Extremes for the inradius in the Poisson line tessellation.

Nicolas Chenavier, Ross Hemsley

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Nicolas Chenavier* Ross Hemsley†

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Abstract

A Poisson line tessellation is observed in the window $\mathbf{W}_\rho := B(0, \pi^{-1/2}\rho^{1/2})$, for $\rho > 0$. With each cell of the tessellation, we associate the inradius, which is the radius of the largest ball contained in the cell. Using Poisson approximation, we compute the limit distributions of the largest and smallest order statistics for the inradii of all cells whose nuclei are contained in $\mathbf{W}_\rho$ as ρ goes to infinity. We additionally prove that the limit shape of the cells minimising the inradius is a triangle.

Keywords line tessellations, Poisson point process, extreme values, order statistics.

AMS 2010 Subject Classifications 60D05 – 60G70 – 60G55 – 60F05 – 62G32

1 Introduction

The Poisson line tessellation Let $\hat{\mathbf{X}}$ be a stationary and isotropic Poisson line process of intensity $\hat{\gamma} = \pi$ in $\mathbf{R}^2$ endowed with its scalar product $\langle \cdot, \cdot \rangle$ and its Euclidean norm $|\cdot|$. By $\mathcal{A}$, we shall denote the set of affine lines which do not pass through the origin $0 \in \mathbf{R}^2$. Each line can be written as

$$H(u,t) := \Big\{ x \in \mathbf{R}^2,\ \langle x, u \rangle = t \Big\}, \tag{1}$$

for some $t \in \mathbf{R}$, $u \in \mathbf{S}$, where $\mathbf{S}$ is the unit sphere in $\mathbf{R}^2$. When $t > 0$, this representation is unique. The intensity measure of $\hat{\mathbf{X}}$ is then given by

$$\mu(\mathcal{E}) := \int_{\mathbf{S}} \int_{\mathbf{R}_+} 1_{H(u,r)\in\mathcal{E}}\, dr\, \sigma(du), \tag{2}$$

for all Borel subsets $\mathcal{E} \subseteq \mathcal{A}$, where $\mathcal{A}$ is endowed with the Fell topology (see for example Schneider and Weil [21], p563) and where $\sigma(\cdot)$ denotes the uniform measure on $\mathbf{S}$ with the normalisation $\sigma(\mathbf{S}) = 2\pi$. The set of closures of the connected components of $\mathbf{R}^2 \setminus \hat{\mathbf{X}}$ defines a stationary and isotropic random

*Université du Littoral Côte d'Opale, LMPA, Calais, France. nicolas.chenavier@lmpa.univ-littoral.fr

†Inria Sophia Antipolis – Méditerranée, mail@ross.click

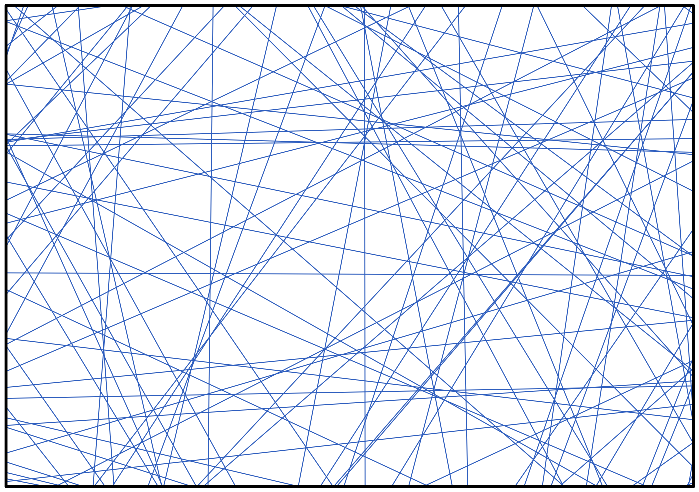

Figure 1: A realisation of the Poisson line tessellation truncated to a window.

tessellation with intensity $\gamma^{(2)} = \pi$ (see for example (10.46) in Schneider and Weil [21]) which is the so-called *Poisson line tessellation*, $\mathfrak{m}_{\text{PHT}}$. By a slight abuse of notation, we also write $\hat{\mathbf{X}}$ to denote the union of lines. An example of the Poisson line tessellation in $\mathbf{R}^2$ is depicted in Figure 1. Let $B(z,r)$ denote the (closed) disc of radius $r \in \mathbf{R}_+$, centred at $z \in \mathbf{R}^2$ and let $\mathcal{K}$ be the family of convex bodies (i.e. convex compact sets in $\mathbf{R}^2$ with non-empty interior), endowed with the Hausdorff topology. With each convex body $K \in \mathcal{K}$, we may now define the *inradius*,

$$r(K) := \sup\left\{ r : B(z,r) \subset K,\, z \in \mathbf{R}^2,\, r \in \mathbf{R}_+ \right\}.$$

When there exists a unique $z' \in \mathbf{R}^2$ such that $B(z', r(K)) \subset K$, we define $z(C) := z'$ to be the *incentre* of K. If no such z' exists, we take $z(K) := 0 \in \mathbf{R}^2$. Note that each cell $C \in \mathfrak{m}_{\text{PHT}}$ has a unique z' almost surely. In the rest of the paper we shall use the shorthand $B(K) := B(z(K), r(K))$. To describe the mean behaviour of the tessellation, we recall the definition of the typical cell as follows. Let W be a Borel subset of $\mathbf{R}^2$ such that $\lambda_2(W) \in (0,\infty)$, where λ_2 is the 2-dimensional Lebesgue measure. The *typical cell* $\mathcal{C}$ of a Poisson line tessellation, $\mathfrak{m}_{\text{PHT}}$ is a random polytope whose distribution is characterised by

$$\mathrm{E}[f(\mathcal{C})] = \frac{1}{\pi\lambda_2(W)} \cdot \mathrm{E}\left[\sum_{\substack{C \in \mathfrak{m}_{\text{PHT}}, \\ z(C) \in W}} f(C - z(C)) \right], \tag{3}$$

for all bounded measurable functions on the set of convex bodies $f\colon \mathcal{K} \to \mathbf{R}$. The typical cell of the Poisson line tessellation has been studied extensively in the literature, including calculations of mean values [15, 16] and distributional results [2] for a number of different geometric characteristics. A long standing conjecture due to D.G. Kendall concerning the asymptotic shape of the typical cell conditioned to be large is proved in Hug et al. [12]. The shape of small cells is also considered in Beermann et al. [1] for a rectangular Poisson line tessellation. Related results have also been obtained by Hug and Schneider [11] concerning the approximate properties of random polytopes formed by the Poisson hyperplane process. Global properties of the tessellation have also been established including, for example, central limit theorems [8, 9].

In this paper, we focus on the extremal properties of geometric characteristics for the cells of a Poisson line tessllation whose incentres are contained in a window. The general theory of extreme values deals with stochastic sequences [10] or random fields [14] (more details may be found in the reference works by de Haan and Ferreira [6] and Resnick [19].) To the best of the authors' knowledge, it appears that the first application of extreme value theory in stochastic geometry was given by Penrose (see Chapters 6,7 and 8 in Penrose [17]). More recently, Schulte and Thäle [23] established a theorem to derive the order statistics of a general functional, $f_k(x_1, \ldots, x_k)$ of k points of a homogeneous Poisson point process, a work which is related to the study of U-statistics. Calka and Chenavier [3] went on to provide a series of results for the extremal properties of cells in the Poisson-Voronoi tessellation, which were then extended by Chenavier [5], who gave a general theorem for establishing this type of limit theorem in tessellations satisfying a number of conditions. Unfortunately, none of these methods are directly applicable to the study of extremes for the geometric properties of cells in the Poisson line tessellation, due in part to the fact that even cells which are arbitrarily spatially separated may share lines.

Potential applications We remark that in addition to the classical references, such as the work by Goudsmit [7] concerning the trajectories of particles in bubble chambers, a number of new and interesting applications of random line processes are emerging in the field of Computer Science. Recent work by Plan and Vershynin [18] concerns the use of random hyperplane tessellations for dimension reduction with applications to high dimensional estimation. Plan and Vershynin [18] in particular point to a lack of results concerning the global properties of cells in the Poisson line tessellation in the traditional stochastic geometry literature. Other interesting applications for random hyperplanes may also be found in context of locality sensitive hashing [4]. We believe that our techniques will provide useful tools for the analysis of algorithms in these contexts and others. Finally, we note that investigating the extremal properties of cells could also provide a way to describe the regularity of tessellations.

1.1 Contributions

Formally, we shall consider the case in which only a part of the tessellation is observed in the *window* $\mathbf{W}_\rho := B\left(0, \pi^{-1/2}\rho^{1/2}\right)$, for $\rho > 0$. Given a measurable function $f\colon \mathcal{K} \to \mathbf{R}$ satisfying $f(C+x) = f(C)$

for all $C \in \mathcal{K}$ and $x \in \mathbf{R}^2$, we consider the order statistics of $f(C)$ for all cells $C \in \mathfrak{m}_{\text{PHT}}$ such that $z(C) \in \mathbf{W}_\rho$ in the limit as $\rho \to \infty$. In this paper, we focus on the case $f(C) := R(C)$ in particular because the inradius is one of the rare geometric characteristics for which the distribution of $f(\mathcal{C})$ can be made explicit. More precisely, we investigate the asymptotic behaviour of $m_{\mathbf{W}_\rho}[r]$ and $M_{\mathbf{W}_\rho}[r]$, which we use respectively to denote the inradii of the r-th smallest and the r-th largest inballs for fixed $r \geq 1$. Thus for $r = 1$ we have

$$m_{\mathbf{W}_\rho}[1] = \min_{\substack{C \in \mathfrak{m}_{\text{PHT}},\\ z(C) \in \mathbf{W}_\rho}} R(C) \qquad \text{and} \qquad M_{\mathbf{W}_\rho}[1] = \max_{\substack{C \in \mathfrak{m}_{\text{PHT}},\\ z(C) \in \mathbf{W}_\rho}} R(C).$$

The asymptotic behaviours of $m_{\mathbf{W}_\rho}[r]$ and $M_{\mathbf{W}_\rho}[r]$ are given in the following theorem.

Theorem 1. *Let $\mathfrak{m}_{\text{PHT}}$ be a stationary, isotropic Poisson line tessellation in $\mathbf{R}^2$ with intensity π and let $r \geq 1$ be fixed, then*

(i) for any $t \geq 0$,

$$\mathrm{P}\left(m_{\mathbf{W}_\rho}[r] \geq (2\pi^2\rho)^{-1}t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-t}\sum_{k=0}^{r-1}\frac{t^k}{k!},$$

(ii) for any $t \in \mathbf{R}$,

$$\mathrm{P}\left(M_{\mathbf{W}_\rho}[r] \leq \frac{1}{2\pi}(\log(\rho)+t)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-e^{-t}}\sum_{k=0}^{r-1}\frac{(e^{-t})^k}{k!}.$$

When $r = 1$, the limit distributions are of type II and type III, so that $m_{\mathbf{W}_\rho}[1]$ and $M_{\mathbf{W}_\rho}[1]$ belong to the domains of attraction of Weibull and Gumbel distributions respectively. The techniques we employ to investigate the asymptotic behaviours of $m_{\mathbf{W}_\rho}[r]$ and $M_{\mathbf{W}_\rho}[r]$ are quite different. For the cells minimising the inradius, we show that asymptotically, $m_{\mathbf{W}_\rho}[r]$ has the same behaviour as the r-th smallest value associated with a carefully chosen U-statistic. This will allow us to apply the theorem in Schulte and Thäle [22]. The main difficulties we encounter will be in checking the conditions for their theorem, and to deal with boundary effects. The cells maximising the inradius are more delicate, since the random variables in question cannot easily be formulated as a U-statistic. Our solution is to use a Poisson approximation, with the method of moments, in order to reduce our investigation to *finite* collections of cells. We then partition the possible configurations of each finite set using a clustering scheme and conditioning on the inter-cell distance.

The shape of cells with small inradius It was demonstrated that the cell which minimises the circumradius for a Poisson-Voronoi tessellation is a triangle with high probability by Calka and Chenavier [3]. In the following theorem we demonstrate that the analogous result holds for the cells of a Poisson line tessellation with small inradius. We begin by observing that almost surely, there exists a unique cell in $\mathfrak{m}_{\text{PHT}}$ with incentre in $\mathbf{W}_\rho$, say $C_{\mathbf{W}_\rho}[r]$, such that $R(C_{\mathbf{W}_\rho}[r]) = m_{\mathbf{W}_\rho}[r]$. We then consider the random variable $n(C_{\mathbf{W}_\rho}[r])$ where, for any (convex) polygon P in $\mathbf{R}^2$, we use $n(P)$ to denote the number of vertices of P.

Theorem 2. *Let $\mathfrak{m}_{\text{PHT}}$ be a stationary, isotropic Poisson line tessellation in $\mathbf{R}^2$ with intensity π and let $r \geq 1$ be fixed, then*

$$\mathrm{P}\left(\bigcap_{1\leq k\leq r}\left\{n(C_{\mathbf{W}_\rho}[k]) = 3\right\}\right) \underset{\rho\to\infty}{\longrightarrow} 1.$$

Remark. The asymptotic behaviour for the area of all triangular cells with a small area was given in Corollary 2.7 in Schulte and Thäle [23]. Applying similar techniques to those which we use to obtain the limit shape of the cells minimising the inradii, and using the fact that

$$\mathrm{P}(\lambda_2(\mathcal{C}) < v) \ \leq \ \mathrm{P}\left(R(\mathcal{C}) < (\pi^{-1}v)^{1/2}\right)$$

for all $v > 0$, we can also prove that the cells with a small *area* are triangles with high probability. As mentioned in Remark 4 in Schulte and Thäle [23] (where a formal proof is not provided), this implies that Corollary 2.7 in Schulte and Thäle [23] makes a statement not only about the area of the smallest triangular cell, but also about the area of the smallest cell in general.

Remark. Our theorems are given specifically for the two dimensional case with a fixed disc-shaped window, $\mathbf{W}_\rho$ in order to keep our calculations simple. However, Theorem 1 remains true when the window is any convex body. We believe that our results concerning the largest order statistics may be extended into higher dimensions and more general anisotropic (stationary) Poisson processes, using standard arguments. For the case of the smallest order statistics, these generalisations become less evident, and may require alternative arguments in places.

1.2 Layout

In Section 2, we shall introduce the general notation and background which will be required throughout the rest of the paper. In Section 3, we provide the asymptotic behaviour of $m_{\mathbf{W}_\rho}[r]$, proving the first part of Theorem 1 and Theorem 2. In Section 4, we establish some technical lemmas which will be used to derive the asymptotic behaviour of $M_{\mathbf{W}_\rho}[r]$. We conclude in Section 5 by providing the asymptotic behaviour of $M_{\mathbf{W}_\rho}[r]$, finalising the proof of Theorem 1.

2 Preliminaries

Notation

- We shall use $\mathrm{Po}(\tau)$ as a place-holder for a Poisson random variable with mean $\tau > 0$.

