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Generalized Exponential, Polynomial and Trigonometric Theories for Vibration and Stability Analysis of Porous FG Sandwich Beams Resting on Elastic Foundations

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Abstract

The present article investigates the free vibration and elastic stability behaviour of three-dimensional functionally graded sandwich beams featured by two different types of porosity, with arbitrary boundary conditions and resting on Winkler-Pasternak elastic foundations. The investigation is carried out by using the method of series expansion of displacement components. Various hierarchical refined exponential, polynomial, and trigonometric higher-order beam theories are developed in a generalized manner and are validated and assessed against 3D FEM results. The weak-form of the governing equations (GEs) is derived via Hamilton's Principle. The GEs are then solved by using the Ritz method, whose accuracy is significantly enhanced by orthogonalizing the algebraic Ritz functions by virtue of the Gram-Schmidt process. Convergence and accuracy are comprehensively analysed by testing 86 quasi-3D beam theories. Moreover, the effect of significant parameters such as slenderness ratio, volume fraction index, porosity coefficient, elastic foundation coefficients, FG sandwich beam typology as well as boundary conditions, on the circular frequency parameters and critical buckling loads, is discussed.

Keywords: Quasi-3D beam theories, Free Vibration, Stability, FG sandwich beams, Porosities, Elastic foundations, Ritz method.

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1. Introduction

Functionally Graded Materials (FGMs) are a class of inhomogenous composites whose material constituents vary smoothly and continuously from one surface to another. The gradation in properties of the materials results in a reduction of thermal stresses, residual stresses and stress concentration factors which affect laminated composite structures. Moreover, problems such as delamination, fibre failure as well as adverse hygroscopic effects, are effectively eliminated or non-existent. Thus, due to their potential use in several fields, with a focus on thermal engineering applications (thermal barrier structures subjected to sever thermal gradients) there is the need to fully understand their mechanical and above all thermal behaviour.

Many scientific articles have been recently published on the static and dynamic analysis of FG beams, some are given in Refs. [1–10] amongst many others. However, since complex fabrication procedures are used to realized FGM structures [11], micro voids and porosities often occur. In particular, during these processes, due to the large difference in the solidification temperature of the FGM material constituents a certain amount of defects appear. So, during the design procedure, an accurate modelling of such porosities become a mandatory issue, to predict properly both the static and the dynamic response of the FGM structure under investigation. Recently, a relevant amount of papers entirely focused on this research topic has been proposed in the literature. Various successful methods of modelling the porosities have been developed. More specifically, Wattanasakulpong *et al.* [12] studied the linear and nonlinear vibration characteristics of elastically restrained ends FGM beams with porosities. The differential transformation method (DTM) was employed to solve linear and nonlinear vibration responses of FGM beams with different kinds of elastic supports. The same author [13] proposed a comprehensive static analysis of imperfect FGM beam by combining the Timoshenko beam theory and the Chebyshev collocation method. Ebrahimi and Zia [14] dealt with the large-amplitude nonlinear vibration of functionally graded (FG) beams made of porous materials. The forced and free vibration behaviour of FGM porous beams, with non-uniform porosity distribution whose elastic moduli and mass density are nonlinearly graded along the thickness direction, have been investigated by Chen et al. [15]. Timoshenko beam model was employed along with the Lagrange equations method and the Ritz method was used as solution technique. Moreover, the Newmark-β method was applied as time integration scheme. The same author [16] for the same structure typology provided a complete bending and elastic stability analysis. The partial differential equation system, governing the buckling and bending behaviour of porous beams is derived through the Hamilton's principle. The Ritz method was employed to obtain the critical buckling loads and transverse bending deflections, the trial functions were chosen to be simple algebraic polynomials. A probabilistic analysis accounting for the effect of the porosities in functionally graded material nanoplates resting on Winkler-Pasternak elastic foundations has been given by Mechab *et al.* [17]. The small scale effects were introduced using the non local elasticity theory. The governing differential equations (GDEs) were solved analytically. In addition, the Monte Carlo Simulation (MCS) method was used to predict the distribution function of the dynamic response. Atmane *et al.* [18] investigated the effect of both thickness stretching and porosity on mechanical response of functionally graded beams resting on elastic foundations. Murin *et al.* [19] developed a homogenized beam finite element for modal analysis of FGM beams. The shear force deformation effect and the effect of longitudinally varying inertia and rotary inertia were taken into account. Moreover, the effect of the Winkler elastic foundation was also accounted for. Simsek [20] investigated the linearised buckling behaviour of Timoshenko beams composed of two-dimensional functionally graded material having different boundary conditions. Rjoub and Hamad [21] developed an analytical method to study the dynamic behaviour of functionally Euler-Bernoulli and Timoshenko graded beams accounting for porosities with differing boundary conditions. The transfer matrix method (TMM) was used to obtain the natural frequency equations. Porous FGM box have been investigated by Ziane*et al.* [22]. In particular, the authors focused on the thermal effects on the instability characteristics by using the Galerkin's method. FGM structures have also been deeply investigated for free vibration and static problems in Refs. [23–33], amongst many others.

As regards the development of advanced beam theories in the modelling of beam structures, it is worth mentioning those generated by using the Carrera Unified Formulation (CUF), some of these contributions can be found in Refs. [34–38]. In the present article the accurate Hierarchical Ritz Formulation (HRF), extensively employed in the analysis of laminated composite and FGM beams, plates and shells [10, 39–48] has been significantly extended to provide a comprehensive free vibration and stability analysis of FG sandwich beams including porosities. The investigation has been carried out by using the method of series expansion of displacement components. In particular, advanced generalized exponential, polynomial as well as trigonometric quasi-3D beam theories with hierarchical capabilities have been developed. The latter have also been validated and assessed against results available in literature and 3D FEM results obtained by using the commercial software ABAQUS. Orthogonal admissible functions have been used in the Ritz approximation. More specifically, given a polynomial function a set of orthogonal shape functions have been developed over the considered domain by using the Gram-Schmidt process. This recursive procedure increase significantly the computational stability of the adopted admissible functions allowing to obtain a higher accuracy. The effect of some other parameters such as slenderness ratio, volume fraction index, FG sandwich beam typology and boundary conditions, on the dimensionless frequency parameters and the dimensionless critical buckling loads has been commented.

2. Geometric and Constitutive relations

The geometry and related nomenclature of the FG sandwich beam under investigation are shown in Fig. 1. In particular, the problem is defined by using a rectangular Cartesian reference system (xyz), the cross-section area is considered lying in the plane (xy) and is named Ω, while the axial coordinate z is referred to as reference line of the beam. In the present study two different FG beam configurations are investigated. In particular, the type-I is a FG isotropic beam depicted in Fig. 2 (a); the type-II is a sandwich beam with FG face sheets and a ceramic-core, as shown in Fig. 2 (b). The length of the beam is indicated by l while the symbols b and h denote the beam width and thickness, respectively. According to the reference system the stress and strain vectors are indicated as follows

$$\boldsymbol{\sigma} = \left\{ \sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz} \quad \sigma_{zz} \right\}^T, \quad \boldsymbol{\varepsilon} = \left\{ \varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz} \quad \varepsilon_{zz} \right\}^T \tag{1}$$

The strain-displacement relations are

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} \tag{2}$$

where $\mathbf{D}$ and $\mathbf{u}$ are a differential matrix operator and the displacement vector, defined as follows

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix}, \quad \mathbf{u} = \left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\} \tag{3}$$

In the case of one-directional FG beams, the 3D constitutive equations related to thermoelastic applications are given as

$$\boldsymbol{\sigma} = \mathbf{C}\,(x)\,\boldsymbol{\varepsilon} \tag{4}$$

where $\mathbf{C}$ is the constitutive matrix,

$$\mathbf{C}\,(x) = \begin{bmatrix} \lambda\,(x) + 2\,\mu\,(x) & \lambda\,(x) & 0 & 0 & 0 & \lambda\,(x) \\ \lambda\,(x) & \lambda\,(x) + 2\,\mu\,(x) & 0 & 0 & 0 & \lambda\,(x) \\ 0 & 0 & \mu\,(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu\,(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu\,(x) & 0 \\ \lambda\,(x) & \lambda\,(x) & 0 & 0 & 0 & \lambda\,(x) + 2\,\mu\,(x) \end{bmatrix} \tag{5}$$

where λ and μ are the Lamé coefficients. The latter can be expressed in terms of Young's modulus E and Poisson's ratio ν as follows

$$\lambda\,(x) = \frac{\nu\,(x)\;E\,(x)}{[1 + \nu\,(x)]\,[1 - 2\,\nu\,(x)]}; \qquad \mu\,(x) = G\,(x) = \frac{E\,(x)}{2\,[1 + \nu\,(x)]} \tag{6}$$

where G represents the shear modulus.

3. Effective FG material properties

The effective material properties of porous FG beams are derived by careful considerations on the micro-mechanical behaviour of the structure. Various ad-hoc models have been developed in the recent years in order to increase the accuracy of the homogenization process. In particular, if the difference of the material properties of the FG beam constituent is relatively small, as in the present investigation, it is then possible to use Voigt's rule of mixture (ROM) [49] with no loss of accuracy with respect to Mori-Tanaka (MT) homogenization scheme [50]. In this respect, a proof of what above stated, substantiated by a considerable amount of examples, can be found in Ref. [51]. The volume fraction of the ceramic constituent is given according to the FG beam structure under investigation. In particular, in the present article the following two typologies are examined:

1. *FG beam with metallic bottom skin and ceramic top skin (see Fig. 2(a)).*

   The volume fraction of the ceramic phase is defined according to the following power-law:

$$V_c\,(x) = \left(\tfrac{x}{h} + \tfrac{1}{2}\right)^p \qquad x \in [-h/2,\, h/2] \tag{7}$$

2. *FG sandwich beam with metallic top/bottom skins and ceramic core (see Fig 2(b)).*

   The volume fraction of the ceramic phase is defined according to the following power-law:

$$
\begin{aligned}
&V_c^{(1)}(x) = \left(\tfrac{x-h_0}{h_1-h_0}\right)^p && x \in [h_0,\, h_1] \\
&V_c^{(2)}(x) = 1 && x \in [h_1,\, h_2] \\
&V_c^{(3)}(x) = \left(\tfrac{h_3-x}{h_3-h_2}\right)^p && x \in [h_2,\, h_3]
\end{aligned}
\tag{8}
$$

Where, in the case of classical FG isotropic beams, h is the beam total thickness and p is the volume fraction index indicating the material variation through-the-beam-thickness direction. In the case of FG sandwich beams, it is possible to define the following parameters: $h_{fb} = h_1 - h_0$ which represents the bottom FG layer thickness; $h_{tb} = h_3 - h_2$ which indicates the top FG layer thickness and finally, $h_c = h_2 - h_1$ that can be identified with the ceramic core thickness. The volume fraction of the metal phase is give as $V_m^i(z) = 1 - V_c^i(z)$ with $i = 1, 2, 3$.

3.1. Voigt's rule of mixture and inclusion of porosities

In the case of the modified ROM accounting for porosities, as proposed in Refs. [12, 13, 52], Young's modulus $E_f(x)$ and material density $\rho_f(x)$ are computed by the following two law-of-mixtures,

1. *Type-I*, porosities uniformly distributed over the beam cross section.

$$
\begin{aligned}
E_f(x) &= (E_c - E_m)\, V_c^i(x) + E_m - \frac{\beta}{2}\,(E_c + E_m) \\
\rho_f(x) &= (\rho_c - \rho_m)\, V_c^i(x) + \rho_m - \frac{\beta}{2}\,(\rho_c + \rho_m)
\end{aligned}
\tag{9}
$$

2. *Type-II*, porosities unevenly distributed over the beam cross section and mainly concentrated in the central area of the beam.

$$
\begin{aligned}
E_f(x) &= (E_c - E_m)\, V_c^i(x) + E_m - \frac{\beta}{2}\,(E_c + E_m)\left(1 - \frac{2\,|z|}{h}\right) \\
\rho_f(x) &= (\rho_c - \rho_m)\, V_c^i(x) + \rho_m - \frac{\beta}{2}\,(\rho_c + \rho_m)\left(1 - \frac{2\,|z|}{h}\right)
\end{aligned}
\tag{10}
$$

   with $i = 1, 2, 3$ and where β ($\beta << 1$) is the porosity coefficient.

The Poisson's coefficient ν_f is considered constant. The two different models proposed to introduce the effect of the porosities, have been given in Ref. [12]. The porosities in both cases are spread over the cross-section. Depending on the manufacturing-typology used during material production, it is possible to select Type-I or Type-II. More specifically, as explained in Ref. [12], if the FG structure is manufactured, for instance, by using the principle of the multi-step sequential infiltration technique often used in the production of FGM samples, the porosities mostly occur at the middle zone. At this zone, it is difficult to infiltrate the materials completely, while at the top and bottom zones, the process of material infiltration can be performed easier and leaves less porosity. In this case the porosity Type-II could be successfully used to model the FG structure in production.

