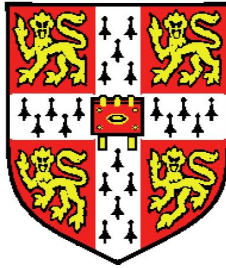# Three Viewpoints on Semi-Abelian Homology

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A thesis submitted for the degree of

Doctor of Philosophy

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I would like to dedicate this thesis to my parents. Although it took them a while to accept that anyone might want to study something without any apparent application to industry, they have always supported me in whatever I wanted to do.

I would also like to dedicate this thesis to my collaborator Tim Van der Linden, without whose constant help and encouragement I would probably have given up research long ago.
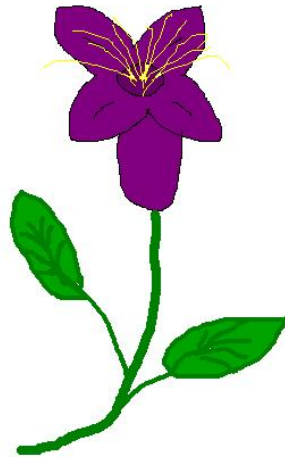
Declaration of Originality

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as detailed below and where specifically indicated in the text.

Chapter 1 gives the background for the thesis and is known material. Chapter 2 contains more background material in the first two sections, but Section 2.3 presents my own work, generalising known results from the abelian context into the semi-abelian setting. Chapter 3 is based on joint work with Tim Van der Linden, published in the Journal of Homotopy and Related Structures in 2007. I contributed roughly 50% to these results. Chapter 4 again presents known material which is needed in Chapter 5. This chapter is based on another joint paper with Tim Van der Linden, which will be published in the Mathematical Proceedings of the Cambridge Philosophical Society. As before I contributed 50% to the paper; for the thesis I rewrote the results using the concept of an axiomatic class of extensions introduced by Tomas Everaert, which makes many of the statements and proofs easier. The results of Chapter 6 are my own work, developing a suggestion given by Marino Gran.

Acknowledgements

I would like to thank my supervisor Peter Johnstone for taking me on as his student even though I did not show much enthusiasm for working with toposes. A very big thank you goes to my collaborator Tim Van der Linden, who is a great pleasure to work with and who supported and advised me throughout my time as a PhD student, and will hopefully carry on doing so in the future. I am also indebted to George Janelidze for pointing Tim and me to his work on satellites as an alternative approach to homology. Marino Gran showed great interest in my work, and his suggestion that I should work on stem extensions resulted in Chapter 6. I am grateful to Tomas Everaert for helpful discussions and comments, and for providing the useful concept of an axiomatic class of extensions, which makes Chapter 5 so much easier.

Furthermore I would like to thank Alexander Shannon, arguably the best proof-reader in the world, for finding (hopefully) all the little mistakes in my thesis (though I take full responsibility for any remaining ones). Thank you also to Martin Hyland, Alexander Frolkin and James Griffin for helpful comments and suggestions throughout my time as a PhD student, and to Ignacio Lopez Franco, Enrico Vitale and Burt Totaro for pointing out some mistakes in my examples which nobody else had picked up on.

I would also like to thank Emmanuel College and the Department of Pure Mathematics and Mathematical Statistics for their help and support during my time at Cambridge, and the EPSRC for funding the research towards my PhD. All my family and friends have also supported me to get to this point through many ups and downs. In particular I would like to thank Marcus Zibrowius for managing to boost my work morale enough so that I could actually write this thesis.

Abstract

The main theme of the thesis is to present and compare three different viewpoints on semi-abelian homology, resulting in three ways of defining and calculating homology objects. Any two of these three homology theories coincide whenever they are both defined, but having these different approaches available makes it possible to choose the most appropriate one in any given situation, and their respective strengths complement each other to give powerful homological tools.

The oldest viewpoint, which is borrowed from the abelian context where it was introduced by Barr and Beck, is comonadic homology, generating projective simplicial resolutions in a functorial way. This concept only works in monadic semi-abelian categories, such as semi-abelian varieties, including the categories of groups and Lie algebras. Comonadic homology can be viewed not only as a functor in the first entry, giving homology of objects for a particular choice of coefficients, but also as a functor in the second variable, varying the coefficients themselves. As such it has certain universality properties which single it out amongst theories of a similar kind. This is well-known in the setting of abelian categories, but here we extend this result to our semi-abelian context.

Fixing the choice of coefficients again, the question naturally arises of how the homology theory depends on the chosen comonad. Again it is well-known in the abelian case that the theory only depends on the projective class which the comonad generates. We extend this to the semi-abelian setting by proving a comparison theorem for simplicial resolutions. This leads to the result that any two projective simplicial resolutions, the definition of which requires slightly more care in the semi-abelian setting, give rise to the same homology. Thus again the homology theory only depends on the projective class.

The second viewpoint uses Hopf formulae to define homology, and works in a non-monadic setting; it only requires a semi-abelian category with enough projectives. Even this slightly weaker setting leads to strong results such as a long exact homology sequence, the Everaert sequence, which is a generalised and extended version of the Stallings-Stammbach sequence known for groups. Hopf formulae use projective presentations of objects, and this is closer to the abelian philosophy of using any projective resolution, rather than a special functorial one generated by a comonad. To define higher Hopf formulae for the higher homology objects the use of categorical Galois theory is crucial. This theory allows a choice of Birkhoff subcategory to generate a class of central extensions, which play a big role not only in the definition via Hopf formulae but also in our third viewpoint.

This final and new viewpoint we consider is homology via satellites or pointwise Kan extensions. This makes the universal properties of the homology objects apparent, giving a useful new tool in dealing with statements about homology. The driving motivation behind this point of view is the Everaert sequence mentioned above. Janelidze's theory of generalised satellites enables us to use the universal properties of the Everaert sequence to interpret homology as a pointwise Kan extension, or limit. In the first instance, this allows us to calculate homology step by step, and it removes the need for projective objects from the definition. Furthermore, we show that homology is the limit of the diagram consisting of the kernels of all central extensions of a given object, which forges a strong connection between homology and cohomology. When enough projectives are available, we can interpret homology as calculating fixed points of endomorphisms of a given projective presentation.

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Introduction

The main theme of this thesis is to present and compare three different viewpoints on semi-abelian homology. As a motivation for the newest of these viewpoints, consider a perfect group A. We know that it has a universal central extension, and the kernel of this universal central extension is isomorphic to the second integral homology group $H_2(A, \mathbb{Z})$. It would be nice if this kind of property could be extended to the homology of all groups, not only perfect ones. One of the main results in this thesis makes this possible: it shows that the homology $H_2(A, \mathbb{Z})$ is the limit of the diagram of kernels of *all* central extensions of A. This is a natural generalisation of the perfect case, as the limit of a diagram with an initial object is just this initial object. Similarly the higher homology groups form limits of diagrams using kernels of higher central extensions. This result emphasises the close connection of homology with central extensions, and thus makes a connection to cohomology.

Semi-abelian categories

The results of this thesis are not limited to the category of groups, but take place in the more general setting of semi-abelian categories. Classically, homological algebra is an area which is studied in the context of abelian categories. Many of the diagram lemmas used in homological algebra, such as the Five Lemma and the Snake Lemma, are proved in this context, but they also hold in the category of groups and other settings close to it, such as Lie algebras or crossed modules. Semi-abelian categories give a wider context for homological algebra which includes these non-abelian examples as well as the traditional abelian categories. Many of the homological algebra results which hold in abelian categories also hold in semi-abelian ones, though not quite all, as can be seen most easily in the category of groups. For example, products and coproducts do not coincide in general semi-abelian categories, and not every monomorphism is a kernel, as for instance not every subgroup is a normal subgroup. We use semi-abelian categories as a framework to study homology in a non-abelian context.

Three viewpoints on semi-abelian homology

The oldest viewpoint on semi-abelian homology, which is borrowed from the abelian context where it was introduced by Barr and Beck, is comonadic homology, generating projective simplicial resolutions in a functorial way. This concept only works in monadic semi-abelian categories, such as semi-abelian varieties, which include the categories of groups and Lie algebras. The second viewpoint uses Hopf formulae to define homology,

and works in a non-monadic setting; it only requires a semi-abelian category with enough projectives. Even this slightly weaker setting leads to strong results such as a long exact homology sequence. Hopf formulae use projective presentations of objects, and this is closer to the abelian philosophy of using any projective resolution, rather than a special functorial one generated by a comonad. The final viewpoint we consider in this thesis is homology via satellites or pointwise Kan extensions. This makes the universal properties of the homology objects apparent, giving a useful new tool in dealing with statements about homology. Any two of these three definitions of homology coincide whenever they are both defined; but having these different viewpoints available makes it possible to choose the most appropriate one in any given situation, and their respective strengths complement each other to give powerful homological tools. There is another viewpoint we do not consider in this thesis, which uses kernel pairs and Galois groupoids. This viewpoint introduced by Janelidze in [Jan2008] is perhaps closest to our second viewpoint using Hopf formulae. We now describe our three viewpoints in slightly more detail.

Viewpoint 1: Comonadic homology

Comonadic homology can be viewed not only as a functor in the first entry, giving homology of objects for a particular choice of coefficients, but also as a functor in the second variable, varying the coefficients themselves. As such it has certain universality properties which single it out amongst theories of a similar kind. This is well-known in the setting of abelian categories, but here we extend this result to our semi-abelian context.

Fixing the choice of coefficients again, the question naturally arises of how the homology theory depends on the chosen comonad. Again it is well-known in the abelian case that the theory only depends on the projective class which the comonad generates. We extend this to the semi-abelian setting by proving a comparison theorem for simplicial resolutions. This leads to the result that any two projective simplicial resolutions, the definition of which requires slightly more care in the semi-abelian setting, give rise to the same homology. Thus again the homology theory only depends on the projective class.

Viewpoint 2: Hopf formulae

The Hopf formula

$$\mathrm{H}_2(A, \mathbb{Z}) \cong \frac{[P, P] \cap \mathrm{K}[p]}{[\mathrm{K}[p], P]}$$

for the second homology group is very well known in the context of integral group homology. Here $p \colon P \longrightarrow A$ is a projective presentation of the group A with kernel $\mathrm{K}[p]$, and $[P, P]$ and $[\mathrm{K}[p], P]$ are commutator subgroups of P. This formula can be generalised both to a semi-abelian setting and also to higher Hopf formulae, which are then used to define higher homology objects. A good way to summarise these higher Hopf formulae is to say that homology measures the difference between the centralisation and the trivialisation

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of a projective presentation of a given object. As hinted above, using this definition any short exact sequence

$$0 \longrightarrow K \rightarrowtail B \overset{f}{\longrightarrow} A \longrightarrow 0$$

gives rise to a long exact homology sequence

$$\cdots \longrightarrow \mathrm{H}_{n+1}(A, I) \overset{\delta_f^{n+1}}{\longrightarrow} \mathrm{K}[\mathrm{H}_n(f, I_1)] \overset{\gamma_f^n}{\longrightarrow} \mathrm{H}_n(B, I) \overset{\mathrm{H}_n(f,I)}{\longrightarrow} \mathrm{H}_n(A, I) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{H}_2(A, I) \underset{\delta_f^2}{\longrightarrow} \mathrm{K}[\mathrm{H}_1(f, I_1)] \underset{\gamma_f^1}{\longrightarrow} \mathrm{H}_1(B, I) \underset{\mathrm{H}_1(f,I)}{\longrightarrow} \mathrm{H}_1(A, I) \longrightarrow 0$$
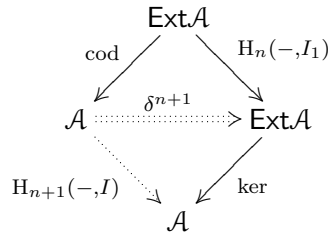
which we call the Everaert sequence. This sequence has a different appearance than its abelian counterpart: instead of being functorial in the objects of the short exact sequence, it is functorial in the extension f. It incorporates not only homology of objects, such as $\mathrm{H}_n(A, I)$, but also homology of an extension $\mathrm{H}_n(f, I_1)$, which is a higher-dimensional version of the same theory. The lowest part of the Everaert sequence is the Stallings-Stammbach sequence, which first appeared in the context of groups, but now it can be extended to a full long exact sequence. Its universal properties play a very important role in a large part of this thesis.

Birkhoff subcategories and abelian objects

The main ingredient of all approaches to semi-abelian homology is a Birkhoff subcategory $\mathcal{B}$ of a semi-abelian category $\mathcal{A}$, as the reflector I takes the role of coefficients of the homology theory. Thus semi-abelian homology calculates the derived functors of this reflector, using different approaches to do this. The most common examples such as integral group homology or homology of Lie algebras use the subcategory of abelian objects as the Birkhoff subcategory, which makes it easy to see that all homology objects are abelian. But there are also many examples where the Birkhoff subcategory is not given by the abelian objects, for instance the categories of nilpotent or solvable groups inside the category of groups, the category of Lie algebras inside that of Leibniz algebras, or the category of crossed modules inside that of precrossed modules. In these cases it can still be shown that all homology objects from the second onwards are abelian, but as the first homology just gives the reflection of an object into the Birkhoff subcategory, these will not in general be abelian objects. In the Everaert sequence, this means that at the lower end the objects stop being abelian. But the map δ_f^2 from the last abelian object to the first non-abelian object turns out to be central in the sense of Huq, meaning that its image commutes with everything in its codomain. This nicely connects the abelian part of the sequence with the non-abelian part, and does not seem to have been realised before.

Viewpoint 3: Homology via satellites

The universal properties of the Everaert sequence are the driving motivation for defining homology via satellites, as a pointwise Kan extension or limit. The connecting homomorphism δ in the Everaert sequence is exactly what makes the Kan extension work. In the first instance, this allows us to calculate homology step by step, and it removes the dependance on projective objects from the definition. For example, the $(n+1)$st homology $H_{n+1}(-, I)$ is the left satellite of the nth homology of extensions $H_n(-, I_1)$.

$$
\begin{array}{ccc}
 & \mathsf{Ext}\mathcal{A} & \\
\text{cod} \swarrow & & \searrow H_n(-,I_1) \\
\mathcal{A} & \xRightarrow{\ \delta^{n+1}\ } & \mathsf{Ext}\mathcal{A} \\
H_{n+1}(-,I) \searrow & & \swarrow \text{ker} \\
 & \mathcal{A} &
\end{array}
$$

For $n = 1$ this can be reformulated to give the result hinted at in the first paragraph of this introduction: homology is the limit of a diagram of kernels of all central extensions of a given object.

$$H_2(A, I) = \lim_{f \in \mathsf{CExt}_A\mathcal{A}} K[f]$$

That is, we consider all central extensions of an object A, and the maps between them. Taking kernels gives us a diagram in the category we are working in, and the second homology $H_2(A, I)$ is the limit of this diagram.
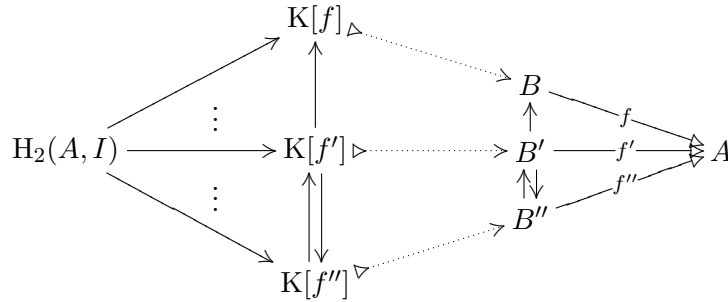
An analogous result using higher central extensions can be obtained for the higher homology objects by concatenation of the original Kan extensions. In the case where enough projectives do exist, this category of central extensions of a given object has a weakly initial object induced by a projective presentation. In this case, homology can be viewed as calculating the common fixed points of the endomorphisms of this weakly initial presentation.

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Structure of the text

In Chapter 1 we set up the semi-abelian context we are working in and state known results that we will use throughout the thesis. Chapter 2 introduces the first viewpoint on comonadic homology and deals with the theory as a functor in the second variable of coefficients. While the first two sections are known material, Section 2.3 contains my own work, generalising a corresponding result from the abelian context. Chapter 3 is devoted to the Comparison Theorem for simplicial resolutions and is joint work with Tim Van der Linden. In Chapter 4 the established theory of homology via Hopf formulae is introduced, which is then needed in Chapter 5. This chapter develops the theory of homology via satellites and is based on another joint paper with Tim Van der Linden; I have rewritten the results using the concept of an axiomatic class of extensions introduced by Tomas Everaert, which simplifies many of the statements and proofs. Finally Chapter 6 explores special cases of central extensions in the context of abelianisation, using the lowest part of the Everaert sequence to give a natural correspondence between isomorphism classes of central extensions of a perfect object B by a fixed abelian object K and maps $\mathrm{H}_2(B, \mathrm{ab}) \longrightarrow K$. The results of this chapter came into existence after a suggestion from Marino Gran.

Chapter 1

The Semi-Abelian Context

As mentioned in the introduction, the framework for this thesis is formed by semi-abelian categories. These include abelian categories as well as many non-abelian examples in which homological algebra can be studied. A good survey of semi-abelian categories can be found in [Bor2004], while the monograph [BB2004] gives a more detailed introduction to semi-abelian categories and the weaker settings connected to them.

In this chapter, we give established definitions and results that will be used throughout the thesis. Section 1.1 introduces semi-abelian categories and many classical results of homological algebra, while Section 1.2 defines the concept of abelian objects, which play a prominent role in our theory. Homology of chain complexes and the main ingredients of semi-abelian homology are set up in Section 1.3. Section 1.4 introduces categorical Galois theory, which forms a crucial base for the theory in Chapter 4 and Chapter 5. Finally in Section 1.5 we compare several different notions of central extensions in the context of abelianisation.

1.1 Semi-abelian categories

This section gives the main definitions that we need for homological algebra in semi-abelian categories, and gathers some known results that will be used throughout the thesis. Two of the first things we need in order to do homological algebra are kernels and cokernels.

1.1.1 Definition (kernels and cokernels): A category $\mathcal{A}$ is **pointed** when it has a **zero-object**, i.e. an object 0 which is both initial and terminal. For any two objects A and B, there is a unique map $B \longrightarrow A$ that factors over the zero object; we call this a **zero map** $0\colon B \longrightarrow A$.

In a pointed category with finite limits we define the **kernel** of a morphism $f\colon B \longrightarrow A$ by the pullback

$$
\begin{array}{ccc}
\mathrm{K}[f] & \overset{\mathrm{Ker}\, f}{\rightarrowtail} & B \\
\downarrow & & \downarrow{\scriptstyle f} \\
0 & \longrightarrow & A
\end{array}
$$

where we refer both to the object $\mathrm{K}[f]$ and to the map $\mathrm{Ker}\, f$ as the kernel of f. Notice that $\mathrm{Ker}\, f$ is a monomorphism, being the pullback of the monomorphism $0 \longrightarrow A$. Equivalently

K[f] is the equaliser of f and the zero map. We say a monomorphism $m\colon K \longrightarrow B$ is **normal** when it is the kernel of some map, and write $K \rightarrowtail B$ as in the diagram above.

In a pointed category with finite colimits we dually define the **cokernel** of a morphism $f\colon B \longrightarrow A$ by the pushout

$$
\begin{array}{ccc}
B & \longrightarrow & 0 \\
\downarrow{\scriptstyle f} & & \downarrow \\
A & \xrightarrow[\text{Coker } f]{} & \mathrm{Q}[f].
\end{array}
$$

If f is a normal monomorphism, we also write $\mathrm{Q}[f] = A/B$. Any cokernel is a regular epimorphism: it is the coequaliser of $f\colon B \longrightarrow A$ and the zero map $0\colon B \longrightarrow A$. We say a regular epimorphism is **normal** when it is the cokernel of some map, and write $A \twoheadrightarrow Q$.

The definition of semi-abelian categories is built up out of many constituent parts, which we introduce now.

1.1.2 Definition: A category $\mathcal{A}$ is called **regular** [Bar1971] when it is finitely complete, every kernel pair in $\mathcal{A}$ has a coequaliser and the class of regular epimorphisms in $\mathcal{A}$ is pullback-stable.

A category $\mathcal{A}$ is called **Barr exact** [Bar1971] when it is regular and every equivalence relation in $\mathcal{A}$ is a kernel pair.

A pointed and regular category is called **Bourn protomodular** [Bou1991] when the **(regular) Short Five Lemma** holds: for every commutative diagram

$$
\begin{array}{ccccc}
\mathrm{K}[f'] & \overset{\mathrm{Ker}\, f'}{\rightarrowtail} & B' & \xrightarrow{f'} & A' \\
{\scriptstyle k}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle a}\downarrow \\
\mathrm{K}[f] & \underset{\mathrm{Ker}\, f}{\rightarrowtail} & B & \xrightarrow[f]{} & A
\end{array}
$$

such that f and f' are regular epimorphisms, if k and a are isomorphisms then b is an isomorphism.

1.1.3 Definition: [JMT2002] A **semi-abelian category** is a category which is pointed, Barr exact and Bourn protomodular, and has binary coproducts.

1.1.4 Example: All abelian categories are semi-abelian. The category Gp of groups is one of the leading examples of semi-abelian categories. More generally, any variety of Ω-groups is semi-abelian. This includes the categories of Lie algebras, Leibniz algebras, non-unital rings and (pre)crossed modules. Other examples are the category of Heyting semi-lattices, the dual of the category of pointed sets, and more generally the dual of the category of pointed objects in any topos.

A semi-abelian category has all finite colimits as well as finite limits. Many results relating to homological algebra that hold in abelian categories are also true in semi-abelian ones, but not quite all of them. Semi-abelian categories are not enriched in abelian groups (only in pointed sets), and finite products and coproducts do not coincide as in abelian categories. In fact, a semi-abelian category with isomorphic binary products and coproducts is abelian. While $K[f] = 0$ if and only if f is a monomorphism, the "dual" statement only holds in one direction: if f is a regular epimorphism, then its cokernel is zero, but not the other way around. As in abelian categories, every regular epimorphism is a cokernel (of its kernel), but not every monomorphism is a kernel. For example in the category of groups, not every subgroup is a normal subgroup. The weaker result holds that every *normal* monomorphism is the kernel of its cokernel.

1.1.5 Remark: As every regular epimorphism is a cokernel, we shall mostly talk only about regular epimorphisms and denote them by $B \longrightarrow\!\!\!\!\!\triangleright A$. We also call such a regular epimorphism an **extension (of A)**.

1.1.6 Definition (Image factorisation): [Bar1971] As a semi-abelian category is regular, any map $f \colon B \longrightarrow A$ in $\mathcal{A}$ can be factored into a regular epimorphism followed by a monomorphism.

$$
\begin{array}{ccc}
B & \xrightarrow{\;\;f\;\;} & A \\
& \searrow \quad \nearrow & \\
& I[f] & {\scriptstyle \mathrm{Im}\,f}
\end{array}
$$

We refer to both the object $I[f]$ and the map $\mathrm{Im}\,f$ as the **image** of f. This image factorisation is stable under pullback, as both regular epimorphisms and monomorphisms are. Notice that the regular epimorphism is a cokernel (as all regular epimorphisms are), but the monomorphism need not be a kernel, as it is in abelian categories. If $\mathrm{Im}\,f$ is a kernel, we call f a **proper** morphism. These play an important role in homological algebra in semi-abelian categories.

1.1.7 Definition (Exact sequences): A sequence of morphisms

$$
A \xrightarrow{\;\;f\;\;} B \xrightarrow{\;\;g\;\;} C
$$

is called **exact (at B)** if $\mathrm{Im}\,f = \mathrm{Ker}\,g$.

A sequence $0 \longrightarrow A \longrightarrow B$ is exact if and only if the map $A \longrightarrow B$ is a monomorphism, and $A \longrightarrow B \longrightarrow 0$ is exact if and only if $A \longrightarrow B$ is a regular epimorphism. Note that a non-proper map can never occur as the first map of an exact sequence, so in a long exact sequence all maps must be proper, except possibly the very last.

As mentioned earlier, there are many weaker settings that are part of the whole semi-abelian context. Later we will wish to consider an example which is not quite semi-abelian, as it does not have coproducts, and many of our results do not need coproducts. But instead of always stating the precise weakest setting in which a result holds, we will generally work in semi-abelian categories and sometimes highlight results which do not need the full strength of the definition. The monograph [BB2004] gives a very detailed description of the weaker settings, some of which we now mention.

Results from homological algebra

A pointed regular and protomodular category is also called **homological**, and many results of homological algebra hold already in this weaker setting. An example of a homological category which is not semi-abelian is the category of topological groups, which is not exact. Any homological category is **Mal'tsev**, which means that it has all finite limits and every internal reflexive relation is an equivalence relation [CLP1991, CKP1993, Bou1996]. Also, for any two objects A and B, the pair of morphisms

$$A \xrightarrow{(1_A,0)} A \times B \xleftarrow{(0,1_B)} B$$

is jointly strongly epic, which means that the category is **unital** [Bou1996, BB2004].

We will now give a collection of results that we will need repeatedly throughout the thesis, and will often use without referring to them.

1.1.8 Lemma (pullback cancellation): [Bou1991, BB2004] *In a homological category, given a commutative diagram*

$$
\begin{array}{ccc}
A' \longrightarrow & B' \longrightarrow & C' \\
\downarrow & \downarrow{\scriptstyle b} & \downarrow \\
A \longrightarrow & B \longrightarrow & C
\end{array}
$$

where b is a regular epimorphism, if the outer rectangle is a pullback, the left hand square is a pullback if and only if the right hand square is a pullback. $\square$

Notice that the direction "right hand square pullback implies left hand square pullback" holds in general, even when b is not a regular epimorphism, but the other direction uses protomodularity.

1.1.9 Lemma: [Bou1991] *In a protomodular category, pullbacks reflect monomorphisms.*
$\square$

The following classical result will be used later on to show that a particular morphism is regular epic.

1.1.10 Lemma: *Let $\mathcal{A}$ be a regular category. A map $y\colon Z_0 \longrightarrow Y$ factors through the image of a map $f\colon X \longrightarrow Y$ if and only if there is a regular epimorphism $z\colon Z \longrightarrow Z_0$ and a map $x\colon Z \longrightarrow X$ with $yz = fx$.* $\qquad\square$

1.1.11 Remark: Of course, if we can show that *every* map $y\colon Z_0 \longrightarrow Y$ factors through the image of a given map $f\colon X \longrightarrow Y$, this map f is a regular epimorphism.

The well-known Five Lemma is an easy consequence of the Short Five Lemma given in the definition of pointed protomodular categories above.

1.1.12 Lemma (Five Lemma): [Bor2004] *In a homological category, given a morphism between five-term exact sequences*

$$
\begin{array}{ccccccccc}
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\
\cong \downarrow a & & \cong \downarrow b & & \downarrow c & & d \downarrow \cong & & e \downarrow \cong \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E
\end{array}
$$

where the outer four morphisms a, b, d and e are isomorphisms, then c is also an isomorphism. $\qquad\square$

1.1.13 Lemma (3×3 Lemma): [Bou2001, BB2004] *In a homological category, given a commutative diagram*

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K'' & \rightarrowtail & B'' & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow b' & & \downarrow & & \\
0 & \longrightarrow & K' & \rightarrowtail & B' & \longrightarrow & A' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow b & & \downarrow & & \\
0 & \longrightarrow & K & \rightarrowtail & B & \longrightarrow & A & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

with short exact rows and $b \circ b' = 0$, if any two columns are short exact, then so is the third. $\qquad\square$

1.1.14 Lemma (Snake Lemma): [Bou2001, BB2004] *In a homological category, a commutative diagram with exact rows as the diagram of solid arrows below,*

where a, b and c are proper maps, gives rise to a six-term exact sequence

$$\mathrm{K}[a] \longrightarrow \mathrm{K}[b] \longrightarrow \mathrm{K}[c] \longrightarrow \mathrm{Q}[a] \longrightarrow \mathrm{Q}[b] \longrightarrow \mathrm{Q}[c]$$

which is natural in a, b and c. Moreover, if the first solid row is short exact, we can add a zero to the front of this six term exact sequence, and if the second solid row is short exact we can add a zero to the end. □

A consequence of the 3×3 Lemma which we shall be using is

1.1.15 Lemma (Noether's Third Isomorphism Theorem): [BB2004] *In a homological category, consider two normal subobjects $A \lhd C$ and $B \lhd C$, with $A \subseteq B$. Then*

- *A is a normal subobject of B,*

- *B/A is a normal subobject of C/A,*

- *the isomorphism $\frac{C/A}{B/A} \cong \frac{C}{B}$ holds.*

$$0 \longrightarrow \frac{B}{A} \rightarrowtail \frac{C}{A} \longrightarrow \frac{C}{B} \longrightarrow 0$$

□

1.1.16 Lemma: [Bou1991, Bou2001] *Given a morphism between short exact sequences in a homological category as follows,*

(1) the left hand square is a pullback if and only if a is a monomorphism,

(2) the right hand square is a pullback if and only if k is an isomorphism.

Proof. The implications from right to left are easy diagram chases which hold in general, but the opposite implication in (1) uses that $K[f] = 0$ if and only if f is a mono (which holds in a pointed protomodular category), while the one in (2) uses Lemma 1.1.8.   □

We will most often be using the easy implications of these two results, but give both directions here for completeness. There is a dual result to the easy implication of (2) which we shall also need.

1.1.17 Lemma: *Let $\mathcal{A}$ be a pointed category. Consider the diagram*

$$
\begin{array}{ccc}
B' \xrightarrow{f'} A' \longrightarrow Q[f'] \\
\downarrow \qquad \downarrow \qquad \downarrow \\
B \xrightarrow{f} A \longrightarrow Q[f]
\end{array}
$$

where $Q[f]$ and $Q[f']$ are the cokernels of f and f' respectively. If the left square is a pushout, then the induced map between the cokernels is an isomorphism.   □

An important concept later on in this thesis will be a **regular pushout** [Bou2003, CKP1993], which is a commutative square of regular epimorphisms where the comparison map to the pullback is also a regular epimorphism.
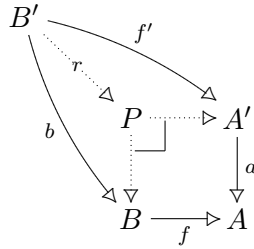
A regular pushout in a semi-abelian category is always a pushout; in fact, a commutative square of regular epimorphisms is a regular pushout if and only if it is a pushout [Bou2003, CKP1993]. The following lemma is a consequence of this.

1.1.18 Lemma: [JMT2002] *In a semi-abelian category, the direct image of a normal subobject along a regular epimorphism is again a normal subobject. That is, given a diagram as below where f is regular epimorphism, b is a normal monomorphism and $a \circ f'$ is the*

image factorisation of $f \circ b$,

$$
\begin{array}{ccc}
B' & \xrightarrow{\ f'\ } & A' \\
b\downarrow & & \downarrow a \\
B & \xrightarrow{\ f\ } & A
\end{array}
$$

then a is also a normal monomorphism. ◻

There is another definition of a regular pushout which is slightly more general than the one above. We will need it in this more general form in Chapter 3.

1.1.19 Definition (generalised regular pushout): [VdL2006, VdL2008] Let $\mathcal{A}$ be a semi-abelian category. A square in $\mathcal{A}$ with horizontal regular epimorphisms

$$
\begin{array}{ccc}
B' & \xrightarrow{\ f'\ } & A' \\
b\downarrow & & \downarrow a \\
B & \xrightarrow{\ f\ } & A
\end{array}
$$

is a **generalised regular pushout** when the comparison map $(b, f') \colon B' \longrightarrow B \times_A A'$ to the pullback $B \times_A A'$ of a along f is a regular epimorphism. (The maps b and a are *not* required to be regular epimorphisms.)

A generalised regular pushout is always a pushout, but a pushout need not be a generalised regular pushout, differing from the first definition of regular pushout above.

1.2 Abelian objects

Just as one has abelian groups in the category of groups, we can define abelian objects in any semi-abelian category. These will play an important role throughout the thesis.

1.2.1 Definition (Abelian objects): [Huq1968, Bor2004] An object A in a semi-abelian category is called an **abelian object** when there exists a (necessarily unique) morphism $m \colon A \times A \longrightarrow A$ satisfying

$$
\begin{array}{ccc}
 & A & \\
{\scriptstyle (1_A,0)}\downarrow & & \\
A \times A & \xrightarrow{\ m\ } & A \\
{\scriptstyle (0,1_A)}\uparrow & & \\
 & A &
\end{array}
$$

called the **multiplication** or **addition** of A (see also [Bou2002] and Section 1.5).

1.2.2 Remark: The multiplication m is unique because the pair $((1_A, 0), (0, 1_A))$ is jointly epic (see Section 1.1).

An abelian object in the category of groups is an abelian group. We have a similar result in any semi-abelian category, saying that the abelian objects are exactly the internal abelian groups.

1.2.3 Lemma: [Bor2004, Bou2000] *Let A be an object in a semi-abelian category. The following are equivalent:*

(1) A is an abelian object;

(2) the diagonal $\Delta \colon A \longrightarrow A \times A$ is a normal subobject;

(3) A carries an internal abelian group structure.

Moreover, the abelian group structure on A is necessarily unique. $\qquad\square$

Abelian objects in a semi-abelian category allow us to establish connections with abelian categories.

1.2.4 Lemma: [BG2002a, Bou2000, JMT2002, Bor2004] *Let $\mathcal{A}$ be a semi-abelian category, and denote the full subcategory of abelian objects by $\mathsf{Ab}\mathcal{A}$. Then*

- *$\mathsf{Ab}\mathcal{A}$ is an abelian category.*

- *$\mathcal{A}$ is abelian if and only if all objects in $\mathcal{A}$ are abelian objects.*

- *$\mathcal{A}$ is abelian if and only if every subobject in $\mathcal{A}$ is normal.*

- *If $\mathcal{A}$ is semi-abelian and $\mathcal{A}^{\mathrm{op}}$ is semi-abelian, then $\mathcal{A}$ is abelian.* $\qquad\square$

The full subcategory of abelian objects will play an important role throughout this thesis. The most important aspect of it is the reflector $\mathrm{ab} \colon \mathcal{A} \longrightarrow \mathsf{Ab}\mathcal{A}$.

1.2.5 Lemma: [BB2004] *Let $\mathcal{A}$ be a semi-abelian category. The inclusion $\mathsf{Ab}\mathcal{A} \longrightarrow \mathcal{A}$ admits a left adjoint. More explicitly, given an object A in $\mathcal{A}$, the coequaliser of the pair*

$$A \xrightarrow[\;(0,1_A)\;]{\;(1_A,0)\;} A \times A$$

*is the **abelianisation** $\mathrm{ab}\, A$ of A.*

Proof. Note that the construction of this abelianisation functor does need coequalisers.

In [BB2004], an object with a multiplication as in Definition 1.2.1 is called **commutative**, and this lemma is proved for the subcategory of those commutative objects. But in a homological category, every commutative object is abelian (meaning it is an internal abelian group), so we can still use this result. □

1.2.6 Example (Abelianisation functors): The abelianisation functor $\mathrm{ab}\colon \mathsf{Gp} \longrightarrow \mathsf{Ab}$ on groups takes a group G to $\mathrm{ab}\, G = G/[G,G]$, where $[G,G]$ is the commutator subgroup of G, the normal subgroup generated by elements of the form $aba^{-1}b^{-1}$. Similarly for $\mathrm{ab}\colon \mathsf{Lie}_K \longrightarrow \mathsf{AbLie}_K$, a Lie algebra $\mathfrak{g}$ is sent to $\mathrm{ab}\,\mathfrak{g} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, where $[\mathfrak{g},\mathfrak{g}]$ now denotes the ideal generated by elements of the form $[x,y]$, the Lie bracket of x and y.

The following result can be a useful tool for determining whether an object is abelian.

1.2.7 Lemma: [Bou2005, Theorem 2.1] *Let $\mathcal{A}$ be a semi-abelian category, and consider two normal subobjects $U \lhd B$ and $V \lhd B$ of an object B in $\mathcal{A}$. Let $f\colon B \longrightarrow A$ be a regular epimorphism. Then the quotient*

$$\frac{f(U) \cap f(V)}{f(U \cap V)}$$

is always an abelian object. □

1.3 Homology in semi-abelian categories

The first step towards any homology theory is usually homology of chain complexes, which we introduce in this section. We also give the basic setting of semi-abelian homology.

As usual a **chain complex** is a sequence of maps $(d_n\colon C_n \longrightarrow C_{n-1})_{n\in\mathbb{Z}}$ satisfying $d_{n-1}\circ d_n = 0$ for all n. A chain complex is called **proper** when all the maps d_n are proper maps.

1.3.1 Definition (Homology of chain complexes): [EVdL2004b] Let C be a proper chain complex in a semi-abelian category. The nth homology object $\mathrm{H}_n C$ is the cokernel of $d'_{n+1}\colon C_{n+1} \longrightarrow \mathrm{Ker}\,[d_n]$, the factorisation of $d_{n+1}\colon C_{n+1} \longrightarrow C_n$ over $\ker d_n$.

For a proper chain complex this is isomorphic to the dually defined K_nC, the kernel of the factorisation of d_n over the cokernel of d_{n+1}.

It is easy to see from the definition that C is exact at C_n if and only if $H_nC = 0$. Thus homology detects failure of exactness as usual. When talking about homology, we only consider proper chain complexes, because otherwise this property is false. Consider, for instance, the following example in the category of groups. Recall that in Gp a monomorphism is normal if and only if it is the inclusion of a normal subgroup. Define a chain complex by taking d_1 to be the inclusion of A_4 into A_5, and all other objects to be zero. Since A_5 is simple, d_1 is not proper, and all objects H_nC are zero, but clearly C is not exact at C_0.

To calculate homology of an ordinary object A in $\mathcal{A}$, the most common way is to form some resolution of A which gives rise to a chain complex, of which we can then take homology. But there are also other ways of calculating the homology of an object in a semi-abelian category. This is the main theme of this thesis.

Semi-abelian homology is based on a semi-abelian category $\mathcal{A}$ with a Birkhoff subcategory $\mathcal{B}$ and a reflector $I\colon \mathcal{A} \longrightarrow \mathcal{B}$. It calculates the derived functors of the reflector I.

1.3.2 Definition (Birkhoff subcategory): [JK1994] By a **reflective subcategory** $\mathcal{B}$ of $\mathcal{A}$ we mean a full and replete subcategory for which the inclusion functor $\subseteq\colon \mathcal{B} \longrightarrow \mathcal{A}$ admits a left adjoint $I\colon \mathcal{A} \longrightarrow \mathcal{B}$, called the **reflector**, or sometimes the reflection.

A **Birkhoff subcategory** $\mathcal{B}$ of a Barr-exact category $\mathcal{A}$ is a reflective subcategory which is closed under subobjects and regular quotients.

To understand this choice of name, recall the classical Birkhoff Theorem of universal algebra, also called Birkhoff's HSP theorem: a variety of algebras is a class of algebraic structures of a given signature satisfying a given set of identities, or equivalently, via Birkhoff's theorem, a class of algebraic structures of the same signature which is closed under taking homomorphic images, subalgebras and products. Notice that the inclusion functor of a reflective subcategory is a right adjoint and so preserves limits, so that Birkhoff subcategories as we have defined them are automatically closed under products as well.

1.3.3 Example: It is immediately clear from the explanation above that a Birkhoff subcategory of a semi-abelian variety of universal algebras is the same as a subvariety. Examples are the category Ab of abelian groups inside Gp, the category of abelian Lie algebras inside Lie algebras, and the category of crossed modules inside precrossed modules. For any semi-abelian category $\mathcal{A}$, the full subcategory of abelian objects $\mathsf{Ab}\mathcal{A}$ forms a Birkhoff subcategory with reflector $\mathrm{ab}\colon \mathcal{A} \longrightarrow \mathsf{Ab}\mathcal{A}$ as in Lemma 1.2.5 (see [Gra2001]). The first two examples given above are of this form (see Example 1.2.6), but the last is not. There

are many more examples where the Birkhoff subcategory is not the subcategory of abelian objects; we will mention three.

A group G is **nilpotent of class** m when $\mathrm{LC}_m G = \gamma_{m+1} G = 0$ (and $\mathrm{LC}_{m-1} G \neq 0$), where $\mathrm{LC}_m G$ is the $(m+1)$st term in the **descending central series** of G

$$\mathrm{LC}_1 G = \gamma_2 G = [G, G]$$
$$\mathrm{LC}_2 G = \gamma_3 G = [[G, G], G]$$
$$\cdots = \cdots = \cdots$$
$$\mathrm{LC}_m G = \gamma_{m+1} G = [\mathrm{LC}_{m-1} G, G].$$

(The unconventional notation with shifted numbering is used to keep the index of the reflector and the kernel of the reflector consistent, see below. We write 0 for the one-element group, as it is the zero-object of the category of groups, but we will later write 1 for the identity element inside a non-abelian group.) The subcategory Nil_m of all nilpotent groups of class at most m is a Birkhoff subcategory of Gp with reflector

$$\mathsf{Gp} \xrightarrow{\quad \mathrm{nil}_m \quad} \mathsf{Nil}_m$$
$$G \longmapsto G/\mathrm{LC}_m G$$

for any m. Similarly, a group G is **solvable of derived length** m when ($D_{m-1} G \neq 0$ and) $D_m G = 0$, the mth term in the **derived series** of G

$$D_1 G = [G, G]$$
$$D_2 G = [[G, G], [G, G]]$$
$$\cdots = \cdots$$
$$D_m G = [D_{m-1} G, D_{m-1} G].$$

The subcategory Sol_m of all solvable groups of class at most m is also a Birkhoff subcategory of Gp, with reflector

$$\mathsf{Gp} \xrightarrow{\quad \mathrm{sol}_m \quad} \mathsf{Sol}_m$$
$$G \longmapsto G/D_m G$$

for any m.

Another example is that of Lie algebras inside Leibniz algebras. A Leibniz algebra is a vector space $\mathfrak{g}$ over a field K with a bilinear **bracket** $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the **Leibniz identity**

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

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When $[x, x] = 0$ for all $x \in \mathfrak{g}$, and thus $[x, y] = -[y, x]$ by bilinearity, this Leibniz identity becomes the **Jacobi identity**

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

and $\mathfrak{g}$ is a Lie algebra. Writing $\mathfrak{g}^{\mathrm{Ann}}$ for the two-sided ideal generated by all elements of the form $[x, x]$, we have a reflector

$$\mathsf{Leib}_K \xrightarrow{\quad \text{lie} \quad} \mathsf{Lie}_K$$
$$\mathfrak{g} \longmapsto \mathfrak{g}/\mathfrak{g}^{\mathrm{Ann}}$$

and Lie_K is a Birkhoff subcategory of Leib_K.

