Computability in Symbolic Dynamics

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Abstract. We give an overview of the interplay between computability and symbolic dynamics.

A multidimensional shift of finite type (SFT) is a set of colorings of $\mathbb{Z}^d$ given by local rules. SFTs are one of the most fundamental objects in symbolic dynamics [LM95], and are well understood when $d = 1$ where they can be studied using finite automata theory. The situation becomes drastically different in dimension 2, where they are sometimes called tilings of the (discrete) plane, as almost any natural question about them becomes undecidable[Ber64,Rob71].

The uncomputability of many properties has for a long time being seen as a hurdle in the study of multidimensional symbolic dynamics. Douglas Lind [Lin04] has in particular described multidimensional SFTs as "The Swamp of Undecidability. It's a place you don't want to go".

In recent years, the position has changed, as many results have proven that it is actually possible to understand quite well many properties of multidimensional dynamical systems, as long as one accepts that the answer might involve computability theory.

We present here a few examples of this phenomenon. The focus of the first few sections is on one-dimensional symbolic dynamical systems given by computable constraints, and we show how these constraints translate into computability obstructions on their dynamics. In the last part, we explain how these results may be translated to multidimensional dynamical systems given by finite means, using strong embedding theorems.

1 Definitions

We start with a few relevant definitions.

Let A be a finite alphabet. We denote by $A^\star$ the set of finite words over the alphabet A, and by $A^{\mathbb{Z}}$ the set of biinfinite words over the alphabet A. The empty word will be denoted by ϵ. Given a finite word $w = w_0 w_1 \dots w_{n-1}$ (whose length n is denoted $|w|$) and a biinfinite word $u = \dots u_{-1} u_0 u_1 \dots$, we say that w appears in u (or that u contains w) if there exists some position k s.t. $u_{i+k} = w_i$ for all $0 \leq i < n$. We will also use the notion "w appears in u" for u a finite word, with a similar definition.

Given a set $\mathcal{F} \subseteq \mathcal{A}^\star$ of words, the *subshift* defined by $\mathcal{F}$ is the set of all biinfinite words where no word of $\mathcal{F}$ appears. We usually denote by $X_{\mathcal{F}}$ the subshift defined by $\mathcal{F}$.

Example 1. Let $A = \{0, 1\}$. Let $\mathcal{F} = \{00\}$. Then $X_{\mathcal{F}}$ is the set of biinfinite words that do not contain two consecutive symbols 0. For $\mathcal{F} = \{0, 01, 11\}$, $X_{\mathcal{F}}$ is evidently the empty set. For an alphabet A, let $\mathcal{F} = \{uu | u \in A^\star, |u| \geq 1\}$. Then $X_{\mathcal{F}}$ is the set of biinfinite words that do not contain any square.

Definition 1. *A subset X of $A^{\mathbb{Z}}$ is a subshift if there exists $\mathcal{F}$ s.t. $X = X_{\mathcal{F}}$. $\mathcal{F}$ will be called a set of forbidden patterns for X. If $\mathcal{F}$ can be chosen finite, X is called a subshift of finite type (SFT for short).*

If $\mathcal{F}$ is recursively enumerable, X is called an effectively closed subshift.

Example 2. The first two previous examples are obviously subshifts of finite type by definition. If A is a two letter alphabet, the set of biinfinite words that do not contain any square is a subshift of finite type. Indeed, in this case, $X = \emptyset = X_{\{\epsilon\}}$. If A has more than two letters, this subshift is nonempty and it is easy to see it is not of finite type. In any case, all these subshifts are effectively closed.

The set S of all words over the alphabet $A = \{0, 1\}$ with exactly one symbol 1 is not a subshift. Indeed, suppose there is $\mathcal{F}$ s.t. $S = X_{\mathcal{F}}$. Then $\mathcal{F}$ cannot contain any word consisting only of the symbol 0, therefore the biinfinite word containing only 0 is in $X_{\mathcal{F}}$, a contradiction.

As made evident by the previous examples, the same subshift can be given by different sets of forbidden patterns. In set theoretical terms, there is a largest set: If X is a subshift, then the set $\mathcal{B}(X)$ of all words that do not appear in any word of X is a set of forbidden patterns for X, that is $X = X_{\mathcal{B}(X)}$ and it is clearly maximal.

