$, so it follows that $\mathbf{x}_1 = \mathbf{z}_j$ for some $1 \leq j \leq k'$. We may assume $j = 1$, so $\mathbf{x}_1 \subseteq \mathbf{y}_1$. By the same argument we have $\mathbf{y}_1 \subseteq \mathbf{x}_i$, for some i. But by disjointness of $\mathbf{x} = \cup_{i=1}^{k} \mathbf{x}_i$, we must have $\mathbf{y}_1 \subseteq \mathbf{x}_1$. Thus after a permutation, we have $\mathbf{x}_1 = \mathbf{y}_1$. Therefore $f_1 = g_1$ up to a nonzero constant.

By fixing some assignment to $\mathbf{x}_1 = \mathbf{y}_1$ such that f_1 and g_1 are not zero, we may cancel this factor, and the proof is completed by induction. Therefore we must have $k = k'$ and $\{f_i\}$ and $\{g_j\}$ are the same up to a permutation (and nonzero factors). □

In fact, we can efficiently find the unique factorization.

Lemma 5.5. *There is an algorithm to compute, for any input signature f of arity n that is not identically zero, the unique factorization of f into irreducible factors with runtime polynomial in N. More specifically, the algorithm computes irreducibles $f_1, \ldots, f_k$ of arities $n_1, \ldots, n_k \in \mathbb{Z}^+$ (for some $k \geq 1$) such that $\sum_{i=1}^{k} n_i = n$ and $f(\mathbf{x_1}, \ldots, \mathbf{x_k}) = \prod_{i=1}^{k} f_i(\mathbf{x_i})$.*

Proof. For the variable set $\mathbf{x}$ of length n, we may partition it into two sets $\mathbf{x}_1$ and $\mathbf{x}_2$ of length n_1 and n_2, respectively, such that $1 \leq n_1, n_2 \leq n-1$, $\mathbf{x}_1 \cup \mathbf{x}_2 = \mathbf{x}$, and $\mathbf{x}_1 \cap \mathbf{x}_2 = \emptyset$. Define a 2^{n_1}-by-2^{n_2} matrix M such that $M_{\mathbf{u}_1,\mathbf{u}_2} = f(\mathbf{u}_1, \mathbf{u}_2)$ for $\mathbf{u}_1 \in \{0,1\}^{n_1}$ and $\mathbf{u}_2 \in \{0,1\}^{n_2}$. Then M is of rank at most 1 iff there exist f_1 and f_2 of arity n_1 and n_2, such that $f(\mathbf{x}) = f_1(\mathbf{x}_1)f_2(\mathbf{x}_2)$.

Therefore, in order to factor f, we only need to run through all distinct partitions, and check if there exists at least one such matrix of rank at most 1. If none exists, then f is irreducible. The total number of possible such partitions is $2^{n-1}-1$. Hence the running time is polynomial in $2^n \leq N$.

Once we have found $f = f_1 \otimes f_2$, we recursively apply the above procedure to f_1 and f_2 until every component is irreducible. The total running time is polynomial in N. □

This factorization algorithm gives a simple algorithm to determine membership in $\mathscr{P}$.

Lemma 5.6. *There is an algorithm to decide, for a given signature f of arity n, whether $f \in \mathscr{P}$ with running time polynomial in N.*

Proof. If f is identically zero, then $f \in \mathscr{P}$. Otherwise, f is not identically zero and we obtain its unique factorization $f = \bigotimes_i f_i$ by Lemma 5.5. Then $f \in \mathscr{P}$ iff for all i, we have $f_i \in \mathscr{E}$. Since membership in $\mathscr{E}$ is easy to check, our proof is complete. □

Let $T \in \mathbf{GL}_2(\mathbb{C})$ be some transformation and f some signature. To check if $f \in T\mathscr{P}$, it suffices to first factor f and then check if each irreducible factor is in $T\mathscr{E}$.

Lemma 5.7. *Suppose $f = \bigotimes_{i=1}^k f_i$ is not identically zero and that f_i is irreducible for all $1 \leq i \leq k$. Let $T \in \mathbf{GL}_2(\mathbb{C})$. Then $f \in T\mathscr{P}$ iff $f_i \in T\mathscr{E}$ for all $1 \leq i \leq k$.*

Proof. Suppose f is of arity n and f_i is of arity n_i so that $\sum_{i=1}^k n_i = n$. If $f_i \in T\mathscr{E}$ for all $1 \leq i \leq k$, then there exists $g_i \in \mathscr{E}$ such that $f_i = T^{\otimes n_i} g_i$. Thus $f = \bigotimes_{i=1}^k f_i = \bigotimes_{i=1}^k T^{\otimes n_i} g_i = T^{\otimes n} \bigotimes_{i=1}^k g_i$. Since $g_i \in \mathscr{E}$, we have $\bigotimes_{i=1}^k g_i \in \mathscr{P}$. Therefore $f \in T\mathscr{P}$.

On the other hand, assume $f \in T\mathscr{P}$. By the definition of $\mathscr{P}$, there exist $g_1, \ldots, g_{k'} \in \mathscr{E}$ of arities $m_1, \ldots, m_{k'} \in \mathbb{Z}^+$, such that $f = T^{\otimes n} g$, where $g = \bigotimes_{i=1}^{k'} g_i$. It is easy to verify that each $g_i \in \mathscr{E}$ is irreducible. Let $f_i' = T^{\otimes m_i} g_i \in T\mathscr{E}$ for all $1 \leq i \leq k'$, which are also irreducible. Then $\bigotimes_{i=1}^{k'} f_i' = f = \bigotimes_{i=1}^k f_i$. By Lemma 5.4, we have $k = k'$ and $\{f_i\}$ and $\{f_i'\}$ are the same up to a permutation. Therefore each $f_i \in T\mathscr{E}$. □

With Lemmas 5.5 and 5.7 in mind, we focus our attention on membership in $\mathscr{E}$. We show how to efficiently decide if there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \mathscr{E}$ when f is irreducible.

Lemma 5.8. *There is an algorithm to decide, for a given irreducible signature f of arity $n \geq 2$, whether there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \mathscr{E}$ with running time polynomial in N. If so, there exist at most eight $H \in \mathbf{SO}_2(\mathbb{C})$ such that $H^{\otimes n} f \in \mathscr{E}$ unless $f = (1,0,0,1)^{\mathtt{T}}$ or $f = (0,1,-1,0)^{\mathtt{T}}$.*

Proof. Assume there exists an $H = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \in \mathbf{SO}_2(\mathbb{C})$ such that $g = H^{\otimes n} f \in \mathscr{E}$, where $a^2 + b^2 = 1$. Then by Lemma 4.4, there exists a diagonal transformation $T = \left[\begin{smallmatrix} a-bi & 0 \\ 0 & a+bi \end{smallmatrix}\right]$ such that $\hat{g} = T^{\otimes n} \hat{f} \in \left[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right] \mathscr{E}$. In particular, $\hat{g}$ and $\hat{f}$ have the same support. For two vectors $\mathbf{u}, \mathbf{x} \in \{0,1\}^n$, the entry indexed by row $\mathbf{u}$ and column $\mathbf{x}$ in the matrix $\left[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right]^{\otimes n}$ is $i^{w(\mathbf{x})}(-1)^{\langle \mathbf{x}, \mathbf{u} \rangle}$, where $w(\cdot)$ denotes Hamming weight and $\langle \cdot, \cdot \rangle$ is the dot product.

Since f is irreducible, g is irreducible as well. Thus g has two nonzero entries with opposite index, say $\mathbf{x}$ and $\overline{\mathbf{x}}$. Hence we have

$$\begin{aligned}\hat{g}_{\mathbf{u}} &= i^{w(\mathbf{x})}(-1)^{\langle \mathbf{x},\mathbf{u}\rangle} g_{\mathbf{x}} + i^{w(\overline{\mathbf{x}})}(-1)^{\langle \overline{\mathbf{x}},\mathbf{u}\rangle} g_{\overline{\mathbf{x}}} \\ &= i^{w(\mathbf{x})}(-1)^{\langle \mathbf{x},\mathbf{u}\rangle} g_{\mathbf{x}} + i^{n-w(\mathbf{x})}(-1)^{w(\mathbf{u})-\langle \mathbf{x},\mathbf{u}\rangle} g_{\overline{\mathbf{x}}} \\ &= (-1)^{\langle \mathbf{x},\mathbf{u}\rangle}\left(i^{w(\mathbf{x})} g_{\mathbf{x}} + i^{n-w(\mathbf{x})}(-1)^{w(\mathbf{u})} g_{\overline{\mathbf{x}}}\right)\end{aligned}$$

for any vector $\mathbf{u} \in \{0,1\}^n$.

For $\mathbf{u}_1, \mathbf{u}_2 \in \{0,1\}^n$, if $w(\mathbf{u}_1) \equiv w(\mathbf{u}_2) \pmod 2$, then

$$\hat{g}_{\mathbf{u}_1} = \pm \hat{g}_{\mathbf{u}_2}. \tag{2}$$

Therefore, if any entry of $\hat{f}$ with even Hamming weight is 0, then all entries with even Hamming weight are 0. This also holds for entries with odd Hamming weight. However, $\hat{f}$ is not identically zero because it is irreducible and of arity $n \geq 2$. Therefore, we know that either all entries of even Hamming weight are not 0 or all entries of odd Hamming weight are not 0. If $n \geq 3$, or if $n = 2$ and all entries of even Hamming weight are not 0, then we can take two nonzero entries of $\hat{f}$ whose Hamming weight differ by 2. Their ratio restricts the possible choices of $a + bi$, as in the proof of Lemma 4.7, because the only possible ratios for $\hat{g}_{\mathbf{u}_1}/\hat{g}_{\mathbf{u}_2}$ are ± 1 by (2). Together with $a^2 + b^2 = 1$, this gives at most 8 possible matrices $H \in \mathbf{SO}_2(\mathbb{C})$.

