Graphical representations of Ising and Potts models

Stochastic geometry of the quantum Ising model and the space–time Potts model

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Preface

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

I would like to thank my PhD supervisor Geoffrey Grimmett. Chapter 3 and Section 4.1 were done in collaboration with him. We have agreed that 65% of the work is mine. This work has appeared in a journal as a joint publication [**15**]. I am the sole author of the remaining material. Section 4.2 has been published in a journal [**14**].

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Summary

Statistical physics seeks to explain macroscopic properties of matter in terms of microscopic interactions. Of particular interest is the phenomenon of phase transition: the sudden changes in macroscopic properties as external conditions are varied. Two models in particular are of great interest to mathematicians, namely the Ising model of a magnet and the percolation model of a porous solid. These models in turn are part of the unifying framework of the random-cluster representation, a model for random graphs which was first studied by Fortuin and Kasteleyn in the 1970's. The random-cluster representation has proved extremely useful in proving important facts about the Ising model and similar models.

In this work we study the corresponding graphical framework for two related models. The first model is the transverse field quantum Ising model, an extension of the original Ising model which was introduced by Lieb, Schultz and Mattis in the 1960's. The second model is the space–time percolation process, which is closely related to the contact model for the spread of disease. In Chapter 2 we define the appropriate 'space–time' random-cluster model and explore a range of useful probabilistic techniques for studying it. The space–time Potts model emerges as a natural generalization of the quantum Ising model. The basic properties of the phase transitions in these models are treated in this chapter, such as the fact that there is at most one unbounded FK-cluster, and the resulting lower bound on the critical value in $\mathbb{Z}$.

In Chapter 3 we develop an alternative graphical representation of the quantum Ising model, called the random-parity representation.

This representation is based on the random-current representation of the classical Ising model, and allows us to study in much greater detail the phase transition and critical behaviour. A major aim of this chapter is to prove sharpness of the phase transition in the quantum Ising model—a central issue in the theory—and to establish bounds on some critical exponents. We address these issues by using the random-parity representation to establish certain differential inequalities, integration of which give the results.

In Chapter 4 we explore some consequences and possible extensions of the results established in Chapters 2 and 3. For example, we determine the critical point for the quantum Ising model in $\mathbb{Z}$ and in 'star-like' geometries.

Contents

List of Notation

$\langle\cdot\|$	Conjugate transpose, page 9
$\langle\cdot\rangle$	Expectation under Ising measure, page 30
$\langle\cdot\rangle^{\pm}$	Ising measure with $\pm$ boundary condition, page 78
$\|\pm\rangle$	Basis of $\mathbb{C}^2$, page 9
$\|\sigma\rangle$	Basis vector in $\mathcal{H}$, page 9
α	Part of Potts boundary condition, page 25
$\mathbb{B}$	Edge set of $\mathbb{H}$, page 151
$\mathcal{B}(\mathbb{K})$	Borel σ-algebra, page 17
B	Process of bridges, page 17
b	Boundary condition, page 21
χ	Magnetic susceptibility, page 136
$\hat{\partial}\Lambda$	(Inner) boundary, page 16
Δ	Process of cuts, page 109
δ	Intensity of D, page 17
$d(v)$	Number of deaths in K_v, page 103
D_v	Deaths in K_v, page 103
$\partial\Lambda$	Outer boundary, page 16
$\partial\psi$	Weight of colouring ψ, page 97
D	Process of deaths, page 17
$\mathbb{E}$	Edge set of $\mathbb{L}$, page 13
$\mathrm{ev}(\psi)$	Set of 'even' points in ψ, page 97
E	Edge set of L, page 14
$E(D)$	Edge set of the graph $G(D)$, page 99
$\mathcal{F}$	Skorokhod σ-algebra on Ω, page 18
$\mathcal{F}_\Lambda$	Restricted σ-algebra, page 23

$\mathbb{F}$	The product $\mathbb{E} \times \mathbb{R}$, page 14
f	Free boundary condition, page 22
ϕ	Random-cluster measure, page 23
ϕ^0	Free random-cluster measure, page 70
ϕ^1	Wired random-cluster measure, page 70
Φ^b	Random-cluster measure on $\mathbb{X}$, page 152
F	Subset of $\mathbb{F}$, page 14
$\mathcal{G}$	σ-algebra for the Potts model, page 25
Γ	Ghost site, page 15
γ	Intensity of G, page 17
G	Process of ghost-bonds, page 17
$G(D)$	Discrete graph constructed from D, page 99
$\mathcal{H}$	Hilbert space $\bigotimes_{v \in V} \mathbb{C}^2$, page 9
$\mathbb{H}$	Hypergraph, page 151
I_i^v	Maximal subinterval of K_v, page 89
$J_{k,l}^e$	Element of $E(D)$, page 99
J_k^v	Subintervals of K bounded by deaths, page 98
$\mathbb{K}$	The product $\mathbb{V} \times \mathbb{R}$, page 14
K	Subset of $\mathbb{K}$, page 14
k_Λ^b	Number of connected components, page 22
Λ	Region, page 14
λ	Intensity of B, page 17
Λ°	Interior of the region Λ, page 16
$\mathbb{L}$	Infinite graph, page 13
$\overline{\Lambda}$	Closure of the region Λ, page 15
L	Finite subgraph of $\mathbb{L}$, page 14
μ	Law of space–time percolation, page 17
μ_δ	Law of D, page 17
μ_γ	Law of G, page 17
μ_λ	Law of B, page 17

$m(v)$	Number of intervals constituting K_v, page 89
$M_\Lambda^{b,\alpha}$	Finite-volume magnetization, page 80
M_+	Spontaneous magnetization, page 85
$M_{B,G}$	Uniform measure on colourings, page 97
$\mathcal{N}$	Potts model configuration space, page 25
$\mathcal{N}(D)$	Potts configurations permitted by D, page 25
ν	Potts configuration, page 25
ν'_x	$(\sigma_x + 1)/2$, page 81
$n(v, D)$	Number of death-free intervals in K_v, page 98
$\mathrm{odd}(\psi)$	Set of 'odd' points in ψ, page 97
Ω	Percolation configuration space, page 17
ω	Percolation configuration, page 17
ω_d	Dual configuration, page 69
π	Potts measure, page 26
$\mathbb{P}$	Edwards–Sokal coupling, page 28
ψ^A	Colouring, page 96
Ψ^b	Dual of Φ^{1-b}, page 152
ρ_c^β	Critical value, page 86
$\rho_\mathrm{c}(q)$	Percolation threshold, page 58
$r(\nu)$	The number of intersection points with W, page 116
Σ	Ising configuration space, page 29
σ	Ising- or Potts spin, page 46
$\Sigma(D)$	Ising configurations permitted by D, page 29
$\sigma^{(1)}$, $\sigma^{(3)}$	Pauli matrices, page 9
sf	'Side free' boundary condition, page 154
$\mathbb{S}_\beta$	Circle of circumference β, page 88
sw	'Side wired' boundary condition, page 154
S	Switching points, page 96
S_n	Region in $\mathbb{Z} \times \mathbb{R}$, page 153
$\mathcal{T}_\Lambda$	Events defined outside Λ, page 23

$\boldsymbol{\Theta}$	The pair $(\mathbb{K}, \mathbb{F})$, page 14
$\boldsymbol{\Theta}_\beta$	Finite-β space, page 17
τ^β	Two-point function, page 93
θ	Percolation probability, page 58
$\mathrm{tr}(\cdot)$	Trace, page 9
T_n	$S_n(n, 0)$, page 153
$\mathbb{V}$	Vertex set of $\mathbb{L}$, page 13
V	Vertex set of L, page 14
$V(D)$	Collection of maximal death-free intervals, page 98
$V_x(\omega)$	Element count of B, G or D, page 52
w	Wired boundary condition, page 22
$w^A(\xi)$	Weight of backbone, page 106
$\mathbb{W}$	Vertices of $\mathbb{H}$, page 151
W	Vertices $v \in V$ such that $K_v = \mathbb{S}$, page 97
$\xi(\psi)$	Backbone, page 105
$\mathbb{X}$	Product $\mathbb{L} \times \mathbb{R}$ for $\mathbb{L}$ star-like, page 151
$\mathbb{Y}$	Dual of $\mathbb{X}$, page 151
ζ^k	Part of a backbone, page 106
Z'	Ising partition function, page 89
Z_Λ^b	Random-cluster model partition function, page 23
Z_K	$E(\partial\psi^\varnothing)$, page 105

CHAPTER 1

Introduction and background

Many physical and mathematical systems undergo a *phase transition*, of which some of the following examples may be familiar to the reader: water boils at 100°C and freezes at 0°C; Erdős-Rényi random graphs produce a 'giant component' if and only if the edge-probability $p > 1/n$; and magnetic materials exhibit 'spontaneous magnetization' at temperatures below the Curie point. In physical terminology, these phenomena may be unified by saying that there is an 'order parameter' M (density, size of largest component, magnetization) which behaves non-analytically on the parameters of the system at certain points. In the words of Alan Sokal: "at a phase transition M may be discontinuous, or continuous but not differentiable, or 16 times differentiable but not 17 times"—any behaviour of this sort qualifies as a phase transition.

Since it is the example closest to the topic of this work, let us look at the case of spontaneous magnetization. For the moment we will stay on an entirely intuitive level of description. If one takes a piece of iron and places it in a magnetic field, one of two things will happen. When the strength of the external field is decreased to nought, the iron piece may retain magnetization, or it may not. Experiments confirm that there is a critical value T_{c} of the temperature T such that: if $T < T_{\mathrm{c}}$ there is a residual ('spontaneous') magnetization, and if $T > T_{\mathrm{c}}$ there is not. See Figure 1.1. Thus the order parameter $M_0(T)$ (residual magnetization) is non-analytic at $T = T_{\mathrm{c}}$ (and it turns out that the phase transition is of the 'continuous but not differentiable' variety, see Theorem 4.1.1). Can we account for this behaviour in terms of the

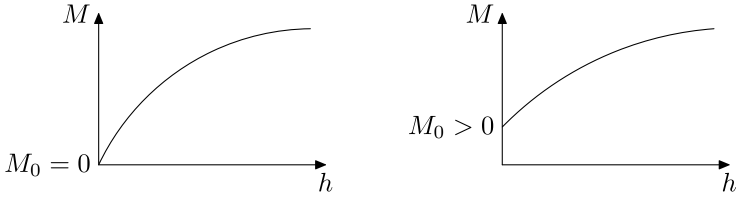

FIGURE 1.1. Magnetization M when $T > T_c$ (left) and when $T < T_c$ (right). The residual magnetization M_0 is zero at high temperature and positive at low temperature.

'microscopic' properties of the material, that is in terms of individual atoms and their interactions?

Considerable ingenuity has, since the 1920's and earlier, gone in to devising mathematical models that strike a good balance between three desirable properties: physical relevance, mathematical (or computational) tractability, and 'interesting' critical behaviour. A whole arsenal of mathematical tools, rigorous as well as non-rigorous, have been developed to study such models. One of the most exciting aspects of the mathematical theory of phase transition is the abundance of amazing conjectures originating in the physics literature; attempts by mathematicians to 'catch up' with the physicists and rigorously prove some of these conjectures have led to the development of many beautiful mathematical theories. As an example of this one can hardly at this time fail to mention the theory of SLE which has finally established some long-standing conjectures in two-dimensional models [**81, 82**].

This work is concerned with the representation of physical models using stochastic geometry, in particular what are called percolation-, FK-, and random-current representations. A major focus of this work is on the *quantum Ising model* of a magnet (described below); on the way to studying this model we will also study 'space–time' random-cluster (or FK) and Potts models. Although a lot of attention has been paid to the graphical representation of classical Ising-like models, this is less true for quantum models, hence the current work. Our methods are rigorous, and mainly utilize the mathematical theory of probability. Although graphical methods may give less far-reaching results than the ‘exact’ methods favoured by mathematical physicists, they are also more robust to changes in geometry: towards the end of this work we will see some examples of results on high-dimensional, and ‘complex one-dimensional’, models where exact methods cannot be used.

1.1. Classical models

1.1.1. The Ising model. The best-known, and most studied, model in statistical physics is arguably the Ising model of a magnet, given as follows. One represents the magnetic material at hand by a finite graph $L = (V, E)$ where the vertices V represent individual particles (or atoms) and an edge is placed between particles that interact (‘neighbours’). A ‘state’ is an assignment of the numbers $+1$ and -1 to the vertices of L; these numbers are usually called ‘spins’. The set $\{-1, +1\}^V$ of such states is denoted Σ, and an element of Σ is denoted σ. The model has two parameters, namely the temperature $T \geq 0$ and the external magnetic field $h \geq 0$. The probability of seeing a particular configuration σ is then proportional to the number

$$\exp\Big(\beta \sum_{e=xy\in E} \sigma_x\sigma_y + \beta h \sum_{x\in V} \sigma_x\Big). \tag{1.1.1}$$

Here $\beta = (k_{\mathrm{B}}T)^{-1} > 0$ is the ‘inverse temperature’, where k_{B} is a constant called the ‘Boltzmann constant’. Intuitively, the number (1.1.1) is bigger if more spins agree, since $\sigma_x\sigma_y$ equals $+1$ if $\sigma_x = \sigma_y$ and -1 otherwise; similarly it is bigger if more spins ‘align with the external field’ in that $\sigma_x = +1$. In particular, the spins at different sites are not in general statistically independent, and the structure of this dependence is subtly influenced by the geometry of the graph L. This is what makes the model interesting.

The Ising model was introduced around 1925 (not originally by but *to* Ising by his thesis advisor Lenz) as a candidate for a model that exhibits a phase transition [**59**]. It turns out that the magnetization M, which is by definition the expected value of the spin at some given vertex, behaves (in the limit as the graph L approaches an infinite graph $\mathbb{L}$) non-analytically on the parameters β, h at a certain point $(\beta = \beta_{\mathrm{c}}, h = 0)$ in the (β, h)-plane.

The Ising model is therefore the second-simplest physical model with an interesting phase transition; the simplest such model is the following. Let $\mathbb{L} = (\mathbb{V}, \mathbb{E})$ be an infinite, but countable, graph. (The main example to bear in mind is the lattice $\mathbb{Z}^d$ with nearest-neighbour edges.) Let $p \in [0, 1]$ be given, and examine each edge in turn, keeping it with probability p and deleting it with probability $1-p$, these choices being independent for different edges. The resulting subgraph of $\mathbb{L}$ is typically denoted ω, and the set of such subgraphs is denoted Ω. The graph ω will typically not be connected, but will break into a number of connected components. Is one of these components infinite? The model possesses a phase transition in the sense that the probability that there exists an infinite component jumps from 0 to 1 at a critical value p_{c} of p.

This model is called *percolation*. It was introduced by Broadbent and Hammersley in 1957 as a model for a porous material immersed in a fluid [**17**]. Each edge in $\mathbb{E}$ is then thought of as a small hole which may be open (if the corresponding edge is present in ω) or closed to the passage of fluid. The existence of an infinite component corresponds to the fluid being able to penetrate from the surface to the 'bulk' of the material. Even though we are dealing here with a countable set of independent random variables, the theory of percolation is a genuine departure from the traditional theory of sequences of independent variables, again since geometry plays such a vital role.

1.1.2. The random-cluster model. At first sight, the Ising- and percolation models seem unrelated, but they have a common generalization. On a finite graph $L = (V, E)$, the percolation configuration ω has probability

$$p^{|\omega|}(1-p)^{|E\setminus\omega|}, \tag{1.1.2}$$

where $|\cdot|$ denotes the number of elements in a finite set, and we have identified the subgraph ω with its edge-set. A natural way to generalize (1.1.2) is to consider absolutely continuous measures, and it turns out that the distributions defined by

$$\phi(\omega) := p^{|\omega|}(1-p)^{|E\setminus\omega|}\frac{q^{k(\omega)}}{Z} \tag{1.1.3}$$

are particularly interesting. Here $q > 0$ is an additional parameter, $k(\omega)$ is the number of connected components in ω, and Z is a normalizing constant. The 'cluster-weighting factor' $q^{k(\omega)}$ has the effect of skewing the distribution in favour of few large components (if $q < 1$) or many small components (if $q > 1$), respectively. This new model is called the random-cluster model, and it contains percolation as the special case $q = 1$. By considering limits as $L \uparrow \mathbb{L}$, one may see that the random-cluster models (with $q \geq 1$) also have a phase transition in the same sense as the percolation model, with associated critical probability $p_\mathrm{c} = p_\mathrm{c}(q)$.

There is also a natural way to generalize the Ising model. This is easiest to describe when $h = 0$, which we assume henceforth. The relative weights (1.1.1) depend (up to a multiplicative constant) only on the number of adjacent vertices with equal spin, so the same model is obtained by using the weights

$$\exp\Big(2\beta \sum_{e=xy\in E} \delta_{\sigma_x,\sigma_y}\Big), \tag{1.1.4}$$

where $\delta_{a,b}$ is 1 if $a = b$ and 0 otherwise. (Note that $\delta_{\sigma_x,\sigma_y} = (\sigma_x\sigma_y + 1)/2$.) In this formulation it is natural to consider the more general model when the spins σ_x can take not only two, but $q = 2, 3, \ldots$ different values, that is each $\sigma_x \in \{1, \ldots, q\}$. Write π for the corresponding distribution on spin configurations; the resulting model is called the q-state Potts model. It turns out that the q-state Potts models is closely related to the random-cluster model, one manifestation of this being the following. (See [**35**], or [**50**, Chapter 1] for a modern proof.)

THEOREM 1.1.1. *If $q \geq 2$ is an integer and $p = 1 - e^{-2\beta}$ then for all $x, y \in V$*

$$\pi(\sigma_x = \sigma_y) - \frac{1}{q} = \Big(1 - \frac{1}{q}\Big)\phi(x \leftrightarrow y)$$

Here $\pi(\sigma_x = \sigma_y)$ denotes the probability that, in the Potts model, the spin at x takes the same value as the spin at y. Similarly, $\phi(x \leftrightarrow y)$ is the probability that, in the random-cluster model, x and y lie in the same component of ω. Since the right-hand-side concerns a typical graph-theoretic property (connectivity), the random-cluster model is called a 'graphical representation' of the Potts model. The close relationship between the random-cluster and Potts models was unveiled by Fortuin and Kasteleyn during the 1960's and 1970's in a series of papers including [**35**]. The random-cluster model is therefore sometimes called the 'FK-representation'. In other words, Theorem 1.1.1 says that the correlation between distant spins in the Potts model is translated to the existence of paths between the sites in the random-cluster model. Using this and related facts one can deduce many important things about the phase transition of the Potts model by studying the random-cluster model. This can be extremely useful since the random-cluster formulation allows geometric arguments that are not present in the Potts model. Numerous examples of this may be found in [**50**]; very recently, in [**82**], the 'loop' version of the random-cluster model was also used to prove conformal invariance for the two-dimensional Ising model, a major breakthrough in the theory of the Ising model.

1.1.3. Random-current representation. For the Ising model there exists also another graphical representation, distinct from the random-cluster model. This is called the 'random-*current* representation' and was developed in a sequence of papers in the late 1980's [**1, 3, 5**], building on ideas in [**48**]. These papers answered many questions for the Ising model on $\mathbb{L} = \mathbb{Z}^d$ with $d \geq 2$ that are still to this day unanswered for general Potts models. Cast in the language of the $q = 2$ random-cluster model, these questions include the following [answers in square brackets].

- If $p < p_\mathrm{c}$, is the expected size of a component finite or infinite? [Finite.]
- If $p < p_\mathrm{c}$, do the connection probabilities $\phi(x \leftrightarrow y)$ go to zero exponentially fast as $|x - y| \to \infty$? [Yes.]
- At $p = p_\mathrm{c}$, does $\phi(x \leftrightarrow y)$ go to zero exponentially fast as $|x - y| \to \infty$? [No.]

In fact, even more detailed information could be obtained, especially in the case $d \geq 4$, giving at least partial answer to the question

- How does the magnetization $M = M(\beta, h)$ behave as the critical point $(\beta_\mathrm{c}, 0)$ is approached?

It is one of the main objectives of this work to develop a random-current representation for the *quantum* Ising model (introduced in the next section), and answer the above questions also for that model.

Here is a very brief sketch of the random-current representation of the classical Ising model. Of particular importance is the normalizing constant or 'partition function' that makes (1.1.1) a probability distribution, namely

$$\sum_{\sigma \in \Sigma} \exp\Big(\beta \sum_{e = xy \in E} \sigma_x \sigma_y\Big) \tag{1.1.5}$$

(we assume that $h = 0$ for simplicity). We rewrite (1.1.5) using the following steps. Factorize the exponential in (1.1.5) as a product over

$e = xy \in E$, and then expand each factor as a Taylor series in the variable $\beta\sigma_x\sigma_y$. By interchanging sums and products we then obtain a weighted sum over vectors $\underline{m}$ indexed by E of a quantity which (by $\pm$ symmetry) is zero if a certain condition on $\underline{m}$ fails to be satisfied, and a positive constant otherwise. The condition on $\underline{m}$ is that: for each $x \in V$ the sum over all edges e adjacent to x of m_e is a multiple of 2.

Once we have rewritten the partition function in this way, we may interpret the weights on $\underline{m}$ as probabilities. It follows that the partition function is (up to a multiplicative constant) equal to the probability that the random graph $G_{\underline{m}}$ with each edge e replaced by m_e parallel edges is *even* in that each vertex has even total degree. Similarly, other quantities of interest may be expressed in terms of the probability that only a given set of vertices fail to have even degree in $G_{\underline{m}}$; for example, the correlation between σ_x and σ_y for $x, y \in V$ is expressed in terms of the probability that only x and y fail to have even degree. By elementary graph theory, the latter event implies the existence of a path from x to y in $G_{\underline{m}}$. By studying connectivity in the above random graphs with restricted degrees one obtains surprisingly detailed information about the Ising model. Much more will be said about this method in Chapter 3, see for example the Switching Lemma (Theorem 3.3.2) and its applications in Section 3.3.2.

1.2. Quantum models and space–time models

There is a version of the Ising model formulated to meet the requirements of quantum theory, introduced in [**68**]. We will only be concerned with the *transverse field* quantum Ising model. Its definition and physical motivation bear a certain level of complexity which it is beyond the scope of this work to justify in an all but very cursory manner. One is given, as before, a finite graph $L = (V, E)$, and one is interested in the properties of certain matrices (or 'operators') acting on the Hilbert space $\mathcal{H} = \bigotimes_{v\in V} \mathbb{C}^2$. The set $\Sigma = \{-1, +1\}^V$ may now be identified with a basis for $\mathcal{H}$, defined by letting each factor $\mathbb{C}$ in the tensor product have basis consisting of the two vectors $|+\rangle := \binom{1}{0}$ and $|-\rangle := \binom{0}{1}$. We write $|\sigma\rangle = \bigotimes_{v\in V} |\sigma_v\rangle$ for these basis vectors. In addition to the inverse temperature $\beta > 0$, one is given parameters $\lambda, \delta > 0$, interpreted as spin-coupling and transverse field intensities, respectively. The latter specify the *Hamiltonian*

$$H = -\tfrac{1}{2}\lambda \sum_{e=uv\in E} \sigma_u^{(3)}\sigma_v^{(3)} - \delta \sum_{v\in V} \sigma_v^{(1)}, \tag{1.2.1}$$

where the 'Pauli spin-$\frac{1}{2}$ matrices' are given as

$$\sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.2.2}$$

and $\sigma_v^{(i)}$ acts on the copy of $\mathbb{C}^2$ in $\mathcal{H}$ indexed by $v \in V$. Intuitively, the matrices $\sigma^{(1)}$ and $\sigma^{(3)}$ govern spins in 'directions' 1 and 3 respectively (there is another matrix $\sigma^{(2)}$ which does not feature in this model). The external field is called 'transverse' since it acts in a different 'direction' to the internal interactions. When $\delta = 0$ this model therefore reduces to the (zero-field) classical Ising model (this will be obvious from the space–time formulation below).

The basic operator of interest is $e^{-\beta H}$, which is thus a (Hermitian) matrix acting on $\mathcal{H}$; one usually normalizes it and studies instead the matrix $e^{-\beta H}/\mathrm{tr}(e^{-\beta H})$. Here the *trace* of the Hermitian matrix A is defined as

$$\mathrm{tr}(A) = \sum_{\sigma\in\Sigma} \langle\sigma|A|\sigma\rangle,$$

where $\langle\sigma|$ is the adjoint, or conjugate transpose, of the column vector $|\sigma\rangle$, and we are using the usual matrix product. An eigenvector of $e^{-\beta H}/\mathrm{tr}(e^{-\beta H})$ may be thought of as a 'state' of the system, and is now a 'mixture' (linear combination) of classical states in Σ; the corresponding eigenvalue (which is real since the matrix is Hermitian) is related to the 'energy level' of the state.

In this work we will not be working directly with this formulation of the quantum Ising model, but a (more probabilistic) 'space–time' formulation, which we describe briefly now. It is by now standard that many properties of interest in the transverse field quantum Ising model may be studied by means of a 'path integral' representation, which maps the model onto a type of classical Ising model on the continuous space $V \times [0, \beta]$. (To be precise, the endpoints of the interval $[0, \beta]$ must be identified for this mapping to hold.) This was first used in [**45**], but see also for example [**7, 8, 20, 24, 54, 74**] and the recent surveys to be found in [**52, 58**]. Precise definitions will be given in Chapter 2, but in essence we must consider piecewise constant *functions* $\sigma : V \times [0, \beta] \to \{-1, +1\}$, which are random and have a distribution reminiscent of (1.1.1). The resulting model is called the 'space–time Ising model'. As for the classical case, it is straightforward to generalize this to a space–time *Potts* model with $q \geq 2$ possible spin values, and also to give a graphical representation of these models in terms of a space–time random-cluster model. Although the partial continuity of the underlying geometry poses several technical difficulties, the corresponding theory is very similar to the classical random-cluster theory. The most important basic properties of the models are developed in detail in Chapter 2. On taking limits as L and/or β become infinite, one may speak of the existence of unbounded connected components, and one finds (when $\beta = \infty$) that there is a critical dependence on the ratio $\rho = \lambda/\delta$ of the probability of seeing such a component. One may also develop, as we do in Chapter 3, a type of random-current representation of the space–time Ising model which allows us to deduce many facts about the critical behaviour of the *quantum* Ising model.

Other models of space–time type have been around for a long time in the probability literature. Of these the most relevant for us is the *contact process* (more precisely, its graphical representation), see for example [**69, 70**] and references therein. In the contact process, one imagines individuals placed on the vertices of a graph, such as $\mathbb{Z}^2$. Initially, some of these individuals may be infected with a contagious disease. As time passes, the individuals themselves stay fixed but the disease may spread: individuals may be infected by their neighbours, or by a 'spontaneous' infection. Infected individuals may recover spontaneously. Infections and recoveries are governed by Poisson processes, and depending on the ratio of infection rate to recovery rate the infection may or may not persist indefinitely. The contact model may be regarded as the $q = 1$ or 'independent' case of the space–time random-cluster model (one difference is that we in the space–time model regard time as 'undirected'). Thus one may get to general space–time random-cluster models in a manner reminiscent of the classical case, by skewing the distribution by an appropriate 'cluster weighting factor'. This approach will be treated in detail in Section 2.1.

1.3. Outline

A brief outline of the present work follows. In Chapter 2, the space–time random-cluster and Potts models are defined. As for the classical theory, one of the most important tools is *stochastic comparison*, or the ability to compare the probabilities of certain events under measures with different parameters. A number of results of this type are presented in Section 2.2. We then consider the issue of defining random-cluster and Potts measures on infinite graphs, and of their phase transitions. We etablish the existence of weak limits of Potts and random-cluster measures as $L \uparrow \mathbb{L}$, and introduce the central question of when there is a unique such limit. It turns out that this question is closely related to the question if there can be an unbounded connected component; this helps us to define a critical value $\rho_c(q)$. In general not a lot can be said about the precise value of $\rho_c(q)$, but in the case when $\mathbb{L} = \mathbb{Z}$ there are additional geometric (duality) arguments that can be used to show that $\rho_c(q) \geq q$.

Chapter 3 deals exclusively with the quantum Ising model in its space–time formulation. We develop the 'random parity representation', which is the space–time analog of the random-current representation, and the tools associated with it, most notably the switching lemma. This representation allows us to represent truncated correlation functions in terms of single geometric events. Since truncated correlations are closely related to the derivatives of the magnetization M, we can use this to prove a number of inequalities between the different partial derivatives of M, along the lines of [**3**]. Integrating these differential inequalities gives the information on the critical behaviour that was referred to in Section 1.1.3, namely the sharpness of the phase transition, bounds on critical exponents, and the vanishing of the mass gap. Chapter 3 (as well as Section 4.1) is joint work with Geoffrey Grimmett, and appears in the article *The phase transition of the quantum Ising model is sharp* [**15**], published by the Journal of Statistical Physics.

Finally, in Chapter 4, we combine the results of Chapter 3 with the results of Chapter 2 in some concrete cases. Using duality arguments we prove that the critical ratio $\rho_c(2) = 2$ in the case $\mathbb{L} = \mathbb{Z}$. We then develop some further geometric arguments for the random-cluster representation to deduce that the critical ratio is the same as for $\mathbb{Z}$ on a much larger class of '$\mathbb{Z}$-like' graphs. These arguments (Section 4.2) appear in the article *Critical value of the quantum Ising model on star-like graphs* [**14**], published in the Journal of Statistical Physics. We conclude by describing some future directions for research in this area.

CHAPTER 2

Space–time models: random-cluster, Ising, and Potts

Summary. We provide basic definitions and facts pertaining to the space–time random-cluster and -Potts models. Stochastic inequalities, a major tool in the theory, are proved carefully, and the notion of phase transition is defined. We also introduce the notion of graphical duality.

2.1. Definitions and basic facts

The space–time models we consider live on the product of a graph with the real line. To define space–time random-cluster and Potts models we first work on bounded subsets of this product space, and then pass to a limit. The continuity of $\mathbb{R}$ makes the definitions of boundaries and boundary conditions more delicate than in the discrete case.

2.1.1. Regions and their boundaries. Let $\mathbb{L} = (\mathbb{V}, \mathbb{E})$ be a countably infinite, connected, undirected graph, which is *locally finite* in that each vertex has finite degree. Here $\mathbb{V}$ is the vertex set and $\mathbb{E}$ the edge set. For simplicity we assume that $\mathbb{L}$ does not have multiple edges or loops. An edge of $\mathbb{L}$ with endpoints u, v is denoted by uv. We write $u \sim v$ if $uv \in \mathbb{E}$. The main example to bear in mind is when $\mathbb{L} = \mathbb{Z}^d$ is the d-dimensional lattice, with edges between points that differ by one in exactly one coordinate.

Let

$$\mathbb{K} := \bigcup_{v\in\mathbb{V}} (v \times \mathbb{R}), \qquad \mathbb{F} := \bigcup_{e\in\mathbb{E}} (e \times \mathbb{R}), \tag{2.1.1}$$

$$\boldsymbol{\Theta} := (\mathbb{K}, \mathbb{F}). \tag{2.1.2}$$

Let $L = (V, E)$ be a finite connected subgraph of $\mathbb{L}$. In the case when $\mathbb{L} = \mathbb{Z}^d$, the main example for L is the 'box' $[-n, n]^d$. For each $v \in V$, let K_v be a finite union of (disjoint) bounded intervals in $\mathbb{R}$. No assumption is made whether the constituent intervals are open, closed, or half-open. For $e = uv \in E$ let $F_e := K_u \cap K_v \subseteq \mathbb{R}$. Let

$$K := \bigcup_{v\in V} (v \times K_v), \quad F := \bigcup_{e\in E} (e \times F_e). \tag{2.1.3}$$

We define a *region* to be a pair

$$\Lambda = (K, F) \tag{2.1.4}$$

for L, K and F defined as above. We will often think of Λ as a subset of $\boldsymbol{\Theta}$ in the natural way, see Figure 2.1. Since a region $\Lambda = (K, F)$ is completely determined by the set K, we will sometimes abuse notation by writing $x \in \Lambda$ when we mean $x \in K$, and think of subsets of K (respectively, $\mathbb{K}$) as subsets of Λ (respectively, $\boldsymbol{\Theta}$).

An important type of a region is a *simple region*, defined as follows. For L as above, let $\beta > 0$ and let K and F be given by letting each $K_v = [-\beta/2, \beta/2]$. Thus

$$K = K(L, \beta) := \bigcup_{v\in V} (v \times [-\beta/2, \beta/2]), \tag{2.1.5}$$

$$F = F(L, \beta) := \bigcup_{e\in E} (e \times [-\beta/2, \beta/2]), \tag{2.1.6}$$

$$\Lambda = \Lambda(L, \beta) := (K, F). \tag{2.1.7}$$

Note that in a simple region, the intervals constituting K are all closed. (Later, in the quantum Ising model of Chapter 3, the parameter β will be interpreted as the 'inverse temperature'.)

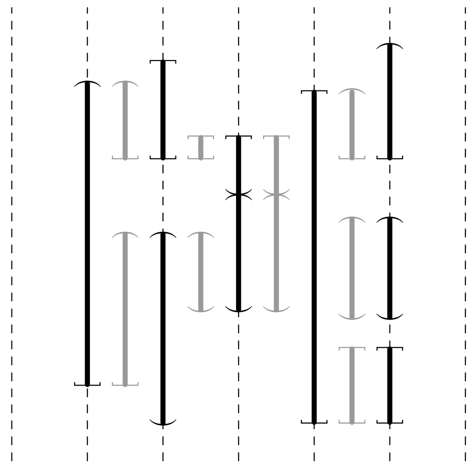

FIGURE 2.1. A region $\Lambda = (K, F)$ as a subset of $\mathbf{\Theta}$ when $\mathbb{L} = \mathbb{Z}$. Here $\mathbb{K}$ is drawn dashed, K is drawn bold black, and F is drawn bold grey. An endpoint of an interval in K (respectively, F) is drawn as a square bracket if it is included in K (respectively, F) or as a rounded bracket if it is not.

Introduce an additional point Γ external to $\mathbf{\Theta}$, to be interpreted as a 'ghost-site' or 'point at infinity'; the use of Γ will be explained below, when the space–time random-cluster and Potts models are defined. Write $\mathbf{\Theta}^\Gamma = \mathbf{\Theta} \cup \{\Gamma\}$, $\mathbb{K}^\Gamma = \mathbb{K} \cup \{\Gamma\}$, and similarly for other notation.

We will require two distinct notions of boundary for regions Λ. For $I \subseteq \mathbb{R}$ we denote the closure and interior of I by $\overline{I}$ and I°, respectively. For Λ a region as in (2.1.4), define the *closure* to be the region $\overline{\Lambda} = (\overline{K}, \overline{F})$ given by

$$\overline{K} := \bigcup_{v \in V} (v \times \overline{K}_v), \quad \overline{F} := \bigcup_{e \in E} (e \times \overline{F}_e); \tag{2.1.8}$$

similarly define the *interior* of Λ to be the region $\Lambda^\circ = (K^\circ, F^\circ)$ given by

$$K^\circ := \bigcup_{v \in V} (v \times K_v^\circ), \quad F^\circ := \bigcup_{e \in E} (e \times F_e^\circ). \tag{2.1.9}$$

Define the *outer boundary* $\partial\Lambda$ of Λ to be the union of $\overline{K} \setminus K^\circ$ with the set of points $(u,t) \in K$ such that $u \sim v$ for some $v \in \mathbb{V}$ such that $(v,t) \notin K$. Define the *inner boundary* $\hat{\partial}\Lambda$ of Λ by $\hat{\partial}\Lambda := (\partial\Lambda) \cap K$. The inner boundary of Λ will often simply be called the *boundary* of Λ. Note that if x is an endpoint of a closed interval in K_v, then $x \in \partial\Lambda$ if and only if $x \in \hat{\partial}\Lambda$, but if x is an endpoint of an open interval in K_v, then $x \in \partial\Lambda$ but $x \notin \hat{\partial}\Lambda$. In particular, if Λ is a simple region then $\partial\Lambda = \hat{\partial}\Lambda$. A word of caution: this terminology is nonstandard, in that for example the interior and the boundary of a region, as defined above, need not be disjoint. See Figure 2.2. We define $\partial\Lambda^\Gamma = \partial\Lambda \cup \{\Gamma\}$ and $\hat{\partial}\Lambda^\Gamma = \hat{\partial}\Lambda \cup \{\Gamma\}$.

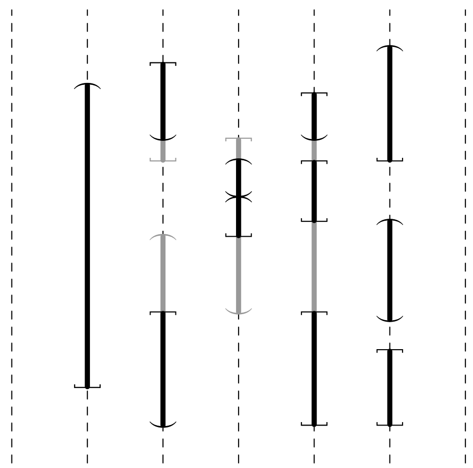

FIGURE 2.2. The (inner) boundary $\hat{\partial}\Lambda$ of the region Λ of Figure 2.1 is marked black, and $K \setminus \hat{\partial}\Lambda$ is marked grey. An endpoint of an interval in $\hat{\partial}\Lambda$ is drawn as a square bracket if it lies in $\hat{\partial}\Lambda$ and as a round bracket otherwise.

A subset S of $\mathbb{K}$ will be called *open* if it equals a union of the form

$$\bigcup_{v \in \mathbb{V}} (v \times U_v),$$

where each $U_v \subseteq \mathbb{R}$ is an open set. Similarly for subsets of $\mathbb{F}$. The σ-algebra generated by this topology on $\mathbb{K}$ (respectively, on $\mathbb{F}$) will be denoted $\mathcal{B}(\mathbb{K})$ (respectively, $\mathcal{B}(\mathbb{F})$) and will be referred to as the Borel σ-algebra.

Occasionally, especially in Chapter 3, we will in place of $\mathbf{\Theta}$ be using the finite β space $\mathbf{\Theta}_\beta = (\mathbb{K}_\beta, \mathbb{F}_\beta)$ given by

$$\mathbb{K}_\beta := \bigcup_{v\in\mathbb{V}} (v \times [-\beta/2, \beta/2]), \qquad \mathbb{F}_\beta := \bigcup_{e\in\mathbb{E}} (e \times [-\beta/2, \beta/2]). \tag{2.1.10}$$

This is because in the quantum Ising model β is thought of as 'inverse temperature', and then both $\beta < \infty$ (positive temperature) and $\beta = \infty$ (ground state) are interesting.

In what follows, proofs will often, for simplicity, be given for simple regions only; proofs for general regions will in these cases be straightforward adaptations. We will frequently be using integrals of the forms

$$\int_K f(x)\,dx \qquad \text{and} \qquad \int_F g(e)\,de. \tag{2.1.11}$$

These are to be interpreted, respectively, as

$$\sum_{v\in V} \int_{K_v} f(v,t)\,dt, \qquad \sum_{e\in E} \int_{F_e} g(e,t)\,dt. \tag{2.1.12}$$

If A is an event, we will write $\mathbb{1}_A$ or $\mathbb{1}\{A\}$ for the indicator function of A.

2.1.2. The space–time percolation model. Write $\mathbb{R}_+ = [0,\infty)$ and let $\lambda : \mathbb{F} \to \mathbb{R}_+$, $\delta : \mathbb{K} \to \mathbb{R}_+$, and $\gamma : \mathbb{K} \to \mathbb{R}_+$ be bounded functions. We assume throughout that λ, δ, γ are all Borel-measurable. We retain the notation λ, δ, γ for the restrictions of these functions to Λ, given in (2.1.4). Let Ω denote the set of triples $\omega = (B, D, G)$ of countable subsets $B \subseteq \mathbb{F}$, $D, G \subseteq \mathbb{K}$; these triples will often be called *configurations*. Let μ_λ, μ_δ, μ_γ be the probability measures associated with independent Poisson processes on $\mathbb{K}$ and $\mathbb{F}$ as appropriate, with respective intensities λ, δ, γ. Let μ denote the probability measure $\mu_\lambda \times \mu_\delta \times \mu_\gamma$ on Ω. Note that, with μ-probability 1, each of the countable sets B, D, G contains no accumulation points; we call such a set *locally finite*. We will sometimes write $B(\omega), D(\omega), G(\omega)$ for clarity.

Remark 2.1.1. *For simplicity of notation we will frequently overlook events of probability zero, and will thus assume for example that Ω contains only triples (B, D, G) of locally finite sets, such that no two points in $B \cup D \cup G$ have the same $\mathbb{R}$-coordinates.*

For the purpose of defining a metric and a σ-algebra on Ω, it is convenient to identify each $\omega \in \Omega$ with a collection of step functions. To be definite, we then regard each $\omega \cap (v \times \mathbb{R})$ and each $\omega \cap (e \times \mathbb{R})$ as an *increasing, right-continuous* step function, which equals 0 at $(v, 0)$ or $(e, 0)$ respectively. There is a metric on the space of right-continuous step functions on $\mathbb{R}$, called the Skorokhod metric, which may be extended in a straightforward manner to a metric on Ω. Details may be found in Appendix A, alternatively see [**11**], and [**31**, Chapter 3] or [**71**, Appendix 1]. We let $\mathcal{F}$ denote the σ-algebra on Ω generated by the Skorokhod metric. Note that the metric space Ω is *Polish*, that is to say separable (it contains a countable dense subset) and complete (Cauchy sequences converge).

However, in the context of percolation, here is how we usually want to think about elements of Ω. Recall the 'ghost site' or 'point at infinity' Γ. Elements of D are thought of as 'deaths', or missing points; elements of B as 'bridges' or line segments between points (u, t) and (v, t), $uv \in \mathbb{E}$; and elements of G as 'bridges to Γ'. See Figure 2.3 for an illustration of this. Elements of B will sometimes be referred to as *lattice bonds* and elements of G as *ghost bonds*. A lattice bond (uv, t) is said to have *endpoints* (u, t) and (v, t); a ghost bond at (v, t) is said to have endpoints (v, t) and Γ.

For two points $x, y \in \mathbb{K}$ we say that there is a *path*, or an *open path*, in ω between x and y if there is a sequence $(x_1, y_1), \ldots, (x_n, y_n)$ of pairs of elements of $\mathbb{K}$ satisfying the following:

- Each pair (x_i, y_i) consists either of the two endpoints of a single lattice bond (that is, element of B) or of the endpoints in $\mathbb{K}$ of two distinct ghost bonds (that is, elements of G),
- Writing $y_0 = x$ and $x_{n+1} = y$, we have that for all $0 \leq i \leq n$, there is a $v_i \in \mathbb{V}$ such that $y_i, x_{i+1} \in (v_i \times \mathbb{R})$,
- For each $0 \leq i \leq n$, the (closed) interval in $v_i \times \mathbb{R}$ with endpoints y_i and x_{i+1} contains no elements of D.

In words, there is a path between x and y if y can be reached from x by traversing bridges and ghost-bonds, as well as subintervals of $\mathbb{K}$ which do not contain elements of D. For example, in Figure 2.3 there is an open path between any two points on the line segments that are drawn bold. By convention, there is always an open path from x to itself. We say that there is a path between $x \in \mathbb{K}$ and Γ if there is a $y \in G$ such that there is a path between x and y. Sometimes we say that $x, y \in \mathbb{K}^\Gamma$ are *connected* if there is an open path between them. Intuitively, elements of D break connections on vertical lines, and elements of B create connections between neighbouring lines. The use of Γ, and the process G, is to provide a 'direct link to ∞'; two points that are joined to Γ are automatically joined to eachother.

We write $\{x \leftrightarrow y\}$ for the event that there is an open path between x and y. We say that two subsets $A_1, A_2 \subseteq \mathbb{K}$ are connected, and write $A_1 \leftrightarrow A_2$, if there exist $x \in A_1$ and $y \in A_2$ such that $x \leftrightarrow y$. For a region Λ, we say that there is an open path between x, y *inside* Λ if y can be reached from x by traversing death-free line segments, bridges, and ghost-bonds that all lie in Λ. Open paths *outside* Λ are defined similarly.

Definition 2.1.2. *With the above interpretation, the measure μ on $(\Omega, \mathcal{F})$ is called the space–time percolation measure on $\boldsymbol{\Theta}$ with parameters λ, δ, γ.*

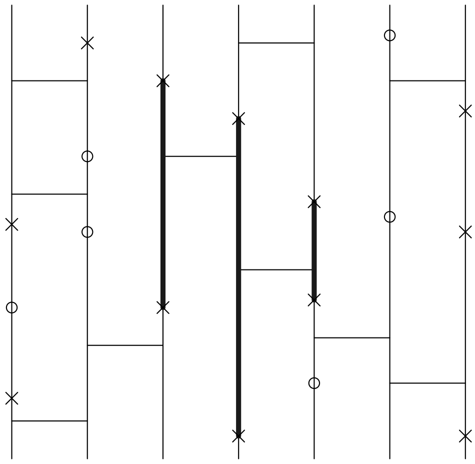

Figure 2.3. Part of a configuration ω when $\mathbb{L} = \mathbb{Z}$. Deaths are marked as crosses and bridges as horizontal line segments; the positions of ghost-bonds are marked as small circles. One of the connected components of ω is drawn bold.

The measure μ coincides with the law of the graphical representation of a contact process with spontaneous infections, see [**6, 11**]. In this work, however, we regard 'time' as undirected, and thus think of ω as a geometric object rather than as a process evolving in time.

2.1.3. Boundary conditions. Any $\omega \in \Omega$ breaks into *components*, where a component is by definition the maximal subset of $\mathbb{K}^{\Gamma}$ which can be reached from a given point in $\mathbb{K}^{\Gamma}$ by traversing open paths. See Figure 2.3. One may imagine $\mathbb{K}$ as a collection of infinitely long strings, which are cut at deaths, tied together at bridges, and also tied to Γ at ghost-bonds. The components are the pieces of string that 'hang together'. The random-cluster measure, which is defined in the next subsection, is obtained by 'skewing' the percolation measure μ in favour of either many small, or a few big, components. Since the total number of components in a typical ω is infinite, we must first, in order to give an analytic definition, restrict our attention to the number of components which intersect a fixed region Λ. We consider a number of different rules for counting those components which intersect the boundary of Λ. Later we will be interested in limits as the region Λ grows, and whether or not these 'boundary conditions' have an effect on the limit.

Let $\Lambda = (K, F)$ be a region. We define a *random-cluster boundary condition* b to be a finite nonempty collection $b = \{P_1, \ldots, P_m\}$, where the P_i are disjoint, nonempty subsets of $\hat{\partial}\Lambda^\Gamma$, such that each $P_i \setminus \{\Gamma\}$ is a finite union of intervals. (These intervals may be open, closed, or half-open, and may consist of a single point.) We require that Γ lies in one of the P_i, and by convention we will assume that $\Gamma \in P_1$. Note that the union of the P_i will in general be a proper subset of $\hat{\partial}\Lambda^\Gamma$. For $x, y \in \Lambda^\Gamma$ we say that $x \leftrightarrow y$ *with respect to* b if there is a sequence $x_1, \ldots, x_l$ (with $0 \leq l \leq m$) such that

- Each $x_j \in P_{i_j}$ for some $0 \leq i_j \leq m$;
- There are open paths inside Λ from x to x_1 and from x_l to y;
- For each $j = 1, \ldots, l-1$ there is some point $y_j \in P_{i_j}$ such that there is a path inside Λ from y_j to x_{j+1}.

See Figure 2.4 for an example.

When Λ and b are fixed and $x, y \in \Lambda^\Gamma$, we will typically without mention use the symbol $x \leftrightarrow y$ to mean that there is a path between x and y in Λ with respect to b. Intuitively, each P_i is thought of as *wired together*; as soon as you reach one point $x_j \in P_{i_j}$ you automatically reach all other points $y_j \in P_{i_j}$. It is important in the definition that each P_i is a subset of the *inner* boundary $\hat{\partial}\Lambda^\Gamma$ and not $\partial\Lambda^\Gamma$.

Here are some important examples of random-cluster boundary conditions.

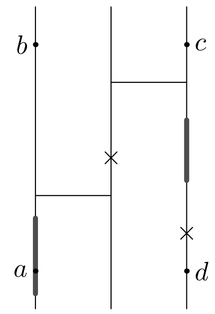

FIGURE 2.4. Connectivities with respect to the boundary condition $b = \{P_1\}$, where $P_1 \setminus \{\Gamma\}$ is the subset drawn bold. The following connectivities hold: $a \leftrightarrow b$, $a \leftrightarrow c$, $a \not\leftrightarrow d$. (This picture does not specify which endpoints of the subintervals of P_1 lie in P_1.)

- If $b = \{\hat{\partial}\Lambda^\Gamma\}$ then the entire boundary $\hat{\partial}\Lambda$ is wired together; we call this the *wired* boundary condition and denote it by $b = \mathrm{w}$;
- If $b = \{\{\Gamma\}\}$ then $x \leftrightarrow y$ with respect to b if and only if there is an open path between x, y inside Λ; we call this the *free* boundary condition, and denote it by $b = \mathrm{f}$.
- Given any $\tau \in \Omega$, the boundary condition $b = \tau$ is by definition obtained by letting the P_i consist of those points in $\hat{\partial}\Lambda^\Gamma$ which are connected by open paths of τ *outside* Λ.
- We may also impose a number of *periodic* boundary conditions on simple regions. One may then regard $[-\beta/2, \beta/2]$ as a circle by identifying its endpoints, and/or in the case $L = [-n, n]^d$ identify the latter with the torus $(\mathbb{Z}/[-n, n])^d$. Notation for periodic boundary conditions will be introduced when necessary. Periodic boundary conditions will be particularly important in the study of the quantum Ising model in Chapter 3.

For each boundary condition b on Λ, define the function $k_\Lambda^b : \Omega \to \{1, 2, \ldots, \infty\}$ to count the number of components of ω in Λ, counted with respect to the boundary condition b. There is a natural partial order on boundary conditions given by: $b' \geq b$ if $k_\Lambda^{b'}(\omega) \leq k_\Lambda^b(\omega)$ for all $\omega \in \Omega$. For example, for any boundary condition b we have $k_\Lambda^{\mathrm{w}} \leq k_\Lambda^b \leq k_\Lambda^{\mathrm{f}}$ and hence $\mathrm{w} \geq b \geq \mathrm{f}$. (Alternatively, $b' \geq b$ if b is a refinement of b'. Note that for $b = \tau \in \Omega$, this partial order agrees with the natural partial order on Ω, defined in Section 2.2.)

2.1.4. The space–time random-cluster model. For $q > 0$ and b a boundary condition, define the (random-cluster) *partition functions*

$$Z_\Lambda^b = Z_\Lambda^b(\lambda, \delta, \gamma, q) := \int_\Omega q^{k_\Lambda^b(\omega)}\, d\mu(\omega). \tag{2.1.13}$$

It is not hard to see that each $Z_\Lambda^b < \infty$.

Definition 2.1.3. *We define the* finite-volume random-cluster measure $\phi_\Lambda^b = \phi_{\Lambda;q,\lambda,\delta,\gamma}^b$ *on* Λ *to be the probability measure on* $(\Omega, \mathcal{F})$ *given by*

$$\frac{d\phi_\Lambda^b}{d\mu}(\omega) := \frac{q^{k_\Lambda^b(\omega)}}{Z_\Lambda^b}.$$

Thus, for any bounded, $\mathcal{F}$-measurable $f : \Omega \to \mathbb{R}$ we have that

$$\phi_\Lambda^b(f) = \frac{1}{Z_\Lambda^b} \int_\Omega f(\omega) q^{k_\Lambda^b(\omega)}\, d\mu(\omega). \tag{2.1.14}$$

We say that an event $A \in \mathcal{F}$ is *defined* on a pair (S, T) of subsets $S \subseteq \mathbb{K}$ and $T \subseteq \mathbb{F}$ if whenever $\omega \in A$, and $\omega' \in \Omega$ is such that $B(\omega) \cap T = B(\omega') \cap T$, $D(\omega) \cap S = D(\omega') \cap S$ and $G(\omega) \cap S = G(\omega') \cap S$, then also $\omega' \in A$. Let $\mathcal{F}_{(S,T)} \subseteq \mathcal{F}$ be the σ-algebra of events defined on (S, T). For $\Lambda = (K, F)$ a region we write $\mathcal{F}_\Lambda$ for $\mathcal{F}_{(K,F)}$; we abbreviate $\mathcal{F}_{(S,\varnothing)}$ and $\mathcal{F}_{(\varnothing,T)}$ by $\mathcal{F}_S$ and $\mathcal{F}_T$, respectively. Let $\mathcal{T}_{(S,T)} = \mathcal{F}_{(\mathbb{K}\setminus S, \mathbb{F}\setminus T)}$ denote the σ-algebra of events defined *outside* S and T. We call $A \in \mathcal{F}$ a *local* event if there is a region Λ such that $A \in \mathcal{F}_\Lambda$ (this is sometimes also called a *finite-volume* event or a *cylinder* event).

Note that the version of $d\phi_\Lambda^b / d\mu$ given in Definition 2.1.3 is $\mathcal{F}_\Lambda$-measurable; thus we may either regard ϕ_Λ^b as a measure on the full space $(\Omega, \mathcal{F})$, or, by restricting consideration to events in $\mathcal{F}_\Lambda$, as a measure on $(\Omega, \mathcal{F}_\Lambda)$.

For $\Delta = (K, F)$ a region and $\omega, \tau \in \Omega$, let

$$B_\Delta(\omega, \tau) = (B(\omega) \cap F) \cup (B(\tau) \cap (\mathbb{F} \setminus F)),$$

$$D_\Delta(\omega, \tau) = (D(\omega) \cap K) \cup (D(\tau) \cap (\mathbb{K} \setminus K)),$$

$$G_\Delta(\omega, \tau) = (G(\omega) \cap K) \cup (G(\tau) \cap (\mathbb{K} \setminus K)).$$

We write

$$(\omega, \tau)_\Delta = (B_\Delta(\omega, \tau), D_\Delta(\omega, \tau), G_\Delta(\omega, \tau))$$

for the configuration that agrees with ω in Δ and with τ outside Δ. The following result is a very useful 'spatial Markov' property of random-cluster measures; it is sometimes referred to as the DLR-, or Gibbs-, property. The proof follows standard arguments and may be found in Appendix B.

PROPOSITION 2.1.4. *Let $\Lambda \subseteq \Delta$ be regions, $\tau \in \Omega$, and $A \in \mathcal{F}$. Then*

$$\phi_\Delta^\tau(A \mid \mathcal{T}_\Lambda)(\omega) = \phi_\Lambda^{(\omega,\tau)_\Delta}(A), \qquad \phi_\Delta^\tau\text{-a.s.}$$

Analogous results hold for $b \in \{\mathrm{f}, \mathrm{w}\}$. The following is an immediate consequence of Proposition 2.1.4.