In terms of computability, it is however the minimal possible description of X, in the following sense.

Definition 2 (enumeration-reducibility [FR59]). *Let $S \subseteq A^\star$ and $S' \subseteq A^\star$ be two sets of finite words.*

We say that S is enumeration-reducible to S', in symbols $S \leq_e S'$, if there is a computable procedure that can enumerate S given any enumeration of S'.

Formally, there exists a partial computable function f that associates to any pair $(u, n) \in A^\star \times \mathbb{N}$ a finite subset of $A^\star$ s.t. $u \in S \iff \exists n, f(u, n) \subseteq S'$.

Enumeration reducibility gives rise naturally to a notion of enumeration-equivalence $\equiv_e$ whose classes are usually called *enumeration degrees.*

Proposition 1. *Let $X = X_{\mathcal{F}}$ be a subshift. Then $\mathcal{B}(X) \leq_e \mathcal{F}$.*

In terms of enumeration reducibility, $\mathcal{B}(X)$ is therefore the smallest possible description of X.

The key to understand this proposition is the compactness property: $u \in \mathcal{B}(X)$ iff there exists a size $k > |u|$ s.t. all elements of $A^\star$ of size k that contain u also contain some element of $\mathcal{F}$. This gives a way to enumerate all finite set F of words that "force" u to be forbidden.

In particular, if X is an effectively closed subshift, its set of forbidden words $\mathcal{B}(X)$ is recursively enumerable. In general, the language $\mathcal{B}(X)$ can be arbitrarily complex. For example given a subset S of $\mathbb{N}$, it is easy to see that the subshift defined by the set of forbidden words $\{10^n1 | n \in S\}$ has the same enumeration degree as S.

Dynamical systems Symbolic dynamics is well equipped to study general dynamical systems. Let $f : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a continuous map. The itinerary of f from point x is the infinite word of $A^{\mathbb{N}}$ defined by $It(f)(x)_n = a$ if the first symbol of $f^n(x)$ is a. If f is bijective, it is more natural to consider biinfinite trajectories, defined similarly for $n \in \mathbb{Z}$.

Then it is easy to see that the set $It(f)$ of all itineraries of f is a subshift. Moreover computability properties of f translate into computability properties of $It(f)$, see [Das08,CDK08] for more details. Interesting examples appear when f is taken to be a map of the interval [Moo91], a cellular automaton, or a Turing machine [Kur97].

2 Computability of subshifts

In this section we investigate computability properties of subshifts, and in particular of points inside a (nonempty) subshift. Typical properties we are interested in is whether a subshift contains a computable point and more generally on the structure of the Turing degrees of subshifts.

Effective subshifts are examples of Π^0_1 *classes* [CR98], a recursion-theoretic concept that appear everywhere in mathematics. Π^0_1 classes (of sets) can be defined using forbidden positioned words, i.e they are given by a (recursively enumerable) list of pairs of the form (i, w), meaning that w is forbidden to appear at position i. This definition, while slightly nonstandard, makes it obvious that effective subshifts are indeed Π^0_1 classes, and from this we can obtain a large number of results on what points of effective subshifts look like [Kre53,Sho60,JS72b].

However, subshifts have the additional property of shift-invariance: if $x \in X$ then the shift $\sigma(x)$ of x (defined formally by $\sigma(x)_i = x_{i+1}$) is also in X. Whether this property translates into computability properties on elements of X is the main question.

While our focus in on effectively closed subshifts, note that the above questions also make sense for general subshifts.

Cenzer, Dashti and King [CDK08] produced an example of an (nonempty) effectively closed subshift with no computable points. Another example is given by Rumyantsev and Ushakov [RU06]: Forbid all words x of Kolmogorov complexity less than $|x|/2 + c$. This subshift is nonempty if c is sufficiently large. More generally, Miller [Mil12] proved that any Π^0_1 set is Medvedev equivalent to an effectively closed subshift.

Definition 3. *Let S, S' two subsets of $A^{\mathbb{Z}}$. We say that $S \leq_M S'$ is Medvedev reducible to S' if there is a Turing functional Φ such that $\Phi(S') \subseteq S$.*

Medvedev equivalence is usually introduced in the context of mass problems [Sim11]: $S \leq_M S'$ if it is easier to find an element of S than to find an element of S', in the sense that, if we find an element y of S', then we also obtain in the same way an element of S (namely $\Phi(y)$).

