Completely regular codes with different parameters giving the same distance-regular coset graphs

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Abstract

We construct several classes of completely regular codes with different parameters, but identical intersection array. Given a prime power q and any two natural numbers a, b, we construct completely transitive codes over different fields with covering radius $\rho = \min\{a, b\}$ and identical intersection array, specifically, one code over $\mathbb{F}_{q^r}$ for each divisor r of a or b. As a corollary, for any prime power q, we show that distance regular bilinear forms graphs can be obtained as coset graphs from several completely regular codes with different parameters.

Keywords: bilinear forms graph, completely regular code, completely transitive code, coset graph, distance-regular graph, distance-transitive graph, Kronecker product construction, lifting of a field, uniformly packed code

1. Introduction

Let $\mathbb{F}_q$ be a finite field of the order q and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. A q-ary linear code C of length n is a k-dimensional subspace of $\mathbb{F}_q^n$. Given any vector $\mathbf{v} \in \mathbb{F}_q^n$, its *distance to the code* C is $d(\mathbf{v}, C) = \min_{\mathbf{x}\in C}\{d(\mathbf{v}, \mathbf{x})\}$, the *minimum distance* of the code is $d = \min_{\mathbf{v}\in C}\{d(\mathbf{v}, C\setminus\{\mathbf{v}\})\}$ and the *covering radius* of the code C is $\rho(C) = \max_{\mathbf{v}\in\mathbb{F}_q^n}\{d(\mathbf{v}, C)\}$. We say that C is a $[n, k, d; \rho]_q$ code. The *Hamming weight* $\mathrm{wt}(\mathbf{v})$ of a vector $\mathbf{v} \in \mathbb{F}_q^n\mathrm{wt}()$ is the number of its nonzero entries, i.e. $\mathrm{wt}(\mathbf{v}) = d(\mathbf{v}, 0)$. Let $D = C + \mathbf{x}$ be a *coset* of C, where $+$ means the component-wise addition in $\mathbb{F}_q$. The *weight* $\mathrm{wt}(D)$ of D is the minimum weight of the vectors in D. The *weight distribution* of D is the the $(n+1)$-tuple $(w_0, w_1, \ldots, w_n)$ of nonnegative integers, where w_i is the number of codewords of D of weight i

For a given q-ary code C with covering radius $\rho = \rho(C)$ define

$$C(i) = \{\mathbf{x} \in \mathbb{F}_q^n : d(\mathbf{x}, C) = i\}, \quad i = 1, 2, ..., \rho.$$

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Say that two vectors $\mathbf{x}$ and $\mathbf{y}$ are *neighbors* if $d(\mathbf{x},\mathbf{y}) = 1$. For two vectors $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n)$ over $\mathbb{F}_q$ denote by $\langle \mathbf{x},\mathbf{y}\rangle$ their inner product over $\mathbb{F}_q$, i.e.

$$\langle \mathbf{x},\mathbf{y}\rangle = x_1y_1 + \ldots + x_ny_n.$$

The linear code $C^\perp = \{\mathbf{v} \mid \langle \mathbf{v},\mathbf{x}\rangle = 0 \text{ for all } \mathbf{x} \in C\}$ is the *dual* code of C. Let $s(C)$ be the *outer distance* of C, i.e. the number of different nonzero weights of codewords in the dual code $C^\perp$.

Definition 1.1. [1] *A q-ary code C with covering radius ρ is called* completely regular *if the weight distribution of any coset D of C is uniquely defined by the minimum weight of D.*

An equivalent definition of completely regular codes is due to Neumaier [2].

Definition 1.2. [2] *A q-ary code C is completely regular, if for all $l \geq 0$ every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$. Define $a_l = (q-1)n - b_l - c_l$ and set $c_0 = b_\rho = 0$. Denote by $\{b_0,\dots,b_{\rho-1}; c_1,\dots,c_\rho\}$ the intersection array of C.*

Let M be a monomial matrix, i.e. a matrix with exactly one nonzero entry in each row and column. If q is a prime, then $\mathrm{Aut}(C)$ consist of all monomial $(n\times n)$-matrices M over $\mathbb{F}_q$ such that $\mathbf{c}M \in C$ for all $\mathbf{c} \in C$. If q is a power of a prime number, then $\mathrm{Aut}(C)$ also contains any field automorphism of $\mathbb{F}_q$ (which can be seen as maps of $\mathbb{F}_q^n$ into itself by acting on each of the coordinates) preserving C. The group $\mathrm{Aut}(C)$ acts on the set of cosets of C in the following way: for all $\sigma \in \mathrm{Aut}(C)$ and for every vector $\mathbf{v} \in \mathbb{F}_q^n$ we have $(\mathbf{v}+C)^\sigma = \mathbf{v}^\sigma + C$.

Definition 1.3. [3, 4] *Let C be a linear code over $\mathbb{F}_q$ with covering radius ρ. Then C is completely transitive if* $\mathrm{Aut}(C)$ *has $\rho+1$ orbits in its action on the cosets of C.*

Since two cosets in the same orbit should have the same weight distribution, it is clear, that any completely transitive code is completely regular.