4. Exponential, Polynomial and Trigonometric quasi-3D beam theories

During the last few decades many efforts have been devoted to the development of refined theories able to describe accurately the kinematic behaviour of beam structures. Some of the main and most significant contributions are following described. The simplest beam theory based on axiomatic assumptions, which may be traced back to Leonardo da Vinci [53], was proposed by Euler [54] and it is usually referred to as da Vinci-Euler-Bernoulli beam theory (DEBBT). The inclusion of transverse shear strains, in the above-mentioned beam model, leads to Timoshenko beam theory (TBT) [55]. Further improvements of these theories came with the introduction of quite questionable warping functions able to partially capture distortion and warping of the beam cross-section. A beam model accounting for the aforementioned features is provided below

$$\begin{aligned} u_x &= u_{x_1} \\ u_y &= u_{y_1} \\ u_z &= u_{z_1} + g\left(x\right)\,\gamma^0_{xz} - y\frac{\partial u_{z_1}}{\partial y} + g\left(y\right)\,\gamma^0_{yz} - x\frac{\partial u_{y_1}}{\partial x} \end{aligned} \tag{11}$$

where $g\left(x\right)$ and $g\left(y\right)$ are the warping functions and γ^0_{xz} and γ^0_{yz} are the shear strains evaluated on the beam reference line. It should be borne in mind that the warping functions are problem-dependent, which represents a significant drawback of this approach. An attempt to generalize the development of higher-order beam theories (HOBTs), avoiding the introduction of cumbersome and controversial warping functions, was provided by Matsunaga [56], who proposed the following displacement field

$$\begin{aligned} u_x &= \sum_{m=0}^{2\,M-1} u_{x_m}\, x^m \\ u_y &= 0 \\ u_z &= \sum_{m=0}^{2\,M-2} u_{z_m}\, x^m \end{aligned} \tag{12}$$

Despite the high accuracy level reached by these HOBTs based on power series expansion, the latter are, however, still featured by some fundamental flaws which lie in the incompleteness of the adopted series expansion given in Eq. (12).

This sort of inconsistency can be completely removed if a full/complete series expansion is taken into account. In Refs. [57, 58] have been proposed various examples of complete and generalized power series expansion to approximate the displacement field with a high level of accuracy. The present work aims to generalise the above mentioned polynomial theories, which have, however, also been developed by the author in Ref. [10], by adopting trigonometric and exponential expansions. In this respect, in all of the proposed beam theories each displacement variable in the displacement field is expanded at any desired order independently from the others and regarding to the results accuracy and the computational cost. The development of the present general HOBTs allows a more accurate and refined description of the beam kinematics. This approach represents a fundamental requisite in order to provide a realistic representation of complex problem in structural mechanics with applications in various engineering sectors and above all those which involve static and dynamic response of beams subjected to multifield loadings. It is, indeed, a matter of fact, that the complications which arise when dealing with such problems make meaningless the use of classical beam theories such as DEBBT and TBT.

More specifically, the employment of HOBTs yields a highly accurate modelling of beam structures that are featured by both in-plane and out-of-plane (cross-sectional warping) deformations, significant distortion, torsion and eventually unpredictable coupling of the spatial directions. Thereby, according to what mentioned above, it is convenient to represent the displacement field related to the beam kinematics in its most general form as follows

$$
\begin{aligned}
u_x(x,y,z,t) &= \sum_{\tau_{u_x}=0}^{N_{u_x}} F_{\tau_{u_x}}(x,y)\; u_{x\tau_{u_x}}(z,t) \\
u_y(x,y,z,t) &= \sum_{\tau_{u_y}=0}^{N_{u_y}} F_{\tau_{u_y}}(x,y)\; u_{y\tau_{u_y}}(z,t) \\
u_z(x,y,z,t) &= \sum_{\tau_{u_z}=0}^{N_{u_z}} F_{\tau_{u_z}}(x,y)\; u_{z\tau_{u_z}}(z,t)
\end{aligned}
\tag{13}
$$

where $F_{\tau_{u_x}}$, $F_{\tau_{u_y}}$ and $F_{\tau_{u_z}}$ are the cross-section functions; $u_{x\tau_{u_x}}$, $u_{y\tau_{u_y}}$ and $u_{z\tau_{u_z}}$ are the displacement vector components and N_{u_x}, N_{u_y} and N_{u_z} are the expansion orders.

4.1. Quasi-3D beam models via polynomial expansion

When the cross-section functions are chosen to be Taylor's series expansion then Eq. (13) can be rewritten as

$$
\begin{aligned}
u_x(x,y,z,t) &= u_{x_0}(z,t) + \sum_{n_{ux}=1}^{N_{ux}} \left[\sum_{n^*_{ux}=1}^{n_{ux}} x^{\left(n_{ux}-n^*_{ux}\right)} y^{n^*_{ux}} u_{x_{\tilde{N}_{ux}}}(z,t) \right] \\
u_y(x,y,z,t) &= u_{y_0}(z,t) + \sum_{n_{uy}=1}^{N_{uy}} \left[\sum_{n^*_{uy}=1}^{n_{uy}} x^{\left(n_{uy}-n^*_{uy}\right)} y^{n^*_{uy}} u_{y_{\tilde{N}_{uy}}}(z,t) \right] \\
u_z(x,y,z,t) &= u_{z_0}(z,t) + \sum_{n_{uz}=1}^{N_{uz}} \left[\sum_{n^*_{uz}=1}^{n_{uz}} x^{\left(n_{uz}-n^*_{uz}\right)} y^{n^*_{uz}} u_{z_{\tilde{N}_{uz}}}(z,t) \right]
\end{aligned}
\tag{14}
$$

where $\tilde{N}_u = \frac{[n_u(n_u+1)+2(n^*_u+1)]}{2}$. The total number of degree of freedoms (DOFs) involved in a generic analysis when using the present models is

$$
\text{DOFs}^{\text{TE}} = \left[\frac{(N_{u_x}+1)(N_{u_x}+2)}{2} + \frac{\left(N_{u_y}+1\right)\left(N_{u_y}+2\right)}{2} + \frac{(N_{u_z}+1)(N_{u_z}+2)}{2} \right] \tag{15}
$$

An example of a possible displacement field according to the present approach and by using expansion orders $N_{u_x}=2$, $N_{u_y}=3$ and $N_{u_z}=1$ is given in Eq. (16) as follows

$$
\begin{aligned}
u_x &= u_{x_0} + x u_{x_1} + y u_{x_2} + x^2 u_{x_3} + xy u_{x_4} + y^2 u_{x_5} \\
u_y &= u_{y_0} + x u_{y_1} + y u_{y_2} + x^2 u_{y_3} + xy u_{y_4} + y^2 u_{y_5} + x^3 u_{y_6} + x^2 y u_{y_7} + x y^2 u_{y_8} + y^3 u_{y_9} \\
u_z &= u_{z_0} + x u_{z_1} + y u_{z_2}
\end{aligned}
\tag{16}
$$

As shown in the results section (Sec. 6), these functions own a good computational stability, allowing generally to reach a high level of accuracy in various structural applications featured by 3D effects.

4.2. Quasi-3D beam models via exponential expansion

Amongst various sets of possible functions that could be used to approximate the beam cross-section kinematics the exponential functions have been chosen to be the second alternative. According to this choice the expansion takes the following form

$$
\begin{aligned}
u_x(x,y,z,t) &= u_{x_0}(z,t) + \sum_{m=1}^{N_{ux}} \left[e^{\left(\frac{m\,x}{h}\right)} u_{x_{2\,m}}(z,t) + e^{\left(\frac{m\,y}{b}\right)} u_{x_{2\,m+1}}(z,t) \right] \\
u_y(x,y,z,t) &= u_{y_0}(z,t) + \sum_{m=1}^{N_{uy}} \left[e^{\left(\frac{m\,x}{h}\right)} u_{y_{2\,m}}(z,t) + e^{\left(\frac{m\,y}{b}\right)} u_{y_{2\,m+1}}(z,t) \right] \\
u_z(x,y,z,t) &= u_{z_0}(z,t) + \sum_{m=1}^{N_{uz}} \left[e^{\left(\frac{m\,x}{h}\right)} u_{z_{2\,m}}(z,t) + e^{\left(\frac{m\,y}{b}\right)} u_{z_{2\,m+1}}(z,t) \right]
\end{aligned}
\tag{17}
$$

The total number of DOFs involved when using the present models is

$$\mathrm{DOFs}^{\mathrm{EX}} = \left[(2\,N_{u_x}+1) + \left(2\,N_{u_y}+1\right) + (2\,N_{u_z}+1)\right] \tag{18}$$

An example of a possible displacement field according to the present approach and by using expansion orders $N_{u_x}=2$, $N_{u_y}=3$ and $N_{u_z}=1$ is given in Eq. (17) as follows

$$\begin{aligned}
u_x &= u_{x0} + e^{\left(\frac{x}{h}\right)} u_{x1} + e^{\left(\frac{y}{b}\right)} u_{x2} + e^{\left(\frac{2\,x}{h}\right)} u_{x3} + e^{\left(\frac{2\,y}{b}\right)} u_{x4} \\
u_y &= u_{y0} + e^{\left(\frac{x}{h}\right)} u_{y1} + e^{\left(\frac{y}{b}\right)} u_{y2} + e^{\left(\frac{2\,x}{h}\right)} u_{y3} + e^{\left(\frac{2\,y}{b}\right)} u_{y4} + e^{\left(\frac{3\,x}{h}\right)} u_{y5} + e^{\left(\frac{3\,y}{b}\right)} u_{y5} \\
u_z &= u_{z0} + e^{\left(\frac{x}{h}\right)} u_{z1} + e^{\left(\frac{y}{b}\right)} u_{z2}
\end{aligned} \tag{19}$$

4.3. Quasi-3D beam models via trigonometric expansion

An other possible alternative is the choice of trigonometric functions. The displacement field can be expanded accordingly as follows

$$\begin{aligned}
u_x(x,y,z,t) &= u_{x0}(z,t) + \sum_{m=1}^{N_{u_x}} \left[\sin\left(\frac{m\,x}{h}\right) u_{x4\,m-2}(z,t) + \sin\left(\frac{m\,y}{h}\right) u_{x4\,m-1}(z,t) + \right. \\
&\qquad \left. \cos\left(\frac{m\,x}{h}\right) u_{x4\,m}(z,t) + \cos\left(\frac{m\,y}{h}\right) u_{x4\,m+1}(z,t)\right] \\
u_y(x,y,z,t) &= u_{y0}(z,t) + \sum_{m=1}^{N_{u_y}} \left[\sin\left(\frac{m\,x}{h}\right) u_{y4\,m-2}(z,t) + \sin\left(\frac{m\,y}{h}\right) u_{y4\,m-1}(z,t) + \right. \\
&\qquad \left. \cos\left(\frac{m\,x}{h}\right) u_{y4\,m}(z,t) + \cos\left(\frac{m\,y}{h}\right) u_{y4\,m+1}(z,t)\right] \\
u_z(x,y,z,t) &= u_{z0}(z,t) + \sum_{m=1}^{N_{u_z}} \left[\sin\left(\frac{m\,x}{h}\right) u_{z4\,m-2}(z,t) + \sin\left(\frac{m\,y}{h}\right) u_{z4\,m-1}(z,t) + \right. \\
&\qquad \left. \cos\left(\frac{m\,x}{h}\right) u_{z4\,m}(z,t) + \cos\left(\frac{m\,y}{h}\right) u_{z4\,m+1}(z,t)\right]
\end{aligned} \tag{20}$$

The total number of DOFs involved in the expansion

$$\mathrm{DOFs}^{\mathrm{TR}} = \left[(4\,N_{u_x}+1) + \left(4\,N_{u_y}+1\right) + (4\,N_{u_z}+1)\right] \tag{21}$$

As for the previous cases, according to the here developed beam model and by selecting the expansion orders as $N_{u_x}=2$, $N_{u_y}=3$ and $N_{u_z}=1$ the displacement field takes the following form

$$\begin{aligned}
u_x = u_{x0} &+ \sin\left(\frac{x}{h}\right) u_{x1} + \sin\left(\frac{y}{b}\right) u_{x2} + \cos\left(\frac{x}{h}\right) u_{x3} + \cos\left(\frac{y}{b}\right) u_{x4} + \\
&\sin\left(\frac{2\,x}{h}\right) u_{x5} + \sin\left(\frac{2\,y}{b}\right) u_{x6} + \cos\left(\frac{2\,x}{h}\right) u_{x7} + \cos\left(\frac{2\,y}{b}\right) u_{x8} \\
u_y = u_{y0} &+ \sin\left(\frac{x}{h}\right) u_{y1} + \sin\left(\frac{y}{b}\right) u_{y2} + \cos\left(\frac{x}{h}\right) u_{y3} + \cos\left(\frac{y}{b}\right) u_{y4} + \\
&\sin\left(\frac{2\,x}{h}\right) u_{y5} + \sin\left(\frac{2\,y}{b}\right) u_{y6} + \cos\left(\frac{2\,x}{h}\right) u_{y7} + \cos\left(\frac{2\,y}{b}\right) u_{y8} \\
&\sin\left(\frac{3\,x}{h}\right) u_{y9} + \sin\left(\frac{3\,y}{b}\right) u_{y10} + \cos\left(\frac{3\,x}{h}\right) u_{y11} + \cos\left(\frac{3\,y}{b}\right) u_{y12}
\end{aligned} \tag{22}$$

$$u_z = u_{z_0}+ \sin\left(\frac{x}{h}\right)u_{z_1} + \sin\left(\frac{y}{b}\right)u_{z_2} + \cos\left(\frac{x}{h}\right)u_{z_3} + \cos\left(\frac{y}{b}\right)u_{z_4}$$