Theorem 1 ([Mil12]). *For any Π^0_1 set S, there is an effectively closed subshift S' that is Medvedev-equivalent to S.*

Note that Medvedev equivalence is a weak notion in the sense that it speaks somehow only about the easiest (in terms of Turing degrees) elements of a set: Two sets S and S' which both contain computable points are always Medvedev equivalent.

We can search for something (somewhat) stronger: Given a Π^0_1 set S, is there a subshift with the same set of Turing degrees ? The answer is negative:

Theorem 2 ([JV13]). *Let S be a subshift. Then either S contains a computable point, or it contains a cone of Turing degrees: There exists a point $x \in S$ s.t. there are points $y \in S$ of arbitrary Turing degree above the degree of x.*

In particular, if S has no computable point, it contains two points of different but comparable Turing degrees. However we can construct some Π^0_1 classes which do not have this property [JS72a], which proves it is rather specific to subshifts. Note that this theorem is true for any subshift, and not only for effectively closed subshifts.

This property is due to the fact that every nonempty subshift contains a point x with a peculiar property, called uniform recurrence: If a word u appears in x, there exists a size n s.t. u occurs in every word of size n that appear in x. An obvious example of an uniformly recurrent word is a periodic word w ($w_i = w_{i+n}$ for some n and all i). Another classical example is the Thue-Morse word.

The situation when S has a computable point is completely understood and subsumed in the following theorem:

Theorem 3 ([JV13]). *For any set S with a computable point, there is a subshift T s.t. S and T have the same Turing degrees.*

Moreover, the set of positioned words that do not appear in S is enumeration-equivalent to the set of words that do not appear in T.

(In particular, if S is a Π^0_1 class, then T is effectively closed).

The situation of sets with cones of Turing degrees is less understood. Hochman and Vanier [HV] produced examples of subshifts for which the Turing degree spectrum is an uncountable union of disjoint cones.

3 Multidimensional subshifts

In the previous section, the subshifts with specific properties that are produced are often effectively closed, but never of finite type. Indeed, the theory of subshifts of finite type is well understood in dimension one, and connected with finite automata theory[LM95].

The situation is dramatically different when dealing with multi-dimensional subshifts. The definition of multi-dimensional subshifts is similar to one-dimensional subshifts, where a configuration is now an element of $A^{\mathbb{Z}^d}$ for some d, and the concept of a pattern (an element of A^{n^d}) replaces the concept of a word.

However, subshifts of finite type now become interesting. Indeed, it is undecidable to know if a subshift of finite type (given by a list of forbidden patterns) is empty [Ber64], and there exist nonempty subshifts of finite type where no configuration is computable [Mye74].

The main reason for these theorems is the ease of coding the space-time diagram of a Turing machine as a two-dimensional configuration. While these results where obtained in the late 60s and early 70s, they are now better understood in the context of the embedding theorems of the next paragraph.

3.1 The embedding theorems

The embedding theorems state that one-dimensional effectively closed subshifts may be encoded into multi-dimensional subshifts of finite type. Using this embedding, many of the previous theorems can be prove to hold for multidimensional subshifts of finite type.

To present the theorems, a few definitions are needed.

If S is a subshift over the alphabet A in dimension d, and π a map from A to B, the *recoloring* $\pi(S)$ is the subset of $B^{\mathbb{Z}^d}$ of all configurations y s.t. there exists $y \in S$ s.t. $y_i = \pi(x)_i$ for all i. The recoloring can be thought of as a way somehow to ignore construction lines by recoloring them. Note that however a recoloring subshift is never more complex than the original subshift: Indeed, $\mathcal{B}(\pi(S)) \leq_e \mathcal{B}(S)$.