COROLLARY 2.1.5 (Deletion-contraction property). *Let $\Lambda \subseteq \Delta$ be regions such that $\hat{\partial}\Lambda \cap \hat{\partial}\Delta = \varnothing$, and let b be a boundary condition on Δ. Let $\mathcal{C}$ be the event that all components inside Λ which intersect $\hat{\partial}\Lambda$ are connected in $\Delta \setminus \Lambda$; let $\mathcal{D}$ be the event that* none *of these components are connected in $\Delta \setminus \Lambda$. Then*

$$\phi_\Delta^b(\cdot \mid \mathcal{C}) = \phi_\Lambda^{\mathrm{w}}(\cdot) \quad \text{and} \quad \phi_\Delta^b(\cdot \mid \mathcal{D}) = \phi_\Lambda^{\mathrm{f}}(\cdot).$$

2.1.5. The space–time Potts model. The classical random-cluster model is closely related to the Potts model of statistical mechanics. Similarly there is a natural 'space–time Potts model' which may be coupled with the space–time random-cluster model. A realization of the space–time Potts measure is a piecewise constant 'colouring' of $\mathbb{K}^\Gamma$. As for the random-cluster model, we will be interested in specifying different boundary conditions, and these will not only tell us which parts of the boundary are 'tied together', but may also specify the precise colour on certain parts of the boundary.

Let us fix a region Λ and $q \geq 2$ an *integer*. Let $\mathcal{N} = \mathcal{N}_q$ be the set of functions $\nu : \mathbb{K}^\Gamma \to \{1, \dots, q\}$ which have the property that their restriction to any $v \times \mathbb{R}$ is piecewise constant and right-continuous. Let $\mathcal{G}$ be the σ-algebra on $\mathcal{N}$ generated by all the functions $\nu \mapsto (\nu(x_1), \dots, \nu(x_N)) \in \mathbb{R}^N$ as N ranges through the integers and $x_1, \dots, x_N$ range through $\mathbb{K}^\Gamma$ (this coincides with the σ-algebra generated by the Skorokhod metric, see Appendix A and [**31**, Proposition 3.7.1]). For $S \subseteq \mathbb{K}$ define the σ-algebra $\mathcal{G}_S \subseteq \mathcal{G}$ of events defined on S^Γ. Although we canonically let $\nu \in \mathcal{N}$ be right-continuous, we will usually identify such ν which agree off sets of Lebesgue measure zero, compare Remark 2.1.1. Thus we will without further mention allow ν to be any piecewise constant function with values in $\{1, \dots, q\}$, and we will frequently even allow ν to be undefined on a set of measure zero. We call elements of $\mathcal{N}$ 'spin configurations' and will usually write ν_x for $\nu(x)$.

Let $b = \{P_1, \dots, P_m\}$ be any random-cluster boundary condition and let $\alpha : \{1, \dots, m\} \to \{0, 1, \dots, q\}$. We call the pair (b, α) a *Potts boundary condition*. We assume that $\Gamma \in P_1$, and write α_Γ for $\alpha(1)$; we also require that $\alpha_\Gamma \neq 0$. Let $D \subseteq K$ be a finite set, and let $\mathcal{N}_\Lambda^{b,\alpha}(D)$ be the set of $\nu \in \mathcal{N}$ with the following properties.

- For each $v \in V$ and each interval $I \subseteq K_v$ such that $I \cap D = \varnothing$, ν is constant on I,
- if $i \in \{1, \dots, m\}$ is such that $\alpha(i) \neq 0$ then $\nu_x = \alpha(i)$ for all $x \in P_i$,

- if $i \in \{1,\dots,m\}$ is such that $\alpha(i) = 0$ and $x, y \in P_i$ then $\nu_x = \nu_y$,
- if $x \notin \Lambda$ then $\nu_x = \alpha_\Gamma$.

Intuitively, the boundary condition b specifies which parts of the boundary are forced to have the same spin, and the function α specifies the *value* of the spin on some parts of the boundary; $\alpha(i) = 0$ is taken to mean that the value on P_i is not specified. (The value of α at Γ is special, in that it takes on the role of an external field, see (2.1.15).)

Let $\lambda : \mathbb{F} \to \mathbb{R}$, $\gamma : \mathbb{K} \to \mathbb{R}$ and $\delta : \mathbb{K} \to \mathbb{R}_+$ be bounded and Borel-measurable; note that λ and γ are allowed to take negative values. For $a, b \in \mathbb{R}$, let $\delta_{a,b} = \mathbb{1}_{\{a=b\}}$, and for $\nu \in \mathcal{N}$ and $e = xy \in \mathbb{E}$, let $\delta_\nu(e) = \delta_{\nu_x,\nu_y}$. Let $\pi_\Lambda^{b,\alpha}$ denote the probability measure on $(\mathcal{N}, \mathcal{G})$ defined by, for each bounded and $\mathcal{G}$-measurable $f : \mathcal{N} \to \mathbb{R}$, letting $\pi_\Lambda^{b,\alpha}(f(\nu))$ be a constant multiple of

(2.1.15)
$$\int d\mu_\delta(D) \sum_{\nu \in \mathcal{N}_\Lambda^{b,\alpha}(D)} f(\nu) \exp\Big(\int_F \lambda(e)\delta_\nu(e)de + \int_K \gamma(x)\delta_{\nu_x,\alpha_\Gamma} dx \Big)$$

(with constant determined by the requirement that $\pi_\Lambda^{b,\alpha}$ be a probability measure). The integrals in (2.1.15) are to be interpreted as in (2.1.12).

Definition 2.1.6. *The probability measure* $\pi_\Lambda^{b,\alpha} = \pi_{\Lambda;q,\lambda,\gamma,\delta}^{b,\alpha}$ *on* $(\mathcal{N}, \mathcal{G})$ *defined by* (2.1.15) *is called the space–time Potts measure with* q *states on* Λ.

Note that, as with ϕ_Λ^b, we may regard $\pi_\Lambda^{b,\alpha}$ as a measure on $(\mathcal{N}, \mathcal{G}_\Lambda)$. Here is a word of motivation for (2.1.15) in the case $b = \mathrm{f}$ and $\alpha_\Gamma = q$; similar constructions hold for other b, α. See Figure 3.2 in Section 3.2.2, and also [**54**]. The set $(v \times K_v) \setminus D$ is a union of maximal death-free intervals $v \times J_v^k$, where $k = 1, 2, \dots, n$ and $n = n(v, D)$ is the number of such intervals. We write $V(D)$ for the collection of all such intervals as v ranges over V, together with the ghost-vertex Γ, to which we assign spin $\nu_\Gamma = q$. The set $\mathcal{N}_\Lambda^{\mathrm{f},\alpha}(D)$ may be identified with $\{1,\dots,q\}^{V(D)}$, and we may think of $V(D)$ as the set of vertices of a graph with edges given as follows. An edge is placed between Γ and each $\bar{v} \in V(D)$. For $\bar{u}, \bar{v} \in V(D)$, with $\bar{u} = u \times I_1$ and $\bar{v} = v \times I_2$ say, we place an edge between $\bar{u}$ and $\bar{v}$ if and only if: (i) uv is an edge of L, and (ii) $I_1 \cap I_2 \neq \varnothing$. Under the space–time Potts measure *conditioned on* D, a spin-configuration $\nu \in \mathcal{N}_\Lambda^{\mathrm{f},\alpha}(D)$ on this graph receives a (classical) Potts weight

$$\exp\left\{\sum_{\bar{u}\bar{v}} J_{\bar{u}\bar{v}}\delta_\nu(\bar{u}\bar{v}) + \sum_{\bar{v}} h_{\bar{v}}\delta_{\nu_{\bar{v}},q}\right\}, \tag{2.1.16}$$

where $\nu_{\bar{v}}$ denotes the common value of ν along $\bar{v}$, and where

$$J_{\bar{u}\bar{v}} = \int_{I_1\cap I_2} \lambda(uv,t)\,dt \qquad \text{and} \qquad h_{\bar{v}} = \int_{\bar{v}} \gamma(x)\,dx.$$

This observation will be pursued further for the Ising model in Section 3.2.2.

The space–time Potts measure may, for special boundary conditions, be coupled to the space–time random-cluster measure, as follows. For α of the form $(\alpha_\Gamma, 0, \dots, 0)$, we call (b,α) a *simple* Potts boundary condition. Thus, under a simple boundary condition, the only spin value which is specified in advance is that of Γ. Let $\omega = (B, D, G) \in \Omega$ be sampled from ϕ_Λ^b and write $\mathcal{N}_\Lambda^{b,\alpha}(\omega)$ for the set of $\nu \in \mathcal{N}$ such that (i) $\nu_x = \alpha_\Gamma$ for $x \notin \Lambda$, and (ii) if $x, y \in \Lambda$ and $x \leftrightarrow y$ in ω under the boundary condition b in Λ then $\nu_x = \nu_y$. In particular, since $\Gamma \notin \Lambda$ we have that $\nu_\Gamma = \alpha_\Gamma$. Note that each $\mathcal{N}_\Lambda^{b,\alpha}(\omega)$ is a finite set. With ω given, we sample $\nu \in \mathcal{N}_\Lambda^{b,\alpha}(\omega)$ as follows. Set $\nu_\Gamma := \alpha_\Gamma$ and set $\nu_x = \alpha_\Gamma$ for all $x \notin \Lambda^\Gamma$; then choose the spins of the other components of ω in Λ uniformly and independently at random. The resulting pair (ω, ν) has a distribution $\mathbb{P}^{b,\alpha}_\Lambda$ on $(\Omega,\mathcal{F})\times(\mathcal{N},\mathcal{G})$ given by

$$\begin{aligned}\mathbb{P}^{b,\alpha}_\Lambda(f(\omega,\nu)) &= \int_\Omega d\phi^b_\Lambda(\omega)\,\frac{1}{q^{k^b_\Lambda(\omega)-1}}\sum_{\nu\in\mathcal{N}^{b,\alpha}_\Lambda(\omega)} f(\omega,\nu)\\ &\propto \int_\Omega d\mu(\omega)\sum_{\nu\in\mathcal{N}^{b,\alpha}_\Lambda(\omega)} f(\omega,\nu),\end{aligned} \tag{2.1.17}$$

for all bounded $f:\Omega\times\mathcal{N}\to\mathbb{R}$, measurable in the product σ-algebra $\mathcal{F}\times\mathcal{G}$. We call the measure $\mathbb{P}^{b,\alpha}_\Lambda$ of (2.1.17) the Edwards–Sokal measure. This definition is completely analogous to a coupling in the discrete model, which was was found in [**28**]. Usually we take $\alpha_\Gamma = q$ and in this case we will often suppress reference to α, writing for example $\mathcal{N}^b_\Lambda(\omega)$ and similarly for other notation.

The marginal of $\mathbb{P}^{b,\alpha}_\Lambda$ on $(\mathcal{N},\mathcal{G})$ is computed as follows. Assume that $f(\omega,\nu)\equiv f(\nu)$ depends only on ν, and let $D\subseteq K$ be a finite set. For $\nu\in\mathcal{N}^{b,\alpha}_\Lambda(D)$, let $\{\nu\sim\omega\}$ be the event that ω has no open paths *inside* Λ that violate the condition that ν be constant on the components of ω. We may rewrite (2.1.17) as

$$\mathbb{P}^{b,\alpha}_\Lambda(f(\nu)) \propto \int d\mu_\delta(D)\int d(\mu_\lambda\times\mu_\gamma)(B,G)\sum_{\nu\in\mathcal{N}^{b,\alpha}_\Lambda(D)} f(\nu)1\!\!1\{\nu\sim\omega\}. \tag{2.1.18}$$

With D fixed, the probability under $\mu_\lambda\times\mu_\gamma$ of the event $\{\nu\sim\omega\}$ is

$$\exp\Big(-\int_F\lambda(e)(1-\delta_\nu(e))de - \int_K\gamma(x)(1-\delta_{\nu_x,\alpha_\Gamma})dx\Big). \tag{2.1.19}$$

Taking out a constant, it follows that $\mathbb{P}^{b,\alpha}_\Lambda(f(\nu))$ is proportional to

$$\int d\mu_\delta(D)\sum_{\nu\in\mathcal{N}^{b,\alpha}_\Lambda(D)} f(\nu)\exp\Big(\int_F\lambda(e)\delta_\nu(e)de + \int_K\gamma(x)\delta_{\nu_x,\alpha_\Gamma}dx\Big). \tag{2.1.20}$$

Comparing this with (2.1.15), and noting that both equations define probability measures, it follows that $\mathbb{P}^{b,\alpha}_\Lambda(f(\nu)) = \pi^{b,\alpha}_\Lambda(f)$.

We may ask for a description of how to obtain an ω with law ϕ_Λ^b from a ν with law $\pi_\Lambda^{b,\alpha}$. In analogy with the discrete case this is as follows:

> Given $\nu \sim \pi_\Lambda^{b,\alpha}(\cdot)$, place a death wherever ν changes spin in Λ, and also place additional deaths elsewhere in Λ at rate δ; place bridges between intervals in Λ of the same spin at rate λ; and place ghost-bonds in intervals in Λ of spin α at rate γ. The outcome ω has law $\phi_\Lambda^b(\cdot)$.

It follows that we have the following correspondence between $\phi = \phi_\Lambda^b$ and $\pi = \pi_{\Lambda,q}^{b,\alpha}$ when (b,α) is simple. The result is completely analogous to the corresponding result for the discrete Potts model (Theorem 1.1.1), and the proof is included only for completeness.

Proposition 2.1.7. *Let $x, y \in \Lambda^\Gamma$. Then*

$$\pi(\nu_x = \nu_y) = \Big(1 - \frac{1}{q}\Big)\phi(x \leftrightarrow y) + \frac{1}{q}.$$

Proof. Writing $\mathbb{P}$ for the Edwards–Sokal coupling, we have that

$$\begin{aligned}
q\pi(\nu_x = \nu_y) - 1 &= \mathbb{P}(q \cdot \mathbb{P}(\nu_x = \nu_y \mid \omega) - 1)\\
&= \mathbb{P}\Big(q\big(\mathbb{I}\{x \leftrightarrow y \text{ in } \omega\} + \frac{1}{q}\mathbb{I}\{x \not\leftrightarrow y \text{ in } \omega\}\big) - 1\Big)\\
&= \mathbb{P}((q-1) \cdot \mathbb{I}\{x \leftrightarrow y \text{ in } \omega\})\\
&= (q-1)\phi(x \leftrightarrow y).
\end{aligned}$$

□

The case $q = 2$ merits special attention. In this case it is customary to replace the states $\nu_x = 1, 2$ by $-1, +1$ respectively, and we thus define $\sigma_x = 2\nu_x - 3$. For α taking values in $\{0, -1, +1\}$, we let $\Sigma, \Sigma_\Lambda^{b,\alpha}(\omega), \Sigma_\Lambda^{b,\alpha}(D)$ denote the images of $\mathcal{N}, \mathcal{N}_\Lambda^{b,\alpha}(\omega), \mathcal{N}_\Lambda^{b,\alpha}(D)$ respectively under the map $\nu \mapsto \sigma$. Reference to α may be suppressed if (b,α) is simple and $\alpha_\Gamma = +1$.

We have that

$$\mathbb{1}\{\sigma_x = \sigma_y\} = \frac{1}{2}(\sigma_x\sigma_y + 1), \qquad \mathbb{1}\{\sigma_x = \alpha_\Gamma\} = \frac{1}{2}(\alpha_\Gamma\sigma_x + 1). \tag{2.1.21}$$

Consequently, $\pi^{b,\alpha}_{\Lambda;q=2}(f(\sigma))$ is proportional to

$$\int d\mu_\delta(D) \sum_{\sigma\in\Sigma^{b,\alpha}_\Lambda(D)} f(\sigma)\exp\Big(\frac{1}{2}\int_F \lambda(e)\sigma_e\,de + \frac{1}{2}\int_K \gamma(x)\alpha_\Gamma\sigma_x\,dx\Big), \tag{2.1.22}$$

where we have written σ_e for $\sigma_x\sigma_y$ when $e = xy$. In this formulation, we call the measure of (2.1.22) the *Ising measure*. Expected values with respect to this measure will typically be written $\langle\cdot\rangle^{b,\alpha}_\Lambda$; thus for example Proposition 2.1.7 says that when $q = 2$ and (b,α) is simple, then

$$\langle\sigma_x\sigma_y\rangle^{b,\alpha}_\Lambda = \phi^b_\Lambda(x \leftrightarrow y). \tag{2.1.23}$$

For later reference, we make a note here of the constants of proportionality in the above definitions. Let

$$Z^b_{\mathrm{RC}} = Z^b_{\mathrm{RC}}(q) = \int_\Omega q^{k^b_\Lambda(\omega)}\,d\mu(\omega) \tag{2.1.24}$$

denote the partition function of the random-cluster model, and

$$Z^{b,\alpha}_{\mathrm{Potts}}(q) = \int d\mu_\delta(D) \sum_{\nu\in\mathcal{N}^{b,\alpha}_\Lambda(D)} \exp\Big(\int_F \delta_\nu(e)\lambda(e)\,de + \int_K \delta_{\nu_x,\alpha_\Gamma}\gamma(x)\,dx\Big) \tag{2.1.25}$$

that of the q-state Potts model. Also, let

$$Z^{b,\alpha}_{\mathrm{Ising}} = \int d\mu_\delta(D) \sum_{\sigma\in\Sigma^{b,\alpha}_\Lambda(D)} \exp\Big(\frac{1}{2}\int_F \lambda(e)\sigma_e\,de + \frac{1}{2}\int_K \gamma(x)\alpha_\Gamma\sigma_x\,dx\Big) \tag{2.1.26}$$

be the partition function of the Ising model. By keeping track of the constants in the above calculations we obtain the following result, which for simplicity is stated only for $\alpha_\Gamma = q$.

PROPOSITION 2.1.8. *Let b be a random-cluster boundary condition. Then*

$$Z^b_{\text{Potts}}(q) = \frac{1}{q} Z^b_{\text{RC}}(q) \cdot \exp\Big(\int_F \lambda(e)\, de + \int_K \gamma(x)\, dx \Big) \tag{2.1.27}$$

$$\begin{aligned} Z^b_{\text{Ising}} &= Z^b_{\text{Potts}}(2) \cdot \exp\Big(-\frac{1}{2} \int_F \lambda(e)\, de - \frac{1}{2} \int_K \gamma(x)\, dx \Big) \\ &= \frac{1}{2} Z^b_{\text{RC}}(2) \cdot \exp\Big(\frac{1}{2} \int_F \lambda(e)\, de + \frac{1}{2} \int_K \gamma(x)\, dx \Big). \end{aligned} \tag{2.1.28}$$

It is easy to check, by a direct computation, that the Potts model behaves in a similar manner to the random-cluster model upon conditioning on the value of ν in part of a region, i.e. that analogs of Proposition 2.1.4 and Corollary 2.1.5 hold. We will not state these results explicitly in full generality, but will record here the following special case for later reference.

LEMMA 2.1.9. *Let $\Lambda \subseteq \Delta$ denote two regions, and consider the boundary condition (w, α). Then for all $\mathcal{G}_\Lambda$-measurable f we have that*

$$\pi^{\mathrm{w},\alpha}_{\Lambda}(f(\nu)) = \pi^{\mathrm{w},\alpha}_{\Delta}(f(\nu) \mid \sigma \equiv \alpha_\Gamma \text{ on } \Delta \setminus \Lambda).$$

2.2. Stochastic comparison

The ability to compare the probabilities of events under a range of different measures is extremely important in the theory of random-cluster measures. In this section we develop in detail the basis for such a methodology in the space–time setting. We also prove versions of the GKS- and FKG inequalities suitable for the space–time Potts and Ising measures, respectively.

Let Λ be a region. Let the pair $(E, \mathcal{E})$ denote one of $(\Omega, \mathcal{F})$, $(\Omega, \mathcal{F}_\Lambda)$, $(\Sigma, \mathcal{G})$ and $(\Sigma, \mathcal{G}_\Lambda)$. Thus E, equipped with the Skorokhod metric, is a Polish metric space. Given a partial order $\geq$ on E, a measurable function $f : E \to \mathbb{R}$ is called *increasing* if for all $\omega, \xi \in E$ such that $\omega \geq \xi$ we have $f(\omega) \geq f(\xi)$. An event $A \in \mathcal{E}$ is increasing if the indicator function $\mathbb{1}_A$ is. We assume that the set $\{(\omega, \xi) \in E^2 : \omega \geq \xi\}$ is closed in the product topology; this will hold automatically in our applications.

Let ψ_1, ψ_2 be two probability measures on $(E, \mathcal{E})$.

DEFINITION 2.2.1. *We say that ψ_1* stochastically dominates *ψ_2, and we write $\psi_1 \geq \psi_2$, if $\psi_1(f) \geq \psi_2(f)$ for all bounded, increasing local functions f.*

By a standard approximation argument using the monotone convergence theorem, $\psi_1 \geq \psi_2$ holds if for all increasing local *events* A we have $\psi_1(A) \geq \psi_2(A)$.

The following general result lies at the heart of stochastic comparison and will be used repeatedly. It goes back to [**83**]; see also [**71**, Theorem IV.2.4] and [**43**, Theorem 4.6].

THEOREM 2.2.2 (Strassen). *Let ψ_1, ψ_2 be probability measures on $(E, \mathcal{E})$. The following statements are equivalent.*

(1) $\psi_1 \geq \psi_2$;
(2) For all continuous *bounded increasing local functions $f : E \to \mathbb{R}$ we have $\psi_1(f) \geq \psi_2(f)$;*
(3) There exists a probability measure P on $(E^2, \mathcal{E}^2)$ such that

$$P(\{(\omega_1, \omega_2) : \omega_1 \geq \omega_2\}) = 1.$$

Note that the equivalence of (1) and (3) extends to *countable* sequences $\psi_1, \psi_2, \psi_3, \ldots$; see [**71**, Theorem IV.6.1].

DEFINITION 2.2.3. *A measure ψ is on $(E, \mathcal{E})$ is called* positively associated *if for all local increasing events A, B we have that $\psi(A \cap B) \geq \psi(A)\psi(B)$.*

The inequality $\psi(A\cap B) \geq \psi(A)\psi(B)$ for local increasing events is sometimes referred to as the FKG-inequality as the systematic study of such inequalities was initiated by Fortuin, Kasteleyn and Ginibre [**36**].

2.2.1. Stochastic inequalities for the random-cluster model. The results in this section are applications, and slight modifications, of stochastic comparison results for point processes that appear in [**78**] and [**44**]. See also [**43**, Theorem 10.4]. Some of the results, such as positive association in the space–time random-cluster model, have been stated before, sometimes with additional assumptions; see for example [**7, 8, 11**]. We do not believe detailed proofs for space–time models have appeared before. The results presented are satisfyingly similar to those for the discrete case, compare [**50**, Chapter 3] and [**51**].

We will follow the method of [**78**] rather than the later (and more general) [**44**]. This is because the former method avoids discretization and is closer to the standard approach of [**56**] (also [**50**, Chapter 2]) for the classical random-cluster model. The method makes use of coupled Markov chains on Ω (specifically, jump-processes, see [**32**, Chapter X]).

For $\omega\in\Omega$, write $B(\omega), D(\omega), G(\omega)$ for the sets of bridges, deaths and ghost-bonds in ω, respectively. We define a partial order on Ω by saying that $\omega\geq\xi$ if $B(\omega)\supseteq B(\xi)$, $D(\omega)\subseteq D(\xi)$ and $G(\omega)\supseteq G(\xi)$.

We will in this section only consider measures on $\mathcal{F}_\Lambda$, that is we take $(E,\mathcal{E}) = (\Omega,\mathcal{F}_\Lambda)$. We will regard B, G, D as subsets of K and F as appropriate. The symbol x will be used to denote a generic point of $\Lambda\equiv K\cup F$, interpreted either as a bridge, a ghost-bond, or a death, as specified. More formally, we may regard x as an element of $F\cup(K\times\{\mathrm{d}\})\cup(K\times\{\mathrm{g}\})$, where the labels d,g allow us to distinguish between deaths and ghost-bonds, respectively. We let $X=(X_t: t\geq 0)$ be a continuous-time stochastic process with state space Ω, defined as follows. If $X_t = (B,G,D)$, there are 6 possible transitions. The process can either jump to one of

$$(B \cup \{x\}, G, D), \quad \text{or } (B, G \cup \{x\}, D), \quad \text{or } (B, G, D \cup \{x\}), \tag{2.2.1}$$

where $x \in \Lambda$; the corresponding move is called a *birth* at x. Alternatively, in the case where $x \in B$, the process can jump to

$$(B \setminus \{x\}, G, D),$$

and similarly for $x \in G$ or $x \in D$; the corresponding move is called a *demise* at x. If $\omega = (B, G, D) \in \Omega$, we will often abuse notation and write ω^x for the configuration (2.2.1) with a point at x added, making it clear from the context whether x is a bridge, ghost-bond, or death. Similarly, if $x \in B \cup G \cup D$, we will write ω_x for the configuration with the bridge, ghost-bond or death at x removed.

The transitions described above happen at the following rates. Let $\mathcal{L}$ denote the Borel σ-algebra on $\Lambda \equiv F \cup (K \times \{\mathrm{d}\}) \cup (K \times \{\mathrm{g}\})$, and let $\mathcal{B} : \Omega \times \mathcal{L} \to \mathbb{R}$ be a given function, such that for each $\omega \in \Omega$, $\mathcal{B}(\omega; \cdot)$ is a finite measure on $(\Lambda, \mathcal{L})$. Also let $\mathcal{D} : \Omega \times \Lambda \to \mathbb{R}$ be such that for all $\omega \in \Omega$ we have that $\mathcal{D}(\omega; x)$ is a non-negative measurable function of x. If for some $t \geq 0$ we have that $X_t = \omega$, then there is a birth in the measurable set $H \subseteq \Lambda$ before time $t + s$ with probability $\mathcal{B}(\omega; H)s + o(s)$. Alternatively, there is a demise at the point $x \in \omega$ before time $t + s$ with probability $\mathcal{D}(\omega_x; x)s + o(s)$.

We may give an equivalent 'jump-hold' description of the chain, as follows. Let

$$\mathcal{A}(\omega) := \mathcal{B}(\omega; \Lambda) + \sum_{x \in \omega} \mathcal{D}(\omega_x; x). \tag{2.2.2}$$

For $A \in \mathcal{F}_\Lambda$ let

$$\mathcal{K}(\omega, A) := \frac{1}{\mathcal{A}(\omega)} \Big(\mathcal{B}(\omega; \{x \in \Lambda : \omega^x \in A\}) + \sum_{\substack{x \in \omega \\ \omega_x \in A}} \mathcal{D}(\omega_x; x) \Big). \tag{2.2.3}$$

Then given that $X_t = \omega$, the holding time until the next transition has the exponential distribution with parameter $\mathcal{A}(\omega)$; once the process jumps it goes to some state $\xi \in A$ with probability $\mathcal{K}(\omega, A)$. Existence and basic properties of such Markov chains are discussed in [**78**].

We will aim to construct such chains X which are in detailed balance with a given probability measure ψ on $(\Omega, \mathcal{F}_\Lambda)$. It will be necessary to make some assumptions on ψ, and these will be stated when appropriate. For now the main assumption we make is the following. Let $\kappa = \mu_{1,1,1,}$ denote the probability measure on $(\Omega, \mathcal{F}_\Lambda)$ given by letting B, G, D all be independent Poisson processes of constant intensity 1.

ASSUMPTION 2.2.4. *The probability measure ψ is absolutely continuous with respect to κ; there exists a version of the density*

$$f = \frac{d\psi}{d\kappa}$$

which has the property that for all $\omega \in \Omega$ and $x \in \Lambda$, if $f(\omega) = 0$ then $f(\omega^x) = 0$.

EXAMPLE 2.2.5. *The space–time percolation measures (restricted to Λ) satisfy Assumption 2.2.4, because by standard properties of Poisson processes, if $\mu = \mu_{\lambda,\delta,\gamma}$ then a version of the density is given by*

$$\frac{d\mu}{d\kappa}(\omega) \propto \prod_{x\in B} \lambda(x) \prod_{y\in D} \delta(y) \prod_{z\in G} \gamma(z). \tag{2.2.4}$$

Moreover, the random-cluster measure $\phi^b_\Lambda = \phi^b_{\Lambda;q,\lambda,\delta,\gamma}$ also satisfies Assumption 2.2.4, having density

$$\frac{d\phi^b_\Lambda}{d\kappa}(\omega) = \frac{d\phi^b_\Lambda}{d\mu}(\omega)\frac{d\mu}{d\kappa}(\omega) \propto q^{k^b_\Lambda(\omega)} \prod_{x\in B} \lambda(x) \prod_{y\in D} \delta(y) \prod_{z\in G} \gamma(z) \tag{2.2.5}$$

against κ.

DEFINITION 2.2.6. *The* Papangelou intensity *of ψ is the function $\iota : \Omega \times \Lambda \to \mathbb{R}$ given by*

$$\iota(\omega, x) = \frac{f(\omega^x)}{f(\omega)} \tag{2.2.6}$$

(where we take $0/0$ to be 0).

The following construction will not itself be used, but serves as a helpful illustration. To construct a birth-and-death chain which has equilibrium distribution ψ we would simply take $\mathcal{D} \equiv 1$ and $\mathcal{B}(\omega; dx) = \iota(\omega, x)dx$. (Here dx denotes Lebesgue measure on $F \cup (K \times \{\mathrm{d}\}) \cup (K \times \{\mathrm{g}\})$.) The corresponding chain X is in detailed balance with ψ, since $d\psi(\omega_x) \cdot \mathcal{B}(\omega_x; dx) = d\kappa(\omega_x) f(\omega^x)dx = d\psi(\omega^x) \cdot 1$. In light of this one may may think of $\iota(\omega, x)$ as the intensity with which the chain X, in equilibrium with ψ, attracts a birth at x.

Example 2.2.7. *For the random-cluster measure* ϕ^b_Λ,

$$\iota(\omega, x) = q^{k^b_\Lambda(\omega^x) - k^b_\Lambda(\omega)} \cdot \begin{cases} \lambda(x), & \text{for } x \text{ a bridge} \\ \delta(x), & \text{for } x \text{ a death} \\ \gamma(x), & \text{for } x \text{ a ghost-bond.} \end{cases} \tag{2.2.7}$$

In the rest of this section we let ψ, ψ_1, ψ_2 be three probability measures satisfying Assumption 2.2.4, and let f, f_1, f_2 and ι, ι_1, ι_2 denote their density functions against κ and their Papangelou intensities, respectively.

Definition 2.2.8. *We say that the pair* (ψ_1, ψ_2) *satisfies the* lattice condition *if the following hold whenever* $\omega \geq \xi$*:*

(1) $\iota_1(\omega, x) \geq \iota_2(\xi, x)$ *whenever* x *is a bridge or ghost-bond such that* $\xi^x \not\leq \omega$*;*
(2) $\iota_2(\xi, x) \geq \iota_1(\omega, x)$ *whenever* x *is a death such that* $\xi \not\leq \omega^x$*.*

We say that ψ *has the* lattice property *if the following hold whenever* $\omega \geq \xi$*:*

(3) $\iota(\omega, x) \geq \iota(\xi, x)$ *whenever* x *is a bridge or ghost-bond such that* $\xi^x \not\leq \omega$*;*
(4) $\iota(\xi, x) \geq \iota(\omega, x)$ *whenever* x *is a death such that* $\xi \not\leq \omega^x$*.*

(We use the term 'lattice' in the above definition in the same sense as [**36**]; 'lattice' is the name for any partially ordered set in which any two elements have a least upper bound and greatest lower bound.)

The next result states that 'well-behaved' measures ψ_1, ψ_2 which satisfy the lattice condition are stochastically ordered, in that $\psi_1 \geq \psi_2$. Intuitively, the lattice condition implies that a chain with equilibrium distribution ψ_1 acquires bridges and ghost-bonds faster than, but deaths slower than, the chain corresponding to ψ_2. Similarly, we will see that measures with the lattice property are positively associated; a similar intuition holds in this case.

THEOREM 2.2.9. *Suppose ψ_1, ψ_2 satisfy the lattice condition, and that the Papangelou intensities ι_1, ι_2 are bounded. Then $\psi_1 \geq \psi_2$.*

THEOREM 2.2.10. *Suppose ψ has the lattice property, and that ι is bounded. Then ψ is positively associated.*

SKETCH PROOF OF THEOREM 2.2.9. This essentially follows from [**78**], the main difference being that our order on Ω is different, in that 'deaths count negative'. The method of [**78**] is to couple two jump-processes X and Y, which have the respective equilibrium distributions ψ_1 and ψ_2. One may define a jump process on the product space $\Omega \times \Omega$ in the same way as described in (2.2.2) and (2.2.3); here is the specific instance we require.

Let $T := \{(\omega, \xi) \in \Omega^2 : \omega \geq \xi\}$, and for $a, b \in \mathbb{R}$ write $a \vee b$ and $a \wedge b$ for the maximum and minimum of a and b, respectively. We let $Z = (X, Y)$ be the birth-and-death process on T started at $(\varnothing, \varnothing)$ and given by the $\mathcal{A}$ and $\mathcal{K}$ defined below. First,

$$\begin{aligned}(2.2.8)\quad \mathcal{A}(\omega, \xi) := \int_\Lambda (\iota_1(\omega, x) \vee \iota_2(\xi, x))\, dx + \\ + (|B(\omega)| \vee |B(\xi)|) + (|D(\omega)| \vee |D(\xi)|) + (|G(\omega)| \vee |G(\xi)|).\end{aligned}$$

Write $\omega \cap \xi$ for the element $(B(\omega) \cap B(\xi), D(\omega) \cap D(\xi), G(\omega) \cap G(\xi))$ of Ω; similarly let $\omega \setminus \xi = (B(\omega) \setminus B(\xi), D(\omega) \setminus D(\xi), G(\omega) \setminus G(\xi))$. For $A \subseteq T$ measurable in the product topology, let

$$\mathcal{K}(\omega, \xi; A) := \frac{1}{\mathcal{A}(\omega, \xi)} \big(\mathcal{K}_{\mathrm{b}}(\omega, \xi; A) + \mathcal{K}_{\mathrm{d}}(\omega, \xi; A)\big) \tag{2.2.9}$$

where

$$\begin{aligned} \mathcal{K}_{\mathrm{d}}(\omega, \xi; A) := {} & |\{x \in \omega \cap \xi : (\omega_x, \xi_x) \in A\}| + \\ & + |\{x \in \omega \setminus \xi : (\omega_x, \xi) \in A\}| + |\{x \in \xi \setminus \omega : (\omega, \xi_x) \in A\}| \end{aligned} \tag{2.2.10}$$

and

$$\begin{aligned} \mathcal{K}_{\mathrm{b}}(\omega, \xi; A) := {} & \int_\Lambda \mathbb{I}_A(\omega^x, \xi^x)(\iota_1(\omega, x) \wedge \iota_2(\xi, x))\, dx + \\ & + \int_\Lambda \mathbb{I}_A(\omega^x, \xi)[\iota_1(\omega, x) - (\iota_1(\omega, x) \wedge \iota_2(\xi, x))]\, dx + \\ & + \int_\Lambda \mathbb{I}_A(\omega, \xi^x)[\iota_2(\xi, x) - (\iota_1(\omega, x) \wedge \iota_2(\xi, x))]\, dx. \end{aligned} \tag{2.2.11}$$

Thanks to the lattice condition, Z is indeed a process on T. In other words, if $\omega \geq \xi$ then $\mathcal{K}(\omega, \xi; T) = 1$. It is also not hard to see that X and Y are birth-and-death processes on Ω with transition intensities $\mathcal{B}_1, \mathcal{D}_1$ and $\mathcal{B}_2, \mathcal{D}_2$ respectively, where $\mathcal{D}_k \equiv 1$ and $\mathcal{B}_k(\omega; dx) = \iota_k(\omega, x)dx$, for $k = 1, 2$.

Define, for $n \geq 0$ and $k \in \{1, 2\}$,

$$\mathcal{B}_k^{(n)} = \sup_{|\omega| = n} \mathcal{B}_k(\omega; \Lambda), \tag{2.2.12}$$

where $|\omega|$ is the total number of bridges, ghost-bonds and deaths in ω. The boundedness of ι_1, ι_2 ensures that the following properties, which appear as conditions in [**78**], hold. First, the expectation

$$\kappa(\mathcal{B}_k(\cdot; \Lambda)) < \infty, \tag{2.2.13}$$

and second,

$$\sum_{n=1}^{\infty} \frac{\mathcal{B}_k^{(0)} \cdots \mathcal{B}_k^{(n-1)}}{n!} < \infty. \tag{2.2.14}$$

Theorems 7.1 and 8.1 of [**78**] therefore combine to give that the chain Z has a unique invariant distribution P such that $Z_t \Rightarrow P$, and such that $P(F \times \Omega) = \psi_1(F)$ and $P(\Omega \times F) = \psi_2(F)$. Since $P(T) = 1$, the result follows: if $A \in \mathcal{F}_\Lambda$ is increasing then

$$\psi_1(A) = P(\omega \in A,\, \omega \geq \xi) \geq P(\xi \in A,\, \omega \geq \xi) = \psi_2(A). \tag{2.2.15}$$

□

REMARK 2.2.11. *The two technical properties* (2.2.13) *and* (2.2.14) *are not strictly necessary for the main results of* [**78**], *as shown in* [**44**], *but they do seem necessary for the proof method in* [**78**]. *See* [**44**, Remark 1.6].

Theorem 2.2.10 follows from Theorem 2.2.9 using the following standard argument [**56**].

PROOF OF THEOREM 2.2.10. Let g, h be two bounded, increasing and $\mathcal{F}_\Lambda$-measurable functions. By adding constants, if necessary, we may assume that g, h are strictly positive. Let $\psi_2 = \psi$ and let ψ_1 be given by

$$f_1(\omega) = \frac{d\psi_1}{d\kappa}(\omega) := \frac{h(\omega)f(\omega)}{\psi(h)}. \tag{2.2.16}$$

We have that

$$\iota_1(\omega, x) = \frac{h(\omega^x)f(\omega^x)}{h(\omega)f(\omega)}, \qquad \iota_2(\xi, x) = \frac{f(\xi^x)}{f(\xi)}. \tag{2.2.17}$$

Clearly ι_1, ι_2 are uniformly bounded; we check that ψ_1, ψ_2 satisfy the lattice condition. Let $\omega \geq \xi$. If x is a bridge or a ghost-bond then $h(\omega^x)/h(\omega) \geq 1$, so by the lattice property of ψ we have that $\iota_1(\omega, x) \geq \iota_2(\xi, x)$. Similarly, if x is a death then $h(\xi^x)/h(\xi) \leq 1$ so $\iota_1(\omega, x) \leq \iota_2(\xi, x)$, as required.

We thus have that

$$\psi(gh) = \psi(h)\psi_1(g) \geq \psi(h)\psi_2(g) = \psi(h)\psi(g). \tag{2.2.18}$$

□

For the next result we let $\lambda, \delta, \gamma, \lambda', \delta', \gamma'$ be non-negative, bounded and Borel-measurable, and write $\lambda' \geq \lambda$ if λ' is pointwise no less that λ (and similarly for other functions). For $a \in \mathbb{R}$, write $a\lambda$ or λa for the function $x \mapsto a \cdot \lambda(x)$ (and similarly for other functions). Recall also the ordering of boundary conditions defined in Section 2.1 (page 23).

Theorem 2.2.12. *If $q \geq 1$ and $0 < q' \leq q$ then for any boundary condition b we have that*

$$\phi^b_{\Lambda;q,\lambda,\delta,\gamma} \leq \phi^b_{\Lambda;q',\lambda',\delta',\gamma'}, \qquad \text{if } \lambda' \geq \lambda,\ \delta' \leq \delta \text{ and } \gamma' \geq \gamma$$

$$\phi^b_{\Lambda;q,\lambda,\delta,\gamma} \geq \phi^b_{\Lambda;q',\lambda',\delta',\gamma'}, \qquad \text{if } \lambda' \leq \lambda q'/q,\ \delta' \geq \delta q/q',\ \text{and } \gamma' \leq \gamma q'/q.$$

Moreover, if $b' \geq b$ are two boundary conditions, then

$$\phi^{b'}_{\Lambda;q,\lambda,\delta,\gamma} \geq \phi^b_{\Lambda;q,\lambda,\delta,\gamma}.$$

Corollary 2.2.13. *Let b be any boundary condition. If $q \geq 1$ then*

$$\phi^b_{\Lambda;q,\lambda,\delta,\gamma} \leq \mu_{\lambda,\delta,\gamma} \quad \text{and} \quad \phi^b_{\Lambda;q,\lambda,\delta,\gamma} \geq \mu_{\lambda/q,q\delta,\gamma/q} \tag{2.2.19}$$

and if $0 < q < 1$ then

$$\phi^b_{\Lambda;q,\lambda,\delta,\gamma} \geq \mu_{\lambda,\delta,\gamma} \quad \text{and} \quad \phi^b_{\Lambda;q,\lambda,\delta,\gamma} \leq \mu_{\lambda/q,q\delta,\gamma/q}. \tag{2.2.20}$$

Proof of Theorem 2.2.12. We prove the first inequality; the rest are similar. The proof (given Theorem 2.2.9) is completely analogous to the one for the discrete random-cluster model, see [**50**, Theorem 3.21]. Recall the formula (2.2.7) for $\iota(\cdot,\cdot)$ in the random-cluster case. Let $\psi_1 = \phi^b_{\Lambda;q',\lambda',\delta',\gamma'}$ and $\psi_2 = \phi^b_{\Lambda;q,\lambda,\delta,\gamma}$. Clearly $\iota_1, \iota_2 \leq qr$ for all ω, x, where r is an upper bound for all of $\lambda, \delta, \gamma, \lambda', \delta', \gamma'$. Let us check the lattice conditions of Definition 2.2.8. Let $\omega \leq \xi$ and let x be a bridge such that $\xi^x \not\leq \omega$. Then $\iota_1(\omega, x) = \lambda'(x)(q')^{k^b_\Lambda(\omega^x)-k^b_\Lambda(\omega)}$ and $\iota_2(\xi, x) = \lambda(x) q^{k^b_\Lambda(\xi^x)-k^b_\Lambda(\xi)}$. Note that the powers of q, q' both take values in $\{0,-1\}$. Since $\lambda' \geq \lambda$ and $q' \leq q$, we are done if we show that $k^b_\Lambda(\omega^x) - k^b_\Lambda(\omega) \geq k^b_\Lambda(\xi^x) - k^b_\Lambda(\xi)$. The left-hand-side is -1 if and only if x ties together two different components of ω. But if it does, then certainly it does the same to ξ since $\xi \leq \omega$; so then also the right-hand-side is -1, as required. It follows that $\iota_1(\omega, x) \geq \iota_2(\xi, x)$. The cases when x is a death or a ghost-bond are similar. $\square$

THEOREM 2.2.14 (Positive association). *Let $q \geq 1$. The random-cluster measure $\phi^b_{\Lambda;q,\lambda,\delta,\gamma}$ is positively associated.*

Presumably positive association fails when $q < 1$, as it does in the discrete random-cluster model.

PROOF. We only have to verify that $\phi^b_{\Lambda;q,\lambda,\delta,\gamma}$ has the lattice property. Since $q \geq 1$ this follows from the fact that $k^b_\Lambda(\omega^x) - k^b_\Lambda(\omega) \geq k^b_\Lambda(\xi^x) - k^b_\Lambda(\xi)$ if $\omega \geq \xi$ and x is a bridge or ghost-bond, and the other way around if x is a death, as in the proof of Theorem 2.2.12. $\square$

The next result is a step towards the 'finite energy property' of Lemma 2.3.4; it provides upper and lower bounds on the probabilities of seeing or not seeing any bridges, deaths or ghost-bonds in small regions. These bounds are useful because they are uniform in Λ. For the statement of the result, we let $q > 0$, let $\Lambda = (K, F)$ be a region and $I \subseteq K$ and $J \subseteq F$ intervals. Define

$$\overline{\lambda} = \sup_{x \in J} \lambda(x), \qquad \underline{\lambda} = \inf_{x \in J} \lambda(x) \tag{2.2.21}$$

and similarly for $\overline{\delta}, \underline{\delta}, \overline{\gamma}, \underline{\gamma}$ with J replaced by I. Write

$$\begin{aligned}
\eta_\lambda &= \min\{e^{-\overline{\lambda}|J|}, e^{-\overline{\lambda}|J|/q}\}, & \eta^\lambda &= \max\{e^{-\underline{\lambda}|J|}, e^{-\underline{\lambda}|J|/q}\},\\
\eta_\delta &= \min\{e^{-\overline{\delta}|I|}, e^{-q\overline{\delta}|I|}\}, & \eta^\delta &= \max\{e^{-\underline{\delta}|I|}, e^{-q\underline{\delta}|I|}\},\\
\eta_\gamma &= \min\{e^{-\overline{\gamma}|I|}, e^{-\overline{\gamma}|I|/q}\}, & \eta^\gamma &= \max\{e^{-\underline{\gamma}|I|}, e^{-\underline{\gamma}|I|/q}\}.
\end{aligned}$$

These are to be interpreted as six distinct quantities.

PROPOSITION 2.2.15. *For any boundary condition b we have that*

$$\eta_\lambda \leq \phi^b_{\Lambda;q,\lambda,\delta,\gamma}(|B \cap J| = 0 \mid \mathcal{F}_{\Lambda\setminus J}) \leq \eta^\lambda$$
$$\eta_\delta \leq \phi^b_{\Lambda;q,\lambda,\delta,\gamma}(|D \cap I| = 0 \mid \mathcal{F}_{\Lambda\setminus I}) \leq \eta^\delta$$
$$\eta_\gamma \leq \phi^b_{\Lambda;q,\lambda,\delta,\gamma}(|G \cap I| = 0 \mid \mathcal{F}_{\Lambda\setminus I}) \leq \eta^\gamma$$

PROOF. Follows from Proposition 2.1.4 and Corollary 2.2.13. □

REMARK 2.2.16. *It is convenient, but presumably not optimal, to deduce finite energy from stochastic ordering as we have done here. For discrete models it is straightforward to prove the analog of Proposition 2.2.15 without using stochastic domination, see* [**50**, Theorem 3.7].

2.2.2. The FKG-inequality for the Ising model. There is a natural partial order on the set $\Sigma^{b,\alpha}_\Lambda$ of space–time Ising configurations, given by: $\sigma \geq \tau$ if $\sigma_x \geq \tau_x$ for all $x \in K$. In Section 2.5.2 we will require a FKG-inequality for the Ising model, and we prove such a result in this section. It will be important to have a result that is valid for all boundary conditions (b, α) of Ising type, and when the function γ is allowed to take negative values. The result will be proved by expressing the space–time Ising measure as a weak limit of discrete Ising measures, for which the FKG-inequality is known. The same approach was used for the space–time percolation model in [**11**]. We let λ, δ denote non-negative functions, as before, and we let $b = \{P_1, \ldots, P_m\}$ and α be fixed.

Recall that K consists of a collection of disjoint intervals I^v_i. Write $\mathcal{E}$ for the set of endpoints x of the I^v_i for which $x \in K$. Similarly, each $P_i \setminus \{\Gamma\}$ is a finite union of disjoint intervals; write $\mathcal{B}$ for the set of endpoints y of these intervals for which $y \in K$. For $\varepsilon > 0$, let

$$K^\varepsilon = \mathcal{E} \cup \mathcal{B} \cup \{(v, \varepsilon k) \in K : k \in \mathbb{Z}\}. \tag{2.2.22}$$

Let Σ^ε denote the set of vectors $\sigma' \in \{-1,+1\}^{K^\varepsilon \cup \{\Gamma\}}$ that respect the boundary condition (b,α); that is, (i) if $x, y \in K^\varepsilon \cup \{\Gamma\}$ are such that $x, y \in P_i$ for some i, then $\sigma'_x = \sigma'_y$, and (ii) if in addition $\alpha(i) \neq 0$ then $\sigma'_x = \alpha(i)$. For each $x = (v,t) \in K^\varepsilon$, let $t' > t$ be maximal such that the interval $I_\varepsilon(x) := v \times [t,t')$ lies in K but contains no other element of K^ε; if no such t' exists let $I_\varepsilon(x) := \{x\}$. See Figure 2.5.

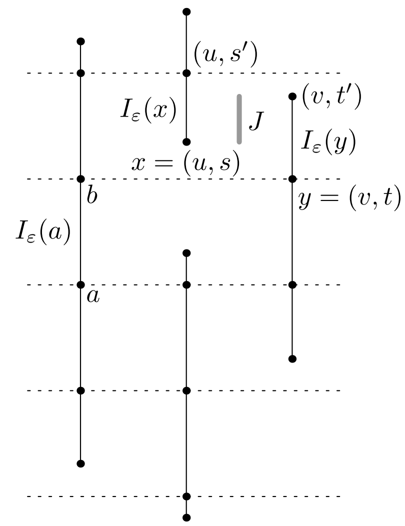

FIGURE 2.5. Discretized Ising model. K is drawn as solid vertical lines, and is the union of four closed, disjoint intervals. Dotted lines indicate the levels $k\varepsilon$ for $k \in \mathbb{Z}$. Elements of K^ε are drawn as black dots. The interval $J = uv \times \{[s,s') \cap [t,t')\}$, which appears in the integral in (2.2.24), is drawn grey. In this illustration $b = \mathrm{f}$.

We now define the appropriate coupling constants for the discretized model. Let $x, y \in K^\varepsilon$, $x \neq y$. First suppose $I_\varepsilon(x)$ and $I_\varepsilon(y)$ share an endpoint, which we may assume to be the right endpoint of $I_\varepsilon(x)$. Then define

$$p^\varepsilon_{xy} = 1 - \int_{I_\varepsilon(x)} \delta(z)\, dz. \tag{2.2.23}$$

Next, suppose $x = (u, s)$ and $y = (v, t)$ are such that $uv \in E$, and such that $I_\varepsilon(x) = \{u\} \times [s, s')$ and $I_\varepsilon(y) = \{v\} \times [t, t')$ satisfy $[s, s') \cap [t, t') \neq \varnothing$. Then let $J = uv \times \{[s, s') \cap [t, t')\}$ and define

$$p^\varepsilon_{xy} = \int_J \lambda(e)\, de. \tag{2.2.24}$$

For all other $x, y \in K^\varepsilon$ we let $p^\varepsilon_{xy} = 0$. Finally, for all $x \in K^\varepsilon$ define

$$p^\varepsilon_{x\Gamma} = \int_{I_\varepsilon(x)} \gamma(z)\, dz. \tag{2.2.25}$$

Note that $p^\varepsilon_{x\Gamma}$ can be negative.

Let J^ε_{xy} and h^ε_x $(x, y \in K^\varepsilon)$ be defined by

$$1 - p^\varepsilon_{xy} = e^{-2J^\varepsilon_{xy}}, \qquad 1 - p^\varepsilon_{x\Gamma} = e^{-2h^\varepsilon_x}. \tag{2.2.26}$$

Let π'_ε be the Ising measure on Σ^ε with these coupling constants, that is

$$\pi'_\varepsilon(\sigma') = \frac{1}{Z^\varepsilon} \exp\Big(\frac{1}{2} \sum_{x,y \in K^\varepsilon} J^\varepsilon_{xy} \sigma'_x \sigma'_y + \sum_{x \in K^\varepsilon} h^\varepsilon_x \sigma'_x \alpha_\Gamma\Big), \tag{2.2.27}$$

where Z^ε is the appropriate normalizing constant. In the special case when $\gamma \geq 0$ and (b, α) is simple, all the p^ε_{xy} and $p^\varepsilon_{x\Gamma}$ lie in $[0, 1]$ for ε sufficiently small, and π'_ε is coupled via the standard Edwards–Sokal measure [**50**, Theorem 1.10] to the $q = 2$ random-cluster measure with these edge-probabilities.

There is a natural way to map each element $\sigma' \in \Sigma^\varepsilon$ to an element σ of $\Sigma^{\mathrm{f}}_\Lambda$, namely by letting σ take the value σ'_x throughout $I_\varepsilon(x)$. Let π_ε denote the law of σ under this mapping. By a direct computation using (2.2.27) (for example by splitting off the factor corresponding to 'vertical' interactions in the sum over x, y) one may see that

$$\pi_\varepsilon \Rightarrow \langle \cdot \rangle^{b,\alpha}_\Lambda \quad \text{as } \varepsilon \downarrow 0, \tag{2.2.28}$$

where $\langle \cdot \rangle^{b,\alpha}_\Lambda$ is the space–time Ising measure defined at (2.1.22).

For $S \in \mathcal{G}_\Lambda$ an event, we write ∂S for the boundary of S in the Skorokhod metric. We say that S is a *continuity set* if $\langle \mathbb{1}_{\partial S} \rangle^{b,\alpha}_\Lambda = 0$. By standard facts about weak convergence, (2.2.28) implies that $\pi_\varepsilon(S) \to \langle \mathbb{1}_S \rangle_\Lambda^{b,\alpha}$ for each continuity set S. Note that $\partial(S \cap T) \subseteq \partial S \cup \partial T$, so if $S, T \in \mathcal{G}_\Lambda$ are continuity sets then so is $S \cap T$.

LEMMA 2.2.17. *Let $S, T \in \mathcal{G}_\Lambda$ be increasing continuity sets. Then*

$$\langle \mathbb{1}_{S\cap T} \rangle_\Lambda^{b,\alpha} \geq \langle \mathbb{1}_S \rangle_\Lambda^{b,\alpha} \langle \mathbb{1}_T \rangle_\Lambda^{b,\alpha}.$$

PROOF. By the standard FKG-inequality for the classical Ising model, we have for each $\varepsilon > 0$ that

$$\pi_\varepsilon(S \cap T) \geq \pi_\varepsilon(S)\pi_\varepsilon(T).$$

The result follows from (2.2.28). □

In the next result, we write $\langle\cdot\rangle_\gamma$ for the space–time Ising measure $\langle\cdot\rangle_\Lambda^{b,\alpha}$ with ghost-field γ.

LEMMA 2.2.18. *Let S be an increasing continuity set, and let $\gamma_1 \geq \gamma_2$ pointwise. Then $\langle \mathbb{1}_S \rangle_{\gamma_1} \geq \langle \mathbb{1}_S \rangle_{\gamma_2}$.*

PROOF. Follows from (2.2.28) and the fact that π'_ε is increasing in γ. □

EXAMPLE 2.2.19. *Here is an example of a continuity set. Let $R \subseteq K$ be a finite union of intervals, some of which may consist of a single point. Let $a \in \{-1, +1\}$. Then the event*

$$S = \{\sigma \in \Sigma : \sigma_x = a \text{ for all } x \in R\}$$

is a continuity set, since $\sigma \in \partial S$ only if σ changes value exactly on an endpoint of one of the intervals constituting R.

The assumption above that S, T be continuity sets is an artefact of the proof method and can presumably be removed. It should be possible to establish versions of Theorems 2.2.9 and 2.2.10 also for Ising spins, using a Markov chain approach. The auxiliary process

D complicates this. The author would like to thank Jeffrey Steif for pointing out an error in an earlier version of this subsection.

2.2.3. Correlation inequalities for the Potts model. A cornerstone in the study of the classical Ising model is provided by the so-called GKS- or Griffiths' inequalities (see [**46, 47, 61**]) which state that certain covariances are non-negative. Recently, in [**41**] and [**51**], it was demonstrated that these inequalities follow from the FKG-inequality for the random-cluster representation, using an argument that also extends to the Potts models. In this section we adapt the methods of [**51**] to the space–time setting.

Let $q \geq 2$ be fixed, Λ a fixed region, and b a fixed random-cluster boundary condition. We let α be such that (b, α) is a simple boundary condition with $\alpha_\Gamma = q$. It is important to note that the proofs in this section are only valid for this choice of α. Therefore, some of the results here are less general than what we require for detailed study of the space–time Ising model, and we will then resort to the results of the previous subsection.