The remaining case is when $n = 2$ and all entires of $\hat{f}$ with even Hamming weight are 0. By (2), we have $\hat{g} = \lambda(0, 1, \pm 1, 0)^{\mathtt{T}}$ for some $\lambda \neq 0$ since $\hat{g}$ and $\hat{f}$ have the same support. Then from $\hat{f} = (T^{-1})^{\otimes 2}\hat{g}$, where $T^{-1} = \left[\begin{smallmatrix} a+bi & 0 \\ 0 & a-bi \end{smallmatrix}\right]$ is diagonal, we calculate that $T^{-1}\left[\begin{smallmatrix} 0 & 1 \\ \pm 1 & 0 \end{smallmatrix}\right](T^{-1})^{\mathtt{T}} = \left[\begin{smallmatrix} 0 & 1 \\ \pm 1 & 0 \end{smallmatrix}\right]$. Hence, up to a nonzero scalar, $\hat{f} = (0, 1, 1, 0)^{\mathtt{T}}$ or $\hat{f} = (0, 1, -1, 0)^{\mathtt{T}}$. Finally $f = (Z'^{-1})^{\otimes 2}\hat{f}$, and we get $f = (1, 0, 0, 1)^{\mathtt{T}}$ or $f = (0, 1, -1, 0)^{\mathtt{T}}$, up to a nonzero scalar. □

Now we give an algorithm that efficiently decides if a set of signatures is $\mathscr{P}$-transformable.

Theorem 5.9. *There is a polynomial time algorithm to decide, for any finite set of signatures $\mathcal{F}$, whether $\mathcal{F}$ is $\mathscr{P}$-transformable. If so, at least one transformation can be found.*

Proof. By Lemma 5.1, we only need to decide if $\mathcal{F} \subseteq \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]\mathscr{P}$ or if there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{P}$. To check if $\mathcal{F} \subseteq \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]\mathscr{P}$, we simply apply Lemma 5.6 to each signature in $\left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]^{-1}\mathcal{F}$.

Now to check if $\mathcal{F} \subseteq H\mathscr{P}$. Any signature in $\mathcal{F}$ that is identically zero is in $T\mathscr{P}$ for any $T \in \mathbf{GL}_2(\mathbb{C})$. Thus, assume that no signature in $\mathcal{F}$ is identically zero. Now we obtain the unique factorization of each signature in $\mathcal{F}$ using Lemma 5.5. If every irreducible factor is either a unary signature, or $(1, 0, 0, 1)^{\mathtt{T}}$, or $(0, 1, -1, 0)^{\mathtt{T}}$, then $\mathcal{F} \subseteq \langle \mathscr{E} \rangle = \mathscr{P}$. Otherwise, let $f \in \mathcal{F}$ be a signature that is not of this form. This means that f has a unique factorization $f = \bigotimes_i f_i$ where some f_i, say f_1, is not unary signature, or $(1, 0, 0, 1)^{\mathtt{T}}$, or $(0, 1, -1, 0)^{\mathtt{T}}$.

By applying Lemma 5.7 to f, we get the necessary condition $f_1 \in H\mathscr{E}$. Then we apply Lemma 5.8 to f_1. If the test passes, then by the definition of f_1, we have at most eight transformations in $\mathbf{SO}_2(\mathbb{C})$ that could work. For each possible transformation H, we apply Lemma 5.6 to every signature in $H^{-1}\mathcal{F}$ to check if it works. □

6 Symmetric $\mathscr{A}$-transformable Signatures

In the next two sections, we consider the case when the signatures are symmetric. The significant difference is that a symmetric signature of arity n is given by $n+1$ values, instead of 2^n values. This exponentially more succinct representation requires us to find a more efficient algorithm.

6.1 A Single Signature

To begin, we provide efficient algorithms to decide membership in each of $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$ for a single signature. If the signature is in one of the sets, then the algorithm also finds at least one corresponding orthogonal transformation satisfying Definition 2.8. By Lemma 2.9, this is enough to check if a single signature is $\mathscr{A}$-transformable.

We say a signature f satisfies a second order recurrence relation, if for all $0 \le k \le n-2$, there exist $a, b, c \in \mathbb{C}$ not all zero, such that $af_k + bf_{k+1} + cf_{k+2} = 0$. For a non-degenerate signature of arity at least 3, these coefficients are unique up to a nonzero scalar.

Lemma 6.1. *Let f be a non-degenerate symmetric signature of arity $n \ge 3$. If f satisfies a second order recurrence relation with coefficients $a, b, c \in \mathbb{C}$ and another one with coefficients $a', b', c' \in \mathbb{C}$, then there exists a nonzero $k \in \mathbb{C}$ such that $(a, b, c) = k(a', b', c')$.*

Proof. Since $f = [f_0, f_1, \ldots, f_n]$ is non-degenerate, the matrix $A = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-1} \\ f_1 & f_2 & \cdots & f_n \end{bmatrix}$ has rank 2. Let $B = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-2} \\ f_1 & f_2 & \cdots & f_{n-1} \\ f_2 & f_3 & \cdots & f_n \end{bmatrix}$. We claim that $\operatorname{rank}(B) \ge 2$, which implies that f satisfies at most one second order recurrence relation up to a nonzero scalar, as desired.

If $(f_1, \ldots, f_{n-1}) = \mathbf{0}$, then $f_0, f_n \neq 0$ since $\operatorname{rank}(A) = 2$, so $\operatorname{rank}(B) = 2$ as well. Otherwise, $(f_1, \ldots, f_{n-1}) \neq \mathbf{0}$. Consider the matrices $A_1 = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-2} \\ f_1 & f_2 & \cdots & f_{n-1} \end{bmatrix}$ and $A_2 = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} \\ f_2 & f_3 & \cdots & f_n \end{bmatrix}$, which are submatrices of both A and B. Both A_1 and A_2 have rank at least 1 since $(f_1, \ldots, f_{n-1}) \neq \mathbf{0}$. We show that either $\operatorname{rank} A_1 = 2$ or $\operatorname{rank}(A_2) = 2$, which implies that $\operatorname{rank}(B) \ge 2$.

For a contradiction, suppose $\operatorname{rank}(A_1) = \operatorname{rank}(A_2) = 1$. Then there exist $\lambda, \mu \in \mathbb{C}$ such that $(f_0, \ldots, f_{n-2}) = \lambda(f_1, \ldots, f_{n-1})$ and $(f_2, \ldots, f_n) = \mu(f_1, \ldots, f_{n-1})$. If $\lambda = 0$, then by $\operatorname{rank}(A) = 2$, we have $f_{n-1} \neq 0$, and $\operatorname{rank}(A_2) = 2$. Similarly if $\mu = 0$, then $\operatorname{rank}(A_1) = 2$. Otherwise $\lambda, \mu \neq 0$ and we get $f_i \neq 0$ for all $0 \le i \le n$, and $\lambda\mu = 1$. This implies that $\operatorname{rank}(A) = 1$, a contradiction. □

Under any holographic transformation, a signature retains both its second order recurrence relation and the condition $b^2 - 4ac \neq 0$.

Lemma 6.2. *Let f be a symmetric signature of arity n and $f' = T^{\otimes n} f$ for some $T \in \mathbf{GL}_2(\mathbb{C})$. Then f satisfies a second order recurrence relation iff f' does. Furthermore, the second order recurrence for f satisfies $b^2 - 4ac \neq 0$ iff the one for f' does.*

For a signature with a second order recurrence relation, the quantity $b^2 - 4ac$ is nonzero precisely when the signature can be expressed as the sum of two degenerate signatures that are linearly independent.

Lemma 6.3. *Let f be a non-degenerate symmetric signature of arity $n \ge 3$. Then f satisfies a second order recurrence relation with coefficients a, b, c satisfying $b^2 - 4ac \neq 0$ iff there exist a_0, b_0, a_1, b_1 (satisfying $a_0 b_1 \neq a_1 b_0$) such that $f = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}^{\otimes n} + \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}^{\otimes n}$.*

The following definition of the θ function is crucial.

Definition 6.4. *For a pair of linearly independent vectors $v_0 = \left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]$ and $v_1 = \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$, we define*

$$\theta(v_0, v_1) = \left(\frac{a_0 a_1 + b_0 b_1}{a_1 b_0 - a_0 b_1}\right)^2.$$

Furthermore, suppose that a signature f of arity $n \geq 3$ can be expressed as $f = v_0^{\otimes n} + v_1^{\otimes n}$, where v_0 and v_1 are linearly independent. Then we define $\theta(f) = \theta(v_0, v_1)$.

Intuitively, this formula is the square of the cotangent of the angle from v_0 to v_1. This notion of cotangent is properly extended to the complex domain. By insisting that v_0 and v_1 be linearly independent, we ensure that $\theta(v_0, v_1)$ is well-defined. The expression is squared so that $\theta(v_0, v_1) = \theta(v_1, v_0)$.