We usually write the unit of the reflection as $\eta\colon 1_{\mathcal{A}} \longrightarrow I$, leaving out the inclusion functor. An alternative characterisation of a Birkhoff subcategory is the following.

1.3.4 Lemma: [JK1994] *A reflective subcategory $\mathcal{B}$ of an exact category $\mathcal{A}$ is a Birkhoff subcategory if and only if the following diagram is a regular pushout for every extension f.*

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & A \\ {\scriptstyle \eta_B}\big\downarrow & & \big\downarrow{\scriptstyle \eta_A} \\ IB & \xrightarrow{\ If\ } & IA \end{array}$$

Proof. In fact any reflective subcategory $\mathcal{B}$ is closed under subobjects if and only if each η_A is a regular epimorphism, and it is also closed under regular quotients if and only if the above square is a regular pushout. $\square$

1.3.5 Remark: Notice that this immediately implies that the functor I preserves extensions.

Given this data, there are several different ways to compute semi-abelian homology, which coincide in situations where they are all defined. The three viewpoints discussed in this thesis are

(1) comonadic homology in a semi-abelian monadic category, in Chapters 2 and 3;

(2) homology via Hopf formulae in a semi-abelian category with enough projectives, in Chapter 4;

(3) homology via Kan extensions or limits in a semi-abelian category, in Chapter 5.

1.3.6 Example: Many examples of Birkhoff subcategories give well-known homology theories in these three different ways above. For example, the subcategory Ab of abelian groups in Gp gives rise to integral group homology, and the subcategory AbLie_K of abelian Lie algebras inside Lie_K gives rise to the Chevalley-Eilenberg homology of Lie algebras.

1.4 Categorical Galois theory

A Birkhoff subcategory $\mathcal{B}$ of a semi-abelian category $\mathcal{A}$ with its reflector $I\colon \mathcal{A} \longrightarrow \mathcal{B}$ gives rise to a Galois structure in the sense of Janelidze [Jan1991]. We will not go into the whole theory of categorical Galois structures and its connection to the Galois theory of field extensions; for this we refer the reader to the monograph [BJ2001] and the papers [Jan1991, JK1994]. We will just recall as much of the theory as we will need.

1.4.1 Definition: (see [Jan1991]) A **Galois structure** $\Gamma = (\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{Z}, I, H)$ consists of two categories $\mathcal{A}$ and $\mathcal{B}$, an adjunction

$$\mathcal{A} \underset{H}{\overset{I}{\underset{\perp}{\rightleftarrows}}} \mathcal{B},$$

and classes $\mathcal{E}$ and $\mathcal{Z}$ of morphisms in $\mathcal{A}$ and $\mathcal{B}$ respectively, such that:

(1) $\mathcal{A}$ has pullbacks along arrows in $\mathcal{E}$,

(2) $\mathcal{E}$ and $\mathcal{Z}$ contain all isomorphisms, are closed under composition and are pullback-stable,

(3) $I(\mathcal{E}) \subset \mathcal{Z}$,

(4) $H(\mathcal{Z}) \subset \mathcal{E}$,

(5) the counit ε is an isomorphism,

(6) each A-component η_A of the unit η belongs to $\mathcal{E}$.

Maps in $\mathcal{E}$ and $\mathcal{Z}$ are called extensions, and we denote them by $B \longrightarrow\!\!\!\!\!\triangleright A$.

1.4.2 Example (Galois structures): Let $\mathcal{A}$ be the category Gp of groups, $\mathcal{B}$ the subcategory Ab of abelian groups, H the inclusion functor and I the abelianisation functor $\mathrm{ab}\colon G \longmapsto G/[G,G]$. Then choosing $\mathcal{E}$ and $\mathcal{Z}$ to be the surjective group homomorphisms in Gp respectively in Ab gives a Galois structure $(\mathsf{Gp}, \mathsf{Ab}, \mathcal{E}, \mathcal{Z}, \mathrm{ab}, \subseteq)$.

A related Galois structure exists on any semi-abelian category $\mathcal{A}$. We take $\mathcal{B}$ to be the subcategory $\mathsf{Ab}\mathcal{A}$ of abelian objects in $\mathcal{A}$, H again the inclusion functor and I the abelianisation functor, and choose for $\mathcal{E}$ and $\mathcal{Z}$ all regular epimorphisms in $\mathcal{A}$ and in $\mathsf{Ab}\mathcal{A}$ respectively. This gives a Galois structure $(\mathcal{A}, \mathsf{Ab}\mathcal{A}, \mathcal{E}, \mathcal{Z}, \mathrm{ab}, \subseteq)$.

An example of a Galois structure not coming from abelianisation is given by the category of Lie algebras inside that of Leibniz algebras, as explained in Example 1.3.3. Taking $\mathcal{E}$ and $\mathcal{Z}$ as the classes of regular epimorphisms in Leib_K and Lie_K respectively, we have a Galois structure $(\mathsf{Leib}_K, \mathsf{Lie}_K, \mathcal{E}, \mathcal{Z}, \mathrm{lie}, \subseteq)$ for a field K. Similarly, we have the structures $(\mathsf{Gp}, \mathsf{Nil}_m, \mathcal{E}, \mathcal{Z}, \mathrm{nil}_m, \subseteq)$ and $(\mathsf{Gp}, \mathsf{Sol}_m, \mathcal{E}, \mathcal{Z}, \mathrm{sol}_m, \subseteq)$ for a fixed $m \in \mathbb{N}$. Another Galois structure not using abelianisation is $(\mathsf{PXMod}, \mathsf{XMod}, \mathcal{E}, \mathcal{Z}, \mathrm{xmod}, \subseteq)$, where xmod is the reflector from precrossed modules into crossed modules, and $\mathcal{E}$ and $\mathcal{Z}$ are the appropriate classes of regular epimorphisms.

In a Galois structure Γ, we define several different sorts of extensions.

1.4.3 Definition: [Jan1991, Jan1990] Let Γ be a Galois structure and $f\colon B \longrightarrow A$ an extension in $\mathcal{E}$. We say that f is

(1) a **trivial extension** (or f is **trivial**) when the square below is a pullback,

$$
\begin{array}{ccc}
B & \xrightarrow{\ f\ } & A \\
\eta_B \downarrow & & \downarrow \eta_A \\
IB & \xrightarrow{\ If\ } & IA
\end{array}
$$

(2) a **central extension** when there exists a map $a\colon A' \longrightarrow A$ in $\mathcal{E}$ such that the pullback a^*f is trivial,

$$
\begin{array}{ccc}
A' \times_A B & \longrightarrow & B \\
a^*f \downarrow & & \downarrow f \\
A' & \xrightarrow{\ a\ } & A
\end{array}
$$

(3) a **normal extension** when the first projection $\pi_1\colon \mathrm{R}[f] \longrightarrow B$ of the kernel pair of f (or equivalently, the second projection π_2) is a trivial extension.

Clearly every normal extension is central. But when the category $\mathcal{A}$ is Mal'tsev, the converse is also true: every central extension is normal (see Theorem 4.8 in [JK1994]). Thus f is central with respect to Γ if and only if either of the two commuting squares in the following diagram is a pullback.

$$
\begin{array}{ccccc}
\mathrm{R}[f] & \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} & B & \xrightarrow{\ f\ } & A \\
\eta_{\mathrm{R}[f]} \downarrow & & \downarrow \eta_B & & \\
I\mathrm{R}[f] & \underset{I\pi_2}{\overset{I\pi_1}{\rightrightarrows}} & IB & &
\end{array}
$$

1.4.4 Example (central extensions): When $\mathcal{A}$ is the category of groups, $\mathcal{B}$ is the Birkhoff subcategory of abelian groups, and $\mathcal{E}$ is the class of all surjections, the central extensions are those whose kernel lies in the centre of the group. The trivial extensions are exactly those surjections $B \longrightarrow A$ where the restriction to the commutator subgroups $[B, B] \longrightarrow [A, A]$ is an isomorphism.

$$\begin{array}{ccc}
[B,B] \rightarrowtail & B \xrightarrow{\;\eta_B\;} & \text{ab}\, B \\
\downarrow{\scriptstyle \cong} & \downarrow{\scriptstyle f} & \downarrow{\scriptstyle \text{ab}\, f} \\
[A,A] \rightarrowtail & A \xrightarrow{\;\eta_A\;} & \text{ab}\, A
\end{array}$$

Similarly, for $\mathcal{A} = \mathsf{Lie}_K$ and $\mathcal{B} = \mathsf{AbLie}_K$, the central extensions are the classical central extensions of Lie algebras, that is, the kernel $\mathrm{K}[f]$ of $f\colon \mathfrak{b} \longrightarrow \mathfrak{a}$ lies in the centre of $\mathfrak{b}$:

$$\mathrm{K}[f] \subseteq Z\mathfrak{b} = \{z \in \mathfrak{b} \mid [z, b] = 0 \text{ for all } b \in \mathfrak{b}\}$$

However, when $\mathcal{A} = \mathsf{Leib}_K$ and $\mathcal{B} = \mathsf{Lie}_K$ for a field K with $\mathrm{char}\, K \neq 2$, the central extensions are those extensions whose kernel lies in the **Lie-centre**

$$Z_{\mathsf{Lie}}(\mathfrak{b}) = \{z \in \mathfrak{b} \mid [b, z] = -[z, b] \text{ for all } b \in \mathfrak{b}\}$$

of $\mathfrak{b}$ (for a proof, see e.g. [CVdL2009]).

A similar result holds for $\mathcal{A} = \mathsf{Gp}$ and $\mathcal{B} = \mathsf{Nil}_m$ or $\mathcal{B} = \mathsf{Sol}_m$ for a fixed $m \in \mathbb{N}$. Write $l_m(b_1, b_2, \ldots, b_{m+1})$ for an element $[[\cdots [b_1, b_2], b_3], \ldots], b_{m+1}] \in \mathrm{LC}_m B$, and similarly $d_m(b_1, b_2, \ldots, b_{2^m}) = [d_{m-1}(b_1, \ldots, b_{2^{m-1}}), d_{m-1}(b_{2^{m-1}+1}, \ldots, d_{2^m})]$ for an element of $D_m B$, thus for example $d_2(b_1, b_2, b_3, b_4) = [[b_1, b_2], [b_3, b_4]]$. It is clear that l_m and d_m give the identity 1 of B whenever any entry is 1. We also write $l_m(K, B, \ldots, B)$ and $d_m(K, B, \ldots, B)$ when K is a normal subgroup of B, for example $l_2(K, B, B) = [[K, B], B]$. Now define the m-**nil-centre** of B by

$$Z_{\mathsf{Nil}_m}(B) = \{z \in B \mid l_m(z, b_2, \ldots, b_{m+1}) = 1 \;\forall\; b_i \in B\}.$$

Notice the nesting

$$ZB = Z_{\mathsf{Nil}_1}(B) \subseteq Z_{\mathsf{Nil}_2}(B) \subseteq \cdots \subseteq Z_{\mathsf{Nil}_m}(B) \subseteq \cdots$$

starting with the usual centre ZB of B. In fact, these subgroups exactly form the **upper central series**

$$0 = Z_0 B \triangleleft Z_1 B \triangleleft \cdots \triangleleft Z_m B \triangleleft \cdots$$

of B defined by $Z_{i+1}B = \{z \in B \mid [z, b] \in Z_i B \;\forall\; b \in B\}$ with $Z_1 B = ZB$. Notice also that $\mathrm{K}[f] \subseteq Z_{\mathsf{Nil}_m} B$ if and only if $l_m(\mathrm{K}[f], B, \ldots, B) = 0$. A group extension $f\colon B \longrightarrow A$ is central with respect to Nil_m if and only if $l_m(\mathrm{K}[f], B, \ldots, B) = 0$, and so if and only

if $\mathrm{K}[f] \subseteq Z_{\mathsf{Nil}_m} B$. This was proved by Everaert and Gran in [EG2006, Corollary 3.5]. Similarly a group extension $f\colon B \longrightarrow A$ is central with respect to Sol_m if and only if

$$\mathrm{K}[f] \subseteq Z_{\mathsf{Sol}_m}(B) = \{z \in B \mid d_m(z, b_2, \ldots, b_{2m}) \forall \, b_i \in B\}$$

if and only if $d_m(\mathrm{K}[f], B, \ldots, B) = 0$ [EG2006, Corollary 4.3]. For better understanding, we give a sketch of our own proof of the result for nilpotent groups. The case of solvable groups works analogously.

Consider the following diagram (here $J = \mathrm{LC}_m$):

$$\begin{array}{ccccc}
J\mathrm{R}[f] \cap \mathrm{K}[\pi_2] & \rightarrowtail & J\mathrm{R}[f] & \overset{\cong}{\longrightarrow} & JB \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{K}[\pi_2] & \rightarrowtail & \mathrm{R}[f] & \overset{\pi_2}{\longrightarrow} & B \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
& & I\mathrm{R}[f] & \longrightarrow & IB
\end{array}$$

By definition, f is central if and only if $J\mathrm{R}[f] \cong JB$, and this isomorphism is induced by the diagonal $\Delta\colon B \longrightarrow \mathrm{R}[f]$. This is the case if and only if $J\mathrm{R}[f] \cap \mathrm{K}[\pi_2] = 0$ (or equivalently if $J\mathrm{R}[f] \cap \mathrm{K}[\pi_1] = 0$). Recall that we write 0 for the one-element group but 1 for the identity element inside a non-abelian group. So assume first that $J\mathrm{R}[f] \cong JB$, and let $k \in \mathrm{K}[f]$. Then $(k, 1)$ and (b_i, b_i) are elements of $\mathrm{R}[f]$, for any $b_i \in B$. So we have

$$l_m\big((k, 1), (b_2, b_2), \ldots, (b_{m+1}, b_{m+1})\big) = \big(l_m(k, b_2 \ldots, b_{m+1}), 1\big) \in \mathrm{LC}_m \mathrm{R}[f] = J\mathrm{R}[f].$$

But as $J\mathrm{R}[f] \cong JB$ via the diagonal, this means

$$l_m(k, b_2, \ldots, b_{m+1}) = 1$$

for any $b_i \in B$, as claimed.

Conversely, suppose $\mathrm{K}[f] \subseteq Z_{\mathsf{Nil}_m} B$, and let $\big(l_m(b_1, \ldots, b_{m+1}), l_m(k_1, \ldots, k_{m+1})\big)$ be an element of $J\mathrm{R}[f] \cap \mathrm{K}[\pi_2]$. This means $l_m(k_1, \ldots, k_{m+1}) = 1$ and each $b_i \cdot k_i^{-1} \in \mathrm{K}[f]$. If we can show that $l_m(b_1, \ldots, b_{m+1}) = 1$, then $J\mathrm{R}[f] \cap \mathrm{K}[\pi_2] = 0$ and f is central. This can be done by tedious calculations using $l_m(k_1, \ldots, k_{m+1}) = 1$ and $l_m(b_i \cdot k_i^{-1}, g_2, \ldots, g_{m+1}) = 1$ for suitable $g_j \in B$, as well as results such as $[[B, B], \mathrm{K}[f]] \subseteq [[\mathrm{K}[f], B], B]$ and correspondingly $l_m(B, \ldots, B, \mathrm{K}[f]) \subseteq l_m(\mathrm{K}[f], B, \ldots, B)$ for any m (see [EG2006, Lemma 3.1]). For higher m these calculations become too long to be done by hand, but for $m = 2$ and $m = 3$ I have checked all the details.

Every trivial extension is central. But some special central extensions also turn out to be trivial, as the next lemma shows. Here a split central extension is just a central extension which is split as an epimorphism.

1.4.5 Lemma: [JK1994, Theorem 4.8] *Let $\mathcal{A}$ be a semi-abelian category with a Galois structure Γ where $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$. Then every split central extension is trivial.*

Proof. In fact every split central extension is trivial if and only if every central extension is normal (Theorem 4.7 in [JK1994]). □

1.5 Central extensions in the context of abelianisation

In Section 1.4 we have met central extensions with respect to a Galois structure. These central extensions will become very important later on in the thesis, in Chapter 5. In the special case of abelianisation, they coincide with other definitions of central extensions which are more algebraic. Here we introduce two of these algebraic definitions, one using *centrality of congruences* introduced by Smith [Smi1976] and generalised in [Ped1995, CPP1992], and the other using the notion of *central arrows* first defined by Huq [Huq1968]. These two definitions of central extensions coincide in pointed protomodular categories (see [BG2002b]), and we will also prove that they do indeed coincide with the Galois-theoretic central extension for abelianisation. Then we can apply results about these algebraic central extensions also to the Galois-theoretic ones we will be using. Even when the Birkhoff subcategory is not that of abelian objects, central morphisms in the sense of Huq appear in the homology theory, as can be seen in Section 4.4.

Huq centrality

We first introduce the concept of cooperating morphisms, using the terminology due to Bourn [Bou2002].

1.5.1 Definition: Two morphisms with common codomain $f \colon B \longrightarrow A$ and $f' \colon B' \longrightarrow A$ are said to **cooperate** when there is a morphism $\phi_{f,f'} \colon B \times B' \longrightarrow A$ such that the following diagram commutes.

$$
\begin{array}{ccc}
B & & \\
{\scriptstyle (1_B,0)}\big\downarrow & \searrow^{f} & \\
B \times B' & \xrightarrow{\ \phi_{f,f'}\ } & A \\
{\scriptstyle (0,1_{B'})}\big\uparrow & \nearrow_{f'} & \\
B' & &
\end{array}
$$

The morphism $\phi_{f,f'}$ is called the **cooperator** of f and f'. We sometimes write ϕ instead of $\phi_{f,f'}$ if the context makes clear which maps are meant.

In Huq's terminology f and f' **commute**.

1.5.2 Remark: Notice that as the pair of morphisms into $A \times A'$ is jointly epic (see Section 1.1), the cooperator is unique as soon as it exists. So for any pair of morphisms f and f', having a cooperator is a property, not a structure.

1.5.3 Example: In the category of groups, two subgroups $H \rightarrowtail G$ and $K \rightarrowtail G$ cooperate precisely when they commute. More generally, two group homomorphisms $f \colon H \longrightarrow G$ and $h \colon K \longrightarrow G$ cooperate when their images in G commute, that is, for all $h \in H$ and $k \in K$, we have $f(h)g(k) = g(k)f(h)$ in G.

If f and g have a cooperator $\phi \colon H \times K \longrightarrow G$, then for all $h \in H$ and $k \in K$ we have

$$\phi(h,k) = \phi((h,1) \cdot (1,k)) = \phi(h,1) \cdot \phi(1,k) = f(h) \cdot g(k)$$

and similarly $\phi(h,k) = \phi((1,k) \cdot (h,1)) = g(k) \cdot f(h)$. Conversely, if the images of f and f' commute, we can define a cooperator ϕ by $\phi(h,k) = f(h) \cdot g(k)$. This becomes a group homomorphism precisely because the images of f and g commute.

From this we define the following notion of central morphism, and that of a central extension.

1.5.4 Definition (Huq central): A morphism $f \colon B \longrightarrow A$ is called **central (in the sense of Huq)** when f and 1_A cooperate. An extension $f \colon B \relbar\joinrel\twoheadrightarrow A$ is called a **central extension** when its kernel is central in the sense of Huq.

1.5.5 Example: When $\mathcal{A}$ is the category of groups, central extensions defined as above coincide with the traditional central extensions, extensions where the kernel lies inside the centre of the group. In the categories Lie_K of Lie algebras or Leib_K of Leibniz algebras, central extensions in this sense are also those $f \colon \mathfrak{b} \longrightarrow \mathfrak{a}$ where the kernel $\mathrm{K}[f]$ lies in the centre $Z\mathfrak{b} = \{z \in \mathfrak{b} \mid [z,b] = 0 \text{ for all } b \in \mathfrak{b}\}$ of $\mathfrak{b}$. Notice that the Huq central extensions in Leib_K are *not* those where $\mathrm{K}[f]$ lies in the *Lie* centre of $\mathfrak{b}$, as seen in Example 1.4.4. As we will see in Proposition 1.5.15, the central extensions in the sense of Definition 1.5.4 coincide with Galois central extensions with respect to abelianisation. As the abelianisation of a Leibniz algebra is not just a Lie algebra but already an abelian Lie algebra, we get the same characterisation of central extensions in Leib_K as in Lie_K in this case.

1.5.6 Remark: Notice the relation of central morphisms to abelian objects as defined in 1.2.1: an object A is abelian when the identity 1_A is central.

For more results about cooperating and central morphisms see for example Sections 1.3 and 1.4 of [BB2004]. In particular, we will use the following two results:

1.5.7 Lemma: [BB2004, Corollary 1.3.8] *Consider the following commutative diagram*

$$
\begin{array}{ccccc}
V & \xrightarrow{\ v\ } & U & \xleftarrow{\ v'\ } & V' \\
\downarrow & & \downarrow u & & \downarrow \\
B & \xrightarrow{\ f\ } & A & \xleftarrow{\ f'\ } & B'
\end{array}
$$

where u is a monomorphism. If f and f' cooperate, then v and v' cooperate as well. $\qquad \square$

1.5.8 Lemma: [Bou2002], also [BB2004, Proposition 1.3.20] *If f is a central morphism, then any morphism of the form $f \circ g$ is central.* $\qquad \square$

Centrality of congruences

One can also define an equivalence relation to be central. This will have a close connection to the central extensions above. We first introduce our notation for equivalence relations.

1.5.9 Notation: We write an equivalence relation $R \xrightarrow{\ r\ } B \times B$ on B as $R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} B$ or simply (R, π_1, π_2), and denote the subdiagonal which is induced by reflexivity by $d_R \colon B \longrightarrow R$; that is, $\pi_1 d_R = \pi_2 d_R = 1_B$.

$$
R \underset{\pi_2}{\overset{\pi_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\underset{d_R}{\longleftarrow}}}}} B
$$

As we are in an exact category, every equivalence relation is the kernel pair of its coequaliser (see Definition 1.1.2). Therefore we will often write an equivalence relation (R, π_1, π_2) on an object B as $(\mathrm{R}[f], \pi_1, \pi_2)$, where f is a regular epimorphism (the coequaliser of π_1 and π_2).

To any equivalence relation $(\mathrm{R}[f], \pi_1, \pi_2)$ on an object B, we can associate a normal subobject of B by taking the composite

$$
\mathrm{K}[\pi_1] \xrightarrow{\ \ker \pi_1\ } \mathrm{R}[f] \xrightarrow{\ \pi_2\ } B.
$$

This composite coincides with the kernel of f and so is indeed a normal subobject of B.

$$
\begin{array}{ccccc}
\mathrm{K}[f] = \mathrm{K}[\pi_1] & \xrightarrow{\ker\pi_1} & \mathrm{R}[f] & \xrightarrow{\pi_2} & B \\
\downarrow & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f} \\
0 & \longrightarrow & B & \xrightarrow{f} & A
\end{array}
$$

There is another notion of a normal subobject of B, defined via equivalence relations.

1.5.10 Definition: [Bou2000] An arrow $k\colon K \longrightarrow B$ is said to be **normal to an equivalence relation R on B** when

- $k^{-1}(R)$ is the largest equivalence relation $(K \times K, \pi_1, \pi_2)$ on K,

- there is a map $\overline{k}\colon K \times K \longrightarrow R$ making the diagram

$$
\begin{array}{ccc}
K \times K & \xdashrightarrow{\ \overline{k}\ } & R \\
{\scriptstyle \pi_1}\downdownarrows{\scriptstyle \pi_2} & & {\scriptstyle \pi_1}\downdownarrows{\scriptstyle \pi_2} \\
K & \xrightarrow{k} & B
\end{array}
$$

commute, and moreover both of the commutative squares are pullbacks. That is, the induced map $(K \times K, \pi_1, \pi_2) \longrightarrow (R, \pi_1, \pi_2)$ in the category $\mathsf{Eq}\mathcal{A}$ of internal equivalence relations in $\mathcal{A}$ is a discrete fibration.

Such an arrow k is necessarily a monomorphism, and when $\mathcal{A}$ is protomodular, k can be normal to at most one equivalence relation, making normality a property rather than a structure. Every kernel is normal, and in a semi-abelian category, normal monomorphisms in this sense and kernels coincide. As we will only be working in this easier situation, we can read *normal mono* as *kernel* as we have been so far.

The association above gives a bijection between the normal subobjects of B and the equivalences on B. For a given equivalence relation R on B, we call the subobject defined above the **associated normal subobject** or **normalisation** k_R. This bijection already exists in the context of pointed protomodular categories (see [Bou2000]). The associated normal subobject to R can be viewed as the equivalence class of zero under R.

1.5.11 Definition (Smith central): [Ped1995, CPP1992] Two equivalence relations R and S on an object B **centralise (in the sense of Smith)** when there is a double

equivalence relation C on R and S such that any commutative square in the diagram

$$C \underset{p_2}{\overset{p_1}{\rightrightarrows}} S$$

is a pullback. When such a C exists, we call it the centralising double relation on R and S.

An equivalence relation R on B is called **central** when R and $B \times B$ (the largest equivalence relation on B) centralise.

Some references say R and S are **connected** when they centralise. This terminology appears for example in [BG2002b].

There is a very useful connection between central equivalence relations in this sense and central morphisms in the sense of Huq.

1.5.12 Proposition: [GVdL2008b, Proposition 2.2] *Let $\mathcal{A}$ be a pointed protomodular category. An equivalence relation R in $\mathcal{A}$ is central if and only if its associated normal subobject k_R is central in the sense of Huq.* $\qquad\square$

It follows immediately that an extension $f \colon B \longrightarrow A$ is a central extension if and only if its kernel pair is central in the sense of Smith.

Coincidence of algebraic and Galois-theoretic definitions

Another useful characterisation of a central equivalence relation is given by the subdiagonal expressing the reflexivity of the relation.

1.5.13 Proposition: *Let B be an object in a semi-abelian category. The equivalence relation (R, π_1, π_2) on B is central if and only if its subdiagonal $d_R \colon B \longrightarrow R$ is a kernel.*

Proof. R is central if and only if it centralises with $B \times B$, the largest equivalence relation on B. The normal subobject associated to $B \times B$ is of course the identity on B. Then Theorem 5.2 in [BG2002b] tells us that R and $B \times B$ are connected (or equivalently, centralise) if and only if the composite $B \overset{k_{B \times B}}{=\!=\!=} B \overset{d_R}{\longrightarrow} R$ is normal, i.e., is a kernel. $\qquad\square$

We will now compare these two algebraic notions of central extensions to the Galois-theoretic one arising from abelianisation. So let $\mathcal{A}$ be a semi-abelian category with subcategory $\mathsf{Ab}\mathcal{A}$ of abelian objects, let $\mathrm{ab} \colon \mathcal{A} \longrightarrow \mathsf{Ab}\mathcal{A}$ be the abelianisation functor and let $\mathcal{E}$ and $\mathcal{Z}$ denote the regular epimorphisms in $\mathcal{A}$ and $\mathsf{Ab}\mathcal{A}$ respectively.

1.5.14 Proposition: [BG2002a, Proposition 3.1] *An extension $f\colon B \longrightarrow A$ is central with respect to the Galois structure $\Gamma = (\mathcal{A}, \mathsf{Ab}\mathcal{A}, \mathcal{E}, \mathcal{Z}, \mathrm{ab}, \subseteq)$ if and only if the subdiagonal $d_{\mathrm{R}[f]}$ of the kernel pair of f is a kernel.*

$$\mathrm{R}[f] \underset{\pi_2}{\overset{\pi_1}{\underset{\Longrightarrow}{\overset{\Longrightarrow}{\leftarrow d}}}} B \xrightarrow{\;f\;} A$$

$\square$

This enables us to compare the two notions of centrality.

1.5.15 Proposition: *In a semi-abelian category $\mathcal{A}$, an extension $f\colon B \longrightarrow A$ is central with respect to $\Gamma = (\mathcal{A}, \mathsf{Ab}\mathcal{A}, \mathcal{E}, \mathcal{Z}, \mathrm{ab}, \subseteq)$ if and only if it is a central extension in the sense Definition 1.5.4.*

Proof. This follows directly from Propositions 1.5.12, 1.5.13 and 1.5.14. $\square$

Central extensions in this sense have a useful property which we will need in Chapter 6.

1.5.16 Proposition: [GVdL2008b, Proposition 2.3] *Let $\mathcal{A}$ be a semi-abelian category and let $f\colon B \longrightarrow A$ be a central extension with respect to abelianisation. Every subobject of the kernel $\mathrm{Ker}\, f\colon K[f] \longrightarrow B$ of f is normal in B.* $\square$

Chapter 2

Comonadic Homology

In this chapter, we introduce the first viewpoint on semi-abelian homology: comonadic homology. This works in a slightly more general setting than that of the usual semi-abelian homology theories introduced in Section 1.3. The concept of comonadic homology was first introduced in an abelian context by Barr and Beck in [BB1969]. Everaert and Van der Linden generalised it to semi-abelian categories in [EVdL2004b]. First we introduce simplicial objects and their homology in Section 2.1, and then Section 2.2 shows how a comonad creates a simplicial object out of any given object, which gives rise to comonadic homology. Section 2.3 treats comonadic homology as a functor in the second variable, that of coefficients. The results of this section are mine, and are mostly straightforward generalisations of the corresponding abelian results in [BB1969].

2.1 Simplicial objects

In semi-abelian categories, simplicial objects play the role which is filled by chain complexes or resolutions in abelian categories. The slightly more complex structure is needed here as it is generally not possible to add or subtract maps in semi-abelian categories.

Given a simplicial object in $\mathcal{A}$, we can form its normalised chain complex. Let us first fix some notation for simplicial objects.

2.1.1 Notation: A **simplicial object** $\mathbb{A} = (A_n)_{n \geq 0}$ in $\mathcal{A}$

$$\cdots \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}} A_2 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} A_1 \mathrel{\substack{\longrightarrow \\ \longrightarrow}} A_0$$

has **face operators** $\partial_i \colon A_n \longrightarrow A_{n-1}$ for $i \in [n] = \{0, \ldots, n\}$ and $n \in \mathbb{N}_{>0}$, and **degeneracy operators** $\sigma_i \colon A_n \longrightarrow A_{n+1}$, for $i \in [n]$ and $n \in \mathbb{N}$, subject to the simplicial identities

$$
\begin{aligned}
\partial_i \circ \partial_j &= \partial_{j-1} \circ \partial_i \quad \text{if } i < j \\
\sigma_i \circ \sigma_j &= \sigma_{j+1} \circ \sigma_i \quad \text{if } i \leq j
\end{aligned}
\qquad
\partial_i \circ \sigma_j =
\begin{cases}
\sigma_{j-1} \circ \partial_i & \text{if } i < j \\
1 & \text{if } i = j \text{ or } i = j+1 \\
\sigma_j \circ \partial_{i-1} & \text{if } i > j+1.
\end{cases}
$$

In a semi-abelian category $\mathcal{A}$, we can define the homology of a simplicial object $\mathbb{A}$ via the *Moore complex* of $\mathbb{A}$.

2.1.2 Definition (Moore complex): Let $\mathbb{A}$ be a simplicial object in a semi-abelian category $\mathcal{A}$. The **Moore complex** or **normalised chain complex** $\mathrm{N}(\mathbb{A})$ has as objects $\mathrm{N}_0\mathbb{A} = A_0$, $\mathrm{N}_{-n}\mathbb{A} = 0$ and

$$\mathrm{N}_n\mathbb{A} = \bigcap_{i=0}^{n-1} \mathrm{K}[\partial_i \colon A_n \longrightarrow A_{n-1}] = \mathrm{K}[(\partial_i)_{i\in[n-1]} \colon A_n \longrightarrow A_{n-1}^n],$$

for $n \geq 1$, and boundary maps $d_n = \partial_n \circ \bigcap_i \mathrm{Ker}\,\partial_i \colon \mathrm{N}_n\mathbb{A} \longrightarrow \mathrm{N}_{n-1}\mathbb{A}$ for $n \geq 1$. This gives rise to a functor $\mathrm{N}\colon \mathcal{S}\mathcal{A} \longrightarrow \mathsf{Ch}\mathcal{A}$ from the category of simplicial objects in $\mathcal{A}$ to the category of chain complexes in $\mathcal{A}$, called the **normalisation functor**.

The **object of n-cycles** is $\mathrm{Z}_n\mathbb{A} = \mathrm{K}[d_n] = \bigcap_{i=0}^{n} \mathrm{K}[\partial_i \colon A_n \longrightarrow A_{n-1}]$ for $n \geq 1$. We write $\mathrm{Z}_0\mathbb{A} = A_0$.

The Moore complex of a simplicial object is always a proper chain complex [EVdL2004b, Theorem 3.6]; thus we can define

$$\mathrm{H}_n\mathbb{A} = \mathrm{H}_n\mathrm{N}(\mathbb{A})$$

for a simplicial object $\mathbb{A}$. In the abelian case, the homology of the Moore complex is the same as the homology of the **unnormalised** chain complex $\mathrm{C}(\mathbb{A})$ of $\mathbb{A}$, where $\mathrm{C}_n\mathbb{A} = A_n$ and $d_n = \partial_0 - \partial_1 + \cdots + (-1)^n\partial_n$.

Notice that the Moore complex and thus the homology of a simplicial object only involve the face maps ∂_i, and not the degeneracies σ_i. So we need only consider semi-simplicial maps between simplicial objects, i.e. maps that commute with the ∂_i but not necessarily with the σ_i. This will be used in Chapter 3.

The homology objects obtained this way are special objects of $\mathcal{A}$.

2.1.3 Lemma: [EVdL2004b, Theorem 5.5] *Let $\mathbb{A}$ be a simplicial object in a semi-abelian category $\mathcal{A}$. For any $n \geq 1$, the object $\mathrm{H}_n\mathbb{A}$ is an abelian object of $\mathcal{A}$.*　　$\square$

An important property of the normalisation functor is the following:

2.1.4 Lemma: [EVdL2004b, Proposition 5.6] *Let $\mathcal{A}$ be a semi-abelian category. The Moore normalisation functor $\mathrm{N}\colon \mathcal{S}\mathcal{A} \longrightarrow \mathsf{Ch}\mathcal{A}$ is exact.*

Proof. A slightly easier proof than that in [EVdL2004b] can be found in [Eve2007].　　$\square$

This, together with the Snake Lemma, implies the following result.

2.1.5 Lemma: [EVdL2004b, Corollary 5.7] *A short exact sequence of simplicial objects*

$$0 \longrightarrow \mathbb{A} \longrightarrow \mathbb{B} \longrightarrow \mathbb{C} \longrightarrow 0$$

in $\mathcal{A}$ gives rise to a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_n\mathbb{A} \longrightarrow \mathrm{H}_n\mathbb{B} \longrightarrow \mathrm{H}_n\mathbb{C}$$
$$\mathrm{H}_{n-1}\mathbb{A} \longrightarrow \cdots \longrightarrow \mathrm{H}_0\mathbb{C} \longrightarrow 0$$

of homology objects which depends naturally on the given short exact sequence. $\qquad\square$

Later we will be considering simplicial resolutions, which are really *augmented* simplicial objects.

2.1.6 Definition: An **augmented** simplicial object is a simplicial object $\mathbb{A}$ together with a map $\partial_0\colon A_0 \longrightarrow A_{-1}$ satisfying $\partial_0\circ\partial_0 = \partial_0\circ\partial_1$, that is, it has equal composite with the two face maps $A_1 \longrightarrow A_0$. A **contraction** of an augmented simplicial object $\mathbb{A}$ is a family of maps $h_n\colon A_n \longrightarrow A_{n+1}$, for $n \geq -1$, which satisfy $\partial_0 h_n = 1_{A_n}$ and $\partial_i h_n = h_{n-1}\partial_{i-1}$ for $i > 0$. A simplicial object that admits a contraction is called **contractible**.

$$\cdots \; \underset{}{\overset{}{\Rrightarrow}} \; A_2 \; \underset{}{\overset{}{\Rightarrow}} \; A_1 \; \underset{}{\overset{}{\Rightarrow}} \; A_0 \longrightarrow A_{-1}$$
$$\qquad\quad h_2 \qquad\quad h_1 \qquad\quad h_0 \qquad\quad h_{-1}$$

When computing the homology of a simplicial object, the following observation is often useful.

2.1.7 Lemma: [EVdL2004b, Proposition 3.9] (cf. [Bou2001]) *Given a diagram*

$$A \underset{\partial_0}{\overset{\partial_1}{\Rightarrow}} B \overset{e}{\longrightarrow} C$$

where $e\circ\partial_0 = e\circ\partial_1$, suppose there is a common splitting $t\colon B \longrightarrow A$ of ∂_0 and ∂_1, that is, $\partial_0\circ t = 1_B = \partial_1\circ t$. Then e is the coequaliser of ∂_0 and ∂_1 if and only if e is the cokernel of $\partial_1\circ\mathrm{Ker}\,\partial_0$. $\qquad\square$

A consequence of this result is:

2.1.8 Lemma: [EVdL2004b, Proposition 3.11] *A contractible augmented simplicial object $\mathbb{A}$ has $\mathrm{H}_0\mathbb{A} = A_{-1}$ and $\mathrm{H}_n\mathbb{A} = 0$ for $n \geq 1$.* $\qquad\square$

A simplicial object in the category of sets is commonly called a simplicial set. A classical property simplicial sets may have is the Kan property. Kan simplicial sets are exactly the fibrant ones (in the usual model structure on $\mathbb{S}\mathsf{Set}$) and may be described as follows.

2.1.9 Definition (Kan property): Let $\mathbb{S}$ be a simplicial set and $n \geq 1$, $k \in [n]$ natural numbers. An (n, k)-**horn** in $\mathbb{S}$ is a sequence $(s_i)_{i \in [n] \setminus \{k\}}$ of elements of S_{n-1} satisfying $\partial_i(s_j) = \partial_{j-1}(s_i)$ for all $i < j$ and $i, j \neq k$. A **filler** of an (n, k)-horn $(s_i)_i$ is an element s of S_n satisfying $\partial_i(s) = s_i$ for all $i \neq k$. A simplicial set $\mathbb{S}$ is **Kan** when every horn in $\mathbb{S}$ has a filler.

This Kan property can be generalised to simplicial objects in a regular category as follows (see [CKP1993]).

2.1.10 Definition (Internal Kan property): Let $\mathbb{A}$ be a simplicial object in a regular category $\mathcal{A}$ and $n \geq 1$, $k \in [n]$ natural numbers. An (n, k)-**horn** in $\mathbb{A}$ is a family of maps $(b_i \colon B \longrightarrow A_{n-1})_{i \in [n] \setminus \{k\}}$ satisfying $\partial_i b_j = \partial_{j-1} b_i$ for all $i < j$ and $i, j \neq k$; we can view this as a map $b \colon B \longrightarrow (A_{n-1})^n$. A **filler** of an (n, k)-horn $b \colon B \longrightarrow (A_{n-1})^n$ is given by a surjection $p \colon Z \longrightarrow B$ and a generalised element $z \colon Z \longrightarrow A_n$ satisfying $\partial_i z = b_i p$ for $i \neq k$. This can be viewed as a filler "up to enlargement of domain".

Carboni, Kelly and Pedicchio show in [CKP1993] that every simplicial object of a regular category $\mathcal{A}$ is Kan if and only if $\mathcal{A}$ is Mal'tsev. Thus when $\mathcal{A}$ is regular Mal'tsev, for example semi-abelian, we can apply the internal Kan property for every simplicial object in $\mathcal{A}$. This property is very powerful and can be used in many situations. To demonstrate this, we present a small new result involving the maps used in the Moore complex. Recall that the object $N_n \mathbb{A}$ is the kernel of the map $(\partial_0, \ldots, \partial_{n-1}) \colon A_n \longrightarrow (A_{n-1})^n$.

2.1.11 Proposition: *Let $\mathbb{A}$ be a simplicial object in a semi-abelian category $\mathcal{A}$. Let I_n denote the image of the map $(\partial_0, \ldots, \partial_{n-1}) \colon A_n \longrightarrow (A_{n-1})^n$. Then the following is an equaliser diagram:*

$$I_n \longrightarrow (A_{n-1})^n \underset{(\partial_0\pi_0, \partial_0\pi_1, \ldots, \partial_0\pi_{n-2}, \partial_1\pi_1, \ldots, \partial_{n-2}\pi_{n-2})}{\overset{(\partial_0\pi_1, \partial_0\pi_2, \ldots, \partial_0\pi_{n-1}, \partial_1\pi_2, \ldots, \partial_{n-2}\pi_{n-1})}{\rightrightarrows}} (A_{n-2})^k$$

where the two parallel maps are constructed to give all the horn conditions $\partial_i \pi_j = \partial_{j-1} \pi_i$ for $i < j$ and $j \leq n-1$; so $k = \frac{1}{2}(n-1)(n-2)$.

Proof. The equaliser of these two maps is an (n, n)-horn through which all other (n, n)-horns factor. So we have to show that all (n, n)-horns factor through I_n.

Let $b \colon B \longrightarrow (A_{n-1})^n$ be an (n, n)-horn. As in a regular Mal'tsev category every simplicial object is Kan [CKP1993, Theorem 4.2], this horn must have a filler

$$(p \colon Z \longrightarrow B, z \colon Z \longrightarrow A_n).$$

This gives the following diagram:

Since p is the coequaliser of some pair of maps, and $iz' = bp$ has equal composite with this pair and i is monic, z' factors through p and we get a factorisation of the horn B through I_n as desired. $\qquad \square$

This fact can be used to prove that the normalisation functor is exact, a result which Tim Van der Linden and Tomas Everaert prove in a different way in [EVdL2004b] (see Lemma 2.1.4).

2.1.12 Lemma: *Given a short exact sequence of simplicial objects*

$$0 \longrightarrow \mathbb{A} \xrightarrow{\ f\ } \mathbb{B} \xrightarrow{\ g\ } \mathbb{C} \longrightarrow 0$$

in a semi-abelian category $\mathcal{A}$, the induced sequence of chain complexes

$$0 \longrightarrow \mathbb{N}\mathbb{A} \longrightarrow \mathbb{N}\mathbb{B} \longrightarrow \mathbb{N}\mathbb{C} \longrightarrow 0$$

is also exact.