- For any pair of functions $f, g\colon \mathbf{R} \to \mathbf{R}$, we write $f(\rho) \underset{\rho\to\infty}{\sim} g(\rho)$ and $f(\rho) = O(g(\rho))$ to respectively mean that $f(\rho)/g(\rho) \to 1$ as $\rho \to \infty$ and $f(\rho)/g(\rho)$ is bounded for ρ large enough.
- By $\mathcal{B}(\mathbf{R}^2)$ we mean the family of Borel subsets in $\mathbf{R}^2$.
- For any $A \in \mathcal{B}(\mathbf{R}^2)$ and any $x \in \mathbf{R}^2$, we write $x+A := \{x+y : y \in A\}$ and $d(x, A) := \inf_{y\in A} |x-y|$.
- Let E be a measurable set and $K \geq 1$.
  - For any K-tuple of points $x_1, \ldots, x_K \in E$, we write $x_{1:K} := (x_1, \ldots, x_K)$.
  - By $E^K_{\neq}$, we mean the set of K-tuples of points $x_{1:K}$ such that $x_i \neq x_j$ for all $1 \leq i \neq j \leq K$.
  - For any function $f\colon E \to F$, where F is a set, and for any $A \subset F$, we write $f(x_{1:K}) \in A$ to imply that $f(x_i) \in A$ for each $1 \leq i \leq K$. In the same spirit, $f(x_{1:K}) > v$ will be used to mean that $f(x_i) > v$ given $v \in \mathbf{R}$.
  - If ν is a measure on E, we write $\nu(dx_{1:K}) := \nu(dx_1) \cdots \nu(dx_K)$.
- Given three lines $H_{1:3} \in \mathcal{A}^3_{\neq}$ in general position (in the sense of Schneider and Weil [21], p128), we denote by $\triangle(H_{1:3})$ the unique triangle that can be formed by the intersection of the halfspaces induced by the lines H_1, H_2 and H_3. In the same spirit, we denote by $B(H_{1:3})$, $R(H_{1:3})$ and $z(H_{1:3})$ the inball, the inradius and the incentre of $\triangle(H_{1:3})$ respectively.
- Let $K \in \mathcal{K}$ be a convex body with a unique inball $B(K)$ such that the intersection $B(K) \cap K$ contains exactly three points, x_1, x_2, x_3. In which case we define T_1, T_2, T_3 to be the lines tangent to the border of $B(K)$ intersecting x_1, x_2, x_3 respectively. We now define $\triangle(K) := \triangle(T_{1:3})$, observing that $B(\triangle(K)) = B(K)$.
- For any line $H \in \mathcal{A}$, we write H^+ to denote the half-plane delimited by H and containing $0 \in \mathbf{R}^2$. According to (1), we have $H^+(u, t) := \{\, x \in \mathbf{R}^2 : \langle x, u\rangle \leq t \,\}$ for given $t > 0$ and $u \in \mathbf{S}$.
- For any $A \in \mathcal{B}(\mathbf{R}^2)$, we take $\mathcal{A}(A) \subset \mathcal{A}$, to be the set $\mathcal{A}(A) := \{H \in \mathcal{A} : H \cap A \neq \varnothing\}$. We also define $\phi\colon\ \mathcal{B}(\mathbf{R}^2) \to \mathbf{R}_+$ as

$$\phi(A) \ := \ \mu(\mathcal{A}(A)) \ = \ \int_{\mathcal{A}(A)} 1_{H\cap A\neq\varnothing}\, \mu(dH) \ = \ \mathrm{E}\left[\#\{H \in \hat{\mathbf{X}} : H \cap A \neq \varnothing\}\right]. \tag{4}$$

Remark. Because $\hat{\mathbf{X}}$ is a Poisson process, we have for any $A \in \mathcal{B}(\mathbf{R}^2)$

$$\mathrm{P}\left(\hat{\mathbf{X}} \cap A = \varnothing\right) = \mathrm{P}\left(\#\hat{\mathbf{X}} \cap \mathcal{A}(A) = 0\right) = e^{-\phi(A)}. \tag{5}$$

Remark. When $A \in \mathcal{B}(\mathbf{R}^2)$ is a convex body, the Crofton formula (Theorem 5.1.1 in Schneider and Weil [21]) gives that

$$\phi(A) = \ell(A), \tag{6}$$

where $\ell(A)$ denotes the perimeter of A. In particular, when $A = B(z,r)$ for some $z \in \mathbf{R}^2$ and $r \geq 0$, we have $\phi(B(z,r)) = \mu(\mathcal{A}(B(z,r))) = 2\pi r$.

A well-known representation of the typical cell The typical cell of a Poisson line tessellation, as defined in (3), can be made explicit in the following sense. For any measurable function $f\colon \mathcal{K} \to \mathbf{R}$, we have from Theorem 10.4.6 in Schneider and Weil [21] that

$$\mathrm{E}[\, f(\mathcal{C})\,] = \frac{1}{24\pi}\int_0^\infty \int_{\mathbf{S}^3} \mathrm{E}\left[\, f\left(C\left(\hat{\mathbf{X}}, u_{1:3}, r\right)\right)\,\right] e^{-2\pi r} a(u_{1:3})\,\sigma(du_{1:3})dr, \tag{7}$$

where

$$C\left(\hat{\mathbf{X}}, u_{1:3}, r\right) := \bigcap_{H \in \hat{\mathbf{X}} \cap (\mathcal{A}(B(0,r)))^c} \left\{ H^+ \cap \bigcap_{j=1}^{3} H^+(u_j, r) \right\} \tag{8}$$

and where $a(u_{1:3})$ is taken to be the area of the convex hull of $\{u_1, u_2, u_3\} \subset \mathbf{S}$ when $0 \in \mathbf{R}^2$ is contained in the convex hull of $\{u_1, u_2, u_3\}$ and 0 otherwise. With standard computations, it may be demonstrated that $\int_{\mathbf{S}^3} a(u_{1:3})\,\sigma(du_{1:3}) = 48\pi^2$, so that when $f(C) = R(C)$, we have the well-known result

$$\mathrm{P}(R(\mathcal{C}) \leq v) = 1 - e^{-2\pi v} \qquad \text{for all } v \geq 0. \tag{9}$$

We note that in the following, we occasionally omit the lower bounds in the ranges of sums and unions, and the arguments of functions when they are clear from context. Throughout the paper we also use c to signify a universal positive constant not depending on ρ but which may depend on other quantities. When required, we assume that ρ is sufficiently large.

3 Asymptotics for cells with small inradii

3.1 Intermediary results

Let $r \geq 1$ be fixed. In order to avoid boundary effects, we introduce a function $q(\rho)$ such that

$$\log\rho \cdot q(\rho) \cdot \rho^{-2} \underset{\rho\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \pi^{-1/2}\left(q(\rho)^{1/2} - \rho^{1/2}\right) - \varepsilon \log\rho \underset{\rho\to\infty}{\longrightarrow} +\infty \tag{10}$$

for some $\varepsilon > 0$. We also introduce two intermediary random variables, the first of which relates collections of 3-tuples of lines in $\hat{\mathbf{X}}$. Let $\overset{\triangle}{m}_{\mathbf{W}_\rho}[r]$ represent the r-th smallest value of $R(H_{1:3})$ over all 3-tuples of lines $H_{1:3} \in \hat{\mathbf{X}}^3_{\neq}$ such that $z(H_{1:3}) \in \mathbf{W}_\rho$ and $\triangle(H_{1:3}) \subset \mathbf{W}_{q(\rho)}$. Its asymptotic behaviour is given in the following proposition.

Proposition 3. *For any $r \geq 1$ and any $t \geq 0$,*

$$\mathrm{P}\left(\overset{\triangle}{m}_{\mathbf{W}_\rho}[r] \geq (2\pi^2\rho)^{-1}t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-t}\sum_{k=0}^{r-1}\frac{t^k}{k!}.$$

The second random variable concerns the cells in $\mathfrak{m}_{\mathrm{PHT}}$. More precisely, we define $\overset{\circ}{m}_{\mathbf{W}_\rho}[r]$ to be the r-th smallest value of the inradius over all cells $C \in \mathfrak{m}_{\mathrm{PHT}}$ such that $z(C) \in \mathbf{W}_\rho$ and $\triangle(C) \subset \mathbf{W}_{q(\rho)}$. We observe that $\overset{\circ}{m}_{\mathbf{W}_\rho}[r] \geq \overset{\triangle}{m}_{\mathbf{W}_\rho}[r]$ and $\overset{\circ}{m}_{\mathbf{W}_\rho}[r] \geq m_{\mathbf{W}_\rho}[r]$. Actually, in the following result we show that the deviation between these quantities is negligible as ρ goes to infinity.

Lemma 4. *For any fixed $r \geq 1$,*

(i) $\mathrm{P}\left(\overset{\circ}{m}_{\mathbf{W}_\rho}[r] \neq \overset{\triangle}{m}_{\mathbf{W}_\rho}[r]\right) \underset{\rho\to\infty}{\longrightarrow} 0$,

(ii) $\mathrm{P}\left(m_{\mathbf{W}_\rho}[r] \neq \overset{\circ}{m}_{\mathbf{W}_\rho}[r]\right) \underset{\rho\to\infty}{\longrightarrow} 0$.

As stated above, Schulte and Thäle established a general theorem to deal with U-statistcs (Theorem 1.1 in Schulte and Thäle [23]). In this work we make use of a new version of their theorem (to appear in Schulte and Thäle [22]), which we modify slightly to suit our requirements. Let $g\colon \mathcal{A}^3 \to \mathbf{R}$ be a measurable symmetric function and take $\overset{\triangle}{m}_{g,\mathbf{W}_\rho}[r]$ to be the r-th smallest value of $g(H_{1:3})$ over all 3-tuples of lines $H_{1:3} \in \hat{\mathbf{X}}^3_{\neq}$ such that $z(H_{1:3}) \in \mathbf{W}_\rho$ and $\triangle(H_{1:}) \subset \mathbf{W}_{q(\rho)}$ (for $q(\rho)$ as in (10).) We now define the following quantities for given $a, t \geq 0$.

$$\alpha_\rho^{(g)}(t) := \frac{1}{6}\int_{\mathcal{A}^3} \mathbb{1}_{z(H_{1:3})\in\mathbf{W}_\rho}\, \mathbb{1}_{\triangle(H_{1:3})\subset\mathbf{W}_{q(\rho)}}\, \mathbb{1}_{g(H_{1:3})<\rho^{-a}t}\, \mu(dH_{1:3}), \tag{11a}$$

$$r_{\rho,1}^{(g)}(t) := \int_{\mathcal{A}}\left(\int_{\mathcal{A}^2} \mathbb{1}_{z(H_{1:3})\in\mathbf{W}_\rho}\, \mathbb{1}_{\triangle(H_{1:3})\subset\mathbf{W}_{q(\rho)}}\, \mathbb{1}_{g(H_{1:3})<\rho^{-a}t}\ \mu(dH_{2:3})\right)^2 \mu(dH_1), \tag{11b}$$

$$r_{\rho,2}^{(g)}(t) := \int_{\mathcal{A}^2}\left(\int_{\mathcal{A}} \mathbb{1}_{z(H_{1:3})\in\mathbf{W}_\rho}\, \mathbb{1}_{\triangle(H_{1:3})\subset\mathbf{W}_{q(\rho)}}\, \mathbb{1}_{g(H_{1:3})<\rho^{-a}t}\ \mu(dH_3)\right)^2 \mu(dH_{1:2}). \tag{11c}$$

Theorem 5 (Schulte and Thäle)**.** *Let $t \geq 0$ be fixed. Assume that $\alpha_\rho(t)$ converges to $\alpha t^\beta > 0$, for some $\alpha, \beta > 0$ and $r_{\rho,1}(t), r_{\rho,2}(t) \underset{\rho\to\infty}{\longrightarrow} 0$, then*

$$\mathrm{P}\left(\overset{\triangle}{m}{}^{(g)}_{\mathbf{W}_\rho}[r] \geq \rho^{-a}t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\alpha t^\beta}\sum_{k=0}^{r-1}\frac{\left(\alpha t^\beta\right)^k}{k!}.$$

Remark. Actually, Theorem 5 is stated in Schulte and Thäle [22] for a Poisson point process in more general measurable spaces with intensity going to infinity. By scaling invariance, we have re-written their result for a fixed intensity (equal to π) and for the window $\mathbf{W}_{q(\rho)} = B(0, \pi^{-1/2}q(\rho)^{1/2})$ with $\rho \to \infty$. We also adapt their result by adding the indicator function $\mathbb{1}_{z(H_{1:3})\in\mathbf{W}_\rho}$ to (11a), (11b) and (11c).

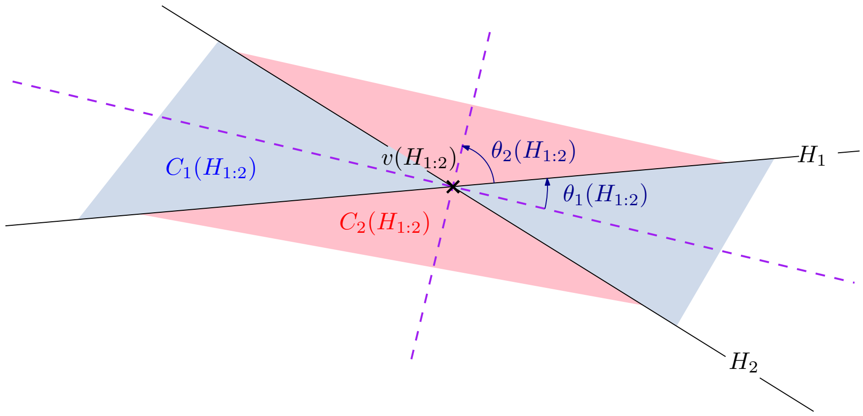

Figure 2: Construction of double cone for change of variables.

Proofs for Proposition 3, Lemma 4, Theorem 1, Part (i) and Theorem 2

Proof of Proposition 3. Let $t \geq 0$ be fixed. We apply Theorem 5 with $g = R$ and $a = 1$. First, we compute the quantity $\alpha_\rho(t) := \alpha_\rho^{(R)}(t)$ as defined in (11a). Applying a Blaschke-Petkantschin type change of variables (see for example Theorem 7.3.2 in Schneider and Weil [21]), we obtain

$$\begin{aligned}\alpha_\rho(t) &= \frac{1}{24}\int_{\mathbf{R}^2}\int_0^\infty\int_{\mathbf{S}^3} a(u_{1:3})\mathbb{1}_{z\in\mathbf{W}_\rho}\,\mathbb{1}_{z+r\triangle(H(u_1),H(u_2),H(u_3))\subset\mathbf{W}_{q(\rho)}}\,\mathbb{1}_{r<\rho^{-1}t}\;\sigma(du_{1:3})drdz\\ &= \frac{1}{24}\int_{\mathbf{R}^2}\int_0^\infty\int_{\mathbf{S}^3} a(u_{1:3})\mathbb{1}_{z\in\mathbf{W}_1}\,\mathbb{1}_{z+r\rho^{-3/2}\triangle(H(u_1),H(u_2),H(u_3))\subset\mathbf{W}_{q(\rho)/\rho}}\,\mathbb{1}_{r<t}\;\sigma(du_{1:3})drdz.\end{aligned}$$

We note that the normalisation of μ_1, as defined in Schneider and Weil [21], is such that $\mu_1 = \frac{1}{\pi}\mu$, where μ is given in (2). It follows from the monotone convergence theorem that

$$\alpha_\rho(t) \underset{\rho\to\infty}{\longrightarrow} \frac{1}{24}\int_{\mathbf{R}^2}\int_0^\infty\int_{\mathbf{S}^3} a(u_{1:3})\mathbb{1}_{z\in\mathbf{W}_1}\,\mathbb{1}_{r<t}\;\sigma(du_{1:3})drdz = 2\pi^2 t \tag{12}$$

since $\lambda_2(\mathbf{W}_1) = 1$ and $\int_{\mathbf{S}^3} a(u_{1:3})\,\sigma(du_{1:3}) = 48\pi^2$. We must now check that

$$r_{\rho,1}(t) \underset{\rho\to\infty}{\longrightarrow} 0, \tag{13}$$

$$r_{\rho,2}(t) \underset{\rho\to\infty}{\longrightarrow} 0, \tag{14}$$

where $r_{\rho,1}(t) := r_{\rho,1}^{(R)}(t)$ and $r_{\rho,2}(t) := r_{\rho,2}^{(R)}(t)$ are defined in (11b) and (11c).