Let $f = v_0^{\otimes n} + v_1^{\otimes n}$ be a non-degenerate signature of arity $n \geq 3$. Since f is non-degenerate, v_0 and v_1 are linearly independent. The next proposition implies that this expression for f via v_0 and v_1 is unique up to a root of unity. Therefore, $\theta(f)$ from Definition 6.4 is well-defined.

Proposition 6.5 (Lemma 9.11 in [24])**.** *Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be four vectors and suppose that $\mathbf{c}, \mathbf{d}$ are linearly independent. If for some $n \geq 3$, we have $\mathbf{a}^{\otimes n} + \mathbf{b}^{\otimes n} = \mathbf{c}^{\otimes n} + \mathbf{d}^{\otimes n}$, then there exist ω_0 and ω_1 satisfying $\omega_0^n = \omega_1^n = 1$ such that either $\mathbf{a} = \omega_0 \mathbf{c}$ and $\mathbf{b} = \omega_1 \mathbf{d}$ or $\mathbf{a} = \omega_0 \mathbf{d}$ and $\mathbf{b} = \omega_1 \mathbf{c}$.*

Proposition 6.5 can be generalized to the following simple lemma.

Lemma 6.6. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be four vectors and suppose that $\mathbf{c}, \mathbf{d}$ are linearly independent. Furthermore, let x_0, x_1, y_0, y_1 be nonzero scalars. If for some $n \geq 3$, we have $x_0 \mathbf{a}^{\otimes n} + x_1 \mathbf{b}^{\otimes n} = y_0 \mathbf{c}^{\otimes n} + y_1 \mathbf{d}^{\otimes n}$, then there exist ω_0 and ω_1, such that either $\mathbf{a} = \omega_0 \mathbf{c}$, $\mathbf{b} = \omega_1 \mathbf{d}$, $x_0 \omega_0^n = y_0$, and $x_1 \omega_1^n = y_1$; or $\mathbf{a} = \omega_0 \mathbf{d}$, $\mathbf{b} = \omega_1 \mathbf{c}$, $x_0 \omega_0^n = y_1$, and $x_1 \omega_1^n = y_0$.*

It is easy to verify that θ is invariant under an orthogonal transformation.

Lemma 6.7. *For two linearly independent vectors v_0, $v_1 \in \mathbb{C}^2$ and $H \in \mathbf{O}_2(\mathbb{C})$, let $\widehat{v_0} = Hv_0$ and $\widehat{v_1} = Hv_1$. Then $\theta(v_0, v_1) = \theta(\widehat{v_0}, \widehat{v_1})$.*

Proof. Within the square in the definition of θ, the numerator is the dot product, which is invariant under any orthogonal transformation. Also, the denominator is the determinant, which is invariant under any orthogonal transformation up to a sign. □

Now we have some necessary conditions for membership in $\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$. Recall that $\mathscr{A}_1 \subseteq \mathscr{P}_1$.

Lemma 6.8. *Let f be a non-degenerate symmetric signature of arity at least 3. Then*

1. *$f \in \mathscr{P}_1 \implies \theta(f) = 0$,*
2. *$f \in \mathscr{A}_2 \implies \theta(f) = -1$, and*
3. *$f \in \mathscr{A}_3 \implies \theta(f) = -\frac{1}{2}$.*

Proof. The result clearly holds when f is in the canonical form of each set. This extends to the rest of each set by Lemma 6.7. □

These results imply the following corollary.

Corollary 6.9. *Let f be a non-degenerate symmetric signature f of arity $n \geq 3$. If f is $\mathscr{A}$-transformable, then f is of the form $v_0^{\otimes n} + v_1^{\otimes n}$, where v_0 and v_1 are linearly independent, and $\theta(v_0, v_1) \in \{0, -1, -\frac{1}{2}\}$.*

The condition given in Lemma 6.8 is not sufficient to determine if $f \in \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$. For example, if $f = v_0^{\otimes n} + v_1^{\otimes n}$ with $v_0 = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]$ and v_1 is not a multiple of $\left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]$, then $\theta(f) = -1$ but f is not in $\mathscr{A}_2$. However, this is essentially the only exceptional case. We achieve the full characterization with some extra conditions.

The next lemma gives an equivalent form for membership in $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$ using transformations in $\mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. Only having to consider transformation matrices in $\mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ is convenient since such matrices are their own inverse.

Lemma 6.10. *Suppose f is a non-degenerate symmetric signature of arity $n \geq 3$ and let $\mathscr{F} \in \{\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\}$. Then $f \in \mathscr{F}$ iff there exists an $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ such that $f \in \mathscr{F}$ with H.*

Proof. Sufficiency is trivial. For necessity, assume that $f \in \mathscr{F}$ with $H \in \mathbf{O}_2(\mathbb{C})$. If $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$, then we are done, so further assume that $H \in \mathbf{SO}_2(\mathbb{C})$. By the definition of $\mathscr{F}$,

$$f = cH^{\otimes n}\left(v_0^{\otimes n} + \beta v_1^{\otimes n}\right),$$

where $c \neq 0$ and v_0, v_1, and β depend on $\mathscr{F}$. Let $H' = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] H^{-1} \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ so that $H'^{\mathsf{T}} = H'^{-1} = H'$. Then

$$\begin{aligned} f &= (H'H')^{\otimes n} f \\ &= cH'^{\otimes n}(H'H)^{\otimes n}\left(v_0^{\otimes n} + \beta v_1^{\otimes n}\right) \\ &= cH'^{\otimes n}\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]^{\otimes n}\left(v_0^{\otimes n} + \beta v_1^{\otimes n}\right) \\ &= cH'^{\otimes n}\left(v_1^{\otimes n} + \beta v_0^{\otimes n}\right) \\ &= c\beta H'^{\otimes n}\left(v_0^{\otimes n} + \beta^{-1} v_1^{\otimes n}\right), \end{aligned}$$

where in the fourth step, we use the fact that $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] v_0 = v_1$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] v_1 = v_0$ for any $\mathscr{F} \in \{\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\}$. To finish, we rewrite β^{-1} in the form of β depending on $\mathscr{F}$.

1. If $\mathscr{F} = \mathscr{A}_1$, then $\beta = \alpha^{tn+2r}$ for some $t \in \{0, 1\}$ and $r \in \{0, 1, 2, 3\}$ and $\beta^{-1} = \alpha^{-tn-2r}$. Pick $r' \in \{0, 1, 2, 3\}$ such that $r' \equiv -tn - r \pmod 4$, so $\beta^{-1} = \alpha^{tn+2r'}$ as required.
2. If $\mathscr{F} = \mathscr{A}_2$, then $\beta = 1$, so $\beta^{-1} = 1 = \beta$ as required.
3. If $\mathscr{F} = \mathscr{A}_3$, then $\beta = i^r$ for some $r \in \{0, 1, 2, 3\}$, so $\beta^{-1} = i^{-r} = i^{4-r}$ as required. □

Before considering $\mathscr{A}_1$, we prove a technical lemma that is also applicable when considering $\mathscr{P}_1$.

Lemma 6.11. *Let $f = v_0^{\otimes n} + v_1^{\otimes n}$ be a symmetric signature of arity $n \geq 3$, where $v_0 = \left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]$ and $v_1 = \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$ are linearly independent. If $\theta(f) = 0$, then there exist an $H \in \mathbf{O}_2(\mathbb{C})$ and a nonzero $k \in \mathbb{C}$ satisfying $a_1 = kb_0$ and $b_1 = -ka_0$ such that*

$$H^{\otimes n} f = \lambda\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + k^n \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n}\right)$$

for some nonzero $\lambda \in \mathbb{C}$.

Proof. Since $\theta(f) = 0$, we have $a_0a_1 + b_0b_1 = 0$. By linear independence, we have $a_1b_0 \neq a_0b_1$. Thus, there exists a nonzero $k \in \mathbb{C}$ such that $a_1 = kb_0$ and $b_1 = -ka_0$. (Note that this is clearly true even if one of a_0 or b_0, but not both, is zero.) Let $c = a_0^2 + b_0^2$, which is nonzero since $a_1b_0 \neq a_0b_1$. Also, let $u_0 = \frac{v_0}{\sqrt{c}}$ and $u_1 = \frac{v_1}{k\sqrt{c}}$ so that the matrix $M = [u_0\ u_1]$ is orthogonal. Then the matrix $H = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] M^{-1}$ is also orthogonal. Under a transformation by H, we have

$$\begin{aligned} H^{\otimes n} f &= H^{\otimes n} \left(c^{\frac{n}{2}} u_0^{\otimes n} + k^n c^{\frac{n}{2}} u_1^{\otimes n} \right) \\ &= \lambda \left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + k^n \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n} \right), \end{aligned}$$

where $\lambda = (c/2)^{\frac{n}{2}} \neq 0$. □

Now we give the characterization of $\mathscr{A}_1$.