Proof. As mentioned above, this is the same as Lemma 2.1.4, but we give an alternate proof here using Proposition 2.1.11.

Given a short exact sequence of simplicial objects as above, we must show that

$$0 \longrightarrow \mathbb{N}_n\mathbb{A} \longrightarrow \mathbb{N}_n\mathbb{B} \longrightarrow \mathbb{N}_n\mathbb{C} \longrightarrow 0$$

is exact in $\mathcal{A}$ for each $n \geq 0$. For $n = 0$ we have $\mathbb{N}_0\mathbb{A} = A_0$, so the result is clear. For $n \geq 1$ we use that $\mathbb{N}_n\mathbb{A}$ is the kernel of

$$(\partial_i)_{i \in [n-1]} \colon A_n \longrightarrow A_{n-1}^n$$

where A_{n-1}^n is the n-fold product of A_{n-1}. Then Proposition 2.1.11 tells us that the image $I_n^\mathbb{A}$ of this map is an equaliser. This implies that the image sequence

$$0 \longrightarrow I_n^\mathbb{A} \longrightarrow I_n^\mathbb{B} \longrightarrow I_n^\mathbb{C} \longrightarrow 0$$

is also exact: we have

where the first row is exact and the columns are image factorisations. Also, as kernels commute with products, f_{n-1}^n is the kernel of g_{n-1}^n. Clearly $\overline{f}$ factors over the kernel $\mathrm{K}[\overline{g}]$ of $\overline{g}$. The kernel property of A_{n-1}^n also induces a map $\mathrm{K}[\overline{g}] \longrightarrow A_{n-1}^n$, which has equal composite with the two maps to $\mathcal{A}_{n-2}^k$, as f_{n-2}^k is a monomorphism. So this map factors over the equaliser $I_n^\mathbb{A}$ and by the universal properties we see that $I_n^\mathbb{A} \cong \mathrm{K}[\overline{g}]$.

Now we can use the 3×3 Lemma:

Here all columns and the last two rows are exact, so the first row is also exact, as desired.

$\square$

2.2 Comonadic Homology

When $\mathcal{A}$ is a semi-abelian monadic category (e.g., a semi-abelian variety; see [GR2004] for a precise characterisation), there is a canonical forgetful/free comonad $\mathbb{G} = (G, \epsilon, \delta)$ on $\mathcal{A}$, which gives rise to a functorial simplicial resolution $\mathbb{G}A$ of any object A, that is, an augmented simplicial object over A with face maps $\partial_i = G^i \epsilon_{G^{n-i}A} \colon G^{n+1}A \longrightarrow G^n A$ and degeneracies $\sigma_i = G^i \delta_{G^{n-i}A} \colon G^{n+1}A \longrightarrow G^{n+2}A$.

$$\cdots \underset{\overline{}}{\overline{\overline{\overline{}}}} G^3 A \xrightarrow[\underset{\epsilon_{G^2 A}}{\overline{G\epsilon_{GA}}}]{\overset{G^2 \epsilon_A}{\overline{}}} G^2 A \xrightarrow[\epsilon_{GA}]{G\epsilon_A} GA \xrightarrow{\epsilon_A} A$$

This gives rise to the following Barr-Beck style [BB1969] notion of homology:

2.2.1 Definition: [EVdL2004b] Let $\mathcal{B}$ be a Birkhoff subcategory of a semi-abelian monadic category $\mathcal{A}$ with reflector $I \colon \mathcal{A} \longrightarrow \mathcal{B}$ and canonical comonad $\mathbb{G}$. For any object A of $\mathcal{A}$ and any $n \geq 0$, we define

$$\mathrm{H}_{n+1}(A, I)_{\mathbb{G}} = \mathrm{H}_n N I \mathbb{G}A. \tag{$\mathbf{A}$}$$

In fact, comonadic homology can be defined in a more general context than that of a semi-abelian category with a Birkhoff subcategory.

2.2.2 Definition (Comonadic homology): [EVdL2004b] Let $\mathcal{C}$ be any category with a comonad

$$\mathbb{G} = (G \colon \mathcal{C} \longrightarrow \mathcal{C}, \quad \delta \colon G \Longrightarrow G^2, \quad \epsilon \colon G \Longrightarrow 1_{\mathcal{C}})$$

and let $E \colon \mathcal{C} \longrightarrow \mathcal{A}$ be a functor to a semi-abelian category $\mathcal{A}$. For $n \geq 1$, the object

$$\mathrm{H}_n(A, E)_{\mathbb{G}} = \mathrm{H}_{n-1} N E \mathbb{G}A$$

is called the n**th homology object of** A **(with coefficients in** E**) relative to the comonad** $\mathbb{G}$. This defines a functor $\mathrm{H}_n(-, E)_{\mathbb{G}} \colon \mathcal{C} \longrightarrow \mathcal{A}$, for every $n \geq 1$.

The dimension shift here is not present in Barr and Beck's original definition, but was introduced in [EVdL2004b] to make it better adjusted to the non-abelian examples (homology of groups, Lie algebras, crossed modules) which traditionally have a shifted numbering.

When E is a contravariant functor, we get **comonadic cohomology** in a similar way.

2.2.3 Example (Comonads and functors of coefficients): The most common example of a comonad used for comonadic homology is that of a forgetful/free comonad on a variety of algebras, such as the free group comonad on the category of groups. Using

this comonad, the abelianisation functor $E = \mathrm{ab}\colon \mathsf{Gp} \longrightarrow \mathsf{Ab}$ gives rise to integral group homology. The forgetful/free comonad on the category $R\text{-}\mathsf{Mod}$ of R-modules gives rise to two well-known homology theories. When $E = N \otimes_R - \colon R\text{-}\mathsf{Mod} \longrightarrow \mathsf{Ab}$ for a fixed module N, we get the Tor groups as homology, that is

$$\mathrm{H}_n(M, N \otimes_R -)_{\mathbb{G}} = \mathrm{Tor}_n^R(M, N).$$

The contravariant functor $E = \mathrm{Hom}_R(-, N)\colon R\text{-}\mathsf{Mod} \longrightarrow \mathsf{Ab}$ gives the Ext groups:

$$\mathrm{H}^n(M, \mathrm{Hom}_R(-, N))_{\mathbb{G}} = \mathsf{Ext}_R^n(M, N)$$

However, if we take the covariant Hom-functor $\mathrm{Hom}_R(N, -)$, we obtain the Eckmann-Hilton homotopy groups (see [BB1969, Example 1.1]). Notice that when taking comonadic homology of the covariant Hom-functor, we are still using *projective* (in fact free) resolutions, and not injective resolutions, which is why we don't get the Ext groups in this case.

There are also other comonads on the category of R-modules: given a ring homomorphism $\phi\colon S \longrightarrow R$, we can view any R-module as an S-module, and any S-module can be turned into an R-module by tensoring it with R over S. This adjunction gives rise to another comonad on $R\text{-}\mathsf{Mod}$.

$$
\begin{array}{ccc}
R\text{-}\mathsf{Mod} & \xrightarrow{\quad G_\phi \quad} & R\text{-}\mathsf{Mod} \\
& U \searrow \quad \nearrow R \otimes_S (-) & \\
& S\text{-}\mathsf{Mod} &
\end{array}
$$

Using this so-called *relative* comonad, the functors given by tensoring and homing as above give rise to Hochschild's *relative* Tor and Ext groups. This comonad will be used in Example 3.3.14.

Many more examples of comonads and functors of coefficients exist, see for example [BB1969].

2.3 $\mathrm{H}_n(-, E)_{\mathbb{G}}$ as a functor in the variable E

Let $\mathbb{G} = (G, \delta, \epsilon)$ be a comonad on the category $\mathcal{C}$, and $E\colon \mathcal{C} \longrightarrow \mathcal{A}$ a functor into a semi-abelian category $\mathcal{A}$. We can view the homology functor $\mathrm{H}_n(-, \ )_{\mathbb{G}}$ as a functor $[\mathcal{C}, \mathcal{A}] \longrightarrow [\mathcal{C}, \mathcal{A}]$, taking E to $\mathrm{H}_n(-, E)_{\mathbb{G}}$. As such it has certain universal properties, which we discuss in this section. The results in this section are straightforward generalisations from the abelian case discussed in [BB1969], and in most parts the proofs carry over without much change. We still give them here in our own notation for completeness.

2.3.1 Proposition (G-acyclicity): *For $n \geq 1$, we have*

$$\mathrm{H}_n(-, EG)_{\mathbb{G}} = 0,$$

and for $n = 0$ the map

$$\lambda \colon \mathrm{H}_0(-, EG)_{\mathbb{G}} \xrightarrow{\ \cong\ } EG$$

is an isomorphism.

Proof. For any object $X \in \mathcal{C}$, the augmented simplicial object $EG\mathbb{G}X$ is contractible, as we have $EG^n\delta_X \colon EG\mathbb{G}^nX \longrightarrow EG\mathbb{G}^{n+1}X$ for $n \geq 0$, which satisfies $EG\mathbb{G}^n\epsilon_X \circ EG^n\delta_X = 1_{EG\mathbb{G}^nX}$ and $EG\mathbb{G}^i\epsilon_{\mathbb{G}^{n-i}X} \circ EG^n\delta_X = EG^{n-1}\delta_X \circ EG\mathbb{G}^i\epsilon_{\mathbb{G}^{n-i-1}X}$. Thus using Lemma 2.1.8 the result follows. $\qquad\square$

We define a **G-exact sequence** to be a sequence of functors

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

such that

$$0 \longrightarrow E_1G \longrightarrow E_2G \longrightarrow E_3G \longrightarrow 0$$

is exact. We then get

2.3.2 Proposition (G-connectedness): *Any $\mathbb{G}$-exact sequence*

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

gives rise to a long exact sequence on homology:

$$\cdots \longrightarrow \mathrm{H}_n(-, E_1)_{\mathbb{G}} \longrightarrow \mathrm{H}_n(-, E_2)_{\mathbb{G}} \longrightarrow \mathrm{H}_n(-, E_3)_{\mathbb{G}}$$

$$\mathrm{H}_{n-1}(-, E_1)_{\mathbb{G}} \xrightarrow{\quad\partial\quad} \cdots \longrightarrow \mathrm{H}_0(-, E_3)_{\mathbb{G}} \longrightarrow 0$$

Proof. For any object $X \in \mathcal{C}$ the $\mathbb{G}$-exact sequence gives rise to a short exact sequence of simplicial objects

$$0 \longrightarrow E_1\mathbb{G}X \longrightarrow E_2\mathbb{G}X \longrightarrow E_3\mathbb{G}X \longrightarrow 0$$

which in turn gives rise to the desired long exact sequence on homology using Lemma 2.1.5. $\qquad\square$

Analogously to [BB1969], we can define a theory of $\mathbb{G}$-left derived functors by the above properties:

2.3.3 Definition: $\mathbb{L} = (L_n, \lambda, \partial)$ is a **theory of $\mathbb{G}$-left derived functors** if the following are satisfied:

(1) Each L_n is a functor $L_n \colon [\mathcal{C}, \mathcal{A}] \longrightarrow [\mathcal{C}, \mathcal{A}]$.

(2) $\lambda \colon L_0 \longrightarrow 1_{[\mathcal{C}, \mathcal{A}]}$ is a natural transformation from L_0 to the identity functor on the functor category $[\mathcal{C}, \mathcal{A}]$.

(3) ($\mathbb{G}$-acyclicity) For a functor of the form EG we have

$$\lambda \colon L_0(EG) \overset{\cong}{\longrightarrow} EG \qquad\qquad \text{is an isomorphism,}$$
$$L_n(EG) = 0 \qquad\qquad \text{for } n \geq 1.$$

(4) ($\mathbb{G}$-connectedness) Every $\mathbb{G}$-exact sequence $0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$ gives rise to a long exact sequence:

$$\cdots \longrightarrow L_n(E_1) \longrightarrow L_n(E_2) \longrightarrow L_n(E_3)$$
$$\overset{\partial}{\longleftarrow}$$
$$L_{n-1}(E_1) \longrightarrow \cdots \longrightarrow L_0(E_3) \longrightarrow 0$$

where ∂ depends of the given sequence, and

$$
\begin{array}{ccc}
L_n E_3 & \overset{\partial}{\longrightarrow} & L_{n-1} E_1 \\
\downarrow & & \downarrow \\
L_n F_3 & \underset{\partial}{\longrightarrow} & L_{n-1} F_1
\end{array}
$$

commutes for any map of sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & 0.
\end{array}
$$

We will show that the homology functor above is special amongst these theories of $\mathbb{G}$-left derived functors. To do this, we first prove a result which is needed in the proof of the next theorem.

2.3.4 Lemma: *The following is a coequaliser diagram:*

$$L_0(EG^2) \overset{L_0(EG\epsilon)}{\underset{L_0(E\epsilon_G)}{\rightrightarrows}} L_0(EG) \overset{L_0(E\epsilon)}{\longrightarrow} L_0 E$$

Proof. Notice that this diagram is the image under L_0 of the lowest part of the augmented simplicial object $E\mathbb{G}$. Similarly to Lemma 2.1.7, it is enough to show that $L_0(E\epsilon)$ is the cokernel of $L_0(d_1)\colon L_0(\mathrm{N}_1(E\mathbb{G})) \longrightarrow L_0(EG)$, which is the composite

$$L_0(\mathrm{N}_1(E\mathbb{G})) \xrightarrow{L_0(\mathrm{Ker}\,(E\epsilon_G))} L_0(EG^2) \xrightarrow{L_0(EG\epsilon)} L_0(EG).$$

For this let M be the kernel of $E\epsilon\colon EG \longrightarrow E$, and form the following diagram, where the bottom row is $\mathbb{G}$-exact (as $E\epsilon_G\colon EG^2 \longrightarrow EG$ is split epic and so regular epic):

$$
\begin{array}{ccccccc}
& & & \mathrm{N}_1(E\mathbb{G}) & & & \\
& & \swarrow^{\nu} & \downarrow{d_1} & & & \\
0 \longrightarrow & M & \xrightarrow{\;i\;} & EG & \xrightarrow{\;E\epsilon\;} & E \longrightarrow & 0
\end{array}
$$

The morphism ν is induced by the kernel property of i. We now apply L_0 to this diagram and get

$$
\begin{array}{ccccccc}
& & & L_0(\mathrm{N}_1(E\mathbb{G})) & & & \\
& & {}^{L_0(\nu)}\swarrow & \downarrow{L_0(d_1)} & & & \\
& L_0(M) & \xrightarrow{L_0(i)} & L_0(EG) & \xrightarrow{L_0(E\epsilon)} & L_0(E) \longrightarrow & 0.
\end{array}
$$

Now the bottom row is exact, as the bottom row of the previous diagram was $\mathbb{G}$-exact. Note that we do not require L_0 to preserve 0, but we still have $L_0(E\epsilon)L_0(d_1) = 0$, as the exact sequence

$$0 \longrightarrow \mathrm{N}_1 E\mathbb{G} \longrightarrow EG^2 \xrightarrow{E\epsilon_G} EG \longrightarrow 0$$

gives rise to the diagram

$$
\begin{array}{ccccccccc}
0 \longrightarrow & L_0(\mathrm{N}_1 E\mathbb{G}) & \longrightarrow & L_0(EG^2) & \xrightarrow{L_0(E\epsilon_G)} & L_0(EG) & \longrightarrow & 0 \\
& & {}_{L_0(d_1)}\searrow & \downarrow{L_0(EG\epsilon)} & & \downarrow{L_0(E\epsilon)} & & \\
& & & L_0(EG) & \xrightarrow{L_0(E\epsilon)} & L_0(E) & &
\end{array}
$$

where the top row is short exact by $\mathbb{G}$-connectedness and $\mathbb{G}$-acyclicity.

Thus if we denote the images of $L_0(i)$ and $L_0(d_1)$ by I and I' respectively, we get a factorisation $I' \longrightarrow I$, which is of course monic. If we can show it is regular epic as well, we have shown that $L_0(d_1)$ followed by $L_0(E\epsilon)$ is also exact. For this it is sufficient to show that $L_0(\nu)$ is a regular epi.

Let $K = \mathrm{K}[\nu]$; then it is sufficient to show that

$$0 \longrightarrow KG \longrightarrow \mathrm{N}_1(E\mathbb{G}G) \xrightarrow{\nu_G} MG \longrightarrow 0$$

is exact, as then $\mathbb{G}$-connectedness implies that

$$L_0(K) \longrightarrow L_0(N_1(E\mathbb{G})) \xrightarrow{L_0(\nu)} L_0(M) \longrightarrow 0$$

is exact. So we only need to prove that ν_G is regular epic. In the following diagram both rows are exact, and the left downwards square and right upwards square commute:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_1(E\mathbb{G}G) & \xrightarrow{\,j\,} & EG^3 & \xrightarrow{E\epsilon_{G^2}} & EG^2 & \longrightarrow & 0 \\
 & & \big\updownarrow{\scriptstyle h}\,\big\downarrow{\scriptstyle \nu_G} & & EG\epsilon_G \big\updownarrow\big\downarrow{\scriptstyle EG\delta} & & \big\uparrow{\scriptstyle E\delta} & & \\
0 & \longrightarrow & MG & \xrightarrow{\,i_G\,} & EG^2 & \xrightarrow{E\epsilon_G} & EG & \longrightarrow & 0
\end{array}
$$

The dotted arrow h is induced by $E\epsilon_{G^2}\circ EG\delta\circ i_G = 0$. Then

$$i_G\nu_G h = EG\epsilon_G\circ j\circ h = EG\epsilon_G\circ EG\delta\circ i_G = i_G$$

so as i_G is monic, ν_G is split epic, so in particular regular epic. Thus $L_0(E\epsilon)$ is the cokernel of $L_0(d_1)$, as asserted.

This implies that $L_0(E\epsilon)$ is the coequaliser of $L_0(E\epsilon_G) = h_0$ and $L_0(EG\epsilon) = h_1$ (we rename them for convenience), in an analogous way to Lemma 2.1.7. Let $k = L_0(\operatorname{Ker} E\epsilon_G)$. In the following diagram, the outer rectangle is a pushout, and both squares commute.

$$
\begin{array}{ccccc}
L_0(N_1E\mathbb{G}) & \xrightarrow{L_0(\operatorname{Ker} E\epsilon_G)=k} & L_0(EG^2) & \xrightarrow{L_0(EG\epsilon)=h_1} & L_0(EG) \\
\big\downarrow & & {\scriptstyle L_0(E\epsilon_G)=h_0}\big\downarrow & & {\scriptstyle L_0(E\epsilon)}\big\downarrow \\
0 & \longrightarrow & L_0(EG) & \xrightarrow{L_0(E\epsilon)} & L_0(E) \\
\end{array}
$$

To show that $L_0(E\epsilon)$ is the required coequaliser, it is enough to show that if $fh_0 = fh_1$, then $fh_1k = 0$. But this is clear as $h_0k = 0$. Thus f factors through $L_0(E\epsilon)$. $\qquad\square$

We can now prove the following Uniqueness Theorem:

2.3.5 Theorem: *Let $\mathbb{L}$ be a theory of $\mathbb{G}$-left derived functors. Then there exists a unique family of natural isomorphisms*

$$L_n \xrightarrow[\cong]{\sigma_n} H_n(-,\ )_{\mathbb{G}}$$

for $n \geq 0$, (i.e. natural isomorphisms $L_n E \longrightarrow H_n(-, E)_{\mathbb{G}}$ which are also natural in E), which are compatible with the augmentation λ and the connecting homomorphism ∂:

$$
\begin{array}{ccc}
L_0 E \xrightarrow{\ \sigma_0\ } H_0(-, E)_{\mathbb{G}} & \qquad & L_n E_3 \xrightarrow{\ \partial\ } L_{n-1} E_1 \\
\lambda \searrow \quad \swarrow \lambda & & \sigma_n \downarrow \qquad \qquad \downarrow \sigma_{n-1} \\
E & & H_n(-, E_3)_{\mathbb{G}} \xrightarrow[\partial]{} H_{n-1}(-, E_1)_{\mathbb{G}}
\end{array}
$$

Proof. The proof from [BB1969] for the abelian case carries over almost completely to the semi-abelian case, but we give it in full in our own notation for convenience.

We start by constructing σ_0.

$$
\begin{array}{ccc}
L_0(EG^2) \overset{L_0(EG\epsilon)}{\underset{L_0(E\epsilon_G)}{\rightrightarrows}} L_0(EG) \xrightarrow{\ L_0(E\epsilon)\ } L_0(E) \\
\cong \downarrow \lambda \qquad\qquad \cong \downarrow \lambda \qquad\qquad \downarrow \sigma_0 \\
EG^2 \overset{EG\epsilon}{\underset{E\epsilon_G}{\rightrightarrows}} EG \xrightarrow{\qquad\qquad} H_0(-, E)_{\mathbb{G}}
\end{array}
$$

From Lemma 2.3.4 we know that the top row of this diagram is a coequaliser, as is the bottom row (by definition of homology and Lemma 2.1.7). Thus we get an induced map σ_0, which is an isomorphism, since both λ occurring in the diagram are isomorphisms (by the acyclicity property). It is clear that σ_0 also commutes with the augmentations λ to E, since $L_0(E\epsilon)$ is a regular epi.

Now we construct the other σ_n inductively. From the construction of σ_0 it is clear that it is also natural in E. We take $M = K[E\epsilon \colon EG \longrightarrow E]$, so that

$$
0 \longrightarrow M \xrightarrow{\ i\ } EG \xrightarrow{\ E\epsilon\ } E \longrightarrow 0
$$

is $\mathbb{G}$-exact.

For $n = 1$ we then have

$$
\begin{array}{ccccc}
L_1(EG) = 0 \longrightarrow & L_1 E & \xrightarrow{\ \partial_L\ } & L_0 M & \xrightarrow{\ L_0 i\ } & L_0(EG) \\
& \sigma_1 \downarrow & & \cong \downarrow \sigma_0 & & \cong \downarrow \sigma_0 \\
H_1(-, EG)_{\mathbb{G}} = 0 \longrightarrow & H_1(-, E)_{\mathbb{G}} & \xrightarrow{\ \partial_H\ } & H_0(-, M)_{\mathbb{G}} & \xrightarrow{\ H_0(-, i)_{\mathbb{G}}\ } & H_0(-, EG)_{\mathbb{G}}
\end{array}
$$

where both rows are exact, by $\mathbb{G}$-connectedness. This induces the map σ_1, which is also an isomorphism (by uniqueness of limits, or the Five Lemma).

For $n > 1$ we use the diagram

$$
\begin{array}{ccccccc}
L_n(EG) = 0 & \longrightarrow & L_n E & \xrightarrow[\cong]{\partial_L} & L_{n-1} M & \longrightarrow & 0 = L_{n-1}(EG) \\
& & \Big\downarrow{\scriptstyle \sigma_n} & & \cong \Big\downarrow{\scriptstyle \sigma_{n-1}} & & \\
\mathrm{H}_n(-, EG)_{\mathbb{G}} = 0 & \longrightarrow & \mathrm{H}_n(-, E)_{\mathbb{G}} & \xrightarrow[\cong]{\partial_{\mathrm{H}}} & \mathrm{H}_{n-1}(-, M)_{\mathbb{G}} & \longrightarrow & 0 = \mathrm{H}_{n-1}(-, EG)_{\mathbb{G}}
\end{array}
$$

Again both rows are exact, so $\sigma_n = \partial_{\mathrm{H}}^{-1} \sigma_{n-1} \partial_L$ is also an isomorphism. This defines the natural isomorphisms σ_n for all n. Now we must verify compatibility with the connecting homomorphisms.

Let $0 \longrightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \longrightarrow 0$ be $\mathbb{G}$-exact. Let K be the kernel of the composite $\gamma \colon E_2 G \xrightarrow{E_2 \epsilon} E_2 \xrightarrow{\beta} E_3$, and M_3 the kernel of $E_3 \epsilon$ as before. Then we get a map of $\mathbb{G}$-exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{j} & E_2 G & \xrightarrow{\gamma} & E_3 & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle \beta_G} & & \Big\| & & \\
0 & \longrightarrow & M_3 & \xrightarrow{i_3} & E_3 G & \xrightarrow{E_3 \epsilon} & E_3 & \longrightarrow & 0
\end{array} \tag{B}
$$

where κ is induced by the kernel property of i_3.

This gives rise to a diagram

Here both rows are exact, and the triangles commute by naturality of the connecting homomorphism. The rightmost square and the front right square commute by naturality of σ_0. Now since ∂_{H} is the kernel of $\mathrm{H}_0(-, j)_{\mathbb{G}}$ (as $\mathrm{H}_1(-, E_2 G)_{\mathbb{G}} = 0$), we get a factorisation $L_1 E_3 \longrightarrow \mathrm{H}_1(-, E_3)_{\mathbb{G}}$ making the front left square commute. This morphism also makes the back rectangle commute. But that rectangle defines σ_1 uniquely, so the front left square must commute with σ_1 substituted in.

Now let K_β be the kernel of β, then we also have the following map of $\mathbb{G}$-exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{j} & E_2 G & \xrightarrow{\gamma} & E_3 & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \kappa_2} & & \Big\downarrow{\scriptstyle E_2 \epsilon} & & \Big\| & & \\
0 & \longrightarrow & K_\beta & \longrightarrow & E_2 & \xrightarrow{\beta} & E_3 & \longrightarrow & 0
\end{array} \tag{C}
$$

with the obvious induced κ_2. We also have an induced morphism of $\mathbb{G}$-exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & 0 \\
& & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & K_\beta & \longrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & 0
\end{array}
$$

which implies that $L_n(K_\beta) \cong L_n(E_1)$ and $H_n(-, K_\beta) \cong H_n(-, E_1)$ for $n \geq 0$, using $\mathbb{G}$-connectedness and the Five Lemma.

The map of $\mathbb{G}$-exact sequences (**C**) induces the following prism, where we can substitute E_1 for K_β using the above isomorphisms:

The triangles again commute by naturality of the connecting homomorphisms, the right front square commutes by naturality of σ_0, and the back square is the square whose commutativity we have shown above. Thus the front left square also commutes.

The case $n \geq 2$ works similarly, again using diagram (**B**) to get

This time it is clear that the front left square commutes with σ_n substituted in, as everything in sight is an isomorphism (by $\mathbb{G}$-connectedness and $\mathbb{G}$-acyclicity). Then we can again use (**C**) to get a similar prism to the above, which proves that

$$
\begin{array}{ccc}
L_n E_3 & \xrightarrow{\partial_L} & L_{n-1} E_1 \\
\sigma_n \downarrow & & \downarrow \sigma_{n-1} \\
H_n(-, E_3)_{\mathbb{G}} & \xrightarrow[\partial_H]{} & H_{n-1}(-, E_1)_{\mathbb{G}}
\end{array}
$$

commutes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Thus we have shown that the comonadic homology theories are "the only" such theories of $\mathbb{G}$-left derived functors, up to isomorphism.

Chapter 3

A Comparison Theorem for Simplicial Resolutions

Introduction

In Section 2.3 we studied comonadic homology as a functor in the second variable. This chapter addresses a different question: how does the comonadic homology theory depend on the given comonad $\mathbb{G}$? Barr and Beck showed in [BB1969] that, in the abelian setting, two comonads $\mathbb{G}$ and $\mathbb{K}$ give rise to the same comonadic homology theory when they generate the same class of projective objects. In this chapter, we prove a suitable extension of their result to the semi-abelian case. That is, we show that given any category $\mathcal{C}$ with two comonads $\mathbb{G}$ and $\mathbb{K}$, and any functor $E \colon \mathcal{C} \longrightarrow \mathcal{A}$ to a semi-abelian category $\mathcal{A}$, the functors $\mathrm{H}_n(-, E)_{\mathbb{G}}$ and $\mathrm{H}_n(-, E)_{\mathbb{K}} \colon \mathcal{C} \longrightarrow \mathcal{A}$ are isomorphic for $n \in \mathbb{N}$ when $\mathbb{G}$ and $\mathbb{K}$ generate the same Kan projective class. The condition on the projective class ensures that homming from a ($\mathbb{G}$- or $\mathbb{K}$-)projective object into a ($\mathbb{G}$- or $\mathbb{K}$-)simplicial resolution gives a Kan simplicial set. It is exactly the condition needed to prove a comparison theorem for these simplicial resolutions.

The examples reveal that this condition is not too strong. In an additive category, homming from *any* object into *any* simplicial object results in a Kan simplicial set, and when $\mathcal{C}$ is a regular Mal'tsev category, which includes semi-abelian ones, then the condition is fulfilled as soon as the $\mathbb{G}$-projective objects are also regular projectives.

For their proof in the abelian case, Barr and Beck make heavy use of additive structure via a free additive completion of the category $\mathcal{C}$. As a semi-abelian category is only additive when it is abelian, we have to use a different approach to extend their result to the semi-abelian setting. We prove a comparison theorem which shows that, given a projective class $\mathcal{P}$, any two $\mathcal{P}$-resolutions are homotopy equivalent and consequently have the same homology. The advantage of our method is that it shows that *any* $\mathcal{P}$-resolution of a given object will give the same homology, so that it is also possible to use resolutions not coming from a comonad, should this turn out to be more convenient. The only subtlety lies in the definition of a $\mathcal{P}$-resolution of an object A: this is an augmented simplicial object $\mathbb{A} = (A_n)_{n \geq -1}$ where $A_{-1} = A$, all other $A_n \in \mathcal{P}$, and for any object $P \in \mathcal{P}$ the augmented simplicial set $\mathrm{Hom}(P, \mathbb{A})$ is Kan and contractible. This is the reason for our condition

on the projective class stated above: we must make sure that a $\mathbb{G}$-resolution is also a $\mathcal{P}$-resolution.

Section 3.1 sets the scene by explaining the definition of a $\mathcal{P}$-resolution in detail. Section 3.2 is devoted to the Comparison Theorem 3.2.3: if $\mathbb{P}$ is a simplicial object over B with each $P_i \in \mathcal{P}$, and $\mathbb{A}$ is a simplicial object over A such that all augmented simplicial sets $\mathrm{Hom}(P_i, \mathbb{A})$ are contractible and Kan, then any map $f \colon B \longrightarrow A$ extends to a semi-simplicial map $f \colon \mathbb{P} \longrightarrow \mathbb{A}$, and any two such extensions are simplicially homotopic. In this section we also relate our comparison theorem to that of Tierney and Vogel [TV1969], which uses a different definition of resolution, in a category with finite limits.

The Comparison Theorem is used in Section 3.3 to prove the main result of this chapter, Theorem 3.3.11: under the condition on $\mathcal{C}$ mentioned above, any two comonads $\mathbb{G}$ and $\mathbb{K}$ that generate the same class of projectives induce isomorphic homology theories. We obtain it as an immediate consequence of Corollary 3.3.10 which states that, in a semi-abelian category, simplicially homotopic maps have the same homology: if $f \simeq g$ then, for any $n \in \mathbb{N}$, $\mathrm{H}_n f = \mathrm{H}_n g$.

All results in this chapter are joint work with Tim Van der Linden and also appear in our paper [GVdL2007].

3.1 Simplicial resolutions

To obtain the comonadic homology of a given object, we need to consider simplicial resolutions relative to a chosen class of projectives. Here we recall the definition of a *projective class* and give some examples.

3.1.1 Definition (Projective class): Let $\mathcal{C}$ be a category, P an object and $e \colon B \longrightarrow A$ a morphism of $\mathcal{C}$. Then P is called e-**projective**, and e is called P-**epic**, if the induced map

$$\mathrm{Hom}(P, e) = e \circ (\cdot) \colon \mathrm{Hom}(P, B) \longrightarrow \mathrm{Hom}(P, A)$$

is a surjection. That is, for every map $P \longrightarrow A$, there is a (not necessarily unique) map making the following diagram commute:

$$
\begin{array}{ccc}
 & & P \\
 & \swarrow & \downarrow \\
B & \xrightarrow{\ e\ } & A
\end{array}
$$

Let $\mathcal{P}$ be a class of objects of $\mathcal{C}$. A morphism e is called $\mathcal{P}$-**epic** if it is P-epic for every $P \in \mathcal{P}$; the class of all $\mathcal{P}$-epimorphisms is denoted by $\mathcal{P}$-epi. Let $\mathcal{E}$ be a class of morphisms in $\mathcal{C}$. An object P is called $\mathcal{E}$-**projective** if it is e-projective for every e in $\mathcal{E}$; the class of

$\mathcal{E}$-projective objects is denoted $\mathcal{E}$-proj. $\mathcal{C}$ is said **to have enough $\mathcal{E}$-projectives** if for every object Y there is a morphism $P \longrightarrow Y$ in $\mathcal{E}$ with P in $\mathcal{E}$-proj.

A **projective class on** $\mathcal{C}$ is a pair $(\mathcal{P}, \mathcal{E})$, $\mathcal{P}$ a class of objects of $\mathcal{C}$, $\mathcal{E}$ a class of morphisms of $\mathcal{C}$, such that $\mathcal{P} = \mathcal{E}$-proj, $\mathcal{P}$-epi $= \mathcal{E}$ and $\mathcal{C}$ has enough $\mathcal{E}$-projectives. Since, given a projective class $(\mathcal{P}, \mathcal{E})$, $\mathcal{P}$ and $\mathcal{E}$ determine each other, we will sometimes abusively write **the projective class** $\mathcal{P}$ or **the projective class** $\mathcal{E}$.

It is easy to see that any retract of a projective object is also projective, as is any coproduct of projectives.

3.1.2 Example: If $\mathcal{E}$ is the class of regular epimorphisms, $\mathcal{P}$ is called the class of **regular projectives**. In a variety, the class of regular projectives is generated by the free objects, hence there are enough projectives.

The regular projectives in a variety $\mathcal{C}$ are also generated by the values of the canonical comonad $\mathcal{C}$, induced by the forgetful functor to Set. More generally, *any* comonad on a category $\mathcal{C}$ generates a projective class:

3.1.3 Definition (Projective class generated by a comonad): Let $\mathbb{G} = (G, \epsilon, \delta)$ be a comonad on a category $\mathcal{C}$. An object P in $\mathcal{C}$ is called $\mathbb{G}$**-projective** if it is in the projective class $(\mathcal{P}_{\mathbb{G}}, \mathcal{E}_{\mathbb{G}})$ generated by the objects of the form GA. A map in $\mathcal{E}_{\mathbb{G}}$ is called a $\mathbb{G}$**-epimorphism**.

The A-component $\epsilon_A \colon GA \longrightarrow A$ of the counit ϵ is always a $\mathbb{G}$-epimorphism. Indeed, any map $f \colon GB \longrightarrow A$ factors over ϵ_A as $Gf \circ \delta_B$, because $\epsilon_A \circ Gf \circ \delta_B = f \circ \epsilon_{GB} \circ \delta_B = f$. It is now clear that $\mathcal{C}$ has enough projectives of this class, since for any A we have $\epsilon_A \colon GA \longrightarrow A$.

This definition coincides with the definition of $\mathbb{G}$-projectives in [BB1969]. There a $\mathbb{G}$-*projective object* is an object P which admits a map $s \colon P \longrightarrow GP$ such that $\epsilon_P s = 1_P$. Indeed, if $P \in \mathcal{P}$, then the identity on P factors over the $\mathcal{P}$-epimorphism ϵ_P, which gives the splitting s.

A simplicial resolution or a simplicial object in a semi-abelian category $\mathcal{A}$ gives rise to simplicial sets, for example by homing into the simplicial object from a fixed object of the category. As we saw in Section 2.1, an important classical property simplicial sets may satisfy is the Kan property defined in 2.1.9. We need the simplicial objects in the category $\mathcal{C}$ to satisfy a similar property, but relative to a chosen projective class $\mathcal{P}$ on $\mathcal{C}$. For the purposes of this chapter, we will call this the *relative Kan property*.

3.1.4 Definition (relative Kan property): A simplicial object $\mathbb{A}$ is **Kan (relative to $\mathcal{P}$)** when for every object $P \in \mathcal{P}$ the simplicial set $\text{Hom}(P, \mathbb{A})$ is Kan.

3.1.5 Example: If $\mathcal{C}$ is regular with enough regular projectives and $\mathcal{P}$ the induced projective class, saying that $\mathbb{A}$ is Kan relative to $\mathcal{P}$ is the same as saying that the simplicial object $\mathbb{A}$ is Kan, in the internal sense of Definition 2.1.10. Every simplicial object of $\mathcal{C}$ has this Kan property if and only if $\mathcal{C}$ is a Mal'tsev category [CKP1993, Theorem 4.2]. Thus when $\mathcal{C}$ is semi-abelian, every simplicial object is Kan with respect to the class of regular projectives.

Note, however, that $\mathcal{C}$ need not have enough projectives for the internal Kan condition to make sense. The projective objects in the definition of the relative Kan property given here may be replaced by an enlargement of domain as in Definition 2.1.10. In case there are enough projectives, of course the two notions do coincide.

3.1.6 Example: It is well known that the underlying simplicial set of a simplicial group is always Kan. This may be seen as a consequence of the previous example, because the category Gp is a Mal'tsev variety and the forgetful functor $U\colon \mathsf{Gp} \longrightarrow \mathsf{Set}$ is represented by the group of integers $\mathbb{Z}$.

Since $\mathcal{C}$ is an arbitrary category (without any extra structure) and $\mathcal{A}$ is just semi-abelian (rather than abelian), we have to be careful when considering simplicial resolutions of objects of $\mathcal{C}$. Definition 3.1.7 seems to suit our purposes.

3.1.7 Definition (Simplicial resolution): Let $\mathcal{P}$ be a projective class. A $\mathcal{P}$-**resolution of** A is an augmented simplicial object $\mathbb{A} = (A_n)_{n \geq -1}$ with $A_{-1} = A$, where $A_n \in \mathcal{P}$ for $n \geq 0$, and for every object $P \in \mathcal{P}$ the augmented simplicial set $\mathrm{Hom}(P, \mathbb{A})$ is Kan and contractible.

In this chapter, we focus on simplicial resolutions in a category $\mathcal{C}$ which are generated by a comonad $\mathbb{G}$ on $\mathcal{C}$. For any $\mathbb{G}$-projective object P, the simplicial set $\mathrm{Hom}(P, \mathbb{G}A)$ is contractible: choose a splitting s for $\epsilon_P\colon GP \longrightarrow P$; given a map $f\colon P \longrightarrow G^{n+1}A$, define $h_n(f) = Gf{\circ}s\colon P \longrightarrow G^{n+2}A$. The morphisms $h_n\colon \mathrm{Hom}(P, G^{n+1}A) \longrightarrow \mathrm{Hom}(P, G^{n+2}A)$ then satisfy $\partial_0 h_n = 1_{\mathrm{Hom}(P,G^{n+1}A)}$ and $\partial_i h_n = h_{n-1}\partial_{i-1}$ for $i > 0$. Thus they give a contraction of the simplicial set $\mathrm{Hom}(P, \mathbb{G}A)$. Later we assume that the category $\mathcal{C}$ and the projective class $\mathcal{P}$ generated by $\mathbb{G}$ are such that $\mathbb{G}A$ is Kan relative to $\mathcal{P}$ for any object A, so that $\mathbb{G}A$ is a $\mathcal{P}$-resolution of A.

In the case when $\mathcal{C}$ is a category with finite limits, there exists another definition of simplicial resolution, using *simplicial kernels*. We give the definition of simplicial kernels here so that we can relate our Comparison Theorem of the next section to that of Tierney and Vogel [TV1969].

3.1.8 Definition (Simplicial kernels): [TV1969] Let

$$(f_i \colon B \longrightarrow A)_{0 \leq i \leq n}$$

be a sequence of $n+1$ morphisms in the category $\mathcal{C}$. A **simplicial kernel** of $(f_0, \ldots, f_n)$ is a sequence

$$(k_i \colon K \longrightarrow B)_{0 \leq i \leq n+1}$$

of $n+2$ morphisms in $\mathcal{C}$ satisfying $f_i k_j = f_{j-1} k_i$ for $0 \leq i < j \leq n+1$, which is universal with respect to this property. In other words, it is the limit for a certain diagram in $\mathcal{C}$.

For example, the simplicial kernel of one map is just its kernel pair. If $\mathcal{C}$ has finite limits, simplicial kernels always exist. We can then factor any augmented simplicial object through its simplicial kernels as follows:
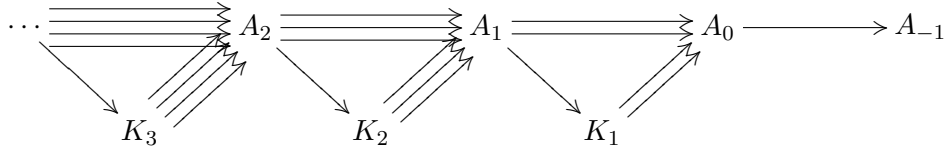
Here the K_{n+1} are the simplicial kernels of the maps $(\partial_i)_i \colon A_n \longrightarrow A_{n-1}$. This gives a definition of $\mathcal{P}$-exact simplicial objects:

3.1.9 Definition: [TV1969] Let $\mathcal{P}$ be a projective class. An augmented simplicial object $\mathbb{A} = (A_n)_{n \geq -1}$ is called $\mathcal{P}$**-exact** when the comparison maps to the simplicial kernels and the map $\partial_0 \colon A_0 \longrightarrow A_{-1}$ are $\mathcal{P}$-epimorphisms.

3.1.10 Remark: It can be shown that for any $\mathcal{P}$-exact simplicial object $\mathbb{A}$, the simplicial set $\mathrm{Hom}(P, \mathbb{A})$ is contractible for any $P \in \mathcal{P}$. We will call this property of $\mathbb{A}$ **relative contractibility**.

A resolution in the Tierney-Vogel sense is then a $\mathcal{P}$-exact augmented simplicial object $\mathbb{A}$ in which all objects A_n for $n \geq 0$ are in the projective class $\mathcal{P}$. For their definition they need the presence of simplicial kernels, so they have to assume for example that the category $\mathcal{C}$ has finite limits. In our definition all assumptions are on the comonad $\mathbb{G}$ or rather the induced projective class $\mathcal{P}$, and not on the category $\mathcal{C}$. In the next section we will make clear the connections between our definition and theirs.

3.2 The Comparison Theorem

Let $\mathcal{P}$ be a projective class on $\mathcal{C}$.