Definition 1.4. [5] *Let C be a q-ary code of length n and let ρ be its covering radius. We say that C is* uniformly packed *in the wide sense if there exist rational numbers $\alpha_0, \ldots, \alpha_\rho$ such that for any $\mathbf{v} \in \mathbb{F}_q^n$*

$$\sum_{k=0}^{\rho} \alpha_k\, f_k(\mathbf{v}) \;=\; 1, \tag{1}$$

where $f_k(\mathbf{v})$ is the number of codewords at distance k from $\mathbf{v}$.

Completely regular and completely transitive codes are classical subjects in algebraic coding theory, which are closely connected with graph theory, combinatorial designs and algebraic combinatorics. Existence, construction and enumeration of all such codes are open hard problems (see [6, 7, 8, 2] and references there).

In this paper we extend our previous construction [9] connecting it with [10] and, as a result, we obtain, for any prime power q, several different infinite classes of completely regular codes with different parameters n, k, q and with identical intersection arrays. This gives different presentations, as coset graphs, of distance-regular bilinear form graphs.

Under the same conditions, an explicit construction of an infinite family of q-ary uniformly packed codes (in the wide sense) with covering radius ρ, which are not completely regular, is also given.

2. Preliminary results

Lemma 2.1. [2] *Let C be a linear completely regular code with intersection array $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$, and let μ_i be the number of cosets of C of weight i. Then*

$$\mu_{i-1} b_{i-1} = \mu_i c_i.$$

Definition 2.2. *For two matrices $A = [a_{r,s}]$ and $B = [b_{i,j}]$ over $\mathbb{F}_q$ define a new matrix H which is the Kronecker product $H = A \otimes B$, where H is obtained by replacing any element $a_{r,s}$ in A by the matrix $a_{r,s}B$.*

Consider the matrix $H = A \otimes B$ and let C, C_A and C_B be the codes over $\mathbb{F}_q$ which have, respectively, H, A and B as parity check matrices. Assume that A and B have size $m_a \times n_a$ and $m_b \times n_b$, respectively. Clearly, the codewords in code C are presented as matrices $[\mathbf{c}]$ of size $n_b \times n_a$:

$$[\mathbf{c}] = \begin{bmatrix} c_{1,1} & \ldots & c_{1,n_a} \\ c_{2,1} & \ldots & c_{2,n_a} \\ \vdots & \vdots & \vdots \\ c_{n_b,1} & \ldots & c_{n_b,n_a} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_{n_b} \end{bmatrix} = \left[\mathbf{c}^{(1)}\, \mathbf{c}^{(2)} \, \ldots \, \mathbf{c}^{(n_a)} \right], \qquad (2)$$

where $c_{i,j} = a_{r,j} b_{s,i}$, $\mathbf{c}_r$ is the rth row vector of the matrix C and $\mathbf{c}^{(\ell)}$ is its ℓth column.

The following result was obtained in [9].

Theorem 2.3. *Let $C(H)$ be the $[n, k, d; \rho]_q$ code with parity check matrix $H = A \otimes B$ where A and B are parity check matrices of Hamming $[n_a, k_a, 3]_q$ and $[n_b, k_b, 3]_q$ codes, C_A and C_B, respectively, where $n_a = (q^{m_a} - 1)/(q-1) \geq 3$, $n_b = (q^{m_b} - 1)/(q-1) \geq 3$, $k_a = n_a - m_a$, $k_b = n_b - m_b$ and where*

$$n = n_a n_b, \;\; k = n - m_a m_b, \;\; d = 3, \;\; \rho = \min\{m_a, m_b\}.$$

Then the code C is completely transitive and, therefore, completely regular with covering radius $\rho = \min\{m_a, m_b\}$ and intersection numbers

$$\begin{aligned} b_\ell =& \frac{(q^{m_a} - q^\ell)(q^{m_b} - q^\ell)}{(q-1)}, \;\; \ell = 0, \ldots, \rho - 1, \\ c_\ell =& q^{\ell-1} \cdot \frac{q^\ell - 1}{q-1}, \;\; \ell = 1, \ldots, \rho. \end{aligned} \qquad (3)$$

Definition 2.4. *Let C be the $[n,k,d;\rho]_q$ code with parity check matrix H where $1 \leq k \leq n-1$ and $d \geq 3$. Denote by C_r the $[n,k,d]_{q^r}$ code over $\mathbb{F}_{q^r}$ with the same parity check matrix H. Say that code C_r is obtained by lifting C to $\mathbb{F}_{q^r}$.*

In [10] we proved the following result

Theorem 2.5. *Let $C_r(H^q_m)$ be the $[n,n-m,3;\rho]_{q^r}$ code of length $n=(q^m-1)/(q-1)$ over the field $\mathbb{F}_{q^r}$ obtained by lifting a q-ary perfect $[n,n-m,3]_q$ code $C(H^q_m)$ with parity check matrix H^q_m. Then, the code $C_r(H^q_m)$ is completely regular with covering radius $\rho = \min\{m,r\}$ and intersection numbers given by (3) taking $m_a = m$ and $m_b = r$.*

3. Extending the Kronecker product construction

Recall that by $C(H)$ we denote the code defined by the parity check matrix H, by H^q_m we denote the parity check matrix of the q-ary Hamming $[n,n-m,3;1]_q$ code $C = C(H^q_m)$ of length $n=(q^m-1)/(q-1)$, and by $C_r(H^q_m)$ we denote the code (of the same length $n=(q^m-1)/(q-1)$) obtained by lifting $C(H^q_m)$ to the field $\mathbb{F}_{q^r}$.