5. Theoretical Formulation

In the derivation of what follows Hamilton's principle is employed along with the Hierarchical Ritz Formulation (HRF). In its classical from Hamilton's principle can be written as

$$\delta \int_{t_1}^{t2} \mathcal{L}\,\mathrm{d}t = 0 \tag{23}$$

where t_1 and t_2 are the initial and the generic instant of time; $\mathcal{L}$ is the Lagrangian which assumes the following form

$$\mathcal{L} = T - \Pi \tag{24}$$

where

$$\Pi = \Phi_e + \Phi_f^{wp} + \Phi_f^{nl} \tag{25}$$

T is the kinetic energy and Π is the total potential energy of the system; Φ_e, Φ_f^{wp} and Φ_f^{nl} are the potential strain energy, the potential energy related to the Winkler-Pasternak elastic foundation and the potential energy due to the the initial stresses, respectively. Their explicit expression is given as follows

$$\begin{aligned}
T &= \frac{1}{2}\int_V \rho\left[(\dot{u}_x)^2 + (\dot{u}_y)^2 + (\dot{u}_z)^2\right]\mathrm{d}V \\
\Phi_e &= \frac{1}{2}\int_V \left(\sigma_{xx}\,\varepsilon_{xx} + \sigma_{yy}\,\varepsilon_{yy} + \tau_{xy}\,\gamma_{xy} + \tau_{xz}\,\gamma_{xz} + \tau_{yz}\,\gamma_{yz} + \sigma_{zz}\,\varepsilon_{zz}\right)\mathrm{d}V \\
\Phi_f^{wp} &= \frac{1}{2}\int_V \Big[\kappa_{xx}^{w-}\,u_x|_{x=-\frac{h}{2}} + \kappa_{xx}^{p-}\,u_{x,z}|_{x=-\frac{h}{2}}\kappa_{xx}^{w+}\,u_x|_{x=\frac{h}{2}} + \kappa_{xx}^{p+}\,u_{x,z}|_{x=\frac{h}{2}} + \\
&\qquad \kappa_{yy}^{w-}\,u_y|_{y=-\frac{b}{2}} + \kappa_{yy}^{p-}\,u_{y,z}|_{y=-\frac{b}{2}}\kappa_{yy}^{w+}\,u_y|_{y=\frac{b}{2}} + \kappa_{yy}^{p+}\,u_{y,z}|_{y=\frac{b}{2}} + \\
&\qquad \kappa_{zz}^{w0}\,u_z|_{z=0} + \kappa_{zz}^{p0}\left(u_{z,x}|_{z=0} + u_{z,y}|_{z=0}\right)\kappa_{zz}^{wl}\,u_z|_{z=l} + \kappa_{zz}^{pl}\left(u_{z,x}|_{z=l} + u_{z,y}|_{z=l}\right)\Big]\mathrm{d}\Omega \\
\Phi_f^{nl} &= \frac{1}{2}\int_V \sigma_{zz}^{(0)}\left[\left(u_{x,z}\right)^2 + \left(u_{y,z}\right)^2 + \left(u_{z,z}\right)^2\right]\mathrm{d}V
\end{aligned} \tag{26}$$

where ρ is the material density; κ_{ii}^{w-}, κ_{ii}^{w+}, κ_{ii}^{p-} and κ_{ii}^{p+} with $ii = xx, yy, zz$ represent the stiffness constant values of the Winkler-springs and Pasternak layers distributed all around the beam boundary and $\sigma_{zz}^{(0)}$ is the mechanical pre-stress. Introducing Eq.(26) in Eq.(23) and using both the geometric relationships and the constitutive equations given in Eqs. (2)

and (4), respectively, Hamilton's principle can be rewritten more conveniently in terms of displacements as follows

$$
\begin{aligned}
\int_{t_1}^{t^2} \Bigg\{ & \int_V \Big[\rho\left(\delta \dot{u}_x\, \dot{u}_x + \delta \dot{u}_y\, \dot{u}_y + \delta \dot{u}_z\, \dot{u}_z\right)\Big] \mathrm{d}V + \\
& \int_V \Big[\left[\lambda(x) + 2\,\mu(x)\right] \delta u_{x,x}\, u_{x,x} + \mu(x)\, \delta u_{x,y}\, u_{x,y} + \mu(x)\, \delta u_{x,z}\, u_{x,z} + \\
& \quad \lambda(x)\, \delta u_{x,x}\, u_{y,y} + \mu(x)\, \delta u_{x,y}\, u_{y,x} + \\
& \quad \lambda(x)\, \delta u_{x,x}\, u_{z,z} + \mu(x)\, \delta u_{x,z}\, u_{z,x} + \\
& \quad \mu(x)\, \delta u_{y,x}\, u_{x,y} + \lambda(x)\, \delta u_{y,y}\, u_{x,x} + \\
& \quad \left[\lambda(x) + 2\,\mu(x)\right] \delta u_{y,y}\, u_{y,y} + \mu(x)\, \delta u_{y,x}\, u_{y,x} + \mu(x)\, \delta u_{y,z}\, u_{y,z} + \\
& \quad \lambda(x)\, \delta u_{y,y}\, u_{z,z} + \mu(x)\, \delta u_{y,z}\, u_{z,y} + \\
& \quad \mu(x)\, \delta u_{z,x}\, u_{x,z} + \lambda(x)\, \delta u_{z,z}\, u_{x,x} + \\
& \quad \mu(x)\, \delta u_{z,y}\, u_{y,z} + \lambda(x)\, \delta u_{z,z}\, u_{y,y} + \\
& \quad \left[\lambda(x) + 2\,\mu(x)\right] \delta u_{z,z}\, u_{z,z} + \mu(x)\, \delta u_{z,x}\, u_{z,x} + \mu(x)\, \delta u_{z,y}\, u_{z,y} \Big] \mathrm{d}V + \\
& \int_V \Big[\kappa_{xx}^{w-} \left(\delta u_x\, u_x\right)_{x=-\frac{h}{2}} - \kappa_{xx}^{p-} \left(\delta u_{x,z}\, u_{x,z}\right)_{x=-\frac{h}{2}} + \kappa_{xx}^{w+} \left(\delta u_x\, u_x\right)_{x=\frac{h}{2}} - \kappa_{xx}^{p+} \left(\delta u_{x,z}\, u_{x,z}\right)_{x=\frac{h}{2}} + \\
& \quad \kappa_{yy}^{w-} \left(\delta u_y\, u_y\right)_{y=-\frac{b}{2}} - \kappa_{yy}^{p-} \left(\delta u_{y,z}\, u_{y,z}\right)_{y=-\frac{b}{2}} + \kappa_{yy}^{w+} \left(\delta u_y\, u_y\right)_{y=\frac{b}{2}} - \kappa_{yy}^{p+} \left(\delta u_{y,z}\, u_{y,z}\right)_{y=\frac{b}{2}} + \\
& \quad \kappa_{zz}^{w0} \left(\delta u_z\, u_z\right)_{z=0} - \kappa_{zz}^{p0} \left[\left(\delta u_{z,x}\, u_{z,x}\right) + \left(\delta u_{z,y}\, u_{z,y}\right)\right]_{z=0} + \\
& \quad \kappa_{zz}^{wl} \left(\delta u_z\, u_z\right)_{z=l} - \kappa_{zz}^{pl} \left[\left(\delta u_{z,x}\, u_{z,x}\right) + \left(\delta u_{z,y}\, u_{z,y}\right)\right]_{z=l} \Big] \mathrm{d}V \\
& \int_V \Big[\sigma_{zz}^{(0)} \left(\delta u_{x,z}\, u_{x,z} + \delta u_{y,z}\, u_{y,z} + \delta u_{z,z}\, u_{z,z}\right) \Big] \mathrm{d}V \Bigg\} \mathrm{d}t = 0
\end{aligned}
\tag{27}
$$

5.1. *The Hierarchical Ritz Formulation*

In the Ritz method the displacement amplitude vector components $u_{x_{\tau_{u_x}}}$, $u_{y_{\tau_{u_y}}}$ and $u_{z_{\tau_{u_z}}}$, appearing in Eq. (13), are expressed in series expansion as follows

$$
\begin{aligned}
u_{x_{\tau_{u_x}}}(z,t) &= \sum_{i}^{\mathcal{N}} U_{x_{\tau_{u_x}} i}\, \psi_{x_i}(z)\; e^{\imath\, \omega_{ij}\, t} \\
u_{y_{\tau_{u_y}}}(z,t) &= \sum_{i}^{\mathcal{N}} U_{y_{\tau_{u_y}} i}\, \psi_{y_i}(z)\; e^{\imath\, \omega_{ij}\, t} \\
u_{z_{\tau_{u_z}}}(z,t) &= \sum_{i}^{\mathcal{N}} U_{z_{\tau_{u_z}} i}\, \psi_{z_i}(z)\; e^{\imath\, \omega_{ij}\, t}
\end{aligned}
\tag{28}
$$

where $\imath = \sqrt{-1}$, t is the time and ω_{ij} the circular frequency; $\mathcal{N}$ indicates the order of expansion in the Ritz approximation; $U_{x_{\tau_{u_x}} i}$, $U_{y_{\tau_{u_y}} i}$ and $U_{z_{\tau_{u_z}} i}$ are the unknown coefficients and ψ_{x_i}, ψ_{y_i} and ψ_{z_i} are the Ritz functions appropriately selected with respect to the features of the problem under investigation. Convergence to the exact solution is guaranteed if the Ritz functions are admissible functions in the used variational principle [10, 39, 59, 60]. Finally, the displacement field, expressed in terms of general cross-section functions and Ritz functions assumes the following form

$$
\begin{aligned}
u_x(x,y,z,t) &= \sum_{i=1}^{\mathcal{N}} \sum_{\tau_{u_x}=0}^{N_{u_x}} U_{x_{\tau_{u_z}}i}\, F_{\tau_{u_x}}(x,y)\; \psi_{x_i}(z)\; e^{\imath\,\omega_{ij}\,t} \\
u_y(x,y,z,t) &= \sum_{i=1}^{\mathcal{N}} \sum_{\tau_{u_y}=0}^{N_{u_y}} U_{y_{\tau_{u_z}}i}\, F_{\tau_{u_y}}(x,y)\; \psi_{y_i}(z)\; e^{\imath\,\omega_{ij}\,t} \\
u_z(x,y,z,t) &= \sum_{i=1}^{\mathcal{N}} \sum_{\tau_{u_z}=0}^{N_{u_z}} U_{z_{\tau_{u_z}}i}\, F_{\tau_{u_z}}(x,y)\; \psi_{z_i}(z)\; e^{\imath\,\omega_{ij}\,t}
\end{aligned}
\tag{29}
$$

5.2. Admissible functions

Various admissible functions have been used in the literature for solving a wide range of problems related to beam, plate and shell structural analysis. In particular, simple polynomial functions [61, 62], transcendental functions [63, 64] and hybrid admissible functions [65, 66] (generated by the combination of both polynomial and trigonometric functions) have been successfully employed. Some more information and interesting insights, in this respect, can be found in Ref. [67]. In the present study a set of characteristic orthogonal polynomials has been employed. The latter are generated by using a Gram-Schmidt process [61, 68, 69]. The first member of the orthogonal polynomial set $\psi_{x_1}(z)$ is chosen as the simplest polynomial of the least order that satisfies the geometrical boundary conditions of the beam. The other members of the orthogonal set in the interval $l_0 \leq z \leq l$ are generated by using the following recursive procedure

$$
\begin{aligned}
&\psi_{x_2}(z) = (z - B_2)\,\psi_{x_1}(z)\,, \\
&\psi_{x_3}(z) = (z - B_3)\,\psi_{x_2}(z) - C_3\,\psi_{x_1}(z)\,, \\
&\vdots \\
&\psi_{x_i}(z) = (z - B_i)\,\psi_{x_{i-1}}(z) - C_i\,\psi_{x_{i-2}}(z) \\
&\vdots \\
&\psi_{x_{\mathcal{N}}}(z) = (z - B_{\mathcal{N}})\,\psi_{x_{\mathcal{N}-1}}(z) - C_{\mathcal{N}}\,\psi_{x_{\mathcal{N}-2}}(z)
\end{aligned}
\tag{30}
$$

where

$$B_i = \frac{\left[\int_{l_0}^{l} w\left(z\right)\, z\, \psi_{x_{i-1}}^2\left(z\right)\,\mathrm{d}z\right]}{\left[\int_{l_0}^{l} w\left(z\right)\, \psi_{x_{i-1}}^2\left(z\right)\,\mathrm{d}z\right]}; \quad C_i = \frac{\left[\int_{l_0}^{l} w\left(z\right)\, z\, \psi_{x_{i-1}}\left(z\right)\, \psi_{x_{i-2}}\left(z\right)\,\mathrm{d}z\right]}{\left[\int_{l_0}^{l} w\left(z\right)\, \psi_{x_{i-2}}^2\left(z\right)\,\mathrm{d}z\right]} \tag{31}$$

The polynomials ψ_{x_i} satisfy the orthogonality condition

$$\int_{l_0}^{l} w\left(z\right)\, \psi_{x_i}\left(z\right)\, \psi_{x_j}\left(z\right)\mathrm{d}z = \begin{cases} 0 & \text{if } i \neq j \\ l_{ij} & \text{if } i = j \end{cases} \tag{32}$$

where $w\left(z\right)$ is a weight function. In the particular case of uniform beams $w\left(z\right) = 1$. The same process has been applied for the orthogonalization of the Ritz functions $\psi_{y_i}\left(z\right)$ and $\psi_{z_i}\left(z\right)$ with $i = 1, 2, 3, \cdots, \mathcal{N}$.