3.2.1 Lemma: *Let $P \in \mathcal{P}$, and let $\mathbb{A}$ be an augmented simplicial object for which the augmented simplicial set $\mathrm{Hom}(P, \mathbb{A})$ is contractible and Kan. Let $n \geq 0$. Given a sequence of maps $(a_i \colon P \longrightarrow A_{n-1})_{i \in [n]}$ satisfying $\partial_i a_j = \partial_{j-1} a_i$ for $i < j$, there is a map $a \colon P \longrightarrow A_n$ with $\partial_i a = a_i$.*

Proof. Define the maps $b_{i+1} = h_{n-1}(a_i)$, where $(h_n)_{n \geq -1}$ is the contraction of the simplicial set $\mathrm{Hom}(P, \mathbb{A})$. These maps satisfy $\partial_0 b_{i+1} = a_i$, and also $\partial_j b_{i+1} = \partial_{i+1} b_{j+1}$ for $i < j \leq n$, since $(\partial_j h_{n-1})(a_i) = h_{n-2}(\partial_{j-1} a_i)$, and $\partial_{j-1} a_i = \partial_i a_j$ for $i < j$.
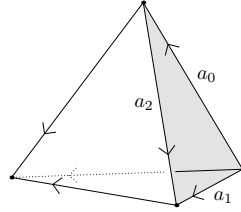
Thus they form an $(n + 1, 0)$-horn in the simplicial set $\mathrm{Hom}(P, \mathbb{A})$, and since we are assuming that this simplicial set is Kan, this horn has a filler $b \colon P \longrightarrow A_{n+1}$. This gives the required map $a = \partial_0 b$. $\qquad\qquad\square$

3.2.2 Remark: This lemma shows that in the presence of finite limits our $\mathcal{P}$-resolutions are also simplicial resolutions in the sense of Tierney and Vogel [TV1969]; that is, the comparison maps to the simplicial kernels are $\mathcal{P}$-epimorphisms. Together with Remark 3.1.10 we see that $\mathcal{P}$-exactness and relative contractibility are equivalent in the situation when we have finite limits and any simplicial object is Kan relative to $\mathcal{P}$. So if $\mathcal{C}$ has finite limits, the Comparison Theorem 2.4 from [TV1969] is more general than the one following in this section, but in the absence of finite limits our Comparison Theorem still works.

We now prove our Comparison Theorem using the above lemma.

3.2.3 Theorem (Comparison Theorem): *Let $\mathbb{P}$ be a simplicial object over B with each $P_i \in \mathcal{P}$, and let $\mathbb{A}$ be a simplicial object over A, for which all the augmented simplicial sets $\mathrm{Hom}(P_i, \mathbb{A})$ are contractible and Kan. Then any map $f \colon B \longrightarrow A$ can be extended to a semi-simplicial map $f \colon \mathbb{P} \longrightarrow \mathbb{A}$, and any two such extensions are simplicially homotopic.*

Proof. We construct this semi-simplicial map inductively, using Lemma 3.2.1.

$$
\begin{array}{ccccccc}
\cdots \Rrightarrow & P_2 & \Rrightarrow & P_1 & \Rightarrow & P_0 & \longrightarrow & B \\
& \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}=f} \\
\cdots \Rrightarrow & A_2 & \Rrightarrow & A_1 & \Rightarrow & A_0 & \longrightarrow & A
\end{array}
$$

Suppose the maps $f_j \colon P_j \longrightarrow A_j$ are given for $-1 \leq j < n$, and commute appropriately with the ∂_i. This gives us $n + 1$ maps $a_i \colon P_n \longrightarrow A_{n-1}$, where $i \in [n]$, by composing the $\partial_i \colon P_n \longrightarrow P_{n-1}$ with f_{n-1}. These maps satisfy $\partial_i a_j = \partial_{j-1} a_i$ for $i < j$, since $\partial_i f_{n-1} = f_{n-2} \partial_i$, and the ∂_j in $\mathbb{P}$ satisfy the simplicial identities. Thus we can use Lemma 3.2.1 to obtain the map $f_n \colon P_n \longrightarrow A_n$ such that $\partial_i f_n = a_i = f_{n-1} \partial_i$.

Now suppose $f \colon \mathbb{P} \longrightarrow \mathbb{A}$ and $g \colon \mathbb{P} \longrightarrow \mathbb{A}$ are two semi-simplicial maps commuting with $f \colon B \longrightarrow A$. We construct a homotopy $h_i^n \colon P_n \longrightarrow A_{n+1}$ for $n \geq 0$ and $0 \leq i \leq n$, which satisfies $\partial_0 h_0^n = f_n$, $\partial_{n+1} h_n^n = g_n$ and

$$
\partial_i h_j^n = \begin{cases} h_{j-1}^{n-1} \partial_i & \text{for } i < j \\ \partial_i h_{i-1}^n & \text{for } i = j \neq 0 \\ h_j^{n-1} \partial_{i-1} & \text{for } i > j + 1. \end{cases}
$$

h_0^0 can be constructed using Lemma 3.2.1. Suppose the h_j^k exist for $k < n$ and commute appropriately with the ∂_i. Then h_0^n must satisfy $\partial_0 h_0^n = f_n$, $\partial_1 h_0^n = \partial_1 h_1^n$ and $\partial_i h_0^n = h_0^{n-1} \partial_{i-1}$ for $i > 1$. Of these maps, all are known except for $\partial_1 h_1^n$. Setting $a_0^0 = f_n$ and $a_i^0 = h_0^{n-1} \partial_{i-1}$ for $i > 1$, we form an $(n + 1, 1)$-horn in $\mathrm{Hom}(P_n, A)$. A filler for this horn gives h_0^n, and also $a_1^0 = \partial_1 h_1^n$, which is needed for the next step. Now suppose h_j^n are given for $j < l$, and we have $a_l^{l-1} = \partial_l h_l^n = \partial_l h_{l-1}^n$. Then $a_i^l = h_{l-1}^{n-1} \partial_i$ for $i < l$, $a_l^l = a_l^{l-1}$ and $a_i^l = h_l^{n-1} \partial_{i-1}$ for $i > l + 1$ form an $(n + 1, l + 1)$-horn. A filler for this gives h_l^n and $a_{l+1}^l = a_{l+1}^{l+1}$ for the next step. In the last step we have $a_i^n = h_{n-1}^{n-1} \partial_i$ for $i < n$, $a_n^n = a_n^{n-1} = \partial_n h_{n-1}^n$ and $a_{n+1}^n = g_n$. Then we use Lemma 3.2.1 again to get h_n^n. $\qquad\square$

3.3 Comonads generating the same projective class

In this section we will need an assumption on the category $\mathcal{C}$ and the projective class $\mathcal{P}$ generated by the comonad $\mathbb{G}$.

3.3.1 Definition (Kan projective class): Let $\mathbb{G}$ be a comonad on a category $\mathcal{C}$ and let $\mathcal{P}$ be the projective class generated $\mathbb{G}$. The projective class $\mathcal{P}$ is called a **Kan projective class on** $\mathcal{C}$ when any augmented simplicial object $\mathbb{A}$ which is relatively contractible is also Kan relative to $\mathcal{P}$.

In particular, when $\mathbb{G}$ generates a Kan projective class, the simplicial object $\mathbb{G}A$ is Kan relative to $\mathcal{P}$ for any object A. Thus if $\mathbb{K}$ is a second comonad which generates the same projective class, the simplicial object $\mathbb{K}A$ is automatically also Kan relative to $\mathcal{P}$.

3.3.2 Example (Additive categories): If $\mathcal{C}$ is an additive category, any projective class $\mathcal{P}$ in $\mathcal{C}$ is a Kan projective class, since then for any simplicial object $\mathbb{A}$ and any object P, the simplicial set $\mathrm{Hom}(P, \mathbb{A})$ is actually a simplicial group and thus Kan (cf. Example 3.1.6). For example, any comonad on the category $R\text{-}\mathsf{Mod}$ of (left) R-modules generates a Kan projective class, including both the "absolute" comonad generated by the forgetful/free adjunction to Set and the "relative" comonad induced by a ring homomorphism $\phi\colon R \longrightarrow S$ as in Example 2.2.3.

3.3.3 Example (Regular projectives): When $\mathcal{C}$ is a regular category and the projective class $\mathcal{P}$ is the class of regular projectives, as remarked in Example 3.1.5, saying that a simplicial object $\mathbb{A}$ is Kan relative to $\mathcal{P}$ is the same as saying $\mathbb{A}$ is internally Kan in $\mathcal{C}$. Thus when $\mathcal{C}$ is also Mal'tsev, every simplicial object is Kan [CKP1993, Theorem 4.2], and $\mathbb{G}A$ is a $\mathcal{P}$-resolution. This includes the forgetful/free comonads on the categories Gp of groups, Rng of non-unital rings, XMod of crossed modules, Comm of commutative rings, $K\text{-}\mathsf{Alg}$ of associative K-algebras, etc. In fact it includes any variety $\mathcal{C}$ with the comonad generated by the forgetful functor to Set, as the following argument shows.

Given a comonad $\mathbb{G}$ on a category $\mathcal{C}$ which comes from an adjunction

$$\mathcal{C} \xrightarrow{\quad G \quad} \mathcal{C}$$
$$U \searrow \qquad \nearrow F$$
$$\mathcal{D}$$

we can determine the class of morphisms of the projective class $(\mathcal{P}, \mathcal{E})$ generated by $\mathbb{G}$ in the following way:

Given an object A and a morphism $e\colon B \longrightarrow C$ in $\mathcal{C}$, the diagram

$$
\begin{array}{c}
GA \\
\downarrow{f} \\
B \xrightarrow{\ e\ } C
\end{array}
$$

corresponds via the adjunction to

$$
\begin{array}{c}
UA \\
\downarrow \\
UB \xrightarrow{\ Ue\ } UC
\end{array}
$$

If e is in $\mathcal{E}$, by choosing $A = C$ and $f = \epsilon_C$, we see that Ue must be split in $\mathcal{D}$, since ϵ_C corresponds under the adjunction to 1_{UC}. Conversely if Ue is a split epimorphism in $\mathcal{D}$, we can factor any map $UA \longrightarrow UC$ over Ue, which implies that we can factor any map $f \colon GA \longrightarrow C$ over e in $\mathcal{C}$, thus $e \in \mathcal{E}$. Therefore the class $\mathcal{E}$ is exactly the class of morphisms whose images under U are split in $\mathcal{D}$. Thus when $\mathcal{C}$ is a variety and U is the forgetful functor to Set, we will always get the class of regular projectives on $\mathcal{C}$.

The first step towards our goal is to show that the two simplicial resolutions $\mathbb{G}A$ and $\mathbb{K}A$ are homotopy equivalent.

3.3.4 Lemma: *Let $\mathbb{G}$ and $\mathbb{K}$ be two comonads on $\mathcal{C}$ which generate the same Kan projective class $\mathcal{P}$. Then the simplicial objects $\mathbb{G}A$ and $\mathbb{K}A$ are homotopy equivalent for any object A.*

Proof. Our assumptions on $\mathcal{C}$ and $\mathcal{P}$ imply that for any object A, the simplicial objects $\mathbb{G}A$ and $\mathbb{K}A$ are both $\mathcal{P}$-resolutions of A. Thus we can use the Comparison Theorem 3.2.3 to get semi-simplicial maps $f \colon \mathbb{G}A \longrightarrow \mathbb{K}A$ and $g \colon \mathbb{K}A \longrightarrow \mathbb{G}A$ which commute with the identity on A.

$$
\begin{array}{ccccccc}
\cdots \Rrightarrow G^3 A & \Rightarrow & G^2 A & \Rightarrow & GA & \longrightarrow & A \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1_A} \\
\cdots \Rrightarrow K^3 A & \Rightarrow & K^2 A & \Rightarrow & KA & \longrightarrow & A \\
\downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle 1_A} \\
\cdots \Rrightarrow G^3 A & \Rightarrow & G^2 A & \Rightarrow & GA & \longrightarrow & A
\end{array}
$$

Using the second part of the Comparison Theorem we see that both fg and gf are homotopic to the identity on $\mathbb{K}A$ and $\mathbb{G}A$ respectively. Thus $\mathbb{G}A$ and $\mathbb{K}A$ are homotopy equivalent. $\qquad\square$

3.3.5 Remark: In this case we don't actually need the full strength of the second half of Theorem 3.2.3. For any semi-simplicial map $f \colon \mathbb{G}A \longrightarrow \mathbb{G}A$ which commutes with the identity on A, we can use the homotopy $h_i^n = (G^{i+1} f_{n-i}) \sigma_i$ to see that it is homotopic to the identity on $\mathbb{G}A$.

Given a functor $E \colon \mathcal{C} \longrightarrow \mathcal{A}$, the simplicial objects $E\mathbb{G}A$ and $E\mathbb{K}A$ are still homotopy equivalent. We now show that, when $\mathcal{A}$ is a semi-abelian category, two simplicially homotopic semi-simplicial maps induce the same map on homology (see also [VdL2006]). For this we need to define a special simplicial object, so that all the maps that form a simplicial homotopy are taken together to form a single semi-simplicial map. We do this by defining the following limit objects A_n^I.

3.3.6 Notation: Suppose that $\mathcal{A}$ has finite limits and let $\mathbb{A}$ be a simplicial object in $\mathcal{A}$. Put $A_0^I = A_1$ and, for $n > 0$, let A_n^I be the limit (with projections $\mathrm{pr}_1, \ldots, \mathrm{pr}_{n+1} \colon A_n^I \longrightarrow A_{n+1}$) of the zigzag

in $\mathcal{A}$.

Let $\epsilon_0(\mathbb{A})_n, \epsilon_1(\mathbb{A})_n \colon A_n^I \longrightarrow A_n$ and $s(\mathbb{A})_n \colon A_n \longrightarrow A_n^I$ denote the morphisms respectively defined by

$$\begin{aligned} \epsilon_0(\mathbb{A})_0 &= \partial_0 & \epsilon_1(\mathbb{A})_0 &= \partial_1 \\ \epsilon_0(\mathbb{A})_n &= \partial_0\mathrm{pr}_1, & \epsilon_1(\mathbb{A})_n &= \partial_{n+1}\mathrm{pr}_{n+1}, \end{aligned} \quad \text{and} \quad s(\mathbb{A})_n = (\sigma_0, \ldots, \sigma_n).$$

3.3.7 Proposition: *Let $\mathbb{A}$ be a simplicial object in a finitely complete category $\mathcal{A}$. Then the faces $\partial_i^I \colon A_n^I \longrightarrow A_{n-1}^I$ and degeneracies $\sigma_i^I \colon A_n^I \longrightarrow A_{n+1}^I$ given by*

$$\begin{aligned} \partial_0^I &= \partial_0\mathrm{pr}_2 \colon A_1^I \longrightarrow A_0^I \\ \partial_1^I &= \partial_2\mathrm{pr}_1 \colon A_1^I \longrightarrow A_0^I \qquad \mathrm{pr}_j\partial_i^I = \begin{cases} \partial_{i+1}\mathrm{pr}_j & \text{if } j \leq i \\ \partial_i\mathrm{pr}_{j+1} & \text{if } j > i \end{cases} \colon A_n^I \longrightarrow A_n \\ \sigma_0^I &= (\sigma_1, \sigma_0) \colon A_0^I \longrightarrow A_1^I \end{aligned}$$

$$\mathrm{pr}_k\sigma_i^I = \begin{cases} \sigma_{i+1}\mathrm{pr}_k & \text{if } k \leq i+1 \\ \sigma_i\mathrm{pr}_{k-1} & \text{if } k > i+1 \end{cases} \colon A_n^I \longrightarrow A_{n+2},$$

for $i \in [n]$, $1 \leq j \leq n$ and $1 \leq k \leq n+2$, determine a simplicial object $\mathbb{A}^I$. The morphisms mentioned in Notation 3.3.6 above form simplicial morphisms

$$\epsilon_0(\mathbb{A}), \epsilon_1(\mathbb{A}) \colon \mathbb{A}^I \longrightarrow \mathbb{A} \qquad \text{and} \qquad s(\mathbb{A}) \colon \mathbb{A} \longrightarrow \mathbb{A}^I$$

such that $\epsilon_0(\mathbb{A}) \circ s(\mathbb{A}) = 1_{\mathbb{A}} = \epsilon_1(\mathbb{A}) \circ s(\mathbb{A})$. In other words, $(\mathbb{A}^I, \epsilon_0(\mathbb{A}), \epsilon_1(\mathbb{A}), s(\mathbb{A}))$ forms a cocylinder on $\mathbb{A}$.

Two semi-simplicial maps $f, g \colon \mathbb{B} \longrightarrow \mathbb{A}$ are simplicially homotopic if and only they are homotopic with respect to the cocylinder $(\mathbb{A}^I, \epsilon_0(\mathbb{A}), \epsilon_1(\mathbb{A}), s(\mathbb{A}))$: there exists a semi-simplicial map $h \colon \mathbb{B} \longrightarrow \mathbb{A}^I$ satisfying $\epsilon_0(\mathbb{A}) \circ h = f$ and $\epsilon_1(\mathbb{A}) \circ h = g$. $\qquad\square$

Using a Kan property argument, we now give a direct proof that homotopic semi-simplicial maps have the same homology.

3.3.8 Proposition: *Let $\mathbb{A}$ be a simplicial object in a semi-abelian category $\mathcal{A}$; consider*

$$\epsilon_0(\mathbb{A})\colon \mathbb{A}^I \longrightarrow \mathbb{A}.$$

For every $n \in \mathbb{N}$, $\mathrm{H}_n\epsilon_0(\mathbb{A})$ is an isomorphism.

Proof. Recall that in a semi-abelian category every simplicial object is Kan, relative to the class of regular epimorphisms. Using the Kan property, we show that the commutative diagram

$$
\begin{array}{ccc}
\mathrm{N}_{n+1}\mathbb{A}^I & \xrightarrow{\ \mathrm{N}_{n+1}\epsilon_0(\mathbb{A})\ } & \mathrm{N}_{n+1}\mathbb{A} \\
{\scriptstyle d'_{n+1}}\downarrow & & \downarrow{\scriptstyle d'_{n+1}} \\
\mathrm{Z}_n\mathbb{A}^I & \xrightarrow[\ \mathrm{Z}_n\epsilon_0(\mathbb{A})\ ]{} & \mathrm{Z}_n\mathbb{A}
\end{array}
$$

is a generalised regular pushout (see Definition 1.1.19); then it is also a pushout, and Lemma 1.1.17 implies that the induced map $\mathrm{H}_n\mathbb{A}^I \longrightarrow \mathrm{H}_n\mathbb{A}$ is an isomorphism. Consider morphisms $z\colon Y_0 \longrightarrow \mathrm{Z}_n\mathbb{A}^I$ and $a\colon Y_0 \longrightarrow \mathrm{N}_{n+1}\mathbb{A}$ that satisfy $d'_{n+1}{\circ}a = \mathrm{Z}_n\epsilon_0(\mathbb{A}){\circ}z$. It is enough to show that there exist a regular epimorphism $y\colon Y \longrightarrow Y_0$ and a morphism $h\colon Y \longrightarrow \mathrm{N}_{n+1}\mathbb{A}^I$ satisfying $d'_{n+1}{\circ}h = z{\circ}y$ and $\mathrm{N}_{n+1}\epsilon_0(\mathbb{A}){\circ}h = a{\circ}y$: this implies that the comparison map to the pullback is a regular epimorphism, by Lemma 1.1.10 and the fact that the morphisms of a limit cone form a jointly monic family.

We first sketch the geometric idea of this in the case $n = 0$. Consider $a = a_0$ and $z = z_0$ as in Figure 3.1; then (up to enlargement of domain) using the Kan property twice yields the needed (h_0, h_1) in $\mathrm{N}_1\mathbb{A}^I$.
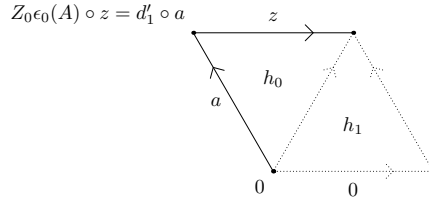
Figure 3.1: Using the Kan property twice to obtain (h_0, h_1) in $\mathrm{N}_1\mathbb{A}^I$.

For arbitrary n, write

$$a_0 = \bigcap_j \ker\partial_j{\circ}a\colon Y_0 \longrightarrow A_{n+1},$$

and $(z_0, \ldots, z_n) = \bigcap_j \ker\partial_j{\circ}z$. Note that as $z\colon Y_0 \longrightarrow \mathrm{Z}_n\mathbb{A}^I$, we have $\partial_i^I z = 0$ for $i \in [n]$, which implies $\partial_i z_{j-1} = 0$ for $i < j - 1$ and $i > j$, where $1 \leq j \leq n + 1$. We also have $\partial_j z_{j-1} = \partial_j z_j$ for $1 \leq j \leq n$ from the definition of the objects A_n^I. The map a_0 in turn satisfies $\partial_i a_0 = 0$ for $i \in [n]$, and $\partial_{n+1} a_0 = \partial_0 z_0$. This last equality follows from $d'_{n+1}{\circ}a = \mathrm{Z}_n\epsilon_0(\mathbb{A}){\circ}z$.

Suppose we have regular epimorphisms $y_k \colon Y_k \longrightarrow Y_{k-1}$ for $1 \leq k \leq n+2$, and morphisms $h_{k-1} \colon Y_k \longrightarrow A_{n+2}$ satisfying $\partial_i \circ h_{k-1} = 0$ for $1 \leq k \leq n+2$ and $i \notin \{k-1, k, n+2\}$, and $\partial_{n+2} h_{k-1} = z_{k-1} y_1 \cdots y_k$ for $1 \leq k \leq n+1$, and also $\partial_0 h_0 = a_0 y_1$. We set $y = y_1 \circ \cdots \circ y_{n+2}$. This gives us the required map

$$h = (h_0 \circ y_2 \circ \cdots \circ y_{n+2}, \ldots, h_{n+1}) \colon Y_{n+2} \longrightarrow \mathrm{N}_{n+1} \mathbb{A}^I$$

which satisfies $d'_{n+1} \circ h = z \circ y$ and $\mathrm{N}_{n+1} \epsilon_0(\mathbb{A}) \circ h = a \circ y$.

We construct these maps y_k and h_k inductively. To get h_0 we form an $(n+2, 1)$-horn $(b_i \colon Y_0 \longrightarrow A_{n+1})_i$ of A by setting $b_0 = a_0$, $b_{n+2} = z_0$ and $b_i = 0$ for $1 < i < n+2$. A filler for this horn gives $y_1 \colon Y_1 \longrightarrow Y_0$ and $h_0 \colon Y_1 \longrightarrow A_{n+2}$ satisfying $\partial_0 h_0 = a_0 y_1$. Now suppose for $1 \leq k \leq n+1$ we have $a_k = \partial_k h_{k-1} \colon Y_k \longrightarrow A_{n+1}$ with $\partial_i \circ a_k = 0$ for $i \in [n]$, and $\partial_{n+1} a_k = \partial_k z_{k-1} y_1 y_2 \cdots y_k$. Then we can form an $(n+2, k+1)$-horn by setting $b_k = a_k$, $b_{n+2} = z_k y_1 \cdots y_k$ and $b_i = 0$ for $i < k$ and $k+1 < i < n+2$, which, when filled, induces y_{k+1} and h_k with the desired properties. $\qquad \square$

3.3.9 Remark: A homology functor H_n involves an implicit choice of colimits: the cokernels involved in the construction of the $\mathrm{H}_n \mathbb{A}$. We may, and from now on we will, assume that these colimits are chosen in such a way that $\mathrm{H}_n \epsilon_0(\mathbb{A})$ is an *identity* instead of just an isomorphism. This gives us the equality in the next corollary.

3.3.10 Corollary: *If $f \simeq g$ then, for any $n \in \mathbb{N}$, $\mathrm{H}_n f = \mathrm{H}_n g$.*

Proof. Proposition 3.3.8 states that $\mathrm{H}_n \epsilon_0(\mathbb{A})$ is an isomorphism; by a careful choice of colimits in the definition of H_n, we may assume that $\mathrm{H}_n \epsilon_0(\mathbb{A}) = 1_{\mathrm{H}_n \mathbb{A}} = \mathrm{H}_n \epsilon_1(\mathbb{A})$.

If now h is a homotopy $f \simeq g$, then $\mathrm{H}_n f = \mathrm{H}_n \epsilon_0(\mathbb{A}) \circ \mathrm{H}_n h = \mathrm{H}_n \epsilon_1(\mathbb{A}) \circ \mathrm{H}_n h = \mathrm{H}_n g$. $\qquad \square$

Using the above, we can now prove our Main Theorem.

3.3.11 Theorem: *Let $\mathbb{G}$ and $\mathbb{K}$ be two comonads on $\mathcal{C}$ which generate the same Kan projective class $\mathcal{P}$. Let $E \colon \mathcal{C} \longrightarrow \mathcal{A}$ be a functor into a semi-abelian category. Then the functors $\mathrm{H}_n(-, E)_{\mathbb{G}}$ and $\mathrm{H}_n(-, E)_{\mathbb{K}}$ from $\mathcal{C}$ to $\mathcal{A}$ are isomorphic for all $n \geq 1$.*

Proof. It follows from Lemma 3.3.4 that the simplicial objects $E\mathbb{G}A$ and $E\mathbb{K}A$ are homotopy equivalent. Thus Corollary 3.3.10 implies that $\mathrm{H}_n(A, E)_{\mathbb{G}} \cong \mathrm{H}_n(A, E)_{\mathbb{K}}$. Given a map $f\colon B \longrightarrow A$, the two semi-simplicial maps

$$\mathbb{G}B \longrightarrow \mathbb{K}B \xrightarrow{\mathbb{K}f} \mathbb{K}A \qquad \text{and} \qquad \mathbb{G}B \xrightarrow{\mathbb{G}f} \mathbb{G}A \longrightarrow \mathbb{K}A$$

are both semi-simplicial extensions of f, so they are homotopic by the Comparison Theorem 3.2.3. Again using Corollary 3.3.10, we see that the square

$$\begin{array}{ccc} \mathrm{H}_n\mathbb{G}B & \xrightarrow{\mathrm{H}_n\mathbb{G}f} & \mathrm{H}_n\mathbb{G}A \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathrm{H}_n\mathbb{K}B & \xrightarrow[\mathrm{H}_n\mathbb{K}f]{} & \mathrm{H}_n\mathbb{K}A \end{array}$$

commutes, which proves that the isomorphisms are natural. $\qquad\square$

3.3.12 Remark: In fact, the above isomorphism is also natural in the second variable. If $\alpha\colon E \Longrightarrow F$ is a natural transformation, then the square

$$\begin{array}{ccc} \mathrm{H}_n(A, E)_{\mathbb{G}} & \xrightarrow{\cong} & \mathrm{H}_n(A, E)_{\mathbb{K}} \\ {\scriptstyle \mathrm{H}_n(A,\alpha)_{\mathbb{G}}}\downarrow & & \downarrow{\scriptstyle \mathrm{H}_n(A,\alpha)_{\mathbb{G}}} \\ \mathrm{H}_n(A, F)_{\mathbb{G}} & \xrightarrow[\cong]{} & \mathrm{H}_n(A, F)_{\mathbb{K}} \end{array}$$

also commutes, since

$$\begin{array}{ccc} E\mathbb{G}A & \longrightarrow & E\mathbb{K}A \\ {\scriptstyle \alpha_{\mathbb{G}A}}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathbb{K}A}} \\ F\mathbb{G}A & \longrightarrow & F\mathbb{K}A \end{array}$$

already commutes.

3.3.13 Remark: We could define homology just using a projective class instead of a comonad, since the Comparison Theorem and Corollary 3.3.10 imply that *any* $\mathcal{P}$-resolution of A will give the same homology. Consider for example the following (Tierney-Vogel) resolution in a category with finite limits:

Given an object A, we can find a $\mathcal{P}$-projective object A_0 with a $\mathcal{P}$-epimorphism $\partial_0\colon P_0 \longrightarrow A$, since there are enough $\mathcal{P}$-projectives. We call this a presentation of A. Take the kernel pair of ∂_0, and take the presentation of the resulting object to get P_1. Composition gives two maps ∂_0 and ∂_1 to P_0, and we can take the simplicial kernel of

these and the presentation of the resulting object to get P_3 and so on. This gives a resolution in the Tierney-Vogel sense [TV1969]. When $\mathcal{P}$ is a Kan projective class, it is also a $\mathcal{P}$-resolution in our sense and thus gives the same homology. This resolution is often easier to work with than the functorial $\mathbb{G}A$.

3.3.14 Example (Two comonads on R-Mod): Given a ring homomorphism $\phi\colon S \longrightarrow R$, every R-module can also be considered as an S-module by restricting the R-action to S via ϕ. This gives rise to an adjunction $R \otimes_S (-) \dashv \operatorname{Hom}_R(R, -)$ between the categories of modules, where R is viewed as an S-module.

Now consider two rings S_1 and S_2 with a surjective ring homomorphism $\psi\colon S_1 \longrightarrow S_2$. Let R be another ring, with ring homomorphisms as below which make the diagram commute:

$$
\begin{array}{ccc}
S_1 & & \\
\Big\downarrow{\psi} & \searrow^{\phi_1} & \\
& \phi_2 & R \\
S_2 & \nearrow &
\end{array}
$$

For each $i = 1, 2$ this gives us a comonad on R-Mod using the adjunction above:

$$
\begin{array}{ccc}
R\text{-Mod} & \xrightarrow{\ \ \ G_i\ \ \ } & R\text{-Mod} \\
\searrow_{U_i} & & \nearrow_{R\otimes_{S_i}(-)} \\
& S_i\text{-Mod} &
\end{array}
$$

We write U_i for the forgetful functor $\operatorname{Hom}_R(R, -)$ from R-modules to S_i-modules.

As seen in Example 3.3.3, the projective class generated by $\mathbb{G}_i$ is given by the class of maps in R-Mod which are split as S_i-module maps. An R-module map $e\colon B \longrightarrow C$ is split as an S_i-module map if and only if there exists a function $f\colon C \longrightarrow B$ with $f(sc + s'c') = sf(c) + s'f(c')$ for $s \in \mathrm{I}[\phi_i]$. Since ψ is a surjection, we have $\mathrm{I}[\phi_1] = \mathrm{I}[\phi_2]$, so any R-module map e has the property that $U_1(e)$ is split if and only if $U_2(e)$ is split. Thus the comonads $\mathbb{G}_1$ and $\mathbb{G}_2$ induce the same projective class, and thus give rise to the same homology on R-Mod, for any functor $E\colon R$-Mod $\longrightarrow \mathcal{A}$ to a semi-abelian category $\mathcal{A}$. As mentioned in [BB1969], when using the functors $E = N \otimes_R -$ or $E = \operatorname{Hom}_R(N, -)$ for an R-module N, this homology is Hochschild's S-relative Tor or Ext respectively; so we see we get the same relative Tor or Ext functor for a ring S and any quotient of S. This means we can replace S by the image of ϕ as a submodule of R.

Chapter 4

Homology via Hopf Formulae

In the early 1940s, Hopf proved the well-known formula for the second integral homology of a special kind of topological space, which calculates the homology purely in terms of the fundamental group of the space. This led to the definition of *group* homology independent of a topological space. So today we can write his famous formula as

$$H_2 A = \frac{[P, P] \cap K[p]}{[K[p], P]}$$

where $p \colon P \longrightarrow A$ is a projective presentation of a group A. The group commutators used here can be generalised so that this formula makes sense in any semi-abelian category. Furthermore, it can be extended to higher dimensional Hopf formulae which give the higher homology objects $H_n A$. This course was taken by Everaert, Gran and Van der Linden in [EGVdL2008] and further pursued by Everaert in his thesis [Eve2007]. We give the main background and results here, as they will be needed later on in the thesis.

Though we will work in semi-abelian categories, most of the results in this section do not need coproducts. In particular, all constructions borrowed from [EGVdL2008] and [Eve2007] which take place in a semi-abelian category still work in pointed exact protomodular ones: though these categories need not have coproducts, they still have cokernels of kernels (see [BB2004, Corollary 4.1.3]). This allows us to apply our results to examples where the category being considered is pointed exact protomodular but lacks coproducts, such as the category of finite groups.

In Section 4.1 we introduce the concept of an axiomatically defined class of extensions and the associated higher extensions which are the starting point for the whole theory. Section 4.2 defines strongly $\mathcal{E}$-Birkhoff subcategories, which are a generalisation of the ordinary Birkhoff subcategories we met in Section 1.3. The Galois structures and the induced centralisation and trivialisation functors on which the Hopf formulae crucially build are discussed in Section 4.3. Finally in Section 4.4 we define homology via the Hopf formulae and exhibit the Everaert sequence: a long exact homology sequence which generalises and extends the Stallings-Stammbach sequence known in the case of groups. This sequence and its universal properties play a crucial role in Chapter 5.

The content of this chapter is known material, and mainly taken from the work of Everaert, Gran and Van der Linden [EGVdL2008] and Everaert's thesis [Eve2007]. We will need the results and concepts introduced here in Chapter 5.

4.1 Extensions and higher extensions

The main ingredient for higher Hopf formulae is the concept of higher-dimensional extensions. To arrive at this notion, we will first introduce higher-dimensional arrows. Here $\mathcal{A}$ always denotes a semi-abelian category, unless stated otherwise.

4.1.1 Definition (Higher-dimensional arrows): The category $\mathsf{Arr}^k\mathcal{A}$ consists of k-**dimensional arrows** in $\mathcal{A}$: $\mathsf{Arr}^0\mathcal{A} = \mathcal{A}$, $\mathsf{Arr}^1\mathcal{A} = \mathsf{Arr}\mathcal{A}$ is the category of arrows $\mathsf{Fun}(2, \mathcal{A})$ where 2 is generated by a single map $\varnothing \longrightarrow \{\varnothing\}$, and $\mathsf{Arr}^{k+1}\mathcal{A} = \mathsf{Arr}\mathsf{Arr}^k\mathcal{A}$. Thus a **double arrow** is a commutative square in $\mathcal{A}$, a 3-arrow is a commutative cube, and a k-arrow is a commutative k-cube. Clearly, $\mathsf{Arr}^k\mathcal{A}$ is also semi-abelian. The functor $\ker\colon \mathsf{Arr}^{k+1}\mathcal{A} \longrightarrow \mathsf{Arr}^k\mathcal{A}$ maps a $(k+1)$-arrow a to its kernel $\mathrm{K}[a]$, and a morphism (f', f) between $(k+1)$-arrows b and a to the induced morphism between their kernels.

$$
\begin{array}{ccccc}
\mathrm{K}[b] & \overset{\mathrm{Ker}\,b}{\rightarrowtail} & B' & \overset{b}{\longrightarrow} & B \\
{\scriptstyle \ker(f',f)}\big\downarrow & & {\scriptstyle f'}\big\downarrow & \Downarrow & \big\downarrow{\scriptstyle f} \\
\mathrm{K}[a] & \underset{\mathrm{Ker}\,a}{\rightarrowtail} & A' & \underset{a}{\longrightarrow} & A
\end{array}
$$

Repeating it n times gives a functor $\ker^n\colon \mathsf{Arr}^{k+n}\mathcal{A} \longrightarrow \mathsf{Arr}^k\mathcal{A}$ which sends a $(k+n)$-arrow a to the object $\mathrm{K}^n[a]$ of $\mathsf{Arr}^k\mathcal{A}$.

We now axiomatically define a *class of extensions* as in [Eve2007]. The definitions and most of the results and proofs in this section are taken from Tomas Everaert's thesis [Eve2007].

Given a class of morphisms $\mathcal{E}$ in a semi-abelian category $\mathcal{A}$, we write $\mathrm{ob}\mathcal{E}$ for the class of objects $A \in |\mathcal{A}|$ that occur as domains or codomains of the arrows in $\mathcal{E}$: $A \in \mathrm{ob}\mathcal{E}$ if and only if there is at least one $f \in \mathcal{E}$ with $f\colon A \longrightarrow B$ or $f\colon C \longrightarrow A$. We also write $\mathcal{A}_{\mathcal{E}}$ for the full subcategory of $\mathcal{A}$ determined by the objects of $\mathrm{ob}\mathcal{E}$.

4.1.2 Definition (Extensions): [Eve2007] Let $\mathcal{E}$ be a class of regular epimorphisms in $\mathcal{A}$ with $0 \in \mathrm{ob}\mathcal{E}$. Then $\mathcal{E}$ is called a **class of extensions** when it satisfies the following properties:

(1) $\mathcal{E}$ contains all split epimorphisms $f\colon B \longrightarrow A$ with A and B in $\mathrm{ob}\mathcal{E}$;

(2)   (a) if $f\colon B \longrightarrow A$ and $g\colon C \longrightarrow B$ are in $\mathcal{E}$, then so is their composite $f\circ g$;

      (b) if $f\circ g$ is in $\mathcal{E}$ and B is in $\mathrm{ob}\mathcal{E}$, then g is in $\mathcal{E}$;

(3) morphisms $f \in \mathcal{E}$ are stable under pullback along arrows $h \colon D \longrightarrow A$ in $\mathcal{A}$ with $D \in \mathrm{ob}\mathcal{E}$, i.e. in the diagram below, h^*f is again in $\mathcal{E}$ whenever $D \in \mathrm{ob}\mathcal{E}$;

$$
\begin{array}{ccc}
D \times_A B & \dashrightarrow & B \\
{\scriptstyle h^*f}\downarrow & & \downarrow{\scriptstyle f} \\
D & \xrightarrow{\;h\;} & A
\end{array}
$$

(4) given a short exact sequence in $\mathcal{A}$ as below with $B \in \mathrm{ob}\mathcal{E}$, we have $f \in \mathcal{E}$ whenever $K \in \mathrm{ob}\mathcal{E}$;

$$
0 \longrightarrow K \longrightarrow B \xrightarrow{\;f\;} A \longrightarrow 0
$$

(5) given a commutative diagram as below with short exact rows in $\mathcal{A}$, whenever both g and k are in $\mathcal{E}$ and $B \in \mathrm{ob}\mathcal{E}$, then we also have $b \in \mathcal{E}$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & C & \xrightarrow{\;g\;} & A & \longrightarrow & 0 \\
& & {\scriptstyle k}\downarrow & & {\scriptstyle b}\downarrow & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & B & \xrightarrow{\;f\;} & A & \longrightarrow & 0
\end{array}
$$

We call an arrow $f \in \mathcal{E}$ an $\mathcal{E}$-extension, or just an extension, and write $B \longrightarrow\!\!\!\!\!> A$.

4.1.3 Remark: As we have $0 \in \mathrm{ob}\mathcal{E}$, condition (3) implies that the kernel $\mathrm{K}[f]$ of any extension f is in $\mathrm{ob}\mathcal{E}$. This gives the converse to (4), so that we have: for any short exact sequence in $\mathcal{A}$ as in (4), $K \in \mathrm{ob}\mathcal{E}$ if and only if $f \in \mathcal{E}$.

4.1.4 Example: The leading example of a class of extensions is the class of all regular epimorphisms in $\mathcal{A}$. In this case we have $\mathrm{ob}\mathcal{E} = |\mathcal{A}|$, all objects of the category $\mathcal{A}$. In fact, it follows from (4) that the class of all regular epimorphisms is the *only* class $\mathcal{E}$ with this property.

We also introduce a concept of *higher extensions*.

4.1.5 Definition (Higher extensions): Given a class of extensions $\mathcal{E}$ in a semi-abelian category $\mathcal{A}$, we define the class of n-**fold ($\mathcal{E}$-)extensions** (called n-**extensions** when $\mathcal{E}$ is understood) inductively as follows:

a 0-**extension** is an object in $\mathrm{ob}\mathcal{E}$, a 1-**extension** is an arrow in $\mathcal{E}$, and for $n \geq 1$, an $(n+1)$-**extension** is a morphism (f', f) in $\mathsf{Arr}^n\mathcal{A}$ such that all arrows in the induced

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diagram

(**D**)

are n-extensions. Here P is the pullback of a and f. We will say **double ($\mathcal{E}$-)extension** for a 2-fold extension. We denote the class of $(n+1)$-fold $\mathcal{E}$-extensions by $\mathcal{E}^n$, thus $\mathcal{E}^0 = \mathcal{E}$ and $\mathcal{E}^1$ denotes the double $\mathcal{E}$-extensions.

To justify the name of $(n+1)$-extension, we will have to show that the class $\mathcal{E}^n$ really is a class of extensions in the sense of 4.1.2. Clearly it is enough to show that $\mathcal{E}^1$ is a class of extensions, and the rest follows by induction, as we can then view $\mathcal{E}^n$ as $(\mathcal{E}^{n-1})^1$. So we will now concentrate on double extensions.

4.1.6 Remark: Notice that a morphism $(f', f)\colon b \longrightarrow a$ in $\mathsf{Arr}\mathcal{A}$ is a double extension if and only if $(b, a)\colon f' \longrightarrow f$ is a double extension.

We can see that, as any extension is a regular epimorphism, a double extension gives a square in $\mathcal{A}$ which is a regular pushout (see Section 1.1). In particular, a double extension is a regular epimorphism in $\mathsf{Arr}\mathcal{A}$, both viewed as $(f', f)\colon a \longrightarrow b$ and as $(a, b)\colon f' \longrightarrow f$.

We will now prove a property relating double extensions to extensions, reminiscent of condition (4) in Definition 4.1.2. This will be a key ingredient in proving that $\mathcal{E}^1$ is a class of extensions.

4.1.7 Lemma: *Given a commutative diagram in $\mathcal{A}$ with short exact rows as below, where f', f, a and b are extensions,*

the right hand square is a double extension if and only if k is an extension.

Proof. We can decompose the above diagram as follows, with pullback squares as indicated.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \rightarrowtail & B' & \xrightarrow{\ f'\ } & A' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle r} & & \| & & \\
0 & \longrightarrow & K & \rightarrowtail & P & \dashrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle a} & & \\
0 & \longrightarrow & K & \rightarrowtail & B & \xrightarrow{\ f\ } & A & \longrightarrow & 0
\end{array}
$$

By definition, the right hand square of the original diagram is a double extension if and only if the factorisation r to the pullback is an extension. Notice that by Condition (3) of Definition 4.1.2 the pullback $P \longrightarrow A$ of f along a is again an extension and so P is in ob$\mathcal{E}$, and by Remark 4.1.3 both K and K' are also in ob$\mathcal{E}$. If r is an extension, then k is an extension by 4.1.2 (3). Conversely, if k is an extension, then so is r by 4.1.2 (5). $\qquad\square$

4.1.8 Proposition: *Given a class of extensions $\mathcal{E}$ in $\mathcal{A}$, the induced class $\mathcal{E}^1$ is a class of extensions in $\mathsf{Arr}\mathcal{A}$.*

Proof. We remarked earlier that a double extension is indeed a regular epimorphism in $\mathsf{Arr}\mathcal{A}$. Notice that ob$\mathcal{E}^1 = \mathcal{E}$, and clearly $1_0 \in \mathcal{E}$. We have to show that $\mathcal{E}^1$ satisfies conditions (1) to (5) of Definition 4.1.2. To distinguish between these conditions applied to $\mathcal{E}$ and to $\mathcal{E}^1$, we will denote the conditions referring to $\mathcal{E}^1$ by $(1)^1$ to $(5)^1$.