Considering the above Kronecker construction (Theorem 2.3) we could see that the alphabets of both matrices $A=[a_{i,j}]$ and B should be compatible to each other in the sense that the multiplication $a_{i,j}B$ can be carried out. To have this compatibility it is enough that, say, the matrix A is over $\mathbb{F}_{q^u}$ and B is over $\mathbb{F}_q$. First, we consider the covering radius of the resulting codes.

Lemma 3.1. *Let $C(H^{q^u}_{m_a})$ and $C(H^q_{m_b})$ be two Hamming codes with parameters $[n_a, n_a-m_a, 3]_{q^u}$ and $[n_b, n_b-m_b, 3]_q$, respectively, where $n_a = (q^{u\,m_a}-1)/(q^u-1)$, $n_b = (q^{m_b}-1)/(q-1)$, q is a prime power, $m_a, m_b \geq 2$, and $u \geq 1$. Then the code C with parity check matrix $H = H^{q^u}_{m_a} \otimes H^q_{m_b}$, the Kronecker product of $H^{q^u}_{m_a}$ and $H^q_{m_b}$, has covering radius $\rho = \min\{u\,m_a,\, m_b\}$.*

Proof. Assume that the matrices H, $H^{q^u}_{m_a}$, and $H^q_{m_b}$ have columns $\mathbf{h}_i$, $\mathbf{a}_j$, and $\mathbf{b}_s$, respectively, i.e.

$$H = [\mathbf{h}_1|\cdots|\mathbf{h}_n],\;\; H^{q^u}_{m_a} = [\mathbf{a}_1|\cdots|\mathbf{a}_{n_a}],\;\; H^q_{m_b} = [\mathbf{b}_1|\cdots|\mathbf{b}_{n_b}].$$

We have to prove that any column vector $\mathbf{x} \in (\mathbb{F}_{q^u})^{m_a m_b}$ can be presented as a linear combination of not more than ρ columns of H.

By construction the column $\mathbf{h}_i$ is

$$\mathbf{h}_i^T = [a_{1,j}\mathbf{b}_s^T, a_{2,j}\mathbf{b}_s^T, \ldots, a_{m_a,j}\mathbf{b}_s^T],$$

where $i=1,\ldots,n$, $n=n_a n_b$, $j=1,\ldots,n_a$ $s=1,\ldots,n_b$ and $(\cdot)^T$ means transposition. By definition, the matrix $H^q_{m_b}$ contains as column vectors any vector $\mathbf{y} \in (\mathbb{F}_q)^{m_b}$ over the ground field $\mathbb{F}_q$, up to multiplication by scalars of $\mathbb{F}_q$. Vectors $\mathbf{x}$ are arbitrary vectors over the extended field $\mathbb{F}_{q^u}$ and each one can be presented as a linear combination of u or less vectors from $H^q_{m_b}$. Hence for any choice of $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_{m_a})$ we can always take not more than $u\,m_a$ columns of H to have m_a equalities of the type

$$\mathbf{x}_i = \sum_{s=1}^{u} \alpha_s \mathbf{b}_{i_s}, \quad i = 1, \ldots, m_a,\ \alpha_i \in \mathbb{F}_{q^u},$$

implying that

$$\rho \leq u\,m_a.$$

From the other side, by permuting the rows of H, the column $\mathbf{h}_i$ can be presented (with other index, say, i') as follows:

$$\mathbf{h}_{i'}^T = [b_{1,s}\mathbf{a}_j, b_{2,s}\mathbf{a}_j, \ldots, b_{m_b,s}\mathbf{a}_j].$$

and with the same argumentation as before we obtain

$$\rho \leq m_b.$$

Since in both cases the bounds can be reached by appropriate choices of vector $\mathbf{x}$, we obtain the result. □

We also give several simple facts from [9, 10], which will be used in the proof of the forthcoming theorem. As we said before (2), any codeword $\mathbf{c} \in C(H)$, where $H = A \otimes B$, can be seen as a $(n_b \times n_a)$-matrix $[\mathbf{c}]$.

For any vector $\mathbf{v}$ of length $n = n_b \times n_a$ presented as a $(n_b \times n_a)$-matrix $[\mathbf{v}]$ define its *syndrome* which, in a matrix representation, is a $(m_b \times m_a)$ matrix

$$S_{\mathbf{v}} = [(A \otimes B)\mathbf{v}^T] = B[\,\mathbf{v}\,]A^T.$$

If we compute the syndrome of a codeword $[\mathbf{c}]$ of the code $C(H)$ we obtain $S_{\mathbf{c}} = [(A \otimes B)\mathbf{c}^T] = B[\,\mathbf{c}\,]A^T = [0]$.

Fix a $1-1$ mapping ν from $\mathbb{F}_{q^u}$ to $(\mathbb{F}_q)^u$ writing for any element $a \in \mathbb{F}_{q^u}$ its $\mathbb{F}_q$-presentation $\nu(a)$:

$$\nu(a) = [a_0, a_1, \ldots, a_{u-1}] \quad \longleftrightarrow \quad a = \sum_{i=0}^{u-1} a_i \nu_i,$$

where $\nu_0, \nu_1, \ldots, \nu_{u-1}$ is a fixed basis in $\mathbb{F}_{q^u}$ over $\mathbb{F}_q$. Finally, extend the map ν to vectors $\mathbf{v} = (v_1, \ldots, v_n) \in (\mathbb{F}_{q^u})^n$ by the obvious way: $\nu(\mathbf{v}) = [\nu(v_1)\,|\,\cdots\,|\,\nu(v_n)]$.