5.3. Weak-form of the governing equations

Once Eq.(29) is substitute in Eq.(27), the weak-form of the governing equations is derived, and can be generally written as follows

$$\delta \begin{Bmatrix} U_{x_{\tau_{u_x}\,i}} \\ U_{y_{\tau_{u_y}\,i}} \\ U_{z_{\tau_{u_z}\,i}} \end{Bmatrix} : \left(\begin{bmatrix} K_{\tau_{u_x}\,s_{u_x}\,i\,j} & K_{\tau_{u_x}\,s_{u_y}\,i\,j} & K_{\tau_{u_x}\,s_{u_z}\,i\,j} \\ K_{\tau_{u_y}\,s_{u_x}\,i\,j} & K_{\tau_{u_y}\,s_{u_y}\,i\,j} & K_{\tau_{u_y}\,s_{u_z}\,i\,j} \\ K_{\tau_{u_z}\,s_{u_x}\,i\,j} & K_{\tau_{u_z}\,s_{u_y}\,i\,j} & K_{\tau_{u_z}\,s_{u_z}\,i\,j} \end{bmatrix} + \right.$$

$$\begin{bmatrix} K^{(wp)}_{\tau_{u_x}\,s_{u_x}\,i\,j} & 0_{\tau_{u_x}\,s_{u_y}\,i\,j} & 0_{\tau_{u_x}\,s_{u_z}\,i\,j} \\ 0_{\tau_{u_y}\,s_{u_x}\,i\,j} & \delta_{wp_y}\, K^{(wp)}_{\tau_{u_y}\,s_{u_y}\,i\,j} & 0_{\tau_{u_y}\,s_{u_z}\,i\,j} \\ 0_{\tau_{u_z}\,s_{u_x}\,i\,j} & 0_{\tau_{u_z}\,s_{u_y}\,i\,j} & \delta_{wp_z}\, K^{(wp)}_{\tau_{u_z}\,s_{u_z}\,i\,j} \end{bmatrix} +$$

$$\lambda_{ij} \begin{bmatrix} K^{(\sigma)}_{\tau_{u_x}\,s_{u_x}\,i\,j} & 0_{\tau_{u_x}\,s_{u_y}\,i\,j} & 0_{\tau_{u_x}\,s_{u_z}\,i\,j} \\ 0_{\tau_{u_y}\,s_{u_x}\,i\,j} & \delta_{vK}\, K^{(\sigma)}_{\tau_{u_y}\,s_{u_y}\,i\,j} & 0_{\tau_{u_y}\,s_{u_z}\,i\,j} \\ 0_{\tau_{u_z}\,s_{u_x}\,i\,j} & 0_{\tau_{u_z}\,s_{u_y}\,i\,j} & \delta_{vK}\, K^{(\sigma)}_{\tau_{u_z}\,s_{u_z}\,i\,j} \end{bmatrix} -$$

$$\left. \omega_{ij}^2 \begin{bmatrix} M_{\tau_{u_x}\,s_{u_x}\,i\,j} & 0_{\tau_{u_x}\,s_{u_y}\,i\,j} & 0_{\tau_{u_x}\,s_{u_z}\,i\,j} \\ 0_{\tau_{u_y}\,s_{u_x}\,i\,j} & M_{\tau_{u_y}\,s_{u_y}\,i\,j} & 0_{\tau_{u_y}\,s_{u_z}\,i\,j} \\ 0_{\tau_{u_z}\,s_{u_x}\,i\,j} & 0_{\tau_{u_z}\,s_{u_y}\,i\,j} & M_{\tau_{u_z}\,s_{u_z}\,i\,j} \end{bmatrix} \right) \begin{Bmatrix} U_{x_{s_{u_x}\,j}} \\ U_{y_{s_{u_y}\,j}} \\ U_{z_{s_{u_z}\,j}} \end{Bmatrix} = \begin{Bmatrix} 0_{s_{u_x}\,j} \\ 0_{s_{u_y}\,j} \\ 0_{s_{u_z}\,j} \end{Bmatrix} \tag{33}$$

The tracers δ_{wp} and δ_{vK} are introduced in order to retain and/or discard the contribution of secondary fundamental nuclei appearing in the leading diagonal of both Winkler-Pasternak and initial stress primary fundamental nuclei. More specifically, The tracer δ_{wp} accounts for the presence of springs all around the beam or only the beam thickness direction, while the

δ_{vK} accounts for the full nonlinear terms or the von kármán approximation. The complete expression of each single term of the matrices involved in Eq.(33) is provided in Appendix A. Equation (33) can be written in a more compact form as

$$\delta \{U_{\tau i}\}: \quad \left([K_{\tau sij}] + \left[K^{(wp)}_{\tau sij}\right] + \lambda_{ij} \left[K^{(\sigma)}_{\tau sij}\right] - \omega^2_{ij} \, [M_{\tau sij}]\right) \{U_{sj}\} = \{0_{sj}\} \tag{34}$$

From Eq. (34) a free vibration and stability analysis can be performed in order to investigate the modal and buckling characteristics of the structure under examination. More specifically, the following two eigenvalue problems can be considered

$$\left|\left([K_{\tau sij}] + \left[K^{(wp)}_{\tau sij}\right]\right) - \omega^2_{ij} \, [M_{\tau sij}]\right| = 0, \quad \left|\left([K_{\tau sij}] + \left[K^{(wp)}_{\tau sij}\right]\right) + \lambda_{ij} \left[K^{(\sigma)}_{\tau sij}\right]\right| = 0 \tag{35}$$

6. Numerical results and discussion

In this section the developed unconventional quasi-3D beam theories are validated and assessed by virtue of results available in literature. A comprehensive analysis of both FG isotropic and FG sandwich structures is carried out. The effect of two different type of porosity on the stability and free vibration characteristics of the analysed structures is taken into account. The FG constituents are aluminium (Al) as metal and alumina (Al_2O_3) as ceramic. The material properties of the Aluminium are $E_m = 70\,GPa$, $\nu_m = 0.30$, and $\rho_m = 2702\,Kg/m^3$, and those of the alumina are $E_c = 380\,GPa$, $\nu_c = 0.30$, and $\rho_c = 3960\,Kg/m^3$. The results are provided in terms of dimension circular frequency parameters, dimensionless buckling loads and dimensionless elastic foundation coefficients, which are defined, respectively, as follows

$$\hat{\omega} = \omega \left(\frac{l^2}{h}\right) \sqrt{\frac{\rho_m}{E_m}}; \qquad \hat{P}_{cr} = \frac{P_{cr}\, l^2}{E_m\, h^3}; \qquad \mathcal{K}^{w-} = \frac{\kappa^{w-}_{xx}\, l^2}{E_m\, h}; \qquad \mathcal{K}^{p-} = \frac{\kappa^{p-}_{xx}}{E_m\, h}; \tag{36}$$

Moreover, the acronyms $TE_{N_{u_x} N_{u_y} N_{u_z}}$, $EX_{N_{u_x} N_{u_y} N_{u_z}}$ and $TR_{N_{u_x} N_{u_y} N_{u_z}}$related to Taylor's series expansion, exponential expansion and trigonometric expansion, respectively, are used to identify the various beam theories used in the present investigation. These functions are generally used to describe the displacement field over the beam cross-section. The three independent expansion orders used to generate a generic beam model, are given as N_{u_x}, N_{u_y} and N_{u_z}.

6.1. Convergence analysis

A thorough convergence analysis of 86 unconventional and quasi-3D beam theories is carried out. In all of the addressed cases, convergence in terms of Ritz functions is reached for $i = j = 18$. The first set of results, in terms of fundamental frequency, shown in Table 1 is obtained by using classical Taylor's polynomial (TE) to describe the beam cross section kinematic. More specifically, 36 beam theories are assessed against results published in literature showing an excellent agreement. In Table 2 the assessment has been carried out for 30 exponential beam theories and in Table 3 for 20 trigonometric beam theories. Both exponential (EX) and trigonometric (TR) beam theories turned out to have the same level of accuracy, which is however, much lower than that reached by using TE based beam theories. The proposed set of beam theories has been deliberately restricted to a six order expansion in all of the displacement components for both TE and EX beam theories while the forth order, for cross-sectional displacements, and the sixth order for the axial displacement, have been selected for the TR beam theories. The reason which lies behind this choice is directly related to the computational cost. However, as it can be seen from the convergence analysis proposed in Tables 1, 2 and 3 the maximum level of accuracy, between the proposed beam theories, is already obtained with the theories TE_{445}, EX_{445} and TR_{225}, which means that for the proposed analysis any further refinement of the displacement components does not generate any improvement in the results accuracy. This statement can be proved by evaluating, for example, the difference between the theory EX_{666} (702 DOFs) and the theory EX_{445} (522 DOFs). The former leads to a dimensionless fundamental circular frequency parameter equal to $\hat{\omega}_1 = 1.527634$, while the latter gives $\hat{\omega}_1 = 1.528060$ the percentage difference is 0.02 %. A similar consideration can be drawn for the TE and TR theories.

6.2. Free vibration and buckling analysis of porous FG beams resting on two parameters elastic foundations

The free vibration behaviour of Porous FG isotropic beam is investigated in Table 4. Various boundary conditions such as CC (Clamped-clamped), CF (Clamped-Free) and FF (Free-Free) have been taken into account in the analysis. The effect of the volume fraction index on the dimensionless circular frequency parameters is also evaluated. Two different type of porosities spread over the cross-section, and generated by the material production procedures, as already explained in Sec. 3, are considered. The analysis for several slenderness ratio (l/h) is carried out. More specifically, as expected the dimensionless fundamental frequency increases when increasing the length-to-thickness ratio and decreases when increasing the volume fraction index. For low values of the volume fraction index $p = 0.2$, the porosity type-I slightly increases the fundamental frequency a further increment is obtained by considering the porosity type-II, for all of the considered boundary conditions. For higher values of the volume fraction index $p = 1.0$ and $p = 5.0$, the porosity type-I generates a significant reduction of the fundamental frequency. Instead, the FG beams featured by the porosity type-II, are characterized by a different behaviour. In particular, the fundamental frequency increases even for a volume fraction index of $p = 1.0$ for CC, CF and FF boundary conditions, but decreases, for all of the boundary conditions, for $p = 5.0$, basically, when the FG beam is mainly made up of the metal constituent. The effect of the Winkler-Pasternak foundations is taken into account in Table 5. According to the dimensionless two-parameters elastic foundations (see Eqs. (36)), as expected, the fundamental frequency increases by considering the Winkler foundation and a further slight increase is obtained by adding the Pasternak layer. The effect of the Pasternak layer is more prominent in the range of short beam where the effect of the shear deformation is significant. It should also be borne in mind that the effect of the Pasternak foundation is highly affected by the value of the stiffness coefficient used for the Winkler springs. In Table 6 the dimensionless critical buckling load for isotropic porous FG beam structures is evaluated. As can be seen from the results shown in the same table, the latter increases when increasing the length to thickness ratio and decreases when increasing the volume fraction index. For lower value of the latter, $p = 0.5$, the effect of the porosity slightly decreases the dimensionless critical buckling load. For higher values, $p = 1.0$ and $p = 5.0$, the porosity dramatically decreases the buckling load of the structures under investigation. In Table 7 the effect of the Winkler-Pasternack elastic foundation is considered. Both foundation act in such a way to increase the critical buckling loads of the structures. As for the case of the dimensionless frequency parameter even for buckling analysis the introduction of the Pasternak foundation as a more significant effect at lower value of the slenderness ratio.

6.3. Free vibration and Buckling analysis of porous FG sandwich beams resting on two parameters elastic foundations

In the present section the free vibration and buckling analysis of porous FG sandwich structures have been carried out. More specifically, in Table 8 the first 6 dimensionless circular frequency parameters of a symmetric FG sandwich beam $1-2-1$, setting the volume fraction index $p = 1.0$, for CC, CF and FF boundary conditions and slenderness ratios $l/h = 5$ and $l/h = 20$ are computed. The results obtained by using the advanced beam models TE_{445} and EX_{445} have been compared with those evaluated by using the FEM commercial software ABAQUS. The comparison showed an excellent agreement and indeed the average difference for all of the considered boundary conditions is always below the 0.2% reaching the 0.01% in the case of FF boundary condition. Similar consideration can be drawn in Table 9 where the asymmetric FG sandwich beam $2-2-1$ is investigated. For both symmetric and asymmetric configuration the mesh details are provided in Fig. 2, the element used is the C3D20R. The first six mode shapes for the two different FG sandwich beam structures above mentioned with slenderness ratio $l/h = 5$ are shown in Figs. 3 and 4, respectively. Similarly, Figs. 5 and 6 depict the first six mode shapes of the same structures but with a slenderness ratio $l/h = 20$. In Tables 10, 11 and 12 an extension of the results given in Tables 8 and 9 has been provided. In particular, the first three dimensionless circular frequency parameter are computed for the CC, CF and FF boundary conditions, respectively. The analysis has been carried out by considering various values of the volume fraction index p, as expected the natural frequencies decrease when increasing p. The effect of the porosity has also been taken into account in the FG beam structures configuration in Table 13. From all of the analysed symmetric and asymmetric FG sandwich configurations the highest value of the dimensionless circular frequency parameter is obtained by the scheme $2-3-1$ being the latter featured by the higher quantity of the ceramic constituent. The effect of the Winkler-Pasternak foundation has been evaluated in Table 14 considering the CC boundary condition and the porosity type-I and type-II with a porosity coefficient $\beta = 0.2$. Moreover, the analysis has been carried out for short beam ($l/h = 5$) with a volume fraction index of $p = 1.0$. Once again as expected for all of the considered FG sandwich beam configuration scheme, both symmetric and asymmetric, the introduction of the Winkler-Pasternak foundation generates an overall increase of the fundamental circular frequency parameter. The dimensionless critical buckling loads of the FG sandwich beam structures featured by a symmetric distribution of the metallic and ceramic material constituents is provided in Table 15. The highest vale of the dimensionless critical buckling loads is obtained by the lamination scheme $1-5-1$, which is the FG scheme with the highest amount, between those proposed, of the ceramic constituent. As expected for both porosity type-I and type-II the critical buckling load decreases significantly while increasing the porosity coefficient β. Even for this case the effect of the elastic foundation have been taken into account in Table 16 and a general increase of the critical buckling load is observed.