Lemma 6.12. *Let $f = v_0^{\otimes n} + v_1^{\otimes n}$ be a symmetric signature of arity $n \geq 3$, where $v_0 = \left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]$ and $v_1 = \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$ are linearly independent. Then $f \in \mathscr{A}_1$ iff $\theta(f) = 0$ and there exist an $r \in \{0, 1, 2, 3\}$ and $t \in \{0, 1\}$ such that $a_1^n = \alpha^{tn+2r} b_0^n \neq 0$ or $b_1^n = \alpha^{tn+2r} a_0^n \neq 0$.*

Proof. Suppose $f \in \mathscr{A}_1$. By Lemma 6.10, after a suitable normalization, there exists a transformation $H = \left[\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right] \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ such that

$$f = H^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n} \right),$$

where $\beta = \alpha^{tn+2r}$ for some $r \in \{0, 1, 2, 3\}$ and some $t \in \{0, 1\}$. Since $H \in \mathbf{O}_2(\mathbb{C})$, we have $x^2 + y^2 = 1$. By Lemma 6.8, $\theta(f) = 0$.

Now we have two expressions for f, which are

$$\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]^{\otimes n} + \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]^{\otimes n} = f = \left[\begin{smallmatrix} x+y \\ y-x \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} x-y \\ y+x \end{smallmatrix}\right]^{\otimes n}.$$

Since v_0 and v_1 are linearly independent, we know that a_0 and a_1 cannot both be 0. Suppose $a_0 \neq 0$. By Lemma 6.6, we have two cases.

1. Suppose $a_0 = \omega_0(x + y)$ and $b_1 = \omega_1(x + y)$ where $\omega_0^n = 1$ and $\omega_1^n = \beta$. Then we have $b_1^n = \beta(x + y)^n = \beta a_0^n \neq 0$. Since $\beta = \alpha^{tn+2r}$, we are done.
2. Suppose $a_0 = \omega_0(x - y)$ and $b_1 = \omega_1(y - x)$ where $\omega_0^n = \beta$ and $\omega_1^n = 1$. Then we have $a_0^n = \beta(x - y)^n = \alpha^{tn+2r}(-1)^n(y - x)^n = \alpha^{tn+2r+4n} b_1^n$, so $b_1^n = \alpha^{-tn-2r-4n} a_0^n \neq 0$. Pick $r' \in \{0, 1, 2, 3\}$ such that $r' \equiv -tn - r - 2n \pmod 4$. Then $\alpha^{-tn-2r-4n} = \alpha^{tn+2r'}$ is of the desired form.

Otherwise, $a_1 \neq 0$, in which case, similar reasoning shows that $a_1^n = \alpha^{tn+2r} b_0^n \neq 0$.

For sufficiency, we apply Lemma 6.11, which gives

$$H^{\otimes n} f = \lambda \left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + k^n \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n} \right)$$

for some $H \in \mathbf{O}_2(\mathbb{C})$, some nonzero $\lambda \in \mathbb{C}$, and some nonzero $k \in \mathbb{C}$ satisfying $a_1 = kb_0$ and $b_1 = -ka_0$. The ratio of these coefficients is k^n. We consider two cases.

1. Suppose $a_1^n = \alpha^{tn+2r} b_0^n \neq 0$. Then $k^n = \alpha^{tn+2r}$, so $f \in \mathscr{A}_1$.
2. Suppose $b_1^n = \alpha^{tn+2r} a_0^n \neq 0$. Then $k^n = (-1)^n \alpha^{tn+2r}$. Pick $r' \in \{0, 1, 2, 3\}$ such that $r' \equiv r + 2n \pmod 4$. Then $k^n = \alpha^{tn+2r'}$, so $f \in \mathscr{A}_1$. □

Now we give the characterization of $\mathscr{A}_3$.

Lemma 6.13. *Let $f = v_0^{\otimes n} + v_1^{\otimes n}$ be a symmetric signature of arity $n \geq 3$, where $v_0 = \left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]$ and $v_1 = \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$ are linearly independent. Then $f \in \mathscr{A}_3$ iff there exist an $\varepsilon \in \{1, -1\}$ and $r \in \{0, 1, 2, 3\}$ such that $a_1\left(\sqrt{2}a_0 + \varepsilon i b_0\right) = b_1\left(\varepsilon i a_0 - \sqrt{2}b_0\right)$, $a_1^n = i^r\left(\varepsilon i a_0 - \sqrt{2}b_0\right)^n$, and $b_1^n = i^r\left(\sqrt{2}a_0 + \varepsilon i b_0\right)^n$.*

Proof. Suppose $f \in \mathscr{A}_3$. By Lemma 6.10, after a suitable normalization, there exists a transformation $H = \left[\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right] \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ such that

$$f = H^{\otimes n}\left(\left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n}\right)$$

for some $r \in \{0, 1, 2, 3\}$. Since $H \in \mathbf{O}_2(\mathbb{C})$, we have $x^2 + y^2 = 1$. By Lemma 6.8, $\theta(f) = -\frac{1}{2}$, which implies $\frac{a_0a_1 + b_0b_1}{a_0b_1 - a_1b_0} = \pm\frac{i}{\sqrt{2}}$. After rearranging terms, we get

$$a_1\left(\sqrt{2}a_0 + \varepsilon i b_0\right) = b_1\left(\varepsilon i a_0 - \sqrt{2}b_0\right),$$

for some $\varepsilon \in \{1, -1\}$. Since v_0 and v_1 are linearly independent, we know that a_1 and b_1 cannot both be 0. Also, if $\sqrt{2}a_0 + \varepsilon i b_0$ and $\varepsilon i a_0 - \sqrt{2}b_0$ are both 0, then we have $-\sqrt{2}a_0 = \varepsilon i b_0$ and $\varepsilon i a_0 = \sqrt{2}b_0$, which implies $a_0 = b_0 = 0$, a contradiction. Therefore, we have

$$a_1 = c(\varepsilon i a_0 - \sqrt{2}b_0) \qquad \text{and} \qquad b_1 = c(\sqrt{2}a_0 + \varepsilon i b_0) \tag{3}$$

for some $c \neq 0$. To prove necessity, it remains to show that c^n is a power of i.

Now using $H^{-1} = H$, we have two expressions for $(H^{-1})^{\otimes n} f$, which are

$$\left[\begin{smallmatrix} xa_0 + yb_0 \\ ya_0 - xb_0 \end{smallmatrix}\right]^{\otimes n} + \left[\begin{smallmatrix} xa_1 + yb_1 \\ ya_1 - xb_1 \end{smallmatrix}\right]^{\otimes n} = H^{\otimes n}\left(\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]^{\otimes n} + \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]^{\otimes n}\right) = \left(H^{-1}\right)^{\otimes n} f = \left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n}.$$

By Lemma 6.6, there are two cases to consider, each of which has two more cases depending on ε.

1. Suppose $ya_0 - xb_0 = \alpha(xa_0 + yb_0)$, $ya_1 - xb_1 = -\alpha(xa_1 + yb_1)$, $(xa_0 + yb_0)^n = 1$, and $(xa_1 + yb_1)^n = i^r$. By rearranging the first two equations, we get

$$(y - \alpha x)a_0 = (x + \alpha y)b_0 \qquad \text{and} \qquad (y + \alpha x)a_1 = (x - \alpha y)b_1. \tag{4}$$

   It cannot be the case that $a_0 = b_0 = 0$ or $y - \alpha x = x + \alpha y = 0$. If $a_0 = 0$, then $x + \alpha y = 0$, so $a_1 = -\sqrt{2}ib_1$ by (4) and $y \neq 0$ lest $x = 0$ as well. If $b_0 = 0$, then $y - \alpha x = 0$, so $\sqrt{2}ia_1 = b_1$, by the same argument. Now we consider the different cases for ε.

   (a) If $\varepsilon = 1$, then $a_1 = c(ia_0 - \sqrt{2}b_0)$ and $b_1 = c(\sqrt{2}a_0 + ib_0)$ by (3). If $a_0 = 0$, then $a_1 = -c\sqrt{2}b_0$ and $b_1 = cib_0$, which contradicts $a_1 = -\sqrt{2}ib_1$; if $b_0 = 0$, then $a_1 = cia_0$ and $b_1 = c\sqrt{2}a_0$, which contradicts $\sqrt{2}ia_1 = b_1$. Thus, $(y - \alpha x)a_0 = (x + \alpha y)b_0 \neq 0$ by (4). Also from (4), $(y + \alpha x)a_1 = (x - \alpha y)b_1$. Then since $c \neq 0$ and using (3) with $\varepsilon = 1$, we get

$$(y + \alpha x)\left(ia_0 - \sqrt{2}b_0\right) = (x - \alpha y)\left(\sqrt{2}a_0 + ib_0\right).$$

   Using $(y - \alpha x)a_0 = (x + \alpha y)b_0 \neq 0$, we get

$$(y + \alpha x)\left(i(x + \alpha y) - \sqrt{2}(y - \alpha x)\right) = (x - \alpha y)\left(\sqrt{2}(x + \alpha y) + i(y - \alpha x)\right).$$

   This equation simplifies to $x^2 + y^2 = 0$, which is a contradiction.

(b) If $\varepsilon = -1$, then $a_1 = c(-ia_0 - \sqrt{2}b_0)$ and $b_1 = c(\sqrt{2}a_0 - ib_0)$, from (3). Then we get

$$\begin{aligned} xa_1 + yb_1 &= xc\left(-ia_0 - \sqrt{2}b_0\right) + yc\left(\sqrt{2}a_0 - ib_0\right) \\ &= c\left(-i(xa_0 + yb_0) + \sqrt{2}(ya_0 - xb_0)\right) \\ &= c(xa_0 + yb_0), \end{aligned}$$

where in the third step, we used $ya_0 - xb_0 = \alpha(xa_0 + yb_0)$ from (4). Raising this equation to the nth power and using $(xa_0 + yb_0)^n = 1$ and $(xa_1 + yb_1)^n = i^r$, we conclude that $c^n = i^r$.