Let π, ϕ denote the Potts- and random-cluster measures with the given parameters, respectively. We will be using the complex variables

$$\sigma_x = \exp\left(\frac{2\pi i \nu_x}{q}\right), \tag{2.2.29}$$

where $i = \sqrt{-1}$. Note that when $q = 2$ this agrees with the previous definition on page 30. (In [**51**] many alternative possibilities for σ are explored; similar results hold at the same level of generality here, but we refrain from treating this added generality for simplicity of presentation.)

Define for $A \subseteq K$ a finite set

$$\sigma_A := \prod_{x \in A} \sigma_x. \tag{2.2.30}$$

More generally, if $\underline{r} = (r_x : x \in A)$ is a vector of integers indexed by A, define

$$\sigma_A^{\underline{r}} := \prod_{x \in A} \sigma_x^{r_x}. \tag{2.2.31}$$

Thus $\sigma_A \equiv \sigma_A^{\underline{1}}$ where $\underline{1}$ is a constant vector of 1's. The set B in the following should not be confused with the bridge-set $B = B(\omega)$.

LEMMA 2.2.20 (GKS inequalities). *Let $A, B \subseteq K$ be finite sets, not necessarily disjoint, and let $\underline{r} = (r_x : x \in A)$ and $\underline{s} = (s_y : y \in B)$. Then*

$$\pi(\sigma_A^{\underline{r}}) \geq 0 \tag{2.2.32}$$

and

$$\pi(\sigma_A^{\underline{r}}; \sigma_B^{\underline{s}}) := \pi(\sigma_A^{\underline{r}} \sigma_B^{\underline{s}}) - \pi(\sigma_A^{\underline{r}})\pi(\sigma_B^{\underline{s}}) \geq 0. \tag{2.2.33}$$

In particular, $\pi(\sigma_A) \geq 0$ and $\pi(\sigma_A; \sigma_B) \geq 0$.

A result similar to Lemma 2.2.20 holds for $A, B \subseteq \overline{K}$, but then care must be taken to define σ_x appropriately for points $x \in \partial\Lambda$ that do not lie in Λ. For example, if $x = (v, t)$ is an isolated point in $\mathbb{K} \setminus K$ then the corresponding result holds if we replace σ_x by one of σ_{x+} or σ_{x-}, where $\sigma_{x+} = \lim_{\varepsilon \downarrow 0} \sigma_{(v,t+\varepsilon)}$ and $\sigma_{x-} = \lim_{\varepsilon \downarrow 0} \sigma_{(v,t-\varepsilon)}$ (these limits exist almost surely but are in general different for such x).

For $\omega \in \Omega$ let $k = k_\Lambda^b(\omega)$, and let $C_1(\omega), \dots, C_k(\omega)$ denote the components of ω in Λ, defined according to the boundary condition b. We assume that $\Gamma \in C_k(\omega)$, and thus $C_1(\omega), \dots, C_{k-1}(\omega)$ are the 'Γ-free' components of ω. Lemma 2.2.20 will follow from Theorems 2.2.12 and 2.2.14 using the following representation.

LEMMA 2.2.21. *Let $\underline{r} = (r_x : x \in A)$ and write $r_j = \sum_{x \in A \cap C_j} r_x$ (for $j = 1, \dots, k-1$). Then*

$$\pi(\sigma_A^{\underline{r}}) = \phi(r_j \equiv 0 \ (\textit{mod } q), \textit{ for } j = 1, \dots, k-1).$$

Note that the event on the right-hand-side is increasing; also note that if $r_x = 1$ for all x then $r_j = |A \cap C_j|$.

Proof. Let $U_1, U_2, \ldots$ be independent random variables with the uniform distribution on $\{e^{2\pi i m/q} : m = 1, \ldots, q\}$, and let $\mathbb{P}$ denote the Edwards–Sokal coupling (2.1.17) of π and ϕ. We have that

$$\mathbb{P}(\sigma_A^{\underline{r}} \mid \omega) = E\Big(1 \cdot \prod_{j=1}^{k-1} U_j^{r_j}\Big) = \prod_{j=1}^{k-1} E(U_j^{r_j}), \tag{2.2.34}$$

where E denotes expectation over the U_j (recall that $\nu_\Gamma = q$, so $\sigma_\Gamma = 1$). Since U_j is uniform we have that

$$E(U_j^r) = \frac{1}{q} \sum_{m=1}^{q} \big(e^{2\pi i m/q}\big)^r = \begin{cases} 1, & \text{if } r \equiv 0 \pmod q, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.35}$$

The result follows on taking the expectation of (2.2.34). □

Proof of Lemma 2.2.20. It is immediate from Lemma 2.2.21 that $\pi(\sigma_A^{\underline{r}}) \geq 0$, which is (2.2.32). For (2.2.33) we note that $\sigma_A^{\underline{r}} \sigma_B^{\underline{s}} = \sigma_{A\cup B}^{\underline{t}}$, where $\underline{t}$ is the vector indexed by $A \cup B$ given by $t_x = r_x + s_x$ if $x \in A \cap B$, $t_x = r_x$ if $x \in A \setminus B$, and $t_x = s_x$ if $x \in B \setminus A$. Thus, with the obvious abbreviations,

$$\begin{aligned} \pi(\sigma_A^{\underline{r}} \sigma_B^{\underline{s}}) &= \phi(t_j \equiv 0 \ \forall j) \\ &\geq \phi(r_j \equiv 0 \ \forall j \text{ and } s_j \equiv 0 \ \forall j) \\ &\geq \phi(r_j \equiv 0 \ \forall j) \phi(s_j \equiv 0 \ \forall j) \\ &= \pi(\sigma_A^{\underline{r}}) \pi(\sigma_B^{\underline{s}}), \end{aligned}$$

where the second inequality follows from positive association of ϕ, Theorem 2.2.14. □

In the Ising model, the covariance (2.2.33) is related to the derivative of $\langle \sigma_A \rangle$ with respect to the coupling strengths; thus it follows from (2.2.33) that $\langle \sigma_A \rangle$ is increasing in these quantities. Here is the corresponding result for the Potts model.

Let $A \subseteq K$ be a finite set, and let $R \subseteq K$ be a finite union of positive length intervals whose interiors are disjoint from A. We write Λ' for the region corresponding to $K' = K \setminus R$. If $b = (P_1, \dots, P_m)$ we define the boundary condition $b' = (P_1', \dots, P_m')$, where $P_i' = P_i \setminus R$. Thus b' agrees with b on $\hat{\partial}\Lambda$, but is 'free' on $\hat{\partial}\Lambda' \setminus \hat{\partial}\Lambda$. See Figure 2.6. Similar results hold for other b'.

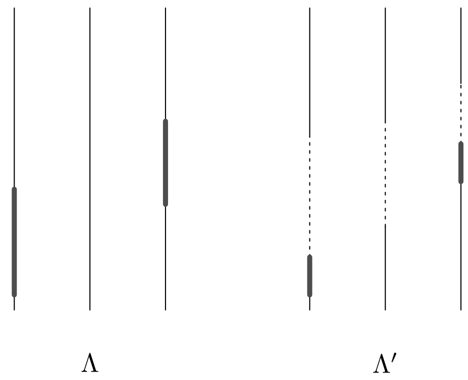

FIGURE 2.6. Left: a region Λ with the boundary condition $b = \{P_1\}$, where $P_1 \setminus \{\Gamma\}$ is drawn bold. Right: the corresponding region Λ' when the set R, drawn dashed, has been removed; the boundary condition is $b' = \{P_1'\}$ where $P_1' = P_1 \setminus R$ and $P_1' \setminus \{\Gamma\}$ is drawn bold. In this picture we have not specified which endpoints of R belong to R.

LEMMA 2.2.22. *The average $\pi_\Lambda^b(\sigma_A^r)$ is increasing in λ and γ and decreasing in δ. Moreover,*

$$\pi_{\Lambda'}^{b'}(\sigma_A^r) \leq \pi_\Lambda^b(\sigma_A^r). \tag{2.2.36}$$

We interpret $\pi_{\Lambda'}^{b'}(\sigma_A^r)$ as 0 when A intersects the interior of R.

PROOF. The claim about monotonicity in λ, γ, δ follows from the stochastic ordering of random-cluster measures, Theorem 2.2.12, and the representation in Lemma 2.2.21. Let us prove (2.2.36). It suffices to consider the case when $R = I$ is a single interval. First note that

$$\pi^b_\Lambda(\sigma^{\underline{r}}_A) = \phi^b_\Lambda(T) \geq \tilde{\phi}^b_\Lambda(T), \tag{2.2.37}$$

where T is the event on the right-hand-side of Lemma 2.2.21, and $\tilde{\phi}^b_\Lambda$ is the measure ϕ^b_Λ with γ set to zero on I, and $\lambda(e)$ set to zero whenever $e \notin F'$. Hence, using also Corollary 2.1.5,

$$\pi^{b'}_{\Lambda'}(\sigma^{\underline{r}}_A) = \phi^{b'}_{\Lambda'}(T) = \tilde{\phi}^b_\Lambda(T \mid D \cap I \neq \varnothing) \leq \frac{\tilde{\phi}^b_\Lambda(T)}{1 - e^{-\delta(I)}} \leq \frac{\pi^b_\Lambda(\sigma^{\underline{r}}_A)}{1 - e^{-\delta(I)}}, \tag{2.2.38}$$

where

$$\delta(I) = \int_I \delta(x) dx.$$

The left-hand-side of (2.2.38) does not depend on the value of δ on I, so we may let $\delta \to \infty$ on I to deduce the result. □

Example 2.2.23. *Here is a consequence of Lemma 2.2.20 when $\underline{r}$ is not constant. Let $x, y \in K$, and write $\tau_{xy} = \sigma_x \sigma_y^{-1}$. Then τ_{xy} is a qth root of unity, and it follows that*

$$\mathbb{1}\{\nu_x = \nu_y\} = \mathbb{1}\{\sigma_x = \sigma_y\} = \frac{1}{q} \sum_{r=0}^{q-1} \tau^r_{xy}. \tag{2.2.39}$$

So if $z, w \in K$ too then

$$\begin{aligned} \pi^b_\Lambda(\nu_x = \nu_y,\, \nu_z = \nu_w) &= \frac{1}{q^2} \sum_{r,s=0}^{q-1} \pi^b_\Lambda(\tau^r_{xy} \tau^s_{zw}) \\ &\geq \frac{1}{q^2} \sum_{r,s=0}^{q-1} \pi^b_\Lambda(\tau^r_{xy}) \pi^b_\Lambda(\tau^s_{zw}) \\ &= \pi^b_\Lambda(\nu_x = \nu_y) \pi^b_\Lambda(\nu_z = \nu_w). \end{aligned} \tag{2.2.40}$$

This inequality does not quite follow from the correlation/connection property of Proposition 2.1.7 when $q > 2$. In the case when $\gamma = 0$ it follows straight away from the Edwards–Sokal coupling, without using stochastic domination properties of the random-cluster model; see [**43**, Corollary 6.5].

2.3. Infinite-volume random-cluster measures

In this section we define random-cluster measures on the *unbounded* spaces $\mathbf{\Theta}, \mathbf{\Theta}_\beta$ of (2.1.4) and (2.1.10), for which Definition 2.1.3 cannot make sense (since k will be infinite). One standard approach in statistical physics is to study the class of measures which satisfy a conditioning property similar to that of Proposition 2.1.4 for all bounded regions; the first task is then to show that this class is nonempty. The book [**42**] is dedicated to this approach for classical models. We will instead follow the route of proving weak convergence as the bounded regions Λ grow. In doing so we follow standard methods (see [**50**, Chapter 4]), adapted to the current setting. See also [**8**] for results of this type.

Central to the topic of infinite-volume measures is the question when there is a unique such measure. There may in general be multiple such measures, obtainable by passing to the limit using different boundary conditions. Non-uniqueness of infinite-volume measures is intimately related to the concept of phase transition described in the Introduction. Intuitively, if there is not a unique limiting measure this means that the boundary conditions have an 'infinite range' effect, and that the system does not know what state to favour, indicating a transition from one preferred state to another.

2.3.1. Weak limits. We fix $q \geq 1$ and non-negative bounded measurable functions λ, δ, γ. Let L_n be a sequence of subgraphs of $\mathbb{L}$ and β_n a sequence of positive numbers. Writing Λ_n for the simple region given by L_n and β_n as in (2.1.7), we say that $\Lambda_n \uparrow \mathbf{\Theta}$ if $L_n \uparrow \mathbb{L}$ and $\beta_n \to \infty$. We assume throughout that L_n and β_n are strictly increasing. Versions of the results in this section are valid also when $\beta < \infty$ is kept fixed as $L_n \uparrow \mathbb{L}$ so that $\Lambda_n \uparrow \mathbf{\Theta}_\beta$ given in (2.1.10). We will only supply proofs in the $\beta_n \to \infty$ case as the $\beta < \infty$ case is similar.

Recall that a sequence ψ_n of probability measures on $(\Omega, \mathcal{F})$ is *tight* if for each $\varepsilon > 0$ there is a compact set A_ε such that $\psi_n(A_\varepsilon) \geq 1 - \varepsilon$ for all n. Here compactness refers, of course, to the Skorokhod topology outlined in Section 2.1 and defined in detail in Appendix A.

Let $\phi_n^b := \phi_{\Lambda_n}^b$. The proof of the following result is given in Appendix A.

LEMMA 2.3.1. *For any sequence of boundary conditions b_n on Λ_n, the sequence of measures $\{\phi_n^{b_n} : n \geq 1\}$ is tight.*

For $x = (e, t) \in \mathbb{F}$ with $t \geq 0$ (respectively $t < 0$), let $V_x(\omega)$ denote the number of elements of the set $B \cap (\{e\} \times [0, t])$ (respectively $B \cap (\{e\} \times (-t, 0]))$. Similarly, for $x \in \mathbb{K} \times \{\mathrm{d}\}$ and $x \in \mathbb{K} \times \{\mathrm{g}\}$, define V_x to count the number of deaths and ghost-bonds between x and the origin, respectively. An event of the form

$$R = \{\omega \in \Omega : V_{x_1}(\omega) \in A_1, \ldots, V_{x_m}(\omega) \in A_m\} \in \mathcal{F}$$

for $m \geq 1$ and the $A_i \subseteq \mathbb{Z}$ is called a *finite-dimensional cylinder event.* For $z = (z_1, \ldots, z_m)$ and $z' = (z'_1, \ldots, z'_m)$ elements of $\mathbb{Z}^m$, we write $z' \geq z$ if $z'_i \geq z_i$ for all $i = 1, \ldots, m$; we write $z' > z$ if $z' \geq z$ and $z' \neq z$.

THEOREM 2.3.2. *Let $b \in \{\mathrm{f}, \mathrm{w}\}$ and $q \geq 1$. The sequence of measures ϕ_n^b converges weakly to a probability measure. The limit measure does not depend on the choice of sequence $\Lambda_n \uparrow \boldsymbol{\Theta}$.*

The limiting measure in Theorem 2.3.2 will be denoted ϕ^b, or $\phi_{q,\lambda,\delta,\gamma}^{b,\beta}$ if the parameters need to be emphasized; here $\beta \in (0, \infty]$.

PROOF. Consider the case $b = \mathrm{w}$. Let Λ be a simple region and $f : \Omega \to \mathbb{R}$ an increasing, $\mathcal{F}_\Lambda$-measurable function. Let n be large enough so that $\Lambda_n \supseteq \Lambda$ and let $\mathcal{C}$ be the event that all components inside Λ_n which intersect $\hat{\partial}\Lambda_n$ are connected in Λ_{n+1}. Then by Corollary 2.1.5

and the FKG-property we have that

$$\phi_n^{\mathrm{w}}(f) = \phi_{n+1}^{\mathrm{w}}(f \mid \mathcal{C}) \geq \phi_{n+1}^{\mathrm{w}}(f), \tag{2.3.1}$$

which is to say that $\phi_n^{\mathrm{w}} \geq \phi_{n+1}^{\mathrm{w}}$. At this point we could appeal to Corollary IV.6.4 of [**71**], which proves that a sequence of probability measure which is tight and stochastically ordered as in (2.3.1) necessarily converges weakly. However, we shall later need to know that the finite dimensional distributions converge, so we prove this now; it then follows from tightness and standard properties of the Skorokhod topology that the sequence converges weakly.

Let $x_1, \dots, x_k \in F \cup (K \times \{\mathrm{g}\})$ and let $x_{k+1}, \dots, x_m \in K \times \{\mathrm{d}\}$. For $z = (z_1, \dots, z_m) \in \mathbb{Z}^m$, write

$$\tilde{z} = (z_1, \dots, z_k, -z_{k+1}, \dots, -z_m).$$

Let $V = V(\omega) = (V_{x_1}(\omega), \dots, V_{x_m}(\omega))$ and for $A \subseteq \mathbb{Z}^m$ consider the finite-dimensional cylinder event $R = \{V \in A\}$. We have that

$$\begin{aligned} \phi_n^{\mathrm{w}}(R) &= \sum_{z\in A} \phi_n^{\mathrm{w}}(V = z) = \sum_{z \in A} \phi_n^{\mathrm{w}}(\tilde{V} = \tilde{z}) \\ &= \sum_{z\in A} [\phi_n^{\mathrm{w}}(\tilde{V} \geq \tilde{z}) - \phi_n^{\mathrm{w}}(\tilde{V} > \tilde{z})]. \end{aligned} \tag{2.3.2}$$

The events $\{\tilde{V} \geq \tilde{z}\}$ and $\{\tilde{V} > \tilde{z}\}$ are both increasing, so by (2.3.1) the limits

$$\overline{\phi}(\tilde{V} \geq \tilde{z}) = \lim_{n\to\infty} \phi_n^{\mathrm{w}}(\tilde{V} \geq \tilde{z}) \quad \text{and} \quad \overline{\phi}(\tilde{V} > \tilde{z}) = \lim_{n\to\infty} \phi_n^{\mathrm{w}}(\tilde{V} > \tilde{z})$$

exist. Define $\overline{\phi}$ by

$$\overline{\phi}(R) := \sum_{z\in A} [\overline{\phi}(\tilde{V} \geq \tilde{z}) - \overline{\phi}(\tilde{V} > \tilde{z})].$$

Then, by the bounded convergence theorem, $\overline{\phi}$ defines a probability measure on the algebra of finite-dimensional cylinder events in $\mathcal{F}_\Lambda$. Thus $\overline{\phi}$ extends to a unique probability measure ϕ^{w} on $\mathcal{F}_\Lambda$ (see [**12**, Theorem 3.1]). Since $\phi_n^{\mathrm{w}}(R) \to \phi^{\mathrm{w}}(R)$ for all finite-dimensional cylinder events in $\mathcal{F}_\Lambda$ and since the sequence $(\phi_n^{\mathrm{w}} : n \geq 1)$ is tight, it follows that $\phi_n^{\mathrm{w}} \Rightarrow \phi^{\mathrm{w}}$ on $(\Omega, \mathcal{F}_\Lambda)$. Since Λ was arbitrary and the $\mathcal{F}_\Lambda$ generate $\mathcal{F}$, the convergence for $b = \mathrm{w}$ follows.

For the independence of the choice of sequence Λ_n, let also $\Delta_n \uparrow \mathbf{\Theta}$. Let m be an integer, and choose $l = l(m)$ and $n = n(m)$ so that $\Lambda_l \subseteq \Delta_m \subseteq \Lambda_n$. We have that

$$\phi^{\mathrm{w}}_{\Lambda_l} \geq \phi^{\mathrm{w}}_{\Delta_m} \geq \phi^{\mathrm{w}}_{\Lambda_n},$$

so letting $m \to \infty$ tells us that the limits are equal (see Remark 2.3.3).

The arguments for $b = \mathrm{f}$ are similar. $\square$

REMARK 2.3.3. *If ψ_1, ψ_2 are two probability measures on $(\Omega, \mathcal{F})$ such that both $\psi_1 \geq \psi_2$ and $\psi_2 \geq \psi_1$ then $\psi_1 = \psi_2$. To see this, note that for R any finite-dimensional cylinder event, we may as in* (2.3.2) *write*

$$\psi_j(R) = \sum_{z \in A} [\psi_j(\tilde{V} \geq \tilde{z}) - \psi_j(\tilde{V} > \tilde{z})], \quad j = 1, 2.$$

It follows that $\psi_1(R) = \psi_2(R)$ for all such R, and hence that $\psi_1 = \psi_2$ (see Appendix A).

For any sequence b_n of boundary conditions, *if* the sequence of measures $(\phi_n^{b_n} : n \geq 1)$ has a weak limit ϕ, then $\phi^{\mathrm{f}} \leq \phi \leq \phi^{\mathrm{w}}$; this follows from the second part of Theorem 2.2.2. Hence there is a unique random-cluster measure if and only if $\phi^{\mathrm{f}} = \phi^{\mathrm{w}}$. It turns out that the set of real triples $(\lambda, \delta, \gamma)$ such that there is *not* a unique random-cluster measure has Lebesgue measure zero, see Theorem 2.3.13.

2.3.2. Basic properties. Some further properties of the measures ϕ^b, for $b \in \{\mathrm{f}, \mathrm{w}\}$, follow, all being straightforward adaptations of standard results, as summarized in [**50**, Section 4.3]. First, recall the upper and lower bounds on the probabilities of seeing no bridges, deaths or ghost-bonds in small regions which is provided by Proposition 2.2.15, as well as the notation introduced there.

LEMMA 2.3.4 (Finite energy property). *Let $q \geq 1$ and let $I \subseteq \mathbb{K}$ and $J \subseteq \mathbb{F}$ be bounded intervals. Then for $b \in \{\mathrm{f}, \mathrm{w}\}$ we have that*

$$\eta_\lambda \leq \phi^b(|B \cap J| = 0 \mid \mathcal{T}_J) \leq \eta^\lambda$$

$$\eta_\delta \leq \phi^b(|D \cap I| = 0 \mid \mathcal{T}_I) \leq \eta^\delta$$

$$\eta_\gamma \leq \phi^b(|G \cap I| = 0 \mid \mathcal{T}_I) \leq \eta^\gamma$$

The same result holds for any weak limit of random-cluster measures with $q > 0$; we assume that $q \geq 1$ and $b \in \{\mathrm{f}, \mathrm{w}\}$ only because then we know that the measures ϕ^b_Λ converge.

PROOF. Recall the notation $V_x(\omega)$ introduced before Theorem 2.3.2, and note that the event $\{|B \cap J| = 0\}$ is a finite-dimensional cylinder event. For $J \subseteq \mathbb{F}$ as in the statement, let $x_1, x_2, \ldots$ be an enumeration of the points in $(\mathbb{K} \times \{\mathrm{d}\}) \cup (\mathbb{K} \times \{\mathrm{g}\}) \cup (\mathbb{F} \setminus J)$ with rational $\mathbb{R}$-coordinate. We have that $\mathcal{T}_J = \sigma(V_{x_1}, V_{x_2}, \ldots)$ so by the martingale convergence theorem

$$\phi^b(|B \cap J| = 0 \mid \mathcal{T}_J) = \lim_{n\to\infty} \phi^b(|B \cap J| = 0 \mid V_{x_1}, \ldots, V_{x_n}).$$

For $\underline{z} \in \mathbb{Z}^n$, let $A_{\underline{z}} = \{(V_{x_1}, \ldots, V_{x_n}) = \underline{z}\}$. Then

$$\begin{aligned}
\phi^b(|B \cap J| = 0 \mid \mathcal{F}_n) &= \sum_{\underline{z}\in\mathbb{Z}^n} \frac{\phi^b(A_{\underline{z}}, \{|B \cap J| = 0\})}{\phi^b(A_{\underline{z}})} \mathbb{I}_{A_{\underline{z}}} \\
&= \lim_{\Delta} \sum_{\underline{z}\in\mathbb{Z}^n} \frac{\phi^b_\Delta(A_{\underline{z}}, \{|B \cap J| = 0\})}{\phi^b_\Delta(A_{\underline{z}})} \mathbb{I}_{A_{\underline{z}}} \\
&= \lim_{\Delta} \phi^b_\Delta(|B \cap J| = 0 \mid \mathcal{F}_n).
\end{aligned}$$

The result now follows from Proposition 2.2.15. A similar argument holds for $\{|D \cap I| = 0\}$ and $\{|G \cap I| = 0\}$. $\square$

Define an *automorphism* on $\boldsymbol{\Theta}$ to be a bijection $T : \boldsymbol{\Theta} \to \boldsymbol{\Theta}$ of the form $T = (\alpha, g) : (x, t) \mapsto (\alpha(x), g(t))$ where $\alpha : \mathbb{V} \to \mathbb{V}$ is an automorphism of the graph $\mathbb{L}$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous bijection. Thus α

has the property that $\alpha(x)\alpha(y) \in \mathbb{E}$ if and only if $xy \in \mathbb{E}$. For T an automorphism and $\omega = (B, D, G) \in \Omega$, let $T(\omega) = (T(B), T(D), T(G))$. For $f : \Omega \to \mathbb{R}$ measurable, let $(f \circ T)(\omega) = f(T(\omega))$, and for ϕ a measure on $(\Omega, \mathcal{F})$ define $\phi \circ T(f) = \phi(T(f))$.

LEMMA 2.3.5. *Let $b \in \{\mathrm{f}, \mathrm{w}\}$ and let T be an automorphism of $\mathbf{\Theta}$ such that $\lambda = \lambda \circ T$, $\gamma = \gamma \circ T$ and $\delta = \delta \circ T$. Then ϕ^b is invariant under T, that is $\phi^b = \phi^b \circ T$.*

PROOF. Let f be a measurable function. Under the given assumptions, we have that for any region Λ,

$$\phi^b_\Lambda(f \circ T) = \int f(T(\omega))\, d\phi^b_\Lambda(\omega) = \int f(\omega)\, d\phi^b_{T^{-1}(\Lambda)}(\omega) = \phi^b_{T^{-1}(\Lambda)}(f).$$

The result now follows from Theorem 2.3.2. □

PROPOSITION 2.3.6. *The tail σ-algebra $\mathcal{T}$ is trivial under the measures ϕ^{f} and ϕ^{w}, in that $\phi^b(A) \in \{0, 1\}$ for all $A \in \mathcal{T}$.*

PROOF. Let $\Lambda \subseteq \Delta$ be two regions. We treat the case when $b = \mathrm{f}$, the case $b = \mathrm{w}$ follows similarly on reversing several of the inequalities below. Let $A \in \mathcal{F}_\Lambda$ be an increasing finite-dimensional cylinder event, and let $B \in \mathcal{F}_{\Delta\setminus\Lambda} \subseteq \mathcal{T}_\Lambda$ be an arbitrary finite-dimensional cylinder event. We may assume without loss of generality that $\phi^{\mathrm{f}}_\Delta(B) > 0$. By the conditioning property Proposition 2.1.4 and the stochastic ordering of Theorem 2.2.12, we have that

$$\phi^{\mathrm{f}}_\Delta(A \cap B) = \phi^{\mathrm{f}}_\Delta(A \mid B)\phi^{\mathrm{f}}_\Delta(B) \geq \phi^{\mathrm{f}}_\Lambda(A)\phi^{\mathrm{f}}_\Delta(B). \tag{2.3.3}$$

Let $\mathcal{R}$ denote the set of finite-dimensional cylinder events in $\mathcal{T}_\Lambda$. Letting $\Delta \uparrow \mathbf{\Theta}$ implies that

$$\phi^{\mathrm{f}}(A \cap B) \geq \phi^{\mathrm{f}}_\Lambda(A)\phi^{\mathrm{f}}(B) \tag{2.3.4}$$

for all $B \in \mathcal{R}$ and all increasing finite-dimensional cylinder events $A \in \mathcal{F}_\Lambda$. The set $\mathcal{R}$ is an algebra, so for fixed A the difference between the left and right sides of (2.3.4) extends to a finite measure ψ on $\mathcal{T}_\Lambda$, and by the uniqueness of this extension it follows that $0 \le \psi(B) = \phi^{\mathrm{f}}(A \cap B) - \phi^{\mathrm{f}}_\Lambda(A)\phi^{\mathrm{f}}(B)$ for all $B \in \mathcal{T}_\Lambda \subseteq \mathcal{T}$. Thus we may let $\Lambda \uparrow \boldsymbol{\Theta}$ to deduce that

$$\phi^{\mathrm{f}}(A \cap B) \ge \phi^{\mathrm{f}}(A)\phi^{\mathrm{f}}(B) \tag{2.3.5}$$

for all increasing finite-dimensional cylinder events $A \in \mathcal{F}_\Lambda$ and all $B \in \mathcal{T}$. However, (2.3.5) also holds with B replaced by its complement B^c; since

$$\phi^{\mathrm{f}}(A \cap B) + \phi^{\mathrm{f}}(A \cap B^c) = \phi^{\mathrm{f}}(A)\phi^{\mathrm{f}}(B) + \phi^{\mathrm{f}}(A)\phi^{\mathrm{f}}(B^c)$$

it follows that

$$\phi^{\mathrm{f}}(A \cap B) = \phi^{\mathrm{f}}(A)\phi^{\mathrm{f}}(B) \tag{2.3.6}$$

for all increasing finite-dimensional cylinder events $A \in \mathcal{F}_\Lambda$ and all $B \in \mathcal{T}$. For fixed B, the left and right sides of (2.3.6) are finite measures which agree on all increasing events $A \in \mathcal{F}_\Lambda$. Using the reasoning of Remark 2.3.3, it follows that (2.3.6) holds for all $A \in \mathcal{F}_\Lambda$, and hence also for all $A \in \mathcal{F}$. Setting $A = B \in \mathcal{T}$ gives the result. $\Box$

In the case when $\mathbb{L} = \mathbb{Z}^d$ and λ, δ, γ are constant, define the automorphisms T_x, for $x \in \mathbb{Z}^d$, by

$$T_x(y,t) = (y + x, t).$$

The T_x are called *translations*. An event $A \in \mathcal{F}$ is called T_x-invariant if $A = T_x^{-1}A$. The following ergodicity result is a standard consequence of Proposition 2.3.6, see for example [**42**, Proposition 14.9] (here 0 denotes the element $(0,\dots,0)$ of $\mathbb{Z}^d$).

LEMMA 2.3.7. *Let $x \in \mathbb{Z}^d \setminus \{0\}$ and $b \in \{\mathrm{f}, \mathrm{w}\}$. If $A \in \mathcal{F}$ is T_x-invariant then $\phi^b(A) \in \{0,1\}$.*

2.3.3. Phase transition. In the random-cluster model, the probability that there is an unbounded connected component serves as 'order parameter': depending on the values of the parameters λ, δ, γ this probability may be zero or positive. We show in this section that one may define a critical point for this probability, and then establish some very basic facts about the phase transition. We assume throughout this section that $\gamma = 0$, that $q \geq 1$, that $\lambda \geq 0$, $\delta > 0$ are constant, and that $\mathbb{L} = \mathbb{Z}^d$ for some $d \geq 1$. Some of the results hold for more general $\mathbb{L}$, but we will not pursue this here. The boundary condition b will denote either f or w throughout.

Let $\{0 \leftrightarrow \infty\}$ denote the event that the origin lies in an unbounded component. Define for $0 < \beta \leq \infty$,

$$\theta^{b,\beta}(\lambda, \delta, q) := \phi^{b,\beta}_{q,\lambda,\delta}(0 \leftrightarrow \infty). \tag{2.3.7}$$

When $\beta = \infty$ a simple rescaling argument implies that $\theta^{b,\infty}(\lambda, \delta, q)$ depends on λ, δ through the ratio $\rho = \lambda/\delta$ only. Hence we will often in what follows set $\delta = 1$ and $\lambda = \rho$, and define for $0 < \beta \leq \infty$

$$\theta^{b,\beta}(\rho) = \theta^{b,\beta}(\rho, q) := \phi^{b,\beta}_{q,\rho,1}(0 \leftrightarrow \infty). \tag{2.3.8}$$

By the stochastic monotonicity of Theorem 2.2.12, and a small argument justifying its application to the event $\{0 \leftrightarrow \infty\}$, the quantity $\theta^b(\rho)$ is increasing in ρ.

Definition 2.3.8. *For $b \in \{\mathrm{f}, \mathrm{w}\}$ and $0 < \beta \leq \infty$ we define the* critical value

$$\rho^{b,\beta}_{\mathrm{c}}(q) := \sup\{\rho \geq 0 : \theta^{b,\beta}(q, \rho) = 0\}.$$

In what follows we will usually suppress reference to β. We will see in Section 2.3.4 that $\rho^{\mathrm{f}}(q) = \rho^{\mathrm{w}}(q)$ for all $q \geq 1$. Therefore we will write $\rho_{\mathrm{c}}(q)$ for their common value. We write ϕ^b_ρ for $\phi^{b,\beta}_{q,\rho,1}$.

One may adapt standard methods (see [**50**, Theorem 5.5]) to prove the following:

THEOREM 2.3.9. *Unless* $d = 1$ *and* $\beta < \infty$ *we have that*

$$0 < \rho_c(q) < \infty.$$

(If $d = 1$ and $\beta < \infty$ then a standard zero-one argument, involving comparison to percolation and the second Borel–Cantelli lemma, implies that $\rho_{\mathrm{c}} = 0$.)

Fix $\rho > 0$ and for $\omega \in \Omega$ let $N = N(\omega)$ denote the number of distinct unbounded components in ω. By Lemma 2.3.7, using for example the translation $T : (x, t) \mapsto (x + 1, t)$, we have that N is almost surely constant under the measures $\phi_\rho^b(\cdot)$, $b \in \{\mathrm{f}, \mathrm{w}\}$.

THEOREM 2.3.10. *The number* N *of unbounded components is either* 0 *or* 1 *almost surely under* ϕ_ρ^b.

PROOF. We follow the strategy of [**18**], and as previously we provide details only in the $\beta = \infty$ case. We first show that $N \in \{0, 1, \infty\}$ almost surely. Suppose to the contrary that there exists $2 \le m < \infty$ such that $N = m$ almost surely. Then we may choose (deterministic) n, β sufficiently large that the corresponding simple region $\Lambda_n = \Lambda_n(\beta)$, regarded as a subset of $\mathbf{\Theta}$, has the property that $\phi_\rho^b(A) > 0$, where A is the event that the m distinct unbounded components all meet $\partial\Lambda_n$. Let C be the event that all points in $\partial\Lambda_n$ are connected inside Λ_n. By the finite energy property, Lemma 2.3.4, we have that $\phi_\rho^b(C \mid A) > 0$, and hence $\phi_\rho^b(C \cap A) > 0$. But on $\{C \cap A\}$ we have $N = 1$, a contradiction. Thus $N \in \{0, 1, \infty\}$.

Now suppose that $N = \infty$ almost surely. Let $\beta = 2n$, and for $v \in \mathbb{V}$ and $r \in \mathbb{Z}$ let

$$I_{v,r} = \{v\} \times [r, r + 1] \subseteq \mathbb{K}. \tag{2.3.9}$$

We call $I_{v,r}$ a *trifurcation* if (i) it is contained in exactly one unbounded component, and (ii) if one removes all bridges incident on $I_{v,r}$ and places a least one death in $I_{v,r}$, then the unbounded component containing it breaks into three distinct unbounded components. See Figure 2.7.

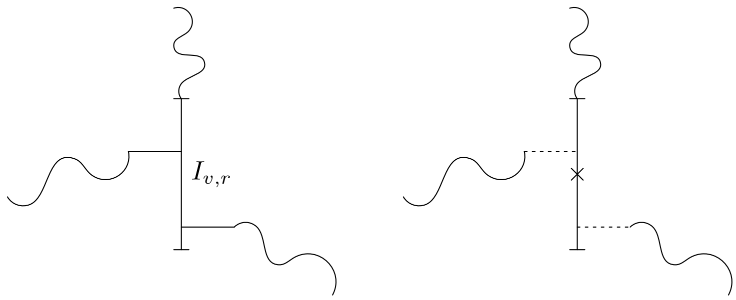

FIGURE 2.7. A trifurcation interval (left); upon removing all incident bridges and placing a death in the interval, the unbounded cluster breaks in three (right).

We claim that

$$\phi_\rho^b(I_{0,0} \text{ is a trifurcation}) > 0. \tag{2.3.10}$$

To see this let n be large enough so that $\partial\Lambda_n$ meets three distinct unbounded components with positive probability. Conditional on $\mathcal{T}_{\Lambda_n}$, the finite energy property Lemma 2.3.4 allows us to modify the configuration inside Λ_n so that, with positive probability, $I_{0,0}$ is a trifurcation.

We note from translation invariance, Lemma 2.3.5, that the number T_n of trifurcations in Λ_n satisfies

$$\begin{aligned}\phi_\rho^b(T_n) &= \sum_{\substack{v\in[-n,n]^d\\ r=-n,\dots,n-1}} \phi_\rho^b(I_{v,r} \text{ is a trifurcation})\\ &= 2n(2n+1)^d\phi_\rho^b(I_{0,0} \text{ is a trifurcation}).\end{aligned} \tag{2.3.11}$$

Define the *sides* of Λ_n to be the union of all intervals $v\times[-n,n]$ where v has at least one coordinate which is $\pm n$. Topological considerations imply that T_n is bounded from above by the total number of deaths on the sides of Λ_n plus twice the number of vertices in $[-n,n]^d$. (Each trifurcation needs at least one unique point of exit from Λ_n). Using the stochastic domination in Corollary 2.2.13 or otherwise, it follows that $\phi_\rho^b(T_n) \le 2(2n+1)^d+\delta\cdot 4dn(2n+1)^{d-1}$. In view of (2.3.10) and (2.3.11) this is a contradiction. See [**16**, Chapter 5] for more details on the topological aspects of this argument. □

It follows from Theorem 2.3.10 that $N = 0$ almost surely under ϕ_{c}^{b} if $\rho < \rho_c$ and that $N = 1$ almost surely if $\rho > \rho_c$. It is crucial for the proof that $\mathbb{L} = \mathbb{Z}^d$ is 'amenable' in the sense that the boundary of $[-n, n]^d$ is an order of magnitude smaller than the volume. The result fails, for example, when $\mathbb{L}$ is a tree, in which case $N = \infty$ may occur; see [**75**] for the corresponding phenomenon in the contact process.

2.3.4. Convergence of pressure. In this section we adapt the well-known 'convergence of pressure' argument to the space–time random-cluster model. By relating the question of uniqueness of measures to that of the existence of certain derivatives, we are able to deduce that there is a unique infinite-volume measure at almost every $(\lambda, \delta, \gamma)$, see Theorem 2.3.13 below. Arguments of this type are 'folklore' in statistical physics, and appear in many places such as [**29, 42, 60**]. We follow closely the corresponding method for the discrete random-cluster model given in [**50**, Chapter 4].

Let $\lambda, \delta, \gamma > 0$ be constants. We will for simplicity of presentation be treating only the case when $\gamma > 0$ and $q \geq 1$, though similar arguments hold when $\gamma = 0$ and when $0 < q < 1$. The partition function

$$Z_\Lambda^b(\lambda, \delta, \gamma, q) = \int_\Omega q^{k_\Lambda^b(\omega)}\, d\mu_{\lambda,\delta,\gamma}(\omega) \tag{2.3.12}$$

is now a function $\mathbb{R}_+^4 \to \mathbb{R}$. In this section we will study the related *pressure functions*

$$P_\Lambda^b(\lambda, \delta, \gamma, q) = \frac{1}{|\Lambda|} \log Z_\Lambda^b(\lambda, \delta, \gamma, q). \tag{2.3.13}$$

Here, and in what follows, we have abused notation by writing $|\Lambda|$ for the (one-dimensional) Lebesgue measure $|K|$ of K, where $\Lambda = (K, F)$. We will be considering limits of P_Λ^b as the region Λ grows. To be concrete we will be considering regions of the form

$$\Lambda = \Lambda_{\underline{n},\beta} \equiv \{1,\dots,n_1\} \times \cdots \{1,\dots,n_d\} \times [0,\beta] \tag{2.3.14}$$

and limits when $\Lambda \uparrow \mathbf{\Theta}$, that is to say all $n_1,\dots,n_d,\beta \to \infty$ (simultaneously). Strictly speaking such regions do not tend to $\mathbf{\Theta}$, but the P^b_Λ are not affected by translating Λ. It will be clear from the arguments that one may deal in the same way with limits as $\Lambda \uparrow \mathbf{\Theta}_\beta$ with $\beta < \infty$ fixed. When $\underline{n}$ and β need to be emphasized we will write $\Lambda_{\underline{n},\beta} = (K_{\underline{n},\beta}, F_{\underline{n},\beta})$.

Here is a simple observation about Z^b_Λ. Writing

$$r = \log\lambda, \quad s = \log\delta, \quad t = \log\gamma, \quad u = \log q, \tag{2.3.15}$$

and

$$D_\Lambda = D \cap K, \quad G_\Lambda = G \cap K, \quad B_\Lambda = B \cap F, \tag{2.3.16}$$

we have that

(2.3.17)
$$\begin{aligned} Z^b_\Lambda(r,s,t,u) &\equiv Z^b_\Lambda(\lambda,\delta,\gamma,q) \\ &= \int_\Omega d\mu_{1,1,1}(\omega)\exp\big(r|B_\Lambda| + s|D_\Lambda| + t|G_\Lambda| + uk^b_\Lambda\big). \end{aligned}$$

(Where $\mu_{1,1,1}$ is the percolation measure where B, D, G all have rate 1.) This follows from basic properties of Poisson processes. It will sometimes be more convenient to work with $Z^b_\Lambda(r,s,t,u)$ in this form. We will also write $P^b_\Lambda(r,s,t,u)$ for the pressure (2.3.13) using these parameters (2.3.15).

Let $\underline{h} = (h_1,\dots,h_4)$ be a unit vector in $\mathbb{R}^4$, and let $y \in \mathbb{R}$. It follows from a simple computation that the function $f(y) = P^b_\Lambda((r,s,t,u)+yh)$ has non-negative second derivative. Indeed, $f''(y)$ is the variance under the appropriate random-cluster measure of the quantity

$$h_1|B_\Lambda| + h_2|D_\Lambda| + h_3|G_\Lambda| + h_4 k^b_\Lambda.$$

Since variances are non-negative, have proved

LEMMA 2.3.11. *Each* $P^b_\Lambda(r,s,t,u)$ *is a convex function* $\mathbb{R}^4 \to \mathbb{R}$.

Our first objective in this section is the following result.

THEOREM 2.3.12. *The limit*

$$P(r,s,t,u) = \lim_{\Lambda \uparrow \mathbf{\Theta}} P^b_\Lambda(r,s,t,u)$$

exists for all $r,s,t,u \in \mathbb{R}$ *and all sequences* $\Lambda \uparrow \mathbf{\Theta}$ *of the form* (2.3.14), *and is independent of the boundary condition* b.

The function P is usually called the *specific Gibbs free energy*, or *free energy* for short. It follows that P is a convex function $\mathbb{R}^4 \to \mathbb{R}$, and hence that the set $\mathcal{D}$ of points in $\mathbb{R}^4$ at which one or more partial derivative of P fails to exist has zero Lebesgue measure. We will return to this observation after the proof of Theorem 2.3.12.

PROOF OF THEOREM 2.3.12. We first prove convergence of P^{f}_Λ with free boundary, and then deduce the result for general b. For each $i = 1,\dots,d$ let $0 < m_i \le n_i$ and also let $0 < \alpha < \beta$. Write $|\underline{m}| = m_1 \cdots m_d$. We may regard the region $\Lambda_{\underline{m},\alpha}$ as a subset of $\Lambda_{\underline{n},\beta}$. Write $T^{\underline{n},\beta}_{\underline{m},\alpha}$ for the set of points in $F_{\underline{n},\beta} \setminus F_{\underline{m},\alpha}$ adjacent to at least one point in $K_{\underline{m},\alpha}$. We have that

$$k^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}} \begin{cases} \le k^{\mathrm{f}}_{\Lambda_{\underline{m},\alpha}} + k^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}\setminus\Lambda_{\underline{m},\alpha}} \\ \ge k^{\mathrm{f}}_{\Lambda_{\underline{m},\alpha}} + k^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}\setminus\Lambda_{\underline{m},\alpha}} - |\underline{m}| - |B \cap T^{\underline{n},\beta}_{\underline{m},\alpha}| - 1. \end{cases} \tag{2.3.18}$$

The lower bound follows because the number of 'extra' components created by 'cutting out' $\Lambda_{\underline{m},\alpha}$ from $\Lambda_{\underline{n},\beta}$ is bounded by the number of intervals constituting $K_{\underline{m},\alpha}$, plus the number of bridges that are cut, plus 1 (for the component of Γ). The upper bound is similar but simpler. Thus

(2.3.19)

$$\log Z^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}} = \log \mu_{\lambda,\delta,\gamma}(q^{k^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}}})$$

$$\begin{cases} \leq \log Z^{\mathrm{f}}_{\Lambda_{\underline{m},\alpha}} + \log Z^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}\setminus\Lambda_{\underline{m},\alpha}} \\ \geq \log Z^{\mathrm{f}}_{\Lambda_{\underline{m},\alpha}} + \log Z^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}\setminus\Lambda_{\underline{m},\alpha}} - \\ \quad -(\log q)|\underline{m}| - \lambda(1-1/q)\alpha d|\underline{m}| \sum_{i=1}^{d} \frac{1}{m_i} - \log q. \end{cases}$$

We have used the fact that

$$|T^{\underline{n},\beta}_{\underline{m},\alpha}| \leq \alpha d|\underline{m}| \sum_{i=1}^{d} \frac{1}{m_i}.$$

There are $\prod_{i=1}^{d} \lfloor n_i/m_i \rfloor \cdot \lfloor \beta/\alpha \rfloor$ 'copies' of $\Lambda_{\underline{m},\alpha}$ in $\Lambda_{\underline{n},\beta}$, each being a translation of $\Lambda_{\underline{m},\alpha}$ by a vector

$$\underline{l} \in \{(b_1 m_1, \ldots, b_d m_d, c\alpha) : b_i = 1, \ldots, \lfloor n_i/m_i \rfloor,\ c = 1, \ldots, \lfloor \beta/\alpha \rfloor\}.$$

Write

(2.3.20)
$$\Lambda = \Big(\bigcup_{\underline{l}} (\Lambda_{\underline{m},\alpha} + \underline{l})\Big) \cup \Lambda';$$

this union is disjoint up to a set of measure zero. Let $\Lambda' = (K', F')$. Repeating the argument leading up to (2.3.19) once for each 'copy' of $\Lambda_{\underline{m},\beta}$ we deduce that $Z^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}}$ is bounded above by

(2.3.21)
$$\Big(\prod_{i=1}^{d} \lfloor n_i/m_i \rfloor \cdot \lfloor \beta/\alpha \rfloor\Big) \log Z^{\mathrm{f}}_{\Lambda_{\underline{m},\alpha}} + \log Z^{\mathrm{f}}_{\Lambda'}$$

and below by the same quantity (2.3.21) minus

(2.3.22)

$$\prod_{i=1}^{d} \lfloor n_i/m_i \rfloor \cdot \lfloor \beta/\alpha \rfloor \Big((\log q)|\underline{m}| + \lambda(1-1/q)\alpha d|\underline{m}| \sum_{i=1}^{d} \frac{1}{m_i} + \log q\Big).$$

We will prove shortly that

(2.3.23)
$$\lim_{n_i,\beta\to\infty} \frac{1}{|\Lambda_{\underline{n},\beta}|} \log Z^{\mathrm{f}}_{\Lambda'} = 0;$$

once this is done it follows on dividing by $|\Lambda_{\underline{n},\beta}| = \beta \cdot |\underline{n}|$ and letting all $n_i, \beta \to \infty$ that

$$\begin{aligned} \frac{1}{|\Lambda_{\underline{m},\alpha}|} \log Z^{\mathrm{f}}_{\Lambda_{\underline{m},\alpha}} &\le \liminf_{n_i,\beta\to\infty} P^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}} \le \limsup_{n_i,\beta\to\infty} P^{\mathrm{f}}_{\Lambda_{\underline{n},\beta}} \\ &\le \frac{1}{|\Lambda_{\underline{m},\alpha}|} \log Z^{\mathrm{f}}_{\Lambda_{\underline{m},\alpha}} + \frac{1}{\alpha} \log q + \\ &\quad + \lambda(1-1/q) d \sum_{i=1}^{d} \frac{1}{m_i} + \frac{1}{|\Lambda_{\underline{m},\alpha}|} \log q, \end{aligned} \tag{2.3.24}$$

and hence that $\lim_\Lambda P^{\mathrm{f}}_\Lambda$ exists and is finite.

Let us prove the claim (2.3.23). The set $K_{\Lambda'}$ consists of a number of disjoint intervals, of which

$$\prod_{i=1}^{d} m_i \lfloor n_i/m_i \rfloor$$

have length $\beta - \alpha\lfloor \beta/\alpha \rfloor$, and

$$\prod_{i=1}^{d} n_i - \prod_{i=1}^{d} m_i \lfloor n_i/m_i \rfloor$$

have length β. The number $k^{\mathrm{f}}_{\Lambda'}$ of components is bounded above by the sum over all such intervals L of $|D \cap L| + 2$ (we have added 1 for the component of Γ). Hence

(2.3.25)

$$\begin{aligned} 0 \le \log Z^{\mathrm{f}}_{\Lambda'} &= \mu_{\lambda,\delta,\gamma}(q^{k^{\mathrm{f}}_{\Lambda'}}) \\ &\le \Big(\prod_{i=1}^{d} m_i \lfloor n_i/m_i \rfloor \Big) \cdot (q-1)\delta(\beta - \alpha\lfloor\beta/\alpha\rfloor) + \\ &\quad + \Big(\prod_{i=1}^{d} n_i - \prod_{i=1}^{d} m_i \lfloor n_i/m_i \rfloor \Big) \cdot (q-1)\delta\beta + 2\log q. \end{aligned}$$

Equation (2.3.23) follows.

Finally, we must prove convergence with arbitrary boundary condition. It is clear that for any boundary condition b we have

$$k^{\mathrm{w}}_\Lambda \le k^{b}_\Lambda \le k^{\mathrm{f}}_\Lambda.$$

On the other hand

$$k_\Lambda^{\mathrm{w}} \geq k_\Lambda^{\mathrm{f}} - 2|\underline{n}| - |D \cap \partial\Lambda| - 1.$$

The result follows. $\square$

We now switch parameters to r, s, t, u, given in (2.3.15). For fixed u (i.e. fixed q) let $\mathcal{D}_u = \mathcal{D}_q$ be the set of points $(r, s, t) \in \mathbb{R}^3$ at which at least one of the partial derivatives

$$\frac{\partial P}{\partial r}, \quad \frac{\partial P}{\partial s}, \quad \frac{\partial P}{\partial t}$$

fails to exist. Since P is convex, $\mathcal{D}_q$ has zero (three-dimensional) Lebesgue measure. By general properties of convex functions, the partial derivatives

$$\frac{\partial P_\Lambda^b}{\partial r}, \quad \frac{\partial P_\Lambda^b}{\partial s}, \quad \frac{\partial P_\Lambda^b}{\partial t}$$

converge to the corresponding derivatives of P whenever $(r, s, t) \notin \mathcal{D}_q$, for any b. Now observe that

$$\begin{aligned}\frac{\partial P_\Lambda^{\mathrm{f}}}{\partial r} &= \frac{1}{|\Lambda|}\phi_\Lambda^{\mathrm{f}}(|B_\Lambda|) \leq \frac{1}{|\Lambda|}\phi^{\mathrm{f}}(|B_\Lambda|) \\ &\leq \frac{1}{|\Lambda|}\phi^{\mathrm{w}}(|B_\Lambda|) \leq \frac{1}{|\Lambda|}\phi_\Lambda^{\mathrm{w}}(|B_\Lambda|) = \frac{\partial P_\Lambda^{\mathrm{w}}}{\partial r},\end{aligned} \tag{2.3.26}$$

so if $(r, s, t) \notin \mathcal{D}_q$ then

$$\lim_{\Lambda \uparrow \mathbf{\Theta}} \frac{1}{|\Lambda|}\phi^{\mathrm{f}}(|B_\Lambda|) = \lim_{\Lambda \uparrow \mathbf{\Theta}} \frac{1}{|\Lambda|}\phi^{\mathrm{w}}(|B_\Lambda|) = \frac{\partial P}{\partial r}. \tag{2.3.27}$$

Recall from Lemma 2.3.5 that ϕ^{f} and ϕ^{w} are both invariant under translations. The set B is a point process on $\mathbb{F}$, which is therefore stationary under both ϕ^{f} and ϕ^{w}, and hence has constant intensities under these measures. Said another way, the *mean measures* $m^{\mathrm{f}}, m^{\mathrm{w}}$ on $(\mathbb{F}, \mathcal{B}(\mathbb{F}))$, given respectively by

$$m^{\mathrm{f}}(F) := \phi^{\mathrm{f}}(|B \cap F|), \quad \text{and} \quad m^{\mathrm{w}}(F) := \phi^{\mathrm{w}}(|B \cap F|)$$

are translation invariant measures. It is therefore a general fact that there are constants $c_{\mathrm{b}}^{\mathrm{f}}$ and $c_{\mathrm{b}}^{\mathrm{w}}$ such that for all regions $\Lambda = (K, F)$,

$$m^{\mathrm{f}}(F) = \phi^{\mathrm{f}}(|B_\Lambda|) = c_{\mathrm{b}}^{\mathrm{f}}|F|, \quad \text{and} \quad m^{\mathrm{w}}(F) = \phi^{\mathrm{w}}(|B_\Lambda|) = c_{\mathrm{b}}^{\mathrm{w}}|F|,$$

where $|\cdot|$ denotes Lebesgue measure. Similarly, there are constants $c_{\mathrm{d}}^{\mathrm{f}}$, $c_{\mathrm{d}}^{\mathrm{w}}$, $c_{\mathrm{g}}^{\mathrm{f}}$ and $c_{\mathrm{g}}^{\mathrm{w}}$ such that

$$\phi^{\mathrm{f}}(|D_\Lambda|) = c_{\mathrm{d}}^{\mathrm{f}}|K|, \quad \text{and} \quad \phi^{\mathrm{w}}(|D_\Lambda|) = c_{\mathrm{d}}^{\mathrm{w}}|K|,$$

and

$$\phi^{\mathrm{f}}(|G_\Lambda|) = c_{\mathrm{g}}^{\mathrm{f}}|K|, \quad \text{and} \quad \phi^{\mathrm{w}}(|G_\Lambda|) = c_{\mathrm{g}}^{\mathrm{w}}|K|,$$

for all regions $\Lambda = (K, F)$.

Note that

$$\lim_{n_i, \beta \to \infty} \frac{|F_{\underline{n},\beta}|}{|K_{\underline{n},\beta}|} = d.$$

It follows from (2.3.27), and similar calculations for D and G, that

$$\text{(2.3.28)} \quad c_{\mathrm{b}}^{\mathrm{f}} = c_{\mathrm{b}}^{\mathrm{w}}, \quad c_{\mathrm{d}}^{\mathrm{f}} = c_{\mathrm{d}}^{\mathrm{w}}, \quad \text{and} \quad c_{\mathrm{g}}^{\mathrm{f}} = c_{\mathrm{g}}^{\mathrm{w}} \quad \text{whenever } (r, s, t) \notin \mathcal{D}_q.$$

Recall the condition given at the end of Section 2.3.1 for the uniqueness of the infinite-volume random-cluster measures, namely that $\phi^{\mathrm{f}} = \phi^{\mathrm{w}}$. We will use the facts listed above to prove

THEOREM 2.3.13. *There is a unique random-cluster measure, in that* $\phi^{\mathrm{f}} = \phi^{\mathrm{w}}$*, whenever* $(r, s, t) \notin \mathcal{D}_q$.