A split epimorphism in $\mathsf{Arr}\mathcal{A}$ is a morphism $(f', f) \colon b \longrightarrow a$ such that both f' and f are split in $\mathcal{A}$ by s' and s respectively, and $(s', s) \colon a \longrightarrow b$ is also a morphism in $\mathsf{Arr}\mathcal{A}$. Using Remark 4.1.6, to show that $(1)^1$ holds it is enough to show that $(b, a) \colon f' \longrightarrow f$ is a double extension whenever a and b are extensions and (f', f) is a split epimorphism in $\mathsf{Arr}\mathcal{A}$ as above. This follows from (1) and Lemma 4.1.7, as the map between the kernels $\mathrm{K}[b]$ and $\mathrm{K}[a]$ is also a split epimorphism.

For the next two conditions we again use Remark 4.1.6, which makes it easy to verify that $(2a)^1$ and $(2b)^1$ follow from Lemma 4.1.7 and $(2a)$ or $(2b)$ respectively.

Condition $(3)^1$ follows from (3) and (2a). The key point here is that when pulling back $(f', f) \colon b \longrightarrow a$ to say $(g', g) \colon e \longrightarrow d$, the square formed by the comparison to the pullbacks

$$
\begin{array}{ccc}
E' & \longrightarrow & Q \\
\downarrow & & \downarrow \\
B' & \dashrightarrow & P
\end{array}
$$

is also a pullback square, and this implies that (g', g) is also an extension.

Condition $(4)^1$ follows easily from (4), (2) and Lemma 4.1.7, and finally Condition $(5)^1$ follows from (5) and (3). Again the squares induced by comparison maps to the pullbacks play a crucial role. $\qquad\square$

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Thus the classes $\mathcal{E}^{n-1}$ are indeed classes of extensions for any $n \geq 1$. We have $\mathrm{ob}\mathcal{E}^n = \mathcal{E}^{n-1}$. The n-extensions $\mathcal{E}^{n-1}$ determine a full subcategory $\mathsf{Ext}^n\mathcal{A}$ of $\mathsf{Arr}^n\mathcal{A}$. Notice that $\mathsf{Ext}^n\mathcal{A} = (\mathsf{Arr}^n\mathcal{A})_{\mathcal{E}^n}$. We sometimes analogously write $\mathsf{Ext}^0\mathcal{A} = \mathcal{A}_{\mathcal{E}}$. When we say that a sequence

$$0 \longrightarrow K[f] \rightarrowtail B \xrightarrow{\ f\ } A \longrightarrow 0$$

is exact in $\mathsf{Ext}^n\mathcal{A}$, we mean that it is an exact sequence in $\mathsf{Arr}^n\mathcal{A}$, and the objects are n-extensions. Recall from Remark 4.1.3 that f is an $n+1$-extension if and only if all three objects are n-extensions.

Roughly, the idea behind this definition of k-extensions is the following: suppose we are given a double extension (f', f) of an object A of $\mathcal{A}$ as in Diagram (**D**), and let α be any element of A. Then in addition to the existence of elements β of B and α' of A' such that $f(\beta) = \alpha$ and $a(\alpha') = \alpha$, there is also an element $\beta' \in B'$ such that $b(\beta') = \beta$ and $f'(\beta') = \alpha'$, whichever β and α' were chosen.

4.2 Strongly ($\mathcal{E}$-)Birkhoff subcategories

Given a class of extensions as in Section 4.1, we will now generalise the notion of Birkhoff subcategory given in Definition 1.3.2. Again this concept is taken from [EGVdL2008] and [Eve2007]. This more general definition will allow us to handle higher extensions at the same time as ordinary Birkhoff subcategories, which makes for clearer statements and proofs later on.

4.2.1 Definition: Given a class of extensions $\mathcal{E}$ in a semi-abelian category $\mathcal{A}$, and a reflective subcategory $\mathcal{B}$ of $\mathcal{A}_{\mathcal{E}}$, we write $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$ for the reflector and denote the unit of the adjunction by η. We call $\mathcal{B}$ a **strongly $\mathcal{E}$-Birkhoff subcategory of** $\mathcal{A}$ if for every ($\mathcal{E}$-)extension $f\colon B \longrightarrow A$ the induced square

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & A \\ {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ IB & \xrightarrow{\ If\ } & IA \end{array}$$

is a double ($\mathcal{E}$-)extension.

4.2.2 Remark: Notice that this immediately implies that the reflector I takes extensions to extensions, and also that η_A is an extension (and so a regular epimorphism) for each A. This in turn implies that $\mathcal{B}$ is closed under subobjects: let $A \xrightarrow{\ m\ } B$ be a mono in $\mathcal{A}$

with $B \in \mathcal{B}$. Then as η is a natural transformation we have

$$
\begin{array}{ccc}
A & \overset{m}{\rightarrowtail} & B \\
{\scriptstyle \eta_A} \downarrow & & \| {\scriptstyle \eta_B} \\
IA & \underset{Im}{\longrightarrow} & IB
\end{array}
$$

and so η_A is both a mono and a regular epi, and thus an isomorphism. In fact it is well known that a reflective subcategory is closed under subobjects if and only if each η_A is a regular epimorphism, but we only need this one direction.

Note that $\mathcal{B}$ is also closed under any limits that exist in $\mathcal{A}_\mathcal{E}$, as it is a reflective subcategory (see for example Proposition 3.5.3 in [Bor1994]).

4.2.3 Example (strongly $\mathcal{E}$-Birkhoff subcategories): When $\mathcal{E}$ is the class of regular epimorphisms in $\mathcal{A}$, strongly $\mathcal{E}$-Birkhoff subcategories of $\mathcal{A}$ coincide with the usual Birkhoff subcategories (see Lemma 1.3.4), and $\mathcal{A}_\mathcal{E} = \mathcal{A}$. Thus the category Ab of abelian groups is a strongly (regular epi)-Birkhoff subcategory of Gp. Later we will meet special classes of higher extensions, the central n-extensions, which form a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathsf{Arr}^n\mathcal{A}$ when $\mathcal{E}$ is the class of n-extensions, so $\mathcal{A}_\mathcal{E} = \mathsf{Ext}^n\mathcal{A}$. This allows us to state results that work at the same time for a semi-abelian category $\mathcal{A}$ with a usual Birkhoff subcategory and for a category of higher extensions $\mathsf{Ext}^n\mathcal{A}$ in $\mathcal{A}$.

There is another criterion for a reflective subcategory $\mathcal{B}$ to be strongly $\mathcal{E}$-Birkhoff, which uses the kernel of the unit η_A for a given object A. We first introduce some notation.

We can view the reflector I as a functor $I: \mathcal{A}_\mathcal{E} \longrightarrow \mathcal{A}_\mathcal{E}$. Then we have another functor $J: \mathcal{A}_\mathcal{E} \longrightarrow \mathcal{A}_\mathcal{E}$, given by $JA = \mathrm{K}[\eta_A]$, which fits into the following short exact sequence of functors.

$$
0 \longrightarrow J \overset{\mu}{\rightarrowtail} 1_{\mathcal{A}_\mathcal{E}} \overset{\eta}{\longrightarrow} I \longrightarrow 0
$$

4.2.4 Proposition: *Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{A}_\mathcal{E}$ such that the unit $\eta_A : A \longrightarrow IA$ is a regular epimorphism for any object $A \in \mathcal{A}$. Then $\mathcal{B}$ is a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ if and only if the functor $J: \mathcal{A}_\mathcal{E} \longrightarrow \mathcal{A}_\mathcal{E}$ preserves extensions.*

Proof. Consider the following diagram with short exact rows:

$$
\begin{array}{ccc}
JB & \rightarrowtail B & \overset{\eta_B}{\longrightarrow} IB \\
{\scriptstyle Jf} \downarrow & {\scriptstyle f} \downarrow & \downarrow {\scriptstyle If} \\
JA & \rightarrowtail A & \underset{\eta_A}{\longrightarrow} IA
\end{array}
$$

By 4.1.7 it follows that if the right hand square is a double extension, then Jf is an extension. Conversely, if Jf is an extension, then 4.1.2(4) implies that η_B and η_A are

extensions, and so by 4.1.2(2) If is as well. Then again by 4.1.7 the right hand square is a double extension for any extension f, and so $\mathcal{B}$ is strongly $\mathcal{E}$-Birkhoff. □

4.3 The Galois structures Γ_n, centralisation and trivialisation

The strongly $\mathcal{E}$-Birkhoff subcategories give rise to Galois structures Γ as defined in 1.4.1. In fact, we will see that a strongly $\mathcal{E}$-Birkhoff subcategory $\mathcal{B}$ induces a whole sequence Γ_n of Galois structures, each one giving rise to the next.

4.3.1 Proposition: *Let $\mathcal{E}$ be a class of extensions (in the sense of 4.1.2) in a semi-abelian category $\mathcal{A}$, and $\mathcal{B}$ a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$. Let $\mathcal{Z}$ be the class of arrows f in $\mathcal{B}$ such that $f \in \mathcal{E}$, and let $I : \mathcal{A}_\mathcal{E} \longrightarrow \mathcal{B}$ be the reflector and $\subseteq : \mathcal{B} \longrightarrow \mathcal{A}_\mathcal{E}$ the inclusion functor. Then $(\mathcal{A}_\mathcal{E}, \mathcal{B}, \mathcal{E}, \mathcal{Z}, I, \subseteq)$ is a Galois structure.*

Proof. This is an immediate consequence of the definitions of a class of extensions 4.1.2 and a strongly $\mathcal{E}$-Birkhoff subcategory 4.2.1. □

4.3.2 Remark: Notice that we take the extensions $\mathcal{E}$ of the Galois structure to be a class of extensions in the sense of 4.1.2, so there is no clash of terminology.

This Galois structure $\Gamma = (\mathcal{A}_\mathcal{E}, \mathcal{B}, \mathcal{E}, \mathcal{Z}, I, \subseteq)$ gives rise to central extensions as defined in 1.4.3. We denote the full subcategory of $\mathsf{Arr}\mathcal{A}$ determined by these central extensions by $\mathsf{CExt}_\mathcal{B}\mathcal{A}$. This gives a subcategory of the category $\mathsf{Ext}\mathcal{A}$ determined by the class of extensions $\mathcal{E}$, which is dependant on the strongly $\mathcal{E}$-Birkhoff subcategory $\mathcal{B}$. When $\mathcal{B}$ is clear from the context, we might also write $\mathsf{CExt}\mathcal{A}$. We will show that this category of central extensions forms a strongly $\mathcal{E}^1$-Birkhoff subcategory of $\mathsf{Arr}\mathcal{A}$, thus giving us another Galois structure as above. To show this, we must first construct a reflector $I_1 : \mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{CExt}_\mathcal{B}\mathcal{A}$.

Recall the short exact sequence of functors $\mathcal{A}_\mathcal{E} \longrightarrow \mathcal{A}_\mathcal{E}$ induced by the reflector $I = I_0$:

$$0 \longrightarrow J \rightarrowtail^{\mu} 1_\mathcal{A} \xrightarrow{\eta} I \longrightarrow 0$$

From this, we build a similar short exact sequence of functors $\mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{Ext}\mathcal{A}$ as follows. (The construction is made pointwise in $\mathsf{Arr}\mathcal{A}$, which has good categorical properties, but the result turns out to be an extension.)

4.3.3 Definition (Centralisation functor): Consider an extension $f \colon B \longrightarrow A$ and its kernel pair $(\mathrm{R}[f], \pi_1, \pi_2)$. Write $J_1[f] = \mathrm{K}[J\pi_1]$ and $J_1 f \colon J_1[f] \longrightarrow 0$.

$$
\begin{array}{ccccc}
J_1[f] = \mathrm{K}[J\pi_1] & \overset{\mathrm{Ker}\, J\pi_1}{\rightarrowtail} & J\mathrm{R}[f] & \overset{J\pi_1}{\underset{J\pi_2}{\rightrightarrows}} & JB \\
\downarrow & & \downarrow{\scriptstyle \mu_{\mathrm{R}[f]}} & & \downarrow{\scriptstyle \mu_B} \\
\mathrm{K}[f] = \mathrm{K}[\pi_1] & \underset{\mathrm{Ker}\, \pi_1}{\rightarrowtail} & \mathrm{R}[f] & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & B
\end{array}
$$

This clearly determines a functor $J_1 \colon \mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{Ext}\mathcal{A}$. Note that $\pi_2 \circ \mathrm{Ker}\, \pi_1 = \mathrm{Ker}\, f$, and the left hand square is a pullback. We define the map $\mu_f^1 \colon J_1 f \longrightarrow f$ as in the left hand square below.

$$
\begin{array}{ccc}
J_1[f] & \overset{\mu_B \circ J\pi_2 \circ \mathrm{Ker}\, J\pi_1}{\longrightarrow} & B \\
{\scriptstyle J_1 f}\downarrow & \overset{\mu_f^1}{\Longrightarrow} & \downarrow{\scriptstyle f} \\
0 & \longrightarrow & A
\end{array}
\qquad
\begin{array}{ccc}
B & \overset{\rho_f^1}{\dashrightarrow} & I_1[f] \\
{\scriptstyle f}\downarrow & \overset{\eta_f^1}{\Longrightarrow} & \downarrow{\scriptstyle I_1 f} \\
A & =\!=\!= & A
\end{array}
$$

The composition $\mu_B \circ J\pi_2 \circ \mathrm{Ker}\, J\pi_1$ is a normal monomorphism, so we can take cokernels, yielding the right hand square. Since μ_f^1 is the kernel of its cokernel, we obtain the short exact sequence

$$
0 \longrightarrow J_1 \overset{\mu^1}{\rightarrowtail} 1_{\mathsf{Ext}\mathcal{A}} \overset{\eta^1}{\dashrightarrow} I_1 \longrightarrow 0
$$

of functors $\mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{Ext}\mathcal{A}$.

4.3.4 Proposition: *The functor $I_1 \colon \mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{Ext}\mathcal{A}$ corestricts to $I_1 \colon \mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{CExt}_{\mathcal{B}}\mathcal{A}$.*

Proof. For a proof see for example [Eve2007, Lemma 1.4.2]. □

This justifies the name of **centralisation functor** for I_1.

4.3.5 Theorem: *Given a strongly $\mathcal{E}$-Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$, the category $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$ is a strongly $\mathcal{E}^1$-Birkhoff subcategory of $\mathsf{Arr}\mathcal{A}$. The reflector is given by $I_1 \colon \mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{CExt}_{\mathcal{B}}\mathcal{A}$.*

Proof. For a proof see for example [Eve2007, Theorem 1.4.3]. □

Now Proposition 4.3.1 implies that $\Gamma_1 = (\mathsf{Ext}\mathcal{A}, \mathsf{CExt}_{\mathcal{B}}\mathcal{A}, \mathcal{E}^1, \mathcal{Z}^1, I_1, \subseteq)$ forms another Galois structure, where $\mathcal{Z}^1$ is defined analogously to $\mathcal{Z}$ above as those maps in $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$ that lie in $\mathcal{E}^1$. This process may be repeated inductively to obtain Galois structures Γ_n and functors $J_n \colon \mathsf{Ext}^n\mathcal{A} \longrightarrow \mathsf{Ext}^n\mathcal{A}$ and $I_n \colon \mathsf{Ext}^n\mathcal{A} \longrightarrow \mathsf{CExt}_{\mathcal{B}}^n\mathcal{A}$. For $n \geq 1$ and an n-extension f, we often call the extension $I_n f$ the **centralisation of f**. We now give some examples of centralisation functors I_1 for the different Birkhoff subcategories we have met.

4.3.6 Remark (Property of group commutators): In the next example, we will be using a property of group commutators which is easy to check: for normal subgroups M and N of a group B with $N \subseteq M$, we have

$$\left[\frac{M}{N}, \frac{B}{N}\right] = \frac{[M, B]}{N}.$$

4.3.7 Example (centralisation functors): When $\mathcal{A} = \mathsf{Gp}$ and $\mathcal{B} = \mathsf{Ab}$, the category of abelian groups, we saw in Example 1.2.6 that $I = \mathrm{ab}$ takes a group G to $G/[G, G]$. We also saw in Example 1.4.4 that an extension $f \colon B \longrightarrow A$ is central if and only if $\mathrm{K}[f] \subseteq ZB$, or equivalently if and only if $[\mathrm{K}[f], B] = 0$. It easily follows, using Remark 4.3.6, that the functor $I_1 = \mathrm{centr}$ takes an extension $f \colon B \longrightarrow A$ to

$$I_1 f = \mathrm{centr}\, f \colon \frac{B}{[\mathrm{K}[f], B]} \longrightarrow A,$$

that is, $J_1[f] = [\mathrm{K}[f], B]$.

When $\mathcal{A} = \mathsf{Gp}$ and $\mathcal{B} = \mathsf{Nil}_m$, the category of nilpotent groups of class at most m, we saw in Example 1.3.3 that $I = \mathrm{nil}_m$ takes a group G to $G/\mathrm{LC}_m G$, for example nil_2 takes G to $G/[[G, G], G]$. Here the centralisation functor I_1 takes an extension $f \colon B \longrightarrow A$ to

$$I_1 f \colon \frac{B}{l_m(\mathrm{K}[f], B, \dots, B)} \longrightarrow A$$

with l_m as in Example 1.4.4. Thus, when $m = 2$, we have $I_1 f \colon B/[[\mathrm{K}[f], B], B] \longrightarrow A$. Again this follows from Example 1.4.4 and Remark 4.3.6.

Similarly when $\mathcal{A} = \mathsf{Gp}$ and $\mathcal{B} = \mathsf{Sol}_m$, the category of solvable groups of class at most m, the reflector $I = \mathrm{sol}_m$ takes a group G to $G/D_m G$, for example sol_2 takes G to $G/[[G, G], [G, G]]$, and I_1 takes an extension $f \colon B \longrightarrow A$ to

$$I_1 f \colon \frac{B}{d_m(\mathrm{K}[f], B, B, \dots, B)} \longrightarrow A.$$

For example, when $m = 2$, we have $I_1 f \colon B/[[\mathrm{K}[f], B], [B, B]] \longrightarrow A$.

When $\mathcal{A} = \mathsf{Leib}_K$ and $\mathcal{B} = \mathsf{Lie}_K$ for a field K (of characteristic $\neq 2$), we saw in Example 1.3.3 the reflector lie: $\mathfrak{g} \longmapsto \mathfrak{g}/\mathfrak{g}^{\mathrm{Ann}}$, where $\mathfrak{g}^{\mathrm{Ann}}$ is the two-sided ideal generated by elements of the form $[x, x]$. Given an extension $f \colon \mathfrak{b} \longrightarrow \mathfrak{a}$, let $[\mathrm{K}[f], \mathfrak{b}]_{\mathrm{lie}}$ be the ideal generated by elements of the form $([k, b] + [b, k])$ for $k \in \mathrm{K}[f]$ and $b \in \mathfrak{b}$. Notice that $[k, b] + [b, k] = [b + k, b + k] - [b, b] - [k, k]$ is an element of $\mathfrak{b}^{\mathrm{Ann}}$. Then the centralisation functor sends $f \colon \mathfrak{b} \longrightarrow \mathfrak{a}$ to

$$I_1 f \colon \frac{\mathfrak{b}}{[\mathrm{K}[f], \mathfrak{b}]_{\mathrm{lie}}} \longrightarrow \mathfrak{a}.$$

We see that $([b, b], [b + k, b + k]) - ([b, b], [b, b]) - ([0, 0], [k, k])$ is an element of $\mathrm{R}[f]^{\mathrm{Ann}}$ which is sent to 0 in $\mathfrak{b}$ by the first projection, so it (or isomorphically, its second projection

$[k, b] + [b, k])$ is an element of $[\mathrm{K}[f], \mathfrak{b}]_{\mathrm{lie}} = J_1[f] = \mathrm{K}[J\pi_1]$. Clearly, for any $k \in \mathrm{K}[f]$, we have $[k, k] = \frac{1}{2}([k, k] + [k, k]) \in [\mathrm{K}[f], \mathfrak{b}]_{\mathrm{lie}}$. Conversely, any element of $\mathrm{R}[f]^{\mathrm{Ann}}$ which maps to 0 under the first projection must be made up of elements of the following sort: either the first entry is of the form $[b, b] - [b, b]$, which must have in the second entry $[b + k, b + k] - [b + k', b + k'] = [b, k] + [k, b] + [k, k] - [b, k'] - [k', b] - [k', k']$ for some $k, k' \in \mathrm{K}[f]$, which is an element of $[\mathrm{K}[f], \mathfrak{b}]_{\mathrm{lie}}$; or the first entry is of the form $[a, a] = 0$, accompanied by $[b, b]$ in the second entry, such that $f(a) = f(b)$. But then $b - a \in \mathrm{K}[f]$ and we have $[b - a, b + a] + [b + a, b - a] = 2[b, b]$ so $[b, b] \in [\mathrm{K}[f], \mathfrak{b}]_{\mathrm{lie}}$. Compare this with the Lie-centre

$$Z_{\mathsf{Lie}}(\mathfrak{b}) = \{z \in \mathfrak{b} \mid [b, z] = -[z, b] \ \forall \ b \in \mathfrak{b}\}$$

from Example 1.4.4.

As mentioned earlier, Theorem 4.3.5 allows us to apply results about strongly $\mathcal{E}$-Birkhoff subcategories at the same time to an ordinary Birkhoff subcategory of a semi-abelian category $\mathcal{A}$ and to the higher central extensions viewed as strongly $\mathcal{E}$-Birkhoff subcategories of $\mathsf{Arr}^n\mathcal{A}$, where $\mathcal{E}$ is the class of n-extensions. Thus when we say "$\mathcal{B}$ is a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$", we have one of these two cases in mind.

4.3.8 Remark: Given an n-extension A, for $n \geq 0$, the centralisation of the $(n + 1)$-extension $!_A \colon A \longrightarrow 0$ turns out to be $I_{n+1}!_A \colon I_n A \longrightarrow 0$.

The following is also often useful, and quite easy to show using the 3×3-Lemma and the fact that $\mathsf{CExt}^n_{\mathcal{B}}\mathcal{A}$ is strongly $\mathcal{E}^n$-Birkhoff.

4.3.9 Lemma: *For an $(n + 1)$-extension $f \colon B \longrightarrow A$, we have*

$$I_n I_{n+1} f = I_n f \colon I_n B \longrightarrow I_n A,$$

i.e., $I_n(I_{n+1}[f]) = I_n B$.

Proof. This proof is taken from [EGVdL2008, Lemma 6.2]. Consider the following diagram, in which the rows and the middle column are exact sequences (here π_1 and π_2 are

the projections of the kernel pair $R[f]$ to B, as in Definition 4.3.3):

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & J_1[f] & = & J_1[f] & \longrightarrow & 0 & & \\
& & \downarrow {\scriptstyle J\pi_2\circ\ker J\pi_1} & \quad & \downarrow {\scriptstyle \mu_f^1} & & \downarrow & & \\
0 & \longrightarrow & JB & \overset{\mu_B}{\rightarrowtail} & B & \overset{\eta_B}{\longrightarrow} & IB & \longrightarrow & 0 \\
& & \downarrow {\scriptstyle J\rho_f} & & \downarrow {\scriptstyle \rho_f} & & \downarrow {\scriptstyle I\rho_f} & & \\
0 & \longrightarrow & JI_1[f] & \overset{\mu_{I_1[f]}}{\rightarrowtail} & I_1[f] & \overset{\eta_{I_1[f]}}{\longrightarrow} & II_1[f] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

The top left square commutes by definition of μ_f^1. Note that this square is a pullback, as μ_B is a monomorphism. Thus $J\pi_2\circ\ker J\pi_1$ is the kernel of $J\rho_f$, and the first column is also an exact sequence. Thus by the 3×3 Lemma, the last column is also exact, which makes $I\rho_f$ an isomorphism. This gives

$$
\begin{array}{ccc}
II_1[f] & \overset{II_1 f}{\longrightarrow} & IA \\
\downarrow {\scriptstyle \cong} & & \| \\
IB & \overset{If}{\longrightarrow} & IA
\end{array}
$$

$\square$

4.3.10 Remark: Given an n-extension f, the only object of $J_n f$ which is non-zero is $\operatorname{dom}^n J_n f$, the "initial" object of the n-cube $J_n f$. This follows easily from the inductive construction of $J_n f$. Thus we have $\operatorname{dom}^n J_n f = \mathrm{K}^n[J_n f]$ for any n-extension f. This also implies that the only object of the n-cube $I_n f$ which differs from f is the initial object $\operatorname{dom}^n I_n f$.

The Galois structures Γ_n also give rise to trivial extensions as defined in 1.4.3. Similarly to the central extensions, the trivial $(n+1)$-extensions of $\mathcal{A}$ form a reflective subcategory $\mathsf{TExt}^{n+1}\mathcal{A}$ of $\mathsf{Ext}^{n+1}\mathcal{A}$; the reflector

$$
T_{n+1}\colon \mathsf{Ext}^{n+1}\mathcal{A} \longrightarrow \mathsf{TExt}^{n+1}\mathcal{A}
$$

maps an extension f to the pullback $T_{n+1}f \colon T_{n+1}[f] \longrightarrow A$ of $I_n f$ along η_A^n, the **trivialisation of** f.
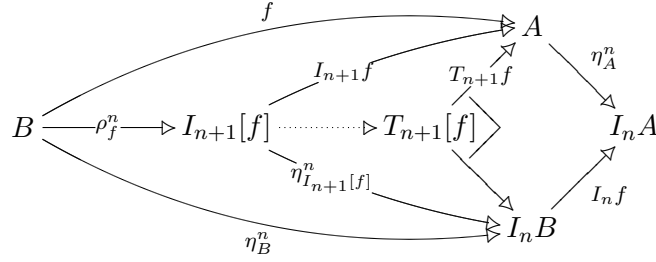
Thus we obtain a comparison map $r_f^{n+1} \colon I_{n+1}[f] \longrightarrow T_{n+1}[f]$, which is a $(n+1)$-extension by the strong $\mathcal{E}^n$-Birkhoff property of the reflector I_n (see 4.2.1) and 4.1.2(2b). This gives an $(n+2)$-extension $I_{n+1}f \longrightarrow T_{n+1}f$.

4.3.11 Remark: Recalling Remark 4.3.10, we see that again the only object of the $(n+1)$-cube $T_{n+1}f$ which differs from f is the initial object $\mathrm{dom}^{n+1} T_{n+1}f$. This implies that the comparison map r_f^{n+1} is the identity everywhere except for on this initial object.

4.4 Hopf formulae

As in the groups case, the Hopf formulae in the general categorical context use projective presentations, so we must define what exactly we mean by a higher projective presentation.

4.4.1 Definition (Projective presentations): An object of $\mathsf{Arr}^k \mathcal{A}$ is called **extension-projective** if it is projective with respect to the class of $(k+1)$-extensions. A k-extension $f \colon B \longrightarrow A$ is called a **(projective) presentation** of A when the object B is extension-projective. A k-extension $f \colon B \longrightarrow A$ is called a k-**fold presentation**, or just k-**presentation**, when the object B is extension-projective and A is a $(k-1)$-presentation. (A 0-**presentation** is an object of $\mathcal{A}_\mathcal{E}$.) Given an object A of $\mathcal{A}_\mathcal{E}$, a k-**fold presentation** p **of** A is a k-fold presentation with $\mathrm{cod}^k p = A$, i.e. the "terminal object" of the k-cube p in $\mathcal{A}$ is A. We will often denote the "initial object" of a k-presentation p by P_k.

We can now state the theorem connecting comonadic homology to the Hopf formulae.

4.4.2 Theorem (Hopf formula): [EGVdL2008, Eve2007] *Let $\mathcal{E}$ be a class of extensions in a semi-abelian monadic category $\mathcal{A}$, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector I. Given an n-presentation p of an object A of $\mathcal{A}_\mathcal{E}$ with initial object P_n, we have*

$$\mathrm{H}_{n+1}(A, I)_{\mathbb{G}} \cong \frac{J P_n \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]}.$$

Proof. The case where $\mathcal{B}$ is a straightforward Birkhoff category of $\mathcal{A}$ is proved in Theorem 8.1 of [EGVdL2008]. For the axiomatic extensions version see Theorem 3.6.10 in [Eve2007]. Notice that different notation is used there. The functor J is denoted by $[\cdot]$, and $\mathrm{K}^n[J_n\cdot]$ by $[\cdot]_n$. The n-fold kernel $\mathrm{K}^n[p] = \bigcap_{i=0}^n \mathrm{K}[p_i]$ is the intersection of all maps p_i with domain P_n in p. $\square$

In their paper [EG2007], Everaert and Gran *define* homology in a semi-abelian category with enough projectives via these higher Hopf formulae, which has been shown to give the same result as comonadic homology in a monadic setting. But for this theory no monadicity conditions are required. We will follow this strategy here as well.

4.4.3 Definition (Hopf homology): Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$ with enough projectives, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector I. We define

$$\mathrm{H}_{n+1}(A, I)_{\mathcal{E}} = \frac{JP_n \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]}$$

with notation and $p \colon P \longrightarrow A$ as in Theorem 4.4.2 above. We also write

$$\mathrm{H}_1(A, I)_{\mathcal{E}} = IA.$$

Notice we have changed the subscript $\mathbb{G}$ of the comonadic homology to $\mathcal{E}$ to distinguish between these two different definitions of homology. As Theorem 4.4.2 shows, the two definitions coincide as soon as they are both defined, namely in a semi-abelian monadic category.

4.4.4 Example: In the category Gp of groups, the functor J becomes the commutator subgroup functor, that is $JA = [A, A]$. So in the case $n = 1$ we recover the well-known formula

$$\mathrm{H}_2(A, \mathbb{Z}) = \frac{[P, P] \cap \mathrm{K}[p]}{[\mathrm{K}[p], P]}$$

for integral group homology. We could write the functor J as a commutator in a general semi-abelian category, as it really is a commutator in many examples, but this would give too many different sorts of square brackets in our notation, so we prefer to call it J in most cases.

When $\mathcal{A} = \mathsf{Lie}_K$ is the category of Lie algebras over a field K, with the Birkhoff subcategory $\mathcal{B} = \mathsf{AbLie}$ of abelian Lie algebras, the homology defined as in Definition 4.4.3 is the Chevalley-Eilenberg homology of Lie algebras.

We can give an alternative formulation of the Hopf formulae, involving centralisation and trivialisation of the n-fold presentation p (cf. [Eve2008, Remark 5.12]).

4.4.5 Proposition: *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$ with enough projectives, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector I. Given an n-presentation p of an object A of $\mathcal{A}_{\mathcal{E}}$ with initial object P_n, we have*

$$\frac{JP_n \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_np]} \cong \mathrm{K}^{n+1}[I_np \longrightarrow T_np]$$

and thus

$$\mathrm{H}_{n+1}(A,I)_{\mathcal{E}} \cong \mathrm{K}^{n+1}[I_np \longrightarrow T_np].$$

Proof. Notice that as I_np and T_np coincide in all but their initial object, this $(n+1)$-fold kernel is really just the initial object of the n-cube $\mathrm{K}[I_np \longrightarrow T_np]$.

We first prove the case $n = 1$, then we use the strongly (extension)-Birkhoff properties to apply this case to higher n and use induction to prove the whole statement. For ease of notation let $p\colon P_1 = P \longrightarrow A$ have kernel K. Writing $I_1p\colon I_1[p] \longrightarrow A$ and $T_1p\colon T_1[p] \longrightarrow A$, we see that

$$\mathrm{K}^2[I_1p \longrightarrow T_1p] = \mathrm{K}[I_1[p] \longrightarrow T_1[p]]$$

as remarked above.

So if we manage to prove

$$I_1[p] = \frac{P}{\mathrm{K}[J_1p]} \qquad \text{and} \qquad T_1[p] = \frac{P}{K \cap JP},$$

then Noether's Isomorphism Theorem 1.1.15 will give the required result.

By definition

$$I_1[p] = \frac{P}{J_1[p]}$$

(see 4.3.3), and we have $J_1[p] = \mathrm{K}[J_1 p]$ as $J_1 p\colon J_1[p] \longrightarrow 0$. For $T_1[p]$, consider the following diagram, where we are taking kernels to the left:

$$
\begin{array}{ccccc}
D & \rightarrowtail & JP & \longrightarrow & JA \\
\downarrow & & \downarrow{\scriptstyle \mu_P} & & \downarrow{\scriptstyle \mu_A} \\
K & \rightarrowtail & P & \xrightarrow{\ p\ } & A \\
\downarrow & & \downarrow & & \downarrow \\
& & IP & \xrightarrow{\ Ip\ } & IA
\end{array}
$$

As μ_A is a mono, the upper left square is a pullback, so $D = K \cap JP$. Now consider

$$
\begin{array}{ccccccc}
K \cap JP & \rightarrowtail & JP & \longrightarrow & JA & =\!= & JA \\
\| & & \downarrow & & \downarrow & & \downarrow \\
K \cap JP & \rightarrowtail & P & \xrightarrow{\ \tau_p^1\ } & T_1[p] & \xrightarrow{\ T_1 p\ } & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & IP & =\!= & IP & \xrightarrow{\ Ip\ } & IA
\end{array}
$$

where the three columns are short exact. As the bottom right square is a pullback, the kernels of its two vertical maps coincide. Now the middle square in the top line is also a pullback, since the map $IP \longrightarrow IP$ at the bottom is a monomorphism. Thus $K \cap JP$ is also the kernel of τ_p^1 and thus $T_1[p] = P/K \cap JP$ as required.

For higher n we make use of the fact that $\mathsf{CExt}^{n-1}\mathcal{A}$ is a strongly $\mathcal{E}^{n-1}$-Birkhoff subcategory of $\mathsf{Arr}^{n-1}\mathcal{A}$. If we write $p\colon P \longrightarrow Q$ with P and Q in $\mathsf{Ext}^{n-1}\mathcal{A}$, we can apply the above to get

$$
\mathrm{K}^2[I_n p \longrightarrow T_n p] = \frac{J_{n-1}P \cap \mathrm{K}[p]}{\mathrm{K}[J_n p]}.
$$

Thus taking more kernels gives

$$
\mathrm{K}^{n+1}[I_n p \longrightarrow T_n p] = \frac{\mathrm{K}^{n-1}[J_{n-1}P] \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]} \tag{E}
$$

as $\mathrm{K}[B/C] = \mathrm{K}[B]/\mathrm{K}[C]$ by the 3×3-Lemma, and kernels commute with intersections as both are limits. So we only have to prove that $\mathrm{K}^{n-1}[J_{n-1}P] \cap \mathrm{K}^n[p] = JP_n \cap \mathrm{K}^n[p]$. This holds because $\mathrm{K}^{n-1}[J_{n-1}P]$ is a subobject of JP_n and $JP_n \cap \mathrm{K}^n[p]$ is a subobject of $\mathrm{K}^{n-1}[J_{n-1}P]$, which we will now show.

Notice that the $(n-1)$-extension P has projective codomain and hence is split. This means that its centralisation and trivialisation coincide (see Lemma 1.4.5). Using (E) for P we see that

$$
\mathrm{K}^n[I_{n-1}P \longrightarrow T_{n-1}P] = \frac{\mathrm{K}^{n-2}[J_{n-2}(\mathrm{dom}\, P)] \cap \mathrm{K}^{n-1}[P]}{\mathrm{K}^{n-1}[J_{n-1}P]} = 0
$$

and so $K^{n-2}[J_{n-2}(\operatorname{dom} P)] \cap K^n[p] \subseteq K^{n-2}[J_{n-2}(\operatorname{dom} P)] \cap K^{n-1}[P] = K^{n-1}[J_{n-1}P]$ as clearly $K^n[p] \subseteq K^{n-1}[P]$. For $n = 2$ this immediately gives

$$JP_2 \cap K^2[p] \subseteq K[J_1P].$$

Now we use induction. By the induction hypothesis we have

$$JP_n \cap K^{n-1}[P] \subseteq K^{n-2}[J_{n-2}(\operatorname{dom} P)]$$

and so

$$JP_n \cap K^n[p] \subseteq K^{n-2}[J_{n-2}(\operatorname{dom} P)].$$

We also have

$$K^{n-2}[J_{n-2}(\operatorname{dom} P)] \cap K^n[p] \subseteq K^{n-1}[J_{n-1}P]$$

from above. Thus we can fit all these together to give

$$JP_n \cap K^n[p] \subseteq K^{n-2}[J_{n-2}(\operatorname{dom} P)] \cap K^n[p] \subseteq K^{n-1}[J_{n-1}P]$$

as desired. $\qquad\square$

The crucial point here is that the information in the higher homology objects is entirely contained in higher-dimensional versions $I_k \colon \mathsf{Ext}^k\mathcal{A} \longrightarrow \mathsf{CExt}^k_{\mathcal{B}}\mathcal{A}$ of the reflector $I \colon \mathcal{A} \longrightarrow \mathcal{B}$. One could say that *homology measures the difference between the centralisation and the trivialisation of an n-presentation p of A.*

We now give a long exact homology sequence, which plays a crucial role in our subsequent theory in Chapter 5. For this we use the homology defined via the Hopf formulae as in 4.4.3 above, so no monadicity conditions are needed. Even so we here only give the proof for the monadic case and refer the reader to [Eve2007] for a proof in full generality.

4.4.6 Theorem (Everaert sequence): [Eve2007, Theorem 2.4.2] *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$, and $\mathcal{B}$ a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I \colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. Then any short exact sequence*

$$0 \longrightarrow K[f] \overset{\operatorname{Ker} f}{\rightarrowtail} B \overset{f}{\longrightarrow} A \longrightarrow 0$$

in $\mathcal{A}_{\mathcal{E}}$ induces a long exact homology sequence

$$\cdots \to H_{n+1}(A,I)_{\mathcal{E}} \overset{\delta_f^{n+1}}{\longrightarrow} K[H_n(f,I_1)_{\mathcal{E}^1}] \overset{\gamma_f^n}{\longrightarrow} H_n(B,I)_{\mathcal{E}} \overset{H_n(f,I)_{\mathcal{E}}}{\longrightarrow} H_n(A,I)_{\mathcal{E}} \to \cdots$$

$$\cdots \longrightarrow H_2(A,I)_{\mathcal{E}} \xrightarrow[\delta_f^2]{} K[H_1(f,I_1)_{\mathcal{E}^1}] \xrightarrow[\gamma_f^1]{} H_1(B,I)_{\mathcal{E}} \xrightarrow[H_1(f,I)_{\mathcal{E}}]{} H_1(A,I)_{\mathcal{E}} \longrightarrow 0$$

$\quad$ **(F)**

in $\mathcal{B}$. This sequence is natural in f.

Proof. A proof of this theorem in its full generality is given in [Eve2007]. However, when we restrict ourselves to the monadic case it becomes relatively easy to understand why the sequence takes this shape. So suppose that we are in a semi-abelian monadic setting, $\mathcal{A}_{\mathcal{E}} = \mathsf{Ext}^k\mathcal{A}$ with $\mathcal{B} = \mathsf{CExt}^k_{\mathcal{B}}\mathcal{A}$, and $\mathbb{G}$ is the induced comonad on $\mathsf{Ext}^k\mathcal{A}$. This comonad produces canonical simplicial resolutions $\mathbb{G}A$ and $\mathbb{G}B$ of A and B and, by functoriality, also a simplicial resolution $\mathbb{G}f$ of f. The Everaert sequence (**F**) is the long exact homology sequence (see [EVdL2004b, Corollary 5.7]) obtained from the short exact sequence of simplicial objects

$$0 \longrightarrow \mathrm{K}[I_k\mathbb{G}f] \rightarrowtail I_k\mathbb{G}B \xrightarrow{I_k\mathbb{G}f} I_k\mathbb{G}A \longrightarrow 0;$$

it remains to be shown that $\mathrm{H}_{n-1}\mathrm{K}[I_k\mathbb{G}f] = \mathrm{K}[\mathrm{H}_n(f, I_{k+1})_{\mathbb{G}}]$ for all $n \geq 1$. (Remember the dimension shift in Equation (**A**).) Now degree-wise, the $(k+1)$-extension

$$I_{k+1}\mathbb{G}f \colon I_{k+1}[\mathbb{G}f] \longrightarrow \mathbb{G}A$$

is a split epimorphic central extension: it is a centralisation, and $\mathbb{G}A$ is degree-wise projective. Via [EGVdL2008, Proposition 4.5], this implies that, degree-wise, it is a trivial extension. This means that $I_{k+1}\mathbb{G}f$ is the pullback of $I_k\mathbb{G}f$ along the unit $\eta^k_{\mathbb{G}A} \colon \mathbb{G}A \longrightarrow I_k\mathbb{G}A$, which in turn implies that $\mathrm{K}[I_k\mathbb{G}f]$ is the kernel $\mathrm{K}[I_{k+1}\mathbb{G}f]$ of $I_{k+1}\mathbb{G}f$. Since $\mathrm{H}_n\mathbb{G}A = 0$ for all $n \geq 1$, $\mathbb{G}A$ being a simplicial resolution, the long exact homology sequence induced by the short exact sequence of simplicial objects

$$0 \longrightarrow \mathrm{K}[I_{k+1}\mathbb{G}f] \rightarrowtail I_{k+1}[\mathbb{G}f] \xrightarrow{I_{k+1}\mathbb{G}f} \mathbb{G}A \longrightarrow 0$$

gives the needed isomorphism $\mathrm{H}_{n-1}\mathrm{K}[I_{k+1}\mathbb{G}f] \cong \mathrm{K}[\mathrm{H}_n(f, I_{k+1})_{\mathbb{G}}]$. $\qquad\square$

Note that in [Eve2007], this sequence has a slightly different appearance: there it contains the objects $\mathrm{dom}\,\mathrm{H}_n(f, I_1)_{\mathcal{E}^1}$ instead of $\mathrm{K}[\mathrm{H}_n(f, I_1)_{\mathcal{E}^1}]$ for $n \geq 2$. But the codomain of $\mathrm{H}_n(f, I_1)_{\mathcal{E}^1}$ is zero (because J_1f has zero codomain, hence I_1 only changes the domain of an extension), so its domain coincides with its kernel. For us, the sequence in its present, more uniform, shape will be easier to work with.