Given a subshift $S \subseteq A^{\mathbb{Z}^d}$ in dimension d, one can define naturally higher and lower dimensional versions of S. The higher dimensional version $S^{\mathbb{Z}^{d'-d}}$ is the set of all configurations x of $A^{\mathbb{Z}^{d'}}$ for which there exists $y \in S$ s.t. $x_{(i,j)} = y_i$ for all $i \in \mathbb{Z}^d$ and $j \in \mathbb{Z}^{d'-d}$. If $d = 1$ and $d' - d = 1$, $S^{\mathbb{Z}}$ is therefore the set of all two-dimensional configurations where all rows are identical to some element of S. It is easy to see that $\mathcal{B}(S^{\mathbb{Z}^{d'}}) \equiv_e \mathcal{B}(S)$.

Then we have the following theorem:

Theorem 4 ([Hoc09,AS13,DRS10]). *Let S be an effectively closed subshift of dimension d over the alphabet A. Then there exist d', a recoloring π, and a subshift of finite type S' s.t. $S^{\mathbb{Z}^{d'}} = \pi(S')$.*

In this theorem, we can take $d' = 1$.

There is a relativized version of this statement. Say that S' (over the alphabet B) is of finite type over S (over the alphabet $A \subseteq B$) if there exists a finite set $\mathcal{F}$ of forbidden patterns s.t. S' is obtained from S by adding these forbidden patterns: $S' = X_{\mathcal{F} \cup \mathcal{B}(S)}$. Note that S' may have a larger alphabet than S. Again it is easy to see that $\mathcal{B}(S') \leq_e \mathcal{B}(S)$.

Theorem 5 ([AS09]). *Let S_1 and S_2 be two subshifts.*

Then $\mathcal{B}(S_1) \leq_e \mathcal{B}(S_2)$ iff there exist integers d_1, d_2, a recoloring π, and S' of finite type over $S_2^{\mathbb{Z}^{d_2}}$ s.t. $S_1^{\mathbb{Z}^{d_1}} = \pi(S')$

If we start from $S_2 = \{0\}^{\mathbb{Z}}$, we recover the previous theorem. It is interesting to note that these theorems are analogues of respectively the Highman embedding theorem [Hig61] and the relative Highman embedding theorems for groups [HS88], with subshifts (of finite type/effectively closed) playing the role of groups (finitely presented/recursively presented).

This gives a way to produce subshifts of finite type with complex behaviours: starting from a effectively closed subshift in dimension one with a given property, we obtain this way a subshift *of finite type* with the same property. Not all properties are preserved by recolorings and higher-dimensional versions, but enough are.

As an example, if we start from a one-dimensional effectively closed subshift with no computable point, we obtain a two-dimensional subshift of finite type with no computable point. If we start from an effectively closed subshift that is Medvedev equivalent to some Π^0_1 set S, we obtain a two-dimensional subshift of finite type that is Medvedev equivalent to S, a result originally from Simpson[Sim14] using a method from Myers[Mye74].

It is therefore reasonable to think of multi-dimensional subshifts of finite type as having similar computational properties as one-dimensional effectively closed subshifts.

3.2 Peculiarities of Subshifts of finite type

We finish this section by presenting some results that cannot be proven by the embedding theorem, either due to the nature of the subshift S' that is constructed, or due to the fact that our computational properties are not invariant under recoloring.

The first property deals with countable subshifts. In the proof of the embedding theorem, the subshift S' is uncountable, and this cannot be corrected. In particular, the theorem cannot be used to prove results on countable subshifts of finite type. Nevertheless, we may obtain:

Theorem 6 ([JV13]). *Let S be a countable Π^0_1 set. Then there exists a countable subshift of finite type T s.t. S and T have the same set of Turing degrees.*

The second property has to do with periodic configurations. In a subshift of finite type, it is algorithmically decidable to know whether there exists a configuration periodic of period n in all directions, as we only have to test all possible hypercubes of size n. This is not true anymore of the recoloring of a subshift of finite type: $\pi(x)$ might be periodic of period n without x being periodic. In fact it is easy to prove (using e.g. the embedding theorem) that the problem has now become undecidable.

However it is possible to obtain a strong characterization of what may happen for a subshift of finite type

Theorem 7 ([JV15]). *Let X be a subshift of finite type. Then the set of all n s.t. X contains a configuration of period exactly n in all directions is in* **NP** *(when n is encoded in unary).*

Conversely, given a unary language L in **NP***, there exists a subshift of finite type X s.t. the set of all n s.t. X contains a configuration of period exactly n in all directions is exactly L.*

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