Only distances are required to reconstruct submanifolds

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Abstract

In this paper, we give the first algorithm that outputs a faithful reconstruction of a submanifold of Euclidean space without maintaining or even constructing complicated data structures such as Voronoi diagrams or Delaunay complexes. Our algorithm uses the witness complex and relies on the stability of *power protection*, a notion introduced in this paper. The complexity of the algorithm depends exponentially on the intrinsic dimension of the manifold, rather than the dimension of ambient space, and linearly on the dimension of the ambient space. Another interesting feature of this work is that no explicit coordinates of the points in the point sample is needed. The algorithm only needs the *distance matrix* as input, i.e., only distance between points in the point sample as input.

Keywords. Witness complex, power protection, sampling, manifold reconstruction

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Contents

1 Introduction

We present an algorithm for reconstructing a submanifold of Euclidean space, from an input point sample, that does not require Delaunay complexes, unlike previous algorithms, which either had to maintain a subset of the Delaunay complex in the ambient space [CDR05, BGO09], or a family of m-dimensional Delaunay complexes [BG14]. Maintaining these highly structured data structures is challenging and in addition, the methods are limited as they require explicit coordinates of the points in the input point sample. One of the goals of this work was to develop a procedure to reconstruct submanifolds that only uses elementary data structures.

We use the witness complex to achieve this goal. The witness complex was introduced by Carlsson and de Silva [CdS04]. Given a point cloud W, their idea was to carefully select a subset L of landmarks on top of which the witness complex is built, and to use the remaining data points to drive the complex construction. More precisely, a point $w \in W$ is called a *witness* for a simplex $\sigma \in 2^L$ if no point of $L \setminus \sigma$ is closer to w than are the vertices of σ, i.e., if there is a closed ball centered at w that includes the vertices of σ, but contains no other points of L in its interior. The witness complex is then the largest abstract simplicial complex that can be assembled using only witnessed simplices. The geometric test for being a witness can be viewed as a simplified version of the classical Delaunay predicate, and its great advantage is to only require mere comparisons of (squared) distances. As a result, witness complexes can be built in arbitrary metric spaces, and the construction time is bound to the size of the input point cloud rather than to the dimension d of the ambient space.

Since its introduction, the witness complex has attracted interest, which can be explained by its close connection to the Delaunay triangulation and the restricted Delaunay complex [AEM07, BGO09, CIdSZ08, CO08, CdS04, GO08]. In his seminal paper [dS08], de Silva showed that the witness complex is always a subcomplex of the Delaunay triangulation $\mathrm{Del}(L)$, provided that the data points lie in some Euclidean space or more generally in some Riemannian manifold of constant sectional curvature. With applications to reconstruction in mind, Attali, Edelsbrunner, and Mileyko [AEM07], and Guibas and Oudot [GO08] considered the case where the data points lie on or close to some m-submanifold of $\mathbb{R}^d$. They showed that the witness complex is equal to the restricted Delaunay complex when $m = 1$, and a subset of it when $m = 2$. Unfortunately, the case of 3-manifolds is once again problematic, and it is now a well-known fact that the restricted Delaunay and witness complexes may differ significantly (no respective inclusion, different topological types, etc) when $m \geq 3$ [BGO09]. To overcome this issue, Boissonnat, Guibas and Oudot [BGO09] resorted to the sliver removal technique on some superset of the witness complex, whose construction incurs an exponential dependence on d, the dimension of the ambient space. The state of affairs as of now is that the complexity of witness complex based manifold reconstruction is exponential in d, and whether it could be made only polynomial in d (while still exponential in m) was an open question, which this paper answers affirmatively.

Our contributions

Our paper builds on recent results on the stability of Delaunay triangulations [BDG13b] which we extend in the context of Laguerre geometry where points are weighted. We introduce the notion of power protection of Delaunay simplices and show that the weighting mechanism already used in [CDE$^+$00, CDR05] and [BGO09] can be adapted to our context. As a result, we get an algorithm that constructs a (weighted) witness complex that is a *faithful reconstruction*, i.e. homeomorphic and a close geometric approximation, of the manifold. Differently from previous reconstruction algorithms [CDR05, BGO09, BG14], our algorithm can be simply adapted to work when we don't have explicit coordinates of the points but just the interpoint distance matrix.

2 Definitions and preliminaries

2.1 General notations

We will mainly work in d-dimensional Euclidean space $\mathbb{R}^d$ with the standard ℓ_2-norm, $\|\cdot\|$. The distance between $p \in \mathbb{R}^d$ and a set $X \subset \mathbb{R}^d$, is

$$d(p, X) = \inf_{x \in X} \|x - p\|.$$

We refer to the distance between two points a and b as $\|b - a\|$ or $d(a, b)$ as convenient.

A ball $B(c, r) = \{x : d(x, c) < r\}$ is open, and $\overline{B}(c, r) = \{x : d(x, c) \leq r\}$ is closed.

Generally, we denote the convex hull of a set X by $\mathrm{conv}(X)$, and the affine hull by $\mathrm{aff}(X)$. The cardinality of X, and not its measure, is denoted by $\#X$. If $X \subseteq \mathbb{R}$, $\mu(X)$ denotes the standard Lebesgue measure of X.

For given vectors u and v in $\mathbb{R}^d$, $\langle u, v\rangle$ denotes the *Euclidean inner product* of the vectors u and v.

For given U and V vector spaces of $\mathbb{R}^d$, with $\dim U \leq \dim V$, the *angle* between them is defined by

$$\angle(U, V) = \max_{u \in U} \min_{v \in V} \angle(u, v).$$

By angle between affine spaces, we mean the angle between corresponding parallel vector spaces.

The following result is a simple consequence of the above definition. For a proof refer to [BG14].

Lemma 1 *Let U and V be vector subspaces of $\mathbb{R}^d$ with $\dim(U) \leq \dim(V)$.*

1. *If $U^\perp$ and $V^\perp$ are the orthogonal complements of U and V in $\mathbb{R}^d$, then $\angle(U, V) = \angle(V^\perp, U^\perp)$.*
2. *If $\dim(U) = \dim(V)$ then $\angle(U, V) = \angle(V, U)$.*

Let $s_i(A)$ denote the i^{th} singular value of matrix A. The singular values are non-negative and ordered by decreasing order of magnitude. The largest singular value $s_1(A)$ is equal to the norm $\|A\|$ of the matrix, i.e.,

$$s_1(A) = \|A\| = \sup_{\|x\|=1} \|Ax\|.$$

If A is an $r \times c$ matrix, its smallest singular value is

$$s_j(A) = \inf_{\|x\|=1} \|Ax\|, \quad \text{where} \quad j = \min\{r, c\}.$$

It is easy to see that:

Lemma 2 *If A is an invertible $j \times j$ matrix, then*

$$s_1(A^{-1}) = s_j(A)^{-1}.$$

2.2 Simplices

Given a set of $j+1$ points $p_0, \ldots, p_j$ in $\mathbb{R}^d$, a *j-simplex*, or just simplex, $\sigma = [p_0, \ldots, p_j]$ denotes the set $\{p_0, \ldots, p_j\}$. The points p_i are called the *vertices* of σ and j denotes the *combinatorial dimension* of the simplex σ. Sometimes we will use an additional superscript, like σ^j, to denote a j-simplex. A simplex σ^j is called *degenerate* if $j > \dim \operatorname{aff}(\sigma)$.

We will denote by $R(\sigma)$, $L(\sigma)$, $\Delta(\sigma)$ the lengths of the smallest circumradius, the smallest edge, and the longest edge of the simplex σ respectively. The circumcentre of the simplex σ will be denoted by $C(\sigma)$ and $N(\sigma)$ denotes the affine space, passing through $C(\sigma)$ and of dimension $d - \dim \operatorname{aff}(\sigma)$, orthogonal to $\operatorname{aff}(\sigma)$.

Any subset $\{p_{i_0}, \ldots, p_{i_k}\}$ of $\{p_0, \ldots, p_j\}$ defines a k-simplex which we call a *face* of σ. We will write $\tau \leq \sigma$ if τ is a face of σ, and $\tau < \sigma$ if τ is a *proper face* of σ.

For a given vertex p of σ, σ_p denotes the subsimplex of σ with the vertex set $\{p_0, \ldots, p_j\} \setminus p$. If τ is a j-simplex, and p is not a vertex of τ, we can get a $(j+1)$-simplex $\sigma = p * \tau$, called the *join* of p and τ. We will denote τ by σ_p and will also write $\sigma = \sigma_p * p$.

The *altitude* of a vertex p in σ is $D(p, \sigma) = d(p, \operatorname{aff}(\sigma_p))$. A poorly-shaped simplex can be characterized by the existence of a relatively small altitude. The *thickness* of a j-simplex σ of diameter $\Delta(\sigma)$ is defined as

$$\Upsilon(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{p \in \sigma} \frac{D(p,\sigma)}{j\Delta(\sigma)} & \text{otherwise.} \end{cases}$$

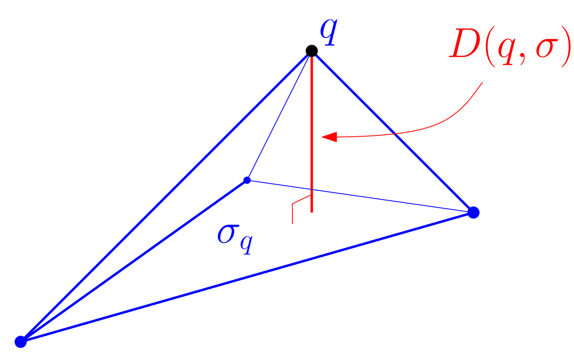


Figure 1: Figure show altitude $D(q, \sigma)$ of the point q in the simplex σ.

Boissonnat, Dyer and Ghosh [BDG13b] connected the geometric properties of a simplex to the largest and smallest singular values of the associated matrix:

Lemma 3 (Thickness and singular value [BDG13b]) *Let $\sigma = [p_0, \ldots, p_j]$ be a non-degenerate j-simplex in $\mathbb{R}^m$, with $j > 0$, and let P be the $m \times j$ matrix whose i^{th} column is $p_i - p_0$. Then*

1. $s_1(P) \leq \sqrt{j}\Delta(\sigma)$, *and*
2. $s_j(P) \geq \sqrt{j}\Upsilon(\sigma)\Delta(\sigma)$.

A simplex that is not thick has a relatively small altitude, but we want to characterize bad simplices for which *all* the altitudes are relatively small.This motivates the definition of Γ_0-slivers.

Definition 4 (Γ_0-good simplices and Γ_0-slivers) Let Γ_0 be a positive real number smaller than one. A simplex σ is Γ_0-*good* if $\Upsilon(\sigma^j) \geq \Gamma_0^j$ for all j-simplices $\sigma^j \leq \sigma$. A simplex is Γ_0-*bad* if it is not Γ_0-good. A Γ_0-*sliver* is a Γ_0-bad simplex in which all the proper faces are Γ_0-good.