The corresponding results holds when $\gamma \geq 0$ is fixed, in that $\phi^{\mathrm{f}} = \phi^{\mathrm{w}}$ except on a set of points (r, s) of zero (two-dimensional) Lebesgue measure. For also $\delta > 0$ fixed, the corresponding set of λ where uniqueness fails is countable, again by general properties of convex functions. Presumably this latter set consists of a single point, namely the point corresponding to $\rho = \rho_{\mathrm{c}}$, but this has not been proved even for the discrete models.

Proof. Since $\phi^{\mathrm{w}} \geq \phi^{\mathrm{f}}$, there is by Theorem 2.2.2 a coupling $\mathbb{P}$ of the two measures such that

$$\mathbb{P}(\{(\omega^{\mathrm{w}}, \omega^{\mathrm{f}}) \in \Omega^2 : \omega^{\mathrm{w}} \geq \omega^{\mathrm{f}}\}) = 1,$$

and such that ω^{w} and ω^{f} have marginal distributions ϕ^{w} and ϕ^{f} under $\mathbb{P}$, respectively. Write B^b, $b \in \{\mathrm{f}, \mathrm{w}\}$ for the bridges of ω^b, and similarly for deaths and ghost-bonds. Let $A \in \mathcal{F}_\Lambda$ be an increasing event. Then

(2.3.29)

$$\begin{aligned}
0 \leq \phi^{\mathrm{w}}(A) - \phi^{\mathrm{f}}(A) &\leq \mathbb{P}(\omega^{\mathrm{w}} \in A, \omega^{\mathrm{w}} \neq \omega^{\mathrm{f}} \text{ in } \Lambda) \\
&\leq \mathbb{P}(|B^{\mathrm{w}}_\Lambda \setminus B^{\mathrm{f}}_\Lambda| + |D^{\mathrm{f}}_\Lambda \setminus D^{\mathrm{w}}_\Lambda| + |G^{\mathrm{w}}_\Lambda \setminus G^{\mathrm{f}}_\Lambda|) \\
&= \phi^{\mathrm{w}}(|B_\Lambda|) - \phi^{\mathrm{f}}(|B_\Lambda|) + \phi^{\mathrm{f}}(|D_\Lambda|) - \phi^{\mathrm{w}}(|D_\Lambda|) + \\
&\qquad + \phi^{\mathrm{w}}(|G_\Lambda|) - \phi^{\mathrm{f}}(|G_\Lambda|) \\
&= |\Lambda|(c^{\mathrm{w}}_{\mathrm{b}} - c^{\mathrm{f}}_{\mathrm{b}} + c^{\mathrm{f}}_{\mathrm{d}} - c^{\mathrm{w}}_{\mathrm{d}} + c^{\mathrm{w}}_{\mathrm{g}} - c^{\mathrm{f}}_{\mathrm{g}}) \\
&= 0,
\end{aligned}$$

so $\phi^{\mathrm{w}} = \phi^{\mathrm{f}}$ as required. □

Here is a consequence when $\gamma = 0$. Recall that we set $\lambda = \rho$ and $\delta = 1$. Suppose $0 < \rho < \rho'$ are given. We may pick $\lambda_1 = \rho_1$ so that $\rho < \rho_1 < \rho'$ and so that there is a unique infinite-volume measure with parameters $\lambda_1 = \rho_1, \delta_1 = 1$ and $\gamma = 0$. Hence

(2.3.30)
$$\phi^{\mathrm{w}}_\rho \leq \phi^{\mathrm{w}}_{\rho_1} = \phi^{\mathrm{f}}_{\rho_1} \leq \phi^{\mathrm{f}}_{\rho'}.$$

It follows that the critical values $\rho^{\mathrm{f}}_{\mathrm{c}}(q)$ and $\rho^{\mathrm{w}}_{\mathrm{c}}(q)$ of Definition 2.3.8 are equal for all $q \geq 1$.

2.4. Duality in $\mathbb{Z} \times \mathbb{R}$

In this section we let $\mathbb{L} = \mathbb{Z}$. Thanks to the notion of planar duality for graphs, much more is known about the discrete random-cluster model in two dimensions than in general dimension. In particular, the critical value for $q = 1, 2$ and $q \geq 25.72$ has been calculated in two dimensions, see [**3, 62, 63, 64**]. In the space-time setting, the $d = 1$ model occupies the two-dimensional space $\mathbb{Z} \times \mathbb{R}$, so we may adapt duality arguments to this case; that is the objective of this section. Such arguments have been applied when $q = 1$ to prove that $\rho_c(1) = 1$, see [**11**]. We will see in Chapter 4 that $\rho_c(2) = 2$, and Theorem 2.4.3 in the present section is a first step towards this result.

Throughout this section we assume that $\gamma = 0$, and hence suppress reference to both γ and G. We also assume that $q \geq 1$ and that λ, δ are positive constants. In light of Theorem 2.3.9 we may disregard the $\beta < \infty$ case, hence we deal in this section only with the $\beta \to \infty$ case. We think of $\mathbf{\Theta} \equiv \mathbb{Z} \times \mathbb{R}$ as embedded in $\mathbb{R}^2$ in the natural way.

We write $\mathbb{L}_\mathrm{d}$ for $\mathbb{Z}+1/2$; of course $\mathbb{L}$ and $\mathbb{L}_\mathrm{d}$ are isomorphic graphs. With any $\omega = (B, D) \in \Omega$ we associate the 'dual' configuration $\omega_\mathrm{d} := (D, B)$ regarded as a configuration in $\mathbf{\Theta}_\mathrm{d} = \mathbb{L}_\mathrm{d} \times \mathbb{R}$. Thus each bridge in ω corresponds to a death in ω_d, and each death in ω corresponds to a bridge in ω_d. This correspondence is illustrated in Figure 2.8. We identify $\omega_\mathrm{d} = (D, B)$ with the element $(D - 1/2, B - 1/2)$ of Ω. Under this identification we may for any measurable $f : \Omega \to \mathbb{R}$ define $f_\mathrm{d} : \Omega \to \mathbb{R}$ by $f_\mathrm{d}(\omega) = f(\omega_\mathrm{d})$.

In the case when $q = 1$ it is clear that for any measurable function $f : \Omega \to \mathbb{R}$, we have the relation $\mu_{\lambda,\delta}(f_\mathrm{d}) = \mu_{\delta,\lambda}(f)$, since the roles of λ and δ are swapped under the duality transformation. We will see that a similar result holds when $q > 1$.

Definition 2.4.1. *Let ψ_1, ψ_2 be probability measures on $(\Omega, \mathcal{F})$. We say that ψ_2 is dual to ψ_1 if for all measurable $f : \Omega \to \mathbb{R}$ we have that*

$$\psi_1(f_\mathrm{d}) = \psi_2(f). \tag{2.4.1}$$

Thus the dual of $\mu_{\lambda,\delta}$ is $\mu_{\delta,\lambda}$. Clearly it is enough to check (2.4.1) on some determining class of functions, such as the local functions.

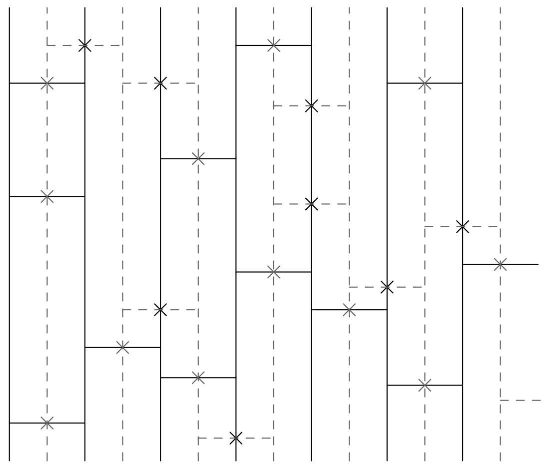

FIGURE 2.8. An illustration of duality. The primal configuration ω is drawn solid black, the dual ω_{d} dashed grey.

It will be convenient in what follows to denote the free and wired random-cluster measures on a region Λ by $\phi^0_{\Lambda;q,\lambda,\delta}$ and $\phi^1_{\Lambda;q,\lambda,\delta}$ respectively, instead of $\phi^{\mathrm{f}}_{\Lambda;q,\lambda,\delta}$ and $\phi^{\mathrm{w}}_{\Lambda;q,\lambda,\delta}$. The following result is stated in terms of infinite-volume measures, but from the proof we see that an analogous result holds also in finite volume.

THEOREM 2.4.2. *Let $b \in \{0,1\}$. The dual of the measure $\phi^b_{q,\lambda,\delta}$ is $\phi^{1-b}_{q,q\delta,\lambda/q}$.*

PROOF. Fix $\beta > 0$ and $q \geq 1$; later we will let $\beta \to \infty$. We write $[m,n]$ for the graph $L \subseteq \mathbb{L}$ induced by the set $\{m, m+1, \dots, n\} \subseteq \mathbb{Z}$ and $\Lambda_{m,n} = (K_{m,n}, F_{m,n})$ for the corresponding simple region. We write $\phi^b_{m,n;\lambda,\delta}$ for the random-cluster measure on the region $\Lambda_{m,n}$, with similar adjustments to other notation.

In what follows it will be useful to restrict attention to the bridges and deaths of $\omega \in \Omega$ that fall in $\Lambda_{m,n}$ only. It is then most natural to consider only those (dual) bridges and deaths of ω_{d} that fall in $\Lambda_{m,n-1} + 1/2$. In line with this we define

$$B_{m,n}(\omega) := B(\omega) \cap F_{m,n}, \qquad D_{m,n}(\omega) := D(\omega) \cap K_{m,n}; \tag{2.4.2}$$

and for the dual

(2.4.3)

$$B_{m,n-1}(\omega_{\mathrm{d}}) := D(\omega) \cap K_{m+1,n-1}, \qquad D_{m,n-1}(\omega_{\mathrm{d}}) := B(\omega) \cap F_{m,n}.$$

The first step is to establish an analog of the Euler equation for planar graphs. We claim that

$$k^1_{m,n}(\omega) - k^0_{m,n-1}(\omega_{\mathrm{d}}) + |B_{m,n}(\omega)| - |D_{m,n}(\omega)| = 1 - n + m. \tag{2.4.4}$$

(A similar result was obtained in [**8**, Lemma 3.3].) This is best proved inductively by successively adding elements to the sets $B_{m,n}(\omega)$ and $D_{m,n}(\omega)$. If both sets are empty, the claim follows on inspection. For each bridge you add to $B_{m,n}(\omega)$, either $k^1_{m,n}(\omega)$ decreases by one or $k^0_{m,n-1}(\omega_{\mathrm{d}})$ increases by one, but never both. Similarly when you add deaths to $D_{m,n}(\omega)$, either $k^1_{m,n}(\omega)$ increases by one or $k^0_{m,n-1}(\omega_{\mathrm{d}})$ decreases by one for each death, but never both. That establishes (2.4.4).

Let $\mu_{m,n;\lambda,\delta}$ denote the percolation measure restricted to $\Lambda_{m,n}$. For $f : \Omega \to \mathbb{R}$ any $\mathcal{F}_{\Lambda_{m,n-1}}$-measurable, bounded and continuous function, we have, using (2.4.4), that

(2.4.5)

$$\begin{aligned}
\phi^1_{m,n;\lambda,\delta}(f_{\mathrm{d}}) &\propto \int d\mu_{m,n;\lambda,\delta}(\omega) q^{k^1_{m,n}(\omega)} f(\omega_{\mathrm{d}}) \\
&\propto \int d\mu_{m,n;\lambda,\delta}(\omega) q^{k^0_{m,n-1}(\omega_{\mathrm{d}})} q^{|D_{m,n}(\omega)|} q^{-|B_{m,n}(\omega)|} f(\omega_{\mathrm{d}}) \\
&\propto \int d\mu_{m,n-1;\delta,\lambda}(\omega_{\mathrm{d}}) q^{k^0_{m,n-1}(\omega_{\mathrm{d}})} q^{|B_{m,n-1}(\omega_{\mathrm{d}})|} q^{-|D_{m,n-1}(\omega_{\mathrm{d}})|} f(\omega_{\mathrm{d}}) \\
&\propto \int d\mu_{m,n-1;q\delta,\lambda/q}(\omega_{\mathrm{d}}) q^{k^0_{m,n-1}(\omega_{\mathrm{d}})} f(\omega_{\mathrm{d}}) \\
&\propto \phi^0_{m,n-1;q\delta,\lambda/q}(f).
\end{aligned}$$

We have used the fact that

$$\frac{d\mu_{m,n-1;q\delta,\lambda/q}}{d\mu_{m,n-1;\delta,\lambda}}(\omega) \propto q^{|B_{m,n-1}(\omega)|} q^{-|D_{m,n-1}(\omega)|}, \tag{2.4.6}$$

a simple statement about Poisson processes.

Since both sides of (2.4.5) are probability measures, it follows that

$$\phi^1_{m,n;\lambda,\delta}(f_{\mathrm{d}}) = \phi^1_{m,n-1;q\delta,\lambda/q}(f). \tag{2.4.7}$$

Letting $m, n, \beta \to \infty$ in (2.4.7) and using Theorem 2.3.2, the result follows. □

Note that if $\lambda/\delta = \rho$ then the corresponding ratio for the dual measure is $q\delta/(\lambda/q) = q^2/\rho$. We therefore say that the space–time random-cluster model is *self-dual* if $\rho = q$. This self-duality was referred to in [**8**, Proposition 3.4].

2.4.1. A lower bound on ρ_c when $d = 1$. In this section we adapt Zhang's famous and versatile argument (published in [**50**, Chapter 6]) to the space-time setting. See [**11**] for the special case of this argument when $q = 1$.

Theorem 2.4.3. *If $d = 1$ and $\rho = q$ then $\theta^{\mathrm{f}}(\rho, q) = 0$; hence the critical ratio $\rho_{\mathrm{c}} \geq q$.*

Proof. Assume for a contradiction that with $\rho = q$ we have that $\theta^{\mathrm{f}}(\rho, q) > 0$. Then by Theorem 2.3.10 there is almost surely a unique unbounded component in ω under ϕ^{f}. It follows from self-duality and the fact that $\theta^{\mathrm{w}} \geq \theta^{\mathrm{f}}$ that there is almost surely also a unique unbounded component in ω_{d}. Let $D_n = \{(x, t) \in \mathbb{R}^2 : |x + 1/2| + |t| \leq n\}$ be the 'lozenge', and $D^{\mathrm{d}}_n = \{(x, t) \in \mathbb{R}^2 : |x| + |t| \leq n\}$ its 'dual', as in Figure 2.9. Number the four sides of each of D_n and D^{d}_n counterclockwise, starting in each case with the north-east side. For $i = 1, \ldots, 4$ let A_i be the event that the ith side of D_n is attached to an unbounded path of ω, which does not otherwise intersect D_n. Similarly let A^{d}_i be the event that the ith side of the dual D^{d}_n is attached to an unbounded path of ω_{d}. Clearly $\phi^{\mathrm{f}}(\cup_{i=1}^4 A_i) \to 1$ as $n \to \infty$. However, all the A_i are increasing, and by symmetry under reflection they carry equal

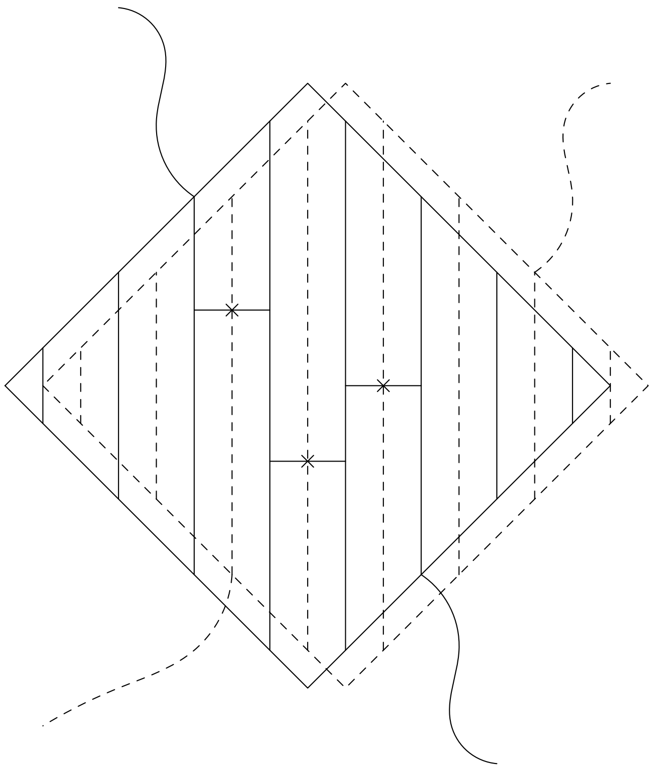

FIGURE 2.9. On the event $A_2 \cap A_4 \cap A_1^{\mathrm{d}} \cap A_3^{\mathrm{d}}$ either the unbounded primal cluster breaks into 2 parts, or the dual one does.

probability. It follows from positive association, Theorem 2.2.14, that

$$\phi^{\mathrm{f}}(\cup_{i=1}^{4} A_i) \leq 1 - \phi^{\mathrm{f}}(A_2^{\mathrm{c}})^4 = 1 - (1 - \phi^{\mathrm{f}}(A_2))^4,$$

and hence $\phi^{\mathrm{f}}(A_2) \to 1$ too. Hence for n large enough we have that $\phi^{\mathrm{f}}(A_2) = \phi^{\mathrm{f}}(A_4) \geq 5/6$, so by positive association again $\phi^{\mathrm{f}}(A_2 \cap A_4) \geq (5/6)^2 > 5/8$ for n large enough. In the same way it follows that for large n we have $\phi^{\mathrm{f}}(A_1^{\mathrm{d}} \cap A_3^{\mathrm{d}}) > 5/8$. But then

$$\phi^{\mathrm{f}}(A_2 \cap A_4 \cap A_1^{\mathrm{d}} \cap A_3^{\mathrm{d}}) \geq \frac{10}{8} - 1 = \frac{1}{4}.$$

Now a glance at Figure 2.9 should convince the reader that this contradicts the uniqueness of the unbounded cluster, either in ω or ω_{d}. This contradiction shows that $\theta^{\mathrm{f}}(\rho, q) = 0$ as required. $\square$

Remark 2.4.4. *It is natural to suppose that the critical value equals the self-dual value $\lambda/\delta = q$. For $q = 1$ this is proved in* [**11**] *and in* [**6**]*; for $q = 2$ it is proved in Theorem 4.1.1 (see also* [**15**]*).*

2.5. Infinite-volume Potts measures

Using the convergence results in Section 2.3, we will in this section construct infinite-volume weak limits of Potts measures. We will also provide more details about uniqueness of infinite-volume measure in the space-time Ising model, extending in that case the arguments of Section 2.3.4. The results in this section will form the foundation for our study of the quantum Ising model in Chapter 3.

2.5.1. Weak limits of Potts measures. Let $q \geq 2$ be an integer, and let $\alpha_\Gamma = q$; we will suppress reference to the simple boundary condition (b, α) throughout this subsection. Recall the two random-cluster measures $\phi^{\mathrm{w}}_\Lambda$ and $\phi^{\mathrm{f}}_\Lambda$ as well as their Potts counterparts π^{w}_Λ and π^{f}_Λ, connected via the coupling (2.1.17). For simplicity we assume in this section that $\mathbb{L} = \mathbb{Z}^d$ for some $d \geq 1$; similar arguments are valid in greater generality, but we do not pursue this here. All regions in this section will be simple, as in (2.1.7). We let $\Lambda_n = (K_n, F_n)$ denote a strictly increasing sequence of simple regions, containing the origin and increasing to either $\mathbf{\Theta}$ or $\mathbf{\Theta}_\beta$. Denote by ϕ^{w}_n, ϕ^{f}_n, π^{w}_n and π^{f}_n the corresponding random-cluster and Potts measures. Proofs will be given for the $\beta = \infty$ case, the case $\beta < \infty$ is similar.

Throughout this subsection we will be making use of the concept of *lattice components*: given $\omega = (B, D, G)$ the lattice components of ω are the connected components in $\mathbb{K}$ of the configuration $(B, D, \varnothing)$. We will think of the points in G as green points, and of any lattice component containing an element of G as *green*. In this subsection we will only use the notation $x \leftrightarrow y$ to mean that x, y lie in the same *lattice* component. We write $C_x(\omega)$ for the lattice component of x in ω.

The following convergence result is an adaptation of arguments in [**4**], see also [**50**, Theorem 4.91].

THEOREM 2.5.1. *The weak limits*

$$\pi^{\mathrm{f}} = \lim_{n\to\infty} \pi_n^{\mathrm{f}} \qquad \text{and} \qquad \pi^{\mathrm{w}} = \lim_{n\to\infty} \pi_n^{\mathrm{w}} \tag{2.5.1}$$

exist and are independent of the manner in which $\Lambda_n \uparrow \mathbf{\Theta}$. *Moreover,* π^{f} *and* π^{w} *are given as follows:*

- *Let* $\omega \sim \phi^{\mathrm{f}}$ *and assign to each green component of* ω *spin* q, *and assign to the remaining components uniformly independent spins from* $1, \dots, q$; *then the resulting spin configuration has law* π^{f}.
- *Let* $\omega \sim \phi^{\mathrm{w}}$ *and assign to each unbounded component and each green component of* ω *spin* q, *and assign to the remaining components uniformly independent spins from* $1, \dots, q$; *then the resulting spin configuration has law* π^{w}.

PROOF. We will make use of a certain total order on $\mathbb{K} = \mathbb{Z}^d \times \mathbb{R}$. The precise details are not important, except that the ordering be such that every (topologically) closed set contains an earliest point. We define such an ordering as follows. We say that $x = (x_1, \dots, x_d, t) < (x_1', \dots, x_d', t') = x'$ if (a) for $k \in \{1, \dots, d\}$ minimal with $x_k x_k' < 0$ we have $x_k > 0$; or if (a) fails but (b) $tt' < 0$ with $t > 0$; or if (a) and (b) fail but (c) $|x| < |x'|$ lexicographically, where $|x| = (|x_1|, \dots, |x_d|, |t|)$.

Slightly different arguments are required for the two boundary conditions. We give the argument only for free boundary. It will be necessary to modify the probability space $(\Omega, \mathcal{F})$, as follows (we omit some details). For each $n \geq 1$ and each $\omega = (B, D, G) \in \Omega$, let $\tilde{\omega}_n = (\tilde{B}_n, \tilde{D}_n, \tilde{G}_n)$ be given by

$$\tilde{B}_n = B \cap F_n, \quad \tilde{D}_n = (D \cap K_n) \cup (\mathbb{K} \setminus K_n), \quad \tilde{G}_n = G \cap K_n.$$

Thus, in $\tilde{\omega}_n$, no two points in $\mathbb{K} \setminus K_n$ are connected. Let $\tilde{\Omega} = \Omega \cup \{\tilde{\omega}_n : \omega \in \Omega, n \geq 1\}$, and define connectivity in elements of $\tilde{\Omega}$ in the obvious way. Define the functions V_x as before Theorem 2.3.2; if $x \in \mathbb{K} \times \{\mathrm{d}\}$ then V_x may now take the value $+\infty$. Let $\tilde{\mathcal{F}}$ denote the σ-algebra generated by the V_x's. (Alternatively, $\tilde{\mathcal{F}}$ is the σ-algebra generated by the appropriate Skorokhod metric when the associated step functions are allowed to take the values $\pm\infty$.) Let $\tilde{\phi}^{\mathrm{f}}_n$ denote the law of $\tilde{\omega}_n$ when ω has law ϕ^{f}_n. Note that the number of components of $\tilde{\omega}_n$ equals $k^{\mathrm{f}}_n(\omega)$.

Extending the partial order on Ω to $\tilde{\Omega}$ in the natural way, we see that for each n we have $\tilde{\phi}^{\mathrm{f}}_n \leq \tilde{\phi}^{\mathrm{f}}_{n+1}$. (It is here that we need to use $\tilde{\Omega}$, since the stochastic ordering $\phi^{\mathrm{f}}_n \leq \phi^{\mathrm{f}}_{n+1}$ holds only on $\mathcal{F}_n$, not on the full σ-algebra $\mathcal{F}$.) Hence there exists by Strassen's Theorem 2.2.2 a probability measure P on $(\tilde{\Omega}^{\mathbb{N}}, \tilde{\mathcal{F}}^{\mathbb{N}})$ such that in the sequence $(\tilde{\omega}_1, \tilde{\omega}_2, \dots)$ the nth component has marginal distribution $\tilde{\phi}^{\mathrm{f}}_n$, and such that $\tilde{\omega}_n \leq \tilde{\omega}_{n+1}$ for all n, with P-probability one. The sequence $\tilde{\omega}_n$ increases to a limiting configuration $\tilde{\omega}_\infty$, which has law ϕ^{f}. We have that $\phi^{\mathrm{f}}(\Omega) = 1$.

For each fixed (bounded) region Δ, if n is large enough then $\tilde{\omega}_n$ agrees with $\tilde{\omega}_\infty$ throughout Δ. Let Λ be a fixed region, and let $\Delta = \Delta(\tilde{\omega}_\infty) \supset \Lambda$ be large enough so that the following hold:

(1) Each bounded lattice-component of $\tilde{\omega}_\infty$ which intersects Λ is entirely contained in Δ;

(2) Any two points $x, y \in \Lambda$ which are connected in $\tilde{\omega}_\infty$ are connected inside Δ;

(3) Any lattice-component of $\tilde{\omega}_\infty$ which is green has a green point inside Δ.

It is (almost surely) possible to choose such a Δ because only finitely many lattice components intersect Λ. We choose $n = n(\tilde{\omega}_\infty)$ large enough so that $\tilde{\omega}_n, \tilde{\omega}_{n+1}, \dots$ all agree with $\tilde{\omega}_\infty$ throughout Δ.

Claim: for all $x, y \in \Lambda$, we have that $x \leftrightarrow y$ in $\tilde{\omega}_n$ if and only if $x \leftrightarrow y$ in $\tilde{\omega}_\infty$. To see this, first note that $C_x(\tilde{\omega}_n) \subseteq C_x(\tilde{\omega}_\infty)$ since

$\tilde{\omega}_n \le \tilde{\omega}_\infty$, proving one of the implications. Suppose now that $x \leftrightarrow y$ in $\tilde{\omega}_\infty$. Then by our choice of Δ, there is a path from x to y inside Δ. But $\tilde{\omega}_\infty$ and $\tilde{\omega}_n$ agree on Δ, so it follows that also $x \leftrightarrow y$ in $\tilde{\omega}_n$.

Let $\tilde{\omega} \in \tilde{\Omega}$, and let C be a lattice component of $\tilde{\omega}$. The (topological) closure of C contains an earliest point in the order defined above. Order the lattice components $C_1(\tilde{\omega}), C_2(\tilde{\omega}), \ldots$ according to the earliest point in their closure; this ordering is almost surely well-defined under any of $\tilde{\phi}^{\mathrm{f}}_n, \tilde{\phi}^{\mathrm{f}}$. Note that the claim above implies that this ordering agrees for those lattice components of $\tilde{\omega}_n$ and $\tilde{\omega}_\infty$ which intersect Λ.

Let $S_1, S_2, \ldots$ be independent and uniform on $\{1, \ldots, q\}$, and define for $x \in \boldsymbol{\Theta}$,

$$\tau_x(\tilde{\omega}) = \begin{cases} q, & \text{if } C_x(\tilde{\omega}) \text{ is green,} \\ S_i, & \text{otherwise, where } C_x(\tilde{\omega}) = C_i. \end{cases} \tag{2.5.2}$$

Then $\tau(\tilde{\omega}_\infty)$ has the law π^{f} described in the statement of the theorem, and $\tau(\tilde{\omega}_n)$ has the law π^{f}_n on events in $\mathcal{G}_\Lambda$. Moreover, from the claim it follows that $\tau_x(\tilde{\omega}_\infty) = \tau_x(\tilde{\omega}_n)$ for any $x \in \Lambda$. Hence for all continuous, bounded f, measurable with respect to $\mathcal{G}_\Lambda$, we have that $f(\tau(\tilde{\omega}_n)) \to f(\tau(\tilde{\omega}_\infty))$ almost surely. It follows from the bounded convergence theorem that

$$\pi^{\mathrm{f}}_n(f) = E(f(\tau(\tilde{\omega}_n))) \to E(f(\tau(\tilde{\omega}_\infty))) = \pi^{\mathrm{f}}(f). \tag{2.5.3}$$

Since such f are convergence determining it follows that $\pi^{\mathrm{f}}_n \Rightarrow \pi^{\mathrm{f}}$. $\square$

Remark 2.5.2. *From the representation given in Theorem 2.5.1 it follows that the correlation/connectivity relation of Proposition 2.1.7 holds also for infinite-volume random-cluster and Potts measures. In particular, when $q = 2$, it follows (using the obvious notation) that the analogue of* (2.1.23) *holds, namely*

$$\langle \sigma_x \sigma_y \rangle^b = \phi^b(x \leftrightarrow y),$$

for $b \in \{\mathrm{f}, \mathrm{w}\}$*. Note also that when* $\gamma = 0$ *then, as in Proposition 2.1.7, we have for for* $b \in \{\mathrm{f}, \mathrm{w}\}$ *that*

$$\langle \sigma_x \rangle^b = \phi^b(x \leftrightarrow \infty). \tag{2.5.4}$$

2.5.2. Uniqueness in the Ising model. We turn our attention now to the space–time Ising model on $\mathbb{L} = \mathbb{Z}^d$ with constant λ, δ, γ. In this section we continue our discussion, started in Section 2.3.4, about uniqueness of infinite-volume measures. More information can be obtained in the case of the Ising model, partly thanks to the so-called GHS-inequality which allows us to show the absence of a phase transition when $\gamma \neq 0$. In contrast, using only results obtained via the random-cluster representation one can say next to nothing about uniqueness when $\gamma \neq 0$ since there is no useful way of combining a $+1$ external field with a -1 'lattice-boundary'. The arguments in this section follow very closely those for the classical Ising model, as developed in [**66**] and [**77**] (see also [**29**, Chapters IV and V]). We provide full details for completeness.

As remarked earlier, the Ising model admits more boundary conditions than the corresponding random-cluster model. It will therefore seem like some of the arguments presented below repeat what was said in Section 2.3.4. It should be noted, however, that the arguments in this section can deal with all boundary conditions that occur in the Ising model. It will be particularly useful to consider the $+$ and $-$ boundary conditions, defined as follows. Let $b = \{P_1, P_2\}$ where $P_1 = \{\Gamma\}$ and $P_2 = \hat{\partial}\Lambda$. We define the $+$ *boundary condition* by letting $\alpha_1 = \alpha_2 = +1$; when $\gamma \geq 0$ this equals the wired random-cluster boundary condition with $\alpha_\Gamma = +1$. We define the $-$ *boundary condition* by letting $\alpha_1 = +1$ and $\alpha_2 = -1$. The measure $\langle \cdot \rangle_\Lambda^-$ does not have a satisfactory random-cluster representation when $\gamma > 0$. (See [**25**] for an in-depth treatment of some difficulties associated with the graphical representation of the

Ising model in an arbitrary external field.) In line with physical terminology we will sometimes in this section refer to the measures $\langle\cdot\rangle_\Lambda^{b,\alpha}$ as 'states'.

For simplicity of notation we will in this section replace λ and γ by 2λ and 2γ throughout. We will be writing $Z_\Lambda^{b,\alpha}$ for the Ising partition function (2.1.26), which therefore becomes

$$Z_\Lambda^{b,\alpha} = \int d\mu_\delta(D) \sum_{\sigma\in\Sigma_\Lambda^{b,\alpha}(D)} \exp\Big(\int_F \lambda(e)\sigma_e de + \int_K \gamma(x)\sigma_x dx\Big). \tag{2.5.5}$$

We will similarly write $P_\Lambda^{b,\alpha} = (\log Z_\Lambda^{b,\alpha})/|\Lambda|$. Thanks to Proposition 2.1.8, the $P_\Lambda^{b,\alpha}$ thus defined converge to a function P which is a multiple of the original P in Theorem 2.3.12. Straightforward modifications of the argument in Theorem 2.3.12 let us deduce that this convergence holds for all boundary conditions b of Ising-type.

We assume throughout this section that $\Lambda = \Lambda_n \uparrow \mathbf{\Theta}$ in such a way that

$$\frac{|K_n \setminus K_{n-1}|}{|K_n|} \to 0, \qquad \text{as } n\to\infty, \tag{2.5.6}$$

where $\Lambda_n = (K_n, F_n)$ and $|\cdot|$ denotes Lebesgue measure. As previously, straightforward modifications of the argument are valid when $\beta < \infty$ is fixed and $\Lambda \uparrow \mathbf{\Theta}_\beta$.

Here are some general facts about convex functions; some facts like these were already used in Section 2.3.4. See e.g. [**29**, Chapter IV] for proofs. Recall that for a function $f : \mathbb{R} \to \mathbb{R}$, the left and right derivatives of f are given respectively by

(2.5.7)
$$\frac{\partial f}{\partial \gamma^+} := \lim_{h\downarrow 0} \frac{f(\gamma+h)-f(\lambda)}{h} \quad \text{and} \quad \frac{\partial f}{\partial \gamma^-} := \lim_{h\downarrow 0} \frac{f(\gamma-h)-f(\lambda)}{-h}$$

provided these limits exist.

Proposition 2.5.3. *Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ a convex function; also let $f_n : I \to \mathbb{R}$ be a sequence of convex functions. Then*

- *The left and right derivatives of f exist throughout I; the right derivative is right-continuous and the left derivative is left-continuous.*
- *The derivative f' of f exists at all but countably many points in I.*
- *If all the f_n are differentiable and $f_n \to f$ pointwise then the derivatives f'_n converge to f' whenever the latter exists.*
- *If the f_n are uniformly bounded above and below then there exists a sub-sequence f_{n_k} and a (necessarily convex) function f such that $f_{n_k} \to f$ pointwise.*

We will usually keep λ, δ fixed and regard $P = P(\gamma)$ as a function of γ, and similarly for other functions. Note that P is an even function of γ: we have for all $\gamma > 0$ that $P^+_\Lambda(-\gamma) = P^-_\Lambda(\gamma)$, and since the limit P is independent of boundary condition it follows that $P(-\gamma) = P(\gamma)$.

Let

$$\bar{M}^{b,\alpha}_\Lambda := \frac{\partial P^{b,\alpha}_\Lambda}{\partial \gamma} = \frac{1}{|\Lambda|} \int_\Lambda dx \langle \sigma_x \rangle^{b,\alpha}_\Lambda, \tag{2.5.8}$$

where we abuse notation to write $x \in \Lambda$ (respectively, $|\Lambda|$) in place of the more accurate $x \in K$ (respectively, $|K|$). Also let

$$M^{b,\alpha}_\Lambda := \langle \sigma_0 \rangle^{b,\alpha}_\Lambda. \tag{2.5.9}$$

Note that (2.5.8) together with the first GKS-inequality (2.2.32) imply that P^{w}_Λ, and hence also P, is increasing for $\gamma > 0$ (and hence decreasing for $\gamma < 0$). Moreover, we see that

$$\frac{\partial^2 P^{\mathrm{w}}_\Lambda}{\partial \gamma^2} = \frac{1}{|\Lambda|} \int_\Lambda \int_\Lambda dx dy \langle \sigma_x ; \sigma_y \rangle^{\mathrm{w}}_\Lambda \geq 0, \tag{2.5.10}$$

from the second GKS-inequality (2.2.33). Thus P is convex in γ.

LEMMA 2.5.4. *The states $\langle \cdot \rangle^+_\Lambda$ and $\langle \cdot \rangle^-_\Lambda$ converge weakly as $\Lambda \uparrow \mathbf{\Theta}$. The limiting states $\langle \cdot \rangle^+$ and $\langle \cdot \rangle^-$ are independent of the way in which $\Lambda \uparrow \mathbf{\Theta}$ and are translation invariant.*

Remark 2.5.5. *The convergence result for + boundary follows from Theorem 2.5.1 and Remark 2.5.2, since when $q = 2$ the measure $\pi^{\mathrm{w}}_{\Lambda}$ there is precisely the state $\langle\cdot\rangle^{+}_{\Lambda}$. However, the result for − boundary does not follow from that result since the random-cluster representation as employed there does not admit the spin at Γ to be different from that at $\partial\Lambda$. (One would have to condition on the event that, in the random-cluster model, the boundary is disconnected from Γ, and then one loses desired monotonicity properties.)*

In the proof of Lemma 2.5.4 we will be applying the FKG-inequality, Lemma 2.2.17. For each $x \in \mathbb{K}$, let $\nu'_x = (\sigma_x + 1)/2$ and for $A \subseteq \mathbb{K}$ finite, write

$$\nu'_A = \prod_{x\in A} \nu'_x. \tag{2.5.11}$$

Note that $\nu'_A = \mathbb{I}_S$, where S is the event that $\sigma_x = +1$ for all $x \in A$. This is an increasing event, and a continuity set by Example 2.2.19. Similarly, if $\Lambda \subseteq \Delta$ are regions and T is the event that $\sigma = +1$ on $\Delta \setminus \Lambda$, then T is an increasing event and a continuity set, also by Example 2.2.19.

Proof of Lemma 2.5.4. It is easy to check that the variables ν'_A, as A ranges over the finite subsets of $\mathbb{K}$, form a convergence determining class. By Lemma 2.1.9 and Lemma 2.2.17 we therefore see that for any regions $\Lambda \subseteq \Delta$ we have that

$$\langle \nu'_A\rangle^{+}_{\Lambda} = \langle \nu'_A \mid \sigma \equiv +1 \text{ on } \Delta\setminus\Lambda\rangle^{+}_{\Delta} \geq \langle \nu'_A\rangle^{+}_{\Delta} \tag{2.5.12}$$

and

$$\langle \nu'_A\rangle^{-}_{\Lambda} = \langle \nu'_A \mid \sigma \equiv -1 \text{ on } \Delta\setminus\Lambda\rangle^{-}_{\Delta} \leq \langle \nu'_A\rangle^{-}_{\Delta}. \tag{2.5.13}$$

Hence $\langle \nu'_A\rangle^{+}_{\Lambda}$ and $\langle \nu'_A\rangle^{-}_{\Lambda}$ converge for all finite $A \subseteq \mathbb{K}$, as required. □

The proof of Lemma 2.5.4 shows in particular that

$$\langle\sigma_0\rangle^+_\Lambda \downarrow \langle\sigma_0\rangle^+ \qquad \text{and} \qquad \langle\sigma_0\rangle^-_\Lambda \uparrow \langle\sigma_0\rangle^-, \tag{2.5.14}$$

and indeed that all the $\langle\sigma_A\rangle^\pm_\Lambda$ converge to the corresponding $\langle\sigma_A\rangle^\pm$. Recall that by convexity, the left and right derivatives of P exist at all $\gamma \in \mathbb{R}$.

LEMMA 2.5.6. *For all $\gamma \in \mathbb{R}$ we have that*

$$\frac{\partial P}{\partial \gamma^+} = \langle\sigma_0\rangle^+ \qquad \textit{and} \qquad \frac{\partial P}{\partial \gamma^-} = \langle\sigma_0\rangle^-. \tag{2.5.15}$$

PROOF. As a preliminary step we first show that $\bar{M}^\pm_\Lambda$ has the same infinite-volume limit as $M^\pm_\Lambda$, that is to say

$$\lim_{\Lambda\uparrow\Theta} \bar{M}^\pm_\Lambda = \langle\sigma_0\rangle^\pm. \tag{2.5.16}$$

We prove this in the case of + boundary, the case of − boundary being similar. First note that

$$\bar{M}^+_\Lambda = \frac{1}{|\Lambda|}\int_\Lambda dx\langle\sigma_x\rangle^+_\Lambda \geq \frac{1}{|\Lambda|}\int_\Lambda dx\langle\sigma_x\rangle^+ = \langle\sigma_0\rangle^+, \tag{2.5.17}$$

by (2.5.14) and translation invariance. Thus $\liminf_\Lambda \bar{M}^+_\Lambda \geq \langle\sigma_0\rangle^+$. Next let $\varepsilon > 0$ and let Λ be large enough so that $\langle\sigma_0\rangle^+_\Lambda \leq \langle\sigma_0\rangle^+ + \varepsilon$. If $x \in \mathbb{K}$ and Δ is large enough that the translated region $\Lambda + x \subseteq \Delta$ then

$$\langle\sigma_x\rangle^+_\Delta \leq \langle\sigma_x\rangle^+_{\Lambda+x} = \langle\sigma_0\rangle^+_\Lambda \leq \langle\sigma_0\rangle^+ + \varepsilon. \tag{2.5.18}$$

Let $\Delta' := \{x \in \Delta : \Lambda + x \in \Delta\}$. Then

$$\begin{aligned}\bar{M}^+_\Delta = \frac{1}{|\Delta|}\int_\Delta dx\langle\sigma_x\rangle^+_\Delta &\leq \frac{1}{|\Delta|}\Big(\int_{\Delta'} dx\langle\sigma_x\rangle^+_\Delta + |\Delta\setminus\Delta'|\Big)\\ &\leq \frac{1}{|\Delta|}\Big(|\Delta'|\big(\langle\sigma_x\rangle^+ + \varepsilon\big) + |\Delta\setminus\Delta'|\Big).\end{aligned} \tag{2.5.19}$$

It therefore follows from the assumption (2.5.6) that $\limsup_\Lambda \bar{M}^+_\Lambda \leq \langle\sigma_0\rangle^+ + \varepsilon$, which gives (2.5.16).

Next we claim that $\langle\sigma_0\rangle^+$ and $\langle\sigma_0\rangle^-$ are right- and left continuous in γ, respectively. First consider $+$ boundary. Then for $\gamma' > \gamma$, we have for any Λ from Lemma 2.2.18 that $\langle\sigma_0\rangle^+_{\Lambda,\gamma'} \geq \langle\sigma_0\rangle^+_{\Lambda,\gamma}$. Thus

$$\begin{aligned}\langle\sigma_0\rangle^+_\gamma &\leq \liminf_{\gamma'\downarrow\gamma}\langle\sigma_0\rangle^+_{\gamma'} \leq \limsup_{\gamma'\downarrow\gamma}\langle\sigma_0\rangle^+_{\gamma'} \\ &\leq \limsup_{\gamma'\downarrow\gamma}\langle\sigma_0\rangle^+_{\Lambda,\gamma'} = \langle\sigma_0\rangle^+_{\Lambda,\gamma} \xrightarrow[\Lambda\uparrow\boldsymbol{\Theta}]{} \langle\sigma_0\rangle^+_\gamma.\end{aligned} \tag{2.5.20}$$

(We have used the fact that $\langle\sigma_0\rangle^+_\Lambda$ is continuous in γ.) A similar calculation holds for $-$ boundary.

Now, by convexity of P, the right derivative $\frac{\partial P}{\partial\gamma^+}$ is right-continuous, and also $\lim_\Lambda \bar{M}^\pm_\Lambda = \frac{\partial P}{\partial\gamma}$ whenever the right side exists. But it exists for all but countably many γ, so given γ there is a sequence $\gamma_n \downarrow \gamma$ such that $\frac{\partial P}{\partial\gamma}(\gamma_n) = \langle\sigma_0\rangle^+_{\gamma_n}$ for all n, and similarly for $-$ boundary. The result follows. □

We say that there is a *unique state* at γ (or at λ,δ,γ) if for all finite $A \subseteq \mathbb{K}$, the limit $\langle\sigma_A\rangle := \lim_\Lambda\langle\sigma_A\rangle^{b,\alpha}_\Lambda$ exists and is independent of the boundary condition (b,α). Note that, by linearity, it is equivalent to require that all the limits $\langle\nu'_A\rangle := \lim_\Lambda\langle\nu'_A\rangle^{b,\alpha}_\Lambda$ exist and are independent of the boundary condition. Alternatively, there is a unique state if and only if the measures $\langle\cdot\rangle^{b,\alpha}_\Lambda$ all converge weakly to the same limiting measure.

Lemma 2.5.7. *There is a unique state at $\gamma \in \mathbb{R}$ if and only if P is differentiable at γ. There is a unique state at any $\gamma \neq 0$.*

Proof. We have that

$$f_A := \sum_{x\in A}\nu'_x - \nu'_A \tag{2.5.21}$$

is increasing in σ. By the FKG-inequality, Lemma 2.2.17, we have that $\langle f_A\rangle^+_\Lambda \geq \langle f_A\rangle^-_\Lambda$. It follows on letting $\Lambda \uparrow \boldsymbol{\Theta}$, and using translation invariance as well as Lemma 2.5.6, that

(2.5.22)
$$0 \le \langle \nu'_A \rangle^+ - \langle \nu'_A \rangle^- \le \frac{1}{2} \sum_{x \in A} (\langle \sigma_x \rangle^+ - \langle \sigma_x \rangle^-) = \frac{|A|}{2} \Big(\frac{\partial P}{\partial \gamma^+} - \frac{\partial P}{\partial \gamma^-} \Big),$$

where $|A|$ is the number of elements in A. Hence $\langle \nu'_A \rangle^+ = \langle \nu'_A \rangle^-$ whenever $\frac{\partial P}{\partial \gamma}$ exists. Since $\langle \nu'_A \rangle^- \le \langle \nu'_A \rangle^{b,\alpha} \le \langle \nu'_A \rangle^+$ for all (b, α) (a consequence of Lemma 2.2.17), the first claim follows.

The next part makes use of the facts about convex functions stated above; this part of the argument originates in [**77**]. Let $\gamma > 0$, and use the free boundary condition. We already know that P and each P^{f}_Λ is convex. The GHS-inequality, which is standard for the classical Ising model and proved for the current model in Lemma 3.3.4, implies that each $\bar{M}^{\mathrm{f}}_\Lambda$ has non*positive* second derivative for $\gamma > 0$, and hence that each $\bar{M}^{\mathrm{f}}_\Lambda$ is concave. Moreover, each $\bar{M}^{\mathrm{f}}_\Lambda$ lies between -1 and 1. There therefore exists a sequence Λ_n of simple regions such that the sequence $\bar{M}^{\mathrm{f}}_{\Lambda_n}$ converges pointwise to a limiting function which we denote by M^{f}_∞. If $0 < \gamma < \gamma'$ then by the fundamental theorem of calculus and the bounded convergence theorem, we have that

(2.5.23)
$$\begin{aligned} P(\gamma') - P(\gamma) &= \lim_{n \to \infty} \big(P^{\mathrm{f}}_{\Lambda_n}(\gamma') - P^{\mathrm{f}}_{\Lambda_n}(\gamma) \big) \\ &= \lim_{n \to \infty} \int_\gamma^{\gamma'} \bar{M}^{\mathrm{f}}_{\Lambda_n}(\gamma)\, d\gamma = \int_\gamma^{\gamma'} M^{\mathrm{f}}_\infty(\gamma)\, d\gamma. \end{aligned}$$

The function M^{f}_∞ is concave, and hence continuous, in $\gamma > 0$. It therefore follows from the above that P is in fact differentiable at each $\gamma > 0$ (with derivative M^{f}_∞). The result follows since $P(-\gamma) = P(\gamma)$ for all $\gamma > 0$. □

Whenever there is a unique infinite-volume state at γ, we will denote it by $\langle \cdot \rangle = \langle \cdot \rangle_\gamma$.

Lemma 2.5.8. *For each $\gamma \neq 0$ and each (b, α), we have that*

(2.5.24)
$$M := \frac{\partial P}{\partial \gamma} = \lim_{\Lambda \uparrow \mathbf{\Theta}} M^{b,\alpha}_\Lambda = \lim_{\Lambda \uparrow \mathbf{\Theta}} \bar{M}^{b,\alpha}_\Lambda.$$

PROOF. The proof of Lemma 2.5.7 shows that at each $\gamma \neq 0$ the derivative of P is M^{f}_{∞}. Since for all (b, α) and Λ, the function $P^{b,\alpha}_{\Lambda}(\gamma)$ is convex and differentiable with

$$\frac{\partial P^{b,\alpha}_{\Lambda}}{\partial \gamma} = \bar{M}^{b,\alpha}_{\Lambda} \tag{2.5.25}$$

it follows from the properties of convex functions that $\bar{M}^{b,\alpha}_{\Lambda}(\gamma) \to M(\gamma)$ at all $\gamma \neq 0$. That also $M^{b,\alpha}_{\Lambda} \to M$ for $\gamma \neq 0$ follows from the the fact that $M^{-}_{\Lambda} \leq M^{b,\alpha}_{\Lambda} \leq M^{+}_{\Lambda}$ and the fact that $\lim M^{\pm}_{\Lambda} = \lim \bar{M}^{\pm}_{\Lambda}$ as we saw at (2.5.16). $\square$

Lemma 2.5.8 implies in particular that

$$M = \lim_{\Lambda \uparrow \boldsymbol{\Theta}} \langle \sigma_0 \rangle^{\pm}_{\Lambda} \tag{2.5.26}$$

at all $\gamma \neq 0$. We know from Lemma 2.5.4 that the limits

$$M_{\pm} := \lim_{\Lambda \uparrow \boldsymbol{\Theta}} \langle \sigma_0 \rangle^{\pm}_{\Lambda} \tag{2.5.27}$$

exist also at $\gamma = 0$. By Lemma 2.5.6 there is a unique state at $\gamma = 0$ if and only if $M_{+}(0) = M_{-}(0)$. We sometimes call $M_{+}(0)$ the *spontaneous magnetization*.

Note that for all Λ and all $\gamma > 0$ we have $M^{+}_{\Lambda}(-\gamma) = -M^{-}_{\Lambda}(\gamma)$, so that $\lim M^{+}_{\Lambda}(-\gamma) = -M(\gamma)$. Hence M is an *odd* function of $\gamma \neq 0$. Note also that

$$M_{+}(0) = \lim_{\gamma \downarrow 0} M(\gamma).$$

Indeed, rather more is true: by repeating the argument at (2.5.20) with σ_A in place of σ_0, it follows that the state $\langle \cdot \rangle^{+}$ of Lemma 2.5.4 may be written as the weak limit

$$\langle \cdot \rangle^{+}_{\gamma=0} = \lim_{\gamma \downarrow 0} \langle \cdot \rangle_{\gamma} \tag{2.5.28}$$

where $\langle \cdot \rangle_{\gamma}$ is the unique state at $\gamma > 0$. Thus we may summarize the results of this section as follows.

THEOREM 2.5.9. *There is a unique state at all $\gamma \neq 0$ and there is a unique state at $\gamma = 0$ if and only if*

$$M_+(0) \equiv \lim_{\gamma \downarrow 0} M(\gamma) = 0. \tag{2.5.29}$$

We now recall the remaining parameters λ, δ and β. As previously, we set $\delta = 1$, $\rho = \lambda/\delta$, and write

$$M^\beta(\rho, \gamma) = M^\beta(\rho, 1, \gamma).$$

It follows from Lemma 2.2.22 that $M_+^\beta(\rho, 0)$ is an increasing function of ρ. This motivates the following definition.

DEFINITION 2.5.10. *We define the* critical value

$$\rho_c^\beta := \inf\{\rho > 0 : M_+^\beta(\rho, 0) > 0\}.$$

From Remark 2.5.2 and (2.5.28) it follows that this ρ_c^β coincides with the 'percolation threshold' $\rho_c(2)$ for the $q = 2$ space–time random-cluster model as defined in Definition 2.3.8. More information about ρ_c^β and the behaviour of M^β and related quantities near the critical point may be found in Section 3.5.

CHAPTER 3

The quantum Ising model: random-parity representation and sharpness of the phase transition

Summary. We develop a 'random-parity' representation for the space–time Ising model; this is the space–time analog of the random-current representation. The random-parity representation is then used to derive a number of differential inequalities, from which one can deduce many important properties of the phase transition of the quantum Ising model, such as sharpness of the transition.

3.1. Classical and quantum Ising models

Recall from the Introduction that the (transverse field) quantum Ising model on the finite graph L is given by the Hamiltonian

$$H = -\tfrac{1}{2}\lambda \sum_{e=uv\in E} \sigma_u^{(3)}\sigma_v^{(3)} - \delta \sum_{v\in V} \sigma_v^{(1)}, \tag{3.1.1}$$

acting on the Hilbert space $\mathcal{H} = \bigotimes_{v\in V} \mathbb{C}^2$. We refer to that chapter for definitions of the notation used. In the quantum Ising model the number $\beta > 0$ is thought of as the 'inverse temperature'. We define the positive temperature states

$$\nu_{L,\beta}(Q) = \frac{1}{Z_L(\beta)} \mathrm{tr}(e^{-\beta H} Q), \tag{3.1.2}$$

where $Z_L(\beta) = \mathrm{tr}(e^{-\beta H})$ and Q is a suitable matrix. The *ground state* is defined as the limit ν_L of $\nu_{L,\beta}$ as $\beta \to \infty$. If $(L_n : n \geq 1)$ is an increasing sequence of graphs tending to the infinite graph $\mathbb{L}$, then we may also make use of the *infinite-volume* limits

$$\nu_{L,\beta} = \lim_{n\to\infty} \nu_{L_n,\beta}, \qquad \nu_L = \lim_{n\to\infty} \nu_{L_n}.$$

The existence of such limits is discussed in [**7**], see also the related discussion of limits of space–time Ising measures in Section 2.5.

The quantum Ising model is intimately related to the space–time Ising model, one manifestation of this being the following. Recall that if $|\psi\rangle$ denotes a vector then $\langle\psi|$ denotes its conjugate transpose. The state $\nu_{L,\beta}$ of (3.1.2) gives rise to a probability measure μ on $\{-1,+1\}^V$ by

$$\mu(\sigma) = \frac{\langle\sigma|e^{-\beta H}|\sigma\rangle}{\mathrm{tr}(e^{-\beta H})}, \qquad \sigma \in \{-1,+1\}^V. \tag{3.1.3}$$

When $\gamma = 0$, it turns out that μ is the law of the vector $(\sigma_{(v,0)} : v \in V)$ under the space–time Ising measure of (2.1.22) (with periodic boundary, see below). See [**7**] and the references therein. It therefore makes sense to study the phase diagram of the quantum Ising model via its representation in the space–time Ising model. Note, however, that in our analysis it is crucial to work with $\gamma > 0$, and to take the limit $\gamma \downarrow 0$ later. The role played in the classical model by the external field will in our analysis be played by the 'ghost-field' γ rather than the 'physical' transverse field δ. (In fact, γ corresponds to a $\sigma^{(3)}$-field, see [**26**].)

In most of this chapter we will be working with periodic boundary conditions in the $\mathbb{R}$-direction. That is to say, for simple regions of the form (2.1.7) we will identify the endpoints of the the 'time' interval $[-\beta/2, \beta/2]$, and think of this interval as the circle of circumference β. We will denote this circle by $\mathbb{S} = \mathbb{S}_\beta$ and thus our simple regions will be of the form $L \times \mathbb{S}$ for some finite graph L. We shall generally (until Section 3.5) keep $\beta > 0$ fixed, and thus suppress reference to β. Similarly, we will generally suppress reference to the boundary condition.

Thus we will write for instance $\Sigma(D)$ for the set of spin configurations permitted by D (see the discussion before (2.1.22)).