Definition 3.2. *Given a vector $\mathbf{v} \in (\mathbb{F}_{q^u})^n$, with syndrome $S_{\mathbf{v}}$, which is a $(m_b \times m_a)$ matrix over $\mathbb{F}_{q^u}$ denote by $\nu_{\mathbf{v}}$ the $(m_b \times (u m_a))$ matrix obtained from $S_{\mathbf{v}}$ using the map ν in its rows.*

We need the following results.

Lemma 3.3. *Let* $\mathbf{x}, \mathbf{y} \in (\mathbb{F}_{q^u})^n$ *be two vectors with syndromes* $S_\mathbf{x}$ *and* $S_\mathbf{y}$ *and corresponding matrices* $\nu_\mathbf{x}$ *and* $\nu_\mathbf{y}$*, respectively. Then, for any non singular* $m_b \times m_b$ *matrix* K *over* $\mathbb{F}_q$*, the two equalities*

$$(\nu_\mathbf{x})^T K = \nu_\mathbf{y}^T \text{ and } (S_\mathbf{x})^T K = S_\mathbf{y}^T$$

are valid (or do not valid), simultaneously.

Proof. Straightforward.

Lemma 3.4. *If* $\mathbf{v} \in C(\ell)$*, then* $\mathrm{rank}(\nu_\mathbf{v}) = \ell$*.*

Proof. The proof is similar to the case of codes over the same alphabet [9].

Lemma 3.5. *Let* $C = C(H)$ *be a code with parity check matrix* $H = H^{q^u}_{m_a} \otimes H^q_{m_b}$*. Let* μ_i *be the number of cosets in* $C(i)$*. Then*

$$\mu_i \;=\; \frac{Q_i(q^{m_b})Q_i(q^{um_a})}{Q_i(q^i)},$$

where $Q_i(q^j) = (q^j - 1)(q^j - q)\cdots(q^j - q^{i-1})$*.*

Proof. From Lemma 3.4 for all vectors $\mathbf{v} \in C(\ell)$ we have $\mathrm{rank}(\nu_\mathbf{v}) = \ell$. All vectors in the same coset $C + \mathbf{v}$ have the same syndrome $\nu_\mathbf{v}$. Therefore, the number of different cosets in $C(i)$ is equal to the number of different syndromes and so the number of different $(m_b \times (um_a))$ matrices over $\mathbb{F}_q$ of rank i. This number is well known [11] and gives the statement.

The following theorem generalizes the results of [9, 10].

Theorem 3.6. *Let* $C(H^{q^u}_{m_a})$ *and* $C(H^q_{m_b})$ *be two Hamming codes with parameters* $[n_a, n_a - m_a, 3]_{q^u}$ *and* $[n_b, n_b - m_b, 3]_q$*, respectively, where* $n_a = (q^{u\,m_a} - 1)/(q^u - 1)$*,* $n_b = (q^{m_b} - 1)/(q - 1)$*,* q *is a prime power,* $m_a, m_b \geq 2$*, and* $u \geq 1$*.*

(i) *The code* C *with parity check matrix* $H = H^{q^u}_{m_a} \otimes H^q_{m_b}$*, the Kronecker product of* $H^{q^u}_{m_a}$ *and* $H^q_{m_b}$*, is a completely transitive, and so completely regular,* $[n, k, d; \rho]_{q^u}$ *code with parameters*

$$n = n_a\, n_b, \quad k = n - m_a\, m_b, \quad d = 3, \quad \rho = \min\{u\, m_a,\ m_b\}. \qquad (4)$$

(ii) *The code* C *has the intersection numbers:*

$$b_\ell \;=\; \frac{(q^{u\,m_a} - q^\ell)(q^{m_b} - q^\ell)}{(q-1)}, \quad \ell = 0, 1, \ldots, \rho - 1,$$

and

$$c_\ell \;=\; q^{\ell-1}\frac{q^\ell - 1}{q - 1}, \quad \ell = 1, 2, \ldots, \rho.$$

(iii) *The lifted code* $C_{m_b}(H^q_{um_a})$ *is a completely regular code with the same intersection array as* C.

Proof. The proof follows from similar arguments which we used in the two previous papers [9, 10].

First, from Lemma 3.1, we have that $\rho = \min\{m_b, u\,m_a\}$.

The next step is to prove that the code $C = C(A \otimes B)$ is completely transitive. Here we use Lemma 3.3 in order to guaranty the existence of the invertible $m_b \times m_b$ matrix K over $\mathbb{F}_q$ such that the equality $S_x^T K = S_y^T$ holds where $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ are the syndromes of vectors $\mathbf{x}$ and $\mathbf{y}$ over $\mathbb{F}_{q^u}$.

Denote by C_A and C_B the codes over $\mathbb{F}_{q^u}$, with parity check matrices A and B, respectively.