Proof of Convergence (13). Let H_1 be fixed and define

$$G_\rho(H_1) := \int_{\mathcal{A}^2} \mathbb{1}_{z(H_{1:3})\in\mathbf{W}_\rho}\,\mathbb{1}_{\triangle(H_{1:3})\subset\mathbf{W}_{q(\rho)}}\,\mathbb{1}_{R(H_{1:3})<\rho^{-1}t}\;\mu(dH_{2:3}).$$

Bounding $\mathbb{1}_{\triangle(H_{1:3})\subset\mathbf{W}_{q(\rho)}}$ by 1, and applying Lemma 12, Part (i) (given in appendix) to $R := \rho^{-1}t$, $R' := \pi^{-1/2}\rho^{1/2}$ and $z' = 0$, we get for ρ large enough

$$G_\rho(H_1) \leq c\cdot\rho^{-1/2}\mathbb{1}_{d(0,H_1)<\rho^{1/2}}\,.$$

Noting that $r_{\rho,1}(t) = \int_{\mathcal{A}} G_\rho(H_1)^2 \mu(dH_1)$, it follows from (2) that

$$\begin{aligned} r_{\rho,1}(t) &\leq c \cdot \rho^{-1} \int_{\mathcal{A}} 1_{d(0,H_1)<\rho^{1/2}} \ \mu(dH_1) \\ &= O\left(\rho^{-1/2}\right). \end{aligned} \tag{15}$$

□

Proof of Convergence (14). Let H_1 and H_2 be such that H_1 intersects H_2 at a unique point, $v(H_{1:2})$. The set $H_1 \cup H_2$ divides $\mathbf{R}^2$ into two double-cones with supplementary angles, $C_i(H_{1:2})$, $1 \leq i \leq 2$ (see Figure 2.) We then denote by $\theta_i(H_{1:2}) \in [0, \frac{\pi}{2})$ the half-angle of $C_i(H_{1:2})$ so that $2(\theta_1(H_{1:2}) + \theta_2(H_{1:2})) = \pi$. Moreover, we write

$$E_i(H_{1:2}) = \left\{ H_3 \in \mathcal{A} : z(H_{1:3}) \in \mathbf{W}_\rho \cap C_i(H_{1:2}),\ \triangle(H_{1:3}) \subset \mathbf{W}_{q(\rho)},\ R(H_{1:3}) < \rho^{-1}t \right\}.$$

We provide below a suitable upper bound for $G_\rho(H_1, H_2)$ defined as

$$\begin{aligned} G_\rho(H_1, H_2) &:= \int_{\mathcal{A}} 1_{z(H_{1:3})\in \mathbf{W}_\rho} 1_{\triangle(H_{1:3})\subset \mathbf{W}_{q(\rho)}} 1_{R(H_{1:3})<\rho^{-1}t} \ \mu(dH_3) \\ &= \sum_{i=1}^{2} \int_{\mathcal{A}} 1_{H_3 \in E_i(H_{1:2})} \ \mu(dH_3). \end{aligned} \tag{16}$$

To do this, we first establish the following lemma.

Lemma 6. *Let $H_1, H_2 \in \mathcal{A}$ be fixed and let $H_3 \in E_i(H_{1:2})$ for some $1 \leq i \leq 2$, then*

(i) $H_3 \cap W_{c\cdot\rho} \neq \varnothing$, *for some* c,

(ii) $H_3 \cap B\left(v(H_{1:2}), \frac{c\cdot\rho^{-1}}{\sin\theta_i(H_{1:2})}\right) \neq \varnothing$,

(iii) $|v(H_{1:2})| \leq c \cdot q(\rho)^{1/2}$, *for some* c.

Proof of Lemma 6. The first statement is a consequence of the fact that

$$d(0, H_3) \ \leq \ |z(H_{1:3})| + d(z(H_{1:3}), H_3) \ \leq \ \pi^{-1/2}\rho^{1/2} + \rho^{-1}t \ \leq \ c \cdot \rho^{1/2}.$$

For the second statement, we have

$$d(v(H_{1:2}), H_3) \ \leq \ |v(H_{1:2}) - z(H_{1:3})| + d(z(H_{1:3}), H_3) \ \leq \ \frac{R(H_{1:3})}{\sin\theta_i(H_{1:2})} + \rho^{-1}t \ \leq \ \frac{c\cdot\rho^{-1}}{\sin\theta_i(H_{1:2})}$$

since $R(H_{1:3}) = |v(H_{1:2}) - z(H_{1:3})| \cdot \sin\theta_i(H_{1:2})$. Finally, the third statement comes from the fact that $v(H_{1:2}) \in \mathbf{W}_{q(\rho)}$ since $\triangle(H_{1:3}) \subset \mathbf{W}_{q(\rho)}$. □

We apply below the first statement of Lemma 6 when $\theta_i(H_{1:2})$ is small enough and the second one otherwise. More precisely, it follows from (16) and Lemma 6 that

$$G_\rho(H_1,H_2) \leq \sum_{i=1}^{2} \int_{\mathcal{A}} \mathbb{1}_{H_3\cap \mathbf{W}_{c\cdot\rho}\neq\varnothing}\, \mathbb{1}_{|v(H_{1:2})|\leq c\cdot q(\rho)^{1/2}}\, \mathbb{1}_{\sin\theta_i(H_{1:2})\leq \rho^{-3/2}}\ \mu(dH_3)$$
$$+ \int_{\mathcal{A}} \mathbb{1}_{H_3\cap B\left(v(H_{1:2}),\frac{c\cdot\rho^{-1}}{\sin\theta_i(H_{1:2})}\right)\neq\varnothing}\, \mathbb{1}_{|v(H_{1:2})|\leq c\cdot q(\rho)^{1/2}}\, \mathbb{1}_{\sin\theta_i(H_{1:2})> \rho^{-3/2}}\ \mu(dH_3). \tag{17}$$

Integrating over H_3 and applying (6) to

$$B := \mathbf{W}_{c\cdot\rho} = B(0, c^{1/2}\rho^{1/2}) \quad \text{and} \quad B' := B\left(v(H_{1:2}), \frac{c\cdot\rho^{-1}}{\sin\theta_i(H_{1:2})}\right),$$

we obtain

$$G_\rho(H_1,H_2) \leq c\cdot \sum_{i=1}^{2}\left(\rho^{1/2}\mathbb{1}_{\sin\theta_i(H_{1:2})\leq\rho^{-3/2}} + \frac{\rho^{-1}}{\sin\theta_i(H_{1:2})}\mathbb{1}_{\sin\theta_i(H_{1:2})>\rho^{-3/2}}\right)$$
$$\times\, \mathbb{1}_{|v(H_{1:2})|\leq c\cdot q(\rho)^{1/2}}\,. \tag{18}$$

Applying the fact that

$$r_{\rho,2}(t) = \int_{\mathcal{A}} G_\rho(H_1,H_2)^2\mu(dH_{1:2}) \qquad \text{and} \qquad \left(\sum_{i=1}^{2}(a_i+b_i)\right)^2 \leq 4\sum_{i=1}^{2}\left(a_i^2+b_i^2\right)$$

for any $a_1,a_2,b_1,b_2\in\mathbf{R}$, it follows from (18) that

$$r_{\rho,2}(t) \leq c\cdot\sum_{i=1}^{2}\int_{\mathcal{A}^2}\left(\rho\mathbb{1}_{\sin\theta_i(H_{1:2})\leq\rho^{-3/2}} + \frac{\rho^{-2}}{\sin^2\theta_i(H_{1:2})}\mathbb{1}_{\sin\theta_i(H_{1:2})>\rho^{-3/2}}\right)$$
$$\times\, \mathbb{1}_{|v(H_{1:2})|\leq c\cdot q(\rho)^{1/2}}\ \mu(dH_{1:2})$$

For any couple of lines $(H_1,H_2)\in\mathcal{A}^2$ such that $H_1 = H(u_1,t_1)$ and $H_2 = H(u_2,t_2)$ for some $u_1,u_2\in\mathbf{S}$ and $t_1,t_2>0$, let $\theta(H_1,H_2)\in[-\frac{\pi}{2},\frac{\pi}{2})$ be the oriented half angle between the vectors u_1 and u_2. In particular, the quantity $|\theta(H_{1:2})|$ is equal to $\theta_1(H_{1:2})$ or $\theta_2(H_{1:2})$. This implies that

$$r_{\rho,2}(t) \leq 4c\cdot\int_{\mathcal{A}^2}\left(\rho\mathbb{1}_{\sin\theta(H_{1:2})\leq\rho^{-3/2}} + \frac{\rho^{-2}}{\sin^2\theta(H_{1:2})}\mathbb{1}_{\sin\theta(H_{1:2})>\rho^{-3/2}}\right)\mathbb{1}_{\theta(H_{1:2})\in[0,\frac{\pi}{2})}$$
$$\times\, \mathbb{1}_{|v(H_{1:2})|\leq c\cdot q(\rho)^{1/2}}\ \mu(dH_{1:2}). \tag{19}$$

With each $v=(v_1,v_2)\in\mathbf{R}^2$, $\beta\in[0,2\pi)$ and $\theta\in[0,\pi/2)$, we associate two lines H_1 and H_2 as follows. We first define $L(v_1,v_2,\beta)$ as the line containing $v=(v_1,v_2)$ with normal vector $\vec{\beta}$, where for any $\alpha\in[0,2\pi)$, we write $\vec{\alpha}=(\cos\alpha,\sin\alpha)$. Then we define H_1 and H_2 as the lines containing $v=(v_1,v_2)$ with angles θ and $-\theta$ with respect to $L(v_1,v_2,\beta)$ respectively. These lines can be written as $H_1 = H(u_1, t_1)$ and $H_2 = H(u_2, t_2)$ with

$$u_1 := u_1(\beta, \theta) := \overrightarrow{\beta - \theta},$$

$$t_1 := t_1(v_1, v_2, \beta, \theta) := \left| -\sin(\beta - \theta)v_1 + \cos(\beta - \theta)v_2 \right|,$$

$$u_2 := u_2(\beta, \theta) := \overrightarrow{\beta + \theta},$$

$$t_2 := t_2(v_1, v_2, \beta, \theta) := \left| \sin(\beta + \theta)v_1 + \cos(\beta + \theta)v_2 \right|.$$

Denoting by $\overline{\alpha}$, the unique real number in $[0, 2\pi)$ such that $\overline{\alpha} \equiv \alpha \mod 2\pi$, we define

$$\psi \colon \mathbf{R}^2 \times [0, 2\pi) \times [0, \tfrac{\pi}{2}) \longrightarrow \mathbf{R}_+ \times [0, 2\pi) \times \mathbf{R}_+ \times [0, 2\pi)$$
$$(v_1, v_2, \beta, \theta) \longmapsto \left(t_1(v_1, v_2, \beta, \theta),\ \overline{\beta - \theta},\ t_2(v_1, v_2, \beta, \theta),\ \overline{\beta + \theta}\right).$$

Modulo null sets, ψ is a $\mathcal{C}^1$ diffeomorphism with Jacobian $J\psi$ given by $|J\psi(v_1, v_2, \beta, \theta)| = 2\sin 2\theta$ for any point $(v_1, v_2, \beta, \theta)$ where ψ is differentiable. Taking the change of variables as defined above, we deduce from (19) that

$$\begin{aligned} r_{\rho,2}(t) &\leq c \cdot \int_{\mathbf{R}^2} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin(2\theta) \left(\rho \mathbb{1}_{\sin\theta \leq \rho^{-3/2}} + \frac{\rho^{-2}}{\sin^2\theta} \mathbb{1}_{\sin\theta > \rho^{-3/2}} \right) \mathbb{1}_{|v| \leq c \cdot q(\rho)^{1/2}} \, d\theta d\beta dv \\ &= O\left(\log\rho \cdot q(\rho) \cdot \rho^{-2}\right). \end{aligned} \tag{20}$$

As a consequence of (10), the last term converges to 0 as ρ goes to infinity. □

The above combined with (12), (15) and Theorem 5 concludes the proof of Proposition 3. □

Proof of Lemma 4, (i). Almost surely, there exists a unique triangle with incentre contained in $\mathbf{W}_{q(\rho)}$, denoted by $\Delta_{\mathbf{W}_\rho}[r]$, such that

$$z(\triangle_{\mathbf{W}_\rho}[r]) \in \mathbf{W}_\rho \qquad \text{and} \qquad R(\triangle_{\mathbf{W}_\rho}[r]) = \overset{\triangle}{m}_{\mathbf{W}_\rho}[r].$$

Also, $z(\triangle_{\mathbf{W}_\rho}[r])$ is the incentre of a cell of $\mathfrak{m}_{\text{PHT}}$ if and only if $\hat{\mathbf{X}} \cap B(\triangle_{\mathbf{W}_\rho}[r]) = \varnothing$. Since $\overset{\circ}{m}_{\mathbf{W}_\rho}[r] \geq \overset{\triangle}{m}_{\mathbf{W}_\rho}[r]$, this implies that

$$\overset{\circ}{m}_{\mathbf{W}_\rho}[r] = \overset{\triangle}{m}_{\mathbf{W}_\rho}[r] \qquad \Longleftrightarrow \qquad \exists 1 \leq k \leq r \quad \text{such that} \quad \hat{\mathbf{X}} \cap B(\triangle_{\mathbf{W}_\rho}[k]) \neq \varnothing.$$

In particular, for any $\varepsilon > 0$, we obtain

$$\begin{aligned} \mathrm{P}\left(\overset{\circ}{m}_{\mathbf{W}_\rho}[r] \neq \overset{\triangle}{m}_{\mathbf{W}_\rho}[r]\right) \leq \sum_{k=1}^{r} \Bigg(&\mathrm{P}\left(\hat{\mathbf{X}} \cap B(\triangle_{\mathbf{W}_\rho}[k]) \neq \varnothing,\ R(\triangle_{\mathbf{W}_\rho}[k]) < \rho^{-1+\varepsilon}\right) \\ &+ \mathrm{P}\left(R(\triangle_{\mathbf{W}_\rho}[k]) > \rho^{-1+\varepsilon}\right)\Bigg). \end{aligned} \tag{21}$$