2. Suppose $ya_0 - xb_0 = -\alpha(xa_0 + yb_0)$, $ya_1 - xb_1 = \alpha(xa_1 + yb_1)$, $(xa_0 + yb_0)^n = i^r$, and $(xa_1 + yb_1)^n = 1$. Now we consider the different cases for ε.
   (a) If $\varepsilon = 1$, then $a_1 = c(ia_0 - \sqrt{2}b_0)$ and $b_1 = c(\sqrt{2}a_0 + ib_0)$ by (3). Using similar reasoning to that in case 1b leads to $(-c)^n i^r = 1$, so c^n is a power of i.
   (b) If $\varepsilon = -1$, then $a_1 = c(-ia_0 - \sqrt{2}b_0)$ and $b_1 = c(\sqrt{2}a_0 - ib_0)$ by (3). Using similar reasoning to that in case 1a leads to a contradiction.

For sufficiency, suppose the three equations hold for some $\varepsilon \in \{1, -1\}$ and some $r \in \{0, 1, 2, 3\}$. Further assume $\varepsilon = 1$, in which case, the equations are

$$a_1\left(\sqrt{2}a_0 + ib_0\right) = b_1\left(ia_0 - \sqrt{2}b_0\right), \tag{5}$$

as well as

$$a_1^n = i^r\left(ia_0 - \sqrt{2}b_0\right)^n \qquad \text{and} \qquad b_1^n = i^r\left(\sqrt{2}a_0 + ib_0\right)^n. \tag{6}$$

From (5), we have

$$a_1 = c(ia_0 - \sqrt{2}b_0) \qquad \text{and} \qquad b_1 = c(\sqrt{2}a_0 + ib_0) \tag{7}$$

for some $c \in \mathbb{C}$. In (5), a_1, b_1 cannot be both zero. Similarly, $\sqrt{2}a_0 + ib_0, ia_0 - \sqrt{2}b_0$ cannot be both zero. Thus at least one equation in (7) has both sides nonzero and we can always find some c even if one factor is zero. We can write (7) as

$$\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right] = c\left[\begin{smallmatrix} i & -\sqrt{2} \\ \sqrt{2} & i \end{smallmatrix}\right]\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right].$$

This implies that $a_0a_1 + b_0b_1 = ci(a_0^2 + b_0^2)$. Using (6) or (7), whichever equation is not zero on both sides, we have $c^n = i^r$. Since (5) implies $\theta(f) = -\frac{1}{2}$, we know that $a_0^2 + b_0^2 \neq 0$ because otherwise v_0 is a multiple of $\left[\begin{smallmatrix} 1 \\ \pm i \end{smallmatrix}\right]$, which makes $\theta(f) = -1$ regardless of v_1.

We now define two orthogonal matrices $T_1 = \frac{1}{\sqrt{1+i}}\left[\begin{smallmatrix} 1 & \alpha \\ -\alpha & 1 \end{smallmatrix}\right]$ and $T_2 = \frac{1}{\sqrt{a_0^2+b_0^2}}\left[\begin{smallmatrix} a_0 & b_0 \\ b_0 & -a_0 \end{smallmatrix}\right]$. Also let $T = T_1T_2 \in \mathbf{O}_2(\mathbb{C})$. For $f = \left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]^{\otimes n} + \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]^{\otimes n}$, we want to calculate $T^{\otimes n}f$. First,

$$T_2\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right] = \sqrt{a_0^2 + b_0^2}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \qquad \text{and} \qquad T\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right] = \gamma\left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right],$$

where $\gamma = \sqrt{\frac{a_0^2+b_0^2}{1+i}}$. Furthermore, $a_1b_0 - a_0b_1 = \sqrt{2}i(a_0a_1 + b_0b_1) = -c\sqrt{2}(a_0^2 + b_0^2)$ by (5). Then

$$T_2\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right] = \frac{1}{\sqrt{a_0^2 + b_0^2}}\left[\begin{smallmatrix} a_0a_1 + b_0b_1 \\ a_1b_0 - a_0b_1 \end{smallmatrix}\right] = c\sqrt{a_0^2 + b_0^2}\left[\begin{smallmatrix} i \\ -\sqrt{2} \end{smallmatrix}\right].$$

It follows that

$$T\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right] = c\gamma \left[\begin{smallmatrix} 1 & \alpha \\ -\alpha & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} i \\ -\sqrt{2} \end{smallmatrix}\right] = c\gamma \left[\begin{smallmatrix} i-\sqrt{2}\alpha \\ -i\alpha-\sqrt{2} \end{smallmatrix}\right] = -c\gamma \left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right].$$

Thus

$$T^{\otimes n} f = \gamma^n \left(\left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n} + (-c)^n \left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} \right).$$

So T transforms f into the canonical form of $\mathscr{A}_3$. If we write out the orthogonal transformation T explicitly, then $T = \left[\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right]$ where

$$x = \frac{a_0 + \alpha b_0}{\sqrt{(i+1)\left(a_0^2 + b_0^2\right)}} \qquad \text{and} \qquad y = \frac{b_0 - \alpha a_0}{\sqrt{(i+1)\left(a_0^2 + b_0^2\right)}}.$$

When $\varepsilon = -1$, the argument is similar. In this case, $a_1 = c(-ia_0 - \sqrt{2}b_0)$ and $b_1 = c(\sqrt{2}a_0 - ib_0)$ for some $c \in \mathbb{C}$ satisfying $c^n = i^r$ and the entries of T are

$$x = \frac{a_0 - \alpha b_0}{\sqrt{(i+1)\left(a_0^2 + b_0^2\right)}} \qquad \text{and} \qquad y = \frac{b_0 + \alpha a_0}{\sqrt{(i+1)\left(a_0^2 + b_0^2\right)}}. \qquad \square$$

Remark: Notice that either $a_1(\sqrt{2}a_0 + ib_0) = b_1(ia_0 - \sqrt{2}b_0)$ or $a_1(\sqrt{2}a_0 - ib_0) = b_1(-ia_0 - \sqrt{2}b_0)$ implies $\theta(f) = -\frac{1}{2}$, unless $\det(\left[\begin{smallmatrix} a_0 & a_1 \\ b_0 & b_1 \end{smallmatrix}\right]) = 0$.

As mentioned before, $\mathscr{A}_2 = \mathscr{P}_2$ requires a stronger condition than just θ. If $f \in \mathscr{A}_2 = \mathscr{P}_2$, then $\theta(f) = -1$, but the reverse is not true. If $f = v_0^{\otimes n} + v_1^{\otimes n}$ with $v_0 = [1, i]$ and v_1 is not a multiple of $[1, -i]$, then $\theta(f) = -1$ but f is not in $\mathscr{A}_2 = \mathscr{P}_2$, since any orthogonal H fixes $\{[1, i], [1, -i]\}$ set-wise, up to a scalar multiple.

The next lemma, which appeared in [11], gives a characterization of $\mathscr{A}_2$. It says that any signature in $\mathscr{A}_2$ is essentially in canonical form. For completeness, we include its proof.

Lemma 6.14 (Lemma 8.8 in [11])**.** *Let f be a non-degenerate symmetric signature. Then $f \in \mathscr{A}_2$ iff f is of the form $c\left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n}\right)$ for some $c, \beta \neq 0$.*

Proof. Assume that $f = c\left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n}\right)$ for some $c, \beta \neq 0$. Consider the orthogonal transformation $H = \left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right]$, where $a = \frac{1}{2}\left(\beta^{\frac{1}{2n}} + \beta^{-\frac{1}{2n}}\right)$ and $b = \frac{1}{2i}\left(\beta^{\frac{1}{2n}} - \beta^{-\frac{1}{2n}}\right)$. We pick a and b in this way so that $a + bi = \beta^{\frac{1}{2n}}$, $a - bi = \beta^{-\frac{1}{2n}}$, and $(a+bi)(a-bi) = a^2 + b^2 = 1$. Also $\left(\frac{a+bi}{a-bi}\right)^n = \beta$. Then

$$\begin{aligned} H^{\otimes n} f &= c\left(\left[\begin{smallmatrix} a+bi \\ -ai+b \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} a-bi \\ ai+b \end{smallmatrix}\right]^{\otimes n}\right) \\ &= c\left((a+bi)^n \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} + (a-bi)^n \beta \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n}\right) \\ &= c\sqrt{\beta}\left(\left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} + \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n}\right), \end{aligned}$$

so f can be written as

$$f = c\sqrt{\beta}(H^{-1})^{\otimes n}\left(\left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} + \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n}\right).$$

Therefore $f \in \mathscr{A}_2$.

On the other hand, the desired form $f = c(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n})$ follows from the fact that $\{\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]\}$ is fixed setwise under any orthogonal transformation up to nonzero constants. $\square$

Remark: Notice that $\theta(v_0, v_1) = -1$ for linearly independent v_0 and v_1 if and only if at least one of v_0, v_1 is $\left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1\\ -i\end{smallmatrix}\right]$, up to a nonzero scalar.