Remark 5 (On the good and bad simplex definitions) 1. Observe that in the definition of Γ_0-*good* simplex the thickness bound goes down exponentially with dimensions. Ideally, one would like to have the thickness bound to be independent of the dimension of the simplex. We have defined it this way because with the current sliver removal technology we cannot guarantee the output triangulation to have thickness lower bound that is independent of the dimension of the simplices in the triangulation.

2. Observe that a sliver must have dimension at least 2, since $\Upsilon(\sigma^j) = 1$ for $j < 2$. Observe also that our definition departs from the standard one since the slivers we consider have no upper bound on their circumradius, and in fact may be degenerate and not even have a circumradius. Also, observe that for a fixed Γ_0 we say a simplex σ is good if $\Upsilon(\sigma^j) \geq \Gamma_0^j$ for all j-simplices $\sigma^j \leq \sigma$.

Ensuring that all simplices are Γ_0-good is the same as ensuring that there are no slivers. Indeed, if σ is Γ_0-bad, then it has a j-face σ^j that is not Γ_0^j-thick. By considering such a face with minimal dimension we arrive at the following important observation:

Lemma 6 *A simplex is Γ_0-bad if and only if it has a face that is a Γ_0-sliver.*

2.3 Weighted points and weighted Delaunay complex

For a finite set of points L in $\mathbb{R}^d$, a weight assignment of L is a non-negative real function from L to $[0, \infty)$, i.e., $\varpi : L \to [0, \infty)$. A pair $(p, \varpi(p))$, $p \in L$, is called a *weighted point*. For simplicity, we denote the weighted point $(p, \varpi(p))$ as p^ϖ. The *relative amplitude* of ϖ is defined as

$$\tilde{\varpi} = \max_{p \in L} \max_{q \in L \setminus p} \frac{\varpi(p)}{\|p - q\|}. \tag{1}$$

Given a point $x \in \mathbb{R}^d$, the *weighted distance* of x from a weighted point $(p, \varpi(p))$ is defined as

$$d(x, p^\varpi) = \|x - p\|^2 - \varpi(p)^2.$$

We say a sphere $S(c, r)$ is *orthogonal* to $p^\varpi = (p, \varpi(p))$ if $d(c, p^\varpi) = r^2$, i.e., if

$$\|p - c\|^2 = \varpi(p)^2 + r^2.$$

For a simplex $\sigma = [p_0, \ldots, p_k]$ with vertices in L and $\varpi : L \to [0, \infty)$ a weight assignment, we define the *ϖ-weighted normal space*, or just weighted normal space, $N_\varpi(\sigma)$ of σ as

$$N_\varpi(\sigma) = \left\{ x \in \mathbb{R}^d : \ d(x, p_i^\varpi) = d(x, p_j^\varpi), \forall \, p_i, \, p_j \in \sigma \right\}.$$

We call $S(c, r)$ a *ϖ-ortho sphere*, or just an *ortho sphere*, of σ if it is orthogonal to the vertices of σ, i.e., if for all $p_i \in \sigma$, we have $r^2 = d(c, p_i^\varpi)$. Every $c \in N_\varpi(\sigma)$ is the center of an ortho sphere $S(c, r)$ with $r^2 = d(c, p_0^\varpi)$, and conversely, every ortho sphere is centered in $N_\varpi(\sigma)$.

We define the *ϖ-weighted* (or just weighted) center of σ as

$$C_\varpi(\sigma) = \mathrm{argmin}_{x \in N_\varpi(\sigma)} \, d(x, p_0^\varpi).$$

$N_\varpi(\sigma)$ is an orthogonal compliment of $\mathrm{aff}(\sigma)$ intersecting $\mathrm{aff}(\sigma)$ at $C_\varpi(\sigma)$.

The *ϖ-weighted* (or just weighted) ortho-radius of σ is defined by

$$R_\varpi(\sigma)^2 = d(C_\varpi(\sigma), p_0^\varpi).$$

Note that weighted othro-radius $R_\varpi(\sigma)^2$ can be negative, i.e., $R_\varpi(\sigma)^2 < 0$.

For a point $p \in L$ we define the *weighted Voronoi cell* $\mathrm{Vor}_\varpi(p)$ of p as

$$\mathrm{Vor}_\varpi(p) = \{ x \in \mathbb{R}^d : \forall \ q \in L \setminus p, \ d(x, p^\varpi) \leq d(x, q^\varpi) \}.$$

For a simplex $\sigma = [p_0, \ldots, p_k]$ with vertices in L, the *weighted Voronoi face* $\mathrm{Vor}_\varpi(\sigma)$ of σ is defined as

$$\mathrm{Vor}_\varpi(\sigma) = \bigcap_{i=0}^{k} \mathrm{Vor}_\varpi(p_i).$$

Observe that the Voronoi faces are convex. We define $\dim \mathrm{Vor}_\varpi(\sigma)$ to be the dimension of $\mathrm{aff}(\mathrm{Vor}_\varpi(\sigma))$.

The weighted Voronoi cells give a decomposition of $\mathbb{R}^d$, denoted $\mathrm{Vor}_\varpi(L)$, called the *weighted Voronoi diagram* of L corresponding to the weight assignment ϖ. Let $c \in \mathrm{Vor}_\varpi(\sigma)$ and $r^2 = d(c, p_i^\varpi)$ where $p_i \in \sigma$. We will call a $S(c, r)$ *ϖ-ortho Delaunay sphere*, or just *Delaunay sphere*, of σ.

The *weighted Delaunay complex* $\mathrm{Del}_\varpi(L)$ is defined as the *nerve* of $\mathrm{Vor}_\varpi(L)$, i.e.,

$$\sigma \in \mathrm{Del}_\varpi(L) \;\text{ iff }\; \mathrm{Vor}_\varpi(\sigma) \neq \emptyset.$$

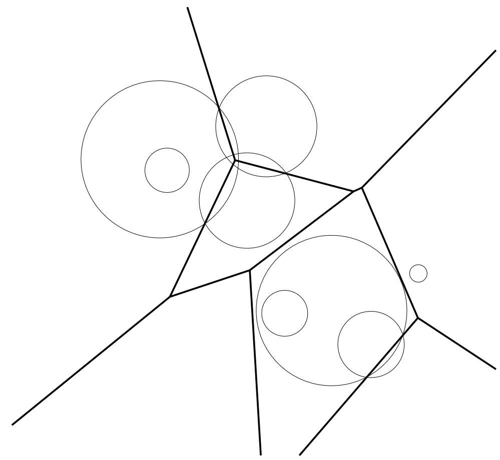


Figure 2: The figure shows the weighted Voronoi diagram of weighted points, denoted by circles with centered at the points and radii equal to the weight of the points, in the plane.

2.4 Manifolds and reach

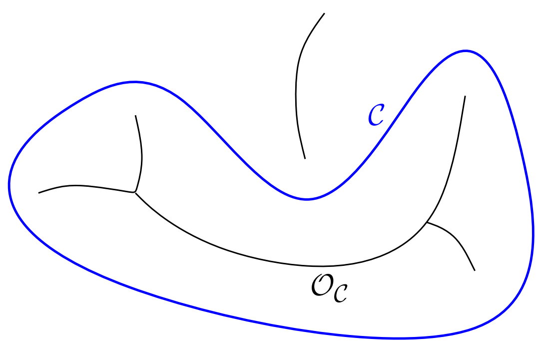


Figure 3: The figure shows the medial axis $\mathcal{O}_\mathcal{C}$, drawn in black, of the blue curve $\mathcal{C}$.

For a given compact submanifold $\mathcal{M}$ of $\mathbb{R}^d$, the *medial axis* $\mathcal{O}_\mathcal{M}$ of $\mathcal{M}$ is defined as the closure of the set of points in $\mathbb{R}^d$ that have more than one closest points in $\mathcal{M}$. The *reach* of $\mathcal{M}$ is defined as

$$\mathrm{rch}(\mathcal{M}) = \inf_{x \in \mathcal{M}} d(x, \mathcal{O}_\mathcal{M}).$$

Federer [Fed59] proved that $\mathrm{rch}(\mathcal{M})$ is (strictly) positive when $\mathcal{M}$ is of class C^2 or even $C^{1,1}$, i.e. the normal bundle is defined everywhere on $\mathcal{M}$ and is Lipschitz continuous. For simplicity, we are anyway assuming that $\mathcal{M}$ is a smooth compact submanifold.

$T_p\mathcal{M}$ and $N_p\mathcal{M}$ denote the tangent space and normal space at $p \in \mathcal{M}$. We will use the following results from [Fed59, GW04, BDG13a]. See [GW04, Lem. 6 & 7] and [BDG13a, Lem. B.3].

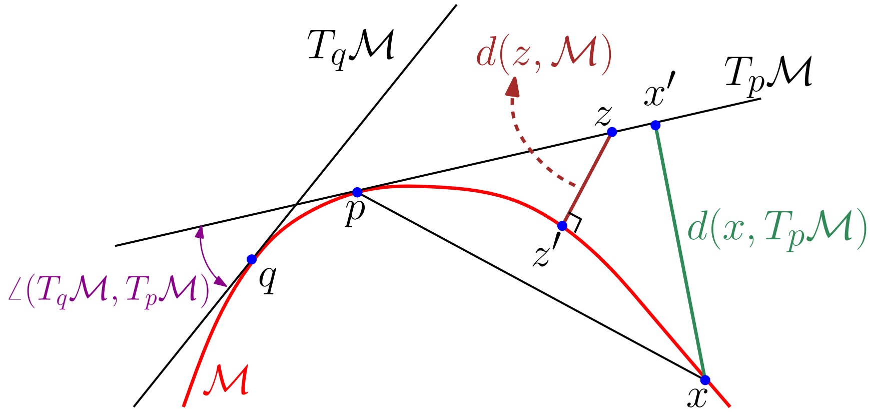


Figure 4: Diagram for the Lemma 7.

Lemma 7 *Let p be a point on the manifold $\mathcal{M}$.*

1. *If $x \in \mathcal{M}$ and $\|p - x\| < \mathrm{rch}(\mathcal{M})$, then $\sin \angle(px, T_p\mathcal{M}) \leq \frac{\|p-x\|}{2\mathrm{rch}(\mathcal{M})}$.*
2. *If $z \in T_p\mathcal{M}$ and $\|p - z\| < \frac{\mathrm{rch}(\mathcal{M})}{4}$ then $d(z, \mathcal{M}) \leq \frac{2\|p-z\|^2}{\mathrm{rch}(\mathcal{M})}$.*
3. *If $q \in \mathcal{M}$ and $\|p - q\| < \frac{\mathrm{rch}(\mathcal{M})}{4}$, then $\sin \angle(T_p\mathcal{M}, T_q\mathcal{M}) < \frac{6\|p-q\|}{\mathrm{rch}(\mathcal{M})}$.*

The following structural result is a restricted version[1] of a result due to Boissonnat, Guibas and Oudot [BGO09, Lem. 4.3 & 4.4].