The second term of the series converges to 0 as ρ goes to infinity thanks to Proposition 3. For the first term, we obtain for any $1 \leq k \leq r$, that

$$\begin{aligned}
&\mathrm{P}\left(\hat{\mathbf{X}} \cap B(\triangle_{\mathbf{W}_\rho}[k]) \neq \varnothing,\ R(\triangle_{\mathbf{W}_\rho}[k]) < \rho^{-1+\varepsilon}\right) \\
&\quad \leq \mathrm{P}\left(\bigcup_{H_{1:4} \in \hat{\mathbf{X}}^4_{\neq}} \{z(H_{1:3}) \in \mathbf{W}_\rho\} \cap \{R(H_{1:3}) < \rho^{-1+\varepsilon}\} \cap \{H_4 \cap B(z(H_{1:3}), \rho^{-1+\varepsilon}) \neq \varnothing\}\right) \\
&\quad \leq \mathrm{E}\left[\sum_{H_{1:4} \in \hat{\mathbf{X}}^4_{\neq}} \mathbb{1}_{z(H_{1:3}) \in \mathbf{W}_\rho} \mathbb{1}_{R(H_{1:3}) < \rho^{-1+\varepsilon}} \mathbb{1}_{H_4 \cap B(z(H_{1:3}), \rho^{-1+\varepsilon}) \neq \varnothing}\right] \\
&\quad = \int_{\mathcal{A}^4} \mathbb{1}_{z(H_{1:3}) \in \mathbf{W}_\rho} \mathbb{1}_{R(H_{1:3}) < \rho^{-1+\varepsilon}} \mathbb{1}_{H_4 \cap B(z(H_{1:3}), \rho^{-1+\varepsilon}) \neq \varnothing} \, \mu(dH_{1:4}),
\end{aligned}$$

where the last line comes from Mecke-Slivnyak's formula (Corollary 3.2.3 in Schneider and Weil [21]). Applying the Blaschke-Petkantschin change of variables, we obtain

$$\begin{aligned}
&\mathrm{P}\left(\hat{\mathbf{X}} \cap B(\triangle_{\mathbf{W}_\rho}[k]) \neq \varnothing,\ R(\triangle_{\mathbf{W}_\rho}[k]) < \rho^{-1+\varepsilon}\right) \\
&\quad \leq c \cdot \int_{\mathbf{W}_\rho} \int_0^{\rho^{-1+\varepsilon}} \int_{\mathbf{S}^3} \int_{\mathcal{A}} a(u_{1:3}) \mathbb{1}_{H_4 \cap B(z, \rho^{-1+\varepsilon}) \neq \varnothing} \, \mu(dH_4) \sigma(du_{1:3}) dr dz.
\end{aligned}$$

As a consequence of (4) and (6), we have

$$\int_{\mathcal{A}} \mathbb{1}_{H_4 \cap B(z, \rho^{-1+\varepsilon}) \neq \varnothing} \, \mu(dH_4) \;=\; c \cdot \rho^{-1+\varepsilon}$$

for any $z \in \mathbf{R}^2$. Integrating over $z \in \mathbf{W}_\rho$, $r < \rho^{-1+\varepsilon}$ and $u_{1:3} \in \mathbf{S}^3$, we obtain

$$\mathrm{P}\left(\hat{\mathbf{X}} \cap B(\triangle_{\mathbf{W}_\rho}[k]) \neq \varnothing,\ R(\triangle_{\mathbf{W}_\rho}[k]) < \rho^{-1+\varepsilon}\right) \leq c \cdot \rho^{-1+2\varepsilon} \tag{22}$$

since $\lambda_2(\mathbf{W}_\rho) = \rho$. Taking $\varepsilon < \frac{1}{2}$, we deduce Lemma 4, (i) from (21) and (22). □

Proof of Theorem 2 . Let $\varepsilon \in (0, \frac{1}{2})$ be fixed. For any $1 \leq k \leq r$, we write

$$\begin{aligned}
&\mathrm{P}\left(n(C_{\mathbf{W}_\rho}[k]) \neq 3\right) \\
&\quad = \mathrm{P}\left(n(C_{\mathbf{W}_\rho}[k]) \geq 4,\ m_{\mathbf{W}_\rho}[k] \geq \rho^{-1+\varepsilon}\right) + \mathrm{P}\left(n(C_{\mathbf{W}_\rho}[k]) \geq 4,\ m_{\mathbf{W}_\rho}[k] < \rho^{-1+\varepsilon}\right).
\end{aligned}$$

According to Proposition 3, Lemma 4, (i) and the fact that $\mathring{m}_{\mathbf{W}_\rho}[k] \geq m_{\mathbf{W}_\rho}[k]$, the first term of the right-hand side converges to 0 as ρ goes to infinity. For the second term, we obtain from (3) that

$$\begin{aligned}
\mathrm{P}\left(n(C_{\mathbf{W}_\rho}[k]) \geq 4,\ m_{\mathbf{W}_\rho}[k] < \rho^{-1+\varepsilon}\right) &\leq \mathrm{P}\left(\min_{\substack{C \in \mathfrak{m}_{\mathrm{PHT}}, \\ z(C) \in \mathbf{W}_\rho,\, n(C) \geq 4}} R(C) < \rho^{-1+\varepsilon}\right) \\
&\leq \mathrm{E}\left[\sum_{\substack{C \in \mathfrak{m}_{\mathrm{PHT}}, \\ z(C) \in \mathbf{W}_\rho}} \mathbb{1}_{R(C) < \rho^{-1+\varepsilon}} \mathbb{1}_{n(C) \geq 4}\right] \\
&= \pi\rho \cdot \mathrm{P}\left(R(\mathcal{C}) < \rho^{-1+\varepsilon},\ n(\mathcal{C}) \geq 4\right).
\end{aligned} \tag{23}$$

We give below an integral representation of $\mathrm{P}(R(\mathcal{C}) < \rho^{-1+\varepsilon},\ n(\mathcal{C}) \geq 4)$. Let $r > 0$ and $u_1, u_2, u_3 \in \mathbf{S}$ be fixed. We denote by $\triangle(u_{1:3}, r)$ the triangle $\triangle(H(u_1, r), H(u_2, r), H(u_3, r))$. Let us notice that the random polygon $C(\hat{\mathbf{X}}, u_{1:3}, r)$, as defined in (8), satisfies $n(C(\hat{\mathbf{X}}, u_{1:3}, r)) \geq 4$ if and only if $\hat{\mathbf{X}} \in \mathcal{A}\left(\triangle(u_{1:3}, r) \setminus B(0, r)\right)$. According to (5) and (7), this implies that

$$\begin{aligned}
&\pi\rho \cdot \mathrm{P}\left(R(\mathcal{C}) < \rho^{-1+\varepsilon},\ n(\mathcal{C}) \geq 4\right) \\
&\quad = \frac{\rho}{24} \int_0^{\rho^{-1+\varepsilon}} \int_{\mathbf{S}^3} \left(1 - e^{-\phi(\triangle(u_{1:3}, r) \setminus B(0,r))}\right) e^{-2\pi r} a(u_{1:3}) \sigma(du_{1:3}) dr.
\end{aligned}$$

Using the fact that $1 - e^{-x} \leq x$ for all $x \in \mathbf{R}$ and the fact that

$$\phi\big(\triangle(u_{1:3}, r) \setminus B(0, r)\big) \ \leq\ \phi(\triangle(u_{1:3}, r)) \ =\ r\ell(\triangle(u_{1:3}))$$

according to (6), we get

$$\begin{aligned}
\pi\rho \cdot \mathrm{P}\left(R(\mathcal{C}) < \rho^{-1+\varepsilon},\ n(\mathcal{C}) \geq 4\right) &\leq \frac{\rho}{24} \int_0^{\rho^{-1+\varepsilon}} \int_{\mathbf{S}^3} r e^{-2\pi r} \ell(\triangle(u_{1:3})) \sigma(du_{1:3}) dr \\
&= O\left(\rho^{-1+2\varepsilon}\right).
\end{aligned}$$

This together with (23) gives that

$$\mathrm{P}\left(n(C_{\mathbf{W}_\rho}[k]) \geq 4,\ m_{\mathbf{W}_\rho}[k] < \rho^{-1+\varepsilon}\right) \underset{\rho \to \infty}{\longrightarrow} 0.$$

□

Proof of Lemma 4, (ii). Since $m_{\mathbf{W}_\rho}[r] \neq \mathring{m}_{\mathbf{W}_\rho}[r]$ if and only if $\triangle\left(C_{\mathbf{W}_\rho}[k]\right) \cap \mathbf{W}^{\mathrm{c}}_{q(\rho)} \neq \varnothing$ for some $1 \leq k \leq r$, we get for any $\varepsilon > 0$

$$\begin{aligned}
&\mathrm{P}\left(m_{\mathbf{W}_\rho}[r] \neq \mathring{m}_{\mathbf{W}_\rho}[r]\right) \\
&\quad \leq \sum_{k=1}^{r} \Bigg(\mathrm{P}\left(R(C_{\mathbf{W}_\rho}[k]) \geq \rho^{-1+\varepsilon}\right) + \mathrm{P}\left(n(C_{\mathbf{W}_\rho}[k]) \neq 3\right) \\
&\qquad\qquad + \mathrm{P}\left(\triangle\left(C_{\mathbf{W}_\rho}[k]\right) \cap \mathbf{W}^{\mathrm{c}}_{q(\rho)} \neq \varnothing,\ n(C_{\mathbf{W}_\rho}[k]) = 3,\ R(C_{\mathbf{W}_\rho}[k]) < \rho^{-1+\varepsilon}\right)\Bigg).
\end{aligned} \tag{24}$$

As in the proof of Theorem 2, the first term of the series converges to zero. The same fact is also true for the second term as a consequence of Theorem 2. Moreover, for any $1 \leq k \leq r$, we have

$$\begin{aligned}
&\mathrm{P}\left(\triangle\left(C_{\mathbf{W}_\rho}[k]\right) \cap \mathbf{W}^{\mathrm{c}}_{q(\rho)} \neq \varnothing,\ n(C_{\mathbf{W}_\rho}[k]) = 3,\ R(C_{\mathbf{W}_\rho}[k]) < \rho^{-1+\varepsilon}\right) \\
&\quad \leq \mathrm{P}\left(\bigcup_{H_{1:3} \in \hat{\mathbf{X}}^3_{\neq}} \left\{ \hat{\mathbf{X}} \cap \triangle(H_{1:3}) = \varnothing,\ z(H_{1:3}) \in \mathbf{W}_\rho,\ \triangle(H_{1:3}) \cap \mathbf{W}^{\mathrm{c}}_{q(\rho)} \neq \varnothing,\ R(H_{1:3}) < \rho^{-1+\varepsilon} \right\} \right) \\
&\quad \leq \int_{\mathcal{A}^3} \mathrm{P}\left(\hat{\mathbf{X}} \cap \triangle(H_{1:3}) = \varnothing\right) \mathbb{1}_{z(H_{1:3}) \in \mathbf{W}_\rho} \mathbb{1}_{\triangle(H_{1:3}) \cap \mathbf{W}^{\mathrm{c}}_{q(\rho)} \neq \varnothing} \mathbb{1}_{R(H_{1:3}) < \rho^{-1+\varepsilon}}\ \mu\left(dH_{1:3}\right) \\
&\quad \leq \int_{\mathcal{A}^3} e^{-\ell(\triangle(H_{1:3}))} \mathbb{1}_{z(H_{1:3}) \in \mathbf{W}_\rho} \mathbb{1}_{\ell(\triangle(H_{1:3})) > \pi^{-1/2}(q(\rho)^{1/2} - \rho^{1/2})} \mathbb{1}_{R(H_{1:3}) < \rho^{-1+\varepsilon}}\ \mu\left(dH_{1:3}\right),
\end{aligned}$$

where the second and the third inequalities come from Mecke-Slivnyak's formula and (5) respectively. Using the fact that

$$e^{-\ell(\triangle(3H_{1:3}))} \leq e^{-\pi^{-1/2}\left(q(\rho)^{1/2}-\rho^{1/2}\right)},$$

and applying the Blaschke-Petkantschin formula, we get

$$\mathrm{P}\left(\triangle\left(C_{\mathbf{W}_\rho}[k]\right)\cap \mathbf{W}^{\mathrm{c}}_{q(\rho)} \neq \varnothing,\ n(C_{\mathbf{W}_\rho}[k])=3\right) \leq c\cdot\rho^{\varepsilon}\cdot e^{-\pi^{-1/2}\left(q(\rho)^{1/2}-\rho^{1/2}\right)}.$$

According to (10), the last term converges to zero. This together with (24) completes the proof of Lemma 4, (ii). □

Proof of Theorem 1, (i). The proof follows immediately from Proposition 3 and Lemma 4. □

Remark. As mentioned on page 7, we introduce an auxiliary function $q(\rho)$ to avoid boundary effects. This addition was necessary to prove the convergence of $r_{\rho,2}(t)$ in (20).

4 Technical results

In this section, we establish two results which will be needed in order to derive the asymptotic behaviour of $M_{\mathbf{W}_\rho}[r]$.

4.1 Poisson approximation

Consider a measurable function $f\colon \mathcal{K}\to\mathbf{R}$ and a *threshold* v_ρ such that $v_\rho\to\infty$ as $\rho\to\infty$. The cells $C\in\mathfrak{m}_{\text{PHT}}$ such that $f(C)>v_\rho$ and $z(C)\in\mathbf{W}_\rho$ are called the *exceedances*. A classical tool in extreme value theory is to estimate the limiting distribution of the number of exceedances by a Poisson random variable. In our case, we achieve this with the following lemma.

Lemma 7. *Let $\mathfrak{m}_{\text{PHT}}$ be a Poisson line tessellation embedded in $\mathbf{R}^2$ and suppose that for any $K\geq 1$,*

$$\mathrm{E}\left[\sum_{\substack{C_{1:K}\in(\mathfrak{m}_{\text{PHT}})^K_{\neq}\\ z(C_{1:K})\in\mathbf{W}_\rho}} \mathbb{1}_{f(C_{1:K})>v_\rho}\right] \underset{\rho\to\infty}{\longrightarrow} \tau^K. \tag{25}$$

Then

$$\mathrm{P}\left(M_{f,\mathbf{W}_\rho}[r]\leq v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} \sum_{k=0}^{r-1}\frac{\tau^k}{k!}e^{-\tau}.$$

Proof of Lemma 7. Let the number of exceedance cells be denoted

$$U(v_\rho) := \sum_{\substack{C\in\mathfrak{m}_{\text{PHT}},\\ z(C)\in\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho}.$$

Let $1 \leq K \leq n$ and let $\left\{ {n \atop K} \right\}$ denote the Stirling number of the second kind. According to (25), we have

$$\begin{aligned}
\mathrm{E}\left[U(v_\rho)^n\right] &= \mathrm{E}\left[\sum_{K=1}^{n} \left\{ {n \atop K} \right\} U(v_\rho)\cdot\big(U(v_\rho)-1\big)\cdot\big(U(v_\rho)-2\big)\cdots\big(U(v_\rho)-K+1\big)\right] \\
&= \sum_{K=1}^{n} \left\{ {n \atop K} \right\} \mathrm{E}\left[\sum_{\substack{C_{1:K}\in \mathfrak{m}_{\mathrm{PHT}}{}^{K}_{\neq},\\ z(C_{1:K})\in \mathbf{W}_\rho}} \mathbb{1}_{f(C_{1:K})>v_\rho}\right] \\
&\underset{\rho\to\infty}{\longrightarrow} \sum_{K=1}^{n} \left\{ {n \atop K} \right\} \tau^K \\
&= \mathrm{E}\left[\mathrm{Po}(\tau)^n\right].
\end{aligned}$$

Thus by the method of moments, $U(v_\rho)$ converges in distribution to a Poisson distributed random variable with mean τ. We conclude the proof by noting that $M_{f,\mathbf{W}_\rho}[r] \leq v_\rho$ if and only if $U(v_\rho) \leq r-1$. □

Lemma 7 can be generalised for any window $\mathbf{W}_\rho$ and for any tessellation in any dimension. A similar method was used to provide the asymptotic behaviour for couples of random variables in the particular setting of a Poisson-Voronoi tessellation (see Proposition 2 in Calka and Chenavier [3]). The main difficulty is applying Lemma 7, and we deal partially with this in the following section.