To prove that C is a completely transitive code it is enough to show that starting from two vectors $\mathbf{x}, \mathbf{y} \in C(\ell),\;\; 1 \leq \ell \leq \rho$, there exists a monomial matrix $\varphi \in \mathrm{Aut}(C)$ such that $\mathbf{x}\varphi \in \mathbf{y} + C$ or, in terms of syndromes,

$$S_{\mathbf{x}\varphi} = [(A \otimes B)(\mathbf{x}\varphi)^T] = [(A \otimes B)(\mathbf{y})^T] = S_{\mathbf{y}}.$$

Let ϕ_1 be any monomial $(n_a \times n_a)$ matrix and ϕ_2 be any monomial $(n_b \times n_b)$ matrix. It is well known [12] that

$$(A\phi_1) \otimes (B\phi_2) = (A \otimes B)(\phi_1 \otimes \phi_2)$$

and $\phi_1 \otimes \phi_2$ is a monomial $(n_a n_b \times n_a n_b)$ matrix.

Note that if $\varphi \in \mathrm{Aut}(C)$ then $H\varphi^T$ is a parity check matrix for C when H is. Therefore, taking the specific case where $\phi_1^T \in \mathrm{Aut}(C_A)$ and $\phi_2^T \in \mathrm{Aut}(C_B)$ we conclude that $(\phi_1^T \otimes \phi_2^T)^T \in \mathrm{Aut}(C)$, or the same $\phi_1 \otimes \phi_2 \in \mathrm{Aut}(C)$.

The two given vectors $\mathbf{x}, \mathbf{y}$ belong to $C(\ell)$ and, from Lemma 3.4, $\mathrm{rank}(S_{\mathbf{x}}) = \mathrm{rank}(S_{\mathbf{y}}) = \ell$, where $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ are the syndrome of $\mathbf{x}$ and $\mathbf{y}$, respectively. To prove that C is a completely transitive code we show that there exists a monomial matrix $\phi^T \in \mathrm{Aut}(C_B)$ such that

$$\begin{aligned}(A \otimes B)\mathbf{y}^T &= (A \otimes B\phi)\mathbf{x}^T \\ &= (A \otimes B)((I_{n_a} \otimes \phi)\mathbf{x}^T)\end{aligned}$$

where I_{n_a} is the $n_a \times n_a$ identity matrix.

Since $\ell \leq \rho \leq m_b$, it is straightforward to find an invertible $(m_b \times m_b)$ matrix K over $\mathbb{F}_q$ such that $\nu_x^T\,K = \nu_y^T$. By Lemma 3.3 we conclude that $S_x^T\,K = S_y^T$. Since B is the parity check matrix of a Hamming code, the matrix $K^T\,B$ is again a parity check matrix for a Hamming code and $K^T B = B\phi$ for some monomial matrix ϕ. Moreover, if G_B is the corresponding generator matrix for this Hamming code, i.e. $B\,G_B^T = [0]$, then $(B\phi)G_B^T = (K^T B)G_B^T = [0]$ and so $\phi^T \in \mathrm{Aut}(C_B)$.

Finally, we have

$$\begin{aligned}(A \otimes B)\mathbf{y}^T &= S_{\mathbf{y}} = K^T\,S_{\mathbf{x}} = K^T(B[\mathbf{x}]A^T) \\ &= B\phi[\mathbf{x}]A^T = (A \otimes B\phi)\mathbf{x}^T \\ &= (A \otimes B)((I_{n_a} \otimes \phi)\mathbf{x}^t).\end{aligned}$$

Since the code C is completely transitive we conclude that C is completely regular with the parameters (4). This gives (i).

Now we have to write down the expressions for all intersection numbers. In this case we use the same approach as in [10]. We begin computing b_0, so the number of vectors in $C(1)$ which are at distance one from one given vector in C. Without loss of generality (since C is a linear code) we can fix the zero codeword $\mathbf{0}$ in C and count how many different vectors of weight one there are in $C(1)$. The answer is immediately

$$b_0 \;=\; n\,(q^u - 1) \;=\; \frac{(q^{u\,m_a} - 1)(q^{m_b} - 1)}{(q-1)}.$$

Since the code C has minimum distance $d > 2$, we have $c_1 = 1$.

In general, let $1 \leq i \leq \rho - 1$. Take a $(u\,m_a \times m_b)$-matrix E of rank i, over F_q, and compute the value b_i as the number of different $(u\,m_a \times m_b)$-matrices $\overline{E}$, over F_q, of rank $i+1 \leq \rho$, such that $E - \overline{E}$ has only one nonzero row. This value is well known ([11, 10]):

$$b_i = \frac{(q^{u\,m_a} - q^i)(q^{m_b} - q^i)}{(q-1)}\,.$$

Now, using the expressions for b_{i-1}, μ_i and μ_{i-1} from Lemmas 2.1 and 3.5, we obtain

$$c_i = \frac{\mu_{i-1} b_{i-1}}{\mu_i} = q^{i-1} \cdot \frac{(q^i - 1)}{q-1},$$

i.e., we have (ii).

The last statement (iii) follows directly from Theorem 2.5. □

Remark 3.7. We have to remark here that in the statement (iii) we can not choose the code $C_{m_b}(H^{q^u}_{m_a})$ (instead of $C_{m_b}(H^q_{um_a})$), which seems to be natural. We emphasize that the codes $C_{m_b}(H^q_{um_a})$ and $C_{m_b}(H^{q^u}_{m_a})$ are not only different completely regular codes, but the corresponding coset graphs are distance-regular graphs with different intersection arrays. So, the code $C_{m_b}(H^q_{um_a})$ suits to the codes from (i) in the sense that it has the same intersection array. For example, the code $C_2(H^{2^2}_3)$ induces a distance-regular graph with intersection array $\{315, 240; 1, 20\}$ and the code $C_2(H^2_6)$ gives a distance-regular graph with intersection array $\{189, 124; 1, 6\}$. To reach these results in both cases we use the same Theorem 2.5.