Lemma 8 *Let $L \subseteq \mathcal{M}$ be a ϵ-sample of $\mathcal{M}$ with $\epsilon < \mathrm{rch}(\mathcal{M})$, and $\varpi : L \to [0, \infty)$ be a weight assignment with $\tilde{\tilde{\varpi}} < \frac{1}{2}$.*

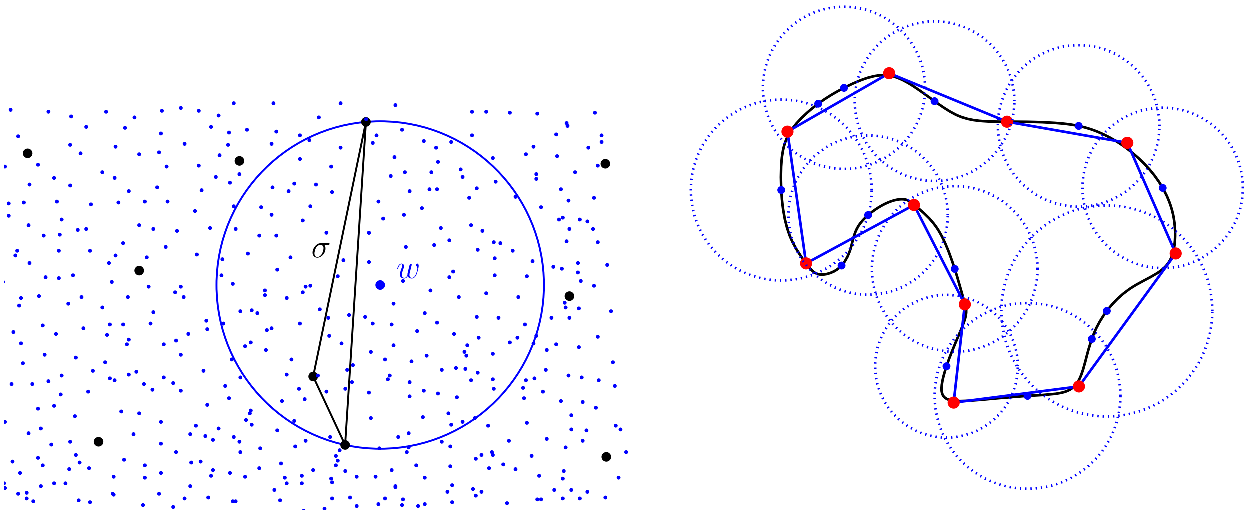
1. *For all $p \in L$, $\varpi(p) \leq 2\tilde{\tilde{\varpi}}\epsilon$.*
2. *If $\epsilon \leq \frac{\mathrm{rch}(\mathcal{M})}{4}$, then, for all $x \in \mathcal{M}$ and $k \in \{0, 1\}$, the Euclidean distance between x and its $(k + 1)$-nearest weighted neighbor in L is at most $(1 + 2\tilde{\tilde{\varpi}} + 2k(1 + 3\tilde{\tilde{\varpi}}))\epsilon$.*

The following result, due to [BDG14], bounds the angle between the affine plane of a simplex with vertices on the manifold $\mathcal{M}$ and the tangent planes to the manifold $\mathcal{M}$ at the vertices of the simplex.

Corollary 9 *Let σ be a k-simplex with $k \leq m$ and the vertices of σ are on the submanifold $\mathcal{M}$ of dimension m. If σ is Γ_0-good and $\Delta(\sigma) < \mathrm{rch}(\mathcal{M})$, then for all $p \in \sigma$ we have*

$$\sin \angle(\mathrm{aff}(\sigma), T_p\mathcal{M}) \leq \frac{\Delta(\sigma)}{\Gamma_0^m \mathrm{rch}(\mathcal{M})}.$$

[1] Boissonnat, Guibas and Oudot [BGO09, Lem. 4.4] proved a more general result bounding the distance between x and its $(k + 1)$-nearest weighted neighbor for all $k \leq d$. In Lemma 8 (2) we only stated the special case when $k \in \{0, 1\}$.



(a) (b)

Figure 5: (a) Blue points are the witnesses W and the black points are the landmarks L. All the landmarks in this example are assigned “zero” weights. The simplex σ is witnessed by w. Note that the figure is taken from the paper [BDG15].
(b) In the figure witness set (blue points) and the landmark set (red) are sampled from the black curve. The landmarks are assigned “zero” weight. The one-dimensional complex (drawn in blue) is the witness complex approximating the black curve

2.5 Witness, cocone and tangential complex

We now recall the definition of the weighted *witness complex* introduced by de Silva [dS08]. Let $W \subset \mathbb{R}^d$, and let $L \subseteq W$ be a finite set, and $\varpi : L \to [0, \infty)$ be a weight assignment of L. The points in the set W are called *witnesses* and the points in L are called *landmarks*.

- We say $w \in W$ is a ϖ-*witness* of a simplex $\sigma = [p_0, \ldots, p_k]$ with vertices in L, if the $p_0, \ldots, p_k$ are among the $k+1$ nearest neighbors of w in the weighted distance, i.e., $p \in \sigma$, $q \in L \setminus \sigma$, $d(w, p^{\varpi}) \leq d(w, q^{\varpi})$. See Figure 5 (a).
- The ϖ-*witness complex* $\mathrm{Wit}_{\varpi}(L, W)$ is the maximum abstract simplicial complex with vertices in L, whose faces are ϖ-witnessed by points of W. When there is no ambiguity, we will call $\mathrm{Wit}_{\varpi}(L, W)$ just witness complex for simplicity. See Figure 5 (b).

For any point p on a smooth submanifold $\mathcal{M}$ and $\theta \in [0, \frac{\pi}{2}]$, we call the θ-*cocone* of $\mathcal{M}$ at p, or $\mathrm{K}^{\theta_0}(p)$ for short, the cocone of semi-aperture θ around the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at p:

$$\mathrm{K}^{\theta}(p) = \left\{ x \in \mathbb{R}^d : \angle(px, T_p\mathcal{M}) \leq \theta \right\}.$$

Given an angle $\theta \in [0, \frac{\pi}{2}]$, a finite point set $\mathsf{P} \subset \mathcal{M}$, and a weight assignment $\varpi : \mathsf{P} \to [0, \infty)$, the *weighted θ-cocone complex* of P, denoted by $\mathrm{K}^{\theta}_{\varpi}(\mathsf{P})$, is defined as

$$\mathrm{K}^{\theta}_{\varpi}(\mathsf{P}) = \left\{ \sigma \in \mathrm{Del}_{\varpi}(\mathsf{P}) : \mathrm{Vor}_{\varpi}(\sigma) \cap \left(\bigcup_{p \in \sigma} \mathrm{K}^{\theta}(p) \right) \neq \emptyset \right\}. \tag{2}$$

The cocone complex was first introduced by Amenta, Choi, Dey and Leekha [ACDL02] in $\mathbb{R}^3$ for reconstructing surfaces and was later generalized by Cheng, Dey and Ramos [CDR05] for reconstructing submanifolds.

The *weighted tangential complex*, or just tangential complex, of P is the weighted θ-cocone complex $\mathrm{K}^{\theta}_{\varpi}(\mathsf{P})$ with θ equal to "zero" and will be denoted by $\mathrm{Del}_{\varpi}(\mathsf{P}, T\mathcal{M})$. The Tangential complex was first defined by Boissonnat and Flötotto [BF04] for getting a coordinate system from a point set sampled from a surface. Boissonnat and Ghosh [BG14] later extended the definition to the weighted setting and using the weighted tangential complex they gave the first manifold reconstruction algorithm whose time complexity depends linearly on the ambient dimension.

Hypothesis 10 For the rest of this paper, we take

$$\begin{aligned}\theta_0 &\stackrel{\text{def}}{=} \frac{\pi}{32},\\ \mathrm{K}(p) &\stackrel{\text{def}}{=} \mathrm{K}^{\theta_0}(p), \text{ and}\\ \mathrm{K}_{\varpi}(\mathsf{P}) &\stackrel{\text{def}}{=} \mathrm{K}^{\theta_0}_{\varpi}(\mathsf{P}).\end{aligned}$$

3 Power protection

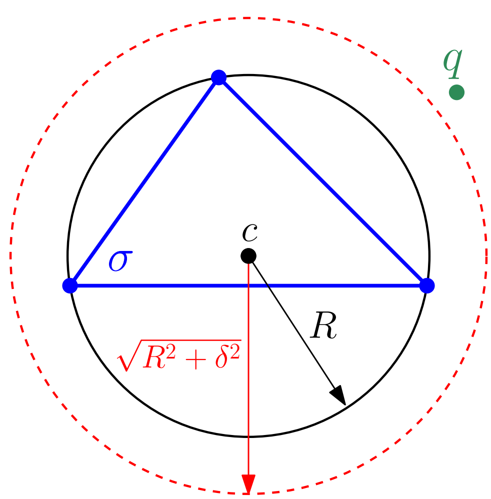


Figure 6: Example of δ^2-power protection when points are not weighted, i.e., the points have "zero" weight.

Let $\mathsf{P} \subset \mathbb{R}^d$ be a finite point sample. A simplex $\sigma \in \mathrm{Del}_{\varpi}(\mathsf{P})$ is δ^2-*power protected* at $c \in \mathrm{Vor}_{\varpi}(\sigma)$ if

$$\|q-c\|^2 - \varpi(q)^2 > \|p-c\|^2 - \varpi(p)^2 + \delta^2 \quad \text{for all } q \in L \setminus \sigma \text{ and } p \in \sigma.$$

For convenience, we will say a simplex $\sigma \in \mathrm{Del}_{\varpi}(\mathsf{P})$ is δ^2-*power protected* if $\exists\, c \in \mathrm{Vor}_{\varpi}(\sigma)$ such that σ is δ^2-*power protected* at c.

The following result shows that power protecting d-simplices implies power protecting lower dimensional subsimplices as well.