4.2 A uniform upper bound for ϕ for the union of discs

Let $\phi : \mathcal{B}(\mathbf{R}^2) \to \mathbf{R}_+$ as in (4). We evaluate $\phi(B)$ in the particular case where $B = \bigcup_{1\leq i\leq K}^{K} B(z_i, r_i)$ is a finite union of balls centred in z_i and with radius r_i, $1 \leq i \leq K$. Closed form representations for $\phi(B)$ could be provided but these formulas are not of practical interest to us. We provide below (see Proposition 8) some approximations for $\phi(\cup_{1\leq i\leq K}^{K} B(z_i, r_i))$ with simple and quasi-optimal lower bounds.

4.2.1 Connected components of cells

Our bound will follow by splitting collections of discs into a set of connected components. Suppose we are given a threshold v_ρ such that $v_\rho \to \infty$ as $\rho \to \infty$ and $K \geq 2$ discs $B(z_i, r_i)$, satisfying $z_i \in \mathbf{R}^2$, $r_i \in \mathbf{R}_+$ and $r_i > v_\rho$, for all $i = 1, \ldots, K$. We take $R := \max_{1\leq i\leq K} r_i$. The *connected components* are constructed from the graph with vertices $B(z_i, r_i), i = 1, \ldots, K$ and edges

$$B(z_i, r_i) \longleftrightarrow B(z_j, r_j) \quad \Longleftrightarrow \quad B(z_i, R^3) \cap B(z_j, R^3) \neq \varnothing. \tag{26}$$

On the right-hand side, we have chosen radii of the form R^3 to provide a simpler lower bound in Proposition 8. The *size* of a component is the number of discs in that component. To refer to these components, we use the following notation which is highlighted for ease of reference.

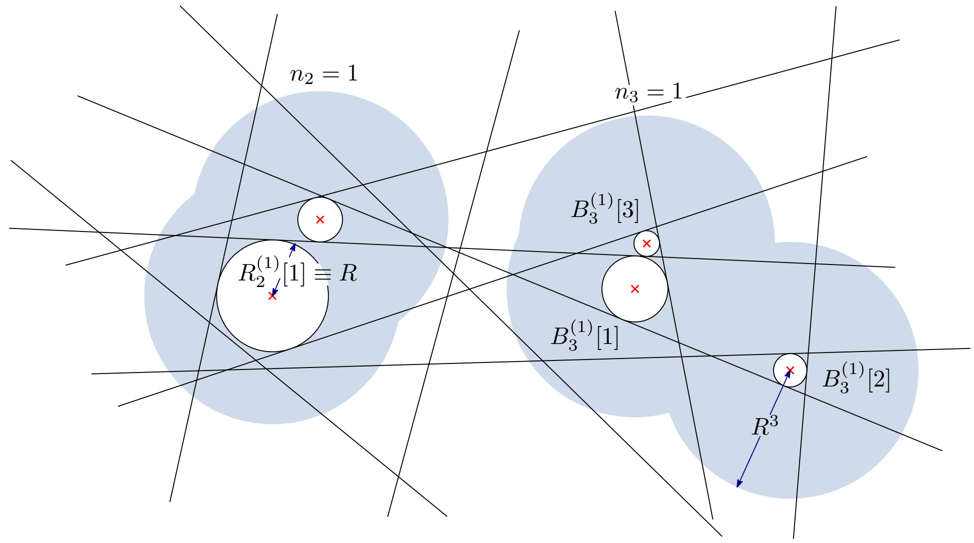

Figure 3: Example connected components for $K = 5$ and $(n_1, \ldots, n_K) = (0, 1, 1, 0, 0)$.

Notation

- For all $k \leq K$, write $n_k := n_k(z_{1:K}, R)$ to denote the number of connected components of size k. Observe that in particular, $\sum_{k=1}^{K} k \cdot n_k = K$.
- Suppose that with each component of size k is assigned a unique label $1 \leq j \leq n_k$. We then write $B_k^{(j)} := B_k^{(j)}(z_{1:K}, R)$, to refer to the union of balls in the jth component of size k.
- Within a component, we write $B_k^{(j)}[r] := B_k^{(j)}(z_{1:K}, R)[r]$, $1 \leq r \leq k$, to refer to the ball having the rth largest radius in the jth cluster of size k. In particular, we have $B_k^{(j)} = \bigcup_{r=1}^{k} B_k^{(j)}[r]$. We also write $z_k^{(j)}[r]$ and $r_k^{(j)}[r]$ as shorthand to refer to the centre and radius of the ball $B_k^{(j)}[r]$.

4.2.2 The uniform upper bound

In extreme value theory, a classical method to investigate the behaviour of the maximum of a sequence of random variables relys on checking two conditions of the sequence. One such set of conditions is given by Leadbetter [13], who defines the conditions $D(u_n)$ and $D'(u_n)$ which represent an asymptotic property and a local property of the sequence respectively. We shall make use of analagous conditions for the Poisson line tessellation, and it is for this reason that we motivate the different cases concerning spatially separated and spatially close balls in Proposition 8.

Proposition 8. *Consider a collection of K disjoint balls, $B(z_i, r_i)$ for $i = 1, \ldots, K$ such that $r_{1:K} > v_\rho$ and $R := \max_{1 \leq i \leq K} r_i$.*

(i) When $n_{1:K} = (K, 0, \ldots, 0)$, i.e. $\min_{1\le i,j\le K} |z_i - z_j| > R^3$, we obtain for ρ large enough

$$\phi\left(\bigcup_{1\le i\le K} B(z_i, r_i)\right) \ge 2\pi \sum_{i=1}^{K} r_i - c \cdot v_\rho^{-1}. \tag{27}$$

(ii) (a) for ρ large enough,

$$\phi\left(\bigcup_{1\le i\le K} B(z_i, r_i)\right) \ge 2\pi R + \left(\sum_{k=1}^{K} n_k - 1\right) 2\pi v_\rho - c \cdot v_\rho^{-1},$$

(b) when $R \le (1+\varepsilon)v_\rho$, for some $\varepsilon > 0$, we have for ρ large enough

$$\phi\left(\bigcup_{1\le i\le K} B(z_i, r_i)\right) \ge 2\pi R + \left(\sum_{k=1}^{K} n_k - 1\right) 2\pi v_\rho + \sum_{k=2}^{K} n_k (4 - \varepsilon\pi) v_\rho - c \cdot v_\rho^{-1}.$$

Remark. Suppose that $n_{1:K} = (K, 0, \ldots, 0)$.

1. We observe that (27) is quasi-optimal since we also have

$$\phi\left(\bigcup_{1\le i\le K} B(z_i, r_i)\right) \quad \le \quad \sum_{i=1}^{K} \phi(B(z_i, r_i)) \quad = \quad 2\pi \sum_{i=1}^{K} r_i. \tag{28}$$

2. Thanks to (5), (27) and (28), we remark that

$$\left| \mathrm{P}\left(\bigcap_{1\le i\le K} \left\{\hat{\mathbf{X}} \cap B(z_i, r_i) = \varnothing\right\}\right) - \prod_{1\le i\le K} \mathrm{P}\left(\hat{\mathbf{X}} \cap B(z_i, r_i) = \varnothing\right) \right| \le c \cdot v_\rho^{-1} \underset{\rho\to\infty}{\longrightarrow} 0.$$

The fact that the events considered in the probabilities above tend to be independent is well-known and is related to the fact that the tessellation $\mathfrak{m}_{\mathrm{PHT}}$ satisfies a mixing property (see, for example the proof of Theorem 10.5.3 in Schneider and Weil [21].) Our contribution is to provide a *uniform rate of convergence* (in the sense that it does not depend on the centres and the radii) when the balls are distant enough (case (i)) and a suitable *uniform* upper bound for the opposite case (case (ii).) Proposition 8 will be used to check (25). Before attacking Proposition 8, we first state two lemmas. The first of which deals with the case of just two balls.

Lemma 9. *Let $z_1, z_2 \in \mathbf{R}^2$ and $R \ge r_1 \ge r_2 > v_\rho$ such that $|z_2 - z_1| > r_1 + r_2$.*

(i) If $|z_2 - z_1| > R^3$, we have for ρ large enough that

$$\mu\left(\mathcal{A}(B(z_1, r_1)) \cap \mathcal{A}(B(z_2, r_2))\right) \le c \cdot v_\rho^{-1}.$$

(ii) If $R \le (1+\varepsilon)v_\rho$ for some $\varepsilon > 0$, then we have for ρ large enough that

$$\mu\left(\mathcal{A}(B(z_1, r_1)) \cap \mathcal{A}(B(z_2, r_2))\right) \le 2\pi r_2 - (4 - \varepsilon\pi) v_\rho$$

Actually, closed formulas for the measure of all lines intersecting two convex bodies can be found in Santaló [20], p33. However, Lemma 9 is more practical since it provides an upper bound which is independent of the centres and the radii. The following lemma is a generalisation of the previous result.

Lemma 10. *Let $z_{1:K} \in \mathbf{R}^{2K}$ and R such that, for all $1 \leq i \neq j \leq K$, we have $R \geq r_i > v_\rho$ and $|z_i - z_j| > r_i + r_j$.*

(i) $\mu\left(\bigcup_{1\leq i\leq K} \mathcal{A}\left(B\left(z_i, r_i\right)\right)\right) \geq \sum_{k=1}^{K}\sum_{j=1}^{n_k} 2\pi \cdot r_k^{(j)}[1] - c \cdot v_\rho^{-1}$.

(ii) If $R \leq (1+\varepsilon)v_\rho$ for some $\varepsilon > 0$, we have the following more precise inequality

$$\mu\left(\bigcup_{1\leq i\leq K} \mathcal{A}\left(B\left(z_i, r_i\right)\right)\right) \geq \sum_{k=1}^{K}\sum_{j=1}^{n_k} 2\pi \cdot r_k^{(j)}[1] + \sum_{k=2}^{K} n_k(4 - \varepsilon\pi)v_\rho - c \cdot v_\rho^{-1}.$$

4.2.3 Proofs

Proof of Proposition 8. The proof of (i) follows immediately from (4) and Lemma 10, (i). Using the fact that $r_k^{(j)}[1] > v_\rho$ for all $1 \leq k \leq K$ and $1 \leq j \leq n_k$ such that $r_k^{(j)}[1] \neq R$, we obtain (iia) and (iib) from Lemma 10, (i) and (ii) respectively. ☐

Proof of Lemma 9. As previously mentioned, Santaló [20] provides a general formula for the measure of all lines intersecting two convex bodies. However, to obtain a more explicit representation of $\mu(\mathcal{A}(B(z_1, r_1)) \cap \mathcal{A}(B(z_2, r_2)))$, we re-write his result in the particular setting of two balls. According to (2) and the fact that μ is invariant under translations, we obtain with standard computations that

$$\begin{aligned}
\mu\left(\mathcal{A}(B(z_1, r_1)) \cap \mathcal{A}(B(z_2, r_2))\right) &= \int_{\mathbf{S}}\int_{\mathbf{R}_+} 1_{H(u,t)\cap B(0,r_1)\neq\varnothing}\, 1_{H(u,t)\cap B(z_2 - z_1, r_2)\neq\varnothing}\, dt\, \sigma(du) \\
&= \int_{\mathbf{S}}\int_{\mathbf{R}_+} 1_{t<r_1}\, 1_{d(z_2-z_1, H(u,t))<r_2}\, dt\, \sigma(du) \\
&= \int_{[0,2\pi)}\int_{\mathbf{R}_+} 1_{t<r_1}\, 1_{|\cos\alpha\cdot|z_2-z_1|-t|<r_2}\, dt d\alpha \\
&= 2 \cdot f(r_1, r_2, |z_2 - z_1|),
\end{aligned}$$

where

$$\begin{aligned}
&f(r_1, r_2, h) \\
&\quad := (r_1 + r_2)\arcsin\left(\tfrac{r_1+r_2}{h}\right) - (r_1 - r_2)\arcsin\left(\tfrac{r_1-r_2}{h}\right) - h\left(\sqrt{1 - \left(\tfrac{r_1-r_2}{h}\right)^2} - \sqrt{1 - \left(\tfrac{r_1+r_2}{h}\right)^2}\right)
\end{aligned}$$

for all $h > r_1 + r_2$. It may be demonstrated that the function $f_{r_1,r_2} : (r_1 + r_2, \infty) \to \mathbf{R}_+,\ h \mapsto f(r_1, r_2, h)$ is positive, strictly decreasing and converges to zero as h tends to infinity. We now consider each of the two cases given above.