Remark 3.8. The above Theorem 3.6 can not be extended to the more general case when the alphabets $\mathbb{F}_{q^a}$ and $\mathbb{F}_{q^b}$ of component codes C_A and C_B, respectively, neither $\mathbb{F}_{q^a}$ is a subfield of $\mathbb{F}_{q^b}$ or vice versa $\mathbb{F}_{q^b}$ is a subfield of $\mathbb{F}_{q^a}$. We illustrate it by considering the smallest nontrivial example. Take two Hamming codes, the $[5, 3, 3; 1]_4$ code C_A over $\mathbb{F}_{2^2}$ with parity check matrix $H^{2^2}_2$, and the $[9, 7, 3; 1]_8$ code C_B over $\mathbb{F}_{2^3}$ with parity check matrix $H^{2^3}_2$. Then the resulting $[45, 41, 3; 3]_{64}$ code $C = C(H^{2^2}_2 \otimes H^{2^3}_2)$ over $\mathbb{F}_{2^6}$ is not even uniformly packed in the wide sense, since it has the covering radius $\rho = 3$ and the outer distance $s = 7$, which can be checked by considering the parity check matrix of C.

4. Completely regular codes with different parameters, but the same intersection array

In [10, Theo. 2.11] it is proved that by lifting a q-ary Hamming code $C(H_m^q)$ to $\mathbb{F}_{q^s}$ we obtain a completely regular code $C_s(H_m^q)$ which is not necessarily isomorphic to the code $C_m(H_s^q)$. However, both codes $C_s(H_m^q)$ and $C_m(H_s^q)$ have the same intersection array. As we saw above, the code obtained by the Kronecker product construction, or our extension for the case when the component codes have different alphabets, can have the same intersection array. The next statements are the main results of our paper.

Theorem 4.1. *Let q be any prime power and let a, b, u be any natural numbers. Then:*

1) *There exist the following completely regular codes with different parameters $[n, k, d; \rho]_{q^r}$, where $d = 3$ and $\rho = \min\{ua, b\}$:*
   - (i) $C_{ua}(H_b^q)$ *over* $\mathbb{F}_{q^{ua}}$ *with* $n = \frac{q^b-1}{q-1}$, $k = n - b$;
   - (ii) $C_b(H_{ua}^q)$ *over* $\mathbb{F}_{q^b}$ *with* $n = \frac{q^{ua}-1}{q-1}$, $k = n - ua$;
   - (iii) $C(H_b^q \otimes H_{ua}^q)$ *over* $\mathbb{F}_q$ *with* $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q-1}$, $k = n - bua$;
   - (iv) $C(H_b^q \otimes H_u^{q^a})$ *over* $\mathbb{F}_{q^a}$ *with* $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^a-1}$, $k = n - bu$;
   - (v) $C(H_b^q \otimes H_a^{q^u})$ *over* $\mathbb{F}_{q^u}$ *with* $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^u-1}$, $k = n - ba$;
2) *All codes above have the same intersection numbers*

$$b_\ell = \frac{(q^b - q^\ell)(q^{ua} - q^\ell)}{(q-1)},\ \ell = 0, \ldots, \rho - 1,\ \ c_\ell = q^{\ell-1} \cdot \frac{q^\ell - 1}{q - 1},\ \ell = 1, \ldots, \rho.$$

3) *All codes above coming from Kronecker constructions are completely transitive.*

Proof. The first two codes are obtained by the known lifting construction of the corresponding perfect codes and they all come from Theorem 2.5. The third code is obtained by the known Kronecker product construction (both components have the same alphabet) and come from Theorem 2.3. The two last codes are obtained by the Kronecker construction when the two component codes have different alphabets (q and, q^a or q^u)) and come from Theorem 3.6.

For every code we find the covering radius and compute the intersection array using the corresponding expressions given in the quoted theorems. All these codes have covering radius $\rho = \min\{ua, b\}$.

Complete transitivity of all codes coming from Kronecker constructions follows from Theorem 2.3 and Theorem 3.6. □

It is easy to see that the number of different completely transitive (and, therefore, completely regular) codes with different parameters and the same intersection array is growing when n is growing and q is fixed. To be more specific we summarize the results in the following Corollary, which comes straightforwardly from the above Theorem 4.1.

Denote by $\tau(n)$ the number of divisors of n.

Corollary 4.2. *Given a prime power q choose any two natural numbers $a, b > 1$. For each divisor r of a or b the following $\tau(a)+\tau(b)$ different codes with identical intersection array and covering radius $\rho = \min\{a, b\}$, are constructed:*

(i) *Completely transitive codes $C(H^{q^r}_{\bar{r}} \otimes H^q_b)$ over $\mathbb{F}_{q^r}$, for any proper divisor r of a and $r\bar{r} = a$.*

(ii) *Completely transitive codes $C(H^q_a \otimes H^{q^r}_{\bar{r}})$ over $\mathbb{F}_{q^r}$, for any proper divisor r of b and $r\bar{r} = b$.*

(iii) *Completely regular codes $C_a(H^q_b)$ over $\mathbb{F}_{q^a}$ and $C_b(H^q_a)$ over $\mathbb{F}_{q^b}$.*

5. Uniformly packed codes

Recall that a trivial repetition q-ary code of length n is a perfect code if and only if $q = 2$ and n is odd. Denote by $R^q_n = [I_{n-1} | -\mathbf{1}^{\mathbf{T}}]$ the parity check matrix of the repetition q-ary code of length n . The following statement generalizes the corresponding result of [9].