General regions of the form (2.1.4) will usually be thought of as subsets of the simple region $L \times \mathbb{S}$. Thus, for $v \in V$, we let $K_v \subseteq \mathbb{S}$ be a finite union of disjoint intervals, and we write $K_v = \bigcup_{i=1}^{m(v)} I_i^v$. As before, no assumption is made on whether the I_i^v are open, closed, or half-open. With the K_v given, we define F and Λ as in (2.1.4).

For simplicity of notation we replace in this chapter the functions λ, γ in (2.1.26) by $2\lambda, 2\gamma$, respectively. Thus the space–time Ising measure on a region $\Lambda = (K, F)$ has partition function

$$Z' = \int d\mu_\delta(D) \sum_{\sigma \in \Sigma(D)} \exp\left\{ \int_F \lambda(e)\sigma_e \, de + \int_K \gamma(x)\sigma_x \, dx \right\}, \tag{3.1.4}$$

where $\sigma_e = \sigma_{(u,t)}\sigma_{(v,t)}$ if $e = (uv, t)$. See (2.1.22). As previously, we write $\langle f \rangle$ for the mean of a $\mathcal{G}_\Lambda$-measurable $f : \Sigma \to \mathbb{R}$ under this measure. Thus for example

$$\langle \sigma_A \rangle = \frac{1}{Z'} \int d\mu_\delta(D) \sum_{\sigma \in \Sigma(D)} \sigma_A \exp\left\{ \int_F \lambda(e)\sigma_e \, de + \int_K \gamma(x)\sigma_x \, dx \right\}. \tag{3.1.5}$$

Note that in this chapter we denote the partition function by Z'.

It is essential for our method in this chapter that we work on general regions of the form given in (2.1.4). The reason for this is that, in the geometrical analysis of currents, we shall at times remove from K a random subset called the 'backbone', and the ensuing domain has the form of (2.1.4). Note that considering this general class of regions also allows us to revert to a 'free' rather than a 'vertically periodic' boundary condition. That is, by setting $K_v = [-\beta/2, \beta/2)$ for all $v \in V$, rather than $K_v = [-\beta/2, \beta/2]$, we effectively remove the restriction that the 'top' and 'bottom' of each $v \times \mathbb{S}$ have the same spin.

Whenever we wish to emphasize the roles of particular K, λ, δ, γ, we include them as subscripts. For example, we may write $\langle\sigma_A\rangle_K$ or $\langle\sigma_A\rangle_{K,\gamma}$ or Z'_γ, and so on.

3.1.1. Statement of the main results. Let 0 be a given point of $V \times \mathbb{S}$. We will be particularly concerned with the *magnetization* and *susceptibility* of the space–time Ising model on $\Lambda = L \times \mathbb{S}$, given respectively by

$$M = M_\Lambda(\lambda, \delta, \gamma) := \langle\sigma_0\rangle, \tag{3.1.6}$$

$$\chi = \chi_\Lambda(\lambda, \delta, \gamma) := \frac{\partial M}{\partial \gamma} = \int_\Lambda \langle\sigma_0; \sigma_x\rangle\, dx, \tag{3.1.7}$$

where we recall that the *truncated* two-point function $\langle\sigma_0; \sigma_x\rangle$ is given by

$$\langle\sigma_A; \sigma_B\rangle := \langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle. \tag{3.1.8}$$

Note that, for simplicity of notation, we will in most of this chapter keep M and χ free from sub- and superscripts even though they refer to finite-volume quantities. Some basic properties of these quantities were discussed in Section 2.5.2.

Our main choice for L is a box $[-n, n]^d$ in the d-dimensional cubic lattice $\mathbb{Z}^d$ where $d \geq 1$, with a periodic boundary condition. That is to say, apart from the usual nearest-neighbour bonds, we also think of two vertices u, v as joined by an edge whenever there exists $i \in \{1, 2, \ldots, d\}$ such that u and v differ by exactly $2n$ in the ith coordinate. Subject to this boundary condition, M and χ do not depend on the choice of origin 0. We shall pass to the infinite-volume limit as $L \uparrow \mathbb{Z}^d$. The model is over-parametrized, and we shall, as before, normally assume $\delta = 1$, and write $\rho = \lambda/\delta$. The critical point $\rho_c = \rho_c^\beta$ is given as in Definition 2.5.10 by

$$\rho_c^\beta := \inf\{\rho : M_+^\beta(\rho) > 0\}, \tag{3.1.9}$$

where

$$M_+^\beta(\rho) := \lim_{\gamma\downarrow 0} M^\beta(\rho,\gamma), \tag{3.1.10}$$

is the magnetization in the limiting state $\langle\cdot\rangle_+^\beta$ as $\gamma \downarrow 0$. As in Theorem 2.3.9, we have that:

$$\begin{aligned} &\text{if } d \geq 2: \quad 0 < \rho_{\mathrm{c}}^\beta < \infty \text{ for } \beta \in (0,\infty], \\ &\text{if } d = 1: \quad \rho_{\mathrm{c}}^\beta = \infty \text{ for } \beta \in (0,\infty),\ 0 < \rho_{\mathrm{c}}^\infty < \infty. \end{aligned} \tag{3.1.11}$$

Complete statements of our main results are deferred until Section 3.5, but here are two examples of what can be proved.

THEOREM 3.1.1. *Let $u, v \in \mathbb{Z}^d$ where $d \geq 1$, and $s, t \in \mathbb{R}$. For $\beta \in (0,\infty]$:*

(i) *if $0 < \rho < \rho_{\mathrm{c}}^\beta$, the two-point correlation function $\langle\sigma_{(u,s)}\sigma_{(v,t)}\rangle_+^\beta$ of the space–time Ising model decays exponentially to 0 as $|u-v| + |s-t| \to \infty$,*

(ii) *if $\rho \geq \rho_{\mathrm{c}}^\beta$, $\langle\sigma_{(u,s)}\sigma_{(v,t)}\rangle_+^\beta \geq M_+^\beta(\rho)^2 > 0$.*

Theorem 3.1.1 is what is called 'sharpness of the phase transition': there is no intermediate regime in which correlations decay to zero slowly. (See for example [**23**] and [**43**] for examples of systems where this does occur).

THEOREM 3.1.2. *Let $\beta \in (0,\infty]$. In the notation of Theorem 3.1.1, there exists $c = c(d) > 0$ such that*

$$M_+^\beta(\rho) \geq c(\rho - \rho_{\mathrm{c}}^\beta)^{1/2} \qquad \text{for } \rho > \rho_{\mathrm{c}}^\beta.$$

These and other facts will be stated and proved in Section 3.5. Their implications for the infinite-volume quantum model will be elaborated around (3.1.14)–(3.1.16).

The approach used here is to prove a family of differential inequalities for the finite-volume magnetization $M(\rho,\gamma)$. This parallels the methods established in [**2, 3**] for the analysis of the phase transitions in percolation and Ising models on discrete lattices, and indeed our arguments are closely related to those of [**3**]. Whereas new problems arise in the current context and require treatment, certain aspects of the analysis presented here are simpler that the corresponding steps of [**3**]. The application to the quantum model imposes a periodic boundary condition in the β direction; some of our conclusions are valid for the space–time Ising model with a free boundary condition.

The following is the principal differential inequality we will derive. (Our results are in fact valid in greater generality, see the statement before Assumption 3.3.7.)

THEOREM 3.1.3. *Let $d \geq 1$, $\beta < \infty$, and $L = [-n, n]^d$ with periodic boundary. Then*

$$M \leq \gamma\chi + M^3 + 2\lambda M^2 \frac{\partial M}{\partial \lambda} - 2\delta M^2 \frac{\partial M}{\partial \delta}. \tag{3.1.12}$$

A similar inequality was derived in [**3**] for the classical Ising model, and our method of proof is closely related to that used there. Other such inequalities have been proved for percolation in [**2**] (see also [**49**]), and for the contact model in [**6, 11**]. As observed in [**2, 3**], the powers of M on the right side of (3.1.12) determine the bounds of Theorems 3.1.1(ii) and 3.1.2 on the critical exponents. The cornerstone of our proof is a 'random-parity representation' of the space–time Ising model.

The analysis of the differential inequalities, following [**2, 3**], reveals a number of facts about the behaviour of the model. In particular, we will show the exponential decay of the correlations $\langle\sigma_0\sigma_x\rangle_+$ when $\rho < \rho_c^\beta$ and $\gamma = 0$, as asserted in Theorem 3.1.1, and in addition certain bounds on two critical exponents of the model. See Section 3.5 for further details.

We draw from [**7, 8**] in the following summary of the relationship between the phase transitions of the quantum and space–time Ising models. Let $u, v \in V$, and

$$\tau_L^\beta(u,v) := \mathrm{tr}\big(\nu_{L,\beta}(Q_{u,v})\big), \qquad Q_{u,v} = \sigma_u^{(3)}\sigma_v^{(3)}.$$

It is the case that

$$\tau_L^\beta(u,v) = \langle \sigma_A \rangle_L^\beta \tag{3.1.13}$$

where $A = \{(u,0),(v,0)\}$, and the role of β is stressed in the superscript. Let τ_L^∞ denote the limit of τ_L^β as $\beta \to \infty$. For $\beta \in (0,\infty]$, let τ^β be the limit of τ_L^β as $L \uparrow \mathbb{Z}^d$. (The existence of this limit may depend on the choice of boundary condition on L, and we return to this at the end of Section 3.5.) By Theorem 3.1.1,

$$\tau^\beta(u,v) \le c' e^{-c|u-v|}, \tag{3.1.14}$$

where c', c depend on ρ, and $c > 0$ for $\rho < \rho_{\mathrm{c}}^\beta$ and $\beta \in (0,\infty]$. Here, $|u-v|$ denotes the L^1 distance from u to v. The situation when $\rho = \rho_{\mathrm{c}}^\beta$ is more obscure, but one has that

$$\limsup_{|v|\to\infty} \tau^\beta(u,v) \le M_+^\beta(\rho), \tag{3.1.15}$$

so that $\tau^\beta(u,v) \to 0$ whenever $M_+^\beta(\rho) = 0$. It is proved at Theorem 4.1.1 that $\rho_{\mathrm{c}}^\infty = 2$ and $M_+^\infty(2) = 0$ when $d = 1$.

By the FKG inequality, and the uniqueness of infinite clusters in the space–time random-cluster model (see Theorem 2.3.10),

$$\tau^\beta(u,v) \ge M_+^\beta(\rho-)^2 > 0, \tag{3.1.16}$$

when $\rho > \rho_{\mathrm{c}}^\beta$ and $\beta \in (0,\infty]$, where $f(x-) := \lim_{y\uparrow x} f(y)$. The proof is discussed at the end of Section 3.5.

The critical value ρ_{c}^β depends of course on the number of dimensions. We shall in the next chapter use Theorem 3.1.1 and planar duality to show that $\rho_{\mathrm{c}}^\infty = 2$ when $d = 1$, and in addition that the transition is of second order in that $M_+^\infty(2) = 0$. See Theorem 4.1.1. The critical point has been calculated by other means in the quantum case, but we believe that the current proof is valuable. Two applications to the work of [**14, 54**] are summarized in Section 4.1.

Here is a brief outline of the contents of this chapter. Formal definitions are presented in Section 3.1. The random-parity representation of the quantum Ising model is described in Section 3.2. This representation may at first sight seem quite different from the random-current representation of the classical Ising model on a discrete lattice. It requires more work to set up than does its discrete cousin, but once in place it works in a very similar, and sometimes simpler, manner. We then state and prove, in Section 3.3.1, the fundamental 'switching lemma'. In Section 3.3.2 are presented a number of important consequences of the switching lemma, including GHS and Simon–Lieb inequalities, as well as other useful inequalities and identities. In Section 3.4, we prove the somewhat more involved differential inequality of Theorem 3.1.3, which is similar to the main inequality of [**3**]. Our main results follow from Theorem 3.1.3 in conjunction with the results of Section 3.3.2. Finally, in Sections 3.5 and 4.1, we give rigorous formulations and proofs of our main results.

This chapter forms the contents of the article [**15**], which has been published in the Journal of Statistical Physics. The quantum mean-field, or Curie–Weiss, model has been studied using large-deviation techniques in [**24**], see also [**53**]. There is a very substantial overlap between the results reported here and those of the independent and contemporaneous article [**26**]. The basic differential inequalities of Theorems 3.1.3 and 3.3.8 appear in both places. The proofs are in essence the same despite some superficial differences. We are grateful to the authors of [**26**] for explaining the relationship between the random-parity representation of Section 3.2 and the random-current representation of [**58**, Section 2.2]. As pointed out in [**26**], the appendix of [**24**] contains a type of switching argument for the mean-field model.

A principal difference between that argument and those of [**26, 58**] and the current work is that it uses the classical switching lemma developed in [**1**], applied to a discretized version of the mean-field system.

3.2. The random-parity representation

The classical Ising model on a discrete graph L is a 'site model', in the sense that configurations comprise spins assigned to the vertices (or 'sites') of L. As described in the Introduction, the classical random-current representation maps this into a bond-model, in which the sites no longer carry random values, but instead the *edges* e (or 'bonds') of the graph are replaced by a random number N_e of parallel edges. The bond e is called *even* (respectively, *odd*) if N_e is even (respectively, odd). The odd bonds may be arranged into paths and cycles. One cannot proceed in the same way in the above space–time Ising model.

There are two possible alternative approaches. The first uses the fact that, conditional on the set D of deaths, Λ may be viewed as a discrete structure with finitely many components, to which the random-current representation of [**1**] may be applied. This is explained in detail around (3.2.12) below. Another approach is to forget about 'bonds', and instead to concentrate on the parity configuration associated with a current-configuration, as follows.

The circle $\mathbb{S}$ may be viewed as a continuous limit of a ring of equally spaced points. If we apply the random-current representation to the discretized system, but only record whether a bond is even or odd, the representation has a well-defined limit as a partition of $\mathbb{S}$ into even and odd sub-intervals. In the limiting picture, even and odd intervals carry different weights, and it is the properties of these weights that render the representation useful. This is the essence of the main result in this section, Theorem 3.2.1. We will prove this result without recourse to discretization.

We now define two additional random processes associated with the space–time Ising measure on Λ. The first is a random colouring of K, and the second is a random (finite) weighted graph. These two objects will be the main components of the random-parity representation.

3.2.1. Colourings. Let $\overline{K}$ be the closure of K. A set of *sources* is a finite set $A \subseteq \overline{K}$ such that: each $a \in A$ is the endpoint of at most one maximal subinterval I_i^v. (This last condition is for simplicity later.) Let $B \subseteq F$ and $G \subseteq K$ be finite sets. Let $S = A \cup G \cup V(B)$, where $V(B)$ is the set of endpoints of bridges of B, and call members of S *switching points*. As usual we shall assume that A, G and $V(B)$ are disjoint.

We shall define a colouring $\psi^A = \psi^A(B, G)$ of $K \setminus S$ using the two colours (or labels) 'even' and 'odd'. This colouring is constrained to be 'valid', where a valid colouring is defined to be a mapping $\psi : K \setminus S \to \{\text{even}, \text{odd}\}$ such that:

(i) the label is constant between two neighbouring switching points, that is, ψ is constant on any sub-interval of K containing no members of S,

(ii) the label always switches at each switching point, which is to say that, for $(u,t) \in S$, $\psi(u, t-) \neq \psi(u, t+)$, whenever these two values are defined,

(iii) for any pair v, k such that $I_k^v \neq \mathbb{S}$, in the limit as we move along $v \times I_k^v$ towards either endpoint of $v \times I_k^v$, the colour converges to 'odd' if that endpoint lies in S, and to 'even' otherwise.

If there exists $v \in V$ and $1 \leq k \leq m(v)$ such that $v \times \overline{I_k^v}$ contains an *odd* number of switching points, then conditions (i)–(iii) cannot be satisfied; in this case we set the colouring ψ^A to a default value denoted $\#$.

Suppose that (i)–(iii) *can* be satisfied, and let

$$W = W(K) := \{v \in V : K_v = \mathbb{S}\}.$$

If $W = \varnothing$, then there exists a unique valid colouring, denoted ψ^A. If $r = |W| \geq 1$, there are exactly 2^r valid colourings, one for each of the two possible colours assignable to the sites $(w, 0)$, $w \in W$; in this case we let ψ^A be chosen uniformly at random from this set, independently of all other choices.

We write $M_{B,G}$ for the probability measure (or expectation when appropriate) governing the randomization in the definition of ψ^A: $M_{B,G}$ is the uniform (product) measure on the set of valid colourings, and it is a point mass if and only if $W = \varnothing$. See Figure 3.1.

Fix the set A of sources. For (almost every) pair B, G, one may construct as above a (possibly random) colouring ψ^A. Conversely, it is easily seen that the pair B, G may (almost surely) be reconstructed from knowledge of the colouring ψ^A. For given A, we may thus speak of a configuration as being either a pair B, G, or a colouring ψ^A. While $\psi^A(B, G)$ is a colouring of $K \setminus S$ only, we shall sometimes refer to it as a colouring of K.

The next step is to assign weights $\partial\psi$ to colourings ψ. The 'failed' colouring $\#$ is assigned weight $\partial\# = 0$. For every valid colouring ψ, let $\mathrm{ev}(\psi)$ (respectively, $\mathrm{odd}(\psi)$) denote the subset of K that is labelled even (respectively, odd), and let

$$\partial\psi := \exp\bigl\{2\delta(\mathrm{ev}(\psi))\bigr\}, \tag{3.2.1}$$

where

$$\delta(U) := \int_U \delta(x)\,dx, \qquad U \subseteq K.$$

Up to a multiplicative constant depending on K and δ only, $\partial\psi$ equals the square of the probability that the odd part of ψ is death-free.

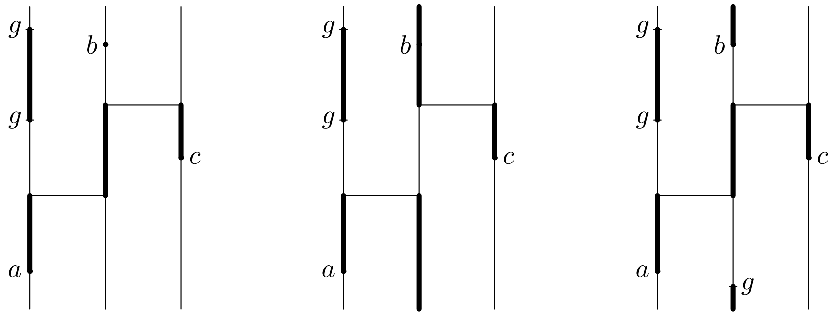

FIGURE 3.1. Three examples of colourings for given $B \subseteq F$, $G \subseteq K$. Points in G are written g. Thick line segments are 'odd' and thin segments 'even'. In this illustration we have taken $K_v = \mathbb{S}$ for all v. *Left and middle*: two of the eight possible colourings when the sources are a, c. *Right*: one of the possible colourings when the sources are a, b, c.

3.2.2. Random-parity representation. The expectation $E(\partial\psi^A)$ is taken over the sets B, G, and over the randomization that takes place when $W \neq \varnothing$, that is, E denotes expectation with respect to the measure $d\mu_\lambda(B)d\mu_\gamma(G)dM_{B,G}$. The notation has been chosen to harmonize with that used in [**3**] in the discrete case: the expectation $E(\partial\psi^A)$ will play the role of the probability $P(\partial\underline{n} = A)$ of [**3**]. The main result of this section now follows.

THEOREM 3.2.1 (Random-parity representation). *For any finite set $A \subseteq \overline{K}$ of sources,*

$$\langle\sigma_A\rangle = \frac{E(\partial\psi^A)}{E(\partial\psi^\varnothing)}. \tag{3.2.2}$$

We introduce a second random object in advance of proving this. Let D be a finite subset of K. The set $(v\times K_v)\setminus D$ is a union of maximal death-free intervals which we write $v \times J^v_k$, and where $k = 1, 2, \ldots, n$ and $n = n(v, D)$ is the number of such intervals. We write $V(D)$ for the collection of all such intervals.

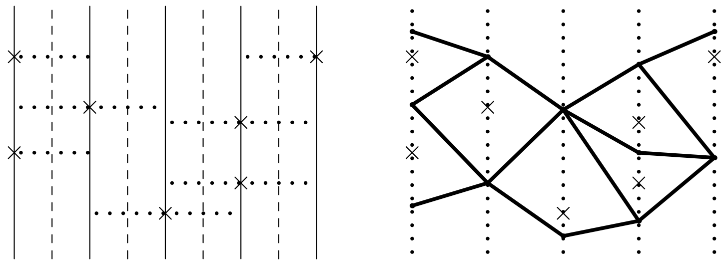

FIGURE 3.2. *Left*: The partition $E(D)$. We have: $K_v = \mathbb{S}$ for $v \in V$, the lines $v \times K_v$ are drawn as solid, the lines $e \times K_e$ as dashed, and elements of D are marked as crosses. The endpoints of the $e \times J^e_{k,l}$ are the points where the dotted lines meet the dashed lines. *Right*: The graph $G(D)$. In this illustration, the dotted lines are the $v \times K_v$, and the solid lines are the edges of $G(D)$.

For each $e = uv \in E$, and each $1 \leq k \leq n(u)$ and $1 \leq l \leq n(v)$, let

$$J^e_{k,l} := J^u_k \cap J^v_l, \tag{3.2.3}$$

and

$$E(D) = \big\{ e \times J^e_{k,l} : e \in E,\ 1 \leq k \leq n(u),\ 1 \leq l \leq n(v),\ J^e_{k,l} \neq \varnothing \big\}. \tag{3.2.4}$$

Up to a finite set of points, $E(D)$ forms a partition of the set F induced by the 'deaths' in D.

The pair

$$G(D) := (V(D), E(D)) \tag{3.2.5}$$

may be viewed as a graph, illustrated in Figure 3.2. We will use the symbols $\bar{v}$ and $\bar{e}$ for typical elements of $V(D)$ and $E(D)$, respectively. There are natural weights on the edges and vertices of $G(D)$: for $\bar{e} = e \times J^e_{k,l} \in E(D)$ and $\bar{v} = v \times J^v_k \in V(D)$, let

$$J_{\bar{e}} := \int_{J^e_{k,l}} \lambda(e,t)\, dt, \qquad h_{\bar{v}} := \int_{J^v_k} \gamma(v,t)\, dt. \tag{3.2.6}$$

Thus the weight of a vertex or edge is its measure, calculated according to λ or γ, respectively. By (3.2.6),

$$\sum_{\bar{e}\in E(D)} J_{\bar{e}} + \sum_{\bar{v}\in V(D)} h_{\bar{v}} = \int_F \lambda(e)\,de + \int_K \gamma(x)\,dx. \tag{3.2.7}$$

Proof of Theorem 3.2.1. With $\Lambda = (K, F)$ as in (2.1.4), we consider the partition function $Z' = Z'_K$ given in (3.1.4). For each $\bar{v} \in V(D)$, $\bar{e} \in E(D)$, the spins σ_v and σ_e are constant for $x \in \bar{v}$ and $e \in \bar{e}$, respectively. Denoting their common values by $\sigma_{\bar{v}}$ and $\sigma_{\bar{e}}$ respectively, the summation in (3.1.4) equals

$$\begin{aligned}\sum_{\sigma\in\Sigma(D)} \exp\left\{\sum_{\bar{e}\in E(D)} \sigma_{\bar{e}} \int_{\bar{e}} \lambda(e)\,de + \sum_{\bar{v}\in V(D)} \sigma_{\bar{v}} \int_{\bar{v}} \gamma(x)\,dx\right\}& \\ = \sum_{\sigma\in\Sigma(D)} \exp\left\{\sum_{\bar{e}\in E(D)} J_{\bar{e}}\sigma_{\bar{e}} + \sum_{\bar{v}\in V(D)} h_{\bar{v}}\sigma_{\bar{v}}\right\}&.\end{aligned} \tag{3.2.8}$$

The right side of (3.2.8) is the partition function of the discrete Ising model on the graph $G(D)$, with pair couplings $J_{\bar{e}}$ and external fields $h_{\bar{v}}$. We shall apply the random-current expansion of [**3**] to this model.

For convenience of exposition, we introduce the extended graph

$$\begin{aligned}\widetilde{G}(D) &= (\widetilde{V}(D), \widetilde{E}(D)) \\ &:= \big(V(D) \cup \{\Gamma\}, E(D) \cup \{\bar{v}\Gamma : \bar{v} \in V(D)\}\big)\end{aligned} \tag{3.2.9}$$

where Γ is the ghost-site. We call members of $E(D)$ *lattice-bonds*, and those of $\widetilde{E}(D)\backslash E(D)$ *ghost-bonds*. Let $\Psi(D)$ be the random multigraph with vertex set $\widetilde{V}(D)$ and with each edge of $\widetilde{E}(D)$ replaced by a random number of parallel edges, these numbers being independent and having the Poisson distribution, with parameter $J_{\bar{e}}$ for lattice-bonds $\bar{e}$, and parameter $h_{\bar{v}}$ for ghost-bonds $\bar{v}\Gamma$.

Let $\{\partial\Psi(D) = A\}$ denote the event that, for each $\bar{v} \in V(D)$, the total degree of $\bar{v}$ in $\Psi(D)$ *plus* the number of elements of A inside $\bar{v}$ (when regarded as an interval) is even. (There is μ_δ-probability 0 that

A contains some endpoint of some $V(D)$, and thus we may overlook this possibility.) Applying the discrete random-current expansion, and in particular [**50**, eqn (9.24)], we obtain by (3.2.7) that

(3.2.10)

$$\sum_{\sigma\in\Sigma(D)} \exp\left\{ \sum_{\bar{e}\in E(D)} J_{\bar{e}}\sigma_{\bar{e}} + \sum_{\bar{v}\in V(D)} h_{\bar{v}}\sigma_{\bar{v}} \right\} = c2^{|V(D)|} P_D(\partial\Psi(D) = \varnothing),$$

where P_D is the law of the edge-counts, and

$$c = \exp\left\{ \int_F \lambda(e)\,de + \int_K \gamma(x)\,dx \right\}. \tag{3.2.11}$$

By the same argument applied to the numerator in (3.1.5) (adapted to the measure on Λ, see the remark after (3.1.4)),

$$\langle\sigma_A\rangle = \frac{E(2^{|V(D)|}\mathbb{1}\{\partial\Psi(D) = A\})}{E(2^{|V(D)|}\mathbb{1}\{\partial\Psi(D) = \varnothing\})}, \tag{3.2.12}$$

where the expectation is with respect to $\mu_\delta \times P_D$. The claim of the theorem will follow by an appropriate manipulation of (3.2.12).

Here is another way to sample $\Psi(D)$, which allows us to couple it with the random colouring ψ^A. Let $B \subseteq F$ and $G \subseteq K$ be finite sets sampled from μ_λ and μ_γ respectively. The number of points of G lying in the interval $\bar{v} = v \times J_k^v$ has the Poisson distribution with parameter $h_{\bar{v}}$, and similarly the number of elements of B lying in $\bar{e} = e \times J_{k,l}^e \in E(D)$ has the Poisson distribution with parameter $J_{\bar{e}}$. Thus, for given D, the multigraph $\Psi(B, G, D)$, obtained by replacing an edge of $\widetilde{E}(D)$ by parallel edges equal in number to the corresponding number of points from B or G, respectively, has the same law as $\Psi(D)$. Using the *same* sets B, G we may form the random colouring ψ^A.

The numerator of (3.2.12) satisfies

(3.2.13)
$$\begin{aligned}E(2^{|V(D)|}\mathbb{I}\{\partial\Psi(D)=A\}) &= \iint d\mu_\lambda(B)\,d\mu_\gamma(G)\int d\mu_\delta(D)\,2^{|V(D)|}\mathbb{I}\{\partial\Psi(B,G,D)=A\}\\ &= \mu_\delta(2^{|V(D)|})\iint d\mu_\lambda(B)\,d\mu_\gamma(G)\,\widetilde{\mu}(\partial\Psi(B,G,D)=A),\end{aligned}$$

where $\widetilde{\mu}$ is the probability measure on $\mathcal{F}$ satisfying

$$\frac{d\widetilde{\mu}}{d\mu_\delta}(D)\propto 2^{|V(D)|}. \tag{3.2.14}$$

Therefore, by (3.2.12),

$$\langle\sigma_A\rangle=\frac{\widetilde{P}(\partial\Psi(B,G,D)=A)}{\widetilde{P}(\partial\Psi(B,G,D)=\varnothing)}, \tag{3.2.15}$$

where $\widetilde{P}$ denotes the probability under $\mu_\lambda\times\mu_\gamma\times\widetilde{\mu}$. We claim that

$$\widetilde{\mu}(\partial\Psi(B,G,D)=A)=sM_{B,G}(\partial\psi^A(B,G)), \tag{3.2.16}$$

for all B, G, where s is a constant, and the expectation $M_{B,G}$ is over the uniform measure on the set of valid colourings. Claim (3.2.2) follows from this, and the remainder of the proof is to show (3.2.16). The constants s, s_j are permitted in the following to depend only on Λ, δ.

Here is a special case:

$$\widetilde{\mu}(\partial\Psi(B,G,D)=A)=0 \tag{3.2.17}$$

if and only if some interval $\overline{I_k^v}$ contains an odd number of switching points, if and only if $\psi^A(B,G)=\#$ and $\partial\psi^A(B,G)=0$. Thus (3.2.16) holds in this case.

Another special case arises when $K_v=[0,\beta)$ for all $v\in V$, that is, the 'free boundary' case. As remarked earlier, there is a unique valid colouring $\psi^A=\psi^A(B,G)$. Moreover, $|V(D)|=|D|+|V|$, whence from standard properties of Poisson processes, $\widetilde{\mu}=\mu_{2\delta}$. It may be seen after some thought (possibly with the aid of a diagram) that, for given B,

G, the events $\{\partial\Psi(B,G,D)=A\}$ and $\{D\cap\mathrm{odd}(\psi^A)=\varnothing\}$ differ by an event of $\mu_{2\delta}$-probability 0. Therefore,

$$\begin{aligned}(3.2.18)\qquad \widetilde{\mu}(\partial\Psi(B,G,D)=A) &= \mu_{2\delta}(D\cap\mathrm{odd}(\psi^A)=\varnothing)\\ &= \exp\{-2\delta(\mathrm{odd}(\psi^A))\}\\ &= s_1\exp\{2\delta(\mathrm{ev}(\psi^A))\} = s_1\partial\psi^A,\end{aligned}$$

with $s_1=e^{-2\delta(K)}$. In this special case, (3.2.16) holds.

For the general case, we first note some properties of $\widetilde{\mu}$. By the above, we may assume that B, G are such that $\widetilde{\mu}(\partial\Psi(B,G,D)=A)>0$, which is to say that each $\overline{I_k^v}$ contains an even number of switching points. Let $W=\{v\in V: K_v=\mathbb{S}\}$ and, for $v\in V$, let $D_v=D\cap(v\times K_v)$ and $d(v)=|D_v|$. By (3.2.14),

$$\begin{aligned}\frac{d\widetilde{\mu}}{d\mu_\delta}(D)\propto 2^{|V(D)|} &= \prod_{w\in W}2^{1\vee d(w)}\prod_{v\in V\setminus W}2^{m(v)+d(v)}\\ &\propto 2^{|D|}\prod_{w\in W}2^{\mathbb{1}\{d(w)=0\}},\end{aligned}$$

where $a\vee b=\max\{a,b\}$, and we recall the number $m(v)$ of intervals I_k^v that constitute K_v. Therefore,

$$(3.2.19)\qquad \frac{d\widetilde{\mu}}{d\mu_{2\delta}}(D)\propto\prod_{w\in W}2^{\mathbb{1}\{d(w)=0\}}.$$

Three facts follow.

(a) The sets D_v, $v\in V$ are independent under $\widetilde{\mu}$.

(b) For $v\in V\setminus W$, the law of D_v under $\widetilde{\mu}$ is $\mu_{2\delta}$.

(c) For $w\in W$, the law μ_w of D_w is that of $\mu_{2\delta}$ skewed by the Radon–Nikodym factor $2^{\mathbb{1}\{d(w)=0\}}$, which is to say that

$$\begin{aligned}(3.2.20)\qquad \mu_w(D_w\in H)=\frac{1}{\alpha_w}\Big[&2\mu_{2\delta}(D_w\in H,\, d(w)=0)\\ &+\mu_{2\delta}(D_w\in H,\, d(w)\geq 1)\Big],\end{aligned}$$

for appropriate sets H, where

$$\alpha_w=\mu_{2\delta}(d(w)=0)+1.$$

Recall the set $S = A \cup G \cup V(B)$ of switching points. By (a) above,

$$\widetilde{\mu}(\partial\Psi(B,G,D) = A) = \widetilde{\mu}(\forall v,k:\ |S \cap \overline{J^v_k}| \text{ is even}) = \prod_{v\in V} \widetilde{\mu}(\forall k:\ |S \cap \overline{J^v_k}| \text{ is even}). \tag{3.2.21}$$

We claim that

$$\widetilde{\mu}(\forall k:\ |S \cap \overline{J^v_k}| \text{ is even}) = s_2(v) M_{B,G}\Big(\exp\big\{2\delta\big(\mathrm{ev}(\psi^A) \cap (v \times K_v)\big)\big\}\Big), \tag{3.2.22}$$

where $M_{B,G}$ is as before. Recall that $M_{B,G}$ is a product measure. Once (3.2.22) is proved, (3.2.16) follows by (3.2.1) and (3.2.21).

For $v \in V \setminus W$, the restriction of ψ^A to $v \times K_v$ is determined given B and G, whence by (b) above, and the remark prior to (3.2.18),

$$\widetilde{\mu}(\forall k:\ |S \cap \overline{J^v_k}| \text{ is even}) = \mu_{2\delta}(\forall k:\ |S \cap \overline{J^v_k}| \text{ is even}) = \exp\big\{-2\delta\big(\mathrm{odd}(\psi^A) \cap (v \times K_v)\big)\big\}. \tag{3.2.23}$$

Equation (3.2.22) follows with $s_2(v) = \exp\{-2\delta(v \times K_v)\}$.

For $w \in W$, by (3.2.20),

$$\begin{aligned}
&\widetilde{\mu}(\forall k:\ |S \cap J^w_k| \text{ is even})\\
&\quad = \frac{1}{\alpha_w}\Big[2\mu_{2\delta}(D_w = \varnothing) + \mu_{2\delta}(D_w \neq \varnothing,\ \forall k:\ |S \cap J^w_k| \text{ is even})\Big]\\
&\quad = \frac{1}{\alpha_w}\Big[\mu_{2\delta}(D_w = \varnothing) + \mu_{2\delta}(\forall k:\ |S \cap J^w_k| \text{ is even})\Big].
\end{aligned}$$

Let $\psi = \psi^A(B,G)$ be a valid colouring with $\psi(w,0) = \text{even}$. The colouring $\overline{\psi}$, obtained from ψ by flipping all colours on $w \times K_w$, is valid also. We take into account the periodic boundary condition, to obtain this time that

$$\begin{aligned}
&\mu_{2\delta}(\forall k:\ |S \cap \overline{J^w_k}| \text{ is even})\\
&\quad = \mu_{2\delta}\big(\{D_w \cap \mathrm{odd}(\psi) = \varnothing\} \cup \{D_w \cap \mathrm{ev}(\psi) = \varnothing\}\big)\\
&\quad = \mu_{2\delta}(D_w \cap \mathrm{odd}(\psi) = \varnothing) + \mu_{2\delta}(D_w \cap \mathrm{ev}(\psi) = \varnothing) - \mu_{2\delta}(D_w = \varnothing),
\end{aligned}$$

whence

$$\begin{aligned}(3.2.24)\qquad \alpha_w\widetilde{\mu}(\forall k:\ |S\cap \overline{J_k^w}| \text{ is even}) &= \mu_{2\delta}(D_w\cap \operatorname{odd}(\psi)=\varnothing)+\mu_{2\delta}(D_w\cap \operatorname{ev}(\psi)=\varnothing)\\ &= 2M_{B,G}\Big(\exp\big\{-2\delta\big(\operatorname{odd}(\psi^A)\cap(w\times K_w)\big)\big\}\Big),\end{aligned}$$

since $\operatorname{odd}(\psi^A)=\operatorname{odd}(\psi)$ with $M_{B,G}$-probability $\frac{1}{2}$, and equals $\operatorname{ev}(\psi)$ otherwise. This proves (3.2.22) with $s_2(w)=2\exp\{-2\delta(w\times K_w)\}/\alpha_w$. □

By keeping track of the constants in the above proof, we arrive at the following statement, which will be useful later.

Lemma 3.2.2. *The partition function $Z'=Z'_K$ of (3.1.4) satisfies*

$$Z'=2^N e^{\lambda(F)+\gamma(K)-\delta(K)}E(\partial\psi^{\varnothing}),$$

where $N=\sum_{v\in V} m(v)$ is the total number of intervals comprising K.

We denote $Z_K=E(\partial\psi^{\varnothing})$, which is thus a constant multiple of Z'.

3.2.3. The backbone. The concept of the backbone is key to the analysis of [**3**], and its definition there has a certain complexity. The corresponding definition is rather easier in the current setting, because of the fact that bridges, deaths, and sources have (almost surely) no common point.

We construct a total order on K by: first ordering the vertices of L, and then using the natural order on $[0,\beta)$. Let $A\subseteq \overline{K}$, $B\subseteq F$ and $G\subseteq K$ be finite. Let ψ be a valid colouring. We will define a sequence of directed odd paths called the *backbone* and denoted $\xi=\xi(\psi)$. Suppose $A=(a_1,a_2,\dots,a_n)$ in the above ordering. Starting at a_1, follow the odd interval (in ψ) until you reach an element of $S=A\cup G\cup V(B)$. If the first such point thus encountered is the endpoint of a bridge, cross it, and continue along the odd interval; continue likewise until you first reach a point $t_1\in A\cup G$, at which point you stop. Note, by the validity of ψ, that $a_1 \neq t_1$. The odd path thus traversed is denoted ζ^1; we take ζ^1 to be closed (when viewed as a subset of $\mathbb{Z}^d \times \mathbb{R}$). Repeat the same procedure with A replaced by $A \setminus \{a_1, t_1\}$, and iterate until no sources remain. The resulting (unordered) set of paths $\xi = (\zeta^1, \dots, \zeta^k)$ is called the *backbone* of ψ. The backbone will also be denoted at times as $\xi = \zeta^1 \circ \cdots \circ \zeta^k$. We define $\xi(\#) = \varnothing$. Note that, apart from the backbone, the remaining odd segments of ψ form disjoint self-avoiding cycles (or 'eddies'). Unlike the discrete setting of [**3**], there is a (a.s.) unique way of specifying the backbone from knowledge of A, B, G and the valid colouring ψ. See Figure 3.3.

The backbone contains all the sources A as endpoints, and the configuration outside ξ may be any sourceless configuration. Moreover, since ξ is entirely odd, it does not contribute to the weight $\partial\psi$ in (3.2.1). It follows, using properties of Poisson processes, that the conditional expectation $E(\partial\psi^A \mid \xi)$ equals the expected weight of any sourceless colouring of $K \setminus \xi$, which is to say that, with $\xi := \xi(\psi^A)$,

$$E(\partial\psi^A \mid \xi) = E_{K\setminus\xi}(\partial\psi^{\varnothing}) = Z_{K\setminus\xi}. \tag{3.2.25}$$

Cf. (3.1.4) and (3.2.2), and recall Remark 2.1.1. We abbreviate Z_K to Z, and recall from Lemma 3.2.2 that the Z_R differ from the partition functions Z'_R by certain multiplicative constants.

Let Ξ be the set of all possible backbones as A, B, and G vary, regarded as sequences of directed paths in K; these paths may, if required, be ordered by their starting points. For $A \subseteq \overline{K}$ and $\nu \in \Xi$, we write $A \sim \nu$ if there exist B and G such that $M_{B,G}(\xi(\psi^A) = \nu) > 0$. We define the *weight* $w^A(\nu)$ by

$$w^A(\nu) = w^A_K(\nu) := \begin{cases} \dfrac{Z_{K\setminus\nu}}{Z} & \text{if } A \sim \nu, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.26}$$

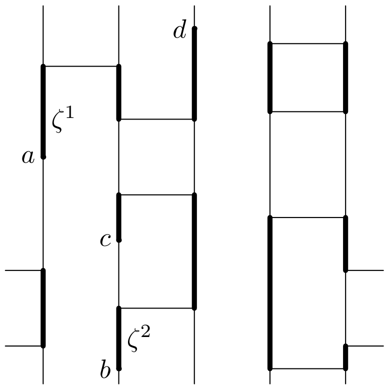

FIGURE 3.3. A valid colouring configuration ψ with sources $A = \{a, b, c, d\}$, and its backbone $\xi = \zeta^1 \circ \zeta^2$. Note that, in this illustration, bridges protruding from the sides 'wrap around', and that there are no ghost-bonds.

By (3.2.25) and Theorem 3.2.1, with $\xi = \xi(\psi^A)$,

$$E(w^A(\xi)) = \frac{E(E(\partial\psi^A \mid \xi))}{Z} = \frac{E(\partial\psi^A)}{E(\partial\psi^\varnothing)} = \langle\sigma_A\rangle. \tag{3.2.27}$$

For $\nu^1, \nu^2 \in \Xi$ with $\nu^1 \cap \nu^2 = \varnothing$ (when viewed as subsets of K), we write $\nu^1 \circ \nu^2$ for the element of Ξ comprising the union of ν^1 and ν^2.

Let $\nu = \zeta^1 \circ \cdots \circ \zeta^k \in \Xi$ where $k \geq 1$. If ζ^i has starting point a_i and endpoint b_i, we write $\zeta^i : a_i \to b_i$, and also $\nu : a_1 \to b_1, \ldots, a_k \to b_k$. If $b_i \in G$, we write $\zeta^i : a_i \to \Gamma$. There is a natural way to 'cut' ν at points x lying on ζ^i, say, where $x \neq a_i, b_i$: let $\bar\nu^1 = \bar\nu^1(\nu, x) = \zeta^1 \circ \cdots \circ \zeta^{i-1} \circ \zeta^i_{\leq x}$ and $\bar\nu^2 = \bar\nu^2(\nu, x) = \zeta^i_{\geq x} \circ \zeta^{i+1} \circ \cdots \circ \zeta^k$, where $\zeta^i_{\leq x}$ (respectively, $\zeta^i_{\geq x}$) is the closed sub-path of ζ^i from a_i to x (respectively, x to b_i). We express this decomposition as $\nu = \bar\nu^1 \circ \bar\nu^2$ where, this time, each $\bar\nu^i$ may comprise a number of disjoint paths. The notation $\overline{\nu}$ will be used only in a situation where there has been a cut.

We note two special cases. If $A = \{a\}$, then necessarily $\xi(\psi^A) : a \to \Gamma$, so

$$\langle\sigma_a\rangle = E\big(w^a(\xi) \cdot \mathbb{1}\{\xi : a \to \Gamma\}\big). \tag{3.2.28}$$

If $A=\{a,b\}$ where $a<b$ in the ordering of K, then

(3.2.29)
$$\langle\sigma_a\sigma_b\rangle = E\big(w^{ab}(\xi)\cdot \mathbb{I}\{\xi : a\to b\}\big) + E\big(w^{ab}(\xi)\cdot \mathbb{I}\{\xi : a\to\Gamma,\, b\to\Gamma\}\big).$$

The last term equals 0 when $\gamma\equiv 0$.

Finally, here is a lemma for computing the weight of ν in terms of its constituent parts. The claim of the lemma is, as usual, valid only 'almost surely'.

LEMMA 3.2.3. *(a) Let $\nu^1,\nu^2\in\Xi$ be disjoint, and $\nu=\nu^1\circ\nu^2$, $A\sim\nu$. Writing $A^i=A\cap\nu^i$, we have that*

$$w^A(\nu) = w^{A^1}(\nu^1)w^{A^2}_{K\setminus\nu^1}(\nu^2). \tag{3.2.30}$$

(b) Let $\nu=\overline{\nu}^1\circ\overline{\nu}^2$ be a cut of the backbone ν at the point x, and $A\sim\nu$. Then

$$w^A(\nu) = w^{B^1}(\overline{\nu}^1)w^{B^2}_{K\setminus\overline{\nu}^1}(\overline{\nu}^2). \tag{3.2.31}$$

where $B^i=A^i\cup\{x\}$.

PROOF. By (3.2.26), the first claim is equivalent to

$$\frac{Z_{K\setminus\nu}}{Z}\mathbb{I}\{A\sim\nu\} = \frac{Z_{K\setminus\nu^1}}{Z}\mathbb{I}\{A^1\sim\nu^1\}\frac{Z_{K\setminus(\nu^1\cup\nu^2)}}{Z_{K\setminus\nu^1}}\mathbb{I}\{A^2\sim\nu^2\}. \tag{3.2.32}$$

The right side vanishes if and only if the left side vanishes. When both sides are non-zero, their equality follows from the fact that $Z_{K\setminus\nu}=Z_{K\setminus(\nu^1\cup\nu^2)}$. The second claim follows similarly, on adding x to the set of sources. □

3.3. The switching lemma

We state and prove next the principal tool in the random-parity representation, namely the so-called 'switching lemma'. In brief, this allows us to take two independent colourings, with different sources, and to 'switch' the sources from one to the other in a measure-preserving way. In so doing, the backbone will generally change. In order to preserve the measure, the *connectivities* inherent in the backbone must be retained. We begin by defining two notions of connectivity in colourings. We work throughout this section in the general set-up of Section 3.2.1.

3.3.1. Connectivity and switching. Let $B \subseteq F$, $G \subseteq K$ be finite sets, let $A \subseteq \overline{K}$ be a finite set of sources, and write $\psi^A = \psi^A(B, G)$ for the colouring given in the last section. In what follows we think of the ghost-bonds as bridges to the ghost-site Γ.

Let $x, y \in K^\Gamma := K \cup \{\Gamma\}$. A *path* from x to y in the configuration (B, G) is a self-avoiding path with endpoints x, y, traversing intervals of K^Γ, and possibly bridges in B and/or ghost-bonds joining G to Γ. Similarly, a *cycle* is a self-avoiding cycle in the above graph. A *route* is a path or a cycle. A route containing no ghost-bonds is called a *lattice-route*. A route is called *odd* (in the colouring ψ^A) if ψ^A, when restricted to the route, takes only the value 'odd'. The failed colouring $\psi^A = \#$ is deemed to contain no odd paths.

Let $B_1, B_2 \subseteq F$, $G_1, G_2 \subseteq K$, and let $\psi_1^A = \psi_1^A(B_1, G_1)$ and $\psi_2^B = \psi_2^B(B_2, G_2)$ be the associated colourings. Let Δ be an auxiliary Poisson process on K, with intensity function $4\delta(\cdot)$, that is independent of all other random variables so far. We call points of Δ *cuts*. A route of $(B_1 \cup B_2, G_1 \cup G_2)$ is said to be *open* in the triple $(\psi_1^A, \psi_2^B, \Delta)$ if it includes no sub-interval of $\mathrm{ev}(\psi_1^A) \cap \mathrm{ev}(\psi_2^B)$ containing one or more elements of Δ. In other words, the cuts break paths, but only when they fall in intervals labelled 'even' in *both* colourings. See Figure 3.4. In particular, if there is an odd path π from x to y in ψ_1^A, then π constitutes an open path in $(\psi_1^A, \psi_2^B, \Delta)$ irrespective of ψ_2^B and Δ. We let

$$\{x \leftrightarrow y \text{ in } \psi_1^A, \psi_2^B, \Delta\} \tag{3.3.1}$$

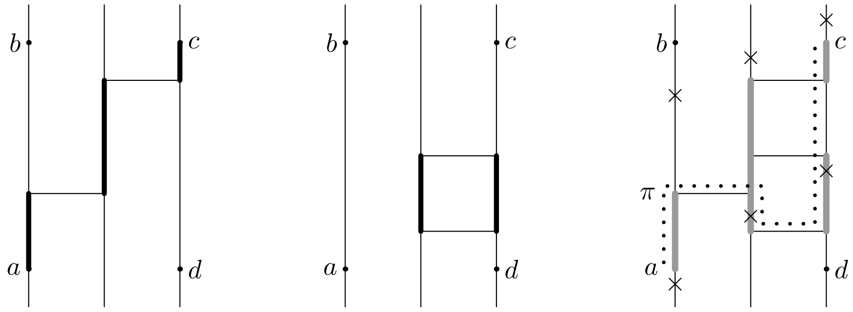

FIGURE 3.4. Connectivity in pairs of colourings. *Left*: ψ_1^{ac}. Middle: $\psi_2^{\varnothing}$. *Right*: the triple $\psi_1^{ac}, \psi_2^{\varnothing}, \Delta$. Crosses are elements of Δ and grey lines are where either ψ_1^{ac} or $\psi_2^{\varnothing}$ is odd. In $(\psi_1^{ac}, \psi_2^{\varnothing}, \Delta)$ the following connectivities hold: $a \not\leftrightarrow b$, $a \leftrightarrow c$, $a \leftrightarrow d$, $b \not\leftrightarrow c$, $b \not\leftrightarrow d$, $c \leftrightarrow d$. The dotted line marks π, one of the open paths from a to c.

be the event that there exists an open path from x to y in $(\psi_1^A, \psi_2^B, \Delta)$. We may abbreviate this to $\{x \leftrightarrow y\}$ when there is no ambiguity.

There is an analogy between open paths in the above construction and the notion of connectivity in the random-current representation of the discrete Ising model. Points labelled 'odd' or 'even' above may be considered as collections of infinitesimal parallel edges, being odd or even in number, respectively. If a point is 'even', the corresponding number of edges may be $2, 4, 6, \dots$ *or* it may be 0; in the 'union' of ψ_1^A and ψ_2^B, connectivity is broken at a point if and only if both the corresponding numbers equal 0. It turns out that the correct law for the set of such points is that of Δ.

Here is some notation. For any finite sequence $(a, b, c, \dots)$ of elements in K, the string $abc\dots$ will denote the subset of elements that appear an odd number of times in the sequence. If $A \subseteq \overline{K}$ is a finite set with odd cardinality, then for any pair (B, G) for which there exists a valid colouring $\psi^A(B, G)$, the number of ghost-bonds must be odd. Thinking of these as bridges to Γ, Γ may thus be viewed as an element of A, and we make the following remark.

Remark 3.3.1. *For $A \subseteq \overline{K}$ with $|A|$ odd, we shall use the expressions ψ^A and $\psi^{A\cup\{\Gamma\}}$ interchangeably.*

We call a function F, acting on $(\psi_1^A, \psi_2^B, \Delta)$, a *connectivity function* if it depends only on the connectivity properties using open paths of $(\psi_1^A, \psi_2^B, \Delta)$, that is, the value of F depends only on the set $\{(x, y) \in (K^\Gamma)^2 : x \leftrightarrow y\}$. In the following, E denotes expectation with respect to $d\mu_\lambda d\mu_\gamma dM_{B,G}dP$, where P is the law of Δ.

Theorem 3.3.2 (Switching lemma). *Let F be a connectivity function and $A, B \subseteq \overline{K}$ finite sets. For $x, y \in K^\Gamma$,*

$$\begin{aligned} E\big(\partial\psi_1^A\partial\psi_2^B \cdot F(\psi_1^A, \psi_2^B, \Delta) \cdot \mathbb{1}\{x \leftrightarrow y \text{ in } \psi_1^A, \psi_2^B, \Delta\}\big) \\ = E\Big(\partial\psi_1^{A\triangle xy}\partial\psi_2^{B\triangle xy} \cdot F(\psi_1^{A\triangle xy}, \psi_2^{B\triangle xy}, \Delta)\cdot \\ \cdot \mathbb{1}\{x \leftrightarrow y \text{ in } \psi_1^{A\triangle xy}, \psi_2^{B\triangle xy}, \Delta\}\Big). \end{aligned} \tag{3.3.2}$$

In particular,

$$E(\partial\psi_1^{xy}\partial\psi_2^B) = E\big(\partial\psi_1^{\varnothing}\partial\psi_2^{B\triangle xy} \cdot \mathbb{1}\{x \leftrightarrow y \text{ in } \psi_1^{\varnothing}, \psi_2^{B\triangle xy}, \Delta\}\big). \tag{3.3.3}$$

Proof. Equation (3.3.3) follows from (3.3.2) with $A = \{x, y\}$ and $F \equiv 1$, and so it suffices to prove (3.3.2). This is trivial if $x = y$, and we assume henceforth that $x \neq y$. Recall that $W = \{v \in V : K_v = \mathbb{S}\}$ and $|W| = r$.

We prove (3.3.2) first for the special case when $F \equiv 1$, that is,

$$\begin{aligned} E\big(\partial\psi_1^A\partial\psi_2^B \cdot \mathbb{1}\{x \leftrightarrow y \text{ in } \psi_1^A, \psi_2^B, \Delta\}\big) \\ = E\big(\partial\psi_1^{A\triangle xy}\partial\psi_2^{B\triangle xy} \cdot \mathbb{1}\{x \leftrightarrow y \text{ in } \psi_1^{A\triangle xy}, \psi_2^{B\triangle xy}, \Delta\}\big), \end{aligned} \tag{3.3.4}$$

and this will follow by conditioning on the pair $Q = (B_1 \cup B_2, G_1 \cup G_2)$.

Let Q be given. Conditional on Q, the law of (ψ_1^A, ψ_2^B) is given as follows. First, we allocate each bridge and each ghost-bond to either ψ_1^A or ψ_2^B with equal probability (independently of one another). If $W \neq \varnothing$, then we must also allocate (uniform) random colours to the points $(w,0)$, $w \in W$, for each of ψ_1^A, ψ_2^B. If $(w,0)$ is itself a source, we work instead with $(w,0+)$. (Recall that the pair (B',G') may be reconstructed from knowledge of a valid colouring $\psi^{A'}(B',G')$.) There are $2^{|Q|+2r}$ possible outcomes of the above choices, and each is equally likely.

The process Δ is independent of all random variables used above. Therefore, the conditional expectation, given Q, of the random variable on the left side of (3.3.4) equals

$$\frac{1}{2^{|Q|+2r}} \sum_{\mathcal{Q}^{A,B}} \partial Q_1 \partial Q_2 \, P(x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta), \tag{3.3.5}$$

where the sum is over the set $\mathcal{Q}^{A,B} = \mathcal{Q}^{A,B}(Q)$ of all possible pairs (Q_1, Q_2) of values of (ψ_1^A, ψ_2^B). The measure P is that of Δ.

We shall define an invertible (and therefore measure-preserving) map from $\mathcal{Q}^{A,B}$ to $\mathcal{Q}^{A\triangle xy, B\triangle xy}$. Let π be a path of Q with endpoints x and y (if such a path π exists), and let $f_\pi : \mathcal{Q}^{A,B} \to \mathcal{Q}^{A\triangle xy, B\triangle xy}$ be given as follows. Let $(Q_1, Q_2) \in \mathcal{Q}^{A,B}$, say $Q_1 = Q_1^A(B_1, G_1)$ and $Q_2 = Q_2^B(B_2, G_2)$ where $Q = (B_1 \cup B_2, G_1 \cup G_2)$. For $i = 1, 2$, let B_i' (respectively, G_i') be the set of bridges (respectively, ghost-bonds) in Q lying in exactly one of B_i, π (respectively, G_i, π). Otherwise expressed, (B_i', G_i') is obtained from (B_i, G_i) by adding the bridges/ghost-bonds of π 'modulo 2'. Note that $(B_1' \cup B_2', G_1' \cup G_2') = Q$.