Proof of (i). Suppose that $|z_2 - z_1| > R^3$. Using the inequalities,

$$r_1 + r_2 \leq 2R, \quad \arcsin\big((r_1+r_2)/(|z_2-z_1|)\big) \leq \arcsin(2/R^2), \quad r_1 \geq r_2$$

we obtain for ρ large enough that,

$$f(r_1, r_2, |z_2 - z_1|) \;<\; f(r_1, r_2, R^3) \;\leq\; 4R \arcsin\left(\tfrac{2}{R^2}\right) \;\leq\; c \cdot R^{-1} \leq c \cdot v_\rho^{-1}.$$

Proof of (ii). Suppose that $R \leq (1+\varepsilon)v_\rho$. Since $|z_2 - z_1| > r_1 + r_2$, we get

$$f(r_1, r_2, |z_2 - z_1|) \;<\; f(r_1, r_2, r_1 + r_2) \;=\; 2\pi r_2 + 2(r_1 - r_2)\arccos\left(\tfrac{r_1 - r_2}{r_1 + r_2}\right) - 4\sqrt{r_1 r_2}.$$

Using the inequalities,

$$r_1 \geq r_2 > v_\rho, \quad \arccos\left(\tfrac{r_1 - r_2}{r_1 + r_2}\right) \leq \frac{\pi}{2}, \quad r_1 \leq R \leq (1+\varepsilon)v_\rho,$$

we have

$$f(r_1, r_2, |z_2 - z_1|) \;<\; 2\pi r_2 + (r_1 - v_\rho)\pi - 4v_\rho \;\leq\; 2\pi r_2 - (4 - \varepsilon\pi)v_\rho.$$

□

Proof of Lemma 10 (i). Using the notation defined in Section 4.2.1, we obtain from Bonferroni inequalities

$$\begin{aligned} \mu\Big(\bigcup_{1\leq i\leq K} \mathcal{A}\left(B(z_i, r_i)\right)\Big) &= \mu\Big(\bigcup_{k\leq K}\bigcup_{j\leq n_k} \mathcal{A}\left(B_k^{(j)}\right)\Big) \\ &\geq \sum_{k=1}^{K}\sum_{j=1}^{n_k} \mu\left(\mathcal{A}\left(B_k^{(j)}\right)\right) - \sum_{(k_1,j_1)\neq(k_2,j_2)} \mu\left(\mathcal{A}\left(B_{k_1}^{(j_1)}\right) \cap \mathcal{A}\left(B_{k_2}^{(j_2)}\right)\right). \end{aligned} \tag{29}$$

We begin by observing that for all $1 \leq k_1 \neq k_2 \leq K$ and $1 \leq j_1 \leq n_{k_1}$, $1 \leq j_2 \leq n_{k_2}$ we have

$$\mu\left(\mathcal{A}\left(B_{k_1}^{(j_1)}\right) \cap \mathcal{A}\left(B_{k_2}^{(j_2)}\right)\right) \;\leq\; \sum_{1\leq \ell_1\leq k_1, 1\leq \ell_2 \leq k_2} \mu\left(\mathcal{A}\left(B_{k_1}^{(j_1)}[\ell_1]\right) \cap \mathcal{A}\left(B_{k_2}^{(j_2)}[\ell_2]\right)\right) \;\leq\; c \cdot v_\rho^{-1} \tag{30}$$

when ρ is sufficiently large, with the final inequality following directly from Lemma 9, (i) taking $r_1 := r_{k_1}^{(j_1)}[\ell_1]$ and $r_2 := r_{k_2}^{(j_2)}[\ell_2]$. In addition,

$$\mu\left(\mathcal{A}\left(B_k^{(j)}\right)\right) \;\geq\; \mu\left(\mathcal{A}\left(B_k^{(j)}[1]\right)\right) \;=\; 2\pi \cdot r_k^{(j)}[1]. \tag{31}$$

We then deduce (i) from (29), (30) and (31). □

Proof of Lemma 10 (ii). We proceed along the same lines as in the proof of (i). The only difference concerns the lower bound for $\mu(\mathcal{A}(B_k^{(j)}))$. We shall consider two cases. For each of the n_1 clusters of size one, we have $\mu(\mathcal{A}(B_1^{(j)})) = 2\pi r_1^{(j)}[1]$. Otherwise, we obtain

$$\begin{aligned}\mu\left(\mathcal{A}\left(B_k^{(j)}\right)\right) &= \mu\left(\bigcup_{\ell=1}^{k}\mathcal{A}\left(B_k^{(j)}[\ell]\right)\right)\\ &\geq \mu\left(\mathcal{A}\left(B_k^{(j)}[1]\right)\cup\mathcal{A}\left(B_k^{(j)}[2]\right)\right)\\ &= 2\pi r_k^{(j)}[1] + 2\pi r_k^{(j)}[2] - \mu\left(\mathcal{A}\left(B_k^{(j)}[1]\right)\cap\mathcal{A}\left(B_k^{(j)}[2]\right)\right)\\ &\geq 2\pi\cdot r_k^{(j)}[1] + (4-\varepsilon\pi)v_\rho\end{aligned}$$

which follows from Lemma 9, (ii). We then deduce (ii) from the previous inequality, (29) and (30). □

5 Asymptotics for cells with large inradii

We begin this section by introducing the following notation. Let $t \geq 0$, be fixed.

Notation

- We shall denote the *threshold* and the mean number of cells having an inradius larger than the threshold respectively as

$$v_\rho \;:=\; v_\rho(t) \;:=\; \frac{1}{2\pi}\big(\log(\pi\rho)+t\big) \qquad \text{and} \qquad \tau \;:=\; \tau(t) \;:=\; e^{-t}. \tag{32}$$

- For any $K \geq 1$ and for any K-tuple of convex bodies $C_1, \ldots, C_K$ such that each C_i has a unique inball, define the events

$$E_{C_{1:K}} := \Big\{ \min_{1\leq i\leq K} R(C_i) \geq v_\rho,\ R\left(C_1\right) = \max_{1\leq i\leq K} C_i \Big\}, \tag{33}$$

$$E^{\circ}_{C_{1:K}} := \Big\{ \forall 1 \leq i \neq j \leq K,\, B(C_i) \cap B(C_j) = \varnothing \Big\}. \tag{34}$$

- For any $K \geq 1$, we take

$$I^{(K)}(\rho) := K\,\mathrm{E}\left[\sum_{\substack{C_{1:K}\in(\mathfrak{m}_{\mathrm{PHT}})^K_{\neq},\\ z(C_{1:K})\in\mathbf{W}_\rho^K}} \mathbb{1}_{E_{C_{1:K}}}\right]. \tag{35}$$

The proof for Theorem 1, Part (ii), will then follow by applying Lemma 7 and showing that $I^{(K)}(\rho) \to \tau^k$ as $\rho \to \infty$, for every fixed $K \geq 1$. To begin, we observe that $I^{(1)}(\rho) \to \tau$ as $\rho \to \infty$ as a consequence of (9) and (32). The rest of this section is devoted to considering the case when $K \geq 2$. Given a K-tuple of cells $C_{1:K}$ in $\mathfrak{m}_{\mathrm{PHT}}$, we use $L(C_{1:K})$ to denote the number lines of $\hat{\mathbf{X}}$ (without repetition) which intersect the inballs of the cells. It follows that $3 \leq L(C_{1:K}) \leq 3K$ since the inball of every cell in $\mathfrak{m}_{\mathrm{PHT}}$ intersects exactly three lines (almost surely.) We shall take

$$\big\{H_1, \ldots, H_{L(C_{1:K})}\big\} \;:=\; \big\{H_1(C_{1:K}), \ldots, H_{L(C_{1:K})}(C_{1:K})\big\}$$

to represent the set of lines in $\hat{\mathbf{X}}$ intersecting the inballs of the cells $C_{1:K}$. We remark that conditional on the event $L(C_{1:K}) = 3K$, none of the inballs of the cells share any lines in common. To apply the bounds we obtained in Section 4.2, we will split the cells up into clusters based on the proximity of their inballs using the procedure outlined in Section 4.2.1. In particular, we define

$$n_{1:K}(C_{1:K}) := n_{1:K}(z(C_{1:K}), R(C_1)).$$

We may now re-write $I^{(K)}(\rho)$ by summing over events conditioned on the number of clusters of each size and depending on whether or not the inballs of the cells *share* any lines of the process,

$$I^{(K)}(\rho) = K \sum_{n_{1:K} \in \mathcal{N}_K} \left(I_{S^c}^{(n_{1:K})}(\rho) + I_S^{(n_{1:K})}(\rho) \right), \tag{36}$$

where the size of each cluster of size k is represented by a tuple contained in

$$\mathcal{N}_K := \Big\{ n_{1:K} \in \mathbf{N}^K : \sum_{k=1}^{K} k \cdot n_k = K \Big\},$$

and where for any $n_{1:K} \in \mathcal{N}_K$ we write

$$I_{S^c}^{(n_{1:K})}(\rho) := \mathrm{E}\left[\sum_{\substack{C_{1:K} \in (\mathfrak{m}_{\mathrm{PHT}})^K_{\neq}, \\ z(C_{1:K}) \in \mathbf{W}_\rho^K}} \mathbb{1}_{E_{C_{1:K}}} \mathbb{1}_{n_{1:K}(C_{1:K}) = n_{1:K}} \mathbb{1}_{L(C_{1:K}) = 3K} \right], \tag{37}$$

$$I_S^{(n_{1:K})}(\rho) := \mathrm{E}\left[\sum_{\substack{C_{1:K} \in (\mathfrak{m}_{\mathrm{PHT}})^K_{\neq}, \\ z(C_{1:K}) \in \mathbf{W}_\rho^K}} \mathbb{1}_{E_{C_{1:K}}} \mathbb{1}_{n_{1:K}(C_{1:K}) = n_{1:K}} \mathbb{1}_{L(C_{1:K}) < 3K} \right]. \tag{38}$$

The following proposition deals with the asymptotic behaviours of these functions.

Proposition 11. *Using the notation given in (37) and (38),*

(i) $I_{S^c}^{(K,0,\dots,0)}(\rho) \underset{\rho\to\infty}{\longrightarrow} \tau^K$,

(ii) for all $n_{1:K} \in \mathcal{N}_K \setminus \{(K, 0, \dots, 0)\}$, *we have* $I_{S^c}^{(n_{1:K})}(\rho) \underset{\rho\to\infty}{\longrightarrow} 0$,

(iii) for all $n_{1:K} \in \mathcal{N}_K$, *we have* $I_S^{(n_{1:K})}(\rho) \underset{\rho\to\infty}{\longrightarrow} 0$.

The convergences in Proposition 11 can be understood intuitively as follows. For (i), the inradii of the cells behave as though they are independent, since they are far apart and no line in the process touches more than one of the inballs in the K-tuple (even though two *cells* in the K-tuple may share a line.) For (ii), we are able to show that with high probability the inradii of neighbouring cells cannot simultaneously exceed the level v_ρ, due to Proposition 8, Part (ii). Finally, to obtain the bound in (iii) we use the fact that the proportion of K-tuples of cells which share at least one line is negligible relative to those that do not.

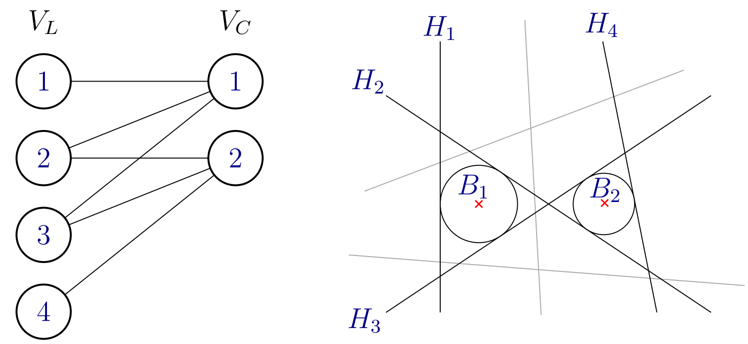

Figure 4: Example of configuration of inballs and lines, with associated configuration graph.

The graph of configurations For Proposition 11, Part (iii), we will need to represent the dependence structure between the *cells* whose inballs share *lines*. To do this, we construct the following *configuration graph*. For $K \geq 2$ and $L \in \{3, \dots, 3K\}$, let $V_C := \{1, \dots, K\}$ and $V_L := \{1, \dots, L\}$. We consider the bipartite graph $\mathbf{G}(V_C, V_L, E)$ with vertices $V := V_C \sqcup V_L$ and edges $E \subset V_C \times V_L$. Let

$$\Lambda_K := \bigcup_{L \leq 3K} \Lambda_{K,L}, \tag{39}$$

where $\Lambda_{K,L}$ represents the collection of all graphs which are isomorphic up to relabling of the vertices and satisfying

1. $\text{degree}(v) = 3, \forall v \in V_C$,
2. $\text{degree}(w) \geq 1, \forall w \in V_L$,
3. $\text{neighbours}(v) \neq \text{neighbours}(v'), \ \forall (v, v') \in (V_C)^2_{\neq}$.

We shall use V_C to represent the cells and V_L to represent the lines in a line process, with each graph edge implying that a line intersects the inball of a cell. The number of such bipartite graphs is finite since $|\Lambda_{K,L}| \leq 2^{KL}$ so that $|\Lambda_K| \leq 3K \cdot 2^{(3K^2)}$.

Proofs

Proof of Proposition 11 (i). For any $1 \leq i \leq K$ and the 3-tuple of lines $H_i^{(1:3)} := (H_i^{(1)}, H_i^{(2)}, H_i^{(3)})$, we recall that $\triangle_i := \triangle_i(H_i^{(1)}, H_i^{(2)}, H_i^{(3)})$ denotes the unique triangle that can be formed by the intersection of the half-spaces induced by the lines $H_i^{(1:3)}$. For brevity, we write $B_i := B(\triangle_i)$ and $H_{1:K}^{(1:3)} := (H_1^{(1:3)}, \dots, H_K^{(1:3)})$. We shall often omit the arguments when they are obvious from context.

Since $1_{E_{C_{1:K}}} = 1_{E_{B_{1:K}}}$ and since the lines of $\hat{\mathbf{X}}$ do not intersect the inballs in their interior, we have

$$\begin{aligned} I_{S^c}^{(K,0,\dots,0)}(\rho) &= \frac{K}{6^K}\,\mathrm{E}\Bigg[\sum_{H_{1:K}^{(1:3)}\in\mathbf{X}_{\neq}^{3K}} 1_{\left\{\hat{\mathbf{X}}\setminus\cup_{i\leq K, j\leq 3}H_i^{(j)}\right\}\cap\left\{\cup_{i\leq K}B_i\right\}=\varnothing} \\ &\qquad\qquad \times 1_{z(B_{1:K})\in\mathbf{W}_\rho^K}\, 1_{E_{B_{1:K}}}\, 1_{n_{1:K}(B_{1:K})=(K,0,\dots,0)}\Bigg] \\ &= \frac{K}{6^K}\int_{\mathcal{A}^{3K}} e^{-\phi\left(\bigcup_{i\leq K}B_i\right)} 1_{z(B_{1:K})\in\mathbf{W}_\rho^K}\, 1_{E_{B_{1:K}}}\, 1_{n_{1:K}(B_{1:K})=(K,0,\dots,0)}\, \mu\left(dH_{1:K}^{(1:3)}\right), \end{aligned}$$

where the last equality comes from (5) and Mecke-Slivnyak's formula. Applying the Blaschke-Petkantschin formula, we get

$$\begin{aligned} I_{S^c}^{(K,0,\dots,0)}(\rho) &= \frac{K}{24^K}\int_{(\mathbf{W}_\rho\times\mathbf{R}_+\times\mathbf{S}^3)^K} e^{-\phi\left(\bigcup_{i\leq K}B(z_i,r_i)\right)}\prod_{i\leq K} a\big(u_i^{(1:3)}\big) 1_{E_{B_{1:K}}} \\ &\qquad \times 1_{n_{1:K}(B_{1:K})=(K,0,\dots,0)}\, dz_{1:K}\, dr_{1:K}\, \sigma\left(du_{1:K}^{(1:3)}\right), \end{aligned}$$

where we recall that $a(u_i^{(1:3)})$ is the area of the triangle spanned by $u_i^{(1:3)}\in\mathbf{S}^3$. From (27) and (28), we have for any $1\leq i\leq K$,

$$e^{-2\pi\sum_{i=1}^K r_i}\cdot 1_{E_{B_{1:K}}} \quad\leq\quad e^{-\phi\left(\bigcup_{i\leq K}B(z_i,r_i)\right)}\cdot 1_{E_{B_{1:K}}} \quad\leq\quad e^{-2\pi\sum_{i=1}^K r_i}\cdot e^{c\cdot v_\rho^{-1}}\cdot 1_{E_{B_{1:K}}}.$$