4.4.7 Corollary: *For any $n \geq 2$ and any projective presentation $p \colon P \longrightarrow A$ of an object $A \in |\mathcal{A}_{\mathcal{E}}|$,*

$$\mathrm{H}_n(p, I_1)_{\mathcal{E}^1} \cong (\mathrm{H}_{n+1}(A, I)_{\mathcal{E}} \longrightarrow 0).$$

Proof. It suffices to note that in the Everaert sequence (**F**), all $\mathrm{H}_{n+1}(P, I)_{\mathcal{E}}$ are zero, because P is projective. $\qquad\square$

This shows how the degree of the homology may be lowered from $n+1$ to n by raising the degree of the reflector.

It is well known that the integral group homology objects are abelian groups. The analogous result holds for the homology objects defined via Hopf formulae.

4.4.8 Lemma: *Let $n \geq 1$. For any object $A \in \mathcal{A}_{\mathcal{E}}$, the homology object $\mathrm{H}_{n+1}(A, I)_{\mathcal{E}}$ is an abelian object.*

Proof. For a proof see [Eve2007, Proposition 2.3.16]. This result uses Lemma 1.2.7. $\square$

When the Birkhoff subcategory $\mathcal{B}$ is the subcategory of abelian objects $\mathsf{Ab}\mathcal{A}$, it is easy to see that all objects occurring in the Everaert sequence (**F**) are abelian objects. However, when $\mathcal{B}$ is not made up of abelian objects, then $\mathrm{H}_1(A, I)_{\mathcal{E}} = IA$ will not in general be abelian. In the Everaert sequence, there is a nice bridge between the abelian and the non-abelian objects: the map from the last abelian object $\mathrm{H}_2(A, I)_{\mathcal{E}}$ to the first non-abelian object $\mathrm{K}[I_1 f]$ is central in the sense of Huq. This is here shown for the first time.

4.4.9 Lemma: *Let $f \colon B \longrightarrow A$ be an extension in $\mathcal{A}$. Then*

$$\delta_f^2 \colon \mathrm{H}_2(A, I)_{\mathcal{E}} \longrightarrow \mathrm{K}[\mathrm{H}_1(f, I_1)_{\mathcal{E}^1}]$$

*in the Everaert sequence (**F**) is central in the sense of Huq.*

Proof. We use some steps leading to the proof of Theorem 2.1 in [Bou2005], which is here quoted as Lemma 1.2.7 (without proof). First of all consider the image

$$\mathrm{Im}\,\delta_f^2 \colon \mathrm{I}[\delta_f^2] \longrightarrow \mathrm{K}[I_1 f] = \mathrm{K}[\mathrm{H}_1(f, I_1)_{\mathcal{E}^1}].$$

By Lemma 1.5.8 it is enough to show that $\mathrm{Im}\,\delta_f^2$ is central. As the Everaert sequence is exact, this image is the kernel of $\gamma_f^1 \colon \mathrm{K}[I_1 f] \longrightarrow IB$ or equivalently the kernel of the corestriction of γ_f^1 to its image, $\mathrm{K}[I_1 f] \longrightarrow \mathrm{K}[If]$. Now recalling the definition of I_1 and J_1 from 4.3.3, we see that

$$\mathrm{K}[I_1 f] = \frac{\mathrm{K}[f]}{J_1[f]} = \frac{\mathrm{K}[f]}{\pi_2(J\mathrm{R}[f] \cap \mathrm{K}[f])}$$

since $J_1[f] = J\mathrm{R}[f] \cap \mathrm{K}[f]$ as a normal subobject of $\mathrm{R}[f]$, and its direct image under π_2 gives us a normal subobject of B (note that $\pi_2(J_1[f]) = J_1[f]$ as μ_f^1 is a normal monomorphism). Similarly

$$\mathrm{K}[If] = \frac{\mathrm{K}[f]}{JB \cap \mathrm{K}[f]} = \frac{\mathrm{K}[f]}{\pi_2(J\mathrm{R}[f]) \cap \pi_2(\mathrm{K}[f])}.$$

Thus, by Noether's Isomorphism Theorem, we have

$$\mathrm{I}[\delta_f^2] = \frac{JB \cap \mathrm{K}[f]}{J_1[f]} = \frac{\pi_2(J\mathrm{R}[f]) \cap \pi_2(\mathrm{K}[f])}{\pi_2(J\mathrm{R}[f] \cap \mathrm{K}[f])}$$

which is an abelian object by Lemma 1.2.7. But in fact, following the proof of this result, we see that

$$\frac{JB \cap \mathrm{K}[f]}{J_1[f]} = \frac{JB}{J_1[f]} \cap \frac{\mathrm{K}[f]}{J_1[f]} = \frac{\pi_2(J\mathrm{R}[f])}{\pi_2(J\mathrm{R}[f] \cap \mathrm{K}[f])} \cap \frac{\pi_2(\mathrm{K}[f])}{\pi_2(J\mathrm{R}[f] \cap \mathrm{K}[f])}$$

as quotienting out by $J_1[f]$ is a regular epimorphism. Now clearly

$$\frac{J\mathrm{R}[f]}{J\mathrm{R}[f] \cap \mathrm{K}[f]} \cap \frac{\mathrm{K}[f]}{J\mathrm{R}[f] \cap \mathrm{K}[f]} = 0$$

so by [Bou2005, Proposition 2.1] these two subobjects of $\mathrm{R}[f]/(J\mathrm{R}[f] \cap \mathrm{K}[f])$ cooperate. Now we take the images under π_2, and [Bou2005, Proposition 1.1] implies that these images $JB/J_1[f]$ and $\mathrm{K}[f]/J_1[f]$ also cooperate, as subobjects of $B/J_1[f]$. Thus, using Lemma 1.5.7, we see that $(JB \cap \mathrm{K}[f])/J_1[f]$ and $\mathrm{K}[f]/J_1[f]$ cooperate as subobjects of $\mathrm{K}[f]/J_1[f] = \mathrm{K}[I_1 f]$, which says exactly that $\mathrm{Im}\,\delta_f^2$ is central. $\qquad\square$

4.4.10 Remark: Notice that γ_f^1 coincides with the composite in the diagram below, showing that the corestriction to the image is in fact $\mathrm{K}[(I_1 f, If)]\colon \mathrm{K}[I_1 f] \longrightarrow \mathrm{K}[If]$.

$$
\begin{array}{ccccc}
\mathrm{K}[I_1 f] & \rightarrowtail & I_1[f] & \xrightarrow{\ I_1 f\ } & A \\
\downarrow & \searrow{\scriptstyle \gamma_f^1} & \downarrow{\scriptstyle \eta_{I_1[f]}} & & \downarrow{\scriptstyle \eta_A} \\
\mathrm{K}[If] & \rightarrowtail & IB & \xrightarrow{\ If\ } & IA
\end{array}
$$

Here the map $(I_1 f, If)\colon \eta_{I_1[f]} \longrightarrow \eta_A$ is a double extension, so its kernel is an extension by Lemma 4.1.7. Thus we have established that

$$\mathrm{K}[\gamma_f^1] = \frac{JB \cap \mathrm{K}[f]}{J_1[f]}$$

which is reminiscent of the Hopf formula; the only difference is that here f is not a projective presentation and so this expression is not independent of the choice of f. The form of this kernel is not surprising, as when f *is* a projective presentation, the kernel of γ_f^1 is exactly $\mathrm{H}_2(A, I)_{\mathcal{E}}$.

Chapter 5

Homology via Satellites

Introduction

Having defined homology via Hopf formulae and introduced the Everaert sequence in the previous chapter, we now use the universal properties of the Everaert sequence to define homology via pointwise Kan extensions or limits. Recall that any short exact sequence

$$0 \longrightarrow \mathrm{K}[f] \overset{\mathrm{Ker}\, f}{\rightarrowtail} B \overset{f}{\longrightarrow} A \longrightarrow 0$$

in $\mathcal{A}$ gives rise to a long exact homology sequence

$$\cdots \longrightarrow \mathrm{H}_{n+1}(A,I)_{\mathcal{E}} \overset{\delta_f^{n+1}}{\longrightarrow} \mathrm{K}[\mathrm{H}_n(f,I_1)_{\mathcal{E}^1}] \overset{\gamma_f^n}{\longrightarrow} \mathrm{H}_n(B,I)_{\mathcal{E}} \overset{\mathrm{H}_n(f,I)_{\mathcal{E}}}{\longrightarrow} \mathrm{H}_n(A,I)_{\mathcal{E}} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{H}_2(A,I)_{\mathcal{E}} \overset{}{\underset{\delta_f^2}{\longrightarrow}} \mathrm{K}[\mathrm{H}_1(f,I_1)_{\mathcal{E}^1}] \overset{}{\underset{\gamma_f^1}{\longrightarrow}} \mathrm{H}_1(B,I)_{\mathcal{E}} \underset{\mathrm{H}_1(f,I)_{\mathcal{E}}}{\longrightarrow} \mathrm{H}_1(A,I)_{\mathcal{E}} \longrightarrow 0$$

which is natural in f. Janelidze's theory of generalised satellites now helps us to compute homology objects step by step: the $(n+1)$st homology functor $\mathrm{H}_{n+1}(-,I)$ is obtained from $\mathrm{H}_n(-,I_1)$ as a pointwise right Kan extension, and the connecting homomorphism δ in the Everaert sequence is exactly what makes this work. This approach removes the dependence on projective objects from the definition of homology. In a further step it also cements the connection between homology and central extensions: gluing all the step by step Kan extensions together we see that homology is the limit of the diagram of kernels of all central extensions of a given object. More precisely, given an object A and the category $\mathsf{CExt}_A^n \mathcal{A}$ of central n-extensions of A, we have

$$\mathrm{H}_{n+1}(A,I) = \lim_{f \in \mathsf{CExt}_A^n \mathcal{A}} \mathrm{K}^n[f]$$

for any $n \geq 1$.

When the category $\mathcal{A}$ does have enough projective objects, we can use projective presentations to cut down the size of the diagram of which the homology object is the limit. Given a projective presentation p of A, the $(n+1)$st homology $\mathrm{H}_{n+1}(A,I)$ forms the limit of the diagram consisting of the n-fold kernel of p and all maps induced by endomorphisms of p over A. This can be interpreted to say that calculating homology

amounts to calculating *common fixed points* of these maps induced by endomorphisms of a projective presentation of A.

The first four sections of this chapter give an analysis of homology in terms of satellites. We start by stating the main definitions in Section 5.1. Then, in Section 5.2, we interpret $H_{n+1}(-, I)_{\mathcal{E}}$ (together with the connecting map δ^{n+1}) as a satellite of $H_n(-, I_1)_{\mathcal{E}^1}$. In Section 5.3 we prove one of the main results of this chapter: a formula which gives H_{n+1} in terms of I_n. Finally in Section 5.4 we explain how the situation is entirely symmetric, in that the connecting map γ^n also arises as a pointwise satellite.

In the last two sections we discuss the theory obtained by *defining* homology via satellites. Section 5.5 gives this definition of homology without projectives and the result that homology is the limit of the diagram of kernels of central extensions, in Corollary 5.5.10. It also establishes that the homology objects are both objects of the Birkhoff subcategory $\mathcal{B}$ and abelian objects of $\mathcal{A}$. In Section 5.6 we investigate the consequences of the new definition when enough projective objects are available. This leads to the interpretation of homology as calculating fixed points of endomorphisms of a projective presentation.

Most material in this chapter is based on joint work with Tim Van der Linden and can also be found in our paper [GVdL2008a], though I use some different concepts and proof techniques here that make many statements and proofs easier.

5.1 Satellites and pointwise satellites

Modulo a minor terminological change, the following definition is due to Janelidze.

5.1.1 Definition (Satellites): [Jan1976, Definition 2] Let $I' \colon \mathcal{A}' \longrightarrow \mathcal{B}'$ be a functor. A **left satellite** (H, δ) **of** I' **(relative to** $F \colon \mathcal{A}' \longrightarrow \mathcal{A}$ **and** $G \colon \mathcal{B}' \longrightarrow \mathcal{B}$**)** is a functor $H \colon \mathcal{A} \longrightarrow \mathcal{B}$ together with a natural transformation $\delta \colon HF \Longrightarrow GI'$
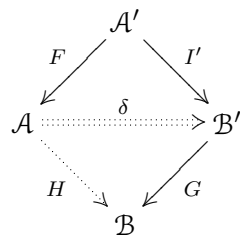
universal amongst such, i.e., if there is another functor $L \colon \mathcal{A} \longrightarrow \mathcal{B}$ with a natural transformation $\lambda \colon LF \Longrightarrow GI'$, then there is a unique natural transformation $\mu \colon L \Longrightarrow H$ satisfying $\delta \circ \mu_F = \lambda$. This means that (H, δ) is the right Kan extension $\mathrm{Ran}_F GI'$ of the functor GI' along $F \colon \mathcal{A}' \longrightarrow \mathcal{A}$.

This makes it possible to compute derived functors in quite diverse situations. The following example, borrowed from [Jan1976], explains how satellites may be used to capture homology in the classical abelian case.

5.1.2 Example: In the abelian context, the $(n + 1)$st homology functor H_{n+1} may be seen as a left satellite of H_n. For instance, let $\mathcal{A} = \mathcal{B}'$ and $\mathcal{B}$ be categories of modules and $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Then $G = \mathrm{H}_0(-, G)$. Let $\mathsf{SESeq}\mathcal{A}$ be the category of short exact sequences

$$0 \longrightarrow K \rightarrowtail^{k} B \xrightarrow{f} A \longrightarrow 0$$

in $\mathcal{A}$, the functor $I'\colon \mathsf{SESeq}\mathcal{A} \longrightarrow \mathcal{A}$ the projection pr_1 that maps a sequence (k, f) to the object K, and $F\colon \mathsf{SESeq}\mathcal{A} \longrightarrow \mathcal{A}$ the projection pr_3 that maps (k, f) to A. Let $H\colon \mathcal{A} \longrightarrow \mathcal{B}$ be the first homology functor $\mathrm{H}_1(-, G)$. We obtain a satellite diagram

where the natural transformation $\delta = (\delta_{(k,f)})_{(k,f)\in|\mathsf{SESeq}\mathcal{A}|}$ consists of the connecting maps from the (classical) long exact homology sequence

$$\cdots \longrightarrow \mathrm{H}_1 K \xrightarrow{\mathrm{H}_1 k} \mathrm{H}_1 B \xrightarrow{\mathrm{H}_1 f} \mathrm{H}_1 A \xrightarrow{\delta_{(k,f)}} \mathrm{H}_0 K \xrightarrow{\mathrm{H}_0 k} \mathrm{H}_0 B \xrightarrow{\mathrm{H}_0 f} \mathrm{H}_0 A \longrightarrow 0.$$

The universality of the Kan extension follows from the universality of the long exact homology sequence amongst similar sequences and may for instance be shown as follows. Given any functor $L\colon \mathcal{A} \longrightarrow \mathcal{B}$ and any natural transformation

$$\lambda\colon L\circ\mathrm{pr}_3 \Longrightarrow \mathrm{H}_0(-, G)\circ\mathrm{pr}_1,$$

we will construct the component at an object $A \in |\mathcal{A}|$ of the required natural transformation

$$L \Longrightarrow \mathrm{H}_1(-, G)$$

by using a projective presentation $p\colon P \longrightarrow A$ of A. Let $k\colon K \longrightarrow P$ be the kernel of this projective presentation of A. Since $\mathrm{H}_1 P$ is zero (as P is projective), the exactness of the long homology sequence induced by (k, p) means that $\delta_{(k,p)}\colon \mathrm{H}_1 A \longrightarrow \mathrm{H}_0 K$ is the kernel of $\mathrm{H}_0 k$. Then the string of equalities

$$\mathrm{H}_0 k\circ\lambda_{(k,p)} \overset{(1)}{=} \lambda_{(1_P, !_P)}\circ L!_A \overset{(2)}{=} \mathrm{H}_0(\mathrm{i}_P)\circ\lambda_{(1_0,1_0)}\circ L!_A \overset{(3)}{=} 0$$

yields the required factorisation $LA \longrightarrow \mathrm{H}_1 A$: (1) expresses the naturality of λ at the upper, downward-pointing morphism of the diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow & K & \overset{k}{\rightarrowtail} & P & \overset{p}{\twoheadrightarrow} & A & \longrightarrow 0 \\
& {\scriptstyle k}\downarrow & {\scriptstyle \Downarrow} & \| {\scriptstyle 1_P \Downarrow} & & \downarrow {\scriptstyle !_A} & \\
0 \longrightarrow & P & \underset{1_P}{=\!=\!=} & P & \underset{!_P}{\twoheadrightarrow} & 0 & \longrightarrow 0 \\
& {\scriptstyle i_P}\uparrow & {\scriptstyle \Uparrow} & \uparrow {\scriptstyle i_P} {\scriptstyle \Uparrow} & & \uparrow {\scriptstyle 1_0} & \\
0 \longrightarrow & 0 & \underset{1_0}{=\!=\!=} & 0 & \underset{1_0}{=\!=\!=} & 0 & \longrightarrow 0
\end{array}
\qquad (\mathbf{G})
$$

in $\mathsf{SESeq}\mathcal{A}$, while (2) follows from $\lambda_{(1_P,!_P)} = \lambda_{(1_P,!_P)} \circ L1_0 = \mathrm{H}_0(i_P) \circ \lambda_{(1_0,1_0)}$, which is the naturality of λ at the lower, upward-pointing morphism; the last equality (3) holds because $\mathrm{H}_0 0 = 0$.

Note that, as such, this example does *not* follow the terminology of Definition 5.1.1. From its point of view one is tempted to call H a left satellite of G (rather than a satellite of I'), and actually this is how the definition appears in the paper [Jan1976]. But the situation we shall be considering in this thesis demands the change in terminology, and the present example may easily be modified to comply with Definition 5.1.1.

Indeed, the functor G may be lifted to a functor

$$\mathsf{SSeqH}_0(-, G) \colon \mathsf{SESeq}\mathcal{A} \longrightarrow \mathsf{SSeq}\mathcal{B}$$

where the latter category consists of short (not necessarily exact) sequences in $\mathcal{B}$. Together with the obvious projection $\mathrm{pr}_1 \colon \mathsf{SSeq}\mathcal{B} \longrightarrow \mathcal{B}$ (s.t. $\mathrm{H}_0(-, G) \circ \mathrm{pr}_1 = \mathrm{pr}_1 \circ \mathsf{SSeqH}_0(-, G)$), this gives us the satellite diagram

Whereas such a viewpoint may seem rather far-fetched in the abelian case, it is the only one still available when the context is widened to semi-abelian categories. In fact, even in the abelian setting, this formulation is slightly reminiscent of the universal property for a derived functor (see for example [Wei1997, Section 10.5]), so it is not all that far-fetched after all.

In practice, satellites may almost always be computed explicitly using limits—namely, as pointwise Kan extensions. Then the definition given above is strengthened as follows.

5.1.3 Notation: Let A be an object of $\mathcal{A}$. We denote by $(A \downarrow F)$ the category of elements of the functor $\mathrm{Hom}(A, F-) \colon \mathcal{A}' \longrightarrow \mathsf{Set}$: its objects are pairs $(A', \alpha \colon A \longrightarrow FA')$, where A' is an object of $\mathcal{A}'$ and α is a morphism in $\mathcal{A}$, and its morphisms are defined in the obvious way (cf. [Bor1994, Theorem 3.7.2]). The forgetful functor $U \colon (A \downarrow F) \longrightarrow \mathcal{A}'$ maps a pair (A', α) to A'. The natural transformation (H, δ) now induces a cone $\overline{\delta}$ on $GI'U \colon (A \downarrow F) \longrightarrow \mathcal{B}$ with vertex HA defined by

$$\overline{\delta}_{(A', \alpha \colon A \longrightarrow FA')} = \delta_{A'} \circ H\alpha \colon HA \xrightarrow{H\alpha} HFA' \xrightarrow{\delta_{A'}} GI'A' = GI'U(A', \alpha).$$

5.1.4 Definition (Pointwise satellites): A left satellite (H, δ) of I' relative to the functors $F \colon \mathcal{A}' \longrightarrow \mathcal{A}$ and $G \colon \mathcal{B}' \longrightarrow \mathcal{B}$ is called **pointwise** when it is pointwise as a Kan extension, i.e., for every object A of $\mathcal{A}$, the cone $(HA, \overline{\delta})$ on $GI'U \colon (A \downarrow F) \longrightarrow \mathcal{B}$ is a limit cone.

To check that a pair (H, δ) is a pointwise satellite it is not necessary to prove its universality as in Definition 5.1.1, but it suffices to check the limit condition from Definition 5.1.4; see, for example, Mac Lane [Mac1998, Theorem X.3.1].

5.2 $H_{n+1}(-, I)_{\mathcal{E}}$ as a satellite of $H_n(-, I_1)_{\mathcal{E}^1}$

We are now ready to prove the first main result of this chapter: we focus on the universal properties of the Everaert sequence (**F**), and prove that they allow us to interpret the $(n+1)$st homology with coefficients in I as a satellite of the nth homology with coefficients in I_1.

For the whole of this section, let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$ with enough projectives, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I \colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$.

5.2.1 Lemma: *For $n \geq 1$ and $A \in |\mathcal{A}_{\mathcal{E}}|$,*

$$K[H_n(!_A \colon A \longrightarrow 0, I_1)_{\mathcal{E}^1}] = H_n(A, I)_{\mathcal{E}}.$$

Proof. This follows from the exactness of the Everaert sequence (**F**) and the fact that all $H_n(0, I)_{\mathcal{E}}$ are zero. $\square$

5.2.2 Lemma: *For all $n \geq 1$ and $f\colon B \longrightarrow A \in |\mathsf{Ext}\mathcal{A}|$,*

$$\gamma_f^n = \ker\left(\mathrm{H}_n\left(\begin{array}{c} \begin{array}{ccc} B & \!\!=\!\!=\!\! & B \\ {\scriptstyle f}\downarrow & \Rightarrow & \downarrow{\scriptstyle !_B} \\ A & \xrightarrow{\ !_A\ } & 0 \end{array}, I_1\end{array}\right)_{\mathcal{E}^1}\right) : \mathrm{K}[\mathrm{H}_n(f,I_1)_{\mathcal{E}^1}] \longrightarrow \mathrm{H}_n(B,I)_{\mathcal{E}}.$$

Proof. This follows from the previous lemma and the naturality of γ^n. Indeed, its naturality square at the map $(1_B, !_A)$ is nothing but

$$
\begin{array}{ccc}
\mathrm{K}[\mathrm{H}_n(f,I_1)_{\mathcal{E}^1}] & \xrightarrow{\ \ker\mathrm{H}_n((1_B,!_A),I_1)_{\mathcal{E}^1}\ } & \mathrm{K}[\mathrm{H}_n(!_B,I_1)_{\mathcal{E}^1}] \\
{\scriptstyle \gamma_f^n}\downarrow & & \downarrow{\scriptstyle \gamma_{!_B}^n} \\
\mathrm{H}_n(B,I)_{\mathcal{E}} & =\!=\!=\!=\!=\!= & \mathrm{H}_n(B,I)_{\mathcal{E}};
\end{array}
$$

and all kernels may be chosen in such a way that $\gamma_{!_B}^n$ is an identity. $\qquad\square$

5.2.3 Proposition: *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$ with enough projectives, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. Let $n \geq 1$. Then $\mathrm{H}_{n+1}(-,I)_{\mathcal{E}}\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{A}_{\mathcal{E}}$ with the connecting natural transformation*

is the pointwise left satellite of $\mathrm{H}_n(-,I_1)_{\mathcal{E}^1}$. That is, for any object A of $\mathcal{A}_{\mathcal{E}}$,

$$\mathrm{H}_{n+1}(A,I)_{\mathcal{E}} = \mathrm{Ran}_{\mathrm{cod}}(\ker\!\circ\mathrm{H}_n(-,I_1)_{\mathcal{E}^1})(A) = \lim_{(f,g)\in|(A\downarrow\mathrm{cod})|} \mathrm{K}[\mathrm{H}_n(f,I_1)_{\mathcal{E}^1}].$$

Proof. Let A be an object of $\mathcal{A}_{\mathcal{E}}$. Let $p\colon P \longrightarrow A$ be a projective presentation of A. We have to show that $(\mathrm{H}_{n+1}(A,I)_{\mathcal{E}}, \overline{\delta^{n+1}})$ is the limit of

$$(A \downarrow \mathrm{cod}) \xrightarrow{\ \ U\ \ } \mathsf{Ext}\mathcal{A} \xrightarrow{\ \mathrm{H}_n(-,I_1)_{\mathcal{E}^1}\ } \mathsf{Ext}\mathcal{A} \xrightarrow{\ \ker\ } \mathcal{A}_{\mathcal{E}}.$$

To do so, let (L, λ) be another cone on $\ker\!\circ\mathrm{H}_n(-,I_1)_{\mathcal{E}^1}\!\circ U$; we use the presentation p of A to construct a map of cones $l\colon L \longrightarrow \mathrm{H}_{n+1}(A,I)_{\mathcal{E}}$.

First we consider the case $n = 1$. Recall from Definition 4.4.3 that $\mathrm{H}_1(-,I)_{\mathcal{E}} = I$ and $\mathrm{H}_1(-,I_1)_{\mathcal{E}^1} = I_1$. Since $p\colon P \longrightarrow A$ is a projective presentation of A, and thus $\mathrm{H}_2(P,I)_{\mathcal{E}} = 0$, the lower end of the Everaert sequence (**F**) of p becomes

$$0 \longrightarrow H_2(A, I)_{\mathcal{E}} \rightarrowtail^{\delta_p^2} K[I_1 p] \xrightarrow{\gamma_p^1} IP.$$

In other words, δ_p^2 is the kernel of γ_p^1. Recalling Diagram (**G**), consider the following two morphisms in $(A \downarrow \mathrm{cod})$:

$$
\begin{array}{ccccc}
P & \xrightarrow{\;p\;} & A & \xrightarrow{\;1_A\;} & A \\
\| & \Downarrow & \downarrow{\scriptstyle !_A} \;\; \Downarrow & & \| \\
P & \xrightarrow{\;!_P\;} & 0 & \xleftarrow{\;!_A\;} & A \\
{\scriptstyle i_P}\uparrow & \Uparrow & {\scriptstyle 1_0}\uparrow \;\; \Uparrow & & \| \\
0 & \overline{\;1_0\;} & 0 & \xleftarrow{\;!_A\;} & A
\end{array}
\qquad\qquad (\mathbf{I})
$$

By Lemma 5.2.2, the naturality of λ at the downward-pointing morphism in Diagram (**I**) means $\gamma_p^1 \circ \lambda_{(p, 1_A)} = \lambda_{(!_P, !_A)}$. This latter morphism is zero, since the naturality of λ at the upward-pointing morphism in (**I**) means $\lambda_{(!_P, !_A)} = I(i_P) \circ \lambda_{(1_0, !_A)}$, and $I0 = 0$. Hence there exists a unique morphism $l \colon L \longrightarrow H_2(A, I)_{\mathcal{E}}$ satisfying $\lambda_{(p, 1_A)} = \delta_p^2 \circ l$.

Higher up in the Everaert sequence (**F**) of p, for $n \geq 2$, Corollary 4.4.7 gives us the isomorphism

$$\delta_p^{n+1} \colon H_{n+1}(A, I)_{\mathcal{E}} \xrightarrow{\;\cong\;} K[H_n(p, I_1)_{\mathcal{E}^1}].$$

Here we may simply put $l = (\delta_p^{n+1})^{-1} \circ \lambda_{(p, 1_A)}$.

It remains to be shown that, in both cases, the constructed map l is a map of cones. Given any object $(f \colon B \longrightarrow C, g \colon A \longrightarrow C)$ of $(A \downarrow \mathrm{cod})$, there is a map

$$
\begin{array}{ccccc}
P & \xrightarrow{\;p\;} & A & = & A \\
\downarrow & \Downarrow & \downarrow{\scriptstyle g} \;\; \Downarrow & & \| \\
B & \xrightarrow{\;f\;} & C & \xleftarrow{\;g\;} & A
\end{array}
$$

as P is projective. Writing h for the image of this morphism under $\ker \circ H_n(-, I_1)_{\mathcal{E}^1} \circ U$, we see that the diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\;\;\;l\;\;\;} & H_{n+1}(A, I)_{\mathcal{E}} \\
{\scriptstyle \lambda_{(p,1_A)}}\downarrow & {\scriptstyle \lambda_{(f,g)}} \searrow \quad \swarrow {\scriptstyle \delta_p} & \downarrow{\scriptstyle \overline{\delta}_{(f,g)}} \\
K[H_n(p, I_1)_{\mathcal{E}^1}] & \xrightarrow{\;\;h\;\;} & K[H_n(f, I_1)_{\mathcal{E}^1}]
\end{array}
$$

commutes: $\lambda_{(f,g)} = h \circ \lambda_{(p,1_A)} = h \circ \delta_p \circ l = \overline{\delta}_{(f,g)} \circ l$. Thus l is indeed a map of cones, and $H_{n+1}(A, I)_{\mathcal{E}}$ is the limit of the given diagram. $\qquad\square$

5.2.4 Remark: This gives a way to derive $H_{n+1}(-, I)_{\mathcal{E}}$ from $H_n(-, I_1)_{\mathcal{E}^1}$ for $n \geq 2$ in exactly the same way as $H_2(-, I)_{\mathcal{E}}$ is derived from $H_1(-, I_1)_{\mathcal{E}^1} = I_1$. In other approaches such as [Eve2007, EGVdL2008] the two cases are formally different.

5.2.5 Remark: Notice that we can apply this proposition for higher extensions as well, making use of the fact that $\mathsf{CExt}^k_{\mathcal{B}}\mathcal{A}$ is a strongly $\mathcal{E}^k$-Birkhoff subcategory of $\mathsf{Arr}^k\mathcal{A}$ for $k \geq 1$. This allows us to use induction, as we will see in the next subsection.

5.3 $\mathrm{H}_{n+1}(-, I)_{\mathcal{E}}$ as a satellite of I_n

Proposition 5.2.3 gives a way to construct $\mathrm{H}_{n+1}(-, I)_{\mathcal{E}}$ from $\mathrm{H}_n(-, I_1)_{\mathcal{E}^1}$. Here, with Theorem 5.3.2, we obtain a one-step construction of $\mathrm{H}_{n+1}(-, I)_{\mathcal{E}}$ out of I_n. To be able to apply Proposition 5.2.3 repeatedly, we have to show that satellite diagrams like Diagram (**H**) may be composed in a suitable way (cf. [Jan1976, Theorem 9]).

The kernel functor

$$\ker \colon \mathsf{Ext}^{k+1}\mathcal{A} \longrightarrow \mathsf{Ext}^k\mathcal{A}$$

that maps an extension $f \colon B \longrightarrow A$ to its kernel $\mathrm{K}[f]$ has a left adjoint, namely the functor $\mathsf{Ext}^k\mathcal{A} \longrightarrow \mathsf{Ext}^{k+1}\mathcal{A}$ that sends an object C of $\mathsf{Ext}^k\mathcal{A}$ to the extension $!_C \colon C \longrightarrow 0$. This allows us to use the following result.

5.3.1 Proposition: *Suppose that* $(I', \delta') = \mathrm{Ran}_{F'}G'I''$ *and* $(H, \delta) = \mathrm{Ran}_F GI'$ *as in the diagrams*

and
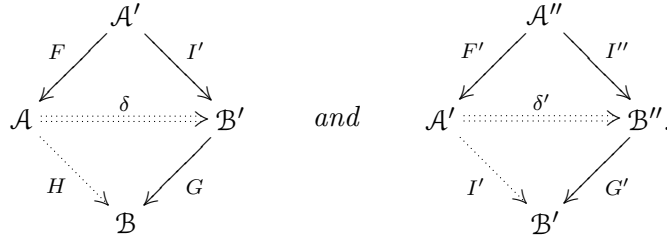
If G is a right adjoint then $(H, G\delta' \circ \delta_{F'}) = \mathrm{Ran}_{FF'}GG'I''$: *the two diagrams may be composed to form a single Kan extension diagram*
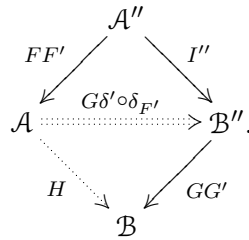
If G preserves limits and (I', δ') and (H, δ) are pointwise satellites then $(H, G\delta' \circ \delta_{F'})$ is also a pointwise satellite.

Proof. We prove the pointwise case. Let A be an object of $\mathcal{A}$, and (C, σ) a cone on the diagram $GG'I''U \colon (A \downarrow FF') \longrightarrow \mathcal{B}$.

For any A' in $\mathcal{A}'$, the pair $(I'A', \overline{\delta'})$ is the limit of the diagram $G'I''U' \colon (A' \downarrow F') \longrightarrow \mathcal{B}'$. Since G preserves limits, $(GI'A', G\overline{\delta'})$ is the limit of $GG'I''U' \colon (A' \downarrow F') \longrightarrow \mathcal{B}$. Now for every $\alpha \colon A \longrightarrow FA'$ the collection

$$\big(\sigma_{(A'', F\alpha' \circ \alpha)}\big)_{(A'', \alpha') \in |(A' \downarrow F')|}$$

also forms a cone on $GG'I''U'$; hence there is a unique map $\mu_{(A', \alpha)} \colon C \longrightarrow GI'A'$ such that $G\delta'_{A''} \circ GI'\alpha' \circ \mu_{(A', \alpha)} = \sigma_{(A'', F\alpha' \circ \alpha)}$.

The collection $(\mu_{(A', \alpha)})_{(A', \alpha) \in |(A \downarrow F)|}$ in turn forms a cone on $GI'U \colon (A \downarrow FF') \longrightarrow \mathcal{B}$. Indeed, if (B', β) is an object of $(A \downarrow F)$ and $f' \colon B' \longrightarrow A'$ is a map in $\mathcal{A}'$ such that $Ff' \circ \beta = \alpha$, then $GI'f' \circ \mu_{(B', \beta)} = \mu_{(A', \alpha)}$, because for every $(A'', \alpha') \in |(A' \downarrow F')|$,

$$G\delta'_{A''} \circ GI'\alpha' \circ GI'f' \circ \mu_{(B', \beta)} = \sigma_{(A'', F(\alpha' \circ f') \circ \beta)}$$
$$= \sigma_{(A'', F\alpha' \circ \alpha)}$$
$$= G\delta'_{A''} \circ GI'\alpha' \circ \mu_{(A', \alpha)},$$

and the $G\delta'_{A''} \circ GI'\alpha'$ are jointly monic.

This cone gives rise to the required unique map $c \colon C \longrightarrow HA$. Since it satisfies $\mu_{(A', \alpha)} = \delta_{A'} \circ H\alpha \circ c$ for all $(A', \alpha) \in |(A \downarrow F)|$, we have that

$$G\delta'_{A''} \circ \delta_{F'A''} \circ H\alpha'' \circ c = G\delta'_{A''} \circ \delta_{F'A''} \circ HF\alpha' \circ H\alpha \circ c$$
$$= G\delta'_{A''} \circ GI'\alpha' \circ \delta_{A'} \circ H\alpha \circ c$$
$$= G\delta'_{A''} \circ GI'\alpha' \circ \mu_{(A', \alpha)}$$
$$= \sigma_{(A'', F\alpha' \circ \alpha)} = \sigma_{(A'', \alpha'')}$$

for all $\alpha'' = F\alpha' \circ \alpha \colon A \longrightarrow FA' \longrightarrow FF'A''$ in $|(A \downarrow FF')|$—and any α'' allows such a decomposition. $\qquad\square$

5.3.2 Theorem: *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$ with enough projectives, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I \colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. Let $n \geq 1$. Then*

$$H_{n+1}(-, I)_{\mathcal{E}} \colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{A}_{\mathcal{E}}$$

with the connecting natural transformation

$$\partial^{n+1} = \ker^{n-1}\delta^2 \circ \cdots \circ \ker\delta^n \circ \delta^{n+1} \colon H_{n+1}(-, I)_{\mathcal{E}} \circ \operatorname{cod}^n \Longrightarrow \ker^n \circ I_n$$

is the pointwise left satellite of I_n.

$$
\begin{array}{ccc}
 & \mathsf{Ext}^n\mathcal{A} & \\
\overset{\mathrm{cod}^n}{\swarrow} & & \overset{I_n}{\searrow} \\
\mathcal{A}_{\mathcal{E}} \dashrightarrow{\partial^{n+1}} & & \mathsf{Ext}^n\mathcal{A} \\
\overset{\mathrm{H}_{n+1}(-,I)_{\mathcal{E}}}{\searrow} & & \overset{\ker^n}{\swarrow} \\
 & \mathcal{A}_{\mathcal{E}} &
\end{array}
$$

This means, for any object A of $\mathcal{A}$,

$$
\mathrm{H}_{n+1}(A,I)_{\mathcal{E}} = \mathrm{Ran}_{\mathrm{cod}^n}(\ker^n \circ I_n)(A) = \lim_{(f,g)\in|(A\downarrow\mathrm{cod}^n)|} \mathrm{K}^n[I_n f].
$$

Proof. This follows from gluing diagrams as in Proposition 5.2.3 together using Proposition 5.3.1 and Remark 5.2.5. ◻

5.4 $\mathrm{H}_n(-,I)_{\mathcal{E}}$ as a right satellite of $\mathrm{H}_n(-,I_1)_{\mathcal{E}^1}$

Proposition 5.2.3 gives an interpretation of the connecting morphisms δ_f^n in the Everaert sequence as left satellites. The connecting morphisms γ_f^n have a dual interpretation: $(\mathrm{H}_n(-,I)_{\mathcal{E}},\gamma^n)$ is a right satellite (left Kan extension) of $\mathrm{H}_n(-,I_1)_{\mathcal{E}^1}$.

5.4.1 Proposition: *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$ with enough projectives, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. Consider $n \geq 1$. Then $(\mathrm{H}_n(-,I)_{\mathcal{E}},\gamma^n)$, i.e., $\mathrm{H}_n(-,I)_{\mathcal{E}}\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{A}_{\mathcal{E}}$ with the connecting natural transformation*

$$
\begin{array}{ccc}
 & \mathsf{Ext}\mathcal{A} & \\
\overset{\mathrm{H}_n(-,I_1)_{\mathcal{E}^1}}{\swarrow} & & \overset{\mathrm{dom}}{\searrow} \\
\mathsf{Ext}\mathcal{A} \dashrightarrow{\gamma^n} & & \mathcal{A}_{\mathcal{E}} \\
\overset{\ker}{\searrow} & & \overset{\mathrm{H}_n(-,I)_{\mathcal{E}}}{\swarrow} \\
 & \mathcal{A}_{\mathcal{E}} &
\end{array}
$$

is the pointwise right satellite of $\mathrm{H}_n(-,I_1)_{\mathcal{E}^1}$.

Proof. For any A, the category $(\mathrm{dom}\downarrow A)$ has a terminal object $(!_A\colon A \longrightarrow 0, 1_A)$, so the colimit object of the diagram

$$
(\mathrm{dom}\downarrow A) \xrightarrow{\ U\ } \mathsf{Ext}\mathcal{A} \xrightarrow{\ \mathrm{H}_n(-,I_1)_{\mathcal{E}^1}\ } \mathsf{Ext}\mathcal{A} \xrightarrow{\ \ker\ } \mathcal{A}_{\mathcal{E}}
$$

is $K[H_n(!_A, I_1)_{\mathcal{E}^1}] = H_n(A, I)_{\mathcal{E}}$. The component of the colimit cocone at

$$(g \colon B \longrightarrow C, f \colon B \longrightarrow A) \in |(\mathrm{dom}{\downarrow}A)|$$

is

$$\ker\left(H_n\left(\begin{array}{ccc} B & \xrightarrow{f} & A \\ g\downarrow & \Rightarrow & \downarrow !_A \\ C & \xrightarrow{!_C} & 0 \end{array}, I_1\right)_{\mathcal{E}^1}\right) = \ker\left(H_n\left(\begin{array}{ccc} B & \xrightarrow{f} & A \\ !_B\downarrow & \Rightarrow & \downarrow !_A \\ 0 & = & 0 \end{array}, I_1\right)_{\mathcal{E}^1}\right) \circ \ker\left(H_n\left(\begin{array}{ccc} B & = & B \\ g\downarrow & \Rightarrow & \downarrow !_B \\ C & \xrightarrow{!_C} & 0 \end{array}, I_1\right)_{\mathcal{E}^1}\right)$$

$$= H_n(f, I)_{\mathcal{E}} \circ \gamma_g^n$$

$$= \overline{\gamma^n}_{(g,f)}$$

by Lemma 5.2.2 and Lemma 5.2.1. $\qquad\square$

5.5 Homology without projectives

In this section we set up a homology theory without projectives by defining homology via pointwise satellites as they appear in Proposition 5.2.3.

5.5.1 Proposition: *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I \colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. Let $k \geq 0$, and consider an object $A \in |\mathcal{A}_{\mathcal{E}}|$. If it exists, write*

$$H_{(2,k)} = \mathrm{Ran}_{\mathrm{cod}}(\ker \circ I_{k+1})$$

for the pointwise left satellite of I_{k+1} relative to the functors cod and $\ker$. Now suppose $H_{(n,k+1)}$ exists for $n \geq 2$, and write

$$H_{(n+1,k)} = \mathrm{Ran}_{\mathrm{cod}}(\ker \circ H_{(n,k+1)})$$

for the pointwise left satellite of $H_{(n,k+1)}$ relative to cod and $\ker$, if this exists. Then $H_{(n+1,k)}$ is also the left satellite of I_n relative to the functors cod^n and $\ker^n$.