We now present the polynomial time algorithm to check if $f \in \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$.

Lemma 6.15. *Given a non-degenerate symmetric signature f of arity at least 3, there is a polynomial time algorithm to decide whether $f \in \mathscr{A}_k$ for each $k \in \{1, 2, 3\}$. If so, k is unique and at least one corresponding orthogonal transformation can be found in polynomial time.*

Proof. First we check if f satisfies a second order recurrence relation. If it does, then the coefficients (a, b, c) of the second order recurrence relation are unique up to a nonzero scalar by Lemma 6.1. If the coefficients satisfy $b^2 - 4ac \neq 0$, then by Lemma 6.3, we can express f as $v_0^{\otimes n} + v_1^{\otimes n}$, where v_0 and v_1 are linearly independent and $\text{arity}(f) = n$. All of this must be true for f to be in $\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$. With this alternate expression for f, we apply Lemma 6.12, Lemma 6.14, and Lemma 6.13 to decide if $f \in \mathscr{A}_k$ for each $k \in \{1, 2, 3\}$ respectively. These sets are disjoint by Lemma 6.8, so there can be at most one k such that $f \in \mathscr{A}_k$. □

6.2 Set of Symmetric Signatures

We first show that if a non-degenerate signature f of arity at least 3 is in $\mathscr{A}_1$ or $\mathscr{A}_3$, then for any set $\mathcal{F}$ containing f, there are only a small constant number of transformations to be checked to decide whether $\mathcal{F}$ is $\mathscr{A}$-transformable. If $f \in \mathscr{A}_2$, then there can be more than a constant number of transformations check. However, this number is at most linear in the arity of f.

Notice that any non-degenerate symmetric signature $f \in \mathscr{A}$ of arity at least 3 is in $\mathscr{F}_{123}$, which contains signatures expressed as a sum of two tensor powers. Therefore $\theta(f)$ is well-defined. By Lemma 2.7, we only need to consider the sets $\mathscr{A}$ and $\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$. In particular,

$$\theta(f) = \begin{cases} 0 & \text{if } f \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{F}_1, \\ -1 & \text{if } f \in \mathscr{F}_3, \\ -\frac{1}{2} & \text{if } f \in \left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right](\mathscr{F}_2 \cup \mathscr{F}_3). \end{cases} \tag{8}$$

Lemma 6.16. *Let $\mathcal{F}$ be a set of symmetric signatures and suppose $\mathcal{F}$ contains a non-degenerate signature $f \in \mathscr{A}_1$ of arity $n \geq 3$ with $H \in \mathbf{O}_2(\mathbb{C})$. Then $\mathcal{F}$ is $\mathscr{A}$-transformable iff $\mathcal{F}$ is a subset of $H\mathscr{A}$, or $H\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\mathscr{A}$, or $H\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$.*

Proof. Sufficiency follows from Lemma 2.7 and both $H, H_2 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right] \in \mathbf{O_2}(\mathbb{C})$.

Before we prove necessity, we first claim that without loss of generality, we may assume $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. If $H \in \mathbf{SO}_2(\mathbb{C})$, we let $\widetilde{H} = H\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right] \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. Then $f \in \mathscr{A}_1$ also with $\widetilde{H}$. From $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right] \in \text{Stab}(\mathscr{A})$, it follows that $\widetilde{H}\mathscr{A} = H\mathscr{A}$. Also $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right] = \left[\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right] = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right] = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right] \in \text{Stab}(\mathscr{A})$. It follows that $\widetilde{H}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A} = H\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$.

Suppose $\mathcal{F}$ is $\mathscr{A}$-transformable. By Lemma 4.3, there exists an $H' \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H'\mathscr{A}$ or $\mathcal{F} \subseteq H'\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$. We only need to show there exists an $M \in \text{Stab}(\mathscr{A})$, such that $H' = HM$ in the first case, and in the second case $H' = H\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]M$, and $M\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right] = \left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]M'$ for some $M' \in \text{Stab}(\mathscr{A})$.

Since $f \in \mathscr{A}_1$ with H, after a suitable normalization by a nonzero scalar, we have

$$f = H^{\otimes n}\left(\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]^{\otimes n} + \beta\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right]^{\otimes n}\right),$$

where $\beta = \alpha^{tn+2r}$ for some $r \in \{0,1,2,3\}$ and $t \in \{0,1\}$. Let $g = (H'^{-1})^{\otimes n} f$ and $T = H'^{-1}H$ so that

$$g = T^{\otimes n}\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^{\otimes n} + \beta\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]^{\otimes n}\right).$$

Note that $T \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ since $H' \in \mathbf{SO}_2(\mathbb{C})$ and $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. Thus $T = T^{-1}$ and $HT = H'$. Let $T = \left[\begin{smallmatrix}a & b\\ b & -a\end{smallmatrix}\right]$ for some $a, b \in \mathbb{C}$ such that $a^2 + b^2 = 1$. There are two possibilities according to $\mathcal{F} \subseteq H'\mathscr{A}$ or $\mathcal{F} \subseteq H'\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$.

1. If $\mathcal{F} \subseteq H'\mathscr{A}$, then $g \in \mathscr{F}_{123}$ since g is symmetric and non-degenerate. By (8), we have $\theta(g) = 0$, so $g \in \mathscr{F}_1$ or $g \in \mathscr{F}_2$. We discuss the two cases of g separately.
   - Suppose $g \in \mathscr{F}_1$. Then we have

     $$T^{\otimes n}\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^{\otimes n} + \beta\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]^{\otimes n}\right) = \lambda\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\otimes n} + i^t\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{\otimes n}\right)$$

     for some $\lambda \neq 0$ and $t \in \{0,1,2,3\}$. Plugging in the expression for T, we have

     $$\left(\left[\begin{smallmatrix}a+b\\ b-a\end{smallmatrix}\right]^{\otimes n} + \beta\left[\begin{smallmatrix}a-b\\ a+b\end{smallmatrix}\right]^{\otimes n}\right) = \lambda\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\otimes n} + i^t\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{\otimes n}\right).$$

     Then by Lemma 6.6, we have $a + b = 0$ or $a - b = 0$. Together with $a^2 + b^2 = 1$, we can solve for $T = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$ or $T = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & -1\\ -1 & -1\end{smallmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$, up to a constant multiple ± 1. Since $\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{A})$, we have $T \in \mathrm{Stab}(\mathscr{A})$, so we are done.
   - Suppose $g \in \mathscr{F}_2$. Then we have

     $$T^{\otimes n}\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^{\otimes n} + \beta\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]^{\otimes n}\right) = \lambda\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^{\otimes n} + i^t\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]^{\otimes n}\right)$$

     for some $\lambda \neq 0$ and $t \in \{0,1,2,3\}$. Plugging in the expression for T, we have

     $$\left(\left[\begin{smallmatrix}a+b\\ b-a\end{smallmatrix}\right]^{\otimes n} + \beta\left[\begin{smallmatrix}a-b\\ a+b\end{smallmatrix}\right]^{\otimes n}\right) = \lambda\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^{\otimes n} + i^t\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]^{\otimes n}\right)$$

     Then by Lemma 6.6, we have $a + b = a - b$ or $a + b = -(a - b)$. Therefore either $a = 0$ or $b = 0$. Thus $T = \pm\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ or $T = \pm\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ and both matrices are in $\mathrm{Stab}(\mathscr{A})$.
2. If $\mathcal{F} \subseteq H'\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{A}$, then we have $g \in \left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{F}_{123}$. By (8), we have $\theta(g) = 0$, so $g \in \left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\mathscr{F}_1$. That is,

   $$T^{\otimes n}\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^{\otimes n} + \beta\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]^{\otimes n}\right) = \lambda\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]^{\otimes n}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\otimes n} + i^t\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{\otimes n}\right) = \lambda\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\otimes n} + i^t\alpha^n\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{\otimes n}\right)$$

   for some $\lambda \neq 0$. This is essentially the same as the case where $g \in \mathscr{F}_1$ above, except that the coefficients are different. However, the coefficients do not affect the argument and our conclusion in this case that $T = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$ or $T = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$, up to a constant multiple ± 1. Notice that $\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{A})$. Moreover,

   $$\begin{aligned}\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right] &= \left[\begin{smallmatrix}0 & -\alpha\\ 1 & 0\end{smallmatrix}\right]\\ &= \left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & -\alpha\\ \alpha^{-1} & 0\end{smallmatrix}\right]\\ &= -\alpha\left[\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & 1\\ i & 0\end{smallmatrix}\right],\end{aligned}$$

   and $\left[\begin{smallmatrix}0 & 1\\ i & 0\end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{A})$. □

Lemma 6.17. *Let $\mathcal{F}$ be a set of symmetric signatures and suppose $\mathcal{F}$ contains a non-degenerate signature $f \in \mathscr{A}_2$ of arity $n \geq 3$. Then there exists a set $\mathcal{H} \subseteq \mathbf{O}_2(\mathbb{C})$ of size $O(n)$ such that $\mathcal{F}$ is $\mathscr{A}$-transformable iff there exists an $H \in \mathcal{H}$ such that $\mathcal{F} \subseteq H\mathscr{A}$. Moreover $\mathcal{H}$ can be computed in polynomial time in the input length of the symmetric signature f.*

Proof. Sufficiency is trivial by Lemma 4.3.