According to (33), this implies that

$$\begin{aligned} I_{S^c}^{(K,0,\dots,0)}(\rho) &\underset{\rho\to\infty}{\sim} \frac{K}{24^K}\int_{(\mathbf{W}_\rho\times\mathbf{R}_+\times\mathbf{S}^3)^K}\prod_{i\leq K} e^{-2\pi\cdot r_i} a\big(u_i^{(1:3)}\big) 1_{r_i>v_\rho}\, 1_{r_1=\max_{j\leq K} r_j}\, 1_{|z_i-z_j|>r_1^3 \text{ for } j\neq i} \\ &\qquad \times dz_{1:K}\, dr_{1:K}\, \sigma\left(du_{1:K}^{(1:3)}\right) \\ &= \frac{K\tau^K}{(24\pi)^K}\int_{(\mathbf{W}_1\times\mathbf{R}_+\times\mathbf{S}^3)^K}\prod_{i\leq K} e^{-2\pi\cdot r_i'} a\big(u_i^{(1:3)}\big) 1_{r_1'=\max_{j\leq K} r_j'}\, 1_{|z_i'-z_j'|>\rho^{-1/2}r_1'^3 \text{ for } j\neq i} \\ &\qquad \times dz_{1:K}'\, dr_{1:K}'\, \sigma\left(du_{1:K}^{(1:3)}\right), \end{aligned}$$

where the last equality comes from (32) and the change of variables $z_i' = \rho^{-1/2}z_i$ and $r_i' = r_i - v_\rho$. It follows from the monotone convergence theorem that

$$\begin{aligned} I_{S^c}^{(K,0,\dots,0)}(\rho) &\underset{\rho\to\infty}{\sim} \frac{K\tau^K}{(24\pi)^K}\int_{(\mathbf{W}_1\times\mathbf{R}_+\times\mathbf{S}^3)^K}\prod_{i\leq K} e^{-2\pi\cdot r_i} a\left(u_i^{(1:3)}\right) 1_{r_1=\max_{j\leq K} r_j}\, dz_{1:K} dr_{1:K}\, \sigma(du_{1:K}^{(1:3)}) \\ &= \frac{\tau^K}{(24\pi)^K}\left(\int_{(\mathbf{W}_1\times\mathbf{R}_+\times\mathbf{S}^3)^K} a(u_{1:3}) e^{-2\pi r} dz\, dr\, \sigma(du_{1:3})\right)^K \\ &\underset{\rho\to\infty}{\longrightarrow} \tau^K, \end{aligned}$$

where the last line follows by integrating over z, r and $u_{1:3}$, and by using the fact that $\lambda_2(\mathbf{W}_1) = 1$ and $\int_{\mathbf{S}^3} a(u_{1:3})\,\sigma(du_{1:3}) = 48\pi^2$.

□

Proof of Proposition 11 (ii). Beginning in the same way as in the proof of (i), we have

$$I_{S^c}^{(n_{1:K})}(\rho) = \frac{K}{24^K} \int_{(\mathbf{W}_\rho \times \mathbf{R}_+ \times \mathbf{S}^3)^K} e^{-\phi\left(\bigcup_{i\le K} B(z_i,r_i)\right)} \prod_{i\le K} a\left(u_i^{(1:3)}\right) 1_{E_{B_{1:K}}} 1_{E^\circ_{B_{1:K}}} \times dz_{1:K}\, dr_{1:K}\ \sigma\left(du_{1:K}^{(1:3)}\right),$$

where the event $E^\circ_{B_{1:K}}$ is defined in (34). Integrating over $u_{1:K}^{(1:3)}$, we get

$$\begin{aligned} I_{S^c}^{(n_{1:K})}(\rho) &= c \cdot \int_{(\mathbf{W}_\rho \times \mathbf{R}_+)^K} e^{-\phi\left(\bigcup_{i\le K} B(z_i,r_i)\right)} \prod_{i\le K} 1_{E_{B_{1:K}}} 1_{E^\circ_{B_{1:K}}} 1_{n_{1:K}(z_{1:K},r_1)=n_{1:K}}\, dz_{1:K} dr_{1:K} \\ &= I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho) + I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho), \end{aligned}$$

where, for any $\varepsilon > 0$, the terms $I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho)$ and $I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho)$ are defined as the term of the first line when we add the indicator that r_1 is larger than $(1+\varepsilon)v_\rho$ in the integral and the indicator for the complement respectively. We provide below a suitable upper bound for these two terms. For $I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho)$, we obtain from Proposition 8 (iia) that

$$I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho) \le c \cdot \int_{(\mathbf{W}_\rho \times \mathbf{R}_+)^K} e^{-\left(2\pi r_1 + \left(\sum_{k=1}^K n_k - 1\right)2\pi v_\rho - c\cdot v_\rho^{-1}\right)} 1_{r_1 > (1+\varepsilon)v_\rho} 1_{r_1 = \max_{j\le K} r_j} \times 1_{n_{1:K}(z_{1:K},r_1)=n_{1:K}}\, dz_{1:K}\, dr_{1:K}.$$

Integrating over $r_{2:K}$ and $z_{1:K}$, we obtain

$$I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho) \le c \cdot \int_{(1+\varepsilon)v_\rho}^{\infty} r_1^{K-1} e^{-\left(2\pi r_1 + \left(\sum_{k=1}^K n_k - 1\right)2\pi v_\rho\right)} \times \lambda_{dK}\left(\left\{\, z_{1:K} \in \mathbf{W}_\rho^K : n_{1:K}(z_{1:K}, r_1) = n_{1:K} \,\right\}\right) dr_1. \tag{40}$$

Furthermore, for each $n_{1:K} \in \mathcal{N}_K \setminus \{(K,0,\dots,0)\}$, we have

$$\lambda_{dK}\left(\left\{\, z_{1:K} \in \mathbf{W}_\rho^K : n_{1:K}(z_{1:K}, r_1) = n_{1:K} \,\right\}\right) \ \le\ c \cdot \rho^{\sum_{k=1}^K n_k} \cdot r_1^{6\left(K - \sum_{k=1}^K n_k\right)}, \tag{41}$$

since the number of connected components of $\bigcup_{i=1}^K B(z_i, r_1^3)$ equals $\sum_{k=1}^K n_k$. It follows from (40) and (41) that there exists a constant $c(K)$ such that

$$\begin{aligned} I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho) &\le c \cdot \left(\rho e^{-2\pi\, v_\rho}\right)^{\left(\sum_{k=1}^K n_k\right)} e^{2\pi v_\rho} \int_{(1+\varepsilon)v_\rho}^{\infty} r_1^{c(K)} e^{-2\pi r_1} dr_1 \\ &= O\left((\log \rho)^{c(K)} \rho^{-\varepsilon}\right), \end{aligned}$$

according to (32). For $I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho)$, we proceed exactly as for $I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho)$, but this time we apply the bound given in Proposition 8 (iib). We obtain

$$\begin{aligned} I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho) &\le c \cdot \left(\rho\, e^{-2\pi v_\rho}\right)^{\left(\sum_{k=1}^K n_k\right)} e^{2\pi v_\rho - \sum_{k=2}^K n_k (4-\varepsilon\pi) v_\rho} \int_{v_\rho}^{(1+\varepsilon)v_\rho} r_1^{c(K)} e^{-2\pi r_1}\, dr_1 \\ &= O\left((\log \rho)^c \cdot \rho^{-\frac{4-\varepsilon\pi}{2\pi}\cdot}\right) \end{aligned}$$

since for all $n_{1:K} \in \mathcal{N}_K \setminus \{(K,0,\dots,0)\}$, there exists a $2 \le k \le K$ such that n_k is non-zero. Choosing $\varepsilon < \frac{4}{\pi}$ ensures that $I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho) \to 0$ as $\rho \to \infty$. □

Proof of Proposition 11 (iii). Let $\mathbf{G} = \mathbf{G}(V_C, V_L, E) \in \Lambda_K$, with $|V_L| = L$ and $|V_C| = K$, be a bipartite graph as in Page 23. With $\mathbf{G}$, we can associate a (unique up to re-ordering of the lines) way to construct K triangles from L lines by taking V_C to denote the set of indices of the triangles, V_L to denote the set of indices of the lines and the edges to represent intersections between them. Besides, let $H_1, \dots, H_L$ be an L-tuple of lines. For each $1 \leq i \leq K$, let $e_i = \{e_i(0), e_i(1), e_i(2)\}$ be the tuple of neighbours of the ith vertex in V_C. In particular,

$$B_i(\mathbf{G}) := B\left(\triangle_i(\mathbf{G})\right) \qquad \text{and} \qquad \triangle_i(\mathbf{G}) := \triangle\left(H_{e_i(0)}, H_{e_i(1)}, H_{e_i(2)}\right)$$

denote the inball and the triangle generated by the 3-tuple of lines with indices in e_i. An example of this configuration graph is given in Figure 4. According to (38), we have

$$I_S^{(n_{1:K})}(\rho) = \sum_{\mathbf{G} \in \Lambda_K} I_{S_{\mathbf{G}}}^{(n_{1:K})}(\rho),$$

where for all $n_{1:K} \in \mathcal{N}_K$ and $\mathbf{G} \in \Lambda_K$, we write

$$\begin{aligned} I_{S_{\mathbf{G}}}^{(n_{1:K})}(\rho) &= \mathrm{E}\left[\sum_{H_{1:L} \in \mathbf{X}_{\neq}^{L}} \mathbb{1}_{\left\{\hat{\mathbf{X}} \setminus \cup_{i \leq L} H_i\right\} \cap \left\{\cup_{i \leq K} B_i(\mathbf{G})\right\} = \varnothing} \mathbb{1}_{z(B_{1:K}(\mathbf{G})) \in \mathbf{W}_\rho^K} \mathbb{1}_{E_{B_{1:K}(\mathbf{G})}} \mathbb{1}_{E^\circ_{B_{1:K}(\mathbf{G})}} \right. \\ &\qquad \left. \times \mathbb{1}_{n_{1:K}(B_{1:K}(\mathbf{G})) = n_{1:K}} \right] \\ &= \int_{\mathcal{A}^{|V_L|}} e^{-\phi\left(\bigcup_{i \leq K} B_i(\mathbf{G})\right)} \mathbb{1}_{z(B_{1:K}(\mathbf{G})) \in \mathbf{W}_\rho^K} \mathbb{1}_{E_{B_{1:K}(\mathbf{G})}} \mathbb{1}_{E^\circ_{B_{1:K}(\mathbf{G})}} \\ &\qquad \times \mathbb{1}_{n_{1:K}(B_{1:K}(\mathbf{G})) = n_{1:K}} \, \mu(dH_{1:L}). \end{aligned} \tag{42}$$

We now prove that $I_{S_{\mathbf{G}}}^{(n_{1:K})}(\rho) \to 0$ as $\rho \to \infty$. Suppose first that $n_{1:K} = (K, 0, \dots, 0)$. In this case, we obtain from (42), Proposition 8 (iia) and (33) and (34) that

$$\begin{aligned} I_{S_{\mathbf{G}}}^{(K,0,\dots,0)}(\rho) &\leq c \cdot \int_{\mathcal{A}^L} e^{-2\pi \cdot (R(B_1(\mathbf{G})) + (K-1)v_\rho)} \mathbb{1}_{z(B_{1:K}(\mathbf{G})) \in \mathbf{W}_\rho} \mathbb{1}_{R(B_1(\mathbf{G})) > v_\rho} \\ &\qquad \times \mathbb{1}_{R(B_1(\mathbf{G})) = \max_{j \leq K} R(B_j(\mathbf{G}))} \mathbb{1}_{n_{1:K}(B_{1:K}(\mathbf{G})) = (K,0,\dots,0)} \, \mu(dH_{1:L}) \\ &\leq c \cdot \rho^{\frac{1}{2}} \int_{v_\rho}^{\infty} r^{c(K)} e^{-2\pi r} dr \\ &= O\left((\log \rho)^{c(K)} \rho^{-\frac{1}{2}}\right), \end{aligned} \tag{43}$$

where the second inequality of (43) is a consequence of (32) and Lemma 13 applied to $f(r) := e^{-2\pi r}$. Suppose now that $n_{1:K} \in \mathcal{N}_K \setminus \{(K, 0, \dots, 0)\}$. In the same spirit as in the proof of Proposition 11 (ii), we shall re-write

$$I_{S_{\mathbf{G}}}^{(n_{1:K})}(\rho) = I_{S_{\mathbf{G}}, a_\varepsilon}^{(n_{1:K})}(\rho) + I_{S_{\mathbf{G}}, b_\varepsilon}^{(n_{1:K})}(\rho) \tag{44}$$

by adding the indicator that $R(B_1(\mathbf{G}))$ is larger than $(1+\varepsilon)v_\rho$ and the opposite in (42). For $I_{S_{\mathbf{G}}, a_\varepsilon}^{(n_{1:K})}(\rho)$, we similarly apply Proposition 8 (iia) to get

$$
\begin{aligned}
I^{(n_{1:K})}_{S_{\mathbf{G}},a_\varepsilon}(\rho) \leq c \cdot \int_{\mathcal{A}^L} e^{-2\pi\left(R(B_1(\mathbf{G}))+\left(\sum_{k=1}^K n_k - 1\right)v_\rho\right)} \mathbb{1}_{z(B_{1:K}(\mathbf{G}))\in \mathbf{W}_\rho} \mathbb{1}_{R(B_1(\mathbf{G}))>(1+\varepsilon)v_\rho} \\
\times \mathbb{1}_{R(B_1(\mathbf{G}))=\max_{j\leq K} R(B_j(\mathbf{G}))} \mathbb{1}_{n_{1:K}(B_{1:K}(\mathbf{G}))=n_{1:K}} \ \mu(dH_{1:L}) \\
\leq c \cdot \left(\rho e^{-2\pi v_\rho}\right)^{\sum_{k=1}^K n_k} \cdot \rho \int_{(1+\varepsilon)v_\rho}^{\infty} r^{c(K)} e^{-2\pi r} dr \qquad (45) \\
= O\left((\log\rho)^{c(K)} \rho^{-\varepsilon}\right),
\end{aligned}
$$

where (45) follows by applying Lemma 13. To prove that $I^{(n_{1:K})}_{S_{\mathbf{G}},b_\varepsilon}(\rho)$ converges to zero, we proceed exactly as before but this time applying Proposition 8 (iib). As for $I^{(n_{1:K})}_{S^c,b_\varepsilon}(\rho)$, we show that

$$
I^{(n_{1:K})}_{S^c,b_\varepsilon}(\rho) = O\left((\log\rho)^{c(K)} \rho^{-\frac{4-\varepsilon\pi}{2\pi}}\right)
$$

by taking $\varepsilon < \frac{4}{\pi}$. This together with (44) and (45) gives that $I^{(n_{1:K})}_{S_{\mathbf{G}}}(\rho)$ converges to zero for any $n_{1:K} \in \mathcal{N}_K \setminus \{(K, 0, \ldots, 0)\}$. □

Proof of Theorem 1 (ii). According to Lemma 7, it is now enough to show that for all $K \geq 1$, we have $I^{(K)}(\rho) \to \tau^K$ as $\rho \to \infty$. This fact is a consequence of (36) and Proposition 11. □

A Technical lemmas

The following technical lemmas are required for the proofs of Proposition 3 and Proposition 11 (iii).