Proof. The proof is the same as the proof of Theorem 5.3.2. $\qquad\square$

5.5.2 Definition (Homology): Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I \colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. Consider an object $A \in |\mathcal{A}_{\mathcal{E}}|$, and let $n \geq 1$. If the functor $H_{(n+1,0)}$ from Proposition 5.5.1 exists, we call it the $(n+1)$**st homology functor**

$$H_{n+1}(-, I) \colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{A}_{\mathcal{E}}$$

(with coefficients in I).

$$
\begin{array}{ccc}
& \mathsf{Ext}\mathcal{A} & \\
\text{cod} \swarrow & & \searrow \mathrm{H}_n(-,I_1) \\
\mathcal{A}_{\mathcal{E}} & \xdashrightarrow{\ \delta^{n+1}\ } & \mathsf{Ext}\mathcal{A} \\
\mathrm{H}_{n+1}(-,I) \searrow & & \swarrow \text{ker} \\
& \mathcal{A}_{\mathcal{E}} &
\end{array}
$$

We also write $\mathrm{H}_1(-, I) = I$.

5.5.3 Remark: As $\mathsf{CExt}^k_{\mathcal{B}}\mathcal{A}$ is strongly $\mathcal{E}^k$-Birkhoff in $\mathsf{Arr}^k\mathcal{A}$, Definition 5.5.2 also gives us functors $\mathrm{H}_{n+1}(-, I_k)\colon \mathsf{Ext}^k\mathcal{A} \longrightarrow \mathsf{Ext}^k\mathcal{A}$. These are sometimes needed directly, but whenever possible we will include these higher-dimensional cases in the setting of strongly (extension)-Birkhoff subcategories, to make statements and proofs easier.

5.5.4 Remark: For any object $A \in |\mathcal{A}_{\mathcal{E}}|$, if $\mathrm{H}_2(A, I)$ exists, it is the limit object of the diagram

$$
(A \downarrow \text{cod}) \xrightarrow{\ U\ } \mathsf{Ext}\mathcal{A} \xrightarrow{\ I_1\ } \mathsf{Ext}\mathcal{A} \xrightarrow{\ \text{ker}\ } \mathcal{A}_{\mathcal{E}}.
$$

Similarly, if $\mathrm{H}_{n+1}(A, I)$ exists, it is the limit object of the diagram

$$
(A \downarrow \text{cod}) \xrightarrow{\ U\ } \mathsf{Ext}\mathcal{A} \xrightarrow{\ \mathrm{H}_n(-,I_1)_{\mathcal{E}1}\ } \mathsf{Ext}\mathcal{A} \xrightarrow{\ \text{ker}\ } \mathcal{A}_{\mathcal{E}}
$$

or equivalently of

$$
(A \downarrow \text{cod}^n) \xrightarrow{\ U\ } \mathsf{Ext}^n\mathcal{A} \xrightarrow{\ I_n\ } \mathsf{Ext}^n\mathcal{A} \xrightarrow{\ \text{ker}^n\ } \mathcal{A}_{\mathcal{E}}. \qquad \textbf{(J)}
$$

Potentially, these limits may exist for a given object A even if the homology functors $\mathrm{H}_{n+1}(-, I)$ do not exist in full.

5.5.5 Example (When the reflection is the identity): If $\mathcal{B} = \mathcal{A}$ then all I_n are identity functors, and the H_n are zero for $n \geq 2$. To see this, we have to prove that the functor $0\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{A}_{\mathcal{E}}$ is a pointwise Kan extension of $\text{ker}\colon \mathsf{Ext}\mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{E}}$ along $\text{cod}\colon \mathsf{Ext}\mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{E}}$, for all $k \geq 0$. This shows that H_2 is zero, which immediately implies that the higher homologies are also zero, being satellites of the zero functor.

Let A be an object of $\mathcal{A}_{\mathcal{E}}$ and (L, λ) a cone on $\ker \circ U \colon (A \downarrow \text{cod}) \longrightarrow \mathcal{A}_{\mathcal{E}}$. Then any map $\lambda_{(f,g)}$, where $(f \colon B \longrightarrow C, g \colon A \longrightarrow C) \in |(A \downarrow \text{cod})|$, fits into the commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\lambda_{(f,g)}} & \mathrm{K}[f] \\
\lambda_{(!_0,!_A)} \downarrow & {\scriptstyle \lambda_{(!_B,!_A)}} \searrow & \downarrow \mathrm{Ker}\, f \\
0 & \longrightarrow & B,
\end{array}
$$

which means that $\lambda_{(f,g)}$ is the zero map. If now (L, λ) is a limit cone, this implies that L is zero.

The category $(A \downarrow \text{cod})$ is rather large, and in a given situation it may be very hard to decide whether the needed limits do indeed exist. Even if they do, they may still be hard to compute. But we may replace the above diagrams with simpler ones, for example using the concept of an *initial subcategory*. Recall its definition as it occurs in [Mac1998, Section IX.3]:

5.5.6 Definition: An **initial functor** is a functor $F \colon \mathcal{D} \longrightarrow \mathcal{C}$ such that for every object C of $\mathcal{C}$, the slice category $(F \downarrow C)$ is non-empty and connected. A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called **initial** when the inclusion of $\mathcal{D}$ into $\mathcal{C}$ is an initial functor, i.e., for every object $C \in |\mathcal{C}|$, the full subcategory $(\mathcal{D} \downarrow C)$ of $(\mathcal{C} \downarrow C)$ determined by the maps $D \longrightarrow C$ with domain D in $\mathcal{D}$ is non-empty and connected.

If $\mathcal{D}$ is initial in $\mathcal{C}$ then limits of diagrams over $\mathcal{C}$ may be computed as the limit of their restriction to $\mathcal{D}$. More generally, if $F \colon \mathcal{D} \longrightarrow \mathcal{C}$ is initial then a diagram $G \colon \mathcal{C} \longrightarrow \mathcal{E}$ has a limit if and only if GF does, in which case it may be computed as the limit of GF.

For any object A of $\mathcal{A}_{\mathcal{E}}$, let $\mathsf{Ext}_A \mathcal{A}$ denote the **category of extensions of A**, the preimage in $\mathsf{Ext} \mathcal{A}$ of the arrow 1_A under the functor $\text{cod} \colon \mathsf{Ext} \mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{E}}$. Then the functor $U' \colon \mathsf{Ext}_A \mathcal{A} \longrightarrow (A \downarrow \text{cod})$ that sends an extension $f \colon B \longrightarrow A$ of A to the pair $(f, 1_A)$ is easily seen to be initial: for every object $(f \colon B \longrightarrow C, g \colon A \longrightarrow C)$ of $(A \downarrow \text{cod})$ there is the natural morphism $U' \overline{f} \longrightarrow (f, g)$

$$
\begin{array}{ccc}
\overline{B} & \xrightarrow{\overline{f}} A \xlongequal{\;1_A\;} A \\
\downarrow \quad {\scriptstyle \Downarrow} \quad {\scriptstyle g} \downarrow \quad {\scriptstyle \Downarrow} \quad \| \\
B & \xrightarrow{\;f\;} C \xleftarrow{\;g\;} A,
\end{array}
$$

where $\overline{f}$ is the pullback of f along g; this $\overline{f}$ is an extension by Definition 4.1.2(3).

Also, any other morphism

$$
\begin{array}{ccc}
D \xrightarrow{\ h\ } A & \xlongequal{\ 1_A\ } & A \\
\downarrow \quad \Downarrow \quad \big\downarrow{g} \quad \Downarrow & & \big\| \\
B \xrightarrow{\ f\ } C & \xleftarrow{\ g\ } & A,
\end{array}
$$

factors over this morphism $U'\overline{f} \longrightarrow (f,g)$, by the universal property of a pullback. This means that the limit of $\ker\circ H_n(-,I_1)_{\mathcal{E}^1}\circ U$ may also be computed as the limit of the diagram $\ker\circ H_n(-,I_1)_{\mathcal{E}^1}\circ UU'$ and moreover, since UU' is just the inclusion of the subcategory $\mathsf{Ext}_A\mathcal{A}$ into $\mathsf{Ext}\mathcal{A}$, as the limit of

$$
\ker\circ H_n(-,I_1)_{\mathcal{E}^1} \colon \mathsf{Ext}_A\mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{E}}.
$$

But even now the diagram of shape $\mathsf{Ext}_A\mathcal{A}$ over which the limit is computed may be too large, in the sense that even if $\mathcal{A}$ is small-complete, it is still unclear whether the limit of $\ker\circ H_n(-,I_1)_{\mathcal{E}^1}$ exists. In the case where $\mathcal{A}$ has enough projectives, however, it is possible to further cut down the size of this diagram. In this case Proposition 5.2.3 shows that the limit of this diagram exists and is equal to the homology object defined via the Hopf formulae. But making the diagram smaller gives a new way to calculate this homology. This situation is discussed in Section 5.6.

5.5.7 Notation: Let $A \in |\mathcal{A}_{\mathcal{E}}|$. Denote by $\mathsf{Ext}^n_A\mathcal{A}$ the category of n-**extensions of** A, defined as the preimage of the arrow 1_A under the functor $\mathrm{cod}^n\colon \mathsf{Ext}^n\mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{E}}$. This generalises the category $\mathsf{Ext}_A\mathcal{A}$ of *extensions of* A defined above. Thus the objects are n-extensions with "terminal object" A, when viewed as diagrams in the category $\mathcal{A}$, and the maps are those maps in $\mathsf{Ext}^n\mathcal{A}$ which restrict to the identity on A under cod^n. Similarly the category $\mathsf{CExt}^n_A\mathcal{A}$ denotes the full subcategory of $\mathsf{Ext}^n_A\mathcal{A}$ determined by those n-extensions which are central. The strongly $\mathcal{E}$-Birkhoff subcategory $\mathcal{B}$ is understood, and not mentioned in the notation.

5.5.8 Remark: The functor $U'\colon \mathsf{Ext}^n_A\mathcal{A} \longrightarrow (A \downarrow \mathrm{cod}^n)$ which sends an n-extension f of A to $(f,1_A)$ is still initial. This may be shown by induction, using the fact that in a category of n-fold extensions, the $(n+1)$-extensions are pullback-stable (see Definition 4.1.2(3)).

5.5.9 Proposition: *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. Consider $n \geq 1$ and $A \in |\mathcal{A}_{\mathcal{E}}|$. If it exists, $H_{n+1}(A,I)$ is also the limit of the diagram*

$$
\ker^n\circ I_n \colon \mathsf{Ext}^n_A\mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{E}}.
$$

Proof. This uses Diagram (**J**) and the fact that $U'\colon \mathsf{Ext}^n_A\mathcal{A} \longrightarrow (A \downarrow \mathrm{cod}^n)$ is initial. $\qquad\square$

5.5.10 Corollary: *For $n \geq 1$ and $A \in |\mathcal{A}_{\mathcal{E}}|$, if it exists, $\mathrm{H}_{n+1}(A, I)$ is the limit of the diagram*

$$\ker^n \colon \mathsf{CExt}_A^n \mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{E}}.$$

Proof. The functor $I_n \colon \mathsf{Ext}_A^n \mathcal{A} \longrightarrow \mathsf{CExt}_A^n \mathcal{A}$ is initial because, for any central extension $f \in |\mathsf{CExt}_A^n \mathcal{A}|$, we have $I_n f = f$, so the slice category $(I_n \downarrow f)$ is non-empty and connected. $\square$

Since limits commute with kernels, Corollary 5.5.10 also says that $\mathrm{H}_{n+1}(A, I)$ may be computed as the n-fold kernel of a certain n-fold arrow in $\mathcal{A}$, namely, the limit in $\mathsf{Arr}^n\mathcal{A}$ of the inclusion of $\mathsf{CExt}_A^n \mathcal{A}$ into $\mathsf{Arr}^n\mathcal{A}$. Sometimes this n-fold arrow in $\mathcal{A}$ itself happens to be an n-fold central extension of A. We say that an n-fold central extension of an object $A \in |\mathcal{A}_{\mathcal{E}}|$ is **universal** when it is an initial object of $\mathsf{CExt}_A^n \mathcal{A}$. We will see in Lemma 6.2.3 that, when $\mathcal{A}$ is a semi-abelian category and $I = \mathrm{ab} \colon \mathcal{A} \longrightarrow \mathsf{Ab}\mathcal{A}$ is the abelianisation functor, then an object A of $\mathcal{A}$ admits a universal central extension p if and only if it is **perfect**: its abelianisation is zero. In fact, this result holds for any reflector I to a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$, not just the abelianisation functor (see [CVdL2009, Theorem 2.9]). In this case, $\mathrm{H}_2(A, I)$ is the kernel of p. This latter property holds in general, also for higher extensions:

5.5.11 Corollary: *Consider $n \geq 1$ and $A \in |\mathcal{A}_{\mathcal{E}}|$. If A has a universal n-fold central extension p then $\mathrm{H}_{n+1}(A, I) = \mathrm{K}^n[p]$. In particular, if $A \in |\mathcal{A}|$ has a universal central extension $p \colon P \longrightarrow A$ then $\mathrm{H}_2(A, I) = \mathrm{K}[p]$.*

Proof. The limit of a functor from a category that has an initial object is the value of the functor at this object. $\square$

5.5.12 Example (The homology of zero is zero): If $A = 0$ then, for any $n \geq 1$, the category $\mathsf{CExt}_A^n \mathcal{A}$ has an initial object, the zero n-cube. Taking kernels as in Corollary 5.5.11 gives $\mathrm{H}_{n+1}(0, I) = 0$.

5.5.13 Remark: Note that in certain special cases a *weakly* universal extension can also determine the homology of an object A. When 1_A is a weakly universal extension of A, i.e., if every extension $f \colon B \longrightarrow A$ of A is split, we have $\mathrm{H}_2(A, I) = 0$. This is because $\mathrm{K}[I1_A] = 0$ for any object A, so if 1_A is weakly initial, every leg of a cone over $\ker \circ I \colon \mathsf{Ext}_A \mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{E}}$ factors over $\mathrm{K}[I1_A]$ and thus is zero. In particular, we get:

5.5.14 Example (The homology of a projective object is zero): For any projective object P and any $n \geq 1$ we have $\mathrm{H}_{n+1}(P, I) = 0$, since 1_P (and also the n-extension only consisting of the maps 1_P) is always weakly initial when P is projective.

5.5.15 Example (Homology of finite groups): For a finite group, we compare its second homology groups with respect to two different adjunctions. On the one hand we have the abelianisation functor $ab \colon \mathsf{Gp} \longrightarrow \mathsf{AbGp}$, where Gp is the category of groups, AbGp is the Birkhoff subcategory of abelian groups, and $ab\, G = G/[G,G]$. This example has been studied in the classical setting in [EVdL2004b] (for lower dimensions) and in [Eve2007, EGVdL2008] (higher dimensions). Here the centralisation functor ab_1 takes an extension $f \colon B \longrightarrow A$ to $\mathrm{centr}\, f \colon B/[\mathrm{K}[f], B] \longrightarrow A$. As mentioned in Example 1.3.6, in this case Definition 5.5.2 gives the classical integral homology of groups.

On the other hand, we could focus on finite groups and let $\mathcal{A} = \mathsf{FinGp}$ be the category of finite groups and $\mathcal{B} = \mathsf{FinAb} = \mathsf{AbFinGp}$ its Birkhoff subcategory of finite abelian groups. Note that FinGp is not semi-abelian and doesn't have enough projectives, but nevertheless it is pointed, Barr exact and Bourn protomodular. All the results that we apply in this example do not use coproducts, so they are still valid in this context. Here $I \colon \mathcal{A} \longrightarrow \mathcal{B}$ again sends a group G to $\mathrm{finab}\, G = G/[G,G]$ and $I_1 \colon \mathsf{ExtFinGp} \longrightarrow \mathsf{ExtFinGp}$ sends an extension $f \colon B \longrightarrow A$ to $\mathrm{fincentr}\, f \colon B/[\mathrm{K}[f], B] \longrightarrow A$. We show that, for any finite group, its second homology groups with respect to the two theories coincide.

For perfect groups this is clear. Recall from Corollary 5.5.11 that if a group G has a universal central extension $p \colon P \longrightarrow G$, then the homology is $\mathrm{H}_2(G, ab) = \mathrm{K}[p]$; this is the case when G is perfect: $ab\, G = 0$. So given a finite perfect group G, we know that it has a universal central extension $p \colon P \longrightarrow G$ in the category Gp of all groups, and that $\mathrm{H}_2(G, \mathbb{Z}) = \mathrm{H}_2(G, ab) = \mathrm{K}[p]$. But we also know that the integral homology of a finite group is a *finite* group, therefore the group P must also be finite, and the universal central extension $p \colon P \longrightarrow G$ lies in the category FinGp of *finite* groups. Thus we also have $\mathrm{H}_2(G, \mathrm{finab}) = \mathrm{K}[p]$. So for a finite perfect group G we have

$$\mathrm{H}_2(G, \mathrm{finab}) = \mathrm{H}_2(G, ab) = \mathrm{H}_2(G, \mathbb{Z}).$$

For a general group, we need a few more steps to prove this equality.

Step 1: First we want to show that, for any finite group G, there is a central extension $G^* \longrightarrow G$ with kernel $\mathrm{H}_2(G, \mathbb{Z})$, such that in the diagram

$$\ker \colon \mathsf{CExt}_G \mathsf{Gp} \longrightarrow \mathsf{Gp}, \qquad (\mathbf{K})$$

the leg from the limit $\mathrm{H}_2(G, ab)$ to this object is an isomorphism. We consider **stem extensions**: central extensions $g \colon H \longrightarrow G$ with $\mathrm{K}[g] \leq [H, H]$. This condition implies that $ab\, H \longrightarrow ab\, G$ is an isomorphism, or equivalently that the map $\mathrm{K}[g] \longrightarrow ab\, H$ is zero. So it follows from exactness in ($\mathbf{F}$) that the leg $\mathrm{H}_2(G, ab) \longrightarrow \mathrm{K}[g]$ is a surjection when g is a stem extension. To find a stem extension with $\mathrm{H}_2(G, \mathbb{Z})$ as its kernel, we use the **Schur multiplier** $M(G)$ of a finite group G introduced in [Sch1904]. Schur proved in [Sch1907]

that for a finite group G, this multiplier $M(G)$ may be expressed in terms of what is now called the Hopf formula (which, in the infinite case, was only introduced in [Hop1942]), and so we have $M(G) \cong \mathrm{H}_2(G, \mathbb{Z})$ (see also, e.g., [Kar1987, Theorem 2.4.6]). In [Sch1904] he showed that, for any finite group G, there is a stem extension $f \colon G^* \longrightarrow G$ of G with kernel $M(G)$ (see also [Kar1987, Theorem 2.1.4]).

Putting these two facts together, we see that $\mathrm{H}_2(G, \mathbb{Z})$ occurs in the diagram (**K**) as the kernel of this stem extension f, and that the leg from $\mathrm{H}_2(G, \mathrm{ab})$ to it must be an isomorphism, being a surjection between finite groups of the same size. From now on we shall assume that this isomorphism is an identity.

Step 2: We now consider the diagram of kernels of *finite* central extensions of G,

$$\ker \colon \mathsf{CExt}_G\mathsf{FinGp} \longrightarrow \mathsf{FinGp}, \tag{$\mathbf{L}$}$$

which is a small diagram and so has a limit in Gp which we denote by L. We shall show in Step 3 that $L \cong \mathrm{H}_2(G, \mathrm{ab})$ and so is actually the limit of (**L**) in the category FinGp as well, as $\mathrm{H}_2(G, \mathrm{ab})$ is a finite group.

$\mathrm{H}_2(G, \mathrm{ab})$ forms a cone on (**L**), using the legs from (**K**). The induced map of cones to L gives a splitting for the leg $p \colon L \longrightarrow \mathrm{H}_2(G, \mathbb{Z}) = \mathrm{K}[f]$. As these are all abelian groups, we have $L \cong \mathrm{H}_2(G, \mathbb{Z}) \oplus E$ for some abelian group E, and $p = \pi_1 \colon L \longrightarrow \mathrm{H}_2(G, \mathbb{Z})$, the first projection. We consider the following central extensions and maps between them:

$$
\begin{array}{ccccc}
\mathrm{H}_2(G,\mathbb{Z}) & \rightarrowtail & G^* & \xrightarrow{\ f\ } & G \\
{\scriptstyle p}\Big\updownarrow & & {\scriptstyle \pi_1}\Big\updownarrow{\scriptstyle (1_{G^*},0)} & & \Big\| \\
\mathrm{H}_2(G,\mathbb{Z}) \oplus E & \rightarrowtail & G^* \times E & \xrightarrow{\ f\circ\pi_1\ } & G \\
{\scriptstyle \pi_2}\Big\downarrow & & {\scriptstyle f\times 1_E}\Big\downarrow & & \Big\| \\
E & \rightarrowtail & G \times E & \xrightarrow[\ \pi_1\ ]{} & G
\end{array}
$$

Since the extension $\pi_1 \colon G \times E \longrightarrow G$ is split, the leg from L to $E = \mathrm{K}[\pi_1]$ must be the zero map. So the leg from L to $\mathrm{K}[f \circ \pi_1]$ is $1_{\mathrm{H}_2(G,\mathbb{Z})} \oplus 0 \colon L \cong \mathrm{H}_2(G, \mathbb{Z}) \oplus E \longrightarrow \mathrm{H}_2(G, \mathbb{Z}) \oplus E$, as $\mathrm{H}_2(G, \mathbb{Z}) \oplus E$ is a product.

Step 3: Finally we consider a third, even smaller diagram. Let $\mathcal{C}$ be the full subcategory of $\mathsf{CExtFinGp}$ containing those extensions g of G for which there exists a map $f \longrightarrow g$ in $\mathsf{CExt}_G\mathsf{FinGp}$. We consider the subdiagram

$$\ker \colon \mathcal{C} \longrightarrow \mathsf{FinGp}, \tag{$\mathbf{M}$}$$

the limit of which is $\mathrm{H}_2(G, \mathrm{ab})$. For any cone D over the diagram (**M**), the two legs $d \colon D \longrightarrow \mathrm{H}_2(G, \mathbb{Z}) = \mathrm{K}[f]$ and $0 \colon D \longrightarrow E = \mathrm{K}[\pi_1]$ again determine the leg to the prod-

uct, $(d, 0) \colon D \longrightarrow \mathrm{H}_2(G, \mathbb{Z}) \oplus E = \mathrm{K}[f \circ \pi_1]$. The leg d also forms the unique cone map $D \longrightarrow \mathrm{H}_2(G, \mathrm{ab})$. Notice that in (**M**) we also have maps from $\mathrm{H}_2(G, \mathbb{Z}) \oplus E$ to any other object, as $p \colon \mathrm{H}_2(G, \mathbb{Z}) \oplus E \longrightarrow \mathrm{H}_2(G, \mathbb{Z})$ is part of the diagram. So as $(1_{\mathrm{H}_2(G, \mathbb{Z})} \oplus 0) \circ (d, 0) = (d, 0)$, the map $(d, 0) \colon D \longrightarrow L$ is a cone map and makes L into a limit of (**M**). So $L \cong \mathrm{H}_2(G, \mathrm{ab})$ as promised, and we have $\mathrm{H}_2(G, \mathrm{finab}) = \mathrm{H}_2(G, \mathrm{ab}) = \mathrm{H}_2(G, \mathbb{Z})$ for any finite group G.

It is well known that all integral homology groups of a group are abelian. More generally, both approaches to homology discussed in Chapter 1 are such that the homology objects are abelian objects of the Birkhoff subcategory $\mathcal{B}$. We now prove that our homology objects $\mathrm{H}_{n+1}(A, I)$ also satisfy these properties.

5.5.16 Lemma: *Consider an object $A \in |\mathcal{A}_{\mathcal{E}}|$. The kernel $\mathrm{K}[f]$ of a central extension $f \colon B \longrightarrow A$ of A is an object of the strongly $\mathcal{E}$-Birkhoff subcategory $\mathcal{B}$.*

Proof. Let $A \in |\mathcal{A}_{\mathcal{E}}|$. First consider a trivial extension $f \colon B \longrightarrow A$. This means f is the pullback of $If \colon IB \longrightarrow IA$ along η_A, so $\mathrm{K}[f]$ is isomorphic to $\mathrm{K}[If]$. This kernel of the extension $If \colon IB \longrightarrow IA$ is an object of $\mathcal{B}$ because $\mathcal{B}$ is closed under subobjects (see Remark 4.2.2). Now for a central extension $f \colon B \longrightarrow A$, recall from Definition 1.4.3 that there exists an extension g such that the pullback $\overline{f}$ of f along g is trivial.

$$
\begin{array}{ccccc}
\mathrm{K}[\overline{f}] & \rightarrowtail & \overline{B} & \xrightarrow{\ \overline{f}\ } & \overline{A} \\
\big\| & & \downarrow{\lrcorner} & & \big\downarrow{g} \\
\mathrm{K}[f] & \rightarrowtail & B & \xrightarrow{\ f\ } & A
\end{array}
$$

But then $\mathrm{K}[f] = \mathrm{K}[\overline{f}]$, which is an object of $\mathcal{B}$ as $\overline{f}$ is trivial. $\qquad\square$

5.5.17 Remark: The converse implication does not hold, as for example in the category of groups not every extension with abelian kernel is central.

5.5.18 Proposition: *Let A be an object of $\mathcal{A}_{\mathcal{E}}$ and $n \geq 0$. Then $\mathrm{H}_{n+1}(A, I)$ is an object of $\mathcal{B}$.*

Proof. If $n = 0$ the result is clear as $\mathrm{H}_1(A, I) = IA$. For $n \geq 1$, we use Lemma 5.5.16 repeatedly to see that the diagram from Corollary 5.5.10 factors over $\mathcal{B}$ and becomes the functor $\ker^n \colon \mathsf{CExt}^n_A \mathcal{A} \longrightarrow \mathcal{B}$. Since $\mathcal{B}$ is closed under limits in $\mathcal{A}$, the limit $\mathrm{H}_{n+1}(A, I)$ of this diagram is still an object of $\mathcal{B}$. $\qquad\square$

5.5.19 Example (When the reflection is zero): If $\mathcal{B} = 0$, the zero subcategory in $\mathcal{A}$, then all homology objects are zero, because they are in $\mathcal{B}$ by Proposition 5.5.18.

The proofs of the next result Proposition 5.5.22 and its lemma were offered to us by Tomas Everaert. Recall that an object A of a homological category $\mathcal{A}$ is abelian if it carries an internal abelian group structure. Such a structure is necessarily unique, and is given by a morphism $m\colon A \times A \longrightarrow A$ satisfying $m \circ (1_A, 0) = 1_A = m \circ (0, 1_A)$, called its addition (see Definition 1.2.1). The abelian objects form a Birkhoff subcategory $\mathsf{Ab}\mathcal{A}$ of $\mathcal{A}$.

5.5.20 Lemma: *For any extension $f\colon B \longrightarrow A$ in $\mathcal{A}$, the image of the connecting morphism*

$$\delta_f^2\colon \mathrm{H}_2(A, I) \longrightarrow \mathrm{K}[\mathrm{H}_1(f, I_1)] = \mathrm{K}[I_1 f]$$

is an abelian object of $\mathcal{A}$.

Proof. We show that $\mathrm{I}[\delta_f^2]$ is a subobject of an abelian object in $\mathcal{A}$, namely the kernel of γ_f^1. Recall from Remark 4.4.10 that

$$\mathrm{K}[\gamma_f^1] = \frac{JB \cap \mathrm{K}[f]}{J_1[f]} = \frac{\pi_2(J\mathrm{R}[f]) \cap \pi_2(\mathrm{K}[f])}{\pi_2(J\mathrm{R}[f] \cap \mathrm{K}[f])}$$

and thus is abelian by Lemma 1.2.7.

Now the composite

$$\mathrm{H}_2(A, I) \xrightarrow{\delta_f^2} \mathrm{K}[I_1 f] \xrightarrow{\gamma_f^1} IB = \mathrm{H}_1(B, I)$$

is zero, as it is the leg from the limit $\mathrm{H}_2(A, I)$ to the kernel $IB = \mathrm{K}[I_1 !_B]$ of a split central extension. Thus the image of δ_f^2 factors over the kernel of γ_f^1, and $\mathrm{I}[\delta_f^2]$ is abelian, as a subobject of the abelian object $\mathrm{K}[\gamma_f^1] = (JB \cap \mathrm{K}[f])/J_1[f]$. $\qquad\square$

5.5.21 Remark: Notice that we can not assume any more that $\mathrm{I}[\delta_f^2] = \mathrm{K}[I_1 f]$, as we did in Lemma 4.4.9. As we are using a different definition of homology, we can not assume that the Everaert sequence is exact. It is however still a complex, and the map γ_f^1 does not change, so $\mathrm{K}[\gamma_f^1]$ is still the same abelian object. Note that Lemma 1.5.8 still implies that δ_f^2 is central in the sense of Huq, as it factors over the central morphism $\mathrm{Ker}\,\gamma_f^1$. Of course, as soon as we are working in a category with enough projectives, the two definitions of homology coincide and the Everaert sequence is exact even when using the definition via limits.

5.5.22 Proposition: *Let A be an object of $\mathcal{A}$ and $n \geq 1$. Then $\mathrm{H}_{n+1}(A, I)$ is an abelian object of $\mathcal{A}$.*

Proof. It suffices to show that, for any $A \in |\mathcal{A}_{\mathcal{E}}|$, the object $\mathrm{H}_2(A, I)$ is abelian in $\mathcal{A}$. We can then use this fact also in higher dimensions when $\mathcal{B} = \mathsf{CExt}^k_{\mathcal{B}}\mathcal{A}$, so then the higher homology objects are limits of a diagram of abelian objects, and thus abelian by induction. To show $\mathrm{H}_2(A, I)$ is abelian, consider the functor

$$\mathrm{H}_2(-, I) \times \mathrm{H}_2(-, I) \colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{A}_{\mathcal{E}}$$

that sends an object A to the product $\mathrm{H}_2(A, I) \times \mathrm{H}_2(A, I)$. The previous lemma gives rise to a natural transformation $(\mathrm{H}_2(-, I) \times \mathrm{H}_2(-, I)) \circ \mathrm{cod} \Longrightarrow \ker \circ I_1$ of functors from $\mathsf{Ext}\mathcal{A}$ to $\mathcal{A}_{\mathcal{E}}$; the component of this natural transformation at an extension $f \colon B \longrightarrow A$ is the composition

$$\mathrm{H}_2(A, I) \times \mathrm{H}_2(A, I) \longrightarrow \mathrm{I}[\delta_f^2] \times \mathrm{I}[\delta_f^2] \longrightarrow \mathrm{I}[\delta_f^2] \longrightarrow \mathrm{K}[I_1 f].$$

Here the first arrow is the corestriction of $\delta_f^2 \times \delta_f^2$, the second arrow is the addition on the abelian object $\mathrm{I}[\delta_f^2]$, and the last arrow is the inclusion of the image into the codomain of δ_f^2. The universal property of the Kan extension $(\mathrm{H}_2(-, I), \delta^2)$ now yields a natural transformation $\mathrm{H}_2(-, I) \times \mathrm{H}_2(-, I) \Longrightarrow \mathrm{H}_2(-, I)$ which is easily seen to define an abelian group structure on all $\mathrm{H}_2(A, I)$. $\qquad\square$

5.5.23 Example (Heyting semi-lattices): As mentioned in Example 1.1.4, Heyting semi-lattices form a semi-abelian category (see [Joh2004]). It would be interesting to study homology there, as traditionally this setting is not connected to any homology theories. However, the previous proposition implies that any semi-abelian homology theory in the category of Heyting semi-lattices is necessarily trivial, as the only abelian object is the zero object. We will prove this fact. For the axioms of Heyting semi-lattices see for example [Joh2004] or [Joh2002, A 1.5.11].

Given a morphism $f \colon A \longrightarrow B$ in the category of Heyting semi-lattices which has equal composite with $(1_A, 0) \colon A \times A \longrightarrow A$ and $(0, 1_A) \colon A \times A \longrightarrow A$, we show that f has to be the zero map. This implies that any abelian object is zero. The condition on f can be written as

$$f(\top, a) = f(a, \top) \quad \forall a \in A.$$

We have

$$f(\top, a) = f((a \Rightarrow a), (\top \Rightarrow a)) = f((a, \top) \Rightarrow (a, a)) = (f(a, \top) \Rightarrow f(a, a))$$

so writing $b = f(\top, a) = f(a, \top)$ and $c = f(a, a)$, we have

$$b = (b \Rightarrow c).$$

But then $b = b \wedge (b \Rightarrow c) = b \wedge c$ and $c \wedge b = c \wedge (b \Rightarrow c) = c$, so $b = c$ and we have

$$b = (b \Rightarrow b) = \top.$$

Therefore, translating back, we have $f(\top, a) = f(a, \top) = \top$ and $f(a, a) = \top$ for all $a \in A$. We then see by

$$f(b, a) = (\top \Rightarrow f(b, a)) = (f(b, \top) \Rightarrow f(b, a)) = f(\top, a) = \top$$

that f is the zero map, as claimed.

5.6 Homology with projectives

In this section we investigate our new definition of homology in the situation when $\mathcal{A}$ does have enough projectives. In this case we know that homology exists, for example via Everaert's definition using the Hopf formulae, and Proposition 5.2.3 shows that it coincides with the notion introduced in Definition 5.5.2. But by reducing the size of the diagram which defines the homology objects, we obtain a new way to calculate homology. Our main aim is to show Theorem 5.6.6 which states that the $(n+1)$st homology of an object A may be computed as a limit over the category $\mathsf{End}p$ of all endomorphisms of an n-presentation p of A.

5.6.1 Notation: For any n-extension f of an object $A \in |\mathcal{A}_{\mathcal{E}}|$, let $\mathsf{End}f$, the **category of endomorphisms of f over** A, be the full subcategory of $\mathsf{Ext}_A^n\mathcal{A}$ determined by the object f. Thus maps in $\mathsf{End}f$ are maps from f to itself which restrict to the identity on A under the functor cod^n.

When $\mathcal{A}$ has enough projectives we can view Proposition 5.2.3 the other way round:

5.6.2 Theorem (Hopf formula): *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$ with enough projectives, and let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ with reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. Let $n \geq 1$. Given an n-fold presentation p of an object $A \in |\mathcal{A}_{\mathcal{E}}|$ with initial object P_n, we have*

$$\mathrm{H}_{n+1}(A, I) \cong \frac{JP_n \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]}.$$

Proof. This is just Proposition 5.2.3 viewed from the perspective of Definition 5.5.2. □

5.6.3 Remark: In [Eve2007, Eve2008] Everaert gives a direct proof that the right hand side of the Hopf formula is a **Baer invariant** of A: an expression independent of the chosen n-fold presentation p of A (see also [EVdL2004a, Frö1963]). More precisely, any morphism $p \longrightarrow p$ over A induces the identity on $(JP_n \cap \mathrm{K}^n[p])/\mathrm{K}^n[J_n p]$.

Of course we can still calculate homology as a limit, as defined in Section 5.5. It turns out that in this case, homology may also be computed as a limit over the small subdiagram of shape $\widehat{\mathsf{End}}p$, which is a subcategory of $(A \downarrow \mathrm{cod}^n)$.

5.6.4 Notation: Let p be an n-presentation of a k-extension A, with initial object P_n. Let $\iota^n P_n$ be the n-cube which has initial object P_n and all other objects zero. The category $\widehat{\mathsf{End}}p$ is inspired by a higher-dimensional variation of Diagram (**I**): it is the subcategory of $(A \downarrow \mathrm{cod}^n)$ that is generated by the objects $(p, 1_A)$, $(\iota_n P_n, !_A)$ and $(0, !_A)$, all endomorphisms of p over A, and the three maps

$$
\begin{array}{ccccc}
p & & A & \overset{1_A}{=\!=\!=} & A \\
{\scriptstyle f}\downarrow & & {\scriptstyle !_A}\downarrow & \Downarrow & \|{\scriptstyle 1_A} \\
\iota_n P_n & & 0 & \overset{!_A}{\longleftarrow} & A \\
\uparrow\downarrow & & \uparrow\downarrow & \Uparrow\Downarrow & \| \\
0 & & 0 & \overset{!_A}{\longleftarrow} & A
\end{array}
$$

in $(A \downarrow \mathrm{cod}^n)$. Here f is the identity on P_n and obvious everywhere else, and the right side of the diagram displays the maps from A to the "terminal" object of the n-cube depicted on the left. Note that there is an obvious inclusion $\mathsf{End}p \longrightarrow \widehat{\mathsf{End}}p$ sending p to $(p, 1_A)$.

5.6.5 Proposition: *Consider $n \geq 1$ and $A \in |\mathcal{A}_\mathcal{E}|$, and let p be an n-fold presentation of A with initial object P_n. Then*

$$
\frac{Jp \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]} = \lim\big(\ker^n \circ I_n \circ U \colon \widehat{\mathsf{End}}p \longrightarrow \mathcal{A}_\mathcal{E}\big).
$$

Proof. The diagram we are considering is

$$
\frac{\mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]} = \mathrm{K}^n[I_n p] \xrightarrow{\ker^n(I_n f) = \bar{f}} \mathrm{K}^n[I_n \iota_n P_n] = I P_n \underset{\longleftarrow}{\overset{\longrightarrow}{\Longrightarrow}} 0
$$

Notice that the initial object of the n-cube $I_n p$ is $P_n/\mathrm{K}^n[J_n p]$. We will show that the object $(Jp \cap \mathrm{K}^n[p])/\mathrm{K}^n[J_n p]$ is the kernel of the map $\bar{f}$, which will in turn imply that it is indeed the limit of our diagram.

Forming the cokernel Q of $\mathrm{K}^n[p] \longrightarrow P_n$, we construct the following maps:

$$
\begin{array}{ccccc}
JP_n \cap \mathrm{K}^n[p] & \rightarrowtail & JP_n & \longrightarrow & JQ \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{K}^n[p] & \rightarrowtail & P_n & \longrightarrow & Q \\
\downarrow & & \downarrow & & \downarrow \\
\frac{\mathrm{K}^n[p]}{JP_n \cap \mathrm{K}^n[p]} & \rightarrowtail & IP_n & \longrightarrow & IQ
\end{array}
$$

Here the columns and the first two rows are short exact sequences, and the top left square is a pullback because $JQ \longrightarrow Q$ is a mono. The last row clearly composes to give the zero-map, so using the 3×3-Lemma we see that the last row is also a short exact sequence. To avoid fractions in the text we will name the map $g \colon IP_n \longrightarrow IQ$ and so refer to its kernel as $\mathrm{K}[g]$, and may use $\mathrm{K}^n[I_n p]$ instead of its explicit description as a fraction.

We now wish to show that $\bar{f}$ factors over $\mathrm{K}[g]$. For this we first see that taking dom^n of the commutative square

$$
\begin{array}{ccc}
p & \xrightarrow{\eta_p^n} & I_n p \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle I_n f} \\
\iota_n P_n & \xrightarrow{\eta_{(\iota_n P_n)}^n} & I_n(\iota_n P_n)
\end{array}
\qquad \text{gives} \qquad
\begin{array}{ccc}
P_n & \longrightarrow & \frac{P_n}{\mathrm{K}^n[J_n p]} \\
\| & & \downarrow \\
P_n & \xrightarrow{\eta_{P_n}} & IP_n.
\end{array}
$$

Also, since $\mathrm{dom}^n I_n(\iota^n P_n) = \mathrm{K}^n[I_n(\iota^n P_n)] = IP_n$, we see that $\bar{f}$ factors as

$$
\begin{array}{ccc}
\mathrm{K}^n[I_n p] = \frac{\mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]} & \rightarrowtail & \frac{P_n}{\mathrm{K}^n[J_n p]} = \mathrm{dom}^n I_n p \; . \\
{\scriptstyle \bar{f}}\downarrow & & \downarrow \\
IP_n & =\!=\!=\!=\!= & IP_n
\end{array}
$$

So in the following diagram, all possible squares and triangles commute.

$$
\begin{array}{ccccc}
\mathrm{K}^n[p] & \rightarrowtail & P_n & \longrightarrow & Q \\
& \searrow & & \searrow & \\
& \frac{\mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]} \rightarrowtail & & \frac{P_n}{\mathrm{K}^n[J_n p]} & \\
\downarrow & {\scriptstyle \bar{f}}\searrow & \downarrow & \swarrow & \downarrow \\
\mathrm{K}[g] & \rightarrowtail & IP_n & \xrightarrow{g} & IQ
\end{array}
$$

By considering the composite $\mathrm{K}^n[p] \longrightarrow\!\!\!\!> \mathrm{K}^n[I_n p] \xrightarrow{\ \bar{f}\ } IP_n \xrightarrow{\ g\ } IQ$ in this diagram, we see that $g \circ \bar{f} = 0$ and so $\bar{f}$ factors over $\mathrm{K}[g]$ by the regular epimorphism

$$\frac{\mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]} \xrightarrow{\quad h \quad}\!\!\!\!\!> \frac{\mathrm{K}^n[p]}{JP_n \cap \mathrm{K}^n[p]} = \mathrm{K}[g].$$

By Noether's Isomorphism Theorem its kernel is

$$\mathrm{K}[\bar{f}] = \mathrm{K}[h] = \frac{JP_n \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]}.$$

Any cone (C, σ) on $\ker^n \circ I_n \circ U \colon \widehat{\mathsf{End}}p \longrightarrow \mathcal{A}_\mathcal{E}$ consists of three maps as shown below.

$$
\begin{array}{ccc}
C & \xrightarrow{\ \sigma_{(p,1_A)}\ } & \mathrm{K}^n[I_n p] \\[2pt]
{\scriptstyle\sigma_{(0,!_A)}}\Big\downarrow \quad {\scriptstyle\sigma_{(\iota^n P_n, !_A)}} & & \Big\downarrow{\scriptstyle\bar{f}} \\[2pt]
0 & \longrightarrow & IP_n
\end{array}
$$

Thus $\sigma_{(p,1_A)}$ factors over $\mathrm{K}[\bar{f}]$, which we claim to be the limit of $\ker^n \circ I_n \circ U \colon \widehat{\mathsf{End}}p \longrightarrow \mathcal{A}_\mathcal{E}$.