7. Conclusions

The free vibration and elastic stability characteristics of porous FG isotropic and sandwich beam structures have been investigated. The analysis has been carried out by considering advanced and refined polynomial, exponential and trigonometric quasi-3D beam models. Algebraic Ritz functions, orthogonalised by using the Gram-Schmidt process, have been employed in the approximation. The effect of significant parameters such as length-to-thickness ratio, volume fraction index, materials, porosity coefficient and boundary conditions, have been commented. A comprehensive convergence analysis of the proposed beam model has been carried out and their accuracy has been evaluated against the results proposed in the literature. Further results have been obtained by considering various FG sandwich beam configurations. In particular, the validation of the proposed models has been carried out by comparison with FEM commercial software such as ABAQUS, and for both a symmetric $(1-2-1)$ and a asymmetric $(2-2-1)$ scheme, respectively. The effect of the two different porosity types on both dimensionless circular frequency parameter and dimensionless critical buckling load has been evaluated. Moreover, the effect of two-parameters elastic foundation has been taken into account. From all of the analysis carried out the following considerations can be drawn:

- Converge is fast for all of the studied cross-section functions and it is slightly affected by the beam order theory.
- Amongst all of the assessed beam theories the TE_{445}, EX_{445} and TR_{225} proved to be the most accurate ones with the lower number of degrees of freedom. However, it must be born in mind that this result is not general but is, of course, highly affected by the type of problem investigated.
- The use of the developed beam models highlight the importance of using advance and refined beam kinematics in order to capture 3D effects such as imperfections due to material porosities through-the-beam-cross-section.

- Between all of the analysed FG sandwich the highest value of the dimensionless frequency parameter, is obtained with the scheme $2-3-1$. The scheme $1-5-1$ turned out to be the one which maximizes the dimensionless critical buckling load.

- For all of the FG beam structures investigated, as expected, the frequencies increases when adding the Winkler foundation and a further increase is obtained by adding the Pasternak-layer. The effect of the latter is more prominent in the range of short beam where the effect of the shear deformation is significant.

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Appendix A

The present appendix proposes the explicit expressions of the matrices given in Eq.(33). The stiffness matrix $[K_{\tau\, s\, i\, j}]$ elements assume the following form

$$
\begin{aligned}
K_{\tau_{u_x}\, s_{u_x}\, i\, j} = & \int_\Omega \left[\lambda\left(x\right) + 2\, G\left(x\right)\right] \left[F_{\tau_{u_{x,x}}}\left(x,y\right)\, F_{s_{u_{x,x}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{x_i}\left(z\right)\, \psi_{x_j}\left(z\right) \mathrm{d}z + \\
& \int_\Omega \mu\left(x\right) \left[F_{\tau_{u_{x,y}}}\left(x,y\right)\, F_{s_{u_{x,y}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{x_i}\left(z\right)\, \psi_{x_j}\left(z\right) \mathrm{d}z + \\
& \int_\Omega \mu\left(x\right) \left[F_{\tau_{u_x}}\left(x,y\right)\, F_{s_{u_x}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{x_{i,z}}\left(z\right)\, \psi_{x_{j,z}}\left(z\right) \mathrm{d}z \\[2ex]
K_{\tau_{u_x}\, s_{u_y}\, i\, j} = & \int_\Omega \lambda\left(x\right) \left[F_{\tau_{u_{x,x}}}\left(x,y\right)\, F_{s_{u_{y,y}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{x_i}\left(z\right)\, \psi_{y_j}\left(z\right) \mathrm{d}z + \\
& \int_\Omega \mu\left(x\right) \left[F_{\tau_{u_{x,y}}}\left(x,y\right)\, F_{s_{u_{y,x}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{x_i}\left(z\right)\, \psi_{y_j}\left(z\right) \mathrm{d}z \\[2ex]
K_{\tau_{u_x}\, s_{u_z}\, i\, j} = & \int_\Omega \lambda\left(x\right) \left[F_{\tau_{u_{x,x}}}\left(x,y\right)\, F_{s_{u_z}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{x_i}\left(z\right)\, \psi_{z_{j,z}}\left(z\right) \mathrm{d}z + \\
& \int_\Omega \mu\left(x\right) \left[F_{\tau_{u_x}}\left(x,y\right)\, F_{s_{u_{z,x}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{x_{i,z}}\left(z\right)\, \psi_{z_j}\left(z\right) \mathrm{d}z \\[2ex]
K_{\tau_{u_y}\, s_{u_x}\, i\, j} = & \int_\Omega \lambda\left(x\right) \left[F_{\tau_{u_{y,y}}}\left(x,y\right)\, F_{s_{u_{x,x}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{y_i}\left(z\right)\, \psi_{x_j}\left(z\right) \mathrm{d}z + \\
& \int_\Omega \mu\left(x\right) \left[F_{\tau_{u_{y,x}}}\left(x,y\right)\, F_{s_{u_{x,y}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{y_i}\left(z\right)\, \psi_{x_j}\left(z\right) \mathrm{d}z \\[2ex]
K_{\tau_{u_y}\, s_{u_y}\, i\, j} = & \int_\Omega \left[\lambda\left(x\right) + 2\, G\left(x\right)\right] \left[F_{\tau_{u_{y,y}}}\left(x,y\right)\, F_{s_{u_{y,y}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{y_i}\left(z\right)\, \psi_{y_j}\left(z\right) \mathrm{d}z + \\
& \int_\Omega \mu\left(x\right) \left[F_{\tau_{u_{y,x}}}\left(x,y\right)\, F_{s_{u_{y,x}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{y_i}\left(z\right)\, \psi_{y_j}\left(z\right) \mathrm{d}z + \\
& \int_\Omega \mu\left(x\right) \left[F_{\tau_{u_y}}\left(x,y\right)\, F_{s_{u_y}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{y_{i,z}}\left(z\right)\, \psi_{y_{j,z}}\left(z\right) \mathrm{d}z \\[2ex]
K_{\tau_{u_y}\, s_{u_z}\, i\, j} = & \int_\Omega \lambda\left(x\right) \left[F_{\tau_{u_{y,y}}}\left(x,y\right)\, F_{s_{u_z}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{y_i}\left(z\right)\, \psi_{z_{j,z}}\left(z\right) \mathrm{d}z + \\
& \int_\Omega \mu\left(x\right) \left[F_{\tau_{u_y}}\left(x,y\right)\, F_{s_{u_{z,y}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_l \psi_{y_{i,z}}\left(z\right)\, \psi_{z_j}\left(z\right) \mathrm{d}z
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
K_{\tau_{u_z}\, s_{u_x}\, i\, j} = & \int_{\Omega} \mu\left(x\right) \left[F_{\tau_{u_{z,x}}}\left(x,y\right)\; F_{s_{u_x}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_i}\left(z\right)\; \psi_{x_{j,z}}\left(z\right) \mathrm{d}z + \\
& \int_{\Omega} \lambda\left(x\right) \left[F_{\tau_{u_z}}\left(x,y\right)\; F_{s_{u_{x,x}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_{i,z}}\left(z\right)\; \psi_{x_j}\left(z\right) \mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
K_{\tau_{u_z}\, s_{u_y}\, i\, j} = & \int_{\Omega} \mu\left(x\right) \left[F_{\tau_{u_{z,y}}}\left(x,y\right)\; F_{s_{u_y}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_i}\left(z\right)\; \psi_{y_{j,z}}\left(z\right) \mathrm{d}z + \\
& \int_{\Omega} \lambda\left(x\right) \left[F_{\tau_{u_z}}\left(x,y\right)\; F_{s_{u_{y,y}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_{i,z}}\left(z\right)\; \psi_{y_j}\left(z\right) \mathrm{d}z
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
K_{\tau_{u_z}\, s_{u_z}\, i\, j} = & \int_{\Omega} \mu\left(x\right) \left[F_{\tau_{u_{z,x}}}\left(x,y\right)\; F_{s_{u_{z,x}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_i}\, \psi_{z_j} \mathrm{d}z + \\
& \int_{\Omega} \mu\left(x\right) \left[F_{\tau_{u_{z,y}}}\left(x,y\right)\; F_{s_{u_{z,y}}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_i}\, \psi_{z_j} \mathrm{d}z + \\
& \int_{\Omega} \left[\lambda\left(x\right) + 2\, G\left(x\right)\right] \left[F_{\tau_{u_z}}\left(x,y\right)\; F_{s_{u_z}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_{i,z}}\; \psi_{z_{j,z}} \mathrm{d}z
\end{aligned}
$$

The three non-zero initial stress matrix $\left[K^{(\sigma)}_{\tau\, s\, i\, j}\right]$ terms assume the following form

$$
\begin{aligned}
K^{(\sigma)}_{\tau_{u_x}\, s_{u_x}\, i\, j} &= \int_{\Omega} \sigma^{(0)}_{zz}\; \left[F_{\tau_{u_x}}\left(x,y\right)\; F_{s_{u_x}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{x_{i,z}}\left(z\right)\; \psi_{x_{j,z}}\left(z\right) \mathrm{d}z \\
K^{(\sigma)}_{\tau_{u_y}\, s_{u_y}\, i\, j} &= \int_{\Omega} \sigma^{(0)}_{zz}\; \left[F_{\tau_{u_y}}\left(x,y\right)\; F_{s_{u_y}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{y_{i,z}}\left(z\right)\; \psi_{y_{j,z}}\left(z\right) \mathrm{d}z \\
K^{(\sigma)}_{\tau_{u_z}\, s_{u_z}\, i\, j} &= \int_{\Omega} \sigma^{(0)}_{zz}\; \left[F_{\tau_{u_z}}\left(x,y\right)\; F_{s_{u_z}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_{i,z}}\left(z\right)\; \psi_{z_{j,z}}\left(z\right) \mathrm{d}z
\end{aligned}
\tag{39}
$$

The three non-zero of the mass matrix $\left[M_{\tau\, s\, i\, j}\right]$ terms assume the following form

$$
\begin{aligned}
M_{\tau_{u_x}\, s_{u_x}\, i\, j} &= \int_{\Omega} \rho\left(x\right) \left[F_{\tau_{u_x}}\left(x,y\right)\; F_{s_{u_x}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{x_i}\left(z\right)\; \psi_{x_j}\left(z\right) \mathrm{d}z \\
M_{\tau_{u_y}\, s_{u_y}\, i\, j} &= \int_{\Omega} \rho\left(x\right) \left[F_{\tau_{u_y}}\left(x,y\right)\; F_{s_{u_y}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{y_i}\left(z\right)\; \psi_{y_j}\left(z\right) \mathrm{d}z \\
M_{\tau_{u_z}\, s_{u_z}\, i\, j} &= \int_{\Omega} \rho\left(x\right) \left[F_{\tau_{u_z}}\left(x,y\right)\; F_{s_{u_z}}\left(x,y\right)\right] \mathrm{d}\Omega \int_{l} \psi_{z_i}\left(z\right)\; \psi_{z_j}\left(z\right) \mathrm{d}z
\end{aligned}
\tag{40}
$$

Finally, the three non-zero terms involved in the stiffness matrix due to the elastic foundations $\left[K^{(wp)}_{\tau s i j}\right]$, are given as

$$\begin{aligned}
K^{(wp)}_{\tau_{u_x} s_{u_x} ij} = & \int_\Omega \kappa^{w-}_{xx} \left[F_{\tau_{u_x}}(\bar{x},y)\ F_{s_{u_x}}(\bar{x},y)\right]_{x=-\frac{h}{2}} \mathrm{d}\Omega \int_l \psi_{x_i}(z)\ \psi_{x_j}(z)\,\mathrm{d}z + \\
& \int_\Omega \kappa^{p-}_{xx} \left[F_{\tau_{u_{x,z}}}(\bar{x},y)\ F_{s_{u_{x,z}}}(\bar{x},y)\right]_{x=-\frac{h}{2}} \mathrm{d}\Omega \int_l \psi_{x_i}(z)\ \psi_{x_j}(z)\,\mathrm{d}z + \\
& \int_\Omega \kappa^{w+}_{xx} \left[F_{\tau_{u_x}}(\bar{x},y)\ F_{s_{u_x}}(\bar{x},y)\right]_{x=\frac{h}{2}} \mathrm{d}\Omega \int_l \psi_{x_i}(z)\ \psi_{x_j}(z)\,\mathrm{d}z + \\
& \int_\Omega \kappa^{p+}_{xx} \left(F_{\tau_{u_{x,z}}}(\bar{x},y)\ F_{s_{u_{x,z}}}(\bar{x},y)\right)_{x=\frac{h}{2}} \mathrm{d}\Omega \int_l \psi_{x_i}(z)\ \psi_{x_j}(z)\,\mathrm{d}z
\end{aligned}$$

$$\begin{aligned}
K^{(wp)}_{\tau_{u_y} s_{u_y} ij} = & \int_\Omega \kappa^{w-}_{yy} \left[F_{\tau_{u_y}}(x,\bar{y})\ F_{s_{u_y}}(x,\bar{y})\right]_{y=-\frac{b}{2}} \mathrm{d}\Omega \int_l \psi_{y_i}(z)\ \psi_{y_j}(z)\,\mathrm{d}z - \\
& \int_\Omega \kappa^{p-}_{yy} \left[F_{\tau_{u_{y,z}}}(x,\bar{y})\ F_{s_{u_{y,z}}}(x,\bar{y})\right]_{y=-\frac{b}{2}} \mathrm{d}\Omega \int_l \psi_{y_i}(z)\ \psi_{y_j}(z)\,\mathrm{d}z + \\
& \int_\Omega \kappa^{w+}_{yy} \left[F_{\tau_{u_y}}(x,\bar{y})\ F_{s_{u_x}}(x,\bar{y})\right]_{y=\frac{b}{2}} \mathrm{d}\Omega \int_l \psi_{x_i}(z)\ \psi_{y_j}(z)\,\mathrm{d}z - \\
& \int_\Omega \kappa^{p+}_{yy} \left[F_{\tau_{u_{y,z}}}(x,\bar{y})\ F_{s_{u_{x,z}}}(x,\bar{y})\right]_{y=\frac{b}{2}} \mathrm{d}\Omega \int_l \psi_{x_i}(z)\ \psi_{y_j}(z)\,\mathrm{d}z
\end{aligned} \tag{41}$$