Suppose $\mathcal{F}$ is $\mathscr{A}$-transformable. By Lemma 4.3, there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{A}$ or $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$. In the first case, we show that the choices of H can be limited to $O(n)$. Then we show that the second case is impossible.

Since $f \in \mathscr{A}_2$, after a suitable normalization by a nonzero scalar, we have

$$f = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \nu \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n}$$

for some $\nu \neq 0$ by Lemma 6.14. Let $g = (H^{-1})^{\otimes n} f$. Then

$$g = (H^{-1})^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \nu \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} \right).$$

There are two possibilities according to $\mathcal{F} \subseteq H\mathscr{A}$ or $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$.

1. Suppose $\mathcal{F} \subseteq H\mathscr{A}$. Therefore $g \in \mathscr{F}_{123}$. By (8), we have $\theta(g) = -1$, so $g \in \mathscr{F}_3$. Then we have
$$(H^{-1})^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \nu \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} \right) = \lambda \left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} \right)$$
for some $\lambda \neq 0$ and $r \in \{0, 1, 2, 3\}$. Because $H^{-1} \in \mathbf{SO}_2(\mathbb{C})$, we may assume that H^{-1} is of the form $\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$ where $a^2 + b^2 = 1$. Therefore
$$\begin{aligned} \lambda \left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} \right) &= \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \nu \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} \right) \\ &= (a + bi)^n \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + \nu (a - bi)^n \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n}. \end{aligned}$$
Comparing the coefficients, by Lemma 6.6, we have
$$\lambda = (a + bi)^n \qquad \text{and} \qquad \lambda i^r = \nu (a - bi)^n.$$
Hence,
$$i^r (a + bi)^n = \nu (a - bi)^n.$$
Since $(a + bi)(a - bi) = a^2 + b^2 = 1$, we know that $(a + bi)^{2n} = \nu i^{-r}$. Therefore $a + bi = \omega_{2n} (\nu i^{-r})^{1/2n}$, where ω_{2n} is a $2n$-th root of unity. There are 4 choices for r, and $2n$ choices for ω_{2n}. However, $a - bi = \frac{1}{a+bi}$, and (a, b) can be solved from $(a + bi, a - bi)$. Hence there are only $O(n)$ many choices for H, depending on f.
2. Suppose $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$. Then $g \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{F}_{123}$. However, $\theta(g) = -1$ by (8), which is a contradiction. □

Lemma 6.18. *Let $\mathcal{F}$ be a set of symmetric signatures and suppose $\mathcal{F}$ contains a non-degenerate signature $f \in \mathscr{A}_3$ of arity $n \geq 3$ with $H \in \mathbf{O}_2(\mathbb{C})$. Then $\mathcal{F}$ is $\mathscr{A}$-transformable iff $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$.*

Proof. Sufficiency is trivial by Lemma 4.3.

Suppose $\mathcal{F}$ is $\mathscr{A}$-transformable. As in the proof of Lemma 6.16, we may assume the given $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. By Lemma 4.3, there exists an $H' \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H'\mathscr{A}$ or

$\mathcal{F} \subseteq H' \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$. We show the first case is impossible. Then in the second case, we show that there exists an M, such that $H' = HM$, where $M \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] M'$ for some $M' \in \mathrm{Stab}(\mathscr{A})$.

Since $f \in \mathscr{A}_3$ with H, after a suitable normalization by a nonzero scalar, we have

$$f = H^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n} \right)$$

for some $r \in \{0, 1, 2, 3\}$. Let $g = (H'^{-1})^{\otimes n} f$ and $T = H'^{-1} H$ so that

$$g = T^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n} \right).$$

Note that $T \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ since $H' \in \mathbf{SO}_2(\mathbb{C})$ and $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. Thus $T = T^{-1}$ and $HT = H'$. Let $T = \left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right]$ for some $a, b \in \mathbb{C}$ such that $a^2 + b^2 = 1$. There are two possibilities according to $\mathcal{F} \subseteq H' \mathscr{A}$ or $\mathcal{F} \subseteq H' \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$.

1. Suppose $\mathcal{F} \subseteq H' \mathscr{A}$. Then $g = (H'^{-1})^{\otimes n} f \in \mathscr{F}_{123}$. However, $\theta(g) = -\frac{1}{2}$ by (8), which is a contradiction.
2. Suppose $\mathcal{F} \subseteq H' \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{A}$. Then $g \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{F}_{123}$, so $\theta(g) = -\frac{1}{2}$ by (8) and $g \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] (\mathscr{F}_2 \cup \mathscr{F}_3)$. We discuss the these two cases separately.
   - Suppose $g \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{F}_2$. Then we have

     $$\begin{aligned} T^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n} \right) &= \lambda \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + i^t \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n} \right) \\ &= \lambda \left(\left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^t \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n} \right) \end{aligned}$$

     for some $\lambda \neq 0$ and $t \in \{0, 1, 2, 3\}$. Plugging in the expression for T, we have

     $$\left(\left[\begin{smallmatrix} a+\alpha b \\ b-\alpha a \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} a-\alpha b \\ b+\alpha a \end{smallmatrix}\right]^{\otimes n} \right) = \lambda \left(\left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^t \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n} \right).$$

     Then by Lemma 6.6, we have either

     $$b - a\alpha = \alpha(a + b\alpha) \qquad \text{and} \qquad b + a\alpha = -\alpha(a - b\alpha)$$

     or

     $$b - a\alpha = -\alpha(a + b\alpha) \qquad \text{and} \qquad b + a\alpha = \alpha(a - b\alpha).$$

     The first case is impossible. In the second case, we have $a = \pm 1$ and $b = 0$. This implies $T = \pm \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{A})$, which commutes with $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]$.
   - Suppose $g \in \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \mathscr{F}_3$. Then we have

     $$\begin{aligned} T^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ \alpha \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix}\right]^{\otimes n} \right) &= \lambda \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes n} + i^t \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]^{\otimes n} \right) \\ &= \lambda \left(\left[\begin{smallmatrix} 1 \\ \alpha i \end{smallmatrix}\right]^{\otimes n} + i^t \left[\begin{smallmatrix} 1 \\ -\alpha i \end{smallmatrix}\right]^{\otimes n} \right) \end{aligned}$$

     for some $\lambda \neq 0$ and $t \in \{0, 1, 2, 3\}$. Plugging in the expression for T, we have

     $$\left(\left[\begin{smallmatrix} a+\alpha b \\ b-\alpha a \end{smallmatrix}\right]^{\otimes n} + i^r \left[\begin{smallmatrix} a-\alpha b \\ b+\alpha a \end{smallmatrix}\right]^{\otimes n} \right) = \lambda \left(\left[\begin{smallmatrix} 1 \\ \alpha i \end{smallmatrix}\right]^{\otimes n} + i^t \left[\begin{smallmatrix} 1 \\ -\alpha i \end{smallmatrix}\right]^{\otimes n} \right).$$

     Then by Lemma 6.6, we have either

     $$b - a\alpha = \alpha i(a + b\alpha) \qquad \text{and} \qquad b + a\alpha = -\alpha i(a - b\alpha)$$

or

$$b - a\alpha = -\alpha i(a + b\alpha) \qquad \text{and} \qquad b + a\alpha = \alpha i(a - b\alpha).$$

The first case is impossible. In the second case, we have $a = 0$ and $b = \pm 1$. This implies that $T = \pm \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Note that $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{smallmatrix}\right] = \alpha^{-1}\left[\begin{smallmatrix} 0 & i \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{A})$. □

Now we are ready to show how to decide if a finite set of signatures is $\mathscr{A}$-transformable. To avoid trivialities, we assume $\mathcal{F}$ contains a non-degenerate signature of arity at least 3. If every non-degenerate signature in $\mathcal{F}$ has arity at most two, then Holant($\mathcal{F}$) is tractable.

Theorem 6.19. *There is a polynomial time algorithm to decide, for any finite input set $\mathcal{F}$ of symmetric signatures containing a non-degenerate signature f of arity $n \geq 3$, whether $\mathcal{F}$ is $\mathscr{A}$-transformable.*

Proof. By Lemma 6.15, we can decide if f is in $\mathscr{A}_k$ for some $k \in \{1, 2, 3\}$. If not, then by Lemma 2.9, $\mathcal{F}$ is not $\mathscr{A}$-transformable. Otherwise, $f \in \mathscr{A}_k$ for some unique k. Depending on k, we apply Lemma 6.16, Lemma 6.17, or Lemma 6.18 to check if $\mathcal{F}$ is $\mathscr{A}$-transformable. □

7 Symmetric $\mathscr{P}$-transformable Signatures

To decide if a signature set is $\mathscr{P}$-transformable, we face the same issue as the $\mathscr{A}$-transformable case. Namely, a symmetric signature of arity n is given by $n + 1$ values, instead of 2^n values. This exponentially more succinct representation requires us to find a more efficient algorithm.

The next lemma tells us how to decide membership in $\mathscr{P}_1$ for signatures of arity at least 3.

Lemma 7.1. *Let $f = v_0^{\otimes n} + v_1^{\otimes n}$ be a symmetric signature of arity $n \geq 3$, where v_0 and v_1 are linearly independent. Then $f \in \mathscr{P}_1$ iff $\theta(f) = 0$.*

Proof. Necessity is clear by Lemma 6.8 and sufficiency follows from Lemma 6.11. □

Since $\mathscr{A}_2 = \mathscr{P}_2$, the membership problem for $\mathscr{P}_2$ is handled by Lemma 6.14. Using Lemma 7.1 and Lemma 6.14, we can efficiently decide membership in $\mathscr{P}_1 \cup \mathscr{P}_2$.