Theorem 5.1. *Let $C(H^{q^u}_m)$ be the q^u-ary (perfect) Hamming $[n, k, 3; 1]_{q^u}$ code of length $n_a = (q^{um} - 1)/(q^u - 1)$ and $C(R^q_{n_b})$ be the repetition q-ary code of length n_b, where q is a prime power, $u \geq 1$, $m \geq 2$, $4 \leq n_b \leq (q^u - 1)n_a + 1$.*

- *The code $C = C(H^{q^u}_m \otimes R^q_{n_b})$ is a q^u-ary uniformly packed (in the wide sense) $[n, k, d; \rho]_{q^u}$ code with parameters*

$$n = n_a\, n_b, \quad k = n - m\,(n_b - 1), \quad d = 3, \quad \rho = n_b - 1. \tag{5}$$

- *The code C is not completely regular.*

Proof. First, we find the outer distance of code C. Using [9, Lemma 4] we see that any linear combination of rows of $H^{q^u}_m$ has weight either $q^{u(m-1)}$ or zero. By the same arguments used in [9, Theo. 3] (any row of the parity check matrix of $C(R^q_{n_b})$ adds one more value to the weight of $C^\perp$) we conclude that $s(C) = n_b - 1$.

Now, about the covering radius of code C, we claim that for the case $n_b \leq (q^u - 1)n_a + 1$ we have $\rho = n_b - 1$.

Since $\rho(C) \leq s(C)$ [1], it is enough to show that $\rho(C) \geq n_b - 1$. Take an arbitrary column vector $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n_b-1})^T$ where $\mathbf{x}_i$ is a vector of length m over $\mathbb{F}_{q^u}$. Present this vector as a linear combination of columns of $H^{q^u}_m \otimes R^q_{n_b}$. For any vector $\mathbf{x}_i$ there is a column of $H^{q^u}_m$ which differs from $\mathbf{x}_i$ by a scalar. Choose as vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n_b-1}$ all possible different vectors of length m over $\mathbb{F}_{q^u}$. This vector can be presented as a linear combination, at least, of $n_b - 1$ columns of $H^{q^u}_m \otimes R^q_{n_b}$. Hence, $\rho \geq n_b - 1$ as we wanted.

Since $(q^u - 1)n_a + 1 = q^{um}$ for the case when $n_b \geq (q^u - 1)n_a + 2$, we can not choose all vectors $\mathbf{x}_i$ such that they are different. So, if $n_b = (q^u - 1)n_a + 2$, for example, then two subvectors $\mathbf{x}_i$ and, say, $\mathbf{x}_j$ should be the same. Now take as columns of $H^{q^u}_m \otimes R^q_{n_b}$ the column $\mathbf{h} = (\mathbf{h}_1, \ldots, \mathbf{h}_{n_b-1})^T$ with the same subcolumns $\mathbf{h}_i = \mathbf{h}_j$. As a result we obtain $\rho = n_b - 2$, but $s = n_b - 1$, implying that the resulting code is not uniformly packed [13].

To complete the proof we have to show that C is not completely regular. It is enough to find an specific counterexample. Take $q = 2, u = 1, m = 3, n_b = 4$ and consider as $C(H_m^{q^u})$ the Hamming $[7, 4, 3; 1]_2$ code and as $C(R_{n_b}^q)$ the repetition $[4, 1, 4; 1]_2]$ code. The resulting $[28, 19, 3; 3]_2$ code is not completely regular [9]. □

6. Coset distance-regular graphs

Following [6], we give some facts on distance-regular graphs. Let Γ be a finite connected simple graph (i.e., undirected, without loops and multiple edges). Let $d(\gamma, \delta)$ be the distance between two vertices γ and δ (i.e., the number of edges in the minimal path between γ and δ). The *diameter* D of Γ is its largest distance. Two vertices γ and δ from Γ are *neighbors* if $d(\gamma, \delta) = 1$. Define

$$\Gamma_i(\gamma) \ = \ \{\delta \in \Gamma : \ d(\gamma, \delta) = i\}.$$

An *automorphism* of a graph Γ is a permutation π of the vertex set of Γ such that, for all $\gamma, \delta \in \Gamma$ we have $d(\gamma, \delta) = 1$ if and only if $d(\pi\gamma, \pi\delta) = 1$. Let Γ_i be the graph with the same vertices of Γ, where an edge (γ, δ) is defined when the vertices γ, δ are at distance i in Γ. Clearly, $\Gamma_1 = \Gamma$. A graph is called *complete* (or a *clique*) if any two of its vertices are adjacent. A connected graph Γ with diameter $D \geq 3$ is called *antipodal* if the graph Γ_D is a disjoint union of cliques [6].