$$\begin{aligned}
K^{(wp)}_{\tau_{u_z} s_{u_z} ij} = & \int_\Omega \kappa^{w0}_{zz} \left[F_{\tau_{u_z}}(x,y)\ F_{s_{u_z}}(x,y)\right] \mathrm{d}\Omega \left[\psi_{z_i}(\bar{z})\ \psi_{z_j}(\bar{z})\right]_{z=0} \\
& \int_\Omega \kappa^{p0}_{zz} \left[F_{\tau_{u_{z,x}}}(x,y)\ F_{s_{u_{z,x}}}(x,y)\right] \mathrm{d}\Omega \left[\psi_{z_i}(\bar{z})\ \psi_{z_j}(\bar{z})\right]_{z=0} + \\
& \int_\Omega \kappa^{p0}_{zz} \left[F_{\tau_{u_{z,y}}}(x,y)\ F_{s_{u_{z,y}}}(x,y)\right] \mathrm{d}\Omega \left[\psi_{z_i}(\bar{z})\ \psi_{z_j}(\bar{z})\right]_{z=0} + \\
& \int_\Omega \kappa^{wl}_{zz} \left[F_{\tau_{u_z}}(x,y)\ F_{s_{u_z}}(x,y)\right] \mathrm{d}\Omega \left[\psi_{z_i}(\bar{z})\ \psi_{z_j}(\bar{z})\right]_{z=l} - \\
& \int_\Omega \kappa^{pl}_{zz} \left[F_{\tau_{u_{z,x}}}(x,y)\ F_{s_{u_{z,x}}}(x,y)\right] \mathrm{d}\Omega \left[\psi_{z_i}(\bar{z})\ \psi_{z_j}(\bar{z})\right]_{z=l} + \\
& \int_\Omega \kappa^{pl}_{zz} \left[F_{\tau_{u_{z,y}}}(x,y)\ F_{s_{u_{z,y}}}(x,y)\right] \mathrm{d}\Omega \left[\psi_{z_i}(\bar{z})\ \psi_{z_j}(\bar{z})\right]_{z=l}
\end{aligned}$$

Tables

Figures

Figure 1: Porous FG isotropic and sandwich beam structures: coordinate system and nomenclature.

Table 1: Convergence analysis of the fundamental frequency of a CF (cantilever) FG short ($l/h = 5$) beam by using polynomial cross-section functions.

Theory	Ritz expansion i, j									$\Delta(\%)^{\dagger}$	DOFs
	2	4	6	8	10	12	14	16	18		
TE_{666}	1.580985	1.509121	1.485747	1.476250	1.471609	1.469043	1.467528	1.466590	1.465981		1512
TE_{665}	1.580986	1.509122	1.485750	1.476253	1.471614	1.469047	1.467532	1.466594	1.465985	0.00	1386
TE_{664}	1.509303	1.509303	1.486019	1.476649	1.472137	1.469648	1.468165	1.467235	1.466623	0.04	1278
TE_{663}	1.581152	1.509320	1.486036	1.476666	1.472154	1.469665	1.468182	1.467252	1.466640	0.05	1188
TE_{662}	1.585452	1.513553	1.490321	1.480886	1.476260	1.473677	1.473677	1.471141	1.470498	0.31	1116
TE_{661}	1.586630	1.514749	1.491551	1.482147	1.477537	1.474960	1.473407	1.472427	1.471785	0.40	1062
TE_{556}	1.581007	1.509155	1.485776	1.476281	1.471652	1.469106	1.467619	1.466709	1.466124	0.01	1260
TE_{555}	1.581008	1.509156	1.485779	1.476284	1.471656	1.469111	1.467623	1.466713	1.466128	0.01	1134
TE_{554}	1.581155	1.509341	1.486048	1.476673	1.472162	1.469680	1.468211	1.467296	1.466698	0.05	1026
TE_{553}	1.581176	1.509359	1.486065	1.476689	1.472178	1.469697	1.468228	1.467314	1.466715	0.05	936
TE_{552}	1.585472	1.513585	1.490344	1.480902	1.476277	1.473699	1.472154	1.471183	1.470549	0.31	864
TE_{551}	1.586650	1.514781	1.491573	1.482163	1.477553	1.474982	1.473439	1.472469	1.471836	0.40	810
TE_{446}	1.581014	1.509166	1.485784	1.476286	1.471656	1.469110	1.467623	1.466713	1.466128	0.01	1044
TE_{445}	1.581015	1.509167	1.485787	1.476290	1.471661	1.469115	1.467627	1.466718	1.466132	0.01	918
TE_{444}	1.581164	1.509353	1.486056	1.476679	1.472167	1.469685	1.468215	1.467301	1.466702	0.05	810
TE_{443}	1.581185	1.509371	1.486074	1.476695	1.472183	1.469701	1.468232	1.467317	1.466718	0.05	720
TE_{442}	1.585480	1.513597	1.490351	1.480907	1.476281	1.473703	1.472157	1.471186	1.470552	0.31	648
TE_{441}	1.586658	1.514791	1.491580	1.482167	1.477556	1.474984	1.473441	1.472471	1.471838	0.40	594
TE_{336}	1.581205	1.509631	1.486362	1.476975	1.472473	1.470025	1.468606	1.467758	1.467236	0.09	864
TE_{335}	1.581206	1.509632	1.486365	1.476980	1.472478	1.470030	1.468611	1.467763	1.467241	0.09	738
TE_{334}	1.581359	1.509826	1.486643	1.477381	1.473002	1.470633	1.469249	1.468408	1.467880	0.13	630
TE_{333}	1.581384	1.509848	1.486663	1.477399	1.473019	1.470649	1.469265	1.468424	1.467896	0.13	540
TE_{332}	1.585668	1.514068	1.490940	1.481599	1.477064	1.474544	1.473036	1.472098	1.471496	0.38	468
TE_{331}	1.586845	1.515263	1.492169	1.482859	1.478341	1.475828	1.474323	1.473386	1.472785	0.46	414
TE_{226}	1.581233	1.509685	1.486401	1.476997	1.472486	1.470034	1.468615	1.467769	1.467249	0.09	720
TE_{225}	1.581235	1.509687	1.486405	1.477001	1.472491	1.470039	1.468620	1.467773	1.467254	0.09	594
TE_{224}	1.581391	1.509884	1.486683	1.477404	1.473016	1.470644	1.469259	1.468419	1.467893	0.13	486
TE_{223}	1.581415	1.509906	1.486704	1.477423	1.473033	1.470660	1.469275	1.468435	1.467909	0.13	396
TE_{222}	1.585696	1.514121	1.490979	1.481623	1.477080	1.474558	1.473050	1.472112	1.471512	0.38	324
TE_{221}	1.586877	1.515323	1.492211	1.482882	1.477080	1.475836	1.474330	1.473393	1.472793	0.46	270
TE_{116}	1.686892	1.675141	1.673843	1.673333	1.673083	1.672942	1.672855	1.672797	1.672757	14.1	612
TE_{115}	1.686892	1.675141	1.673843	1.673333	1.673083	1.672943	1.672856	1.672798	1.672757	14.1	486
TE_{114}	1.686894	1.675146	1.673852	1.673350	1.673106	1.672968	1.672882	1.672824	1.672784	14.1	378
TE_{113}	1.686896	1.675150	1.673857	1.673357	1.673114	1.672977	1.672891	1.672833	1.672793	14.1	288
TE_{112}	1.691330	1.680048	1.678881	1.678409	1.678170	1.678032	1.677945	1.677888	1.677847	14.5	216
TE_{111}	1.692805	1.681676	1.680541	1.680075	1.679838	1.679702	1.679617	1.679560	1.679520	14.6	162

† $\Delta(\%) = \frac{\|f_p - f_o\|}{\|f_o\|} \times 100$ and is evaluated with respect to the TE_{666} beam model.

For comparison purpose it can be noted that $\hat{\omega}_1 = 1.4628$ in Ref. [13].

Figure 2: FEM modelling of symmetric and asymmetric FG sandwich beams.

Table 2: Convergence analysis of the fundamental frequency of a CF (cantilever) FG short ($l/h = 5$) beam by using exponential cross-section functions.

	Ritz expansion i, j										DOFs
Theory	2	4	6	8	10	12	14	16	18	$\Delta(\%)^{\dagger}$	
EX_{666}	1.636116	1.576488	1.552018	1.540673	1.534777	1.531495	1.529573	1.528393	1.527634	4.21	702
EX_{665}	1.636192	1.576573	1.552122	1.540796	1.534915	1.531627	1.529691	1.528500	1.527733	4.21	666
EX_{664}	1.637054	1.577454	1.553044	1.541782	1.535965	1.532725	1.530815	1.529628	1.528852	4.29	630
EX_{663}	1.647552	1.587870	1.563347	1.552008	1.546126	1.542856	1.540938	1.539752	1.538980	4.98	594
EX_{662}	1.773232	1.712964	1.687626	1.675949	1.669857	1.666410	1.664344	1.663049	1.662204	13.39	558
EX_{556}	1.636120	1.576496	1.552028	1.540683	1.534787	1.531505	1.529585	1.528409	1.527656	4.21	630
EX_{555}	1.636195	1.576581	1.552131	1.540806	1.534924	1.531637	1.529703	1.528515	1.527755	4.21	594
EX_{554}	1.637056	1.577459	1.553051	1.541791	1.535977	1.532743	1.530839	1.529659	1.528890	4.29	558
EX_{553}	1.647553	1.587874	1.563353	1.552015	1.546136	1.542871	1.540957	1.539776	1.539010	4.98	522
EX_{552}	1.773234	1.712968	1.687632	1.675956	1.669867	1.666422	1.664359	1.663068	1.662226	13.37	486
EX_{446}	1.636164	1.576587	1.552143	1.540818	1.534951	1.529826	1.529826	1.528682	1.527951	4.23	558
EX_{445}	1.636241	1.576673	1.552245	1.540936	1.531832	1.531832	1.529942	1.528793	1.528060	4.23	522
EX_{444}	1.637102	1.577550	1.553164	1.541920	1.536132	1.532932	1.531065	1.529916	1.529169	4.31	486
EX_{443}	1.647576	1.587935	1.563433	1.552111	1.546251	1.543006	1.541112	1.539948	1.539193	4.99	450
EX_{442}	1.773249	1.713021	1.687699	1.676035	1.669961	1.666530	1.664479	1.663198	1.662364	13.40	414
EX_{336}	1.636718	1.577707	1.553502	1.542313	1.536563	1.533430	1.531650	1.530599	1.529950	4.36	486
EX_{335}	1.636796	1.553603	1.553603	1.674436	1.536695	1.533558	1.531770	1.530714	1.530066	4.37	450
EX_{334}	1.637680	1.578698	1.554541	1.543423	1.537745	1.534655	1.532895	1.531850	1.531200	4.45	414
EX_{333}	1.648156	1.589081	1.564817	1.553620	1.547859	1.544713	1.542923	1.541864	1.541207	5.13	378
EX_{332}	1.773476	1.713686	1.688561	1.677005	1.671006	1.667636	1.665639	1.664406	1.663614	13.48	342
EX_{226}	1.643001	1.590231	1.568571	1.558479	1.553238	1.550312	1.548602	1.547570	1.546932	5.52	414
EX_{225}	1.643080	1.590316	1.568664	1.558581	1.553345	1.550416	1.548701	1.547669	1.547032	5.53	378
EX_{224}	1.643984	1.591242	1.569626	1.559595	1.554413	1.551528	1.549842	1.548823	1.548187	5.61	342
EX_{223}	1.654811	1.602069	1.580427	1.565196	1.565196	1.562318	1.560642	1.559627	1.558992	6.34	306
EX_{222}	1.779669	1.725497	1.702854	1.692184	1.686574	1.683409	1.686574	1.680343	1.679584	14.57	270
EX_{116}	1.683740	1.668891	1.665971	1.664750	1.664184	1.663899	1.663743	1.663650	1.663589	13.48	342
EX_{115}	1.683819	1.668970	1.666052	1.664831	1.663983	1.663983	1.663829	1.663737	1.663677	13.49	306
EX_{114}	1.684724	1.669870	1.666953	1.665173	1.665173	1.664895	1.664745	1.664655	1.664596	13.55	270
EX_{113}	1.695673	1.680612	1.677681	1.676465	1.675920	1.675653	1.675506	1.675415	1.675354	14.28	234
EX_{112}	1.823899	1.805026	1.801813	1.800505	1.799928	1.799640	1.799478	1.799376	1.799307	22.74	198

† The $\Delta(\%)$ is evaluated with respect to the TE_{666} beam model.

Figure 3: The first 6 mode shapes of a square symmetric FG sandwich beam 1-2-1 with CF boundary condition and $l/h = 5$.

Figure 4: The first 6 mode shapes of a square unsymmetric FG sandwich beam 2-2-1 with CF boundary condition and $l/h = 5$.

Table 3: Convergence analysis of the fundamental frequency of a CF (cantilever) FG short ($l/h = 5$) beam by using trigonometric cross-section functions.