Lemma 12. *Let $R, R' > 0$ and let $z' \in \mathbf{R}^d$.*

(i) For all $H_1 \in \mathcal{A}$, we have

$$
G(H_1) := \int_{\mathcal{A}^2} \mathbb{1}_{z(H_{1:3})\in B(z',R')} \mathbb{1}_{R(H_{1:3})<R} \ \mu(dH_{2:3}) \quad \leq \quad c \cdot R \cdot R' \cdot \mathbb{1}_{d(0,H_1)<R+R'} \, .
$$

(ii) For all $H_1, H_2 \in \mathcal{A}$, we have

$$
G(H_1, H_2) := \int_{\mathcal{A}} \mathbb{1}_{z(H_{1:3})\in B(z',R')} \mathbb{1}_{R(H_{1:3})<R} \ \mu(dH_3) \quad \leq \quad c \cdot (R + R').
$$

Lemma 13. *Let $n_{1:K} \in \mathcal{N}_{1:K}$, $f\colon \mathbf{R}_+ \to \mathbf{R}_+$, $\mathbf{G} = \mathbf{G}(V_C, V_L, E) \in \Lambda_K$ with $3 \leq L < 3K$ and let*

$$
\begin{aligned}
F^{(n_{1:K})} := \int_{\mathcal{A}^L} f(R(B_1(\mathbf{G}))) \cdot \mathbb{1}_{z(B_{1:K}(\mathbf{G}))\in \mathbf{W}_\rho} \mathbb{1}_{R(B_1(\mathbf{G}))>v'_\rho} \mathbb{1}_{R(B_1(\mathbf{G}))=\max_{j\leq K} R(B_j(\mathbf{G}))} \\
\times \mathbb{1}_{n_{1:K}(B_{1:K}(\mathbf{G}))=n_{1:K}} \ \mu(dH_{1:L}),
\end{aligned}
$$

where $v'_\rho \to \infty$. Then for some constant $c(K)$, we have

$$
F^{(n_{1:K})} \leq \rho^{\min\left\{\sum_{k=1}^K n_k, K-\frac{1}{2}\right\}} \int_{v'_\rho}^{\infty} r^{c(K)} f(r) dr.
$$

Proof of Lemma 12 (i). The following proof reduces to giving the analagous version of the Blaschke-Petkanschin type change of variables (Theorem 7.3.2 in Schneider and Weil [21]) in which one of the lines is held fixed. We proceed in the same spirit as in the proof of Theorem 7.3.2 in Schneider and Weil [21]. Without loss of generality, we can assume that $z' = 0$ since μ is stationary. Let $H_1 \in \mathcal{A} = H(u_1, t_1)$ be fixed, for some $u_1 \in \mathbf{S}$ and $t_1 \in \mathbf{R}$. We denote by $\mathcal{A}^2_{H_1} \subset \mathcal{A}^2$ the set of pairs of lines (H_2, H_3) such that H_1, H_2 and H_3 are in general position and by $P_{H_1} \subset \mathbf{S}^2$ the set of pairs of unit vectors (u_2, u_3) such that $0 \in \mathbf{R}^2$ belongs to the interior of the convex hull of $\{u_1, u_2, u_3\}$. Then, the mapping

$$\begin{aligned}\phi_{H_1} : \mathbf{R}^2 \times P_{H_1} &\longrightarrow \mathcal{A}_{H_1}\\ (z, u_2, u_3) &\longmapsto (H(u_2, t_2), H(u_3, t_3)),\end{aligned}$$

with $t_i := \langle z, u_i \rangle + r$ and $r := d(z, H_1)$ is bijective. We can easily prove that its Jacobian $J_{\Phi_{H_1}}(z, u_2, u_3)$ is bounded. Using the fact that $d(0, H_1) \leq |z(H_1 1:3)| + R(H_{1:3}) < R + R'$ provided that $z(H_{1:3}) \in B(0, R')$ and $R(H_{1:3}) < R$, it follows that

$$\begin{aligned}G(H_1) &\leq \int_{\mathbf{R}^2 \times P} |J\phi_{H_1}(z, u_2, u_3)| \mathbb{1}_{z \in B(0,R')} \mathbb{1}_{d(z,H_1)<R} \mathbb{1}_{d(0,H_1)<R+R'} \, \sigma(du_{2:3}) dz\\ &\leq c \cdot \lambda_2 \left(B(0, R') \cap (H_1 \oplus B(0, R))\right) \mathbb{1}_{d(0,H_1)<R+R'}\\ &\leq c \cdot R \cdot R' \cdot \mathbb{1}_{d(0,H_1)<R+R'} \,,\end{aligned}$$

where $A \oplus B$ denotes the Minkowski sum between two Borel sets $A, B \in \mathcal{B}(\mathbf{R}^2)$. □

Proof of Lemma 12 (ii). Let H_1 and H_2 be fixed and let H_3 be such that $z(H_{1:3}) \in B(z', R')$ and $R(H_{1:3}) < R$. This implies that

$$d(z', H_3) \quad \leq \quad |z' - z(H_{1:3})| + d(z(H_{1:3}), H_3) \quad \leq \quad R + R'.$$

Integrating over H_3, we get

$$G(H_1, H_2) \quad \leq \quad \int_{\mathcal{A}} \mathbb{1}_{d(z',H_3) \leq R+R'} \, \mu(dH_3) \quad \leq \quad c \cdot (R + R'). \tag{46}$$

□

Proof of Lemma 13. Our proof will follow by re-writing the set of lines $\{1, \ldots, |V_L|\}$, as a disjoint union. We take

$$\Big\{1, \ldots, |V_L|\Big\} = \bigsqcup_{i=1}^{K} e^\star_i \qquad \text{where} \qquad e^\star_i := \big\{e_i(0), e_i(1), e_i(2)\big\} \setminus \bigcup_{j<i} \big\{e_j(0), e_j(1), e_j(2)\big\}.$$

In this way, $\{e^\star_i\}_{i \leq K}$ may understood as associating lines of the process with the inballs of the K cells under consideration, so that no line is associated with more than one inball. In particular, each inball has between zero and three lines associated with it, $0 \leq |e^\star_i| \leq 3$ and $|e^\star_1| = 3$ by definition. We now consider two cases depending on the configuration of the clusters, $n_{1:K} \in \mathcal{N}_K$.

Independent clusters To begin with, we suppose that $n_{1:K} = (K, 0, \ldots, 0)$. For convenience, we shall write

$$\mu(dH_{e_i^\star}) := \prod_{j \in e_i^\star} \mu(dH_j),$$

for some arbitrary ordering of the elements, and defining the empty product to be 1. It follows from Fubini's theorem that

$$\begin{aligned} F^{(K,0,\ldots,0)} = & \int_{\mathcal{A}^3} f(R(B_1(\mathbf{G}))) 1_{z(B_1(\mathbf{G}))\in \mathbf{W}_\rho} 1_{R(\triangle_1(\mathbf{G}))>v'_\rho} \\ & \times \int_{\mathcal{A}^{|e_2^\star|}} 1_{z(B_j(\mathbf{G}))\in \mathbf{W}_\rho} 1_{R(\triangle_2(\mathbf{G}))\le R(B_1(\mathbf{G}))} \\ & \quad \cdots \\ & \quad \times \left[\int_{\mathcal{A}^{|e_K^\star|}} 1_{z(B_K(\mathbf{G}))\in \mathbf{W}_\rho} 1_{R(\triangle_K(\mathbf{G}))\le R(B_1(\mathbf{G}))} \ \mu(dH_{e_K^\star}) \right] \\ & \quad \times \mu(dH_{e_{K-1}^\star}) \cdots \mu(dH_{e_1^\star}). \end{aligned} \tag{47}$$

We now consider three possible cases for the inner-most integral above, (47).

1. If $|e_K^\star| = 3$, the integral equals $c \cdot R(B_1(\mathbf{G}))\rho$ after a Blaschke-Petkanschin change of variables.
2. If $|e_K^\star| = 1, 2$, the integral is bounded by $c \cdot \rho^{1/2} R(B_1(\mathbf{G}))$ thanks to Lemma 12 applied with $R := R(B_1(\mathbf{G}))$, $R' := \pi^{-1/2}\rho^{1/2}$.
3. If $|e_K^\star| = 0$, the integral decays and we may bound the indicators by one. To simplify our notation we just assume the integral is bounded by $c \cdot \rho^{1/2} R(B_1(\mathbf{G}))$.

To distinguish these cases, we define $x_i := 1_{|e_i^\star|<3}$, giving

$$\begin{aligned} F^{(K,0,\ldots,0)} \le & \ c \cdot \rho^{1-\frac{x_K}{2}} \int_{\mathcal{A}^3} R(B_1(\mathbf{G}))^{c(K)} \cdot f(R(B_1(\mathbf{G}))) 1_{R(\triangle_1(\mathbf{G}))>v'_\rho} 1_{z(B_1(\mathbf{G}))\in \mathbf{W}_\rho} \\ & \times \int_{\mathcal{A}^{|e_2^\star|}} 1_{R(\triangle_2(\mathbf{G}))\le R(B_1(\mathbf{G}))} 1_{z(B_j(\mathbf{G}))\in \mathbf{W}_\rho} \\ & \quad \cdots \\ & \quad \times \left[\int_{\mathcal{A}^{|e_{K-1}^\star|}} 1_{R(\triangle_K(\mathbf{G}))\le R(B_1(\mathbf{G}))} 1_{z(B_{K-1}(\mathbf{G}))\in \mathbf{W}_\rho} \ \mu(dH_{e_{K-1}^\star}) \right] \\ & \quad \times \mu(dH_{e_{K-2}^\star}) \cdots \mu(dH_{e_1^\star}) \end{aligned}$$

Recursively applying the same bound, we deduce from the Blaschke-Petkanschin formula that

$$\begin{aligned} F^{(K,0,\ldots,0)} & \le c \cdot \rho^{\sum_{i=2}^K (1-\frac{1}{2}x_i)} \int_{\mathcal{A}^3} R(H_{1:3})^{c(K)} f(R(H_{1:3})) 1_{R(H_{1:3})>v'_\rho} 1_{z(H_{1:3})\in \mathbf{W}_\rho} \ \mu(dH_{1:3}) \\ & = c \cdot \rho^{\left(K - \frac{1}{2}\sum_{i=2}^K x_i\right)} \int_{v'_\rho}^\infty r^{c(K)} \cdot f(r)\, dr \end{aligned}$$

Since, by assumption $|V_L| < 3K$, it follows that $x_i = 1$ for some $i > 1$

$$F^{(K,0,\ldots,0)} \leq c \cdot \rho^{K-\frac{1}{2}} \int_{v'_\rho}^{\infty} r^{c(K)} \cdot f(r)\, dr,$$

as required.

Dependent clusters We now focus on the case in which $n_{1:K} \in \mathcal{N}_K \setminus \{(K, 0, \ldots, 0)\}$. We proceed in the same spirit as before. For any $1 \leq i \neq j \leq K$, we write $B_i(\mathbf{G}) \not\leftrightarrow B_j(\mathbf{G})$ to specify that the balls $B(z(B_i(\mathbf{G})), R(B_1(\mathbf{G}))^3)$ and $B(z(B_j(\mathbf{G})), R(B_1(\mathbf{G}))^3)$ are not in the same connected component of $\bigcup_{l=1}^K B(z(B_l(\mathbf{G})), R(B_1(\mathbf{G}))^3)$. Then we choose a unique 'delegate' convex for each cluster using the following indicator,

$$\alpha_i(B_{1:K}(\mathbf{G})) := 1_{\forall i<j,\, B_j(\mathbf{G}) \not\leftrightarrow B_i(\mathbf{G})}\,.$$

It follows that $\sum_{k=1}^K n_k = \sum_{i=1}^K \alpha_i(B_{1:K}(\mathbf{G}))$ and $\alpha_1(B_{1:K}(\mathbf{G})) = 1$. The set of all possible ways to select the delegates is given by,

$$A_{n_{1:K}} := \Big\{ \alpha_{1:K} \in \{0,1\}^K : \sum_{i=1}^K \alpha_i = \sum_{k=1}^K n_k \Big\}.$$

Then we have,

$$\begin{aligned} F^{(n_{1:K})} = & \sum_{\alpha_{1:K} \in A_{n_{1:K}}} \int_{\mathcal{A}^3} f(R(B_1(\mathbf{G}))) 1_{z(B_1(\mathbf{G})) \in \mathbf{W}_\rho}\, 1_{R(\triangle_1(\mathbf{G})) > v'_\rho} \\ & \times \int_{\mathcal{A}^{|e_2^\star|}} 1_{z(B_j(\mathbf{G})) \in \mathbf{W}_\rho}\, 1_{R(\triangle_2(\mathbf{G})) \leq R(B_1(\mathbf{G}))}\, 1_{\alpha_2(B_{1:K}(\mathbf{G})) = \alpha_2} \\ & \ldots \\ & \quad \times \Big[\int_{\mathcal{A}^{|e_K^\star|}} 1_{z(B_K(\mathbf{G})) \in \mathbf{W}_\rho}\, 1_{R(\triangle_K(\mathbf{G})) \leq R(B_1(\mathbf{G}))}\, 1_{\alpha_K(B_{1:K}(\mathbf{G})) = \alpha_K}\ \mu(dH_{e_K^\star}) \Big] \\ & \quad \times \mu(dH_{e_{K-1}^\star}) \cdots \mu(dH_{e_1^\star}) \end{aligned}$$

For this part, we similarly split into multiple cases and recursively bound the inner-most integral.

1. When $\alpha_K = 1$, the integral equals $c \cdot R(B_1(\mathbf{G}))\rho$ if $e_K^\star = 3$ thanks to the Blaschke-Petkanschin formula and is bounded by $c \cdot \rho^{1/2} R(B_1(\mathbf{G}))$ otherwise thanks to Lemma 12. In particular, we bound the integral by $c \cdot R(B_1(\mathbf{G}))^{c(K)} \rho^{\alpha_K}$.

2. When $\alpha_K = 0$, the integral equals $c \cdot R(B_1(\mathbf{G}))^7$ if $e_K^\star = 3$ and is bounded by $c \cdot R(B_1(\mathbf{G}))^{5/2}$ otherwise for similar arguments. In this case, we can also bound the integral by $c \cdot R(B_1(\mathbf{G}))^{c(K)} \rho^{\alpha_K}$.

Proceeding in the same way and recursively for all $2 \leq i \leq K$, we get

$$
\begin{aligned}
F^{(n_{1:K})} &\leq c \cdot \sum_{\alpha_{1:K} \in A_{n_{1:K}}} \rho^{\sum_{i=2}^{K} \alpha_i} \int_{\mathcal{A}^3} R(B_1(\mathbf{G}))^{c(K)} f(R(H_{1:3})) \mathbb{1}_{z(B_1(\mathbf{G})) \in \mathbf{W}_\rho} \mathbb{1}_{R(B_1(\mathbf{G})) > v'_\rho} \, \mu(dH_{1:3}) \\
&= c \cdot \sum_{\alpha_{1:K} \in A_{n_{1:K}}} \rho^{\sum_{i=2}^{K} \alpha_i} \rho \int_{v'_\rho}^{\infty} r^{c(K)} f(r) dr \\
&\leq c \cdot \rho^{\sum_{k=1}^{K} n_k} \int_{v'_\rho}^{\infty} r^{c(K)} f(r) dr,
\end{aligned}
$$

since

$$
\sum_{i=2}^{K} \alpha_i + 1 \; = \; \sum_{i=1}^{K} \alpha_i \; = \; \sum_{k=1}^{K} n_k.
$$

□

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