Lemma 11 *Let $\mathsf{P} \subset \mathbb{R}^d$ be a set of points, and let $\varpi : \mathsf{P} \to [0, \infty)$ be a weight assignment. In addition, let p be a point of P whose Voronoi cell $\mathrm{Vor}_{\varpi}(p)$ is bounded. Then, if all the d-simplices*

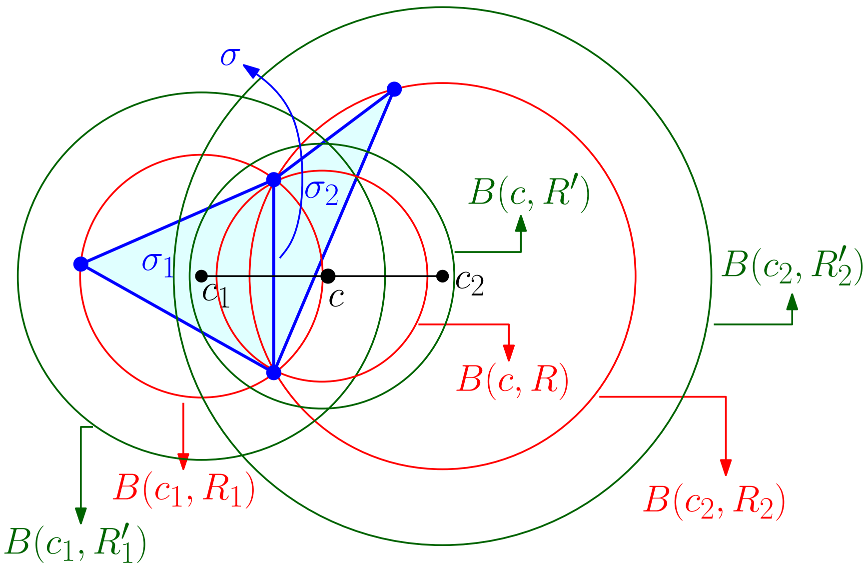


Figure 8: The two triangles σ_1 and σ_2 are δ^2-power protected at c_1 and c_2 respectively. The figure shows that the edge $\sigma = \sigma_1 \cap \sigma_2$ is $\frac{\delta^2}{2}$-power protected at $c = \frac{c_1+c_2}{2}$. Note that the vertices of the triangles have "zero" weight and for $i \in \{1, 2\}$, $R'_i = \sqrt{R_i^2 + \delta^2}$. See, Lemma 11 for a more general result.

incident to p in $\mathrm{Del}_\varpi(\mathsf{P})$ are δ^2-power protected, with $\delta > 0$, then any j-simplex in $\mathrm{Del}_\varpi(\mathsf{P})$ incident to p is

$$\frac{\delta^2}{d-j+1}\text{-power protected.}$$

The proof of the Lemma 11 is done in the lifted $\mathbb{R}^{d+1}$ space where power protection translates to *vertical distance* of points from hyperplanes, see Section 3.3.

The above result in the unweighted case implies that if Voronoi vertices are protected then any Voronoi face contains a point that is far from any of the other Voronoi faces. See Figure 7.

Lemma 11 also implies that if $\mathrm{aff}\,\mathsf{P} = \mathbb{R}^d$ and if all the d-simplices in $\mathrm{Del}_\varpi(\mathsf{P})$ are δ^2-power protected then all the simplices, not on the boundary of $\mathrm{Del}_\varpi(\mathsf{P})$, are also power protected. Another interesting aspect of this result is the fact that the decay in power protection to lower dimensional simplices goes down linearly with the dimension of the ambient space.

To prove Lemma 11 we need the following lemma on power protection.

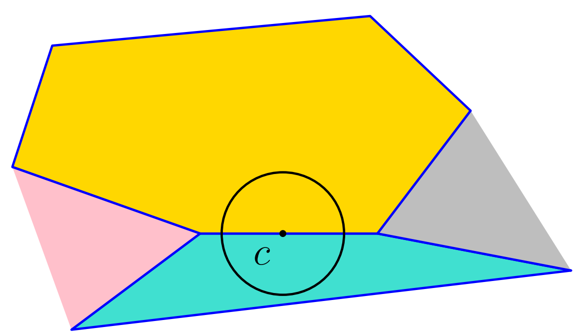


Figure 7: The figure shows existence of a point on a Voronoi face that is far from other Voronoi faces.

Lemma 12 *Let $\mathsf{P} \subset \mathbb{R}^d$ and $\varpi : \mathsf{P} \to [0, \infty)$ be a weight distribution. Let $p \in \mathsf{P}$ such that $\mathrm{Vor}_\varpi(p)$ is bounded and all the d-simplices in $\mathrm{Del}_\varpi(\mathsf{P})$ incident to p are δ^2-power for some $\delta > 0$.*

Then

1. *the dimension of the maximal simplices in* $\mathrm{Del}_\varpi(\mathsf{P})$ *incident to* p *is equal to* d*; and*
2. *for all* j*-simplices* $\sigma^j \in \mathrm{Del}_\varpi(\mathsf{P})$ *incident to* p*,* $\dim \mathrm{Vor}_\varpi(\sigma^j) = d - j$*.*

3.1 Proof of Lemma 12

Remark 13 (On Lemma 12) Observe that since $\mathrm{Vor}_\varpi(p)$ is bounded, we have $\dim \mathrm{aff}(\mathsf{P}) = d$.

The following lemma is analogous to [BDG13b, Lem. 3.2], and the proof is exactly like the proof of that lemma.

Lemma 14 (Maximal simplices) *Every* $\sigma \in \mathrm{Del}_\varpi(\mathsf{P})$ *incident to* p *is a face of a simplex* $\sigma' \in \mathrm{Del}_\varpi(\mathsf{P})$ *with* $\dim \mathrm{aff}(\sigma') = d$*.*

The following lemma is a direct consequence of the above result.

Lemma 15 (No degeneracies) *If every* d*-simplex in* $\mathrm{Del}_\varpi(\mathsf{P})$ *incident to* p *is* δ^2*-power protected for some* $\delta > 0$*, then there are no degenerate (see the definition of degenerate simplex given in Section 2.2) simplices in* $\mathrm{Del}_\varpi(\mathsf{P})$ *that are incident to* p*.*

Like in the case of Lemma 14, following result is analogous to [BDG13b, Lem. 3.3] and can be proved exactly along the same lines.

Lemma 16 (Separation) *If* $\sigma^j \in \mathrm{Del}_\varpi(\mathsf{P})$ *is a* j*-simplex incident to* p *with* $\mathrm{Vor}_\varpi(\sigma^j)$ *bounded, and* $q \in \mathsf{P} \setminus \sigma^j$*, then there is a* d*-simplex* $\sigma^d \in \mathrm{Del}_\varpi(\mathsf{P})$ *incident to* p *such that* $\sigma^j \leq \sigma^d$ *and* $q \notin \sigma^d$*.*

Proof of Lemma 12 The first assertion follows directly from Lemmas 14 and 15.

For the second assertion, we observe that $\dim \mathrm{Vor}_\varpi(\sigma^j) \leq d - j$ since $\mathrm{Vor}_\varpi(\sigma^j) \subseteq N_\varpi(\sigma^j)$, and $\dim N_\varpi(\sigma^j) = d - j$ because σ^j is nondegenerate. In particular, if $j = d$, then we must have $\dim \mathrm{Vor}_\varpi(\sigma^j) = 0$. We obtain the result for all j by showing, by induction on $i = d - j$, that if $\sigma^j \leq \sigma^{j+1}$, then

$$\dim \mathrm{Vor}_\varpi(\sigma^j) > \dim \mathrm{Vor}_\varpi(\sigma^{j+1}). \tag{3}$$

Assume then that $\dim \mathrm{Vor}_\varpi(\sigma^{j+1}) = d - (j + 1)$. We will show that for any facet $\sigma^j < \sigma^{j+1}$ there is a point $c \in \mathrm{Vor}_\varpi(\sigma^j)$ such that $c \notin N_\varpi(\sigma^{j+1})$. The claim (3) then follows since $\mathrm{aff}(\mathrm{Vor}_\varpi(\sigma^{j+1})) \subseteq \mathrm{aff}(\mathrm{Vor}_\varpi(\sigma^j))$ and Lemma 15 implies $\dim N_\varpi(\sigma^{j+1}) = d-(j+1)$, and therefore $N_\varpi(\sigma^{j+1}) = \mathrm{aff}(\mathrm{Vor}_\varpi(\sigma^{j+1}))$ by the hypothesis on the dimension of $\mathrm{Vor}_\varpi(\sigma^{j+1})$.

Let $q \in \sigma^{j+1} \setminus \sigma^j$. From Lemma 16, there exists a d-simplex $\sigma^d \in \mathrm{Del}_\varpi(\mathsf{P})$ such that $\sigma^j < \sigma^d$ and $q \notin \sigma^d$. Since all the d-simplices of $\mathrm{Del}_\varpi(\mathsf{P})$ incident to p are δ^2-protected, the circumcentre $c \in \mathrm{Vor}_\varpi(\sigma^d)$ satisfies

$$d(c, r^\varpi) < d(c, s^\varpi) - \delta^2, \quad \text{for all } r \in \sigma^d \text{ and } s \in \mathsf{P} \setminus \sigma^d. \tag{4}$$

More specifically, this implies

$$d(c, r^\varpi) < d(c, q^\varpi) - \delta^2, \ \forall\, r \in \sigma^j, \tag{5}$$

since $q \notin \sigma^d$ and $\sigma^j < \sigma^d$. Thus $c \notin N_\varpi(\sigma^{j+1})$, and c is the desired point since $c \in \mathrm{Vor}_\varpi(\sigma^d) \subseteq \mathrm{Vor}_\varpi(\sigma^j)$. □

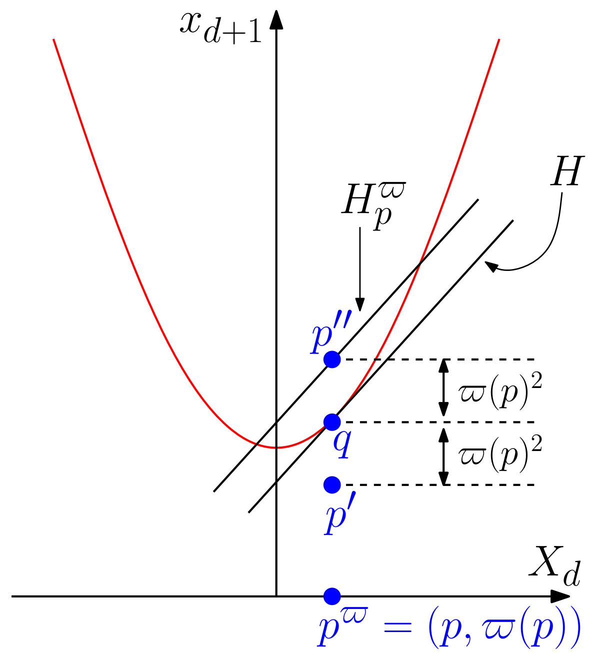


Figure 9: In the figure $p' = \phi(S(p, \varpi(p)))$, $p'' = \phi(S(p, r))$ where $r^2 = -\varpi(p)^2$ and $q = \phi(S(p, 0))$. Note that the hyperplanes H and H_p^{ϖ} are parallel and the hyperplane H is tangential to the paraboloid (drawn in red), i.e., $x_{d+1} = \sum_{i=1}^{d} x_i^2$. Note that $X_d = (x_1, \ldots, x_d)$.

3.2 Lifting map, space of spheres and Voronoi diagram

We are going to argue about the power protection of Delaunay simplices in the "space of spheres" or "lifting space". For our purposes we will be working primarily from the Voronoi perspective. We will give a self-contained summary of the properties of the space of spheres that we will use. Full details can be found in [BY98, Chap. 17].

Since we will be dealing with ortho spheres, see the definition in Section 2.3, in this section a sphere $S(c, r)$ can have $r^2 < 0$.