If $W = \varnothing$, we let $R_1 = R_1^{A\triangle xy}$ (respectively, $R_2^{B\triangle xy}$) be the unique valid colouring of (B_1', G_1') with sources $A\triangle xy$ (respectively, (B_2', G_2') with sources $B\triangle xy$), so $R_1 = \psi^{A\triangle xy}(B_1', G_1')$, and similarly for R_2. When $W \neq \varnothing$ and $i = 1, 2$, we choose the colours of the $(w,0)$, $w \in W$, in R_i in such a way that $R_i \equiv Q_i$ on $K \setminus \pi$.

It is easily seen that the map $f_\pi : (Q_1, Q_2) \mapsto (R_1, R_2)$ is invertible, indeed its inverse is given by the same mechanism. See Figure 3.5.

By (3.2.1),

$$\partial Q_1 \partial Q_2 = \exp\big\{2\delta(\mathrm{ev}(Q_1)) + 2\delta(\mathrm{ev}(Q_2))\big\}. \tag{3.3.6}$$

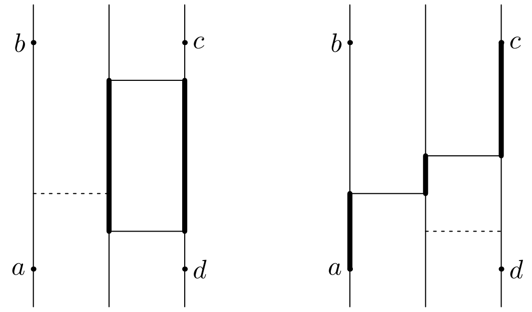

FIGURE 3.5. Switched configurations. Taking Q_1^{ac}, $Q_2^{\varnothing}$ and π to be ψ_1^{ac}, $\psi_2^{\varnothing}$ and π of Figure 3.4, respectively, this figure illustrates the 'switched' configurations $R_1^{\varnothing}$ and R_2^{ac} (left and right, respectively).

Now,

$$
\begin{aligned}
(3.3.7) \qquad \delta(\mathrm{ev}(Q_i)) &= \delta(\mathrm{ev}(Q_i) \cap \pi) + \delta(\mathrm{ev}(Q_i) \setminus \pi) \\
&= \delta(\mathrm{ev}(Q_i) \cap \pi) + \delta(\mathrm{ev}(R_i) \setminus \pi),
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta(\mathrm{ev}(Q_1) \cap \pi) + \delta(\mathrm{ev}(Q_2) \cap \pi) - 2\delta\big(\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi\big) \\
&\qquad = \delta\big(\mathrm{ev}(Q_1) \cap \mathrm{odd}(Q_2) \cap \pi\big) + \delta\big(\mathrm{odd}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi\big) \\
&\qquad = \delta\big(\mathrm{odd}(R_1) \cap \mathrm{ev}(R_2) \cap \pi\big) + \delta\big(\mathrm{ev}(R_1) \cap \mathrm{odd}(R_2) \cap \pi\big) \\
&\qquad = \delta(\mathrm{ev}(R_1) \cap \pi) + \delta(\mathrm{ev}(R_2) \cap \pi) - 2\delta\big(\mathrm{ev}(R_1) \cap \mathrm{ev}(R_2) \cap \pi\big),
\end{aligned}
$$

whence, by (3.3.6)–(3.3.7),

$$
\begin{aligned}
(3.3.8) \qquad \partial Q_1 \partial Q_2 = \partial R_1 \partial R_2 &\exp\big\{-4\delta\big(\mathrm{ev}(R_1) \cap \mathrm{ev}(R_2) \cap \pi\big)\big\} \\
&\times \exp\big\{4\delta\big(\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi\big)\big\}.
\end{aligned}
$$

The next step is to choose a suitable path π. Consider the final term in (3.3.5), namely

$$
(3.3.9) \qquad P(x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta).
$$

There are finitely many paths in Q from x to y, let these paths be $\pi_1, \pi_2, \ldots, \pi_n$. Let $\mathcal{O}_k = \mathcal{O}_k(Q_1, Q_2, \Delta)$ be the event that π_k is the earliest such path that is open in (Q_1, Q_2, Δ). Then

$$
\begin{aligned}
(3.3.10)\quad & P(x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta) \\
&= \sum_{k=1}^{n} P(\mathcal{O}_k) \\
&= \sum_{k=1}^{n} P\big(\Delta \cap [\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi_k] = \varnothing\big) P(\widetilde{\mathcal{O}}_k) \\
&= \sum_{k=1}^{n} \exp\big\{-4\delta\big(\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi_k\big)\big\} P(\widetilde{\mathcal{O}}_k),
\end{aligned}
$$

where $\widetilde{\mathcal{O}}_k = \widetilde{\mathcal{O}}_k(Q_1, Q_2, \Delta)$ is the event that each of $\pi_1, \dots, \pi_{k-1}$ is rendered non-open in (Q_1, Q_2, Δ) through the presence of elements of Δ lying in $K \setminus \pi_k$. In the second line of (3.3.10), we have used the independence of $\Delta \cap \pi_k$ and $\Delta \cap (K \setminus \pi_k)$.

Let $(R_1^k, R_2^k) = f_{\pi_k}(Q_1, Q_2)$. Since $R_i^k \equiv Q_i$ on $K \setminus \pi_k$, we have that $\widetilde{\mathcal{O}}_k(Q_1, Q_2, \Delta) = \widetilde{\mathcal{O}}_k(R_1^k, R_2^k, \Delta)$. By (3.3.8) and (3.3.10), the summand in (3.3.5) equals

$$
\begin{aligned}
& \sum_{k=1}^{n} \partial Q_1 \partial Q_2 \exp\big\{-4\delta\big(\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi_k\big)\big\} P(\widetilde{\mathcal{O}}_k) \\
&\qquad = \sum_{k=1}^{n} \partial R_1^k \partial R_2^k \exp\big\{-4\delta\big(\mathrm{ev}(R_1^k) \cap \mathrm{ev}(R_2^k) \cap \pi_k\big)\big\} P(\widetilde{\mathcal{O}}_k) \\
&\qquad = \sum_{k=1}^{n} \partial R_1^k \partial R_2^k \, P(\mathcal{O}_k(R_1^k, R_2^k, \Delta)).
\end{aligned}
$$

Summing the above over $\mathcal{Q}^{A,B}$, and remembering that each f_{π_k} is a bijection between $\mathcal{Q}^{A,B}$ and $\mathcal{Q}^{A\triangle xy, B\triangle xy}$, (3.3.5) becomes

$$
\begin{aligned}
& \frac{1}{2^{|Q|+2r}} \sum_{k=1}^{n} \sum_{(R_1,R_2) \in \mathcal{Q}^{A\triangle xy, B\triangle xy}} \partial R_1 \partial R_2 \, P(\mathcal{O}_k(R_1, R_2, \Delta)) \\
&\qquad = \frac{1}{2^{|Q|+2r}} \sum_{\mathcal{Q}^{A\triangle xy, B\triangle xy}} \partial R_1 \partial R_2 \, P(x \leftrightarrow y \text{ in } R_1, R_2, \Delta).
\end{aligned}
$$

By the argument leading to (3.3.5), this equals the right side of (3.3.4), and the claim is proved when $F \equiv 1$.

Consider now the case of general connectivity functions F in (3.3.2). In (3.3.5), the factor $P(x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta)$ is replaced by

$$P\big(F(Q_1, Q_2, \Delta) \cdot \mathbb{I}\{x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta\}\big),$$

where P is expectation with respect to Δ. In the calculation (3.3.10), we use the fact that

$$P(F \cdot \mathbb{I}_{\mathcal{O}_k}) = P(F \mid \mathcal{O}_k) P(\mathcal{O}_k)$$

and we deal with the factor $P(\mathcal{O}_k)$ as before. The result follows on noting that, for each k,

$$P\big(F(Q_1, Q_2, \Delta) \,\big|\, \mathcal{O}_k(Q_1, Q_2, \Delta)\big) = P\big(F(R_1^k, R_2^k, \Delta) \,\big|\, \mathcal{O}_k(R_1^k, R_2^k, \Delta)\big).$$

This holds because: (i) the configurations (Q_1, Q_2, Δ) and (R_1^k, R_2^k, Δ) are identical off π_k, and (ii) in each, all points along π_k are connected. Thus the connectivities are identical in the two configurations. $\square$

3.3.2. Applications of switching. In this section are presented a number of inequalities and identities proved using the random-parity representation and the switching lemma. With some exceptions (most notably (3.3.37)) the proofs are adaptations of the proofs for the discrete Ising model that may be found in [**3, 50**].

For $R \subseteq K$ a finite union of intervals, let

$$\widetilde{R} := \{(uv, t) \in F : \text{either } (u, t) \in R \text{ or } (v, t) \in R \text{ or both}\}.$$

Recall that $W = W(K) = \{v \in V : K_v = \mathbb{S}\}$, and $N = N(K)$ is the total number of intervals constituting K.

LEMMA 3.3.3. *Let $R \subseteq K$ be finite union of intervals, and let $\nu \in \Xi$ be such that $\nu \cap R = \varnothing$. If $A \subseteq \overline{K \setminus R}$ is finite and $A \sim \nu$, then*

$$w^A(\nu) \leq 2^{r(\nu) - r'(\nu)} w^A_{K \setminus R}(\nu), \tag{3.3.11}$$

where

$$r(\nu) = r(\nu, K) := |\{w \in W : \nu \cap (w \times K_w) \neq \varnothing\}|,$$
$$r'(\nu) = r(\nu, K \setminus R).$$

Proof. By (3.2.26) and Lemma 3.2.2,

$$w^A(\nu) = \frac{Z_{K\setminus\nu}}{Z_K} = 2^{N(K)-N(K\setminus\nu)} e^{\lambda(\widetilde{\nu})+\gamma(\nu)-\delta(\nu)} \frac{Z'_{K\setminus\nu}}{Z'_K}. \tag{3.3.12}$$

We claim that

$$\frac{Z'_{K\setminus\nu}}{Z'_K} \leq \frac{Z'_{K\setminus(R\cup\nu)}}{Z'_{K\setminus R}}, \tag{3.3.13}$$

and the proof of this follows.

Recall the formula (3.1.4) for Z'_K in terms of an integral over the Poisson process D. The set D is the union of independent Poisson processes D' and D'', restricted respectively to $K \setminus \nu$ and ν. We write P' (respectively, P'') for the probability measure (and, on occasion, expectation operator) governing D' (respectively, D''). Let $\Sigma(D')$ denote the set of spin configurations on $K \setminus \nu$ that are permitted by D'. By (3.1.4),

$$Z'_K = P'\left(\sum_{\sigma' \in \Sigma(D')} Z'_\nu(\sigma') \exp\left\{\int_{F\setminus\widetilde{\nu}} \lambda(e)\sigma'_e\, de + \int_{K\setminus\nu} \gamma(x)\sigma'_x\, dx\right\}\right), \tag{3.3.14}$$

where

$$Z'_\nu(\sigma') = P''\left(\sum_{\sigma'' \in \widetilde{\Sigma}(D'')} \exp\left\{\int_{\widetilde{\nu}} \lambda(e)\sigma_e\, de + \int_\nu \gamma(x)\sigma_x\, dx\right\} \cdot \mathbb{1}_C(\sigma')\right)$$

is the partition function on ν with boundary condition σ', and where σ, $\widetilde{\Sigma}(D'')$, and $C = C(\sigma', D'')$ are given as follows.

The set D'' divides ν, in the usual way, into a collection $V_\nu(D'')$ of intervals. From the set of endpoints of such intervals, we distinguish the subset $\mathcal{E}$ that: (i) lie in K, and (ii) are endpoints of some interval of $K \setminus \nu$. For $x \in \mathcal{E}$, let $\sigma'_x = \lim_{y\to x} \sigma'_y$, where the limit is taken over $y \in K \setminus \nu$. Let $\widetilde{V}_\nu(D'')$ be the subset of $V_\nu(D'')$ containing those intervals with no endpoint in $\mathcal{E}$, and let $\widetilde{\Sigma}(D'') = \{-1, +1\}^{\widetilde{V}_\nu(D'')}$.

Let $\sigma' \in \Sigma(D')$, and let $\mathcal{I}$ be the set of maximal sub-intervals I of ν having both endpoints in $\mathcal{E}$, and such that $I \cap D'' = \varnothing$. Let $C = C(D'')$ be the set of $\sigma' \in \Sigma(D')$ such that, for all $I \in \mathcal{I}$, the endpoints of I have equal spins under σ'. Note that

$$\mathbb{1}_C(\sigma') = \prod_{I\in\mathcal{I}} \tfrac{1}{2}(\sigma'_{x(I)}\sigma'_{y(I)} + 1), \tag{3.3.15}$$

where $x(I)$, $y(I)$ denote the endpoints of I.

Let $\sigma'' \in \widetilde{\Sigma}(D'')$. The conjunction σ of σ' and σ'' is defined except on sub-intervals of ν lying in $V_\nu(D'') \setminus \widetilde{V}_\nu(D'')$. On any such sub-interval with exactly one endpoint x in $\mathcal{E}$, we set $\sigma \equiv \sigma'_x$. On the event C, an interval of ν with both endpoints $x(I)$, $y(I)$ in $\mathcal{E}$ receives the spin $\sigma \equiv \sigma_{x(I)} = \sigma_{y(I)}$. Thus, $\sigma \in \Sigma(D' \cup D'')$ is well defined for $\sigma' \in C$.

By (3.3.14),

$$\frac{Z'_K}{Z'_{K\setminus\nu}} = \langle Z'_\nu(\sigma')\rangle_{K\setminus\nu}.$$

Taking the expectation $\langle\cdot\rangle_{K\setminus\nu}$ inside the integral, the last expression becomes

$$P''\left(\sum_{\sigma''\in\widetilde{\Sigma}(D'')} \left\langle \exp\left\{\int_{\widetilde{\nu}} \lambda(e)\sigma_e\, de\right\} \exp\left\{\int_\nu \gamma(x)\sigma_x\, dx\right\} \cdot \mathbb{1}_C(\sigma')\right\rangle_{K\setminus\nu}\right)$$

The inner expectation may be expressed as a sum over $k, l \geq 0$ (with non-negative coefficients) of iterated integrals of the form

$$\frac{1}{k!}\frac{1}{l!} \iint\limits_{\widetilde{\nu}^k\times\nu^l} \lambda(\mathbf{e})\gamma(\mathbf{x})\langle \sigma_{e_1}\cdots\sigma_{e_k}\sigma_{x_1}\cdots\sigma_{x_l} \cdot \mathbb{1}_C\rangle_{K\setminus\nu}\, d\mathbf{e}\, d\mathbf{x}, \tag{3.3.16}$$

where we have written $\mathbf{e} = (e_1, \ldots, e_k)$, and $\lambda(\mathbf{e})$ for $\lambda(e_1)\cdots\lambda(e_k)$ (and similarly for $\mathbf{x}$). We may write

$$\langle \sigma_{e_1}\cdots\sigma_{e_k}\sigma_{x_1}\cdots\sigma_{x_l} \cdot \mathbb{1}_C\rangle_{K\setminus\nu} = \langle \sigma'_S\sigma''_T \cdot \mathbb{1}_C\rangle_{K\setminus\nu} = \sigma''_T\langle \sigma'_S \cdot \mathbb{1}_C\rangle_{K\setminus\nu},$$

for sets $S \subseteq \overline{K \setminus \nu}$, $T \subseteq \nu$ determined by $e_1, \ldots, e_k, x_1, \ldots, x_l$ and D'' only. We now bring the sum over σ'' inside the integral of (3.3.16). For $T \neq \varnothing$,

$$\sum_{\sigma'' \in \widetilde{\Sigma}(D'')} \sigma''_T \langle \sigma'_S \cdot 1\!\!1_C \rangle_{K \setminus \nu} = 0,$$

so any non-zero term is of the form

$$\langle \sigma'_S \cdot 1\!\!1_C \rangle_{K \setminus \nu}. \tag{3.3.17}$$

By (3.3.15), (3.3.17) may be expressed in the form

$$\sum_{i=1}^{s} 2^{-a_i} \langle \sigma'_{S_i} \rangle_{K \setminus \nu} \tag{3.3.18}$$

for appropriate sets S_i and integers a_i. By Lemma 2.2.22,

$$\langle \sigma'_{S_i} \rangle_{K \setminus \nu} \geq \langle \sigma'_{S_i} \rangle_{K \setminus (R \cup \nu)}.$$

On working backwards, we obtain (3.3.13).

By (3.3.12)–(3.3.13),

$$w^A(\nu) \leq 2^U w^A_{K \setminus R}(\nu),$$

where

$$\begin{aligned} U &= \big[N(K) - N(K \setminus \nu)\big] - \big[N(K \setminus R) - N(K \setminus (R \cup \nu))\big] \\ &= r(\nu) - r'(\nu) \end{aligned}$$

as required. $\square$

For distinct $x, y, z \in K^\Gamma$, let

$$\begin{aligned} \langle \sigma_x; \sigma_y; \sigma_z \rangle := \langle \sigma_{xyz} \rangle &- \langle \sigma_x \rangle \langle \sigma_{yz} \rangle \\ &- \langle \sigma_y \rangle \langle \sigma_{xz} \rangle - \langle \sigma_z \rangle \langle \sigma_{xy} \rangle + 2 \langle \sigma_x \rangle \langle \sigma_y \rangle \langle \sigma_z \rangle. \end{aligned}$$

LEMMA 3.3.4 (GHS inequality). *For distinct* $x, y, z \in K^\Gamma$,

$$\langle \sigma_x; \sigma_y; \sigma_z \rangle \leq 0. \tag{3.3.19}$$

Moreover, $\langle\sigma_x\rangle$ is concave in γ in the sense that, for bounded, measurable functions $\gamma_1, \gamma_2 : K \to \mathbb{R}_+$ satisfying $\gamma_1 \le \gamma_2$, and $\theta \in [0,1]$,

$$\theta\langle\sigma_x\rangle_{\gamma_1} + (1-\theta)\langle\sigma_x\rangle_{\gamma_2} \le \langle\sigma_x\rangle_{\theta\gamma_1+(1-\theta)\gamma_2}. \tag{3.3.20}$$

Proof. The proof of this follows very closely the corresponding proof for the classical Ising model [**48**]. We include it here because it allows us to develop the technique of 'conditioning on clusters', which will be useful later.

We prove (3.3.19) via the following more general result. Let (B_i, G_i), $i = 1,2,3$, be independent sets of bridges/ghost-bonds, and write ψ_i, $i = 1,2,3$, for corresponding colourings (with sources to be specified through their superscripts). We claim that, for any four points $w, x, y, z \in K^\Gamma$,

(3.3.21)
$$E\big(\partial\psi_1^\varnothing\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\big) - E\big(\partial\psi_1^\varnothing\partial\psi_2^{wz}\partial\psi_3^{xy}\big)$$
$$\le E(\partial\psi_1^\varnothing\partial\psi_2^{wx}\partial\psi_3^{yz}) + E(\partial\psi_1^\varnothing\partial\psi_2^{wy}\partial\psi_3^{xz}) - 2E(\partial\psi_1^{wx}\partial\psi_2^{wy}\partial\psi_3^{wz}).$$

Inequality (3.3.19) follows by Theorem 3.2.1 on letting $w = \Gamma$.

The left side of (3.3.21) is

$$E(\partial\psi_1^\varnothing)\big[E(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}) - E(\partial\psi_2^{wz}\partial\psi_3^{xy})\big]$$
$$= Z\,E\big(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot \mathbb{1}\{w \nleftrightarrow z\}\big),$$

by the switching lemma 3.3.2. When $\partial\psi_3^{wxyz}$ is non-zero, parity constraints imply that at least one of $\{w \leftrightarrow x\}\cap\{y \leftrightarrow z\}$ and $\{w \leftrightarrow y\}\cap\{x \leftrightarrow z\}$ occurs, but that, in the presence of the indicator function they cannot both occur. Therefore,

(3.3.22)
$$E(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot \mathbb{1}\{w \nleftrightarrow z\})$$
$$= E\big(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot \mathbb{1}\{w \nleftrightarrow z\}\cdot \mathbb{1}\{w \leftrightarrow x\}\big)$$
$$+ E\big(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot \mathbb{1}\{w \nleftrightarrow z\}\cdot \mathbb{1}\{w \leftrightarrow y\}\big).$$

Consider the first term. By the switching lemma,

(3.3.23)
$$E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{wxyz}\cdot \mathbb{I}\{w\nleftrightarrow z\}\cdot \mathbb{I}\{w\leftrightarrow x\}\big) = E\big(\partial\psi_2^{wx}\partial\psi_3^{yz}\cdot \mathbb{I}\{w\nleftrightarrow z\}\big).$$

We next 'condition on a cluster'. Let $C_z = C_z(\psi_2^{wx}, \psi_3^{yz}, \Delta)$ be the set of all points of K that are connected by open paths to z. Conditional on C_z, define new independent colourings $\mu_2^{\varnothing}$, μ_3^{yz} on the domain $M = C_z$. Similarly, let ν_2^{wx}, $\nu_3^{\varnothing}$ be independent colourings on the domain $N = K \setminus C_z$, that are also independent of the μ_i. It is not hard to see that, if $w \nleftrightarrow z$ in $(\psi_2^{wx}, \psi_3^{yz}, \Delta)$, then, conditional on C_z, the law of ψ_2^{wx} equals that of the superposition of $\mu_2^{\varnothing}$ and ν_2^{wx}; similarly the conditional law of ψ_3^{yz} is the same as that of the superposition of μ_3^{yz} and $\nu_3^{\varnothing}$. Therefore, almost surely on the event $\{w \nleftrightarrow z\}$,

(3.3.24)
$$\begin{aligned} E(\partial\psi_2^{wx}\partial\psi_3^{yz} \mid C_z) &= E'(\partial\mu_2^{\varnothing})E'(\partial\nu_2^{wx})E'(\partial\mu_3^{yz})E'(\partial\nu_3^{\varnothing}) \\ &= \langle\sigma_{wx}\rangle_N E'(\partial\mu_2^{\varnothing})E'(\partial\nu_2^{\varnothing})E'(\partial\mu_3^{yz})E'(\partial\nu_3^{\varnothing}) \\ &\le \langle\sigma_{wx}\rangle_K E(\partial\psi_2^{\varnothing}\partial\psi_3^{yz} \mid C_z), \end{aligned}$$

where E' denotes expectation conditional on C_z, and we have used Lemma 2.2.22. Returning to (3.3.22)–(3.3.23),

$$\begin{aligned} &E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{wxyz}\cdot \mathbb{I}\{w\nleftrightarrow z\}\cdot \mathbb{I}\{w\leftrightarrow x\}\big) \\ &\qquad\le \langle\sigma_{wx}\rangle E(\partial\psi_2^{\varnothing}\partial\psi_3^{yz}\cdot \mathbb{I}\{w\nleftrightarrow z\}). \end{aligned}$$

The other term in (3.3.22) satisfies the same inequality with x and y interchanged. Inequality (3.3.21) follows on applying the switching lemma to the right sides of these two last inequalities, and adding them.

The concavity of $\langle\sigma_x\rangle$ follows from the fact that, if

(3.3.25)
$$T = \sum_{k=1}^{n} a_k \mathbb{I}_{A_k}$$

is a step function on K with $a_k \geq 0$ for all k, and $\gamma(\cdot) = \gamma_1(\cdot) + \alpha T(\cdot)$, then

$$(3.3.26) \qquad \frac{\partial^2}{\partial \alpha^2}\langle \sigma_x \rangle = \sum_{k,l=1}^{n} a_k a_l \iint_{A_k \times A_l} dy\, dz\, \langle \sigma_x; \sigma_y; \sigma_z \rangle \leq 0.$$

Thus, the claim holds whenever $\gamma_2 - \gamma_1$ is a step function. The general claim follows by approximating $\gamma_2 - \gamma_1$ by step functions, and applying the dominated convergence theorem. □

For the next lemma we assume for simplicity that $\gamma \equiv 0$ (although similar results can easily be proved for $\gamma \not\equiv 0$). We let $\bar{\delta} \in \mathbb{R}$ be an upper bound for δ, thus $\delta(x) \leq \bar{\delta} < \infty$ for all $x \in K$. Let $a, b \in K$ be two distinct points. A closed set $T \subseteq K$ is said to *separate* a from b if every lattice path from a to b (whatever the set of bridges) intersects T. Moreover, if $\varepsilon > 0$ and T separates a from b, we say that T is an *ε-fat separating set* if every point in T lies in a closed sub-interval of T of length at least ε.

Lemma 3.3.5 (Simon inequality). *Let $\gamma \equiv 0$. If $\varepsilon > 0$ and T is an ε-fat separating set for $a, b \in K$,*

$$(3.3.27) \qquad \langle \sigma_a \sigma_b \rangle \leq \frac{1}{\varepsilon} \exp(8\varepsilon\bar{\delta}) \int_T \langle \sigma_a \sigma_x \rangle \langle \sigma_x \sigma_b \rangle \, dx.$$

Proof. By Theorems 3.2.1 and 3.3.2,

$$(3.3.28) \qquad \langle \sigma_a \sigma_x \rangle \langle \sigma_x \sigma_b \rangle = \frac{1}{Z^2} E(\partial \psi_1^{\varnothing} \partial \psi_2^{ab} \cdot \mathbb{1}\{a \leftrightarrow x\}),$$

and, by Fubini's theorem,

$$(3.3.29) \qquad \int_T \langle \sigma_a \sigma_x \rangle \langle \sigma_x \sigma_b \rangle \, dx = \frac{1}{Z^2} E(\partial \psi_1^{\varnothing} \partial \psi_2^{ab} \cdot |\widehat{T}|),$$

where $\widehat{T} = \{x \in T : a \leftrightarrow x\}$ and $|\cdot|$ denotes Lebesgue measure. Since $\gamma \equiv 0$, the backbone $\xi = \xi(\psi_2^{ab})$ consists of a single (lattice-) path from a to b passing through T. Let U denote the set of points in K that are separated from b by T, and let X be the point at which ξ exits U for the first time. Since T is assumed closed, $X \in T$. See Figure 3.6.

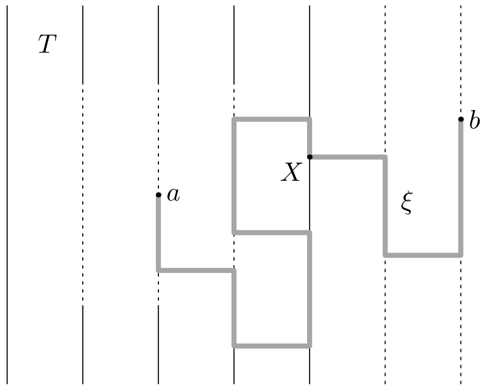

FIGURE 3.6. The Simon inequality. The separating set T is drawn with solid black lines, and the backbone ξ with a grey line.

For $x \in T$, let A_x be the event that there is no element of Δ within the interval of length 2ε centered at x. Thus, $P(A_x) = \exp(-8\varepsilon\bar{\delta})$. On the event A_X, we have that $|\widehat{T}| \geq \varepsilon$, whence

$$\begin{aligned} E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab} \cdot |\widehat{T}|) &\geq E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab} \cdot |\widehat{T}| \cdot \mathbb{1}\{A_X\}) \\ &\geq \varepsilon E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab} \cdot \mathbb{1}\{A_X\}). \end{aligned} \tag{3.3.30}$$

Conditional on X, the event A_X is independent of $\psi_1^{\varnothing}$ and ψ_2^{ab}, so that

$$E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab} \cdot |\widehat{T}|) \geq \varepsilon \exp(-8\varepsilon\bar{\delta}) E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab}), \tag{3.3.31}$$

and the proof is complete. □

Just as for the classical Ising model, only a small amount of extra work is required to deduce the following improvement of Lemma 3.3.5.

LEMMA 3.3.6 (Lieb inequality). *Under the assumptions of Lemma 3.3.5,*

$$\langle\sigma_a\sigma_b\rangle \leq \frac{1}{\varepsilon}\exp(8\varepsilon\bar{\delta}) \int_T \langle\sigma_a\sigma_x\rangle_{\overline{T}} \langle\sigma_x\sigma_b\rangle \, dx, \tag{3.3.32}$$

where $\langle\cdot\rangle_{\overline{T}}$ denotes expectation with respect to the measure restricted to $\overline{T}$.

PROOF. Let $x \in T$, let $\overline{\psi}_1^{ax}$ denote a colouring on the restricted region U, and let ψ_2^{xb} denote a colouring on the full region K as before. We claim that

$$E(\partial\overline{\psi}_1^{ax}\partial\psi_2^{xb}) = E\big(\partial\overline{\psi}_1^{\varnothing}\partial\psi_2^{ab} \cdot \mathbb{1}\{a \leftrightarrow x \text{ in } \overline{T}\}\big). \tag{3.3.33}$$

The use of the letter E is an abuse of notation, since the $\overline{\psi}$ are colourings of U only.

Equation (3.3.33) may be established using a slight variation in the proof of the switching lemma. We follow the proof of that lemma, first conditioning on the set Q of all bridges and ghost-bonds in the two colourings taken together, and then allocating them to the colourings Q_1 and Q_2, uniformly at random. We then order the paths π of Q from a to x, and add the earliest open path to both Q_1 and Q_2 'modulo 2'. There are two differences here: firstly, any element of Q that is not contained in U will be allocated to Q_2, and secondly, we only consider paths π that lie inside U. Subject to these two changes, we follow the argument of the switching lemma to arrive at (3.3.33).

Integrating (3.3.33) over $x \in T$,

$$\int_T \langle\sigma_a\sigma_x\rangle_{\overline{T}}\,\langle\sigma_x\sigma_b\rangle\,dx = \frac{1}{Z_{\overline{T}}Z}E(\partial\overline{\psi}_1^{\varnothing}\partial\psi_2^{ab} \cdot |\widehat{T}|), \tag{3.3.34}$$

where this time $\widehat{T} = \{x \in T : a \leftrightarrow x \text{ in } U\}$. The proof is completed as in (3.3.30)–(3.3.31). □

For the next lemma we specialize to the situation that is the main focus of this chapter, namely the following. Similar results are valid for other lattices and for summable translation-invariant interactions.

ASSUMPTION 3.3.7.

- *The graph $L = [-n, n]^d \subseteq \mathbb{Z}^d$ where $d \geq 1$, with periodic boundary condition.*
- *The parameters λ, δ, γ are non-negative constants.*
- *The set $K_v = \mathbb{S}$ for every $v \in V$.*

Under the periodic boundary condition, two vertices of L are joined by an edge whenever there exists $i \in \{1, 2, \ldots, d\}$ such that their i-coordinates differ by exactly $2n$.

Under Assumption 3.3.7, the model is invariant under automorphisms of L and, furthermore, the quantity $\langle \sigma_x \rangle$ does not depend on the choice of x. Let 0 denote some fixed but arbitrary point of K, and let $M = M(\lambda, \delta, \gamma) = \langle \sigma_0 \rangle$ denote the common value of the $\langle \sigma_x \rangle$.

For $x, y \in K$, we write $x \sim y$ if $x = (u, t)$ and $y = (v, t)$ for some $t \geq 0$ and u, v adjacent in L. We write $\{x \overset{z}{\leftrightarrow} y\}$ for the complement of the event that there exists an open path from x to y not containing z. Thus, $x \overset{z}{\leftrightarrow} y$ if: either $x \not\leftrightarrow y$, or $x \leftrightarrow y$ and every open path from x to y passes through z.

Theorem 3.3.8. *Under Assumption 3.3.7, the following hold.*

(3.3.35)
$$\frac{\partial M}{\partial \gamma} = \frac{1}{Z^2} \int_K dx\, E\big(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \cdot \mathbb{1}\{0 \not\leftrightarrow \Gamma\}\big) \leq \frac{M}{\gamma}.$$

(3.3.36)
$$\frac{\partial M}{\partial \lambda} = \frac{1}{2Z^2} \int_K dx \sum_{y \sim x} E\big(\partial \psi_1^{0xy\Gamma} \partial \psi_2^{\varnothing} \cdot \mathbb{1}\{0 \not\leftrightarrow \Gamma\}\big) \leq 2dM \frac{\partial M}{\partial \gamma}.$$

(3.3.37)
$$-\frac{\partial M}{\partial \delta} = \frac{2}{Z^2} \int_K dx\, E\big(\partial \psi_1^{0\Gamma} \partial \psi_2^{\varnothing} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \leq \frac{2M}{1 - M^2} \frac{\partial M}{\partial \gamma}.$$

Proof. With the exception of (3.3.37), the proofs mimic those of [**3**] for the classical Ising model. For the equality in (3.3.35), note that

$$\frac{\partial M}{\partial \gamma} = \int_K \langle \sigma_0 ; \sigma_x \rangle \, dx.$$

Now

$$\langle \sigma_0 ; \sigma_x \rangle = \langle \sigma_0 \sigma_x \rangle - \langle \sigma_0 \rangle \langle \sigma_x \rangle = \frac{1}{Z^2} (E(\partial \psi_1^{0x} \partial \psi_2^{\varnothing}) - E(\partial \psi_1^{0\Gamma} \partial \psi_2^{x\Gamma}))$$

and the difference $E(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}) - E(\partial\psi_1^{0\Gamma}\partial\psi_2^{x\Gamma})$ on the right hand side equals

$$E(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}) - E(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\cdot \mathbb{1}\{0 \leftrightarrow \Gamma\}) = E(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\cdot \mathbb{1}\{0 \not\leftrightarrow \Gamma\})$$

by the switching lemma. For the inequality in (3.3.35), the concavity of M in γ means that for all $\gamma_2 \geq \gamma_1 > 0$,

$$\frac{\partial M}{\partial \gamma} \leq \frac{M(\lambda,\delta,\gamma_2) - M(\lambda,\delta,\gamma_1)}{\gamma_2 - \gamma_1}. \tag{3.3.38}$$

Letting $\gamma_1 \to 0$ and using the continuity of M and the fact that $M(\lambda,\delta,0) = 0$ for all $\lambda,\delta > 0$, the result follows.

Similarly, for the equality in (3.3.36) we note that

$$\frac{\partial M}{\partial \lambda} = \int_F \langle\sigma_0;\sigma_e\rangle\, de = \frac{1}{2}\int_K dx \sum_{y\sim x}(\langle\sigma_0\sigma_x\sigma_y\rangle - \langle\sigma_0\rangle\langle\sigma_x\sigma_y\rangle).$$

Again

$$\begin{aligned}\langle\sigma_0\sigma_x\sigma_y\rangle - \langle\sigma_0\rangle\langle\sigma_x\sigma_y\rangle &= \frac{1}{Z^2}(E(\partial\psi_1^{0xy\Gamma}\partial\psi_2^{\varnothing}) - E(\partial\psi_1^{0\Gamma}\partial\psi_2^{xy}))\\ &= E(\partial\psi_1^{0xy\Gamma}\partial\psi_2^{\varnothing}\cdot \mathbb{1}\{0 \not\leftrightarrow \Gamma\})\end{aligned}$$

by the switching lemma. For the inequality,

$$\begin{aligned}\frac{\partial M}{\partial \lambda} &= \frac{1}{2}\int_K dx \sum_{y\sim x}\big(\langle\sigma_0\sigma_x\sigma_y\rangle - \langle\sigma_0\rangle\langle\sigma_x\sigma_y\rangle\big)\\ &\leq \frac{1}{2}\int_K dx \sum_{y\sim x}\big(\langle\sigma_x\rangle\langle\sigma_0\sigma_y\rangle + \langle\sigma_y\rangle\langle\sigma_0\sigma_x\rangle - 2\langle\sigma_0\rangle\langle\sigma_x\rangle\langle\sigma_y\rangle\big)\\ &= \int_K dx\, \langle\sigma_0;\sigma_x\rangle \sum_{y\sim x}\langle\sigma_y\rangle\\ &= 2dM\int_K dx\, \langle\sigma_0;\sigma_x\rangle = 2dM\frac{\partial M}{\partial\gamma},\end{aligned} \tag{3.3.39}$$

where we have used the GHS-inequality and translation invariance.

Here is the proof of (3.3.37). Let $|\cdot|$ denote Lebesgue measure. By differentiating

$$M = \frac{E(\partial\psi^{0\Gamma})}{E(\partial\psi^{\varnothing})} = \frac{E(\exp(2\delta|\mathrm{ev}(\psi^{0\Gamma})|))}{E(\exp(2\delta|\mathrm{ev}(\psi^{\varnothing})|))}, \tag{3.3.40}$$

with respect to δ, we obtain that

(3.3.41)

$$\begin{aligned}\frac{\partial M}{\partial \delta} &= \frac{2}{Z^2} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\big[|\mathrm{ev}(\psi_1^{0\Gamma})| - |\mathrm{ev}(\psi_2^{\varnothing})|\big]\big)\\ &= \frac{2}{Z^2}\int dx\, E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\big[\mathbb{1}\{x\in\mathrm{odd}(\psi_2^{\varnothing})\} - \mathbb{1}\{x\in\mathrm{odd}(\psi_1^{0\Gamma})\}\big]\big).\end{aligned}$$

Consider the integrand in (3.3.41). Since $\psi_2^{\varnothing}$ has no sources, all odd routes in $\psi_2^{\varnothing}$ are necessarily cycles. If $x\in\mathrm{odd}(\psi_2^{\varnothing})$, then x lies in an odd cycle. We shall assume that x is not the endpoint of a bridge, since this event has probability 0. It follows that, on the event $\{0\leftrightarrow\Gamma\}$, there exists an open path from 0 to Γ that avoids x (since any path can be re-routed around the odd cycle of $\psi_2^{\varnothing}$ containing x). Therefore, the event $\{0\overset{x}{\leftrightarrow}\Gamma\}$ does not occur, and hence

$$\begin{aligned}(3.3.42)\quad & E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\mathbb{1}\{x\in\mathrm{odd}(\psi_2^{\varnothing})\}\big)\\ &\qquad = E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\mathbb{1}\{x\in\mathrm{odd}(\psi_2^{\varnothing})\}\cdot\mathbb{1}\{0\overset{x}{\leftrightarrow}\Gamma\}^{\mathrm{c}}\big).\end{aligned}$$

We note next that, if $\partial\psi_1^{0\Gamma}\neq 0$ and $0\overset{x}{\leftrightarrow}\Gamma$, then necessarily $x\in\mathrm{odd}(\psi_1^{0\Gamma})$. Hence,

$$\begin{aligned}(3.3.43)\quad & E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\mathbb{1}\{x\in\mathrm{odd}(\psi_1^{0\Gamma})\}\big)\\ &\qquad = E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\mathbb{1}\{x\in\mathrm{odd}(\psi_1^{0\Gamma})\}\cdot\mathbb{1}\{0\overset{x}{\leftrightarrow}\Gamma\}^{\mathrm{c}}\big)\\ &\qquad\qquad + E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\mathbb{1}\{0\overset{x}{\leftrightarrow}\Gamma\}\big).\end{aligned}$$

We wish to switch the sources 0Γ from ψ_1 to ψ_2 in the right side of (3.3.43). For this we need to adapt some details of the proof of the switching lemma to this situation. The first step in the proof of that lemma was to condition on the union Q of the bridges and ghost-bonds of the two colourings; then, the paths from 0 to Γ in Q were listed in a fixed *but arbitrary* order. We are free to choose this ordering in such a way that paths not containing x have precedence, and we assume henceforth that the ordering is thus chosen. The next step is to find the earliest open path π, and 'add π modulo 2' to both $\psi_1^{0\Gamma}$ and $\psi_2^{\varnothing}$. On the event $\{0 \overset{x}{\leftrightarrow} \Gamma\}^{\mathrm{c}}$, this earliest path π does not contain x, by our choice of ordering. Hence, in the new colouring $\psi_1^{\varnothing}$, x continues to lie in an 'odd' interval (recall that, outside π, the colourings are unchanged by the switching procedure). Therefore,

$$E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing} \cdot \mathbb{1}\{x \in \mathrm{odd}(\psi_1^{0\Gamma})\} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}^{\mathrm{c}}\big) = E\big(\partial\psi_1^{\varnothing}\partial\psi_2^{0\Gamma} \cdot \mathbb{1}\{x \in \mathrm{odd}(\psi_1^{\varnothing})\} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}^{\mathrm{c}}\big). \tag{3.3.44}$$

Relabelling, putting the last expression into (3.3.43), and subtracting (3.3.43) from (3.3.42), we obtain

$$\frac{\partial M}{\partial \delta} = -\frac{2}{Z^2}\int dx\, E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \tag{3.3.45}$$

as required.

Turning to the inequality, let C_z^x denote the set of points that can be reached from z along open paths *not containing* x. When conditioning $E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big)$ on C_0^x as in the proof of the GHS inequality, we find that $\psi_1^{0\Gamma}$ is a combination of two independent colourings, one inside C_0^x with sources $0x$, and one outside C_0^x with sources $x\Gamma$. As in (3.3.24), using Lemma 2.2.22 as there,

$$\begin{aligned} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) &= E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\langle\sigma_x\rangle_{K\setminus C_0^x} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \\ &\leq M \cdot E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big). \end{aligned} \tag{3.3.46}$$

We split the expectation on the right side according to whether or not $x \leftrightarrow \Gamma$. Clearly,

$$E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing} \cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\} \cdot \mathbb{1}\{x \not\leftrightarrow \Gamma\}\big) \leq E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing} \cdot \mathbb{1}\{x \not\leftrightarrow \Gamma\}\big). \tag{3.3.47}$$

By the switching lemma 3.3.2, the other term satisfies

(3.3.48)
$$E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\cdot \mathbb{1}\{x\leftrightarrow\Gamma\}\big) = E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{x\Gamma}\cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big).$$

We again condition on a cluster, this time C_Γ^x, to obtain as in (3.3.46) that

$$E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{x\Gamma}\cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \le M\cdot E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot \mathbb{1}\{0 \overset{x}{\leftrightarrow} \Gamma\}\big). \tag{3.3.49}$$

Combining (3.3.46), (3.3.47), (3.3.49) with (3.3.45), we obtain by (3.3.35) that

$$-\frac{\partial M}{\partial\delta} \le 2M\frac{\partial M}{\partial\gamma} + M^2\Big(-\frac{\partial M}{\partial\delta}\Big), \tag{3.3.50}$$

as required. □

3.4. Proof of the main differential inequality

In this section we will prove Theorem 3.1.3, the differential inequality which, in combination with the inequalities of the previous section, will yield information about the critical behaviour of the space–time Ising model. The proof proceeds roughly as follows. In the random-parity representation of $M = \langle\sigma_0\rangle$, there is a backbone from 0 to Γ (that is, to some point $g\in G$). We introduce two new sourceless configurations; depending on how the backbone interacts with these configurations, the switching lemma allows a decomposition into a combination of other configurations which, via Theorem 3.3.8, may be transformed into derivatives of the magnetization.

Throughout this section we work under Assumption 3.3.7, that is, *we work with a translation-invariant model on a cube in the d-dimensional lattice*, while noting that our conclusions are valid for more general interactions with similar symmetries. The arguments in this section borrow heavily from [**3**]. As in Theorem 3.3.8, the main novelty in the proof concerns connectivity in the 'vertical' direction (the term R_v in (3.4.2)–(3.4.3) below).

Proof of Theorem 3.1.3. By Theorem 3.2.1,

$$M = \frac{1}{Z}E(\partial\psi_1^{0\Gamma}) = \frac{1}{Z^3}E(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}). \tag{3.4.1}$$

We shall consider the backbone $\xi = \xi(\psi_1^{0\Gamma})$ and the open cluster C_Γ of Γ in $(\psi_2^{\varnothing}, \psi_3^{\varnothing}, \Delta)$. All connectivities will refer to the triple $(\psi_2^{\varnothing}, \psi_3^{\varnothing}, \Delta)$. Note that ξ consists of a single path with endpoints 0 and Γ. There are four possibilities, illustrated in Figure 3.7, for the way in which ξ, viewed as a directed path from 0 to Γ, interacts with C_Γ:

(i) $\xi \cap C_\Gamma$ is empty,

(ii) $0 \in \xi \cap C_\Gamma$,

(iii) $0 \notin \xi \cap C_\Gamma$, and ξ first meets C_Γ immediately after a bridge,

(iv) $0 \notin \xi \cap C_\Gamma$, and ξ first meets C_Γ at a cut, which necessarily belongs to $\mathrm{ev}(\psi_2^{\varnothing}) \cap \mathrm{ev}(\psi_3^{\varnothing})$.

Thus,

$$M = T + R_0 + R_h + R_v, \tag{3.4.2}$$

where

(3.4.3)

$$T = \frac{1}{Z^3}E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing} \cdot \mathbb{1}\{\xi \cap C_\Gamma = \varnothing\}\big),$$

$$R_0 = \frac{1}{Z^3}E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing} \cdot \mathbb{1}\{0 \leftrightarrow \Gamma\}\big),$$

$$R_h = \frac{1}{Z^3}E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing} \cdot \mathbb{1}\{\text{first point on } \xi \cap C_\Gamma \text{ is a bridge of } \xi\}\big),$$

$$R_v = \frac{1}{Z^3}E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing} \cdot \mathbb{1}\{\text{first point on } \xi \cap C_\Gamma \text{ is a cut}\}\big).$$

We will bound each of these terms separately.

By the switching lemma,

$$\begin{aligned} R_0 &= \frac{1}{Z^3}E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing} \cdot \mathbb{1}\{0 \leftrightarrow \Gamma\}\big) \\ &= \frac{1}{Z^3}E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{0\Gamma}\partial\psi_3^{0\Gamma}\big) = M^3. \end{aligned} \tag{3.4.4}$$

Next, we bound T. The letter ξ will always denote the backbone of the first colouring ψ_1, with corresponding sources. Let X denote the

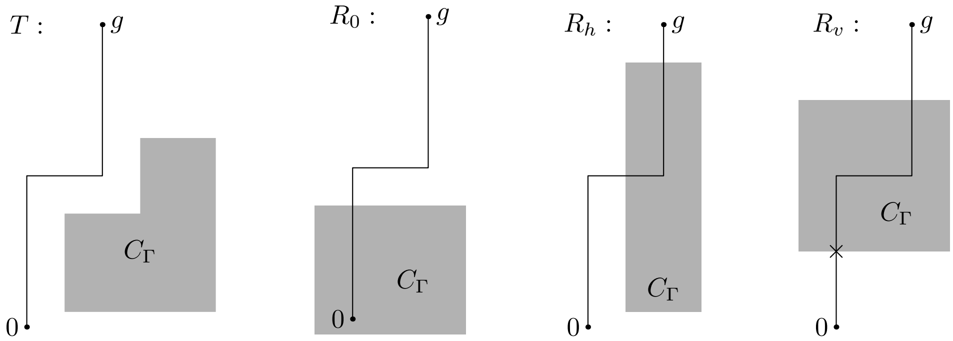

FIGURE 3.7. Illustrations of the four possibilities for $\xi \cap C_\Gamma$. Ghost-bonds in $\psi^{0\Gamma}$ are labelled g. The backbone ξ is drawn as a solid black line, and C_Γ as a grey rectangle.

location of the ghost-bond that ends ξ. By conditioning on X,

(3.4.5)
$$\begin{aligned} T &= \frac{1}{Z^3}\int P(X\in dx)\, E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{\xi\cap C_\Gamma=\varnothing\}\,\big|\, X=x\big)\\ &\le \frac{\gamma}{Z^3}\int dx\, E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{\xi\cap C_\Gamma=\varnothing\}\big). \end{aligned}$$

We study the last expectation by conditioning on C_Γ and bringing one of the factors $1/Z$ inside. By (3.2.25)–(3.2.26) and conditional expectation,

(3.4.6)
$$\begin{aligned} &\frac{1}{Z}E\big(\partial\psi_1^{0x}\cdot \mathbb{1}\{\xi\cap C_\Gamma=\varnothing\}\,\big|\, C_\Gamma\big)\\ &\qquad = E\Big(Z^{-1}E(\partial\psi_1^{0x}\mid \xi, C_\Gamma)\mathbb{1}\{\xi\cap C_\Gamma=\varnothing\}\,\Big|\, C_\Gamma\Big)\\ &\qquad = E\big(w^{0x}(\xi)\cdot \mathbb{1}\{\xi\cap C_\Gamma=\varnothing\}\,\big|\, C_\Gamma\big). \end{aligned}$$

By Lemma 3.3.3,

$$w^{0x}(\xi)\le 2^{r(\xi)-r'(\xi)}w^{0x}_{K\setminus C_\Gamma}(\xi)\quad\text{on}\quad \{\xi\cap C_\Gamma=\varnothing\}, \tag{3.4.7}$$

where

$$r(\xi)=r(\xi,K),\qquad r'(\xi)=r(\xi,K\setminus C_\Gamma).$$

Using (3.2.29) and (3.2.27), we have

(3.4.8)
$$\begin{aligned}
&E\big(w^{0x}(\xi)\cdot \mathbb{1}\{\xi\cap C_\Gamma=\varnothing\}\,\big|\,C_\Gamma\big)\\
&\qquad\qquad\le E\big(2^{r(\xi)-r'(\xi)}w^{0x}_{K\setminus C_\Gamma}(\xi)\cdot \mathbb{1}\{\xi\cap C_\Gamma=\varnothing\}\,\big|\,C_\Gamma\big)\\
&\qquad\qquad\le \langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}.
\end{aligned}$$

The last step merits explanation. Recall that $\xi=\xi(\psi_1^{0x})$, and assume $\xi\cap C_\Gamma=\varnothing$. Apart from the randomization that takes place when ψ_1^{0x} is one of several valid colourings, the law of ξ, $P(\xi\in d\nu)$, is a function of the positions of bridges and ghost-bonds along ν only, that is, the existence of bridges where needed, and the non-existence of ghost-bonds along ν. By (3.4.7) and Lemma 3.3.3, with $\Xi_{K\setminus C}:=\{\nu\in\Xi:\nu\cap C=\varnothing\}$ and P the law of ξ,

$$\begin{aligned}
&E\big(w^{0x}(\xi)\cdot \mathbb{1}\{\xi\cap C_\Gamma=\varnothing\}\,\big|\,C_\Gamma\big)\\
&\qquad\qquad=\int_{\Xi_{K\setminus C_\Gamma}} w^{0x}(\nu)\,P(d\nu)\\
&\qquad\qquad\le\int_{\Xi_{K\setminus C_\Gamma}} 2^{r(\nu)-r'(\nu)}w^{0x}_{K\setminus C_\Gamma}(\nu)\left(\tfrac12\right)^{r(\nu)}\mu(d\nu)
\end{aligned}$$

for some measure μ, where the factor $(\frac12)^{r(\nu)}$ arises from the possible existence of more than one valid colouring. Now, μ is a measure on paths which by the remark above depends only locally on ν, in the sense that $\mu(d\nu)$ depends only on the bridge- and ghost-bond configurations along ν. In particular, the same measure μ governs also the law of the backbone in the *smaller* region $K\setminus C_\Gamma$. More explicitly, by (3.2.27) with $P_{K\setminus C_\Gamma}$ the law of the backbone of the colouring $\psi^{0x}_{K\setminus C_\Gamma}$ defined on $K\setminus C_\Gamma$, we have

$$\begin{aligned}
\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}&=\int_{\Xi_{K\setminus C_\Gamma}} w^{0x}_{K\setminus C_\Gamma}(\nu)\,P_{K\setminus C_\Gamma}(d\nu)\\
&=\int_{\Xi_{K\setminus C_\Gamma}} w^{0x}_{K\setminus C_\Gamma}(\nu)\left(\tfrac12\right)^{r'(\nu)}\mu(d\nu).
\end{aligned}$$

Thus (3.4.8) follows.

Therefore, by (3.4.5)–(3.4.8),

$$
\begin{aligned}
(3.4.9) \qquad T &\le \frac{\gamma}{Z^2}\int dx\, E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\cdot \mathbb{1}\{0 \not\leftrightarrow \Gamma\}\big)\\
&= \gamma\int dx\, \frac{1}{Z^2}E\big(\partial\psi_2^{0x}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{0 \not\leftrightarrow \Gamma\}\big)\\
&= \gamma\frac{\partial M}{\partial\gamma},
\end{aligned}
$$

by 'conditioning on the cluster' C_Γ and Theorem 3.3.8.