It remains to show that $\mathrm{K}[\bar{f}]$ is itself a cone over this diagram. Given any endomorphism e of p over A, we write $\bar{e}$ for the induced endomorphism of $\mathrm{K}^n[I_n p]$ and $e_n \colon P_n \longrightarrow P_n$ for its "top" component. To show $\mathrm{K}[\bar{f}]$ forms a cone over the diagram, we have to prove that $\bar{e} \circ \mathrm{Ker}\,\bar{f} = \mathrm{Ker}\,\bar{f}$. But this follows from the fact that $(JP_n \cap \mathrm{K}^n[p])/\mathrm{K}^n[J_n p]$ is a Baer invariant of A (see Remark 5.6.3). Indeed, in the diagram

$$
\begin{array}{ccccc}
\dfrac{JP_n \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]} & \xrightarrow{\ \mathrm{Ker}\,\bar{f}\ }\!\!\!\!\!> & \mathrm{K}^n[I_n p] & \xrightarrow{\ \bar{f}\ } & IP_n \\[6pt]
{\scriptstyle\bar{e}'}\Big\downarrow & & {\scriptstyle\bar{e}}\Big\downarrow & & \Big\downarrow{\scriptstyle Ie_n} \\[6pt]
\dfrac{JP_n \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]} & \xrightarrow[\ \mathrm{Ker}\,\bar{f}\ ]{}\!\!\!\!\!> & \mathrm{K}^n[I_n p] & \xrightarrow[\ \bar{f}\ ]{} & IP_n
\end{array}
$$

the induced map $\bar{e}'$ is the identity, and the needed equality follows. Thus

$$\frac{JP_n \cap \mathrm{K}^n[p]}{\mathrm{K}^n[J_n p]}$$

forms a cone on $\ker^n \circ I_n \circ U \colon \widehat{\mathsf{End}}p \longrightarrow \mathcal{A}_\mathcal{E}$ and is indeed the limit, as claimed. $\qquad\square$

5.6.6 Theorem: *Consider $n \geq 1$ and $A \in |\mathcal{A}_\mathcal{E}|$. If $\mathcal{A}_\mathcal{E}$ has enough projectives and p is an n-fold presentation of A then*

$$\mathrm{H}_{n+1}(A, I) = \lim\!\big(\ker^n \circ I_n \colon \mathsf{End}p \longrightarrow \mathcal{A}_\mathcal{E}\big).$$

Proof. By Theorem 5.6.2 the $(n+1)$st homology of A is $(JP_n \cap \mathrm{K}^n[p])/\mathrm{K}^n[J_n p]$. Hence

by Proposition 5.6.5 it suffices to show that $\mathsf{End}p$ is initial in $\widehat{\mathsf{End}}_n p$. We must check that the slice categories $(\mathsf{End}p \downarrow (p, 1_A))$, $(\mathsf{End}p \downarrow (\iota_P, !_A))$ and $(\mathsf{End}p \downarrow (1_0, !_A))$ are non-empty and connected (here we view $\mathsf{End}p$ as the full subcategory of $\widehat{\mathsf{End}}p$ determined by $(p, 1_A)$). There is only one possible map from $(p, 1_A)$ to $(1_0, !_A)$, and the other two categories fulfil the needed conditions essentially because $((1_P, 1_Q), (1_A, 1_A))$ is a terminal object of the slice category $(\mathsf{End}p \downarrow (p, 1_A))$, and the only maps in $\widehat{\mathsf{End}}p$ from $(p, 1_A)$ to $(\iota_P, !_A)$ are compositions of an endomorphism of $(p, 1_A)$ with f. $\qquad\square$

5.6.7 Remark: This means that *computing the homology of an object essentially amounts to finding fixed points of endomorphisms of a projective presentation of this object.* The use of this technique will be illustrated in Examples 5.6.9 and 5.6.10.

5.6.8 Remark: We now come back to Remark 5.6.3 and interpret Definition 5.5.2 in terms of Baer invariants. It provides an alternative answer to the following question: "Given a functor $I\colon \mathcal{A} \longrightarrow \mathcal{A}$ and an object A of $\mathcal{A}$, how can we construct an object $\mathrm{H}_{n+1}(A, I)$ out of the n-extensions of A in a manner which is independent of any particular chosen extension of A?" The classical example is the Hopf formula

$$\mathrm{H}_2(A, I)_{\mathbb{G}} \cong \frac{JP \cap \mathrm{K}[p]}{\mathrm{K}[J_1 p]}$$

which expresses $\mathrm{H}_2(A, I)_{\mathbb{G}}$ in terms of a projective presentation $p\colon P \longrightarrow A$ of A. Of course, the very existence of the isomorphism implies that the expression on its right hand side cannot depend on the choice of p. The idea behind Definition 5.5.2 is different but straightforward: simply take the limit of *all* extensions of A. The independence might now be understood as follows. If p is an n-presentation of A then $\mathrm{H}_{n+1}(A, I)$ is the limit of $\ker^n \circ I_n\colon \mathsf{End}p \longrightarrow \mathcal{A}$, which means that $\mathrm{H}_{n+1}(A, I)$ is the universal object with the property that all endomorphisms of p are mapped to the same automorphism of this object, its identity.

Finally we show, as worked out examples, that we can retrieve well-known results in group homology using our new definition.

5.6.9 Example (Finite cyclic groups): We use the methods of our theory to calculate $\mathrm{H}_2(C_n, \mathrm{ab})$ for any $n \in \mathbb{N}$, where C_n is the cyclic group of order n. As $\mathbb{Z}$ is projective and abelian, the map $p\colon \mathbb{Z} \longrightarrow C_n$ which sends $1 \in \mathbb{Z}$ to a generator $c \in C_n$ is a projective presentation of C_n, and central. Thus $\mathrm{H}_2(C_n, \mathrm{ab})$ is the limit of the diagram

$\ker\colon \mathsf{End}\,p \longrightarrow \mathsf{Gp}$. Now any endomorphism of p must be

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\;p\;} & C_n \\
{\scriptstyle \cdot(nk+1)}\big\downarrow & \Downarrow & \big\| \\
\mathbb{Z} & \xrightarrow{\;p\;} & C_n
\end{array}
$$

i.e., multiplication by $(nk+1)$ for some $k \in \mathbb{Z}$. So $\mathrm{H}_2(C_n, \mathrm{ab})$ is the limit of the diagram which has $n\mathbb{Z}$ as the only object , and maps $\cdot(nk+1)\colon n\mathbb{Z} \longrightarrow n\mathbb{Z}$. If $\lambda\colon \mathrm{H}_2(C_n, \mathrm{ab}) \longrightarrow n\mathbb{Z}$ is the leg of the limit cone, we must have $\lambda(x) \cdot (nk+1) = \lambda(x)$ for every element $x \in \mathrm{H}_2(C_n, \mathrm{ab})$ and every k. *So we are looking for fixed points of the map $\cdot(nk+1)$.* But as, in $n\mathbb{Z}$, 0 is the only fixed point of multiplication by $(nk+1)$ for all $k \neq 1$, we have $\lambda(x) = 0$ for all $x \in \mathrm{H}_2(C_n, \mathrm{ab})$. Thus, as λ is a limit cone and so a monomorphism, $\mathrm{H}_2(C_n, \mathrm{ab}) = 0$.

Notice that as C_n and $\mathbb{Z}$ are abelian, they are also nilpotent of any class $m \geq 1$ and solvable of any derived length $m \geq 1$. Therefore $p\colon \mathbb{Z} \longrightarrow C_n$ is also central with respect to Nil_m and Sol_m for any $m \geq 1$, and so we have $\mathrm{H}_2(C_n, \mathrm{nil}_m) = 0$ and $\mathrm{H}_2(C_n, \mathrm{sol}_m) = 0$ for any $m \geq 1$.

5.6.10 Example (Generators and relations): Given a presentation of a group in terms of generators and relations, for example

$$
A = \langle a_1, \ldots, a_n \mid r_i = 1 \rangle
$$

for some relations r_i, the kernel of the free presentation

$$
p\colon F_n \longrightarrow A
$$

is generated by the relations r_i as a *normal* subgroup of F_n. Here F_n is the free group on n generators. But when we go to the centralisation

$$
\mathrm{centr}\,p\colon \frac{F_n}{[\mathrm{K}[p], F_n]} \longrightarrow A,
$$

every element of the kernel commutes with every other element, so now $\mathrm{K}[\mathrm{centr}\,p]$ is generated by the relations r_i as a subgroup of $F_n/[\mathrm{K}[p], F_n]$. Every endomorphism of p over A must send a generator a_i to $a_i k_i$ for some $k_i \in \mathrm{K}[f]$, and any choice of k_i gives such an endomorphism. Thus on $\mathrm{centr}\,p$ we get endomorphisms that send $a_i \in F_n/[\mathrm{K}[p], F_n]$ to $a_i \prod_j r_j^{\alpha_{ij}}$, for some $\alpha_{ij} \in \mathbb{Z}$, and again any choice of α_{ij} gives an endomorphism. Note that $\mathrm{K}[\mathrm{centr}\,p]$ is an abelian group, since it is in the centre of $F_n/[\mathrm{K}[p], F_n]$. From here it is relatively easy to find the fixed points of the induced endomorphism of $\mathrm{K}[\mathrm{centr}\,p]$, given

a specific group in terms of generators and relations. We give as an example

$$C_n \times C_n = \langle a, b \mid a^n = 1 = b^n, aba^{-1}b^{-1} = 1 \rangle.$$

Here $p \colon F_2 \longrightarrow C_n \times C_n$, and $\mathrm{K}[\mathrm{centr}\, p]$ is generated by $x = a^n$, $y = b^n$ and $z = aba^{-1}b^{-1}$. Note that as $aba^{-1}b^{-1}$ commutes with everything, we get $(aba^{-1}b^{-1})^n = ab^na^{-1}b^{-n}$, and as b^n also commutes with everything, we have $z^n = 1$. As described above, any endomorphism of $\mathrm{centr}\, p$ induced by one on p sends $a \in F_2/[\mathrm{K}[p], F_2]$ to $ax^{\alpha_1}y^{\alpha_2}z^{\alpha_3}$ and b to $bx^{\beta_1}y^{\beta_2}z^{\beta_3}$. On $\mathrm{K}[\mathrm{centr}\, p]$ this gives

$$x \longmapsto x^{n\alpha_1+1}y^{n\alpha_2}$$
$$y \longmapsto x^{n\beta_1}y^{n\beta_2+1}$$
$$z \longmapsto z$$

as the x, y and z commute with everything, and $z^n = 1$. For $x^{l_1}y^{l_2}z^{l_3}$ to be a fixed point for any of these endomorphisms, we need

$$l_1\alpha_1 + l_2\beta_1 = 0$$
$$l_1\alpha_2 + l_2\beta_2 = 0$$

for any choice of α_i and β_i, or in other words we need

$$l_1\alpha + l_2\beta = 0$$

for any choice of α and β. Hence $l_1 = l_2 = 0$, and we have fixed points z^{l_3}. Since $z^n = 1$, we get

$$\mathrm{H}_2(C_n \times C_n, \mathrm{ab}) = C_n.$$

Note that we can use the diagram over $\widehat{\mathsf{End}}p$ instead of $\mathsf{End}p$ to see that any fixed point must be of the form $aba^{-1}b^{-1}$ for some a and b (or a product of such), since the fixed point must be sent to the identity in $\mathrm{ab}\, F_n = F_n/[F_n, F_n]$.

$$
\begin{array}{ccc}
\mathrm{H}_2(A, \mathrm{ab}) & \longrightarrow & \mathrm{K}[\mathrm{centr}\, p] \\
\downarrow & \searrow & \downarrow \\
0 & \longrightarrow & \mathrm{ab}\, F_n
\end{array}
$$

Comparing this to the Hopf formula

$$\mathrm{H}_2(A, \mathrm{ab}) = \frac{[F_n, F_n] \cap \mathrm{K}[p]}{[\mathrm{K}[p], F_n]},$$

we see that the calculation using our method is exactly the same as the one using the Hopf

formula; the only thing that is different is the *interpretation* of these elements as fixed points of certain endomorphisms. Note that we of course proved in Proposition 5.6.5 that the limit of the diagram $\ker \circ I_1 \colon \widehat{\mathsf{End}}p \longrightarrow \mathcal{A}$ is the expression of the Hopf formula, so this is exactly what you would expect.

In contrast to the situation using abelianisation, we give another example which uses the Birkhoff subcategory of nilpotent groups.

5.6.11 Example (Gp vs. Nil_2): We calculate the second homology group of $C_2 \times C_2$ with respect to the Birkhoff subcategory Nil_2 of nilpotent groups of class at most 2, using the Hopf formula. We again use the projective presentation

$$p \colon F_2 \longrightarrow C_2 \times C_2$$

with kernel K, but now centralisation gives

$$\frac{K}{[[K,F_2],F_2]} \rightarrowtail \frac{F_2}{[[K,F_2],F_2]} \xrightarrow{I_1 p} C_2 \times C_2.$$

So the Hopf formula gives

$$\mathrm{H}_2(C_2 \times C_2, \mathrm{nil}_2) = \frac{K \cap [[F_2,F_2],F_2]}{[[K,F_2],F_2]} = \frac{K}{[[K,F_2],F_2]} \cap \frac{[[F_2,F_2],F_2]}{[[K,F_2],F_2]}$$

using Examples 1.3.3 and 4.3.7.

If F_2 is generated by a and b, a long (and tedious) calculation shows that $\mathrm{K}[I_1 p]$ is generated by the set

$$\{x_1 = a^2,\ y_1 = b^2,\ z = [a,b],\ x_2 = ba^2 b^{-1},\ y_2 = ab^2 a^{-1},\ x_3 = b^{-1}a^2 b,\ y_3 = a^{-1}b^2 a\}$$

with many relations, some of which we will list. The main fact we need to get these identities is that any element in $[K, F_2]$ commutes with any element of F_2. We get that the element z commutes with all other elements of $\mathrm{K}[I_1 p]$, the x_i commute amongst each other and so do the y_i. We have

$$[x_1, y_1] = x_1 y_1 x_1^{-1} y_1^{-1} = y_1 x_1^{-1} y_1^{-1} x_1 = [y_1, x_1^{-1}] = [x_1^{-1}, y_1^{-1}] = [y_1^{-1}, x_1] = z^4$$

that is, all cyclic permutations of $x_1 y_1 x_1^{-1} y_1^{-1}$ are the same as z^4. Similarly

$$[x_1, y_2] = [x_2, y_1] = [x_1, y_3] = [x_3, y_1] = [x_3, y_2] = [x_2, y_3] = [x_2, y_2] = [x_3, y_3] = z^4$$

with the same cyclic permutations as for $[x_1, y_1]$ above. Thus we see that z^4 commutes with every element of F_2, not just elements of the kernel as z does. We also have

$$[a^2, b]^2 = (x_1 x_2^{-1})^2 = [a^{-2}, b^{-1}]^2 = (x_1^{-1} x_3)^2 = z^4$$

and the corresponding expression for y instead of x. Notice that the elements $[a^2, b]$ etc. already commute with any element in F_2.

The following Witt-Hall identities of commutators are very useful:

$$[g, h_1 h_2] = [g, h_1] h_1 [g, h_2] h_1^{-1}$$
$$[g_1 g_2, h] = g_1 [g_2, h] g_1^{-1} [g_1, h]$$

These identities give for example that $[a, ba] = [a, b]$, and also $[[a, ab], a] = [[a, b], a]$ and $[[a, ab], b] = [[a, b], b]$ (using as always that elements of $[K, F_2]$ commute with every element of F_2). As soon as one each of g, h_2 and g_2, h lie in the kernel K, these identities become

$$[g, h_1 h_2] = [g, h_1][g, h_2]$$
$$[g_1 g_2, h] = [g_2, h][g_1, h].$$

We also see that $[k, g^{-1}] = [k, g]^{-1} = [k^{-1}, g]$ for any $k \in K$ and $g \in F_2$. Thus we find that the elements

$$[z, a] = [[a, b], a] \qquad \text{and} \qquad [z, b] = [[a, b], b]$$

generate $\frac{K \cap [[F_2, F_2], F_2]}{[[K, F_2], F_2]}$, and that

$$[z, a]^4 = [z^4, a] = 1 \qquad \text{and} \qquad [z, b]^4 = [z^4, b] = 1.$$

Thus we have $H_2(C_2 \times C_2, \mathrm{nil}_2) = C_4 \times C_4$.

Chapter 6

Stem Extensions in the Context of Abelianisation

In this chapter we look at certain special central extensions such as trivial and stem extensions, and establish a bijection between the isomorphism classes of stem extensions of an object A and the subobjects of $H_2(A, ab)$. This bijection is induced by the well-known isomorphism $\mathsf{Centr}(A, K) \cong \mathrm{Hom}(H_2 A, K)$ which is usually achieved in two steps, using $\mathsf{Centr}(A, K) \cong H^2(A, K) \cong \mathrm{Hom}(H_2 A, K)$. Here we give an explicit description of the direct isomorphism, without going via the second cohomology group. The Stallings-Stammbach sequence, which is the lower part of the Everaert sequence we have met before, plays a crucial role in these results.

6.1 Stem extensions and stem covers

Let us first review some of the material from Chapters 4 and 5 and restate it appropriately for its use in this chapter.

As we are dealing with a semi-abelian category $\mathcal{A}$ and its Birkhoff subcategory of abelian objects $\mathsf{Ab}\mathcal{A}$, we will denote the kernel of the unit of abelianisation of A by $[A, A]$ instead of JA:

$$0 \longrightarrow [A, A] \rightarrowtail A \xrightarrow{\ \eta_A\ } \mathrm{ab}\, A = \mathrm{H}_1 A \longrightarrow 0$$

This extends to a functor $[-, -] \colon \mathcal{A} \longrightarrow \mathcal{A}$. We also write $[\mathrm{K}[p], P]$ for the object $J_1[p] = \mathrm{K}[J_1 p]$ (see Section 4.3).

We use homology defined via Hopf formulae, as in Definition 4.4.3, but also use properties of homology exhibited by viewing it as a limit, as in Chapter 5. As we will only need the second homology object, we will repeat the definition here with our slightly changed notation.

6.1.1 Definition ([Eve2007]): Given an object A in $\mathcal{A}$, let

$$K \rightarrowtail P \xrightarrow{\ p\ } A$$

be a projective presentation of A. We define the **second homology object of** A by the Hopf Formula

$$\mathrm{H}_2 A = \frac{K \cap [P, P]}{[K, P]}.$$

The Stallings-Stammbach sequence [EVdL2004b, EG2007], which is the lowest part of the Everaert sequence (**F**), plays a big role in our results, but we only use it for central extensions, so we restate it here for this special case.

6.1.2 Proposition: *For any central extension*

$$0 \longrightarrow K \rightarrowtail B \overset{f}{\longrightarrow} A \longrightarrow 0 \tag{N}$$

in $\mathcal{A}$, there is an exact sequence

$$\mathrm{H}_2 B \overset{\mathrm{H}_2 f}{\longrightarrow} \mathrm{H}_2 A \overset{\delta_f}{\longrightarrow} K \overset{\gamma_f}{\longrightarrow} \mathrm{H}_1 B \overset{\mathrm{H}_1 f}{\longrightarrow} \mathrm{H}_1 A \longrightarrow 0 \tag{O}$$

in $\mathsf{Ab}\mathcal{A}$ which is natural in f.

Proof. This is just the lowest part of (**F**) in Theorem 4.4.6. See also [EVdL2004b, EG2007]. Notice that as here f is already central, we get the kernel of f instead of the kernel of $I_1 f = \mathrm{H}_1(f, I_1)_{\varepsilon_1}$. $\qquad \square$

For convenience we also recall Corollary 5.5.10 for the case of H_2.

6.1.3 Proposition: *Given an object A in $\mathcal{A}$, the second homology object $\mathrm{H}_2 A$ is the limit of the diagram*

$$\ker \colon \mathsf{CExt}_A \mathcal{A} \longrightarrow \mathcal{A}.$$

*The legs of the limit cone are the $\delta_f \colon \mathrm{H}_2 A \longrightarrow \mathrm{K}[f]$ as given by the Stallings-Stammbach sequence (**O**) for a central extension f.*

Proof. See Proposition 5.2.3 and Corollary 5.5.10. $\qquad \square$

We now define some special central extensions that we will consider throughout the chapter. Trivial extensions have been defined before, in 1.4.3, but we restate the definition here with more equivalent conditions, along with diagrams which clarify the connection to the new definitions. We have also met stem extensions before in the context of groups, in Example 5.5.15. The lámára extensions, which I have named thus just for the moment, are not particularly important in what follows and are only included for completeness.

6.1.4 Definition: Given a central extension f as in (**N**), we say it is a

(1) **trivial extension** if one of the following equivalent conditions is satisfied:

- $\delta_f = 0$,
- $[B, B] \longrightarrow [A, A]$ is an isomorphism,
- $K \cap [B, B] = 0$;

$$
\begin{array}{ccccc}
0 = K \cap [B,B] & \rightarrowtail & [B,B] & \xrightarrow{\ \cong\ } & [A,A] \\
\downarrow & & \downarrow & & \downarrow \\
K & \rightarrowtail & B & \xrightarrow{\ f\ } & A \\
\parallel & & \downarrow & & \downarrow \\
\mathrm{H}_2 A \xrightarrow{\ 0\ } K & \rightarrowtail & \mathrm{H}_1 B & \longrightarrow & \mathrm{H}_1 A
\end{array}
$$

(2) **lámára extension** if δ_f is a monomorphism;

(3) **stem extension** if one of the following equivalent conditions is satisfied:

- δ_f is a regular epimorphism,
- $\mathrm{H}_1 B \longrightarrow \mathrm{H}_1 A$ is an isomorphism,
- $\gamma_f = 0$,
- $K \leq [B, B]$;

$$
\begin{array}{ccccc}
& & [B,B] & \longrightarrow & [A,A] \\
& \nearrow & \downarrow & & \downarrow \\
K & \rightarrowtail & B & \xrightarrow{\ f\ } & A \\
\parallel & & \downarrow & & \downarrow \\
\mathrm{H}_2 A \xrightarrow{\ \delta_f\ } K & \xrightarrow{\ 0\ } & \mathrm{H}_1 B & \xrightarrow{\ \cong\ } & \mathrm{H}_1 A
\end{array}
$$

(4) **stem cover** if one of the following equivalent conditions is satisfied:

- δ_f is an isomorphism,
- $\mathrm{H}_1 B \longrightarrow \mathrm{H}_1 A$ is an isomorphism and $\mathrm{H}_2 B \longrightarrow \mathrm{H}_2 A$ is the zero map.

$$
\begin{array}{ccccc}
& & [B,B] & \longrightarrow & [A,A] \\
& \nearrow & \downarrow & & \downarrow \\
K & \rightarrowtail & B & \xrightarrow{\ f\ } & A \\
\parallel & & \downarrow & & \downarrow \\
\mathrm{H}_2 B \xrightarrow{0} \mathrm{H}_2 A \xrightarrow{\cong} K & \xrightarrow{0} & \mathrm{H}_1 B & \xrightarrow{\cong} & \mathrm{H}_1 A
\end{array}
$$

Clearly every stem cover is also a stem extension.

6.2 Perfect objects

From now on we only consider central extensions ($\mathbf{N}$) where A is a perfect object. It is well known that a group has a universal central extension if and only if it is perfect. This holds more generally in any semi-abelian category.

6.2.1 Definition (Universal central extension): A universal central extension (with respect to abelianisation) is a central extension $g\colon C \longrightarrow A$ such that for any other central extension $h\colon D \longrightarrow A$ there is a unique map $C \longrightarrow D$ making the following square commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\ g\ } & A \\
\Big\downarrow{\scriptstyle\exists!} & & \Big\| \\
D & \xrightarrow{\ h\ } & A
\end{array}
$$

6.2.2 Definition: An object A of a semi-abelian category $\mathcal{A}$ is called **perfect** when its abelianisation is zero: $\mathsf{Ab}A = 0$.

6.2.3 Lemma: [GVdL2008b, Proposition 4.1] *Let $\mathcal{A}$ be a semi-abelian category with enough projectives. An object A of $\mathcal{A}$ is perfect if and only if A admits a universal central extension.* $\hfill\square$

Recall from Corollary 5.5.11 that if A admits a universal central extension g, then the second homology group is the kernel of this central extension: $\mathrm{H}_2(A, \mathrm{ab}) = \mathrm{K}[g]$. In fact, we can say more:

6.2.4 Lemma: *Any universal central extension is a stem cover.*

Proof. Let $K \rightarrowtail C \xrightarrow{\ g\ } A$ be a universal central extension of A. The limit of the diagram

$$\ker\colon \mathsf{CExt}_A\mathcal{A} \longrightarrow \mathcal{A},$$

is $(\mathrm{H}_2A, \delta_f)_{f \in \mathsf{CExt}_A\mathcal{A}}$. As g is initial in $\mathsf{CExt}_A\mathcal{A}$, the limit H_2A is isomorphic to the kernel K of g, and the isomorphism is given by the leg δ_g. Thus, by definition, g is a stem cover. Notice that we could always make δ_g into an actual identity by changing the kernel of g to an isomorphic one. $\hfill\square$

When A is a perfect object, the special extensions defined in 6.1.4 take a more specific form.

6.2.5 Lemma: *When A is perfect,*

(1) any central extension ($\mathbf{N}$) has $\mathrm{H}_1B \cong K/(K \cap [B, B])$,

(2) *any trivial extension is isomorphic to a product projection $K \times A \xrightarrow{\pi_2} A$,*

(3) *any lámára extension satisfies $\mathrm{H}_2 A \cong K \cap [B, B]$,*

(4) *any stem extension has perfect domain B.*

Proof. (1) If $\mathrm{H}_1 A = 0$, we have the following commuting diagram:

$$
\begin{array}{ccccc}
K \cap [B,B] & \rightarrowtail & [B,B] & \longrightarrow & [A,A] \\
\downarrow & & \downarrow & & \| \\
K & \rightarrowtail & B & \xrightarrow{f} & A \\
\downarrow & & \downarrow & & \downarrow \\
\frac{K}{K \cap [B,B]} & \longrightarrow & \mathrm{H}_1 B & \longrightarrow & 0
\end{array}
$$

Here all columns and the top two rows are exact sequences, so using the 3×3 Lemma [Bou2001], we conclude that the bottom row is also short exact and the result follows. Notice that the map γ_f in the Stallings-Stammbach sequence (**O**) then becomes the canonical quotient $K \twoheadrightarrow K/(K \cap [B,B])$.

(2) If $\mathrm{H}_1 A = 0$, the pullback in 6.1.4 becomes a product:

$$
\begin{array}{ccccc}
 & & K & \rightarrowtail B \xrightarrow{f} A \\
 & & \| & \downarrow \quad \downarrow \\
\mathrm{H}_2 A & \xrightarrow{0} & K & \xrightarrow{\cong} \mathrm{H}_1 B \longrightarrow 0
\end{array}
$$

Thus $B = \mathrm{H}_1 B \times A \cong K \times A$ and f is a product projection.

(3) Combining (1) with the fact that δ_f is a monomorphism, the Stallings-Stammbach sequence (**O**) becomes

$$0 \longrightarrow \mathrm{H}_2 A \rightarrowtail K \longrightarrow \tfrac{K}{K \cap [B,B]} \cong \mathrm{H}_1 B \longrightarrow 0$$

and so $\mathrm{H}_2 A \cong K \cap [B,B]$.

(4) If $\mathrm{H}_1 A = 0$ and $\mathrm{H}_1 B \cong \mathrm{H}_1 A$, clearly $\mathrm{H}_1 B = 0$ as well.

$$
\begin{array}{ccccc}
 & & K & \rightarrowtail B \xrightarrow{f} A \\
 & & \| & \downarrow \quad \downarrow \\
\mathrm{H}_2 A & \xrightarrow{\delta_f} & K & \longrightarrow 0 \xrightarrow{\cong} 0
\end{array}
$$

$\square$

6.3 A natural isomorphism

In this section, we want to define a natural isomorphism

$$\theta \colon \mathsf{Centr}(A, -) \longrightarrow \mathrm{Hom}(\mathrm{H}_2 A, -)$$

of functors $\mathsf{Ab}\mathcal{A} \longrightarrow \mathsf{Ab}$. First we must define the abelian group $\mathsf{Centr}(A, K)$ for an abelian object K.

6.3.1 Remark: Notice that we will be talking about two different kinds of isomorphism classes of central extensions. Given an object A, we say two central extensions of A are isomorphic if there is an isomorphism

of central extensions. However, when we fix the kernel K, we say two central extensions of A by K are isomorphic if there is a morphism

where we demand the kernel part to be an actual identity. It follows from the Short Five Lemma that two central extensions of A by K are isomorphic if and only if there is a morphism between them.

Let $\mathsf{Centr}(A, K)$ denote the set of isomorphism classes of central extensions of A by K, as defined in the remark above. Recall from [Ger1970] how $\mathsf{Centr}(A, K)$ can be made into an abelian group using the Baer sum: given two central extensions

$$K \rightarrowtail^{k} B \xrightarrow{f} A \qquad \text{and} \qquad K \rightarrowtail^{k'} B' \xrightarrow{f'} A,$$

let h be their pullback

 or equivalently

where Δ is the diagonal. Then the extension $f + f'$ is formed as follows, using the multiplication m of the abelian object K:

$$
\begin{array}{ccc}
\mathrm{K}[m] & = & \mathrm{K}[m] \\
\downarrow i & & \downarrow (k \times k') \circ i \\
K \times K \xrightarrow{k \times k'} & B \times_A B' \xrightarrow{h} & A \\
\downarrow m & \mathrm{Coker}\,(k \times k') \circ i & \| \\
K \xrightarrow{\overline{k \times k'}} & D \xrightarrow{f + f'} & A
\end{array}
$$

As h is a central extension with respect to abelianisation, the subobject $(k \times k') \circ i$ is normal (see Proposition 1.5.16). So we can form its cokernel, or equivalently the pushout of $k \times k'$ and m. Then $\overline{k \times k'}$ is a monomorphism as the top square is a pullback. We take its cokernel, the codomain of which is A since the bottom left square is a pushout. Note that this square is also a pullback. This cokernel represents the isomorphism class of $f + f'$; it is central because the category of central extensions in $\mathcal{A}$ is closed under quotients in the category of extensions in $\mathcal{A}$ (see [GVdL2008b] for more details). It is easily checked that this gives a well-defined sum on the isomorphism classes of central extensions of A by K. The zero for this addition is the class of split central extensions, a representative of which is the product projection $K \rightarrowtail K \times A \xrightarrow{\pi_2} B$. Note that any split central extension is trivial [JK1994, Section 4.3], and compare this with 6.2.5(2).

Recall from [GVdL2008b] how $\mathsf{Centr}(A, -)$ is made into a functor $\mathsf{Ab}\mathcal{A} \longrightarrow \mathsf{Ab}$. The main ingredient is the following result:

6.3.2 Lemma: [GVdL2008b, Corollary 3.3] *Given a central extension* (**N**) *and a map from K to an abelian object K', there is a central extension $K' \rightarrowtail B' \longrightarrow A$ making the diagram below commute.*

$$
\begin{array}{ccc}
K \rightarrowtail & B \longrightarrow & A \\
\downarrow & \downarrow & \| \\
K' \rightarrowtail & B' \dashrightarrow & A
\end{array}
$$

Proof. See Corollary 3.3 in [GVdL2008b]. For this construction to work it is crucial that the extension $B \longrightarrow A$ is central in the sense of Huq, which is the main reason why this isomorphism only works in the setting of abelianisation. $\square$

This indeed makes $\mathsf{Centr}(A, -)$ into a functor $\mathsf{Ab}\mathcal{A} \longrightarrow \mathsf{Ab}$ (see Propositions 6.1 and 6.2 in [GVdL2008b]).

For a given abelian object K, the set $\mathrm{Hom}(\mathrm{H}_2 A, K)$ also forms an abelian group. Two maps α and β are added using the multiplication m of K as follows:

$$\alpha + \beta\colon \mathrm{H}_2 A \xrightarrow{(\alpha,\beta)} K \times K \xrightarrow{\quad m \quad} K$$

The zero of the addition is the zero map $\mathrm{H}_2 A \xrightarrow{0} K$. Therefore we can view the functor $\mathrm{Hom}(\mathrm{H}_2 A, -)\colon \mathcal{A} \longrightarrow \mathsf{Set}$ also as a functor $\mathrm{Hom}(\mathrm{H}_2 A, -)\colon \mathsf{Ab}\mathcal{A} \longrightarrow \mathsf{Ab}$.

We can now finally define, for a perfect object A, the required natural transformation.

6.3.3 Definition: Given a perfect object A in $\mathcal{A}$, we define a natural transformation

$$\theta\colon \mathsf{Centr}(A, -) \longrightarrow \mathrm{Hom}(\mathrm{H}_2 A, -)$$

as follows: for each abelian object K in $\mathcal{A}$, θ_K takes the (isomorphism class of the) central extension $K \overset{k}{\rightarrowtail} B \xrightarrow{f} A$ to the map $\delta_f\colon \mathrm{H}_2 A \longrightarrow K$ defined by the Stallings-Stammbach sequence (**O**). This is well-defined: if f is isomorphic to $K \overset{k'}{\rightarrowtail} B' \xrightarrow{f'} A$, the Stallings-Stammbach sequence gives

$$
\begin{array}{ccccccc}
\mathrm{H}_2 A & \xrightarrow{\ \delta_f\ } & K & \longrightarrow & \mathrm{H}_1 B & \longrightarrow & \mathrm{H}_1 A = 0 \\
\| & & \| & & \downarrow{\scriptstyle \cong} & & \| \\
\mathrm{H}_2 A & \xrightarrow{\ \delta_{f'}\ } & K & \longrightarrow & \mathrm{H}_1 B' & \longrightarrow & \mathrm{H}_1 A = 0
\end{array}
$$

so $\delta_f = \delta_{f'}$. Naturality of θ follows from the definition of the functor $\mathsf{Centr}(A, -)$ and the fact that δ itself is a natural transformation. For θ to be a natural transformation between functors to Ab, we must show that each θ_K is a homomorphism of abelian groups. We do this in a separate lemma.

6.3.4 Lemma: *For each abelian object K in $\mathcal{A}$, θ_K is a homomorphism of abelian groups.*

Proof. Given two central extensions f and f' representing two isomorphism classes in $\mathsf{Centr}(A, K)$, we consider their pullback $B \times_A B' \xrightarrow{h} A$ as above in the definition of the sum in $\mathsf{Centr}(A, K)$. We again use the limit cone $(\mathrm{H}_2 A, \delta_f)_{f \in \mathsf{CExt}_A \mathcal{A}}$ of the diagram

$\ker\colon \mathsf{CExt}_A\mathcal{A} \longrightarrow \mathcal{A}$ to see that $\delta_h = (\delta_f, \delta_{f'})\colon \mathrm{H}_2A \longrightarrow K \times K$:

$$
\begin{array}{c}
\mathrm{H}_2A \\
\delta_f \swarrow \quad \downarrow \delta_h \quad \searrow \delta_{f'} \\
K \xleftarrow{\ \pi_1\ } K \times K \xrightarrow{\ \pi_2\ } K \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
B \longleftarrow B \times_A B' \longrightarrow B' \\
\downarrow f \qquad\qquad \downarrow h \qquad\qquad \downarrow f' \\
A =\!=\!= A =\!=\!= A
\end{array}
$$

Remembering that $f + f'$ is given by

$$
\begin{array}{ccc}
K \times K \xrightarrow{\ k \times k'\ } B \times_A B' \xrightarrow{\ h\ } A \\
\downarrow m \qquad\qquad \downarrow \qquad\qquad \| \\
K \rightarrowtail\ \qquad D \xrightarrow{\ f+f'\ } A
\end{array}
$$

the Stallings-Stammbach sequence gives us a commuting diagram

$$
\begin{array}{ccc}
\mathrm{H}_2A & \xrightarrow{\ \delta_h = (\delta_f, \delta_{f'})\ } & K \times K \\
\| & \searrow{\scriptstyle \delta_f + \delta_{f'}} & \downarrow m \\
\mathrm{H}_2A & \xrightarrow[\ \delta_{f+f'}\ ]{} & K
\end{array}
$$

and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

6.3.5 Lemma: *For each abelian object K in $\mathcal{A}$, θ_K is injective.*

Proof. If the central extension f maps to $\delta_f = 0$, it is a trivial extension by definition, and so as A is perfect, f is (isomorphic to) the product projection $K \rightarrowtail K \times A \xrightarrow{\pi_2} A$ (see Lemma 6.2.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

6.3.6 Lemma: *For each abelian object K in $\mathcal{A}$, θ_K is surjective.*

Proof. Given a morphism $\alpha\colon \mathrm{H}_2A \longrightarrow K$, we use a universal central extension of A and Lemma 6.3.2 to construct a central extension of A by K. Let $\mathrm{H}_2A \rightarrowtail C \xrightarrow{g} A$ be a universal central extension of A, which is a stem cover by Lemma 6.2.4. We can choose g such that $\delta_g\colon \mathrm{H}_2A \longrightarrow \mathrm{H}_2A$ is the identity on H_2A. Then as K is an abelian object,

Lemma 6.3.2 gives us a central extension f and a map from g to f as below:

$$
\begin{array}{ccccc}
\mathrm{H}_2A & \rightarrowtail & C & \xrightarrow{\ g\ } & A \\
\downarrow{\scriptstyle\alpha} & & \vdots & & \| \\
K & \rightarrowtail & B & \xrightarrow[\ f\ ]{} & A
\end{array}
$$

Forming the Stallings-Stammbach sequence of f and g, we see that $\delta_f = \alpha$:

$$
\begin{array}{ccc}
\mathrm{H}_2A & \xlongequal{\ \delta_g\ } & \mathrm{H}_2A \\
\| & & \downarrow{\scriptstyle\alpha} \\
\mathrm{H}_2A & \xrightarrow[\ \delta_f\ ]{} & K
\end{array}
$$

Thus the isomorphism class of f really is a preimage of α. $\qquad\square$

We can now put together the above results into our main theorem.

6.3.7 Theorem: *Given a perfect object A in a semi-abelian category $\mathcal{A}$, there is a natural isomorphism*

$$\theta\colon \mathsf{Centr}(A, -) \longrightarrow \mathrm{Hom}(\mathrm{H}_2A, -)$$

of functors $\mathsf{Ab}\mathcal{A} \longrightarrow \mathsf{Ab}$*. Fixing an abelian object K,*

(1) *trivial extensions correspond to the zero map $\mathrm{H}_2A \xrightarrow{\ 0\ } K$ (i.e. there is only one equivalence class of trivial extensions of A by K, one representative of which is the product projection $K \rightarrowtail K \times A \xrightarrow{\ \pi_2\ } A$),*

(2) *lámára extensions correspond to monomorphisms $\mathrm{H}_2A \rightarrowtail K$,*

(3) *stem extensions correspond to regular epimorphisms $\mathrm{H}_2A \twoheadrightarrow K$,*

(4) *stem covers correspond to isomorphisms $\mathrm{H}_2A \xrightarrow{\ \cong\ } K$.*

Proof. θ is a natural isomorphism by Lemmas 6.3.4, 6.3.5 and 6.3.6. The correspondences (1) to (4) all follow directly from Definition 6.1.4 and the definition of θ. $\qquad\square$

6.3.8 Corollary: *There is only one isomorphism class of stem covers of a perfect object.*

Proof. If we only look at isomorphisms of central extensions of a perfect object A by a fixed kernel K, as in $\mathsf{Centr}(A, K)$, Theorem 6.3.7(4) gives us several isomorphism classes

of stem extensions, one for each isomorphism $H_2A \longrightarrow K$. But as soon as we vary K, we see that all stem covers of A are isomorphic to each other (cf. Remark 6.3.1).

$$
\begin{array}{ccc}
H_2A \rightarrowtail B & \xrightarrow{\ g\ } & A \\
{\scriptstyle\delta_f}\downarrow{\scriptstyle\cong} \quad \downarrow{\scriptstyle\cong} & & \| \\
K \rightarrowtail B' & \xrightarrow{\ f\ } & A
\end{array}
$$

$\square$

6.3.9 Corollary: *Any stem cover of a perfect object is a universal central extension.*

Proof. As any universal central extension is a stem cover by Lemma 6.2.4 and there is only one isomorphism class of stem covers, any stem cover is universal. $\square$

6.3.10 Remark: The isomorphism in Theorem 6.3.7 is not new, it exists as a two-step isomorphism $\mathsf{Centr}(A, K) \cong H^2(A, K) \cong \mathrm{Hom}(H_2A, K)$ given for example in [GVdL2008b]. We have merely given a direct correspondence without going via the cohomology group.

6.3.11 Corollary: *There is a bijective correspondence between the isomorphism classes of stem extensions of a perfect object A and subobjects of H_2A.*

Proof. When we vary K in Theorem 6.3.7, we see that the isomorphism classes of stem extensions of A correspond to regular epimorphisms $H_2A \longrightarrow\!\!\!\!\!\gg K$ for any K, which in turn correspond to subobjects U of H_2A by taking the kernels of these regular epimorphisms. $\square$

6.3.12 Lemma: *Every stem extension of a perfect object A is the regular image of a stem cover of A.*

Proof. As A is perfect, it has a universal central extension $H_2A \rightarrowtail C \overset{g}{\longrightarrow\!\!\!\!\!\gg} A$ which is a stem cover. As g is universal, for every stem extension f there is a map

$$
\begin{array}{ccc}
H_2A \rightarrowtail C & \xrightarrow{\ g\ } & A \\
\downarrow{\scriptstyle\delta_f} \quad \downarrow & & \| \\
K \rightarrowtail V & \xrightarrow{\ f\ } & A
\end{array}
$$

where it is easily seen using the Stallings-Stammbach sequence that the induced map $H_2A \longrightarrow K$ is indeed (isomorphic to) δ_f. By the Short Five Lemma, which also holds for regular epimorphisms (see [Bou2001]), the map $C \longrightarrow V$ in the middle is also a regular epimorphism. $\square$

6.3.13 Remark: The results in Corollaries 6.3.8 and 6.3.11 and in Lemma 6.3.12 are extensions to general semi-abelian categories of the corresponding results on crossed modules in [VC2002]. If $\mathcal{C}$ is a semi-abelian category, the category $\mathcal{A} = \mathsf{Grpd}(\mathcal{C})$ of groupoids in $\mathcal{C}$ is also semi-abelian, and its subcategory of abelian objects $\mathsf{Ab}\mathcal{A}$ is $\mathsf{RG}(\mathsf{Ab}\mathcal{C})$, reflective graphs in $\mathsf{Ab}\mathcal{C}$. A special case of this is the category of crossed modules when $\mathcal{C} = \mathsf{Gp}$ the category of groups. This gives back the results in [VC2002].

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