The *lifting map* ϕ takes a sphere $S(c, r)$ in $\mathbb{R}^d$, with centre $c \in \mathbb{R}^d$ and radius r, to the point $(c, \|c\|^2 - r^2) \in \mathbb{R}^{d+1}$, i.e.,

$$\phi(S(c, r)) = (c, \|c\|^2 - r^2).$$

We consider the points in $\mathbb{R}^d$ to be spheres with $r = 0$, and thus $\mathbb{R}^d$ itself is represented as a (hyper-) paraboloid in $\mathbb{R}^{d+1}$, i.e, $x_{d+1} = \sum_{i=1}^{d} x_i^2$.

Let P be a locally finite point set and $\varpi : \mathsf{P} \to [0, \infty)$ be a weight distribution. The set of spheres that are orthogonal to point p, with weight $\varpi(p)$, are represented by a hyperplane $\mathcal{H}_p^{\varpi} \subset \mathbb{R}^{d+1}$ that passes through $\phi(S(p, r))$ where $r^2 = -\varpi(p)^2$. Indeed, for any sphere $S(c, r)$ orthogonal to $p^{\varpi} = (p, \varpi(p))$ satisfies

$$r^2 = \|c - p\|^2 - \varpi(p)^2.$$

This implies

$$\begin{aligned} \phi(S(c, r)) &= (c, \|c\|^2 - r^2) \\ &= (c, 2\langle c, p\rangle + \varpi(p)^2 - \|p\|^2). \end{aligned}$$

So the hyperplane is

$$\mathcal{H}_p^{\varpi} = \{(c,h) \in \mathbb{R}^d \times \mathbb{R} \mid h = 2\langle c, p\rangle + \varpi(p)^2 - \|p\|^2\};$$

see Figure 9.

For any $p \in \mathsf{P} \subset \mathbb{R}^d$, we represent its Voronoi cell $\mathrm{Vor}_{\varpi}(p)$ in the space of spheres by associating to each $c \in \mathrm{Vor}_{\varpi}(p)$ the unique sphere $S(c,r)$, where $r^2 = \|p-c\|^2 - \varpi(p)^2$. Thus $\phi(\mathrm{Vor}_{\varpi}(p)) \subseteq \mathcal{H}_p^{\varpi}$.

For any Delaunay simplex $\sigma \in \mathrm{Del}(\mathsf{P})$, its Voronoi cell $\mathrm{Vor}_{\varpi}(\sigma) = \bigcap_{p\in\sigma} \mathrm{Vor}_{\varpi}(p)$ is mapped in the space of spheres to the intersection of the hyperplanes that support the lifted Voronoi cells of its vertices:

$$\phi(\mathrm{Vor}_{\varpi}(\sigma)) \subset \bigcap_{p\in\sigma} \mathcal{H}_p^{\varpi}.$$

If P is generic and σ is a k-simplex, then $\phi(\mathrm{Vor}_{\varpi}(\sigma))$ lies in a $(d-k)$-dimensional affine space.

We can say more. The lifted Voronoi cell $\phi(\mathrm{Vor}_{\varpi}(\sigma))$ is a convex polytope. Any two points $z, z' \in \phi(\mathrm{Vor}_{\varpi}(\sigma))$ have corresponding points $c, c' \in \mathrm{Vor}_{\varpi}(\sigma) \subset \mathbb{R}^d$, and a line segment between c and c' gets lifted to a line segment between z and z' in $\phi(\mathrm{Vor}_{\varpi}(\sigma))$.

3.3 Proof of Lemma 11: power protection in the "space of spheres" framework

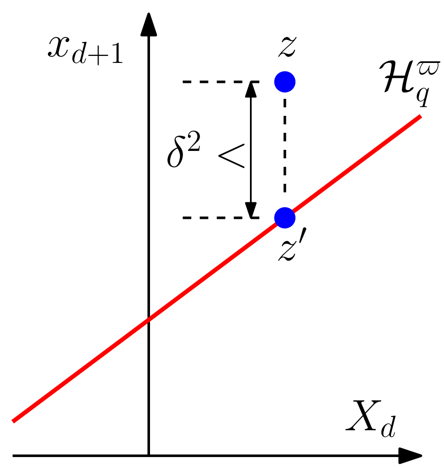


Figure 10: Diagram showing connection between protection and lifting.

We can talk about the power-protection at a point $c \in \mathrm{Vor}_{\varpi}(\sigma)$: it is the power-protection enjoyed by the Delaunay sphere $S(c,r)$ centered at c. For a point $q \in \mathsf{P} \setminus \sigma$, we say that c is *$\check{\delta}^2$-power-protected from q* if

$$\|q-c\|^2 - \varpi(q)^2 - r^2 > \check{\delta}^2.$$

In the lifting space, if $z = \phi(S(c,r))$, then the power protection of c from q is given by the "vertical" distance of z above $\mathcal{H}_q^{\varpi}$, which we will refer to as the *clearance* of z above $\mathcal{H}_q^{\varpi}$. See, Figure 10.

Thus for any $q \in \mathsf{P} \setminus \sigma$ we have a function $f_q : \phi(\mathrm{Vor}_{\varpi}(\sigma)) \to \mathbb{R}$ which associates to each $z \in \phi(\mathrm{Vor}_{\varpi}(\sigma))$ the clearance of z above $\mathcal{H}_q^{\varpi}$. This is a linear function of the sphere centres. Indeed, if $p \in \sigma$ and $z = \phi(S(c,r))$, then $r^2 + \varpi(p)^2 = \|p-c\|^2$, and

$$f_q(z) = 2\langle c,\, p-q\rangle - (\|p\|^2 - \|q\|^2) + (\varpi(p)^2 - \varpi(q)^2).$$

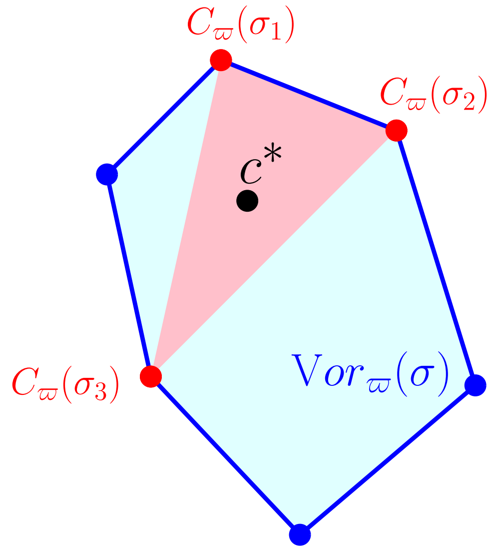


Figure 11: Diagram for Lemma 11. In the figure σ is a $(d-2)$-dimensional simplex with $\sigma < \sigma_i$, with $i \in \{1, 2, 3\}$, where σ_i is a d-dimensional simplex δ^2-power protected at $C_\varpi(\sigma_i)$. From the proof of Lemma 11 we get that σ is $\frac{\delta^2}{3}$-power protected at $c^* = \frac{1}{3}\left(\sum_{i=1}^{3} C_\varpi(\sigma_i)\right)$.

Proof of Lemma 11 We wish to find a bound $h_j(\delta)$ such that if all the d-simplices in $\mathrm{Del}_\varpi(\mathsf{P})$ incident to p are δ^2-power-protected, then the Delaunay j-simplices incident to p will be $h_j(\delta)$-power-protected. Since $\mathrm{Vor}_\varpi(\sigma^j) \subseteq \mathrm{Vor}_\varpi(p)$ and $\mathrm{Vor}_\varpi(p)$ is bounded, we observe that for any j-simplex σ^j its Voronoi cell $\mathrm{Vor}_\varpi(\sigma^j)$ is the convex hull of Voronoi vertices: the ϖ-weighted centres of the Delaunay d-simplices that have σ^j as a face. It follows that $\phi(\mathrm{Vor}_\varpi(\sigma^j))$ is the convex hull of a finite set of points which correspond to these d-simplices. Note that from Lemma 12 we have that $\dim(\mathrm{Vor}_\varpi(\sigma^j))$, which is equal to $\dim(\phi(\mathrm{Vor}_\varpi(\sigma^j)))$, is $d-j$. We choose an affinely independent set $\{z_i\}$, $i \in \{0, 1, \ldots, k\}$, of $k+1$ of these points, where $k = d-j$. See Figure 11.

Let

$$z^* = \frac{1}{k+1}\sum_{i=0}^{k} z_i$$

be the barycenter of these lifted Delaunay spheres, and consider the clearance, $f_q(z^*)$, of z^* above $\mathcal{H}_q^\varpi$, where $q \in \mathsf{P} \setminus \sigma^j$. Observe that z^* is an interior point of $\phi(\mathrm{Vor}_\varpi(\sigma^j))$. Let σ_i be the Delaunay d-simplex corresponding to z_i. There must be a σ_ℓ, $l \in \{0, \ldots, k\}$, which does not contain q, since otherwise the k-simplex defined by the set $\{z_i\}_{i\in\{0,\ldots,k\}}$ would lie in $\phi(\mathrm{Vor}_\varpi(q * \sigma^j))$, implying $\dim\left(\phi(\mathrm{Vor}_\varpi(q*\sigma^j))\right) \geq k$, which contradicts Lemma 12. Since σ_ℓ is δ^2-power-protected, we have $f_q(z_\ell) > \delta^2$, and by the linearity of f_q we get a bound on the clearance of z^* above $\mathcal{H}_q^\varpi$:

$$f_q(z^*) = \frac{1}{k+1}\sum_{i=0}^{k} f_q(z_i) \geq \frac{f_q(z_\ell)}{k+1} > \frac{\delta^2}{k+1}.$$

Since q was chosen arbitrarily from $\mathsf{P} \setminus \sigma^j$, this provides a lower bound on the power protection at $c^* \in \mathrm{Vor}_\varpi(\sigma^j)$, where $z^* = \phi(S(c^*, r^*))$, and hence a lower bound on the power protection of σ^j. $\square$

Remark 17 If we could find two lifted Voronoi vertices z_1 and z_2 such that the line segment between them lies in the relative interior of $\phi(\mathrm{Vor}_\varpi(\sigma^j))$, then the midpoint of that segment would have a power protection of $\frac{\delta^2}{2}$. However, this isn't possible in general, for $\mathrm{Vor}_{\varpi}(\sigma^j)$ could be a $(d-j)$-simplex, when σ^j is not a maximal shared face of any two Delaunay d-simplices.

4 Stability, protection, and the witness complex

Our main structural result, Theorem 19 below, gives conditions that guarantee that $\mathrm{Del}_{\varpi}(L, T\mathcal{M}) = \mathrm{Wit}_{\varpi}(L, W)$. Since the proof of Theorem 19 is quite long and technical, it goes through multiple stages which we will only outline in this section. For full details refer to Appendix A. We begin by introducing some parameters and terminology employed in the statement of the theorem.