Next, we bound R_h. Suppose that the bridge bringing ξ into C_Γ has endpoints X and Y, where we take X to be the endpoint not in C_Γ. When the bridge XY is removed, the backbone ξ consists of two paths: $\zeta^1 : 0 \to X$ and $\zeta^2 : Y \to \Gamma$. Therefore,

$$
\begin{aligned}
R_h &= \frac{1}{Z^3}\int P(X\in dx)\, E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\,\big|\, X = x\big)\\
&\le \frac{\lambda}{Z^3}\int dx \sum_{y\sim x} E\big(\partial\psi_1^{0xy\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{0 \not\leftrightarrow \Gamma,\, y\leftrightarrow\Gamma\}\cdot\mathbb{1}\{J_\xi\}\big),
\end{aligned}
$$

where $\xi = \xi(\psi_1^{0xy\Gamma})$ and

$$J_\xi = \big\{\xi = \zeta^1\circ\zeta^2,\, \zeta^1 : 0\to x,\, \zeta^2 : y\to\Gamma,\, \zeta^1\cap C_\Gamma = \varnothing\big\}.$$

As in (3.4.6),

(3.4.10)

$$R_h \le \frac{\lambda}{Z^2}\int dx \sum_{y\sim x} E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{0 \not\leftrightarrow \Gamma,\, y\leftrightarrow\Gamma\}\cdot w^{0xy\Gamma}(\xi)\cdot\mathbb{1}\{J_\xi\}\big).$$

By Lemmas 3.2.3(a) and 3.3.3, on the event J_ξ,

$$
\begin{aligned}
w^{0xy\Gamma}(\xi) &= w^{0x}(\zeta^1)w^{y\Gamma}_{K\setminus\zeta^1}(\zeta^2)\\
&\le 2^{r-r'}w^{0x}_{K\setminus C_\Gamma}(\zeta^1)w^{y\Gamma}_{K\setminus\zeta^1}(\zeta^2),
\end{aligned}
$$

where $r = r(\zeta^1, K)$ and $r' = r(\zeta^1, K \setminus C_\Gamma)$. By Lemma 2.2.22 and the reasoning after (3.4.8),

$$E\big(w^{0xy\Gamma}(\xi)\cdot \mathbb{I}\{J_\xi\} \,\big|\, \zeta^1, C_\Gamma\big) \le 2^{r-r'} w^{0x}_{K\setminus C_\Gamma}(\zeta^1)\cdot \langle \sigma_y\rangle_{K\setminus\zeta^1}$$
$$\le M\cdot 2^{r-r'} w^{0x}_{K\setminus C_\Gamma}(\zeta^1),$$

so that, similarly,

$$E\big(w^{0xy\Gamma}(\xi)\cdot \mathbb{I}\{J_\xi\} \,\big|\, C_\Gamma\big) \le M\cdot \langle \sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}. \tag{3.4.11}$$

We substitute into the summand in (3.4.10), using the switching lemma, conditioning on the cluster C_Γ, and the bound $\langle\sigma_y\rangle_{C_\Gamma} \le M$, to obtain the upper bound

$$\begin{aligned} &M\cdot E\big(\partial\psi_2^\varnothing \partial\psi_3^\varnothing \cdot \mathbb{I}\{0 \nleftrightarrow \Gamma,\, y\leftrightarrow \Gamma\}\cdot \langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\big) \\ &\qquad = M\cdot E\big(\partial\psi_2^{y\Gamma}\partial\psi_3^{y\Gamma}\cdot \mathbb{I}\{0\nleftrightarrow\Gamma\}\cdot\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\big) \\ &\qquad = M\cdot E\big(\partial\psi_2^{0xy\Gamma}\partial\psi_3^\varnothing\langle\sigma_y\rangle_{C_\Gamma}\cdot\mathbb{I}\{0\nleftrightarrow\Gamma\}\big) \\ &\qquad \le M^2\cdot E\big(\partial\psi_2^{0xy\Gamma}\partial\psi_3^\varnothing\cdot\mathbb{I}\{0\nleftrightarrow\Gamma\}\big). \end{aligned} \tag{3.4.12}$$

Hence, by (3.3.36),

$$\begin{aligned} R_h &\le \lambda M^2 \frac{1}{Z^2}\int dx \sum_{y\sim x} E\big(\partial\psi_2^{0xy\Gamma}\partial\psi_3^\varnothing \mathbb{I}\{0\nleftrightarrow\Gamma\}\big) \\ &= 2\lambda M^2 \frac{\partial M}{\partial\lambda}. \end{aligned}$$

Finally, we bound R_v. Let $X \in \Delta\cap \mathrm{ev}(\psi_2^\varnothing)\cap\mathrm{ev}(\psi_3^\varnothing)$ be the first point of ξ in C_Γ. In a manner similar to that used for R_h at (3.4.10) above, and by cutting the backbone ξ at the point x,

$$R_v \le \frac{1}{Z^2}\int P(X\in dx)\, E\big(\partial\psi_2^\varnothing\partial\psi_3^\varnothing\cdot\mathbb{I}\{0\nleftrightarrow\Gamma,\, x\leftrightarrow\Gamma\}\cdot w^{0\Gamma}(\xi)\cdot\mathbb{I}\{J_\xi\}\big), \tag{3.4.13}$$

where

$$J_\xi = 1\big\{\xi = \overline{\zeta}^1\circ\overline{\zeta}^2,\, \overline{\zeta}^1 : 0\to x,\, \overline{\zeta}^2 : x\to\Gamma,\, \zeta^1\cap C_\Gamma = \varnothing\big\}.$$

As in (3.4.11),

$$\begin{aligned}E(w^{0\Gamma}(\xi)\cdot \mathbb{1}\{J_\xi\}\mid C_\Gamma) &= E\big(E(w^{0\Gamma}(\xi)\cdot \mathbb{1}\{J_\xi\}\mid \overline{\zeta}^1, C_\Gamma)\,\big|\, C_\Gamma\big)\\ &\le E\big(\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\cdot\langle\sigma_x\rangle_{K\setminus\zeta^1}\,\big|\, C_\Gamma\big)\\ &\le \langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\cdot M.\end{aligned}$$

By (3.4.13) therefore,

$$R_v \le M\frac{1}{Z^2}\int P(X\in dx)\, E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{0\not\leftrightarrow\Gamma,\, x\leftrightarrow\Gamma\}\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\big).$$

By removing the cut at x, the origin 0 becomes connected to Γ, but only via x. Thus,

$$R_v \le 4\delta M\frac{1}{Z^2}\int dx\, E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{0\xleftrightarrow{x}\Gamma,\, x\leftrightarrow\Gamma\}\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma^x}\big),$$

where C_Γ^x is the set of points reached from Γ along open paths not containing x. By the switching lemma, and conditioning twice on the cluster C_Γ^x,

$$\begin{aligned}R_v &\le 4\delta M\frac{1}{Z^2}\int dx\, E\big(\partial\psi_2^{x\Gamma}\partial\psi_3^{x\Gamma}\cdot \mathbb{1}\{0\xleftrightarrow{x}\Gamma\}\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma^x}\big)\\ &= 4\delta M\frac{1}{Z^2}\int dx\, E\big(\partial\psi_2^{0\Gamma}\partial\psi_3^{x\Gamma}\cdot \mathbb{1}\{0\xleftrightarrow{x}\Gamma\}\big)\\ &= 4\delta M\frac{1}{Z^2}\int dx\, E\big(\partial\psi_2^{0\Gamma}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{0\xleftrightarrow{x}\Gamma\}\langle\sigma_x\rangle_{C_\Gamma^x}\big)\\ &\le 4\delta M^2\frac{1}{Z^2}\int dx\, E\big(\partial\psi_2^{0\Gamma}\partial\psi_3^{\varnothing}\cdot \mathbb{1}\{0\xleftrightarrow{x}\Gamma\}\big)\\ &= -2\delta M^2\frac{\partial M}{\partial\delta},\end{aligned}$$

by (3.3.37), as required. □

3.5. Consequences of the inequalities

In this section we formulate the principal results of this chapter, and show how the differential inequalities of Theorems 3.1.3 and 3.3.8 may be used to prove them. We will rely in this section on the results in Section 2.5, and we work under Assumption 3.3.7, unless otherwise stated. It is sometimes inconvenient to use periodic boundary conditions, and we revert to the free condition where necessary.

We shall consider the infinite-volume limit as $L \uparrow \mathbb{Z}^d$; the ground state is obtained by letting $\beta \to \infty$ also. Let n be a positive integer, and set $L_n = [-n, n]^d$ with periodic boundary condition. Let $\Lambda_n^\beta := [-n, n]^d \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]$. The symbol β will appear as superscript in the following; the superscript ∞ is to be interpreted as the ground state. Let $0 = (0, 0)$ and

$$M_n^\beta(\lambda, \delta, \gamma) = \langle \sigma_0 \rangle_{L_n}^\beta = \langle \sigma_0 \rangle_{\Lambda_n^\beta}$$

be the magnetization in Λ_n^β, noting that $M_n^\beta \equiv 0$ when $\gamma = 0$.

We have from the results in Section 2.3.4 that the limits

$$M^\beta := \lim_{n\to\infty} M_n^\beta, \quad M^\infty := \lim_{n,\beta\to\infty} M_n^\beta, \tag{3.5.1}$$

exist for all $\gamma \in \mathbb{R}$ (where, in the second limit, $\beta = \beta_n$ is comparable to n in the sense that Assumption 2.5.6 holds). Note that $M^\beta(\lambda, \delta, 0) = 0$ for $\beta \in (0, \infty]$. Recall that we set $\delta = 1$, $\rho = \lambda/\delta$, and write

$$M^\beta(\rho, \gamma) = M^\beta(\rho, 1, \gamma), \qquad \beta \in (0, \infty],$$

with a similar notation for other functions.

Recall the following facts. From Theorem 2.5.9 there is a unique infinite-volume state $\langle\cdot\rangle^\beta$ at every $\gamma > 0$. Letting $\langle\cdot\rangle_+^\beta$ be the limiting state as $\gamma \downarrow 0$, there is a unique state at $(\rho, 0)$ if and only if

$$M_+^\beta(0) := \langle \sigma_0 \rangle_+^\beta = 0.$$

From (2.5.28) the state $\langle\cdot\rangle_+^\beta$ may alternatively be obtained as the infinite volume limit of the + boundary states taken with $\gamma = 0$. The critical value

$$\rho_c^\beta := \inf\{\rho > 0 : M_+^\beta(\rho) > 0\}, \tag{3.5.2}$$

see also (3.1.9) and (3.1.11). We shall have need later for the infinite-volume limit $\langle\cdot\rangle^{\mathrm{f},\beta}$, as $n \to \infty$, with *free* boundary condition in the

$\mathbb{Z}^d$ direction (or in both directions, if $\beta \to \infty$). This limit exists by Theorem 2.5.1. Note from Theorem 2.5.9 that

$$(3.5.3) \qquad \langle\cdot\rangle^{\mathrm{f},\beta}_{\gamma=0} = \langle\cdot\rangle^{\beta}_{\gamma=0} = \langle\cdot\rangle^{\beta}_{+} \quad \text{if} \quad M^{\beta}_{+}(\rho) = 0.$$

The superscript 'f' shall always indicate the free boundary condition.

For $\beta \in (0,\infty]$, let $\phi^{b,\beta}_{\rho}$, $b \in \{\mathrm{f},\mathrm{w}\}$, be the $q = 2$ random-cluster measures of Theorem 2.3.2, with $\gamma = 0$. By Theorem 2.2.12, these measures are non-decreasing in ρ, and, as we saw in (2.3.30),

$$(3.5.4) \qquad \phi^{\mathrm{w},\beta}_{\rho} \leq \phi^{\mathrm{f},\beta}_{\rho'}, \qquad \text{when } 0 \leq \rho < \rho'.$$

As in Remark 2.5.2, for $\beta \in (0,\infty]$,

$$(3.5.5) \qquad \phi^{\mathrm{w},\beta}_{\rho}(x \leftrightarrow y) = \langle\sigma_x\sigma_y\rangle^{\beta}_{+}, \quad \phi^{\mathrm{w},\beta}_{\rho}(0 \leftrightarrow \infty) = M_{+}(\rho).$$

By (3.5.5), the FKG inequality (Theorem 2.2.14), and the uniqueness of the unbounded cluster (Theorem 2.3.10),

$$(3.5.6) \qquad \langle\sigma_x\sigma_y\rangle^{\beta}_{+} \geq \phi^{\mathrm{w},\beta}_{\rho}(x \leftrightarrow \infty)\phi^{\mathrm{w},\beta}_{\rho}(y \leftrightarrow \infty) = M^{\beta}_{+}(\rho)^2.$$

Let $\beta \in (0,\infty)$. Using the concavity of M^{β} implied by Lemma 3.3.4, as well as the properties of convex functions in Proposition 2.5.3, the derivative $\partial M^{\beta}/\partial\gamma$ exists for all $\gamma \in \mathcal{C} \subseteq (0,\infty)$, where $\mathcal{C}$ is a set whose complement has measure zero. When $\gamma \in \mathcal{C}$,

$$(3.5.7) \qquad \chi^{\beta}_{n}(\rho,\gamma) := \frac{\partial M^{\beta}_{n}}{\partial\gamma} \to \chi^{\beta}(\rho,\gamma) := \frac{\partial M^{\beta}}{\partial\gamma} < \infty.$$

The corresponding conclusion holds also as $n, \beta \to \infty$. Furthermore, by the GHS-inequality, Lemma 3.3.4, χ^{β} is decreasing in $\gamma \in \mathcal{C}$, which implies that the limits

$$\chi^{\beta}_{+}(\rho) := \lim_{\gamma\downarrow 0} \chi^{\beta}(\rho,\gamma), \qquad \beta \in (0,\infty].$$

exist when taken along sequences in $\mathcal{C}$.

The limit

$$\chi^{\mathrm{f},\beta}(\rho,0) := \lim_{n\to\infty}\left(\frac{\partial M_n^{\mathrm{f},\beta}}{\partial\gamma}\bigg|_{\gamma=0}\right) \tag{3.5.8}$$
$$= \lim_{n\to\infty}\int_{\Lambda_n^\beta}\langle\sigma_0\sigma_x\rangle_{n,\gamma=0}^{\mathrm{f},\beta}\,dx = \int\langle\sigma_0\sigma_x\rangle_{\gamma=0}^{\mathrm{f},\beta}\,dx$$

exists by monotone convergence, see Lemma 2.2.22. Let

$$\rho_{\mathrm{s}}^\beta := \inf\{\rho>0 : \chi^{\mathrm{f},\beta}(\rho,0)=\infty\}, \qquad \beta\in(0,\infty]. \tag{3.5.9}$$

We shall see in Theorem 3.5.2 that $\chi^{\mathrm{f},\beta}(\rho_{\mathrm{s}}^\beta,0)=\infty$.

It will be useful later to note that

$$\chi_+^\beta(\rho)\geq\chi^{\mathrm{f},\beta}(\rho,0)\quad\text{whenever } M_+^\beta(\rho)=0, \qquad \beta\in(0,\infty]. \tag{3.5.10}$$

To see this, let $\gamma\in\mathcal{C}$ and first note from Fatou's lemma that

$$\chi^\beta(\rho,\gamma)\geq\int\langle\sigma_0;\sigma_x\rangle_\gamma^\beta\,dx, \tag{3.5.11}$$

where we have written $\langle\cdot\rangle_\gamma^\beta$ for the unique state at γ. Hence, using also the monotone convergence theorem and the GHS-inequality,

$$\chi_+^\beta(\rho)=\lim_{\substack{\gamma\downarrow 0\\ \gamma\in\mathcal{C}}}\chi^\beta(\rho,\gamma)\geq\lim_{\substack{\gamma\downarrow 0\\ \gamma\in\mathcal{C}}}\int\langle\sigma_0;\sigma_x\rangle_\gamma^\beta\,dx=\int\langle\sigma_0;\sigma_x\rangle_+^\beta\,dx. \tag{3.5.12}$$

When $M_+(0)=0$ there is a unique state at $\gamma=0$, so that $\langle\sigma_0;\sigma_x\rangle_+^\beta = \langle\sigma_0\sigma_x\rangle_{\gamma=0}^{\mathrm{f},\beta}$ which by (3.5.8) gives (3.5.10). It will follow in particular from Theorem 3.5.2 that $\chi_+^\beta(\rho_{\mathrm{s}}^\beta)=\infty$. Of course, similar arguments are valid for the limit $n,\beta\to\infty$.

By (3.5.8) and Lemma 2.2.22 we have that $\chi^{\mathrm{f},\beta}(\rho,0)$ is increasing in ρ. We claim that

$$\rho_{\mathrm{s}}^\beta\leq\rho_{\mathrm{c}}^\beta; \tag{3.5.13}$$

it will follow that there is a unique equilibrium state when $\gamma=0$ and $\rho<\rho_{\mathrm{s}}^\beta$. First note that, by (3.5.4) and (3.5.5), if $\rho<\rho'<\rho_{\mathrm{s}}^\beta$ then

$$M_+(\rho)=\phi_\rho^{\mathrm{w},\beta}(0\leftrightarrow\infty)\leq\phi_{\rho'}^{\mathrm{f},\beta}(0\leftrightarrow\infty), \tag{3.5.14}$$

so it suffices to show that $\phi_\rho^{\mathrm{f},\beta}(0 \leftrightarrow \infty) = 0$ if $\rho < \rho_{\mathrm{s}}^\beta$. To see this, note that if $\phi_\rho^{\mathrm{f},\beta}(0 \leftrightarrow \infty) > 0$ then certainly

$$\chi^{\mathrm{f},\beta}(\rho, 0) = \int_{\mathbb{Z}^d \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]} \langle \sigma_0 \sigma_x \rangle^{\mathrm{f},\beta} \, dx = \phi_\rho^{\mathrm{f}}(|C_0|) = \infty, \tag{3.5.15}$$

where C_0 denotes the cluster at the origin, and $|\cdot|$ denotes Lebesgue measure.

For $x \in \mathbb{Z}^d \times \mathbb{R}$, let $\|x\|$ denote the supremum norm of x.

THEOREM 3.5.1. *Let $\beta \in (0, \infty]$ and $\rho < \rho_{\mathrm{s}}^\beta$. There exists $\alpha = \alpha(\rho) > 0$ such that*

$$\langle \sigma_0 \sigma_x \rangle^\beta \le e^{-\alpha \|x\|}, \qquad x \in \mathbb{Z}^d \times \mathbb{R}. \tag{3.5.16}$$

PROOF. Fix $\beta \in (0, \infty)$ and $\gamma = 0$, and let $\rho < \rho_{\mathrm{s}}^\beta$, so that (3.5.3) applies. By the uniqueness of the equilibrium state, we have that

$$\chi^{\mathrm{f},\beta}(\rho, 0) = \int_{\mathbb{Z}^d \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]} \langle \sigma_0 \sigma_x \rangle^\beta \, dx = \sum_{k \ge 1} \int_{C_k^\beta} \langle \sigma_0 \sigma_x \rangle^\beta \, dx, \tag{3.5.17}$$

where $C_k^\beta := \Lambda_k^\beta \setminus \Lambda_{k-1}^\beta$. Since $\rho < \rho_{\mathrm{s}}^\beta$, the last summation converges, whence, for sufficiently large k,

$$\int_{C_k^\beta} \langle \sigma_0 \sigma_x \rangle^\beta \, dx < e^{-8}. \tag{3.5.18}$$

The result follows from the the Simon inequality, Lemma 3.3.5, with the 1-fat separating sets C_k^β using standard arguments (see [**50**, Corollary 9.38] for more details on the method). A similar argument holds when $\beta = \infty$. $\square$

Let $\beta \in (0, \infty]$, $\gamma = 0$ and define the *mass*

$$m^\beta(\rho) := \liminf_{\|x\| \to \infty} \left(-\frac{1}{\|x\|} \log \langle \sigma_0 \sigma_x \rangle_\rho^\beta \right) \tag{3.5.19}$$

By Theorem 3.5.1 and (3.5.6),

$$m^\beta(\rho) \begin{cases} > 0 & \text{if } \rho < \rho_{\mathrm{s}}^\beta, \\ = 0 & \text{if } \rho > \rho_{\mathrm{c}}^\beta. \end{cases} \tag{3.5.20}$$

THEOREM 3.5.2. *Except when $d = 1$ and $\beta < \infty$, $m^\beta(\rho_{\mathrm{s}}^\beta) = 0$ and $\chi^{\mathrm{f},\beta}(\rho_{\mathrm{s}}^\beta, 0) = \infty$.*

PROOF. Let $d \geq 2$, $\gamma = 0$, and fix $\beta \in (0, \infty)$. We use the Lieb inequality, Lemma 3.3.6, and the argument of [**67, 80**], see also [**50**, Corollary 9.46]. It is necessary and sufficient for $m^\beta(\rho) > 0$ that

$$\int_{C_n^\beta} \langle \sigma_0 \sigma_x \rangle_{n,\rho}^{\mathrm{f},\beta}\, dx < e^{-8} \quad \text{for some } n. \tag{3.5.21}$$

Necessity holds because the integrand is no greater than $\langle \sigma_0 \sigma_x \rangle^\beta$. Sufficiency follows from Lemma 3.3.6, as in the proof of Theorem 3.5.1.

By (3.1.5),

$$\begin{aligned}\frac{\partial}{\partial \rho} \langle \sigma_0 \sigma_x \rangle_{n,\rho}^{\mathrm{f},\beta} &= \frac{1}{2} \int_{\Lambda_n^\beta} dy \sum_{z \sim y} \langle \sigma_0 \sigma_x ; \sigma_y \sigma_z \rangle_{n,\rho}^{\mathrm{f},\beta} \\ &\leq d\beta(2n+1)^d.\end{aligned}$$

Therefore, if $\rho' > \rho$,

$$\int_{C_n^\beta} \langle \sigma_0 \sigma_x \rangle_{n,\rho'}^{\mathrm{f},\beta}\, dx \leq d[\beta(2n+1)^d]^2(\rho' - \rho) + \int_{C_n^\beta} \langle \sigma_0 \sigma_x \rangle_{n,\rho}^{\mathrm{f},\beta}\, dx. \tag{3.5.22}$$

Hence, if (3.5.21) holds for some ρ, then it holds for ρ' when $\rho' - \rho > 0$ is sufficiently small.

Suppose $m^\beta(\rho_{\mathrm{s}}^\beta) > 0$. Then $m^\beta(\rho') > 0$ for some $\rho' > \rho_{\mathrm{s}}^\beta$, which contradicts $\chi^{\mathrm{f},\beta}(\rho', 0) = \infty$, and the first claim of the theorem follows. A similar argument holds when $d = 1$ and $\beta = \infty$. The second claim follows similarly: if $\chi^{\mathrm{f},\beta}(\rho_{\mathrm{s}}^\beta, 0) < \infty$, then (3.5.21) holds with $\rho = \rho_{\mathrm{s}}^\beta$, whence $m^\beta(\rho') > 0$ and $\chi^{\mathrm{f},\beta}(\rho', 0) < \infty$ for some $\rho' > \rho_{\mathrm{s}}^\beta$, a contradiction. (See also [**9**].) □

We are now ready to state the main results. We will adapt the arguments of [**2**, Lemmas 4.1, 5.1] (see also [**3, 49**]) to prove the following.

THEOREM 3.5.3. *There are constants c_1, $c_2 > 0$ such that, for $\beta \in (0,\infty]$,*

$$M^\beta(\rho_{\mathrm{s}},\gamma) \geq c_1\gamma^{1/3}, \tag{3.5.23}$$

$$M_+^\beta(\rho,0) \geq c_2(\rho-\rho_{\mathrm{s}}^\beta)^{1/2}, \tag{3.5.24}$$

for small, positive γ and $\rho-\rho_{\mathrm{s}}^\beta$, respectively.

This is vacuous when $d = 1$ and $\beta < \infty$; see (3.1.11). The exponents in the above inequalities are presumably sharp in the corresponding mean-field model (see [**3, 5**] and Remark 3.5.5). It is standard that a number of important results follow from Theorem 3.5.3, of which we state the following here.

THEOREM 3.5.4. *For $d \geq 1$ and $\beta \in (0,\infty]$, we have that $\rho_{\mathrm{c}}^\beta = \rho_{\mathrm{s}}^\beta$.*

PROOF. Except when $d = 1$ and $\beta < \infty$, this is immediate from (3.5.13) and (3.5.24). In the remaining case, $\rho_{\mathrm{c}}^\beta = \rho_{\mathrm{s}}^\beta = \infty$. □

PROOF OF THEOREM 3.5.3. We will describe the case when $\beta < \infty$ is fixed; the ground state case is proved by a similar method. The argument is based on [**2**].

We start by proving (3.5.23). If $M_+^\beta(\rho_{\mathrm{s}},0) > 0$ there is nothing to prove, so we assume that $M_+^\beta(\rho_{\mathrm{s}},0) = 0$. The inequalities of Theorems 3.3.8 and 3.1.3 may be combined to obtain

$$M_n^\beta \leq (M_n^\beta)^3 + \chi_n^\beta \cdot \left(\gamma + 4d\lambda(M_n^\beta)^3 + 4\delta\frac{(M_n^\beta)^3}{1-(M_n^\beta)^2}\right). \tag{3.5.25}$$

Set $\delta = 1$ and $\rho = \rho_{\mathrm{s}}^\beta$, and write $f_n(\gamma) = 2M_n^\beta(\rho_{\mathrm{s}}^\beta,\gamma)$. Recall that the sequence $f_n(\gamma)$ converges as $n \to \infty$ to some $f(\gamma)$ for all $\gamma \geq 0$, and that the derivatives $f_n' = 2\chi_n^\beta$ converge for $\gamma \in \mathcal{C}$ to some $g(\gamma)$ which is decreasing in γ. Moreover, from the discussion around (3.5.10) and the assumption that $M_+^\beta(\rho_{\mathrm{s}}, 0) = 0$ it follows that

$$\lim_{\substack{\gamma\downarrow 0\\ \gamma\in\mathcal{C}}} g(\gamma) = \infty. \tag{3.5.26}$$

Multiplying through by $1-(M_n^\beta)^2$ and discarding non-positive terms on the right hand side, we may deduce from (3.5.25) that the functions f_n satisfy the inequality

$$f_n(\gamma) \le \gamma\cdot f_n'(\gamma) + a\cdot f_n'(\gamma) f_n(\gamma)^3 + f_n(\gamma)^3, \qquad \gamma\ge 0, \tag{3.5.27}$$

where $a > 0$ is an appropriate constant depending on λ and d only. For $\gamma > 0$ we may rewrite this as

$$\frac{1}{f_n'(\gamma)}\frac{d}{d\gamma}\Big[\frac{\gamma}{f_n(\gamma)}\Big] \le f_n'(\gamma)\Big(a + \frac{1}{f_n'(\gamma)}\Big). \tag{3.5.28}$$

Letting $\gamma > \varepsilon > 0$ and integrating from ε to γ it follows that

$$\frac{\gamma}{f_n(\gamma)} - \frac{\varepsilon}{f_n(\varepsilon)} \le \int_\varepsilon^\gamma f_n'(x) f_n(x)\Big(a + \frac{1}{f_n'(x)}\Big)\,dx. \tag{3.5.29}$$

Using (3.3.35) of Theorem 3.3.8, it follows on letting $\varepsilon\downarrow 0$ that

$$\frac{\gamma}{f_n(\gamma)} - \frac{1}{f_n'(0)} \le \int_0^\gamma f_n'(x) f_n(x)\Big(a + \frac{1}{f_n'(x)}\Big)\,dx. \tag{3.5.30}$$

Now suppose that $\gamma > 0$ lies in $\mathcal{C}$. If γ is sufficiently small then $g(\gamma)\ge 1.1$, and for such a γ fixed we have for sufficiently large n that $f_n'(\gamma)\ge 1$. Since f_n' is decreasing in γ we may deduce from (3.5.30) that

$$\frac{\gamma}{f_n(\gamma)} - \frac{1}{f_n'(0)} \le (a+1)\int_0^\gamma f_n'(x) f_n(x)\,dx = \frac{a+1}{2} f_n(\gamma)^2 \tag{3.5.31}$$

Letting $n\to\infty$ it follows that

$$\frac{\gamma}{f(\gamma)} \le \frac{a+1}{2} f(\gamma)^2$$

as required.

Let us now turn to (3.5.24). Note first that if $\rho = \lambda/\delta$ then

$$\frac{\partial M_n^\beta}{\partial\lambda} = \frac{1}{\delta}\frac{\partial M_n^\beta}{\partial\rho} \quad\text{and}\quad \frac{\partial M_n^\beta}{\partial\delta} = -\frac{\lambda}{\delta^2}\frac{\partial M_n^\beta}{\partial\rho}, \tag{3.5.32}$$

so that the inequality of Theorem 3.1.3 may be rewritten as

$$M_n^\beta \le \gamma \frac{\partial M_n^\beta}{\partial \gamma} + (M_n^\beta)^3 + 2\rho (M_n^\beta)^2 \frac{\partial M_n^\beta}{\partial \rho}. \tag{3.5.33}$$

This may in turn be rewritten as

$$\frac{\partial}{\partial \gamma}(\log M_n^\beta) + \frac{1}{\gamma}\frac{\partial}{\partial \rho}(\rho (M_n^\beta)^2 - \rho) \ge 0. \tag{3.5.34}$$

We wish to integrate this over the rectangle $[\rho_s^\beta, \rho'] \times [\gamma_0, \gamma_1]$ for $\rho' > \rho_s^\beta$ and $\gamma_1 > \gamma_0 > 0$. Since M_n^β is increasing in ρ and in γ we deduce, after discarding a term $-\rho_s^\beta M_n^\beta(\rho_s^\beta, \gamma)^2$, that

$$(\rho' - \rho_s^\beta) \log\left(\frac{M_n^\beta(\rho', \gamma_1)}{M_n^\beta(\rho_s^\beta, \gamma_0)}\right) + (\rho' M_n^\beta(\rho', \gamma_1)^2 - \rho' + \rho_s^\beta) \log \frac{\gamma_1}{\gamma_0} \ge 0. \tag{3.5.35}$$

We may let $n \to \infty$ in (3.5.35), to deduce that the same inequality is valid with M_n^β replaced by M^β. It follows from (3.5.23) that

$$\liminf_{\gamma_0 \downarrow 0} \frac{\log\left(\frac{M_n^\beta(\rho', \gamma_1)}{M_n^\beta(\rho_s^\beta, \gamma_0)}\right)}{\log(\gamma_1/\gamma_0)} \le \frac{1}{3}. \tag{3.5.36}$$

It follows that

$$\frac{1}{3}(\rho' - \rho_s^\beta) + \rho' M^\beta(\rho', \gamma_1) - (\rho' - \rho_s^\beta) \ge 0, \tag{3.5.37}$$

which on letting $\gamma_1 \downarrow 0$ gives the result. □

Remark 3.5.5. *Let $\beta \in (0, \infty]$. Except when $d = 1$ and $\beta < \infty$, one may conjecture the existence of exponents $a = a(d, \beta)$, $b = b(d, \beta)$ such that*

$$M_+^\beta(\rho) = (\rho - \rho_c^\beta)^{(1+o(1))a} \qquad \text{as } \rho \downarrow \rho_c^\beta, \tag{3.5.38}$$

$$M^\beta(\rho_c^\beta, \gamma) = \gamma^{(1+o(1))/b} \qquad \text{as } \gamma \downarrow 0. \tag{3.5.39}$$

Theorem 3.5.3 would then imply that $a \le \frac{1}{2}$ and $b \ge 3$. In [**24**, Theorem 3.2] *it is proved for the ground-state quantum Curie–Weiss, or mean-field, model that the corresponding $a = \frac{1}{2}$. It may be conjectured that the values $a = \frac{1}{2}$ and $b = 3$ are attained for the space–time*

Ising model on $\mathbb{Z}^d \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]$ *for* d *sufficiently large, as proved for the classical Ising model in* [**5**]*. See also Section 4.3.*

Finally, a note about (3.1.16). The random-cluster measure corresponding to the quantum Ising model is *periodic* in both $\mathbb{Z}^d$ and β directions, and this complicates the infinite-volume limit. Since the periodic random-cluster measure dominates the free random-cluster measure, for $\beta \in (0, \infty)$, as in (3.5.4) and (3.5.6),

$$\begin{aligned} \liminf_{n\to\infty} \tau^{\beta}_{L_n}(u,v) &\geq \langle \sigma_{(u,0)}\sigma_{(v,0)}\rangle^{\beta}_{+,\rho'} && \text{for } \rho' < \rho \\ &\to M_+(\rho-)^2 && \text{as } \rho' \uparrow \rho, \end{aligned}$$

and a similar argument holds in the ground state also.

CHAPTER 4

Applications and extensions

Summary. First we prove that the critical ratio for the ground state quantum Ising model on $\mathbb{Z}$ is $\rho_c^\infty = 2$; we then extend this result to more complicated ('star-like') graphs. Next we discuss the possible applications of 'reflection positivity' to strengthen the results of Chapter 3 when $d \geq 3$, and conclude with a discussion of versions of the random-parity representation of the Potts model.

4.1. In one dimension

The quantum Ising model on $\mathbb{Z}$ has been thoroughly studied in the mathematical physics literature. It is an example of what is called an 'exactly solvable model': using transfer matrices and related techniques, the critical ratio and other important quantities have been computed, see for example [**76**] or [**79**] and references therein. In this section we prove by graphical methods that the critical value coincides with the self-dual value of Section 2.4. The graphical method is valuable in that it extends to more complicated geometries, as in the next section. In the light of (3.1.11), we shall study only the ground state, and we shall suppress the superscript ∞.

THEOREM 4.1.1. *Let* $\mathbb{L} = \mathbb{Z}$. *Then* $\rho_c = 2$, *and the transition is of second order in that* $M_+(2) = 0$.

We mention an application of this theorem. In an account [**54**] of so-called 'entanglement' in the quantum Ising model on the subset

$[-m, m]$ of $\mathbb{Z}$, it was shown that the reduced density matrix ν_m^L of the block $[-L, L]$ satisfies

$$\|\nu_m^L - \nu_n^L\| \leq \min\{2, CL^\alpha e^{-cm}\}, \qquad 2 \leq m < n < \infty,$$

where C and α are constants depending on $\rho = \lambda/\delta$, and $c = c(\rho) > 0$ whenever $\rho < 1$. Using Theorems 3.5.1 and 4.1.1, we have that $c(\rho) > 0$ for $\rho < 2$.

Proof. We adapt the well-known methods [**50**, Chapter 6] for the discrete random-cluster model. Write ϕ_ρ^{f} and ϕ_ρ^{w} for the free and wired $q = 2$ random-cluster measures, respectively. By Theorem 2.5.1 and Remark 2.5.2, and the representation (2.5.28) of the state $\langle \cdot \rangle_+$, we have that

$$\text{(4.1.1)} \qquad \langle \sigma_x \sigma_y \rangle_+ = \phi_\rho^{\mathrm{w}}(x \leftrightarrow y), \qquad \langle \sigma_x \rangle_+ = \phi_\rho^{\mathrm{w}}(x \leftrightarrow \infty).$$

Recall from Theorem 2.4.2 that the measures ϕ_ρ^{f} and $\phi_{4/\rho}^{\mathrm{w}}$ are mutually dual. By Zhang's argument, Theorem 2.4.3, we know of the self-dual point $\rho = 2$ that

$$\text{(4.1.2)} \qquad \phi_2^{\mathrm{f}}(0 \leftrightarrow \infty) = 0$$

and hence that $\rho_{\mathrm{c}} \geq 2$.

We show next that $\rho_{\mathrm{c}} \leq 2$, following the method developed for percolation to be found in [**49, 50**]. Suppose that $\rho_{\mathrm{c}} > 2$. Consider the 'lozenge' D_n of side length n, as illustrated in Figure 2.9 on p. 73, and its 'dual' D_n^{d}. Let A_n denote the event that there is an open path from the bottom left to the top right of D_n in ω, and let A_n^{d} be the 'dual' event that there is in ω_{d} an open path from the top left to the bottom right of D_n^{d}. The events A_n and A_n^{d} are complementary, so we have by duality and symmetry under reflection that

$$\text{(4.1.3)} \qquad 1 = \phi_2^{\mathrm{f}}(A_n) + \phi_2^{\mathrm{f}}(A_n^{\mathrm{d}}) = \phi_2^{\mathrm{f}}(A_n) + \phi_2^{\mathrm{w}}(A_n) \leq 2\phi_2^{\mathrm{w}}(A_n).$$

However, if $2 < \rho_c$ then we have by (4.1.1) and Theorem 3.5.1 that $\phi_2^{\mathrm{w}}(A_n)$ decays to zero in the manner of $Cn^2e^{-\alpha n}$ as $n \to \infty$, a contradiction.

We show that $M_+(2) = 0$ by adapting a simple argument developed by Werner in [**84**] for the classical Ising model on $\mathbb{Z}^2$. Certain geometrical details are omitted. Let π^{f} be the Ising state obtained with free boundary condition, as in Theorem 2.5.1. Recall that π^{f} may be obtained from the random-cluster measures ϕ_2^{f} by assigning to the clusters spin ± 1 independently at random, with probability 1/2 each. By Lemma 2.3.7, π^{f} is ergodic.

The binary relations $\overset{\pm}{\leftrightarrow}$ are defined as follows. A *path* of $\mathbb{Z} \times \mathbb{R}$ is a path of $\mathbb{R}^2$ that: traverses a finite number of line-segments of $\mathbb{Z} \times \mathbb{R}$, and is permitted to connect them by passing between any two points of the form (u, t), $(u \pm 1, t)$. For $x, y \in \mathbb{Z} \times \mathbb{R}$, we write $x \overset{+}{\leftrightarrow} y$ (respectively, $x \overset{-}{\leftrightarrow} y$) if there exists a path with endpoints x, y all of whose elements are labelled $+1$ (respectively, -1). (In particular, for any x we have that $x \overset{+}{\leftrightarrow} x$ and $x \overset{-}{\leftrightarrow} x$.) Let N^+ (respectively, N^-) be the number of unbounded $+$ (respectively, $-$) Ising clusters with connectivity relation $\overset{+}{\leftrightarrow}$ (respectively, $\overset{-}{\leftrightarrow}$). By the Burton–Keane argument, as in Theorem 2.3.10, one may show that either $\pi^{\mathrm{f}}(N^+ = 1) = 1$ or $\pi^{\mathrm{f}}(N^+ = 0) = 1$. The former would entail also that $\pi^{\mathrm{f}}(N^- = 1) = 1$, by the $\pm$ symmetry in the coupling with the random-cluster measure. With an application of Zhang's argument as in Theorem 2.4.3, however, one can show that this is impossible. Therefore,

$$\pi^{\mathrm{f}}(N^{\pm} = 0) = 1. \tag{4.1.4}$$

Recall that $\langle \cdot \rangle^+ = \pi^{\mathrm{w}}$. There is a standard argument for deriving $\pi^{\mathrm{f}} = \langle \cdot \rangle^+$ from (4.1.4), of which the idea is roughly as follows. (See [**4**] or [**50**, Thm 5.33] for examples of similar arguments applied to the random-cluster model.) Let $\Lambda_n = [-n, n]^2 \subseteq \mathbb{Z} \times \mathbb{R}$, and let $m < n$.

We call a set $S \subseteq \Lambda_n$ a *separating set* if any path from Λ_m to $\partial\Lambda_n$ contains an element of S. We adopt the harmless convention that, for any spin-configuration σ, the subset of Λ_n labelled $+1$ is closed, compare Remark 2.1.1. By (4.1.4), for given m, and for $\varepsilon > 0$ and large n, the event $A_{m,n} = \{\Lambda_m \overset{-}{\leftrightarrow} \partial\Lambda_n\}^{\mathrm{c}}$ satisfies $\pi^{\mathrm{f}}(A_{m,n}) > 1 - \varepsilon$. On $A_{m,n}$, there is a separating set labelled entirely $+$; let us call any such separating set a $+$-separating set. Let U denote the set of all points in Λ_n which are $-$-connected to $\partial\Lambda_n$ (note that this includes $\partial\Lambda_n$ itself). Write $S = S(\sigma)$ for $\partial(\Lambda_n \setminus U)$. Note that $S \subseteq \Lambda_n \setminus \Lambda_m$ is a $+$-separating set. See Figure 4.1.

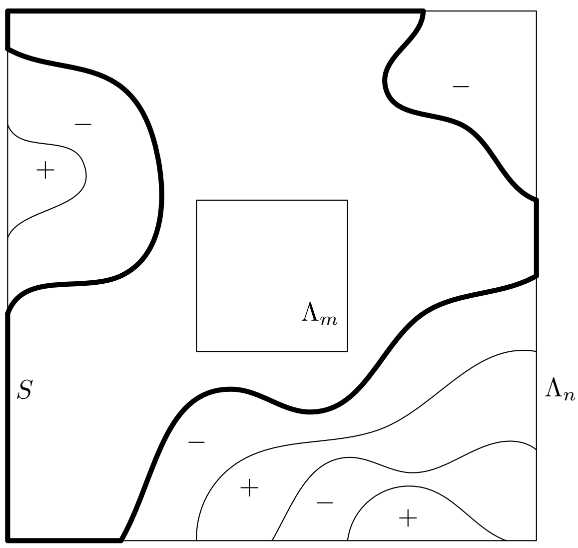

FIGURE 4.1. Sketch of an Ising configuration σ, with the set $S(\sigma)$ drawn bold; S is a $+$-separating set.

For any closed separating set S_1, define $\hat{S}_1$ to be the union of S_1 with the unbounded component of $(\mathbb{Z} \times \mathbb{R}) \setminus S_1$. Also let $\tilde{S}_1$ be the set of points in Λ_n that are separated from $\partial\Lambda_n$ by S_1. The event $\{S(\sigma) = S_1\}$ is $\mathcal{G}_{\hat{S}_1}$-measurable, i.e. it depends only on the restriction of σ to $\hat{S}_1$. Let $B \subseteq \Lambda_m$ be a finite set, and recall the notation ν'_B at (2.5.11). By the DLR-property of Lemma 2.1.9 (the natural extension of which holds also for infinite-volume measures) we deduce that

$$\pi^{\mathrm{f}}(\nu'_B \mid A_{m,n}, S) = \pi^{\mathrm{w}}_{\tilde{S}}(\nu'_B \mid A_{m,n}). \tag{4.1.5}$$

Let $n \to \infty$ to deduce, using also the FKG-inequality of Lemma 2.2.17, that

$$\pi^{\mathrm{f}}(\nu'_B \mid S) = \pi^{\mathrm{w}}_{\tilde{S}}(\nu'_B) \geq \pi^{\mathrm{w}}(\nu'_B). \tag{4.1.6}$$

By integrating, and letting $m \to \infty$, we obtain that $\pi^{\mathrm{f}}(\nu'_B) \geq \pi^{\mathrm{w}}(\nu'_B)$ for all finite sets $B \subseteq \mathbb{Z} \times \mathbb{R}$. Since the reverse inequality $\pi^{\mathrm{f}}(\nu'_B) \leq \pi^{\mathrm{w}}(\nu'_B)$ always holds (by Lemma 2.1.9 and Lemma 2.2.17 again), we deduce that $\pi^{\mathrm{f}} = \pi^{\mathrm{w}}$ as claimed.

One way to conclude that $M_+(2) = 0$ is to use the random-cluster representation again. By (4.1.2) and the above,

$$\phi^{\mathrm{f}}_2(0 \leftrightarrow \infty) = \phi^{\mathrm{w}}_2(0 \leftrightarrow \infty) = 0,$$

whence $M_+(2) = \phi^{\mathrm{w}}_2(0 \leftrightarrow \infty) = 0$. $\square$

4.2. On star-like graphs

We now extend Theorem 4.1.1 of the previous section, to show that the critical ratio $\rho_{\mathrm{c}}(2) = 2$ for a larger class of graphs than just $\mathbb{Z}$. This section forms the contents of the article [**14**].

The class of graphs for which we prove that the critical ratio is 2 includes for example the *star graph*, which is the junction of several copies of $\mathbb{Z}$ at a single point. See Figure 4.2. It also includes many other planar graphs (see Definition 4.2.1). The result for the star is

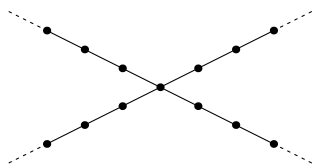

FIGURE 4.2. The star graph has a central vertex of degree $k \geq 3$ and k infinite arms, on which each vertex has degree 2. In this illustration, $k = 4$.

perhaps not unexpected, since the star is only 'locally' different from $\mathbb{Z}$: if you go far enough out on one of the 'arms' then the star 'looks like' $\mathbb{Z}$. However, as pointed out before, the quantum Ising model on the star, unlike on $\mathbb{Z}$, is not exactly solvable, and graphical methods are the only known way to prove this result.

The Ising model on the star-graph has recently arisen in the study of boundary effects in the two-dimensional classical Ising model, see for example [**72, 73**]. Similar geometries have also arisen in different problems in quantum theory, such as transport properties of quantum wire systems, see [**22, 57, 65**].

Throughout this section we consider the ground-state only, that is to say we let $\beta = \infty$; reference to β will be suppressed. We also let $\lambda, \delta > 0$ be constant and $\gamma = 0$. Let $\mathbb{L} = (\mathbb{V}, \mathbb{E})$ be a fixed *star-like graph*:

Definition 4.2.1. *A star-like graph is a countably infinite connected planar graph, in which all vertices have finite degree and only finitely many vertices have degree larger than two.*

Such a graph is illustrated in Figure 4.3; note that the star graph of Figure 4.2 is an example in which exactly one vertex has degree at least three.

The following is the main result of this section.

Theorem 4.2.2. *Let $\mathbb{L}$ be any star-like graph. Then the critical ratio of the ground state quantum Ising model on $\mathbb{L}$ is $\rho_c(2) = 2$.*

Simpler arguments than those presented here can be used to establish the analogous result when $q = 1$, namely that $\rho_c(1) = 1$. Also, the same arguments can be used to calculate the critical probability of the discrete graphs $\mathbb{L} \times \mathbb{Z}$ when $q = 1, 2$. As in the case $\mathbb{L} = \mathbb{Z}$, an

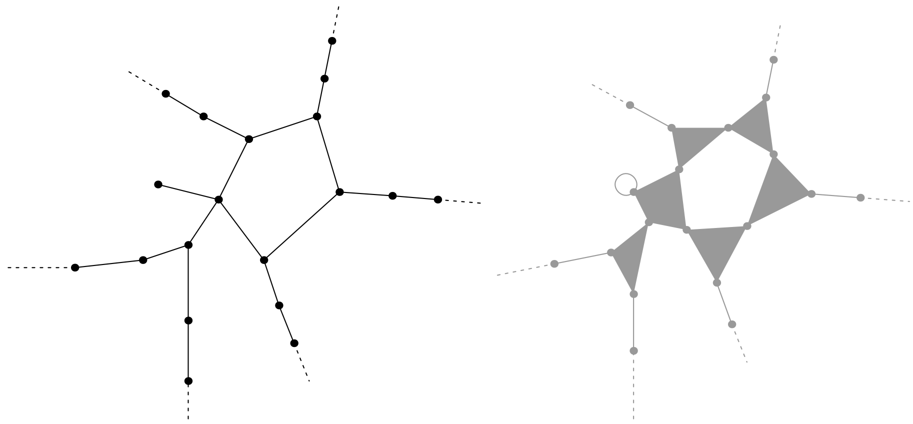

FIGURE 4.3. A star-like graph $\mathbb{L}$ (left) and its line-hypergraph $\mathbb{H}$ (right). Any vertex of degree ≥ 3 in $\mathbb{L}$ is associated with a "polygonal" (hyper)edge in $\mathbb{H}$.

essential ingredient of the proof is the exponential decay of correlations below ρ_c.

Recall that a *hypergraph* is a set $\mathbb{W}$ together with a collection $\mathbb{B}$ of subsets of $\mathbb{W}$, called *edges* (or hyperedges). A graph is a hypergraph in which all edges contain two elements. In our analysis we will use a suitably defined hypergraph 'dual' of $\mathbb{L}$. To be precise, let $\mathbb{H} = (\mathbb{W}, \mathbb{B})$ be the *line-hypergraph* of $\mathbb{L}$, given by letting $\mathbb{W} = \mathbb{E}$ and letting the set $\{e_1, \ldots, e_n\} \subseteq \mathbb{E} = \mathbb{W}$ be an edge (that is, an element of $\mathbb{B}$) if and only if $e_1, \ldots, e_n$ are all the edges of $\mathbb{L}$ adjacent to some particular vertex of $\mathbb{L}$. Note that only finitely many edges of $\mathbb{H}$ have size larger than two, since $\mathbb{L}$ is star-like.

Fix an arbitrary planar embedding of $\mathbb{L}$ into $\mathbb{R}^2$; we will typically identify $\mathbb{L}$ with its embedding. We let $\mathcal{O}$ denote an arbitrary but fixed vertex of $\mathbb{L}$ which has degree at least two; we think of $\mathcal{O}$ as the 'origin'. There is a natural planar embedding of $\mathbb{H}$ defined via the embedding $\mathbb{L}$, in which an edge of size more than two is represented as a polygon. See Figure 4.3. In this section we will use the symbol $\mathbb{X}$ in place of $\boldsymbol{\Theta}$ for $\mathbb{L} \times \mathbb{R}$, and will identify $\mathbb{X}$ with the corresponding subset of $\mathbb{R}^3$. Similarly, we write $\mathbb{Y} = \mathbb{H} \times \mathbb{R}$ for the 'dual' of $\mathbb{X}$, also thought of as a subset of $\mathbb{R}^3$. We will often identify $\omega = (B, D) \in \Omega$ with its embedding, $\omega \equiv (\mathbb{X} \setminus D) \cup B$. We let Λ_n be the simple region corresponding to $\beta = n$ and L the subgraph of $\mathbb{L}$ induced by the vertices at graph distance at most n from $\mathcal{O}$, see (2.1.7). Note that $\Lambda_n \uparrow \mathbb{X}$. In this section we let uppercase Φ_n^b denote the random-cluster measure on Λ_n with parameters $\lambda, \delta > 0$, $\gamma = 0$, $q = 2$ and boundary condition $b \in \{0, 1\}$, where, as in Section 2.4, we let 0 and 1 denote the free and wired boundary conditions, respectively.

Given any configuration $\omega \in \Omega$, one may as in the case $\mathbb{L} = \mathbb{Z}$ associate with it a *dual* configuration on $\mathbb{Y}$ by placing a death wherever ω has a bridge, and a (hyper)bridge wherever ω has a death. Recall Figure 2.8 on p. 70. More precisely, we let Ω_{d} be the set of pairs of locally finite subsets of $\mathbb{B} \times \mathbb{R}$ and $\mathbb{W} \times \mathbb{R}$, and for each $\omega = (B, D) \in \Omega$ we define its dual to be $\omega_{\mathrm{d}} := (D, B)$. As before, we may identify ω_{d} with its embedding in $\mathbb{Y}$, noting that some bridges may be embedded as polygons. We let Ψ_n^b and Ψ^b denote the laws of ω_{d} under Φ_n^{1-b} and Φ^{1-b} respectively.

We will frequently be comparing the random-cluster measures on $\mathbb{X}$ and $\mathbb{Y}$ with the random-cluster measures on $\mathbb{Z} \times \mathbb{R}$; the latter may be regarded as a subset of both $\mathbb{X}$ and $\mathbb{Y}$ (in a sense made more precise below). We will reserve the lower-case symbols ϕ_n^b, ϕ^b for the random-cluster measures on $\mathbb{Z} \times \mathbb{R}$ with the same parameters as Φ_n^b (where ϕ_n^b lives on the simple region given by $\beta = n$ and $L = [-n, n]$). We will write ψ_n^{1-b}, ψ^{1-b} for the dual measures of ϕ_n^b, ϕ^b on $\mathbb{Z} \times \mathbb{R}$; thus by Theorem 2.4.2, the measures ψ_n^{1-b}, ψ^{1-b} are random cluster measures with parameters $q' = q$, $\lambda' = q\delta$ and $\delta' = \lambda/q$, and boundary condition $1 - b$.

Here is a brief outline of the proof of Theorem 4.2.2. First we make the straightforward observation that $\rho_{\mathrm{c}}(2) \leq 2$. Next, we use exponential decay to establish the existence of certain infinite paths in the dual model on $\mathbb{Y}$ when $\lambda/\delta < 2$. Finally, we show how to put these paths together to form 'blocking circuits' in $\mathbb{Y}$, which prevent the existence of infinite paths in $\mathbb{X}$ when $\lambda/\delta < 2$. Parts of the argument are inspired by [**40**].

LEMMA 4.2.3. *For $\mathbb{L}$ any star-like graph, $\rho_c(2) \leq 2$.*

PROOF. Since $\mathbb{L}$ is star-like, it contains an isomorphic copy of $\mathbb{Z}$ as a subgraph. Let Z be such a subgraph; we may assume that $\mathcal{O} \in Z$. We may identify ϕ_n^b, ϕ^b with the random-cluster measures on $Z \times \mathbb{R}$. For each $n \geq 1$, let C_n be the event that no two points in $\Lambda_n \cap (Z \times \mathbb{R})$ are connected by a path which leaves $Z \times \mathbb{R}$. Each C_n is a decreasing event. It follows from the DLR-property, Lemma 2.1.5, that $\Phi_n^b(\cdot \mid C_n) = \phi_n^b(\cdot)$. If A is an increasing local event defined on $Z \times \mathbb{R}$, this means that

$$\phi_n^b(A) = \Phi_n^b(A \mid C_n) \leq \Phi_n^b(A), \tag{4.2.1}$$

i.e. $\phi_n^b \leq \Phi_n^b$ for all n. Letting $n \to \infty$ it follows that $\phi^b \leq \Phi^b$. If $\lambda/\delta > 2$ then $\phi^b((\mathcal{O}, 0) \leftrightarrow \infty) > 0$ and it follows that also

$$\Phi^b((\mathcal{O}, 0) \leftrightarrow \infty) > 0, \tag{4.2.2}$$

which is to say that $\rho_c(2) \leq 2$. $\square$

4.2.1. Infinite paths in the half-plane. Let us now establish some facts about the random-cluster model on the 'half-plane' $\mathbb{Z}_+ \times \mathbb{R}$ which will be useful later. Our notation is as follows: for $n \geq 1$, let

(4.2.3)

$$S_n = \{(a,t) \in \mathbb{Z} \times \mathbb{R} : -n \leq a \leq n, |t| \leq n\}$$

$$S_n(m,s) = S_n + (m,s) = \{(a+m, t+s) \in \mathbb{Z} \times \mathbb{R} : (a,t) \in S_n\}.$$

For brevity write $T_n = S_n(n, 0)$. For $b \in \{0,1\}$ and Δ one of S_n, T_n, we let ϕ_Δ^b denote the $q = 2$ random-cluster measure on the simple region in $\mathbb{X}$ with $K = \Delta$ with boundary condition b and parameters λ, δ. Note that

$$\phi^b = \lim_{n\to\infty} \phi^b_{S_n}, \qquad \psi^b = \lim_{n\to\infty} \psi^b_{S_n}. \tag{4.2.4}$$

We will also be using the limits

$$\phi^{\mathrm{sw}} = \lim_{n\to\infty} \phi^1_{T_n}, \qquad \psi^{\mathrm{sf}} = \lim_{n\to\infty} \psi^0_{T_n}, \tag{4.2.5}$$

which exist by similar arguments to Theorem 2.3.2. (The notation 'sw' and 'sf' is short for 'side wired' and 'side free', respectively.) These are measures on configurations ω on $\mathbb{Z}_+ \times \mathbb{R}$; standard arguments let us deduce all the properties of ϕ^{sw} and ψ^{sf} that we need. In particular ψ^{sf} and ϕ^{sw} are mutually dual (with the obvious interpretation of duality) and they enjoy the positive association property of Theorem 2.2.14 and the finite energy property of Lemma 2.3.4.

Let W be the 'wedge'

$$W = \{(a,t) \in \mathbb{Z}_+ \times \mathbb{R} : 0 \le t \le a/2 + 1\}, \tag{4.2.6}$$

and write 0 for the origin $(0,0)$.

Lemma 4.2.4. *Let $\lambda/\delta < 2$. Then*

$$\psi^{\mathrm{sf}}(0 \leftrightarrow \infty \textit{ in } W) > 0. \tag{4.2.7}$$

Here is some intuition behind the proof of Lemma 4.2.4. The claim is well-known with ψ^0 in place of ψ^{sf}, by standard arguments using duality and exponential decay. However, ψ^{sf} is stochastically smaller than ψ^0, so we cannot deduce the result immediately. Instead we pass to the dual ϕ^{sw} and establish directly a lack of blocking paths. The problem is the presence of the infinite 'wired side'; we get the required fast decay of two-point functions by using the following result.

Proposition 4.2.5. *Let $\lambda/\delta < 2$. There is $\alpha > 0$ such that for all n,*

$$\phi^1_{S_n}(0 \leftrightarrow \partial S_n) \le e^{-\alpha n}. \tag{4.2.8}$$

In words, correlations decay exponentially under finite volume measures if they do so under infinite volume measures. Results of this type for the classical Ising and random-cluster models appear in many places. In [**19**] and [**21**] it is proved for general $q \geq 1$ random-cluster models in two dimensions, and more general results about the two-dimensional case appear in [**10**]. A proof of general results of this type for the classical Ising model in any dimension appears in [**55**]. Below we adapt the argument in [**55**] to the current setting, with the difference that we shorten the proof by using the Lieb inequality, Lemma 3.3.6, in place of the GHS-inequality; use of the Lieb-inequality was suggested by Grimmett (personal communication). Note that the same argument works on $\mathbb{Z}^d$ for any $d \geq 1$.

PROOF. Let $\hat{S}_n \supseteq S_n$ denote the 'tall' box

$$\hat{S}_n = \{(a,t) \in \mathbb{Z} \times \mathbb{R} : -n \leq a \leq n, |t| \leq n+1\}. \tag{4.2.9}$$

We will use a random-cluster measure on $\hat{S}_n$ which has non-constant λ, δ, and nonzero γ. The particular intensities we use are these. Fix n, and fix $m \geq 0$, which we think of as large. Let $\lambda(\cdot)$, $\delta(\cdot)$ and $\gamma_m(\cdot)$ be given by

(4.2.10)

$$\delta(a,t) = \begin{cases} \delta, & \text{if } (a,t) \in S_n \\ 0, & \text{otherwise,} \end{cases}$$

$$\lambda(a+1/2,t) = \begin{cases} \lambda, & \text{if } (a,t) \in S_n \text{ and } (a+1,t) \in S_n \\ 0, & \text{otherwise,} \end{cases}$$

$$\gamma_m(a,t) = \begin{cases} \lambda, & \text{if exactly one of } (a,t) \text{ and } (a+1,t) \text{ is in } S_n \\ m, & \text{if } (a,t) \in \hat{S}_n \setminus S_n \\ 0, & \text{otherwise.} \end{cases}$$

In words, the intensities are as usual 'inside' S_n and in particular there is no external field in the interior; on the left and right sides of S_n, the external field simulates the wired boundary condition; and on top and bottom, the external field simulates an approximate wired boundary (as $m \to \infty$). We let $\tilde{\phi}^b_{m,n}$ denote the random-cluster measure on $\hat{S}_n$ with intensities $\lambda(\cdot), \delta(\cdot), \gamma_m(\cdot)$ and boundary condition $b \in \{0,1\}$. Note that $\tilde{\phi}^0_{m,n}$ and $\phi^0_{S_n}$ agree on events defined on S_n, for any m.