Lemma 7.2. *Given a non-degenerate symmetric signature f of arity at least 3, there is a polynomial time algorithm to decide whether $f \in \mathscr{P}_k$ for some $k \in \{1, 2\}$. If so, k is unique and at least one corresponding orthogonal transformation can be found in polynomial time.*

Proof. First we check if f satisfies a second order recurrence relation. If it does, then the coefficients (a, b, c) of the second order recurrence relation are unique up to a nonzero scalar by Lemma 6.1. If the coefficients satisfy $b^2 - 4ac \neq 0$, then by Lemma 6.3, we can express f as $v_0^{\otimes n} + v_1^{\otimes n}$, where v_0 and v_1 are linearly independent and arity$(f) = n$. All of this must be true for f to be in $\mathscr{P}_1 \cup \mathscr{P}_2$. With this alternate expression for f, we apply Lemma 7.1 and Lemma 6.14 to decide if $f \in \mathscr{P}_k$ for some $k \in \{1, 2\}$ respectively. These sets are disjoint by Lemma 6.8, so there can be at most one k such that $f \in \mathscr{P}_k$. □

Like the symmetric affine case, the following lemmas assume the signature set $\mathcal{F}$ contains a non-degenerate signature of arity at least 3 in $\mathscr{P}_1$ or $\mathscr{P}_2$. Unlike the symmetric affine case, the number of transformations to be checked to decide whether $\mathcal{F}$ is $\mathscr{P}$-transformable is always a small constant.

Lemma 7.3. *Let $\mathcal{F}$ be a set of symmetric signatures and suppose $\mathcal{F}$ contains a non-degenerate signature $f \in \mathscr{P}_1$ of arity $n \geq 3$ with $H \in \mathbf{O}_2(\mathbb{C})$. Then $\mathcal{F}$ is $\mathscr{P}$-transformable iff $\mathcal{F} \subseteq H \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] \mathscr{P}$.*

Proof. Sufficiency is trivial by Lemma 2.10.

Suppose $\mathcal{F}$ is $\mathscr{P}$-transformable. As in the proof of Lemma 6.16, we may assume $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. Then by Lemma 5.1, there exists an $H' \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H'\mathscr{P}$ or $\mathcal{F} \subseteq H' \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P}$, where in the second case we can take $H' = I_2$. In the first case, we show that there exists an $M \in \mathrm{Stab}(\mathscr{P})$ such that $H' = H \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] M$. Then we show that the second case is impossible.

Since $f \in \mathscr{P}_1$ with H, after a suitable normalization by a nonzero scalar, we have

$$f = H^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n} \right)$$

for some $\beta \neq 0$. Let $g = (H'^{-1})^{\otimes n} f$ and $T = H'^{-1} H$ so that

$$g = T^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n} \right).$$

Note that $T \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$ since $H' \in \mathbf{SO}_2(\mathbb{C})$ and $H \in \mathbf{O}_2(\mathbb{C}) - \mathbf{SO}_2(\mathbb{C})$. Thus $T = T^{-1}$ and $HT = H'$.

1. Suppose $\mathcal{F} \subseteq H'\mathscr{P}$. Then g must be a generalized equality since $g \in \mathscr{P}$ with arity $n \geq 3$. The only symmetric non-degenerate generalized equality in $\mathscr{P}$ with arity $n \geq 3$ has the form $\lambda \left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{\otimes n} + \beta' \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^{\otimes n} \right)$, for some $\lambda, \beta' \neq 0$. Thus

   $$T^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]^{\otimes n} \right) = \lambda \left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{\otimes n} + \beta' \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^{\otimes n} \right).$$

   Let $T = \left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right]$ for $a, b \in \mathbb{C}$ such that $a^2 + b^2 = 1$. Then

   $$\left[\begin{smallmatrix} a+b \\ b-a \end{smallmatrix}\right]^{\otimes n} + \beta \left[\begin{smallmatrix} a-b \\ a+b \end{smallmatrix}\right]^{\otimes n} = \lambda \left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{\otimes n} + \beta' \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^{\otimes n} \right).$$

   By Lemma 6.6 we have either $a - b = 0$ or $a + b = 0$. Together with $a^2 + b^2 = 1$, the only solutions are $T = \pm\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ or $T = \pm\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right] = \pm\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Since $\pm\frac{1}{\sqrt{2}} I_2, \pm\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{Stab}(\mathscr{P})$, this case is complete.
2. Suppose $\mathcal{F} \subseteq H' \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P}$. Then $g \in \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P}$, and $\theta(g) = \theta(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]) = 0$ by Lemma 6.8. However, $\theta(h) = -1$ for any non-degenerate $h \in \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right] \mathscr{P}$ of arity at least 3, contradiction. □

Lemma 7.4. *Let $\mathcal{F}$ be a set of symmetric signatures and suppose $\mathcal{F}$ contains a non-degenerate signature $f \in \mathscr{P}_2$ of arity $n \geq 3$. Then $\mathcal{F}$ is $\mathscr{P}$-transformable iff all non-degenerate signatures in $\mathcal{F}$ are contained in $\mathscr{P}_2 \cup \{=_2\}$.*

Proof. Suppose $\mathcal{F}$ is $\mathscr{P}$-transformable. Let $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$. Then by Lemma 5.1, $\mathcal{F} \subseteq Z\mathscr{P}$ or there exists an $H \in \mathbf{SO}_2(\mathbb{C})$ such that $\mathcal{F} \subseteq H\mathscr{P}$. In first case, we show that all the non-degenerate symmetric signatures in $Z\mathscr{P}$ are contained in $\mathscr{P}_2 \cup \{=_2\}$. Then we show that the second case is impossible.

1. Suppose $\mathcal{F} \subseteq Z\mathscr{P}$. Let $g \in Z\mathscr{P}$ be a symmetric non-degenerate signature of arity m. If $(Z^{-1})^{\otimes 2} g = \lambda[0,1,0]$ is the binary disequality signature up to a nonzero scalar $\lambda \in \mathbb{C}$, then

$$g = \lambda Z^{\otimes 2} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is the binary equality signature $=_2$. Otherwise, we can express g as

$$\begin{aligned} g &= cZ^{\otimes m} \left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right]^{\otimes m} + \beta \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right]^{\otimes m} \right) \\ &= c \left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]^{\otimes m} + \beta \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix} \right]^{\otimes m} \right) \end{aligned}$$

for some $c, \beta \neq 0$ with $m \geq 2$. Thus, $g \in \mathscr{P}_2 = \mathscr{A}_2$ by Lemma 6.14. We conclude that the symmetric non-degenerate subset of $Z\mathscr{P}$ is contained in $\mathscr{P}_2 \cup \{=_2\}$. Therefore, the non-degenerate subset of $\mathcal{F}$ is contained in $\mathscr{P}_2 \cup \{=_2\}$.

2. Suppose $\mathcal{F} \subseteq H\mathscr{P}$. Since $f \in \mathscr{P}_2 = \mathscr{A}_2$, after a suitable normalization by a scalar, we have

$$f = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix} \right]^{\otimes n}$$

for some $\beta \neq 0$ by Lemma 6.14. Let $g = (H^{-1})^{\otimes n} f$ so that

$$g = \left(H^{-1}\right)^{\otimes n} \left(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right]^{\otimes n} + \beta \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix} \right]^{\otimes n} \right).$$

In particular, f and g have the same arity $n \geq 3$. By Lemma 6.8, $\theta(g) = \theta(\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix} \right], \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix} \right]) = -1$ since $H^{-1} \in \mathbf{O}_2(\mathbb{C})$. However, $g \in \mathscr{P}$ must be of the form $\left[\begin{smallmatrix} c \\ 0 \end{smallmatrix} \right]^{\otimes n} + \left[\begin{smallmatrix} 0 \\ d \end{smallmatrix} \right]^{\otimes n}$ for some nonzero $c, d \in \mathbb{C}$, which has $\theta(g) = 0$. This is a contradiction.

It is easy to see that all of above is reversible. Therefore sufficiency follows. □

Now we are ready to show how to decide if a finite set of signatures is $\mathscr{P}$-transformable. To avoid trivialities, we assume $\mathcal{F}$ contains a non-degenerate signature of arity at least 3. If every non-degenerate signature in $\mathcal{F}$ has arity at most two, then Holant($\mathcal{F}$) is tractable.

Theorem 7.5. *There is a polynomial time algorithm to decide, for any finite input set $\mathcal{F}$ of symmetric signatures containing a non-degenerate signature f of arity $n \geq 3$, whether $\mathcal{F}$ is $\mathscr{P}$-transformable.*

Proof. By Lemma 7.2, we can decide if f is in $\mathscr{P}_k$ for some $k \in \{1, 2\}$. If not, then by Lemma 2.12, $\mathcal{F}$ is not $\mathscr{P}$-transformable. Otherwise, $f \in \mathscr{P}_k$ for some unique k. Depending on k, we apply Lemma 7.3 or Lemma 7.4 to check if $\mathcal{F}$ is $\mathscr{P}$-transformable. □

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