Let $\mathcal{M}$ be a m-dimensional submanifold of $\mathbb{R}^d$, $\alpha_0 < \frac{1}{2}$ an absolute constant[2], and $\Gamma_0 < 1$ and $\delta_0 < \alpha_0$ parameters to the algorithm satisfying Inequality (12); see Lemma 21 from Section 5. We define

$$\tilde{\alpha}_0 \stackrel{\text{def}}{=} \sqrt{\alpha_0^2 - \delta_0^2}.$$

Definition 18 (Elementary weight perturbations and stable weight assignments) Let $W \subset \mathcal{M}$ be an ε-sample of $\mathcal{M}$, $L \subset W$ a λ-net of W with $\varepsilon \leq \lambda$, and $\varpi : L \to [0, \infty)$ a weight assignment with relative amplitude (1) satisfying $\tilde{\varpi} \leq \tilde{\alpha}_0$. A weight assignment $\xi : L \to [0, \infty)$ will be called an *elementary weight perturbation* of ϖ (*ewp* for short) if

$$\exists\, p \in L,\ \xi(p) \in \left[\varpi(p),\ \sqrt{\varpi(p)^2 + \delta_0^2 \lambda^2}\right] \ \text{ and } \ \xi(q) = \varpi(q) \ \text{ if } \ q \in L \setminus p.$$

We call the weight assignment $\varpi : L \to [0, \infty)$ *stable* (resp., *locally stable at* $p \in L$) if for all ewp ξ of ϖ, $\mathrm{K}_{\xi}(L)$ contains no Γ_0-slivers of dimension $\leq m+1$ (resp., no such slivers incident to p).

Theorem 19 *Let $\mathcal{M}$ be a m-dimensional submanifold of $\mathbb{R}^d$, $W \subset \mathcal{M}$ an ε-sample of $\mathcal{M}$, $L \subset W$ a λ-net of W with $\varepsilon \leq \lambda$, and $\varpi : L \to [0, \infty)$ a stable weight assignment with $\tilde{\varpi} \leq \tilde{\alpha}_0$. If*

$$\lambda < \frac{\mathrm{rch}(\mathcal{M})}{2^{15}(m+1)} \min\left\{\Gamma_0^{2m+1},\ \delta_0^2\right\} \tag{6}$$

and

$$\varepsilon < \frac{\lambda}{24}\left(\frac{\delta_0^2}{m+1} - \frac{2^{15}\lambda}{\mathrm{rch}(\mathcal{M})}\right) \tag{7}$$

then,

$$\mathrm{Del}_{\varpi}(L, T\mathcal{M}) = \mathrm{Wit}_{\varpi}(L, W).$$

In addition, if λ is sufficiently small, then $\mathrm{Wit}_{\varpi}(L, W)$ is homeomorphic to, and a close geometric approximation of, $\mathcal{M}$.

Since ϖ is a stable weight assignment, $\mathrm{K}_{\varpi}(L)$ contains no Γ_0-slivers of dimension $\leq m+1$. The proof of Theorem 19 relies on the following three properties **P1**, **P2** and **P3**, which hold for λ sufficiently small.

P1 For all $\sigma \in \mathrm{K}_{\varpi}(L)$, $\angle(\mathrm{aff}\,\sigma, T_p\mathcal{M}) = O(\lambda)$.

P2 The simplices of $\mathrm{Del}_{\varpi}(L, T\mathcal{M})$ have dimension at most m, and the maximal dimension of simplices in $\mathrm{K}_{\varpi}(L)$ is $\leq m$.

[2] By absolute constant we mean that α_0 is independent of the dimension of the manifold or other parameters of the algorithm. For simplicity, the reader can take $\alpha_0 = \frac{1}{3}$ for the rest of this paper.

P3 For all $\sigma \in \mathrm{Del}_{\varpi}(L, T\mathcal{M})$ and $p \in \sigma$, $\mathrm{Vor}_{\varpi}(\sigma) \cap T_p\mathcal{M} \neq \emptyset$.

The above properties are direct consequence of results from [CDR05, BGO09, BG14]. For the full details see Lemma 32 in Appendix A.

Using properties **P1**, **P2** and **P3**, we will give the outline of the proof of

$$\mathrm{Wit}_{\varpi}(L, W) = \mathrm{Del}_{\varpi}(L, T\mathcal{M}).$$

The part about homeomorphism and close geometric aspect of Theorem 19 will directly follow from a result of Boissonnat and Ghosh [BG14].

We will now give an outline of the proof of $\mathrm{Wit}_{\varpi}(L, W) = \mathrm{Del}_{\varpi}(L, T\mathcal{M})$.

Step 1: $\mathrm{Wit}_{\varpi}(L, W) \subseteq \mathrm{Del}_{\varpi}(L, T\mathcal{M})$. This step is proved by contradiction in Lemma 34. Let $\sigma^k \in \mathrm{Wit}_{\varpi}(L, W)$ be a k-simplex with $\sigma^k \notin \mathrm{Del}_{\varpi}(L, T\mathcal{M})$ and p a vertex of σ^k. We will show that if this is the case then there exists σ^{m+1} with $\sigma^k \leq \sigma^{m+1}$ and $\sigma^{m+1} \in \mathrm{K}_{\varpi}(L)$. We will reach a contradiction via Property P2.

Using the sampling assumptions on L and W, we can show that for any $w \in W$ that is a ϖ-witness of σ^k or of its subfaces, $\|p - w\| = O(\lambda)$ [BGO09, Lem. 4.4]. This implies, from [GW04, Lem. 6],

$$d(w, T_p\mathcal{M}) = O\left(\frac{\lambda^2}{\mathrm{rch}(\mathcal{M})}\right).$$

From [dS08, Thm. 4.1], we know that $\mathrm{Vor}_{\varpi}(\sigma^k)$ intersects the convex hull of the ϖ-witnesses of σ^k and its subfaces. Let $c_k \in \mathrm{Vor}_{\varpi}(\sigma^k)$ be a point in this intersection. We have

$$d(c_k, T_p\mathcal{M}) = O\left(\frac{\lambda^2}{\mathrm{rch}(\mathcal{M})}\right)$$

and, since L is λ-sparse and $\tilde{\varpi} < \frac{1}{2}$,

$$d(p, c_k) \geq d(p, C_{\varpi}(\sigma^k)) \geq \frac{3\lambda}{8}.$$ [3]

Therefore, using the sampling assumption on λ, we get

$$\frac{d(c_k, T_p\mathcal{M})}{d(c_k, p)} = O\left(\frac{\lambda}{\mathrm{rch}(\mathcal{M})}\right) < \sin\theta_0.$$

By the definition of the cocone complex, see Equation (2), this implies that $\sigma^k \in \mathrm{K}_{\varpi}(L)$. Since $\angle(\mathrm{aff}\,\sigma^k, T_p\mathcal{M})$ is small (property P1), there exists $c'_k \in T_p\mathcal{M}$ such that the line segment $[c_k, c'_k]$ is orthogonal to $\mathrm{aff}(\sigma^k)$ and

$$d(c_k, c'_k) = O\left(\frac{\lambda^2}{\mathrm{rch}(\mathcal{M})}\right).$$

[3] Let p and q be distinct vertices of σ^k, and let x be an orthogonal projection of $C_{\varpi}(\sigma^k)$ on the line through p and q. Since $\tilde{\varpi} < \frac{1}{2}$, $x \in pq$. Using the facts that $d(x, p^{\varpi}) = d(x, q^{\varpi})$, $\|p - q\| \geq \lambda$ and $\varpi(q) < \frac{\|p-q\|}{2}$ (as $\tilde{\varpi} < 1/2$), we get

$$\begin{aligned} \|p - x\| &= \frac{\|p-q\|}{2}\left(1 + \frac{\varpi(p)^2 - \varpi(q)^2}{\|p-q\|^2}\right) \\ &\geq \frac{\|p-q\|}{2}\left(1 - \frac{\varpi(q)^2}{\|p-q\|^2}\right) \\ &\geq \frac{3\lambda}{8} \end{aligned} \tag{8}$$

The bound on $d(p, C_{\varpi(\sigma^k)})$ follows from the fact that $d(p, x) \geq d(p, C_{\varpi}(\sigma^k))$.

Again, as λ is small, the line segment $[c_k, c'_k]$ is contained in $\mathrm{K}(p)$. Since $\sigma^k \not\in \mathrm{Del}_\varpi(L, T\mathcal{M})$, $\exists\, c_{k+1} \in [c_k, c'_k]$ and a Delaunay $(k+1)$-simplex σ^{k+1} such that

$$c_{k+1} \in \mathrm{Vor}_\varpi(\sigma^{k+1}) \quad \text{with} \quad \sigma^k < \sigma^{k+1}.$$

Therefore, $\sigma^{k+1} \in \mathrm{K}_\varpi(L)$. If $k = m$, we have reached a contradiction with property P2. Otherwise, using the facts that $\angle(\sigma^{k+1}, T_p\mathcal{M})$ is small, $d(c_{k+1}, T_p\mathcal{M}) = O\left(\frac{\lambda^2}{\mathrm{rch}(\mathcal{M})}\right)$ and $d(p, c_{k+1}) = \Omega(\lambda)$, we will find a $c'_{k+1} \in T_p\mathcal{M}$ such that $[c_{k+1}, c'_{k+1}] \in \mathrm{K}(p)$. Since $\sigma^{k+1} \in \mathrm{K}_\varpi(L)$, $\exists\, c_{k+2} \in [c_{k+1}, c'_{k+1}]$ and $(k+2)$-simplex $\sigma^{k+2} \in \mathrm{K}_\varpi(L)$ such that

$$c_{k+2} \in \mathrm{Vor}_\varpi(\sigma^{k+2}) \quad \text{and} \quad \sigma^{k+1} < \sigma^{k+2}.$$

Continuing this procedure of walking on the Voronoi cell of the simplex from a point, like c_{k+1}, in the intersection the Voronoi cell of the simplex and $\mathrm{K}(p)$ towards $T_p\mathcal{M}$, we will get a sequence of points

$$c_k, \ldots, c_{m+1}$$

and simplies

$$\sigma^k < \cdots < \sigma^{m+1}$$

with

$$c_j \in \mathrm{Vor}_\varpi(\sigma^j) \cap \mathrm{K}(p) \quad \text{and} \quad \sigma^j \in \mathrm{K}_\varpi(L).$$

We have now reached a contradiction via property P2. This concludes the proof of Step 1.