Theory	Ritz expansion i, j										DOFs
	2	4	6	8	10	12	14	16	18	$\Delta(\%)^{\dagger}$	
TR_{446}	1.636110	1.576478	1.551994	1.540622	1.534684	1.531349	1.529373	1.528149	1.527357	4.19	1062
TR_{445}	1.636136	1.576505	1.552021	1.540650	1.534716	1.531386	1.529419	1.528207	1.527427	4.19	990
TR_{444}	1.636686	1.577061	1.552580	1.541215	1.535291	1.531978	1.530035	1.528846	1.528084	4.24	918
TR_{443}	1.650564	1.591079	1.566624	1.555280	1.549393	1.546126	1.544219	1.543048	1.542288	5.21	846
TR_{442}	1.924717	1.864264	1.839417	1.827643	1.821446	1.817958	1.815902	1.814629	1.813797	23.73	774
TR_{336}	1.636113	1.576485	1.552003	1.540631	1.534694	1.531359	1.529385	1.528161	1.527370	4.19	918
TR_{335}	1.636139	1.576512	1.552030	1.540660	1.534726	1.531397	1.529431	1.528219	1.527440	4.19	864
TR_{334}	1.636690	1.577069	1.552590	1.541226	1.535304	1.531993	1.530052	1.528864	1.528102	4.24	774
TR_{333}	1.650571	1.591094	1.566645	1.555307	1.549424	1.546158	1.544252	1.543080	1.542321	5.21	702
TR_{332}	1.924722	1.864275	1.839434	1.827668	1.821477	1.817995	1.815940	1.814666	1.813833	23.73	630
TR_{226}	1.636191	1.576640	1.552187	1.540827	1.534898	1.531571	1.529604	1.528391	1.527612	4.20	774
TR_{225}	1.636217	1.576666	1.552214	1.540856	1.534930	1.531608	1.529650	1.528447	1.527681	4.21	702
TR_{224}	1.636768	1.577224	1.552776	1.541424	1.535510	1.532205	1.530270	1.529091	1.528341	4.25	630
TR_{223}	1.650672	1.591285	1.566884	1.555574	1.549711	1.546462	1.544570	1.543413	1.542672	5.23	558
TR_{222}	1.924892	1.864611	1.839931	1.828269	1.822110	1.818609	1.816525	1.815236	1.814403	23.77	486
TR_{116}	1.637546	1.579331	1.555434	1.544376	1.538678	1.535560	1.533795	1.532770	1.532155	4.51	630
TR_{115}	1.637573	1.579358	1.555462	1.544404	1.538709	1.535596	1.533837	1.532819	1.532213	4.52	558
TR_{114}	1.638125	1.579917	1.556024	1.544972	1.539286	1.536186	1.534446	1.533444	1.532847	4.56	486
TR_{113}	1.652044	1.593987	1.570146	1.559134	1.553492	1.550439	1.548733	1.547748	1.547154	5.54	414
TR_{112}	1.926856	1.867974	1.844313	1.833362	1.827806	1.824799	1.823087	1.822069	1.821438	24.25	342

† The $\Delta(\%)$ is evaluated with respect to the TE_{666} beam model.

Figure 5: The first 6 mode shapes of a square symmetric FG sandwich beam 1-2-1 with CF boundary condition and $l/h = 20$.

Figure 6: The first 6 mode shapes of a square unsymmetric FG sandwich beam 2-2-1 with CF boundary condition and $l/h = 20$.

Table 4: Dimensionless fundamental frequency parameters of a Porous FGM beams, varying the length-to-thickness ratio, the porosity coefficient and the boundary conditions.

Porosity type	β	BCs	p	l/h = 5	10	15	20	50
	0.0	CC	0.2	9.510418	10.902077	11.230206	11.349936	11.478317
			1.0	8.058737	9.157737	9.412606	9.504972	9.603410
			5.0	6.550907	7.730205	8.011787	8.115440	8.230252
		CF	0.2	1.764354	1.795134	1.800197	1.801716	1.802894
			1.0	1.477720	1.502145	1.506082	1.507242	1.508116
			5.0	1.260205	1.286396	1.290818	1.292230	1.293647
		FF	0.2	10.182124	11.066609	11.265521	11.338499	11.419427
			1.0	8.501923	9.252876	9.421404	9.483205	9.551721
			5.0	7.195573	7.922768	8.077752	8.134745	8.198037
I	0.2	CC	0.2	9.699950	11.065078	11.387708	11.506416	11.636060
			1.0	7.738870	8.680612	8.896207	8.974618	9.059443
			5.0	5.274587	6.169857	6.401280	6.489091	6.587350
		CF	0.2	1.788152	1.820249	1.825900	1.827737	1.829459
			1.0	1.395065	1.417595	1.421474	1.422714	1.423856
			5.0	1.003448	1.028710	1.033336	1.034873	1.036384
		FF	0.2	10.369976	11.253417	11.451242	11.523734	11.604068
			1.0	8.058111	8.755981	8.911133	8.967886	9.030714
			5.0	5.613699	6.283913	6.443773	6.503437	6.570281
II	0.2	CC	0.2	9.722932	11.175035	11.521018	11.647945	11.785007
			1.0	8.098449	9.232496	9.499400	9.596846	9.701639
			5.0	6.113225	7.380488	7.667772	7.773786	7.891232
		CF	0.2	1.809381	1.842829	1.848496	1.850252	1.851722
			1.0	1.490613	1.517196	1.521649	1.523015	1.524150
			5.0	1.205732	1.100163	1.237786	1.239359	1.240955
		FF	0.2	10.423056	11.357512	11.568864	11.646528	11.732733
			1.0	8.549013	9.339315	9.518110	9.583823	9.656772
			5.0	6.878703	7.596680	7.749068	7.805064	7.867223

Table 5: Dimensionless fundamental parameter of Porous FG sandwich beams with $l/h = 5$, $\beta = 0.2$, $p = 1.0$, CC boundary condition and including the effect of the Winkler-Pasternak elastic foundations.

Porosity type	$\mathcal{K}^{w-}$	$\mathcal{K}^{p-}$	l/h 5	10	15	20	50
	0.0	0.0	8.058737	9.157737	9.412606	9.504972	9.603410
	0.5	0.0	8.461757	9.866734	10.201078	10.323608	10.457940
	0.5	0.5	8.550889	9.913750	10.231635	10.345025	10.462779
I	0.0	0.0	7.738870	8.680612	8.896207	8.974618	9.059443
	0.5	0.0	8.443457	9.847220	10.179619	10.301723	10.436072
	0.5	0.5	8.542081	9.881555	10.199085	10.314703	10.438870
II	0.0	0.0	8.098449	9.232496	9.499400	9.596846	9.701639
	0.5	0.0	8.456591	9.857315	10.190765	10.313222	10.447571
	0.5	0.5	8.545392	9.897448	10.215661	10.330388	10.451441

Table 6: Dimensionless critical buckling parameter of porous FG beams with CC boundary condition.

Porosity type	β	p	l/h 10	20	40	60	80	100
	0.0	0.5	10.393947	11.240275	11.470924	11.513742	11.528689	11.535599
		1.0	8.060210	8.711658	8.889274	8.922276	8.933803	8.939133
		5.0	5.053081	5.567317	5.711452	5.739002	5.748738	5.753266
I	0.1	0.5	9.367377	10.116714	10.321983	10.360305	10.373706	10.379906
		1.0	6.943418	7.488209	7.637307	7.665153	7.674896	7.679404
		5.0	3.881263	4.321306	4.448823	4.473115	4.481662	4.485627
	0.2	0.5	8.330615	8.983116	9.162496	9.196134	9.207915	9.213369
		1.0	5.787381	6.223798	6.343400	6.365822	6.373678	6.377314
		5.0	2.468453	2.743841	2.824293	2.839707	2.845140	2.847663
II	0.1	0.5	10.103839	10.937603	11.165691	11.208124	11.222944	11.229797
		1.0	7.725177	8.360591	8.534645	8.567062	8.578391	8.583631
		5.0	4.571055	5.048363	5.182247	5.207807	5.216830	5.221023
	0.2	0.5	9.810828	10.632613	10.858304	10.900382	10.915087	10.921888
		1.0	7.381259	8.000770	8.171313	8.203155	8.214290	8.219442
		5.0	4.083984	4.528771	4.653653	4.677474	4.685878	4.689782

Table 7: Dimensionless critical buckling parameter of porous FG with CC boundary condition, $\beta = 0.2$, $p = 1.0$, CC boundary condition and including the effect of the Winkler-Pasternak elastic foundations.

Porosity type	$\mathcal{K}^{w-}$	$\mathcal{K}^{p-}$	l/h 10	20	40	60	80	100
	0.0	0.0	8.060210	8.711658	8.889274	8.922276	8.933803	8.939133
	0.5	0.0	9.389610	10.293907	10.544984	10.592549	10.609219	10.616904
	0.5	0.5	9.432232	10.313325	10.551953	10.595884	10.611117	10.618112
I	0.0	0.0	5.787381	6.223798	6.343400	6.365822	6.373678	6.377314
	0.5	0.0	7.484656	8.215289	8.417632	8.455955	8.469405	8.475620
	0.5	0.5	7.514014	8.224506	8.420716	8.457453	8.470283	8.476199
II	0.0	0.0	7.381259	8.000770	8.171313	8.203155	8.214290	8.219442
	0.5	0.0	8.445030	9.257688	9.483195	9.525870	9.540824	9.547722
	0.5	0.5	8.479001	9.271388	9.488099	9.528261	9.542214	9.548627

Table 8: First six dimensionless frequency parameters of a symmetric FG sandwich beam $1-2-1$ with volume fraction index $p=1$. Comparison with 3D FEM.

l/h	BCs	Theory	Dimensionless frequency parameters						Ave.
			$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$	$\hat{\omega}_4$	$\hat{\omega}_5$	$\hat{\omega}_6$	$\Delta(\%)$
5	CC	ABAQUS†	8.4801	9.4462	15.3722	20.1061	21.5988	28.3755	
		TE_{445}	8.502928	9.452756	15.417823	20.171171	21.604842	28.392491	0.17
		EX_{445}	8.762584	9.721273	16.579740	20.711867	22.158875	28.397865	3.30
	CF	ABAQUS	1.5117	1.7767	7.6466	8.4550	9.5705	14.1281	
		TE_{445}	1.514911	1.778145	7.664768	8.475046	9.576022	14.135771	0.15
		EX_{445}	1.577686	1.852003	8.287879	8.787323	9.919257	14.134923	4.10
	FF	ABAQUS	8.8293	10.2375	15.2106	21.3650	23.9618	28.0351	
		TE_{445}	8.829572	10.237533	15.217165	21.366950	23.962057	28.035155	0.01
		EX_{445}	9.234689	10.704242	16.532178	22.257297	24.946436	28.035652	4.35
20	CC	ABAQUS	9.6804	11.4133	26.2938	30.8000	50.5671	58.7740	
		TE_{445}	9.714835	11.439358	26.375411	30.859164	50.741338	58.891421	0.27
		EX_{445}	10.158474	11.933916	27.551391	32.165846	52.949586	61.299358	4.62
	CF	ABAQUS	1.5329	1.8166	9.5280	11.2537	26.3386	30.4684	
		TE_{445}	1.537508	1.819400	9.555646	11.270017	26.414608	30.548726	0.24
		EX_{445}	1.609230	1.903032	9.996413	11.780976	27.612969	32.370363	5.07
	FF	ABAQUS	9.6765	11.4541	26.3598	31.0688	50.8304	59.5497	
		TE_{445}	9.676732	11.454373	26.360524	31.068917	50.832015	59.549828	0.00
		EX_{445}	10.143466	12.007069	27.614719	32.544195	53.206763	62.318282	4.75

† The total number of DOFs used in the ABAQUS model is 358347.

Table 9: First six dimensionless frequency parameters of an asymmetric FG sandwich beam $2-2-1$ with volume fraction index $p=1$. Comparison with 3D FEM.

l/h	BCs	Theory	Dimensionless frequency parameters $\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$	$\hat{\omega}_4$	$\hat{\omega}_5$	$\hat{\omega}_6$	Ave. $\Delta(\%)$
5	CC	ABAQUS†	8.2595	9.2719	15.0348	19.6331	21.1823	27.8701	
		TE_{445}	8.283298	9.279995	15.107901	19.700357	21.194328	27.893942	0.22
		EX_{445}	8.538180	9.545648	16.212538	20.235428	21.749757	27.896590	3.33
	CF	ABAQUS	1.4667	1.7455	7.4688	8.2212	9.4030	13.8820	
		TE_{445}	1.468755	1.746743	7.490767	8.235801	9.407783	13.889169	0.13
		EX_{445}	1.530923	1.819622	8.095927	8.547320	9.748501	13.889893	4.96
	FF	ABAQUS	8.5685	10.0558	14.8560	20.7790	23.5237	27.5381	
		TE_{445}	8.568678	10.055759	14.863664	20.780463	23.524025	27.537951	0.01
		EX_{445}	9.211065	10.488282	16.046191	22.102757	24.434405	27.467533	5.05
20	CC	ABAQUS	9.3912	11.2129	25.5174	30.2578	49.0983	57.7068	
		TE_{445}	9.424289	11.238615	25.597271	30.315791	49.268977	57.824143	0.27
		EX_{445}	9.855336	11.725406	26.741207	31.602433	51.420583	60.211337	4.63
	CF	ABAQUS	1.4866	1.7848	9.2411	11.0557	25.5543	29.7611	
		TE_{445}	1.489742	1.787310	9.260687	11.070813	25.607445	29.839319	0.20
		EX_{445}	1.561058	1.870424	9.696829	11.575651	26.793522	31.801733	5.19
	FF	ABAQUS	9.3841	11.2537	25.5692	30.5234	49.3215	58.4998	
		TE_{445}	9.384160	11.253879	25.569567	30.523449	49.322969	58.499628	0.00
		EX_{445}	9.837275	11.796980	26.787948	31.973639	51.631991	61.222450	4.75

† The total number of DOFs used in the ABAQUS model is 345255.