Let X denote $\hat{S}_n \setminus S_n$ together with the left and right sides of S_n. By the Lieb inequality, Lemma 3.3.6, we have that

$$\begin{aligned}\tilde{\phi}^1_{m,n}(0 \leftrightarrow \Gamma) &\leq e^{8\delta} \int_X dx\ \tilde{\phi}^0_{m,n}(0 \leftrightarrow x)\tilde{\phi}^1_{m,n}(x \leftrightarrow \Gamma) \\ &\leq e^{8\delta} \int_X dx\ \tilde{\phi}^0_{m,n}(0 \leftrightarrow x),\end{aligned} \tag{4.2.11}$$

since (with these intensities) X separates 0 from Γ. Therefore, by stochastic domination by the infinite-volume measure,

$$\tilde{\phi}^1_{m,n}(0 \leftrightarrow \Gamma) \leq e^{8\delta} \int_X dx\ \phi^0(0 \leftrightarrow x). \tag{4.2.12}$$

All the points $x \in X$ are at distance at least n from the origin. By exponential decay in the infinite volume, Theorem 3.5.1, it follows from (4.2.12) that there is an absolute constant $\tilde{\alpha} > 0$ such that

$$\tilde{\phi}^1_{m,n}(0 \leftrightarrow \Gamma) \leq e^{8\delta}|X|e^{-\tilde{\alpha}n} = e^{8\delta}(8n+2)e^{-\tilde{\alpha}n}. \tag{4.2.13}$$

Now let C be the event that all of $\hat{S}_n \setminus S_n$ belongs to the connected component of Γ, which is to say that all points on $\hat{S}_n \setminus S_n$ are linked to Γ. Then by the DLR-property of random-cluster measures the conditional measure $\tilde{\phi}^1_{m,n}(\cdot \mid C)$ agrees with $\phi^1_{S_n}(\cdot)$ on events defined on S_n. Therefore

$$\begin{aligned}\phi^1_{S_n}(0 \leftrightarrow \partial S_n) = \tilde{\phi}^1_{m,n}(0 \leftrightarrow \partial S_n \mid C) &= \tilde{\phi}^1_{m,n}(0 \leftrightarrow \Gamma \mid C) \\ &\leq \frac{\tilde{\phi}^1_{m,n}(0 \leftrightarrow \Gamma)}{\tilde{\phi}^1_{m,n}(C)} \leq \frac{e^{8\delta}}{\tilde{\phi}^1_{m,n}(C)} \cdot (8n+2)e^{-\tilde{\alpha}n}.\end{aligned} \tag{4.2.14}$$

Since $\tilde{\phi}^1_{m,n}(C) \to 1$ as $m \to \infty$ we conclude that

$$\phi^1_{S_n}(0 \leftrightarrow \partial S_n) \leq e^{8\delta}(8n+2)e^{-\tilde{\alpha}n}. \tag{4.2.15}$$

Since each $\phi^1_{S_n}(0 \leftrightarrow \partial S_n) < 1$ it is a simple matter to tidy this up to get the result claimed. $\square$

PROOF OF LEMMA 4.2.4. Let $T = \{(a, a/2+1) : a \in \mathbb{Z}_+\}$ be the 'top' of the wedge W. We claim that

$$\sum_{n \geq 1} \phi^{\mathrm{sw}}((n,0) \leftrightarrow T \text{ in } W) < \infty. \tag{4.2.16}$$

Once this is proved, it follows from the Borel–Cantelli lemma that with probability one under ϕ^{sw}, at most finitely many of the points $(n,0)$ are connected to T inside W. Hence under the dual measure ψ^{sf} there is an infinite path inside W with probability one, and by the DLR- and positive association properties it follows that

$$\psi^{\mathrm{sf}}(0 \leftrightarrow \infty \text{ in } W) > 0, \tag{4.2.17}$$

as required.

To prove the claim we note that, if n is larger than some constant, then the event '$(n,0) \leftrightarrow T$ in W' implies the event '$(n,0) \leftrightarrow \partial S_{n/3}(n,0)$'. The latter event, being increasing, is more likely under the measure $\phi^1_{S_{n/3}(n,0)}$ than under ϕ^{sw}. But by Proposition 4.2.5,

$$\phi^1_{S_{n/3}(n,0)}((n,0) \leftrightarrow \partial S_{n/3}(n,0)) = \phi^1_{S_{n/3}}(0 \leftrightarrow \partial S_{n/3}) \leq e^{-\alpha n/3}, \tag{4.2.18}$$

which is clearly summable. $\square$

The next lemma uses a variant of standard blocking arguments.

LEMMA 4.2.6. *Let $\lambda/\delta < 2$. There exists $\varepsilon > 0$ such that for each n,*

$$\psi^{\mathrm{sf}}((0, 2n+1) \leftrightarrow (0, -2n-1) \textit{ off } T_n) \geq \varepsilon. \tag{4.2.19}$$

PROOF. Let $L_n = \{(a,n) : a \geq 0)\}$ be the horizontal line at height n, and let $\varepsilon > 0$ be such that $\psi^{\mathrm{sf}}(0 \leftrightarrow \infty \text{ in } W) \geq \sqrt{\varepsilon}$. We claim that

$$\psi^{\mathrm{sf}}((0, -2n-1) \leftrightarrow L_{2n+1} \text{ off } T_n) \geq \sqrt{\varepsilon}. \tag{4.2.20}$$

Clearly ψ^{sf} is invariant under reflection in the x-axis and under vertical translation, see Lemma 2.3.5. Thus once the claim is proved we get that

$$\begin{aligned}\psi^{\mathrm{sf}}&((0,2n+1)\leftrightarrow(0,-2n-1)\text{ off }T_n)\\ &\ge \psi^{\mathrm{sf}}((0,-2n-1)\leftrightarrow L_{2n+1}\text{ off }T_n\\ &\qquad\qquad \text{and }(0,2n+1)\leftrightarrow L_{-2n-1}\text{ off }T_n)\\ &\ge (\sqrt{\varepsilon})^2,\end{aligned} \tag{4.2.21}$$

as required. See Figure 4.4.

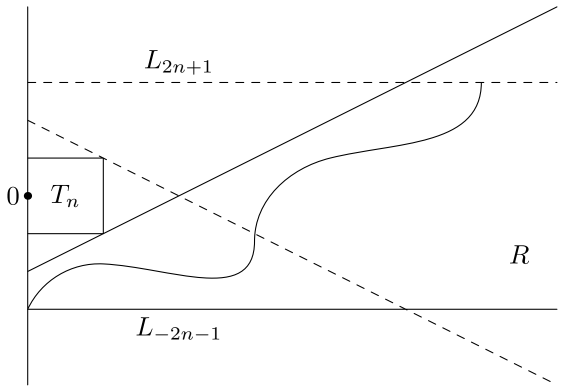

FIGURE 4.4. Construction of a 'half-circuit' in $\mathbb{Z}_+ \times \mathbb{R}$. With probability one, any infinite path in the lower wedge must reach the line L_{2n+1}, and similarly for any infinite path in the upside-down wedge. Any pair of such paths starting on the horizontal axis must cross.

The claim follows if we prove that

$$\psi^{\mathrm{sf}}(0 \leftrightarrow \infty \text{ in } R) = 0, \tag{4.2.22}$$

where R is the strip

$$R = \{(a,t) : a \ge 0, -2n-1 \le t \le 2n+1\}. \tag{4.2.23}$$

However, (4.2.22) follows from the DLR-property, Lemma 2.1.5, the stochastic domination of Theorem 2.2.13, and the Borel–Cantelli lemma; these combine to show that the event 'no bridges between $\{k\}\times[-2n-$

$1, 2n+1]$ and $\{k+1\} \times [-2n-1, 2n+1]$' must happen for infinitely many k with ψ^{sf}-probability one. In more detail: we have that $\psi^{\mathrm{sf}} \leq \mu$, where μ is the percolation measure with parameters λ, δ; under μ the events above are independent, so

$$\psi^{\mathrm{sf}}(0 \leftrightarrow \infty \text{ in } R) \leq \mu(0 \leftrightarrow \infty \text{ in } R) = 0. \tag{4.2.24}$$

□

4.2.2. Proof of Theorem 4.2.2. We may assume that $\mathbb{L} \neq \mathbb{Z}$, since the case $\mathbb{L} = \mathbb{Z}$ is known. Let $\lambda/\delta < 2$, and recall that $\mathbb{L}$ consists of finitely many infinite 'arms', where each vertex has degree two, together with a 'central' collection of other vertices. On each of the arms, let us fix one arbitrary vertex (of degree two) and call it an *exit point*. Let U denote the set of exit points of $\mathbb{L}$.

Given an exit point $u \in U$, call its two neighbours v and w; we may assume that they are labelled so that only v is connected to the origin $\mathcal{O}$ by a path not including u. If the edge uv were removed from $\mathbb{L}$, the resulting graph would consist of two components, where we denote by J_u the component containing w. Let $\hat{\Phi}^b_n, \hat{\Phi}^b$ denote the marginals of Φ^b_n, Φ^b on $X_u := J_u \times \mathbb{R}$; similarly let $\hat{\Psi}^b_n, \hat{\Psi}^b$ denote the marginals of the dual measures. Of course X_u is isomorphic to the half-plane graph considered in the previous subsection. By positive association and the DLR-property of random-cluster measures, $\hat{\Phi}^0_n \leq \phi^1_{T_n(u)}$, so letting $n \to \infty$ also $\hat{\Phi}^0 \leq \phi^{\mathrm{sw}}$. Passing to the dual, it follows that $\hat{\Psi}^1 \geq \psi^{\mathrm{sf}}$. The (primal) edge uv is a *vertex* in the line-hypergraph; denoting it still by uv we therefore have by Lemma 4.2.6 that there is an $\varepsilon > 0$ such that for all n,

$$\Psi^1((uv, -2n-1) \leftrightarrow (uv, 2n+1) \text{ off } T_n(u) \text{ in } X_u) \geq \varepsilon. \tag{4.2.25}$$

Here $T_n(u)$ denotes the copy of the box T_n contained in X_u. Letting A denote the intersection of the events above over all exit points u, and letting $A_1 = A_1(n)$ be the dual event $A_1 = \{\omega_{\rm d} : \omega \in A\}$, it follows from positive association that $\Phi^0(A_1) \geq \varepsilon^k$, where $k = |U|$ is the number of exit points. Note that A_1 is a decreasing event in the primal model.

On A_1, no point in $T_n(u)$ can reach ∞ without passing the line $\{u\} \times [-2n-1, 2n+1]$, since there is a dual blocking path in X_u. Let I denote the (finite) subgraph of $\mathbb{L}$ spanned by the complement of all the J_u for $u \in U$, and let $A_2 = A_2(n)$ denote the event that for all vertices $v \in I$, the intervals $\{v\} \times [2n+1, 2n+2]$ and $\{v\} \times [-2n-1, -2n-2]$ all contain at least one death and the endpoints of no bridges (in the primal model). There is $\eta > 0$ independent of n such that $\Phi^0(A_2) \geq \eta$. So by positive association $\Phi^0(A_1 \cap A_2) \geq \eta\varepsilon^k > 0$. We have that $A_1 \cap A_2 \subseteq A_3$, where A_3 is the event that no point inside the union of $I \times [-n, n]$ with $\cup_{u \in U} T_n(u)$ lies in an unbounded connected component. See Figure 4.5. Taking the intersection of the $A_3 = A_3(n)$ over all n,

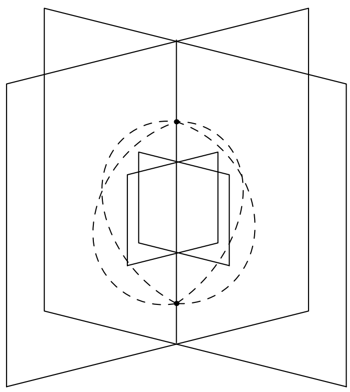

FIGURE 4.5. The dashed lines indicate dual paths that block any primal connection from the interior to ∞. Note that this figure illustrates only the simplest case when $\mathbb{L}$ is a junction of lines at a single point.

it follows that

$$\Phi^0(\text{there is no unbounded connected component}) \geq \eta\varepsilon^k. \tag{4.2.26}$$

The event that there is no unbounded connected component is a tail event. By tail-triviality, Proposition 2.3.6, it follows that whenever $\lambda/\delta < 2$ then

$$\Phi^0(0 \not\leftrightarrow \infty) = 1. \tag{4.2.27}$$

In other words, $\rho_c(2) \geq 2$. Combined with the opposite bound in Lemma 4.2.3, this gives the result. □

One may ask if, as in the case $\mathbb{L} = \mathbb{Z}^d$, the phase transition on star-like graphs is of second order, and if there is exponential decay of correlations below the critical point. We do not know how to prove such results: Zhang's argument (Theorem 2.4.3) fails on star-like graphs, and so do the arguments for Theorem 3.1.3, due to the lack of symmetry.

4.3. Reflection positivity

The theory of reflection positivity was first developed in [**39, 37, 38**], originally as a way to prove the existence of discontinuous phase transitions in a wide range of models in statistical physics. A model which is reflection positive (see definitions below) will satisfy what are called 'Gaussian domination bounds' and 'chessboard estimates'. The latter will not be touched upon here, see the review [**13**] and references therein. One may think of the Gaussian domination bounds, and the related 'infrared bound', as a way of bounding certain quantities in the model by corresponding quantities in another, simpler, model, namely what is called the 'Gaussian free field'. Very roughly, existence of a phase transition in the Gaussian free field therefore implies existence of a phase transition in your reflection positive model.

In [**5**], it was shown that Gaussian domination bounds could also be used in another way for the Ising model. By relating the bounds to quantities that appear naturally in the random-current representation of the Ising model, Aizenman and Fernández were able to establish that the behaviour of the classical Ising model on $\mathbb{Z}^d$ resembles that of the 'mean field' Ising model when d is large, in fact already when $d \geq 4$. In this section we will state more precisely the sense in which 'large d resembles mean field', and give a very brief sketch of the arguments involved. We will also indicate how one might extend the results of [**5**] to the quantum Ising model; this is currently work in progress.

In this section we will only be considering the case when $\mathbb{L} = \mathbb{Z}^d$ for some $d \geq 1$, and $L = (V, E) = [-n, n]^d$ for some n, with periodic boundary (see Assumption 3.3.7). For $j = 1, \dots, d$, write e_j for the element of V whose jth coordinate is 1 and whose other coordinates are zero. For $\sigma \in \{-1, +1\}^V$, we write its classical Ising weight in this section as

$$\exp\Big(\beta \sum_{xy \in E} J_{xy}\sigma_x\sigma_y + \gamma \sum_{x \in V} \sigma_x\Big), \tag{4.3.1}$$

where $\beta, \gamma, J_e \geq 0$. We assume that the model is translation invariant in that $J_{xy} \equiv J_{y-x}$, where, for $z \in V$, $J_z \geq 0$ and $J_z = 0$ unless $z = e_j$ for some j. We also assume that $J_{e_j} = J_{-e_j}$ for all $j = 1, \dots, d$.

The classical Ising model displays a phase-transition in β when $\gamma = 0$, at the critical value β_{c}. As in the quantum Ising model (Theorem 3.5.3), the infinite-volume magnetization $M = M(\beta, \gamma)$ satisfies the inequalities

$$\begin{aligned} M &\geq c_2(\beta - \beta_{\mathrm{c}})^{1/2}, && \text{for } \gamma = 0 \text{ and } \beta \downarrow \beta_{\mathrm{c}}, \\ M &\geq c_1\gamma^{1/3}, && \text{for } \beta = \beta_{\mathrm{c}} \text{ and } \gamma \downarrow 0, \end{aligned} \tag{4.3.2}$$

for some constants c_1, c_2 (this was first proved in [**3**]). As mentioned in Remark 3.5.5, it is conjectured that the limits

$$a = \lim_{\beta \downarrow \beta_{\mathrm{c}}} \frac{\log M(\beta, 0)}{\log(\beta - \beta_{\mathrm{c}})}, \qquad \frac{1}{b} = \lim_{\gamma \downarrow 0} \frac{\log M(\beta_{\mathrm{c}}, \gamma)}{\log \gamma} \tag{4.3.3}$$

exist. Using the random-current representation coupled with results from reflection positivity, [**5**] shows that these limits do indeed exist when $d \geq 4$, and that (4.3.2) is sharp in that $a = 1/2$ and $b = 3$. The values $a = 1/2$ and $b = 3$ are called the 'mean field' values because they are known to be the correct critical exponents for the Ising model on the complete graph (this result is 'well-known', but see [**33, 34**] for reviews). Intuitively, complete graphs are infinite-dimensional, so the higher d is the closer one may expect the behaviour to be to that on the complete graph. The results of [**5**] confirm this, and show that the 'critical dimension' is at most $d = 4$. Their method is roughly as follows.

For $j = 1, \dots, d$ we let $P_i = \{x = (x_1, \dots, x_d) \in V : x_j = 0\}$, and we let $P_j^+ = \{x \in V : x_j > 0\}$ and $P_j^- = \{x \in V : x_j < 0\}$. The symbol θ_i will denote reflection in P_i, thus $\theta_j(x_1, \dots, x_j, \dots, x_d) = (x_1, \dots, -x_j, \dots, x_d)$. Write $\mathcal{F}_{P_j^+}$ and $\mathcal{F}_{P_j^-}$ for the σ-algebras of events defined on P_j^+ and P_j^-, respectively.

Although we will be using the concept of reflection positivity only for the Ising measure (4.3.1), the definition makes sense in greater generality, as follows. Let $S \subseteq \mathbb{R}$ be a compact set, and endow S^V with the product σ-algebra. Fix $j \in \{1, \dots, d\}$, and let ψ denote a probability measure on S^V which is invariant under θ_j. For $s = (s_x : x \in V) \in S^V$, write $\theta_j(s) = (s_{\theta_j(x)} : x \in V)$, and for $f : S^V \to \mathbb{R}$ define $\theta_j f(s) = f(\theta_j(s))$.

Definition 4.3.1. *The probability measure ψ is* reflection positive *with respect to θ_j if for all $\mathcal{F}_{P_j^+}$-measurable $f : S^V \to \mathbb{R}$, we have that*

$$\psi(f \cdot \theta_j f) \geq 0.$$

Lemma 4.3.2.

- *Any product measure on S^V invariant under θ_j is reflection positive with respect to θ_j,*
- *The Ising measure* (4.3.1) *is reflection positive with respect to all the θ_j.*

For a proof of this standard fact, see for example [**13**]. It follows from Lemma 4.3.2 that the Ising model satisfies the following 'Gaussian domination' bounds. For $p \in [-\pi, \pi]^d$, let

$$G(p) := \sum_{x \in V} \langle \sigma_o \sigma_x \rangle_{\gamma=0} e^{ip\cdot x} \tag{4.3.4}$$

be the Fourier transform of $\langle \sigma_0 \sigma_x \rangle_{\gamma=0}$, where $i = \sqrt{-1}$ and $p \cdot x$ denotes the usual dot product. Due to our symmetry assumptions we see that the complex conjugate $\overline{G(p)} = G(-p) = G(p)$ so that $G(p) \in \mathbb{R}$. Also define

$$E(p) := \frac{1}{2} \sum_{x \in V} (1 - e^{ip\cdot x}) J_x; \tag{4.3.5}$$

similarly we see that $E(p) \in \mathbb{R}$.

Proposition 4.3.3 (Gaussian domination).

$$G(p) \leq \frac{1}{2\beta E(p)}.$$

Before we describe how this relates to the random-current representation, we note that a simple calculation shows that $E(p) \geq c \sum_{j=1}^{d} p_j^2$, which at least gives some indication of why Gaussian domination may be particularly useful for large d.

The link to the random-current representation is roughly as follows. Define the *bubble diagram*

$$B_0 = \sum_{x \in V} \langle \sigma_0 \sigma_x \rangle_{\gamma=0}^2. \tag{4.3.6}$$

Recall that $M = \langle \sigma_0 \rangle$ and that we write $\chi = \partial M / \partial \gamma$. We saw in Section 3.3.2 that random-current arguments imply the GHS-inequality, namely that $\partial \chi / \partial \gamma \leq 0$. In [**5**], elaborations of such arguments (for the discrete model) show that in fact

$$\frac{\partial \chi}{\partial \gamma} \leq -\frac{|1 - \tanh(\gamma) B_0 / M|_+^2}{96 B_0 (1 + 2\beta B_0)^2} \tanh(\gamma) \chi^4, \tag{4.3.7}$$

where $|x|_+ = x \vee 0$. The bubble diagram appears here as it becomes necessary to consider the existence of two independent currents between sites 0 and x. Inequality (4.3.7) is an improvement on the GHS-inequality if B_0 is finite; thus the first task is to obtain bounds on B_0. Such bounds are provided primarily by Gaussian domination. The link is provided via Parseval's identity:

$$B_0 = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} G(p)^2 \, dp. \tag{4.3.8}$$

By careful use of Gaussian domination and other bounds, one may establish bounds on B_0 for β close to the critical value β_{c}. More precisely, one may show that there are constants $0 < c_1, c_2 < \infty$ such that

$$\begin{aligned} B_0 &\le c_1, && \text{if } d > 4,\\ B_0 &\le c_2 |\log(\beta_{\mathrm{c}} - \beta)|, && \text{if } d = 4, \end{aligned}$$

as $\beta \uparrow \beta_c$. Careful manipulation and integration of (4.3.7) then gives that there are constants c_1', c_2', c_1'', c_2'' such that the infinite-volume magnetization M satisfies the following. First, as $\beta \downarrow \beta_{\mathrm{c}}$ for $\gamma = 0$,

$$\begin{aligned} M &\le c_1'(\beta - \beta_{\mathrm{c}})^{1/2}, && \text{if } d > 4,\\ M &\le c_2'(\beta - \beta_{\mathrm{c}})^{1/2} |\log(\beta - \beta_{\mathrm{c}})|^{3/2}, && \text{if } d = 4, \end{aligned}$$

and second, for $\beta = \beta_{\mathrm{c}}$ and $\gamma \downarrow 0$,

$$\begin{aligned} M &\le c_1'' \gamma^{1/3}, && \text{if } d > 4,\\ M &\le c_2'' \gamma^{1/3} |\log \gamma|, && \text{if } d = 4. \end{aligned}$$

These are the complementary bounds to (4.3.2) needed to show that the limits (4.3.3) exist and take the values $a = 1/2$ and $b = 3$.

There are two main steps to extending the results of [**5**] to the quantum (or space–time) Ising model: first, to establish reflection positivity and the related Gaussian domination bound, and second, to verify that the random-parity representation can produce an inequality of the form (4.3.7). There is essentially only one known way of showing that a measure is reflection positive, which is to show that it has a density against a product measure which is of a prescribed form [**13**, Lemma 4.4]. Preliminary calculations suggest that this method works also for the space–time Ising model. Although the random-current manipulations in [**5**] leading up to (4.3.7) are considerably more delicate than those presented in Chapter 3 of this work and involve some new ideas such as 'dilution', preliminary calculations again suggest that it should be possible to extend them as required.

4.4. Random currents in the Potts model

The main results of this work have relied on the random-parity representation for the space–time Ising model. It is natural to ask if there is a similar representation for the $q \geq 3$ Potts model. Here we will discuss this question, to start with in the context of the *classical* (discrete) Potts model on a finite graph $L = (V, E)$. For simplicity we will assume free boundary condition and zero external field; it is easy to adapt the results here to positive fields.

It is shown in [**50**, Chapter 9] (see also [**30, 27**]) that the q-state Potts model with $q \geq 3$ possesses a *flow representation*, which is akin to the random-current representation, in that the two-point correlation function may be written as the ratio of two expected values. This representation is as follows.

Let the integer $q \geq 2$ be fixed. For $\underline{n} = (n_e : e \in E)$ a vector of non-negative integers, define the graph $L_{\underline{n}} = (V, E_{\underline{n}})$ by replacing each edge e of L by n_e parallel edges. If $P = (P_e : e \in E)$ is a collection of finite sets with $|P_e| = n_e$, we identify L_P with $L_{\underline{n}}$, and interpret P_e as the set of edges replacing e. We assign to the elements of $E_{\underline{n}}$ arbitrary directions and write $\vec{e}$ for directed elements of $E_{\underline{n}}$; if $\vec{e}$ is adjacent to a vertex $x \in V$ and is directed into x we write $\vec{e} \mapsto x$, and if $\vec{e}$ is directed out of x we write $\vec{e} \hookleftarrow x$. We say that a function $f : E_{\underline{n}} \to \{1, \dots, q-1\}$

is a (nonzero) *mod q flow on $L_{\underline{n}}$* (or q-flow for short) if for all $x \in V$ we have that

$$\sum_{\substack{\vec{e}\in E_{\underline{n}}:\\ e\leftarrow x}} f(\vec{e}) - \sum_{\substack{\vec{e}\in E_{\underline{n}}:\\ e\mapsto x}} f(\vec{e}) \equiv 0 \pmod{q}. \tag{4.4.1}$$

Let $C(L_{\underline{n}}; q)$ denote the number of mod q flows on $L_{\underline{n}}$ (this is called the flow polynomial of $L_{\underline{n}}$). It is easy to see that this number does not depend on the directions chosen on the edges (if the direction of an edge $\vec{e}$ is reversed we can replace $f(\vec{e})$ by $q - f(\vec{e})$).

For each $e \in E$, let $\beta'_e \geq 0$, and recall that the Potts weight of an element $\nu \in \{1, \ldots, q\}^V = \mathcal{N}$ is

$$\exp\Big(\sum_{e=xy\in E} \beta'_e \delta_{\nu_x,\nu_y}\Big), \tag{4.4.2}$$

so that the partition function is

$$Z = \sum_{\nu\in\mathcal{N}} \exp\Big(\sum_{e=xy\in E} \beta'_e \delta_{\nu_x,\nu_y}\Big). \tag{4.4.3}$$

Let $\beta_e = \beta'_e/q$ and let the collection $P = (P_e : e \in E)$ of finite sets be given by letting the $|P_e|$ be independent Poisson random variables, each with parameter β_e. Write $\mathbb{P}_\beta$ for the probability measure governing the P_e and $\mathbb{E}_\beta$ for the corresponding expectation operator.

The flow representation of Z is

$$Z = \exp\Big(2\sum_{e\in E} \beta_e\Big) q^{|V|} \mathbb{E}_\beta[C(L_P; q)]. \tag{4.4.4}$$

In fact, more is true. For $x, y \in V$, let $L^{xy}_{\underline{n}} = (V, E_{\underline{n}} \cup \{xy\})$ denote the graph $L_{\underline{n}}$ with an edge added from x to y. Write $\langle\cdot\rangle$ for the expected value under the q-state Potts measure defined by (4.4.2)–(4.4.3). Then for any $x, y \in V$ we have that

$$q\langle \mathbb{I}\{\nu_x = \nu_y\}\rangle - 1 = \frac{\mathbb{E}_\beta[C(L^{xy}_P; q)]}{\mathbb{E}_\beta[C(L_P; q)]}. \tag{4.4.5}$$

Here is a simple observation that changes the expected value in (4.4.4) into a probability. For $\underline{n} \in \mathbb{Z}^E_+$, let $F_q(\underline{n})$ denote the set of functions

$f : V \to \{1, \dots, q-1\}$. Then

(4.4.6)

$$
\begin{aligned}
\mathbb{E}_\beta[C(L_P;q)] &= \sum_{\underline{n}\in\mathbb{Z}_+^E} \prod_{e\in E} \frac{\beta_e^{n_e}}{n_e!} e^{-\beta_e} \sum_{f\in F_q(\underline{n})} \mathbb{1}\{f \text{ is } q\text{-flow}\} \\
&= \exp\Big((q-2)\sum_{e\in E}\beta_e\Big) \sum_{\underline{n}\in\mathbb{Z}_+^E} \prod_{e\in E} \frac{((q-1)\beta_e)^{n_e}}{n_e!} e^{-(q-1)\beta_e} \\
&\qquad \cdot \frac{1}{(q-1)^{\sum_{e\in E} n_e}} \sum_{f\in F_q(\underline{n})} \mathbb{1}\{f \text{ is } q\text{-flow}\} \\
&= \exp\Big((q-2)\sum_{e\in E}\beta_e\Big) \mathbb{P}(\psi \text{ is } q\text{-flow on } L_{P'}),
\end{aligned}
$$

where, under $\mathbb{P}$, the collection $P' = (P'_e : e \in E)$ is given by letting the $|P'_e|$ be independent Poisson random variables with parameters $(q-1)\beta_e$ respectively, and ψ is, given P', a uniformly chosen element of $F_q(P')$. (As before, arbitrary directions are assigned to the elements of $E_{P'}$, but the probability that ψ is a q-flow does not depend on the choice of directions.)

We now show that a similar representation to (4.4.6) holds for the two-point correlation functions (4.4.5), and indeed for more general correlation functions. As in Section 2.2.3 we will use the variables

$$
\sigma_x = \exp\Big(\frac{2\pi i \nu_x}{q}\Big), \qquad \nu_x = 1, \dots, q.
$$

We write $Q \subseteq \mathbb{C}$ for the set of qth roots of unity, and $\Sigma = Q^V$. For $\underline{r} \in \mathbb{Z}^V$ and $\sigma \in \Sigma$ we let

$$
\sigma^{\underline{r}} = \prod_{x\in V} \sigma_x^{r_x}.
$$

Note that it is equivalent to regard r_x as an element of $\mathbb{Z}/(q\mathbb{Z})$, the integers modulo q. Let $\mathbb{P}$, P' and ψ be as in (4.4.6), and write $\{\psi \equiv 0\}$ for the event that ψ is a q-flow. More generally, write $\{\psi + \underline{r} \equiv 0\}$ for the event that for each $x \in V$,

$$\sum_{\substack{\vec{e} \in E_{P'}: \\ e \hookleftarrow x}} \psi(\vec{e}) - \sum_{\substack{\vec{e} \in E_{P'}: \\ e \hookrightarrow x}} \psi(\vec{e}) \equiv -r_x \pmod q. \tag{4.4.7}$$

(Recall that we have assigned arbitrary directions to the elements of $E_{P'}$.)

THEOREM 4.4.1. *In the discrete Potts model with zero field and coupling constants β'_e,*

$$\langle \sigma^{\underline{r}} \rangle = \frac{\mathbb{P}(\psi + \underline{r} \equiv 0)}{\mathbb{P}(\psi \equiv 0)}.$$

Before proving this, note that if $\sigma \in \mathcal{N}$ and $x, y \in V$, then $\tau_{xy} := \sigma_x \sigma_y^{-1}$ has the property that $\tau_{xy} = 1$ if and only if $\sigma_x = \sigma_y$, and in fact

$$\frac{1}{q} \sum_{r=0}^{q-1} \tau_{xy}^r = \delta_{\sigma_x, \sigma_y}.$$

Thus the partition function (4.4.3) may be written

$$\begin{aligned} Z &= \sum_{\sigma \in \Sigma} \exp\Big(\sum_{e = xy \in E} \beta'_e \delta_{\sigma_x, \sigma_y} \Big) \\ &= \sum_{\sigma \in \Sigma} \exp\Big(\frac{1}{2} \sum_{x,y \in V} \beta_{xy} \sum_{r=1}^{q-1} \tau_{xy}^r \Big) \cdot \exp\Big(\sum_{e \in E} \beta_e \Big), \end{aligned} \tag{4.4.8}$$

where the first sum inside the exponential is over all ordered pairs $x, y \in V$, and we set $\beta_{xy} = \beta_e$ if $e \in E$ is an edge between x and y, and $\beta_{xy} = 0$ otherwise. Note finally that $\tau_{xy} \neq \tau_{yx}$ in general.

PROOF. We perform a calculation on the factor

$$\sum_{\sigma \in \Sigma} \exp\Big(\frac{1}{2} \sum_{x,y \in V} \beta_{xy} \sum_{r=1}^{q-1} \tau_{xy}^r \Big)$$

which appears on the right-hand-side of (4.4.8); this will only re-prove the relation (4.4.6), but it will be clear that a simple extension of the calculation will give the result.

Let us write $\tilde{\beta}_{xy} = \beta_{xy}/2$. We have that

(4.4.9)

$$\sum_{\sigma\in\Sigma} \exp\Big(\frac{1}{2}\sum_{x,y\in V}\beta_{xy}\sum_{r=1}^{q-1}\tau_{xy}^r\Big) = \sum_{\sigma\in\Sigma}\prod_{x,y\in V}\prod_{r=1}^{q-1}\sum_{m\geq 0}\frac{1}{m!}(\tilde{\beta}_{xy}\tau_{xy}^r)^m$$
$$= \sum_{\sigma\in\Sigma}\sum_{\underline{m}} w(\underline{m}) \prod_{x,y\in V}\prod_{r=1}^{q-1}(\tau_{xy}^r)^{m_{x,y,r}},$$

where the vector $\underline{m} = (m_{x,y,r} : x, y \in V, r = 1, \ldots, q-1)$ consists of non-negative integers and

$$w(\underline{m}) = \prod_{x,y\in V}\prod_{r=1}^{q-1}\frac{\tilde{\beta}_{xy}^{m_{x,y,r}}}{m_{x,y,r}!}$$

is an un-normalized Poisson weight on $\underline{m}$. Reordering (4.4.9) we obtain

$$\sum_{\sigma\in\Sigma} \exp\Big(\frac{1}{2}\sum_{x,y\in V}\beta_{xy}\sum_{r=1}^{q-1}\tau_{xy}^r\Big) = \sum_{\underline{m}} w(\underline{m}) \sum_{\sigma\in\Sigma}\prod_{x,y\in V}\tau_{xy}^{M_{xy}} \tag{4.4.10}$$

where

$$M_{xy} = \sum_{r=1}^{q-1} r\cdot m_{x,y,r}.$$

We may interpret $m_{x,y,r}$ as a random number of edges, each of which is directed from x to y and receives flow value r. Then M_{xy} is the total flow from x to y. Up to the constant multiple $\exp\big((q-1)\sum_e \beta_e\big)$, the quantity (4.4.10) equals the expected value of the quantity

$$\sum_{\sigma\in\Sigma}\prod_{x,y\in V}\tau_{xy}^{M_{xy}} \tag{4.4.11}$$

when the $m_{x,y,r}$ have the Poisson distribution with parameter $\tilde{\beta}_{xy}$ and are chosen independently.

The quantity (4.4.11) simplifies, as follows. Let $a \in V$ be fixed, and let $L_a = (V_a, E_a)$ denote L with a removed. Then

$$\begin{aligned}\sum_{\sigma\in\Sigma}\prod_{x,y\in V}\tau_{xy}^{M_{xy}} &= \sum_{\sigma\in\Sigma}\Big(\prod_{b\sim a}\tau_{ab}^{M_{ab}}\tau_{ba}^{M_{ba}}\Big)\prod_{x,y\in V_a}\tau_{xy}^{M_{xy}}\\ &= \sum_{\sigma\in\Sigma}\Big(\prod_{b\sim a}\sigma_a^{M_{ab}-M_{ba}}\sigma_b^{M_{ba}-M_{ab}}\Big)\prod_{x,y\in V_a}\tau_{xy}^{M_{xy}}.\end{aligned} \tag{4.4.12}$$

Write $M_a = \sum_{b\sim a}(M_{ab} - M_{ba})$. We may now take out the factor

$$\sum_{\sigma_a \in Q} \sigma_a^{M_a} = q \cdot \mathbb{1}_{\{M_a \equiv 0 \pmod q\}}. \tag{4.4.13}$$

Proceeding as above with the remaining vertices of L we obtain that

$$\sum_{\sigma\in\Sigma} \prod_{x,y\in V} \tau_{xy}^{M_{xy}} = q^{|V|} \cdot \mathbb{1}\{M_a \equiv 0 \pmod q \text{ for all } a \in V\}. \tag{4.4.14}$$

Thus

$$Z = q^{|V|} \exp\Big(q \sum_{e\in E} \beta_e\Big) \Pr(M_a \equiv 0\ \forall a \in V) \tag{4.4.15}$$

It remains to show that the distribution of M coincides with that of ψ. This is easy: given P', do the following. First, assign for all $e \in E$ each of the $|P'_e|$ edges replacing e a direction uniformly a random; the number of edges directed from x to y then has the Poisson distribution with parameter $(q-1)\beta_e/2$. Next, assign each directed edge a value $1, \dots, q-1$ uniformly at random; the number of edges directed from x to y with value r then has the Poisson distribution with parameter $\tilde{\beta}_e$. The corresponding element of $F_q(P')$ is uniformly chosen given the edge numbers and directions, and since the probability of obtaining a q-flow does not depend on the choice of directions, we are done.

To obtain the full result in the theorem, repeat the above steps with the numerator of $\langle\sigma^{\underline{r}}\rangle$. The quantity M_a in (4.4.13) must then be replaced by $M_a + r_a$, but the rest of the calculation is as before. It follows that

$$\langle\sigma^{\underline{r}}\rangle = \frac{q^{|V|} \exp\Big(q \sum_{e\in E} \beta_e\Big) \mathbb{P}(\psi + \underline{r} \equiv 0)}{q^{|V|} \exp\Big(q \sum_{e\in E} \beta_e\Big) \mathbb{P}(\psi \equiv 0)} = \frac{\mathbb{P}(\psi + \underline{r} \equiv 0)}{\mathbb{P}(\psi \equiv 0)} \tag{4.4.16}$$

□

It is straightforward to extend Theorem 4.4.1 to an analogous representation for the space–time model, and we sketch this here. First, by conditioning on the set D, one obtains (as in (3.2.9)) a discrete graph $G(D) = (V(D), E(D))$. By applying the formulas in the numerator and denominator of (4.4.16) on the graph $G(D)$, one obtains a representation of the form (3.2.12). One may then repeat the procedure in the proof of Theorem 3.2.1 to obtain a formula in terms of weighted labellings; these labellings are defined as follows.

Let $\Lambda = (K, F)$ and β be as in Chapter 3. Fix an arbitrary ordering of the vertices V of L. Let $B \subseteq F$ be a Poisson process with rate $(q-1)\lambda$. We assign directions to the elements of B by letting a bridge between $(u,t) \in K$ and $(v,t) \in K$ be directed from u to v if u comes before v in the ordering of V. We then assign to each element of B a weight from $\{1,\dots,q-1\}$ uniformly at random, these choices being independent.

Let $A \subseteq K^\circ$ be a finite set (which lies in the interior of K only for convenience of exposition). Let $\underline{r} = (r_x : x \in A)$ be a vector of integers, indexed by A, and let $S \subseteq K$ denote the union of A with the set of endpoints of bridges in B. Given the above, a labelling $\psi^{\underline{r}}$ is a map $K \to \mathbb{Z}/(q\mathbb{Z})$, which is constrained to be 'valid' in that:

(1) on each subinterval of each K_v, the label is constant between elements of S,

(2) as we move along a subinterval of K_v $(v \in V)$ in the increasing β direction, the label changes at elements of S; if the label is t before reaching $x \in S$, then the label just after x is
   - $t + r$ if x is the endpoint of a bridge directed into x and which has weight r,
   - $t - r$ if x is the endpoint of a bridge directed out of x with weight r,
   - $t - r_x$ if $x \in A$,

(3) as one moves towards an endpoint of an interval $I_k^v \neq \mathbb{S}$ (in either direction) the label converges to 0.

As for the random-parity representation of the space–time Ising model, these conditions do not uniquely define $\psi^{\underline{r}}$ if there is a $v \in V$ such that

$K_v = \mathbb{S}$. If this is the case, the label at 0 is chosen uniformly at random for each such v, these choices being independent.

A valid labelling is given the weight

$$\partial\psi^{\underline{r}} := \exp(q\delta(\mathcal{L}_0(\psi^{\underline{r}}))),$$

where $\mathcal{L}_0(\psi^{\underline{r}})$ is the set labelled 0 in $\psi^{\underline{r}}$. In the following, $\underline{r} = 0$ denotes the vector which takes the value 0 at all $x \in A$; we let $E(\cdot)$ denote the expectation over B as well as the weights assigned to the elements of B, and the randomization which takes place when there are several valid labellings.

Theorem 4.4.2. *In the space–time Potts model,*

$$\langle\sigma^{\underline{r}}\rangle = \frac{E(\partial\psi^{\underline{r}})}{E(\partial\psi^{0})}.$$

The usefulness of Theorems 4.4.1 and 4.4.2 when $q \geq 3$ is questionable. Mod q flows with $q \geq 3$ are considerable more complicated than mod 2 flows, and there does not seem to be a useful switching lemma (along the lines of Theorem 3.3.2 or its discrete version [**3**]) for general q.

APPENDIX A

The Skorokhod metric and tightness

In this appendix we define carefully the Skorokhod metric on Ω and show that the sequence ϕ_n^b of random-cluster measures in Section 2.3.1 is tight, proving Lemma 2.3.1. We will rely partly on the notation and results in [**31**, Chapter 3]; see also [**71**, Appendix 1].

A function $f : \mathbb{R} \to \mathbb{R}$ is called *càdlàg* if it is right-continuous and has left limits. We let $\mathcal{D}_{\mathbb{Z}}^0(\mathbb{R})$ denote the set of increasing càdlàg step functions on $\mathbb{R}$ with values in $\mathbb{Z}$, and which take the value 0 at 0. It is straightforward to modify the definitions and results of [**31**, Chapter 3], which concern càdlàg functions on $[0,\infty)$ with values in some metric space E, to apply to the set $\mathcal{D}_{\mathbb{Z}}^0(\mathbb{R})$. Specifically, we define the Skorokhod metric on $\mathcal{D}_{\mathbb{Z}}^0(\mathbb{R})$ as follows. Let U denote the set of strictly increasing bijections $u : \mathbb{R} \to \mathbb{R}$ which are Lipschitz continuous and for which the quantity

$$\alpha(u) := \sup_{t>s} \log \Big| \frac{u(t)-u(s)}{t-s} \Big| \tag{A.0.17}$$

is finite. For $a, b \in \mathbb{Z}$ let $r(a,b) = \delta_{a,b}$, and note that r is a metric on $\mathbb{Z}$. The Skorokhod metric on $\mathcal{D}_{\mathbb{Z}}^0(\mathbb{R})$ is by definition given by

$$d'(f,g) = \inf_{u\in U} \Big[\alpha(u) \wedge \int_{-y}^{y} e^{-|y|} d'(f,g,u,y)\, dy \Big], \tag{A.0.18}$$

where

$$d'(f,g,u,y) = \sup_{t\in\mathbb{R}} r(f((t\wedge y)\vee -y), g((u(t)\wedge y)\vee -y)). \tag{A.0.19}$$

It may be checked, as in [**31**, pp. 117], that d' is indeed a metric, and that the metric space $(\mathcal{D}_{\mathbb{Z}}^0(\mathbb{R}), d')$ is complete and separable.

Recall that we are given a countable graph $\mathbb{L} = (\mathbb{V}, \mathbb{E})$. Let $\mathbb{T}$ denote the countable set

$$\mathbb{T} = (\mathbb{V} \times \{\mathrm{d}\}) \cup (\mathbb{V} \times \{\mathrm{g}\}) \cup \mathbb{E},$$

and let $\upsilon : \mathbb{T} \to \{1, 2, \dots\}$ denote an arbitrary bijection. Then we formally define the set Ω to be the product space $\Omega = \mathcal{D}^0_{\mathbb{Z}}(\mathbb{R})^{\mathbb{T}}$. For $\omega \in \Omega$ and $x \in \mathbb{T}$, the restriction ω_x of ω to $x \times \mathbb{R}$ (not to be confused with the ω_x of Section 2.2.1) is to be interpreted as: the process of deaths on $x \times \mathbb{R}$ if $x \in \mathbb{V} \times \{\mathrm{d}\}$, or the process of ghost-bonds on $x \times \mathbb{R}$ if $x \in \mathbb{V} \times \{\mathrm{g}\}$, or the process of bridges on $x \times \mathbb{R}$ if $x \in \mathbb{E}$. In this section we do *not* overlook events of probability zero, that is Remark 2.1.1 does not apply.

DEFINITION A.0.1. *We define the Skorokhod metric d on Ω by*

$$d(\omega, \omega') = \sum_{x \in \mathbb{T}} e^{-\upsilon(x)} d'(\omega_x, \omega'_x).$$

Note that the sum is absolutely convergent since d' is bounded, and in fact also d is bounded. It is straightforward to check that d is indeed a metric on Ω, and (using the dominated convergence theorem) that (Ω, d) is a complete metric space. It is also separable, hence Polish. The σ-algebra $\mathcal{F}$ on Ω generated by d agrees with that generated by all the coordinate functions $\pi_{x,t} : \omega \mapsto \omega_x(t)$ for $x \in \mathbb{T}$ and $t \in \mathbb{R}$, see [**31**, Proposition 3.7.1]. The fact that all finite tuples of such coordinate functions forms a convergence determining class (a fact used in Theorem 2.3.2) follows as in [**31**, Theorem 3.7.8].

In order to establish tightness of the sequence $\phi_n^{b_n}$ we must find compact sets in Ω. Since (Ω, d) is a metric space, compactness is equivalent to sequential compactness. If for each $x \in \mathbb{T}$, the set A_x is (sequentially) compact in $(\mathcal{D}^0_{\mathbb{Z}}(\mathbb{R}), d')$, then by a straightforward diagonal argument the set $A = \bigotimes_{x \in \mathbb{T}} A_x$ is a compact subset of (Ω, d).

Proof of Lemma 2.3.1. As a witness for the tightness of $\{\phi_n^{b_n} : n \geq 1\}$ we will use the product A of the following compact sets A_x. For each $x \in \mathbb{T}$, let $\xi_x : [0, \infty) \to (0, \infty)$ be a strictly positive function, to be specified later. Let A_x be the set of $\omega \in \Omega$ such that for all $t > 0$, all jumps of ω_x in the interval $[-t, t]$ are separated from each other by at least $\xi_x(t)$. It follows from the characterization in [**31**, Theorem 3.6.3] that A_x is compact (alternatively, it is not hard to deduce the sequential compactness of A_x using a diagonal argument).

It remains to show that we can choose the functions ξ_x so as to get a uniform lower bound on $\phi_n^{b_n}(A)$ which is arbitrarily close to 1. We can use stochastic domination, Corollary 2.2.13, to reduce this to checking the tightness of *a single percolation measure*, as follows. If $x \in \mathbb{V} \times \{\mathrm{d}\}$ then the event A_x is increasing, otherwise it is decreasing. Thus $A = \bigcap_{x \in \mathbb{T}} A_x = A^+ \cap A^-$ where

$$A^+ = \bigcap_{x \in \mathbb{V} \times \{\mathrm{d}\}} A_x \qquad \text{and} \qquad A^- = \bigcap_{x \in (\mathbb{V} \times \{\mathrm{g}\}) \cup \mathbb{E}} A_x$$

are increasing and decreasing events, respectively. We have that

$$\phi_n^{b_n}(A) \geq \phi_n^{b_n}(A^+) + \phi_n^{b_n}(A^-) - 1. \tag{A.0.20}$$

The events A^+, A^- are not local events, but by writing them as decreasing limits of local events it is easy to justify the following application of Corollary 2.2.13 to (A.0.20). For suitable choices of the parameters $\lambda_i, \delta_i, \gamma_i$ $(i = 1, 2)$ which are multiples of the original parameters λ, δ, γ we have that

$$\phi_n^{b_n}(A) \geq \mu_{\lambda_1,\delta_1,\gamma_1}(A^+) + \mu_{\lambda_2,\delta_2,\gamma_2}(A^-) - 1. \tag{A.0.21}$$

Clearly, any lower bound on the right-hand-side of (A.0.21) is a uniform lower bound on the $\phi_n^{b_n}(A)$.

Let us focus on A^+, since A^- is similar. Suppose we can, for any $\varepsilon > 0$, choose ξ_x so that

$$\mu_{\lambda_1,\delta_1,\gamma_1}(A_x) \geq e^{-\varepsilon/\upsilon(x)^2}.$$

Then, since the A_x are independent under $\mu_{\lambda_1,\delta_1,\gamma_1}$, we will have that

$$\mu_{\lambda_1,\delta_1,\gamma_1}(A^+) \geq \exp\Big(-\varepsilon\frac{\pi^2}{6}\Big),$$

which is enough. The event A_x concerns only the process D of deaths on $x \times \mathbb{R}$. We may replace δ_1 by a constant upper bound. By adjusting parameters it follows that we are done if we prove the following: for any $\varepsilon > 0$ we have that

$$P(N \in A_x) \geq 1 - \varepsilon, \tag{A.0.22}$$

where P is the measure governing the Poisson process N of rate 1 on $\mathbb{R}$. The proof of (A.0.22) is a straightforward exercise on Poisson processes, but we include it for completeness.

For $I \subseteq \mathbb{R}$ and $a \in \mathbb{R}$ we write $aI = \{at : t \in I\}$. Define $I_1^+ = I_1^- = [-1, 1]$ and for $k \geq 2$ let I_k^+ be the closed interval of length $1/k$ with left endpoint $1 + 1/2 + 1/3 + \cdots + 1/(k-1)$; let $I_k^- = -I_k^+$. Since the series $\sum \frac{1}{k}$ diverges, the $I_k^\pm$ $(k \geq 1)$ cover $\mathbb{R}$. Next let J_k^+ $(k \geq 1)$ be the closed interval whose left and right endpoints are at the midpoints of I_k^+ and I_{k+1}^+ respectively; let $J_k^- = -J_k^+$. Note that $|J_k^\pm| = (|I_k^\pm| + |I_{k+1}^\pm|)/2 \geq \frac{1}{k+1}$. Let $\varepsilon > 0$ and let A' be the event that each $\varepsilon I_k^\pm$ and each $\varepsilon J_k^\pm$ $(k \geq 1)$ contains at most one element of N.

We claim that $A' \subseteq A_x$ for $\xi_x(t) = \varepsilon e^{-t/\varepsilon}/4$. Suppose A' happens and $s \in N$. We may assume $s \in \varepsilon I_k^+$ with $k \geq 2$ (the other cases are similar). Then s also lies in either εJ_{k-1}^+ or εJ_k^+. Hence the closest possible other point of N is a distance at least $\frac{\varepsilon}{2(k+1)}$ from s. Let $t > 0$ and suppose $s \in N \cap [0, t]$. Let k be maximal with $I_k^+ \cap [0, t] \neq \varnothing$.

Then

$$t \geq \varepsilon \sum_{i=1}^{k-1} \frac{1}{i} \geq \varepsilon \log k,$$

and the closest point to s in N is a distance at least

$$\frac{\varepsilon}{2(k+1)} \geq \frac{\varepsilon}{2(e^{t/\varepsilon}+1)} \geq \frac{\varepsilon}{4} e^{-t/\varepsilon}.$$

Similarly if $s < 0$. Hence $A' \subseteq A_x$ as claimed.

It is well-known that there is an absolute constant C such that for $\eta > 0$ small and I a fixed interval of length at most η, we have that $P(|N \cap I| \geq 2) \leq C\eta^2$. Clearly we have

(A.0.23)

$$\begin{aligned} P(N \in A') \geq 1 - 2\sum_{k\geq 2} P(|N \cap \varepsilon I_k^+| \geq 2) - \\ - 2\sum_{k\geq 1} P(|N \cap \varepsilon J_k^+| \geq 2) - P(|N \cap \varepsilon I_1| \geq 2) \\ \geq 1 - \varepsilon^2 C \cdot 2\pi^2/3. \end{aligned}$$

This proves the result. □

APPENDIX B

Proof of Proposition 2.1.4

PROOF OF LEMMA 2.1.4. This is essentially straightforward, but notationally intricate. We write $(\eta, \omega, \tau)_{\Lambda,\Delta}$ for the configuration which equals η inside the smallest set Λ, equals ω in the intermediate region $\Delta \setminus \Lambda$, and equals τ outside Δ. For readability, let us write $k_\Lambda(\cdot;\tau)$ in place of $k_\Lambda^\tau(\cdot)$ in what follows.

Let $A' \in \mathcal{F}_{\Delta\setminus\Lambda}$. Then

(B.0.24)
$$\begin{aligned}\phi_\Delta^\tau(\mathbb{I}_{A'}(\cdot)\phi_\Lambda^{(\cdot,\tau)_\Delta}(A)) &= \iint \mathbb{I}_{A'}(\omega)\mathbb{I}_A((\eta,\omega,\tau)_{\Lambda,\Delta})\, d\phi_\Lambda^{(\omega,\tau)_\Delta}(\eta)d\phi_\Delta^\tau(\omega)\\ &= \iint \mathbb{I}_{A\cap A'}((\eta,\omega,\tau)_{\Lambda,\Delta})\frac{q^{k_\Lambda(\eta;(\omega,\tau)_\Delta)}}{Z_\Lambda^{(\omega,\tau)_\Delta}}\frac{q^{k_\Delta(\omega;\tau)}}{Z_\Delta^\tau}\, d\mu(\eta)d\mu(\omega).\end{aligned}$$

Note that if $(\alpha,\beta)_\Lambda \in \Omega$ then

$$k_\Delta((\alpha,\beta)_\Lambda;\tau) = k_\Lambda(\alpha;(\beta,\tau)_\Delta) + \tilde{k}_\Delta(\beta;\tau), \tag{B.0.25}$$

where $\tilde{k}_\Delta$ counts the number of components in Δ which do not intersect Λ. Let α,β be independent with law μ; then ω has the law of $(\alpha,\beta)_\Lambda$. Use (B.0.25) on each power of q in (B.0.24) to see that

$$\begin{aligned}&\phi_\Delta^\tau\big(\mathbb{I}_{A'}(\cdot)\phi_\Lambda^{(\cdot,\tau)}(A)\big)\\ &\quad= \iiint \mathbb{I}_{A\cap A'}((\eta,\beta,\tau)_{\Lambda,\Delta})\frac{q^{k_\Lambda(\alpha;(\beta,\tau)_\Delta)}q^{k_\Delta((\eta,\beta)_\Lambda;\tau)}}{Z_\Lambda^{(\beta,\tau)_\Delta}Z_\Delta^\tau}\, d\mu(\eta)d\mu(\alpha)d\mu(\beta)\\ &\quad= \int \mathbb{I}_{A\cap A'}((\omega',\tau)_\Delta)\frac{q^{k_\Delta(\omega';\tau)}}{Z_\Delta^\tau}\left(\int \frac{q^{k_\Delta(\alpha;(\omega',\tau))}}{Z_\Lambda^{(\omega',\tau)}}\, d\mu(\alpha)\right) d\mu(\omega')\\ &\quad= \phi_\Delta^\tau(A\cap A'),\end{aligned}$$

where $\omega' = (\eta,\beta)_\Lambda$. This proves the claim. $\square$

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