Step 2: $\mathrm{Del}_\varpi(L, T\mathcal{M}) \subseteq \mathrm{Wit}_\varpi(L, W)$**.** We say that a simplex $\sigma \in \mathrm{Del}_\varpi(L, T\mathcal{M})$ is *δ^2-power protected on $T_p\mathcal{M}$* if it is δ^2-power protected at a point $c \in \mathrm{Vor}_\varpi(\sigma) \cap T_p\mathcal{M}$. In Lemma 35 we show that the stable weight assignment implies that all m-simplices in $\mathrm{Del}_\varpi(L, T\mathcal{M})$ are δ^2-power protected on $T_p\mathcal{M}$, where $\delta = \delta_0\lambda$. To reach a contradiction, let us assume that there exists a m-simplex $\sigma \in \mathrm{Del}_\varpi(L, T\mathcal{M})$ that is not power protected on $T_p\mathcal{M}$ for some $p \in \sigma$. Then for any $c \in \mathrm{Vor}_\varpi(\sigma) \cap T_p\mathcal{M}$ there exists $q \in L \setminus \sigma$ such that

$$d(c, p^\varpi) \geq d(c, q^\varpi) - \delta^2.$$

Consider now the following weight assignment:

$$\xi(x) = \begin{cases} \varpi(x) & \text{if } x \neq q \\ \sqrt{\varpi(q)^2 + \beta^2} & \text{if } x = q \end{cases}$$

where

$$\beta^2 = d(c, q^\varpi) - d(c, p^\varpi).$$

It is easy to see that ξ is an ewp of ϖ and $\tilde{\xi} < 1/2$. Observe that the $(m+1)$-simplex $\sigma' = q * \sigma$ is in $\mathrm{K}_\xi(L)$. Since λ is sufficiently small, σ' is a Γ_0-bad $(m+1)$-simplex; broadly, the idea (see also the proofs of [CDR05, Lem. 13] and [BG14, Lem. 4.9]) is that the thickness of any $(m+1)$-simplex embedded in $\mathbb{R}^m$ is zero, and here σ' is a $(m+1)$-simplex embedded in $\mathbb{R}^d$, but whose vertices belong to a small neighborhood of a m-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^d$ so we can show that its thickness is small. By proving that σ' is Γ_0-bad, we arrive at a contradiction with the fact that ϖ is a stable weight assignment.

In the first part of the proof of Lemma 38 we prove that all simplices (of all dimensions) in $\mathrm{Del}_\varpi(L, T\mathcal{M})$ are $\frac{\delta^2}{m+1}$-power protected on $T_p\mathcal{M}$ for all $p \in \sigma$. To establish this result, we want to use Lemma 11 but we cannot use the lemma directly since it only holds for d-simplices of $\mathbb{R}^d$. To overcome this issue, we resort to Lemma 2.2 of [BG14] which states that $\mathrm{Vor}_\varpi(L) \cap T_p\mathcal{M}$ is identical to a weighted Voronoi diagram $\mathrm{Vor}_\psi(L')$ where L' is the orthogonal projection of L onto $T_p\mathcal{M}$, i.e., $\mathrm{Vor}_\varpi(\sigma) \cap T_p\mathcal{M} = \mathrm{Vor}_\psi(\sigma')$ where σ' is the projection of σ onto $T_p\mathcal{M}$. Also we can prove, using P1, that δ^2-power protection of a simplex $\sigma \in \mathrm{Del}_\varpi(L, T\mathcal{M})$ incident to p on $T_p\mathcal{M}$ implies δ^2-power protection of $\sigma' \in \mathrm{Del}_\psi(L')$. Using this correspondance, we can show that all m-simplices incident to p in $\mathrm{Del}_\psi(L')$ are δ^2-power protected since all the m-simplices incident to p in $\mathrm{Del}_\varpi(L, T\mathcal{M})$ are δ^2-power protected on $T_p\mathcal{M}$. We can now use Lemma 11. Using the bound on λ, we can show that $\mathrm{Vor}_\psi(p) = \mathrm{Vor}_\varpi(p) \cap T_p\mathcal{M}$ is bounded, see [BG14, Lem. 4.4]. From Lemma 11, we then get that all j-simplices in $\mathrm{Del}_\psi(L')$ incident to p' are $\frac{\delta^2}{m+1}$-power protected. This result, together with the correspondence we have established between the power protection of simplices incident to p in $\mathrm{Del}_\psi(L')$ and the power protection on $T_p\mathcal{M}$ of simplices incident to p in $\mathrm{Del}_\varpi(L, T\mathcal{M})$, we deduce that all j-simplices incident to p in $\mathrm{Del}_\varpi(L, T\mathcal{M})$ are $\frac{\delta^2}{m+1}$-power protected on $T_p\mathcal{M}$.

Let σ be $\frac{\delta^2}{m+1}$-power protected at $c \in \mathrm{Vor}_\varpi(\sigma) \cap T_p\mathcal{M}$, where $p \in \sigma$. We can show that there exists $c' \in \mathcal{M}$, such that $\|c - c'\|$ is small compared to $\frac{\delta^2}{m+1}$ and the line passing through c and c' is orthogonal to $\mathrm{aff}(\sigma)$. Using simple triangle inequalities, we can prove that σ is $\Omega\left(\frac{\delta^2}{m}\right)$-power protected at c'. See Lemma 38.

As W is an ε-sample of $\mathcal{M}$, we can find a $w \in W$ such that $\|w - c'\| < \varepsilon$. Using the facts that ε is much smaller than $\delta^2 = \delta_0^2\lambda^2$ and σ is $\Omega\left(\frac{\delta^2}{m}\right)$-power protected at c', we get w to be a ϖ-witness of σ. Since σ is an arbitrary simplex of $\mathrm{Del}_\varpi(L, T\mathcal{M})$, we have proved that $\mathrm{Del}_\varpi(L, T\mathcal{M}) \subseteq \mathrm{Wit}_\varpi(L, W)$. See Lemma 39.

This ends the proof of Theorem 19.

5 Reconstruction algorithm

Let $\mathcal{M}$ be a smooth submanifold with known dimension m, let $W \subset \mathcal{M}$ be an ε-sample of $\mathcal{M}$, and let $L \subset W$ be a λ-net of W for some known λ. We will also assume that $\varepsilon < \lambda$, which implies that L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$. We will discuss the reasonability of these assumptions in Section 5.3.

The primary task of the algorithm is to find a stable weight assignment $\varpi : L \to [0, \infty)$. We will prove that this is possible if Γ_0, δ_0, and the absolute constant $\alpha_0 < \frac{1}{2}$ satisfy Inequality (12) (Lemma 21).

Once we have calculated a stable weight assignment ϖ, we can just output the witness complex $\mathrm{Wit}_\varpi(L, W)$, which is a faithful reconstruction of $\mathcal{M}$ by Theorem 19.

5.1 Outline of the algorithm

We initialize all weights by setting $\varpi_0(q) = 0$ for all $q \in L$. We then process each point $p_i \in L$, $i \in \{1, \ldots, n\}$. At step i, we compute a new weight assignment ϖ_i satisfying the following properties:

C1. $\tilde{\varpi}_i \leq \tilde{\alpha}_0$, and $\forall\ q \in L \setminus \{p_i\}$, $\varpi_i(q) = \varpi_{i-1}(q)$.

C2. ϖ_i is locally stable at p_i.

Once we have assigned weights to all the points of L in the above manner, the algorithm outputs $\mathrm{Wit}_\varpi(L, W)$ where $\varpi = \varpi_n$ is the final weight assignment $\varpi_n : L \to [0, \infty)$.

The crux of our approach is that weight assignments will be done without computing the cocone complex or any other sort of Voronoi/Delaunay subdivision. Rather, we just look at local neighborhoods

$$N(p_i) \stackrel{\text{def}}{=} \left\{x \in L : \#(\overline{B}(p_i, d(p_i, x)) \cap L) \leq N_1\right\}$$

where N_1 is defined in Lemma 22. The main idea is the following. We define the *candidate simplices* of p_i as the Γ_0-slivers σ of dimension $\leq m+1$, with vertices in $N(p_i)$, $p_i \in \sigma$, and of diameter $\Delta(\sigma) \leq 16\lambda$. For such a candidate simplex σ, we compute a *forbidden interval* $I_{\varpi_{i-1}}(\sigma, p_i)$ (to be defined in Section 5.2). We then select a weight for p_i that is outside all the forbidden intervals of the candidate simplices of p_i.

We will denote by $S(p_i)$ the set of candidate simplices of p_i. For a point p in L, we write

$$nn(p) \stackrel{\text{def}}{=} \min_{q \in L \setminus p} \|p - q\|.$$

Algorithm 1 Pseudocode of the algorithm

Input: L, W, Γ_0, δ_0 and m
// let $L = \{p_1, \ldots, p_n\}$
// parameters Γ_0, δ_0 and m satisfy Eq. (12)
Initialization: $\varpi_0 : L \to [0, \infty)$ with $\varpi_0(p) = 0, \forall\, p \in L$;
Compute: $nn(p)$, $N(p)$ for all $p \in L$
for $i = 1$ to n **do**
  Compute: *candidate simplices* $S(p_i)$;
  $I \leftarrow \bigcup_{\sigma \in S(p_i)} I_{\varpi_{i-1}}(\sigma, p_i)$;
  $\varpi_i(q) \leftarrow \varpi_{i-1}(q)$ for all $q \in L \setminus \{p_i\}$;
  $x \leftarrow$ a point from $[0, \tilde{\alpha}_0^2\, nn(p_i)^2] \setminus I$;
  $\varpi_i(p_i) \leftarrow \sqrt{x}$;
end for
Output: $\mathrm{Wit}_{\varpi_n}(L, W)$;

5.2 Analysis

5.2.1 Correctness of the algorithm

Forbidden intervals and elementary weight perturbations are closely related (see Lemma 20 below) and we will prove in Lemma 21 that, if Inequality (12) is satisfied, we can find a locally stable weight assignment ϖ_i at each iteration of the algorithm. Moreover, we will prove that if all ϖ_i are locally stable, then we will end up with a stable weight assignment $\varpi = \varpi_n$ for which Theorem 19 applies. In this respect our algorithm is in the same vein as the seminal work of Cheng *et al.* [CDE+00]. See also [CDR05, BGO09, BG14].

For a given weight assignment $\varpi : L \to [0, \infty)$, and a simplex σ with vertices in L, we define

$$F_\varpi(p, \sigma) \stackrel{\text{def}}{=} D(p, \sigma)^2 + d(p, N_\varpi(\sigma_p))^2 - R_\varpi(\sigma_p)^2. \tag{9}$$

Note that $F_\varpi(p, \sigma)$ depends on the weights of the vertices of σ_p and not on the weight of p. This crucial fact will be used in the analysis of the algorithm.

If σ is a candidate simplex of p, the *forbidden interval* of σ with respect to p is

$$I_\varpi(\sigma, p) \stackrel{\text{def}}{=} \left[F_\varpi(p, \sigma) - \frac{\eta}{2},\, F_\varpi(p, \sigma) + \frac{\eta}{2}\right], \tag{10}$$

where

$$\eta \stackrel{\text{def}}{=} 2^{14}\left(\Gamma_0 + \frac{\delta_0^2}{\Gamma_0^m}\right)\lambda^2. \tag{11}$$

The following result relates candidate simplices, forbidden intervals and stable weight assignments. The proof is included in Appendix B.