Table 10: First three dimensionless frequency parameters of a symmetric $(1-2-1)$ and asymmetric $(2-2-1)$ FG sandwich beam with CC boundary condition.

			Dimensionless frequency parameters					
Sandwich			$l/h = 5$			$l/h = 20$		
type	Theory	p	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$
$1-2-1$	TE_{445}	0.2	9.611444	9.932134	16.742163	11.374419	12.010230	30.761772
		0.5	9.101739	9.708001	16.154702	10.589267	11.743069	28.692985
		1.0	8.502928	9.452756	15.417823	9.714835	11.439358	26.375411
		2.0	7.844135	9.168147	14.551557	8.800275	11.100892	23.938090
		5.0	7.192028	8.855491	13.633579	7.936335	10.728514	21.624176
	EX_{445}	0.2	9.888064	10.205129	18.026249	11.892135	12.527171	32.118610
		0.5	9.371577	9.978764	17.349741	11.072142	12.249526	29.965623
		1.0	8.762584	9.721273	16.579740	10.158474	11.933916	27.551391
		2.0	8.089203	9.433951	15.743282	9.202427	11.582041	25.009727
		5.0	7.417197	9.117268	14.889687	8.298771	11.194505	22.593325
$2-2-1$	TE_{445}	0.2	9.535148	9.881461	16.673606	11.262330	11.947957	30.464998
		0.5	9.182667	9.587173	15.954856	10.396983	11.619158	28.180804
		1.0	8.283298	9.279995	15.107901	9.424289	11.238615	25.597271
		2.0	7.720137	8.880020	14.054597	8.413547	10.805484	22.895705
		5.0	6.980471	8.462504	12.967456	7.506863	10.317148	20.452471
	EX_{445}	0.2	9.810606	10.153796	17.903887	11.775615	12.463052	31.810970
		0.5	9.230259	9.872961	17.118317	10.871749	12.121132	29.433098
		1.0	8.538180	9.545648	16.212538	9.855336	11.725406	26.741207
		2.0	7.773671	9.169704	15.224238	8.798952	11.274800	23.924439
		5.0	7.034262	8.742782	14.230518	7.851995	10.766241	21.378970

Table 11: First three dimensionless frequency parameters of a symmetric (1−2−1) and asymmetric (2−2−1) FG sandwich beam with CF boundary condition.

Sandwich type	Theory	p	Dimensionless frequency parameters					
			$l/h=5$			$l/h=20$		
			$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$
$1-2-1$	TE_{445}	0.2	1.790036	1.871862	8.387679	1.804199	1.909686	11.190657
		0.5	1.655889	1.826922	8.058525	1.677109	1.867307	10.412741
		1.0	1.514911	1.778145	7.664768	1.537508	1.819400	9.555646
		2.0	1.372386	1.725113	7.216210	1.389475	1.765428	8.644127
		5.0	1.238627	1.667093	6.744659	1.251379	1.706308	7.791585
	EX_{445}	0.2	1.843521	1.944073	9.010867	1.890062	1.997634	11.715315
		0.5	1.717999	1.900985	8.672586	1.756882	1.953360	10.901588
		1.0	1.577686	1.852003	8.287879	1.609230	1.903032	9.996413
		2.0	1.430441	1.797392	7.870513	1.455457	1.846921	9.050697
		5.0	1.290928	1.737245	7.356606	1.310749	1.785124	8.158037
$2-2-1$	TE_{445}	0.2	1.751840	1.857246	8.290149	1.786118	1.899799	11.079689
		0.5	1.618790	1.806050	7.939761	1.646229	1.847647	10.222604
		1.0	1.468755	1.746743	7.490767	1.489742	1.787310	9.260687
		2.0	1.312379	1.679204	6.955493	1.327917	1.718648	8.262676
		5.0	1.171623	1.603067	6.384684	1.183659	1.641212	7.369283
	EX_{445}	0.2	1.825649	1.934143	8.948595	1.871699	1.988011	11.600077
		0.5	1.687168	1.881066	8.553310	1.725074	1.933495	10.703421
		1.0	1.530923	1.819622	8.095927	1.561058	1.870424	9.696829
		2.0	1.368051	1.749639	7.594841	1.391488	1.798636	8.652372
		5.0	1.221637	1.670641	6.962036	1.240389	1.717639	7.717892

Table 12: First three dimensionless frequency parameters of a symmetric $(1-2-1)$ and asymmetric $(2-2-1)$ FG sandwich beam with FF boundary condition.

Sandwich type	Theory	p	$l/h=5$ $\hat{\omega}_1$	$l/h=5$ $\hat{\omega}_2$	$l/h=5$ $\hat{\omega}_3$	$l/h=20$ $\hat{\omega}_1$	$l/h=20$ $\hat{\omega}_2$	$l/h=20$ $\hat{\omega}_3$
$1-2-1$	TE_{445}	0.2	10.237512	10.753624	16.585995	11.356939	12.024400	30.859968
		0.5	9.576780	10.512177	15.984356	10.560621	11.757620	28.731764
		1.0	8.829572	10.237533	15.217165	9.676732	11.454373	26.360524
		2.0	8.036172	9.931291	14.306722	8.755087	11.116320	23.879048
		5.0	7.275789	9.594324	13.341578	7.886738	10.744085	21.533303
	EX_{445}	0.2	10.698760	11.236537	17.979316	11.904151	12.603960	32.320462
		0.5	10.012092	10.987281	17.302475	11.069725	12.324613	30.095060
		1.0	9.234689	10.704242	16.532178	10.143466	12.007069	27.614719
		2.0	8.408437	10.387815	15.695594	9.177591	11.653037	25.018227
		5.0	7.615482	10.039037	14.842292	8.267525	11.263125	22.562689
$2-2-1$	TE_{445}	0.2	10.141094	10.697460	16.488107	11.243447	11.962119	30.555603
		0.5	9.407625	10.400296	15.783103	10.366489	11.633769	28.208921
		1.0	8.568678	10.055759	14.863664	9.384160	11.253879	25.569567
		2.0	7.681944	9.663135	13.782544	8.366925	10.821504	22.825110
		5.0	6.869315	9.220381	12.625478	7.458665	10.333752	20.361780
	EX_{445}	0.2	10.598767	11.178701	17.855573	11.785322	12.538708	32.002487
		0.5	9.836164	10.871398	17.065925	10.866534	12.194840	29.548758
		1.0	9.211065	10.488282	16.046191	9.837275	11.796980	26.787948
		2.0	8.250725	10.074927	15.015457	8.771501	11.344103	23.916423
		5.0	7.385332	9.621347	14.001668	7.820231	10.833099	21.339958

Table 13: Dimensionless fundamental frequency parameters of symmetric and asymmetric Porous FG sandwich beams.

Porosity type	β	BCs	p	Symmetric FG sandwich beams $1-0-1$	$1-1-1$	$1-2-1$	Asymmetric FG sandwich beams $2-1-1$	$2-2-1$	$2-3-1$
	0.0	CC	0.2	9.220722	9.466380	9.611444	9.414832	9.535148	9.622962
			1.0	7.465648	8.083267	8.502928	8.223543	8.490899	8.701575
			5.0	5.716416	6.466899	7.192028	6.281963	6.980471	7.249163
		CF	0.2	1.683317	1.735055	1.769130	1.725413	1.751840	1.772106
			1.0	1.311662	1.424760	1.514911	1.404457	1.517348	1.521886
			5.0	1.005918	1.104784	1.238627	1.138115	1.209140	1.282528
		FF	0.2	9.759571	10.050191	10.237512	9.993932	10.141094	10.252750
			1.0	7.664526	8.326181	8.829572	8.503911	8.568678	9.061124
			5.0	5.866260	6.494733	7.275789	6.374405	7.385332	7.342863
I	0.2	CC	0.2	9.349365	9.408477	9.508302	9.413544	9.472611	9.533705
			1.0	7.052799	7.677594	8.174487	7.802001	8.109652	8.366489
			5.0	4.269027	5.412732	6.481925	5.078370	5.883946	6.529032
		CF	0.2	1.694048	1.704908	1.730584	1.708836	1.722401	1.737766
			1.0	1.219611	1.332333	1.436074	1.367249	1.426048	1.479881
			5.0	0.728802	0.900553	1.097837	0.853795	0.987363	1.107257
		FF	0.2	9.876468	9.962117	10.102191	9.971596	10.052206	10.136295
			1.0	7.180869	7.865712	8.455948	8.006816	8.360994	8.671972
			5.0	4.289907	5.358420	6.512623	5.047922	5.852698	6.555031
II	0.2	CC	0.2	9.426713	9.506588	9.616303	9.512501	9.574541	9.637425
			1.0	7.545103	8.049700	8.466983	8.216262	8.459133	8.666846
			5.0	5.553446	6.302483	7.092303	6.076505	6.656104	7.140534
		CF	0.2	1.724546	1.738989	1.766080	1.743121	1.756569	1.771712
			1.0	1.326710	1.414922	1.503429	1.462610	1.460479	1.550985
			5.0	0.984283	1.073333	1.217595	1.055892	1.141978	1.229492
		FF	0.2	9.984572	10.087064	10.235510	10.098641	10.179398	10.263525
			1.0	–	8.283602	8.784558	8.193779	8.533347	8.830408
			5.0	5.723692	6.321464	7.164648	6.172524	6.705887	7.220656

Table 14: Dimensionless fundamental parameter of symmetric and asymmetric Porous FG sandwich beams with $l/h = 5$, $\beta = 0.2$, $p = 1.0$ and CC boundary condition.

Porosity type	$\mathcal{K}^{w-}$	$\mathcal{K}^{p-}$	Symmetric FG sandwich beams 1 − 0 − 1	1 − 1 − 1	1 − 2 − 1	Asymmetric FG sandwich beams 2 − 1 − 1	2 − 2 − 1	2 − 3 − 1
	0.0	0.0	7.465648	8.083267	8.502928	8.223543	8.490899	8.701575
	0.5	0.0	8.527904	9.174725	9.457347	9.013917	9.282618	9.451037
	0.5	0.5	8.575297	9.226232	9.512184	9.067258	9.337178	9.506622
I	0.0	0.0	7.052799	7.677594	8.174487	7.802001	8.109652	8.366489
	0.5	0.0	8.543901	9.278264	9.556342	9.110231	9.391974	9.550970
	0.5	0.5	8.571357	9.311705	9.596150	9.150732	9.431501	9.596831
II	0.0	0.0	7.545103	8.049700	8.466983	8.216262	8.459133	8.666846
	0.5	0.0	8.538178	9.208507	9.481457	9.054144	9.317264	9.478539
	0.5	0.5	8.577565	9.254624	9.532430	9.104339	9.369458	9.532300

Table 15: Dimensionless critical buckling parameter of symmetric Porous FG sandwich beams with $l/h = 20$ and CC boundary condition.

Porosity type	β	p	Symmetric FG sandwich beams 1 − 0 − 1	1 − 1 − 1	1 − 2 − 1	2 − 1 − 2	1 − 5 − 1	5 − 1 − 5
	0.0	0.5	114.862927	132.916027	145.986463	124.588541	168.124159	118.929055
		1.0	81.778616	102.611435	119.445654	92.553668	149.764754	86.135133
		5.0	42.063228	56.580551	75.036652	48.021571	115.738861	44.006783
I	0.1	0.5	102.279809	120.850306	135.069265	112.127479	160.243614	106.362442
		1.0	69.207493	90.534981	108.516541	80.088039	141.883189	73.572108
		5.0	29.539778	44.493211	64.081594	35.565584	107.849463	31.471583
	0.2	0.5	89.748161	108.868184	124.269947	99.724621	152.502081	93.846637
		1.0	56.674573	78.530081	97.688399	67.669411	134.124268	61.047504
		5.0	17.033852	32.447588	53.183603	23.137502	100.035863	18.956125
II	0.1	0.5	111.696599	130.128616	143.840152	121.525204	167.116201	115.779512
		1.0	78.625237	99.821466	117.296100	89.491471	148.756269	82.992394
		5.0	38.921455	53.790649	72.886297	44.961824	114.732380	40.870912
	0.2	0.5	108.543563	127.364174	141.717889	118.478933	166.117690	112.644350
		1.0	75.483569	97.052907	115.168809	86.444585	147.756396	79.862131
		5.0	35.784480	51.016929	70.752194	41.912320	113.732147	37.741271

Table 16: Dimensionless fundamental parameter of symmetric Porous FG sandwich beams with $l/h = 20$, $\beta = 0.2$, $p = 1.0$ CC boundary condition and resting on the Winkler-Pasternak elastic foundation.

Porosity type	$\mathcal{K}^{w-}$	$\mathcal{K}^{p-}$	Symmetric FG sandwich beams					
			$1-0-1$	$1-1-1$	$1-2-1$	$2-1-2$	$1-5-1$	$5-1-5$
	0.0	0.0	81.778616	102.611435	119.445654	92.553668	149.764754	86.135133
	0.5	0.0	123.550479	151.995321	166.237988	140.611894	184.577872	131.303931
	0.5	0.5	123.911761	152.457489	166.764392	141.029861	185.201826	131.689797
I	0.0	0.0	56.674573	78.530081	97.688399	67.669411	134.124268	61.047504
	0.5	0.0	98.620313	135.532165	153.859425	120.872846	177.542714	108.803253
	0.5	0.5	98.788872	135.811339	154.226181	121.101291	178.071449	108.997812
II	0.0	0.0	75.483569	97.052907	115.168809	86.444585	147.756396	79.862131
	0.5	0.0	111.102573	146.544825	163.152904	132.767726	183.561687	121.121840
	0.5	0.5	111.369805	146.947678	163.638027	133.113889	184.169834	121.425061