Definition 6.1. [6] *A simple connected graph Γ is called* distance-regular *if it is regular of valency k, and if for any two vertices $\gamma, \delta \in \Gamma$ at distance i apart, there are precisely c_i neighbors of δ in $\Gamma_{i-1}(\gamma)$ and b_i neighbors of δ in $\Gamma_{i+1}(\gamma)$. Furthermore, this graph is called* distance-transitive*, if for any pair of vertices γ, δ at distance $d(\gamma, \delta)$ there is an automorphism π from $\mathrm{Aut}(\Gamma)$ which moves this pair (γ, δ) to any other given pair γ', δ' of vertices at the same distance $d(\gamma, \delta) = d(\gamma', \delta')$.*

The sequence $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$, where D is the diameter of Γ, is called the *intersection array* of Γ. The numbers c_i, b_i, and a_i, where $a_i = k - b_i - c_i$, are called *intersection numbers*. Clearly $b_0 = k$, $b_D = c_0 = 0$, $c_1 = 1$.

Let C be a linear completely regular code with covering radius ρ and intersection array $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho\}$. Let $\{B\}$ be the set of cosets of C. Define the graph Γ_C, which is called the *coset graph of C*, taking all different cosets $B = C + \mathbf{x}$ as vertices, with two vertices $\gamma = \gamma(B)$ and $\gamma' = \gamma(B')$ adjacent if and only if the cosets B and B' contain neighbor vectors, i.e., there are $\mathbf{v} \in B$ and $\mathbf{v}' \in B'$ such that $d(\mathbf{v}, \mathbf{v}') = 1$.

Lemma 6.2. [6, 14] *Let C be a linear completely regular code with covering radius ρ and intersection array $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho\}$ and let Γ_C be the coset graph of C. Then Γ_C is distance-regular of diameter $D = \rho$ with the same intersection array. If C is completely transitive, then Γ_C is distance-transitive.*

From all different completely transitive codes described above in Theorem 4.1, we obtain distance-transitive graphs with classical parameters (see [6]). These graphs have q^{uab} vertices, diameter $D = \min\{ua, b\}$, and intersection array given by

$$b_l = \frac{(q^{ua} - q^l)(q^b - q^l)}{(q-1)};\; c_l = q^{l-1}\frac{q^l - 1}{q-1},$$

where $0 \leq l \leq D$.

Notice that bilinear forms graphs [6, Sec. 9.5] have the same parameters and are distance-transitive too. These graphs are uniquely defined by their parameters (see [6, Sec. 9.5]). Therefore, all graphs coming from the completely regular and completely transitive codes described in Theorem 4.1 are bilinear forms graphs. We did not find in the literature (in particular in [15], where the association schemes, formed by bilinear forms, have been introduced), the description of these graphs, as many different coset graphs of different completely regular codes. It is also known that these graphs are not antipodal and do not have antipodal covers (see [6, Sec. 9.5]). This can also be easily seen from the proof of Lemma 3.1. Indeed, a given vector $\mathbf{x} \in C(\rho)$ has many neighbors in $C(\rho)$.

Theorem 6.3. *Let C_1, C_2, ..., C_k be a family of linear completely transitive codes constructed by Theorem 3.6 and let $\Gamma_{C_1}, \Gamma_{C_2}, \ldots, \Gamma_{C_k}$ be their corresponding coset graphs. Then:*

(i) *Any graph Γ_{C_i} is a distance-transitive graph, induced by bilinear forms.*
(ii) *If any two codes C_i and C_j have the same intersection array, then the graphs Γ_{C_i} and Γ_{C_j} are isomorphic.*
(iii) *If the graph Γ_{C_i} has q^m vertices, where m is not a prime, then it can be presented as a coset graph by several different ways, depending on the number of factors of m.*

Proof. The proofs are straightforward. Given a completely transitive code C_i, constructed by Theorem 3.6, we conclude that the corresponding coset graph is distance-transitive with the same intersection array (Lemma 6.2). Then, by using [6, Sec. 9.5], we conclude that this graph is uniquely defined by their parameters and, therefore, it is induced by bilinear forms. Since two codes C_i and C_j with the same intersection arrays induce two coset graphs with the same parameters, we conclude that the corresponding graphs Γ_{C_i} and Γ_{C_j} are isomorphic. The last statement follows from Theorem 4.1, since it gives codes with the same intersection array. □

7. Conclusions

In the current paper we use the Kronecker product construction [9] for the case when component codes have different alphabets and connect the resulting completely regular codes with codes obtained by lifting q-ary perfect codes.

This gives several different infinite classes of completely regular codes with different parameters and with identical intersection arrays. Given a prime power q and any two natural numbers a, b, we construct completely transitive codes over different fields with covering radius $\rho = \min\{a, b\}$ and identical intersection array, specifically, one code over $\mathbb{F}_{q^r}$ for each divisor r of a or b. We prove that the corresponding induced distance-regular coset graphs are equivalent. In other words, the large class of distance-regular graphs, induced by bilinear forms [15], can be obtained as coset graphs from different non-isomorphic completely regular codes (either obtained by the Kronecker product construction from perfect codes over different alphabets, or obtained by lifting perfect codes [10]). Similar results are obtained for uniformly packed codes in the wide sense. Under the same conditions, explicit construction of an infinite family of q-ary uniformly packed codes (in the wide sense) with covering radius ρ, which are not completely regular, is also given.

Finally, an open question arises: *are bilinear forms graphs the only distance-transitive graphs which have such many different presentations as coset graphs?*

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