Lemma 20 *Let $L \subset \mathcal{M}$ be a $(\lambda, 2\lambda)$-net of $\mathcal{M}$ with $\lambda < \frac{1}{18}(1-\sin\theta_0)^2 \mathrm{rch}(\mathcal{M})$ and $\varpi : L \to [0,\infty)$ be a weight assignment with $\tilde{\varpi} \leq \tilde{\alpha}_0$. Let, in addition, p be a point of L, and σ a candidate simplex of p. If there exists an ewp ϖ_1 of ϖ satisfying $\tilde{\varpi}_1 \leq \alpha_0$ and $\sigma \in \mathrm{K}_{\varpi_1}(L)$, then $\varpi(p)^2 \in I_\varpi(p,\sigma)$.*

Lemma 20 shows that the emergence of a candidate simplex σ incident to p in the weighted cocone complex under an ewp implies that the original weight of p had to be in the forbidden interval, i.e., $I_\varpi(p,\sigma)$.

The following lemma shows that *good weights*, i.e., weights which do not lie in any forbidden intervals, exist, which ensures that the algorithm will terminate.

Lemma 21 (Existence of good weights) *Assume that $\lambda \leq \frac{\mathrm{rch}(\mathcal{M})}{512}$, and Γ_0, δ_0 and $\tilde{\alpha}_0$ ($=\sqrt{\alpha_0^2 - \delta_0^2}$) satisfy*

$$\Gamma_0 + \frac{\delta_0^2}{\Gamma_0^m} < \frac{\tilde{\alpha}_0^2}{2^{14}N} \tag{12}$$

where $N = 2^{O(m^2)}$ and will be defined explicitly in the proof. Then, at the i^{th} step, one can find a weight $\varpi_i(p_i) \in [0, \tilde{\alpha}_0\, nn(p_i)]$ outside the forbidden intervals of the candidate simplices of $S(p_i)$. Moreover, ϖ_i satisfies properties **C1** *and* **C2**.

Using simple packing arguments and Lemma 7 (1), we get the following bound (similar arguments were used, for example, in [GW04, Lem. 9] and [BG14, Lem. 4.12]).

Lemma 22 *If $\lambda \leq \frac{\mathrm{rch}(\mathcal{M})}{512}$, then for any $p \in L$, $\#(B(p,16\lambda)\cap L) \leq 66^m \stackrel{\text{def}}{=} N_1$.*

Proof of Lemma 21 Write $S(p_i)$ for the set of candidate simplices of p_i. We have

$$\#S(p_i) \leq N \stackrel{\text{def}}{=} \sum_{j=2}^{m+1} N_1^j.$$

For all $\varpi : L \to [0,\infty)$ with $\tilde{\varpi} \leq \alpha_0$, we get from Lemmas 32 (2) and 22 that the set of Γ_0-slivers of dimension $\leq m+1$ in $\mathrm{K}_\varpi(L)$ that are incident to p_i is a subset of $S(p_i)$.

Recall, from Section 2.1, that for $X \subseteq \mathbb{R}$, $\mu(X)$ denotes the standard Lebesgue measure of X. Since

$$\begin{aligned}
\mu\left(\bigcup_{\sigma \in S(p_i)} I_{\varpi_{i-1}}(\sigma, p_i)\right) &\leq \sum_{\sigma\in S(p_i)} \mu(I_{\varpi_{i-1}}(\sigma,p_i)) \\
&\leq N\eta \\
&< \tilde{\alpha}_0^2 \lambda^2 \\
&\leq \tilde{\alpha}_0^2\, nn(p_i)^2,
\end{aligned}$$

we can select $\varpi(p_i) \in [0, \tilde{\alpha}_0\, nn(p_i)]$ such that $\varpi(p_i)^2$ is outside the forbidden intervals of the candidate simplices of p_i, i.e.,

$$\varpi(p_i)^2 \notin \bigcup_{\sigma\in S(p_i)} I_{\varpi_{i-1}}(\sigma, p_i).$$

By Lemma 20, the weight assignment ϖ_i we obtain is a locally stable weight assignment for p_i. $\square$

The following lemma shows that getting a locally stable weight assignment ϖ_i at each iteration of the algorithm gives a globally stable weight assignment ϖ_n at the end of the algorithm.

Lemma 23 *The weight assignment $\varpi_n : L \to [0, \infty)$ is stable.*

Proof It is easy to see that $\tilde{\varpi}_n \leq \tilde{\alpha}_0$, since for all $p \in L$, the weights were chosen from the interval $[0, \tilde{\alpha}_0\, nn(p)]$.

We will prove the stability of ϖ_n by contradiction. Let $\xi : L \to [0, \infty)$ be an ewp of ϖ_n that modifies the weight of $q \in L$, and assume that there exists a Γ_0-sliver $\sigma = [p_{i_0}, \ldots, p_{i_k}] \in \mathrm{K}_\xi(L)$. Note that $\tilde{\xi} \leq \alpha_0$, and that for any $p \in \sigma$, $\sigma \in S(p)$ (from the definition of $S(p)$ and Lemma 22). Without loss of generality assume that

$$i_0 < \cdots < i_k.$$

We will have to consider the following two cases:

Case 1. q is not a vertex of σ. This implies that $\sigma \in \mathrm{K}_{\varpi_n}(L)$ since $\xi(x) = \varpi_n(x)$ for all $x \in L \setminus \{q\}$, and $\xi(q) \geq \varpi_n(q)$. Using the same arguments, we can show that $\sigma \in \mathrm{K}_{\varpi_{i_k}}(L)$. From Lemma 21 and the fact that ϖ_{i_k} is an ewp of itself, we have reached a contradiction as ϖ_{i_k} is a locally stable weight assignment for p_{i_k}.

Case 2. q is a vertex of σ. Using the same arguments as in Case 1 we can show that $\sigma \in \mathrm{K}_{\xi_1}(L)$ where $\xi_1 : L \to [0, \infty)$ is a weight assignment satisfying: $\xi_1(q) = \xi(q)$ and $\xi_1(x) = \varpi_{i_k}(x)$ for all $x \in L \setminus \{q\}$. Observe that ξ_1 is an ewp of ϖ_{i_k}. As in Case 1, we have reached a contradiction since ϖ_{i_k} is a locally stable weight assignment for p_{i_k}. $\square$

5.2.2 Complexity of the algorithm

The following theorem easily follows from the algorithm and the previous analysis.

Theorem 24 *Let $\mathcal{M}$ be a m-dimensional submanifold of $\mathbb{R}^d$, and let $W \subset \mathcal{M}$ be an ε-sample of $\mathcal{M}$ and $L \subseteq W$ be a λ-net of W with $\varepsilon \leq \lambda$. Also, assume that the parameters α_0, δ_0 and Γ_0 satisfy Equation* (12) *from Lemma 21, and ε and λ satisfy Equations* (6) *and* (7) *from Theorem 19. The time and space complexity of Algorithm 1 are*

$$d\#L\left(2^{O(m)}\#L + 2^{O(m^2)} + O(m)\#W\right) + O(m^3\#W)$$

and

$$d\#W + \#L\left(2^{O(m^2)} + d\right) + O(m\#L \times \#W)$$

respectively.

Proof In the initialization phase of the algorithm one needs to compute $nn(p)$ and $N(p)$ for all $p \in L$. Time time complexity for this part of the procedure will be $d2^{O(m)}(\#L)^2$.

Inside the for-loop for each $p_i \in L$, one needs to do the following:

1. compute candidate simplices $S(p_i)$

2. compute $I = \bigcup_{\sigma \in S(p_i)} I_{\varpi_{i-1}}(\sigma, p_i)$

3. Find a point x in $\left[0, \tilde{\alpha}_0^2\, nn(p_i)^2\right] \setminus I$

Observe that for all $\sigma^j \subseteq N(p_i)$ with $p_i \in \sigma$ and $j \leq m+1$, we need to check if σ^j is in $S(p_i)$ and the time complexity for this procedure for a given σ^j is $d\, 2^{O(j)}$. The bound follows from the facts that the number of faces of σ^j is 2^j, and for a given face $\sigma^k \leq \sigma^j$ and a vertex $q \in \sigma^k$, we can compute $D(p, \sigma^k)$ in time complexity $O(d\, \mathrm{poly}(k))$ [4]. If $\sigma^j \in S(p_i)$, then the time complexity of computing $I_{\varpi_{i-1}}(\sigma^j, p_i)$ is $O(d\, \mathrm{poly}(j))$. Time complexity of computing I and finding a point $x \in \left[0, \tilde{\alpha}_0^2\, nn(p_i)^2\right] \setminus I$ will be at most $O((\#S(p_i))^2)$. Since $\#S(p_i) = 2^{O(m^2)}$, therefore the overall time complexity for one execution of the for-loop will be $d 2^{O(m^2)}$.

The above discussion implies the overall time complexity of computing a stable weight assignment $\varpi : L \to [0, \infty)$ will be $d\, 2^{O(m^2)}(\#L)$.

Once a stable weight assignment has been computed then we can use Boissonnat and Maria's simplex-tree based witness complex computation algorithm [BM14, Sec. 3.2] for computing $\mathrm{Wit}_\varpi(L, W)$. Observe that since $\varpi : L \to [0, \infty)$ is a stable weight assignment the m-skeleton[5] of $\mathrm{Wit}_\varpi(L, W)$ is equal to $\mathrm{Wit}_\varpi(L, W)$, see Theorem 19, and therefore we will only compute in the algorithm the m-skeleton of $\mathrm{Wit}_\varpi(L, W)$. Note that for constructing m-skeleton of $\mathrm{Wit}_\varpi(L, W)$, the simplex-tree based algorithm of Boissonnat and Maria [BM14] will need access to $(m+1)$-nearest ϖ-weighted neighbors in L for each $w \in W$. The time complexity for computing this $(m+1)$-nearest ϖ-weighted neighbors in L for all witness in W will be $O(m\#L \times \#W)$. The time complexity of Boissonat and Maria's witness complex construction algorithm will be

$$O\left(m^3\left(\#\mathrm{Wit}_\varpi(L, W) + \#W\right)\right) = O\left(2^{O(m^2)}\#L + m^3\#W\right).$$

The last bound follows from the fact that $\#\mathrm{Wit}_\varpi(L, W) = 2^{O(m^2)}\#L$. Note that the upper bound on $\#\mathrm{Wit}_\varpi(L, W)$ follows from the facts that the dimensions of simplices in $\mathrm{Wit}_\varpi(L, W)$ are at most m, using $\varepsilon \leq \lambda$ and from triangle inequality we have L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$, for all $\sigma \in \mathrm{Wit}_\varpi(L, W)$ we have $\Delta(\sigma) \leq 16\lambda$ (see Lemma 29), and for all $p \in L$ we have $\#\left(B(p, 16\lambda) \cap L\right) = 2^{O(m)}$ (see Lemma 22).

Combining everything, we get that the time complexity of the algorithm is

$$d\,\#L\left(2^{O(m)}\#L + 2^{O(m^2)} + O(m)\#W\right)\#L + O(m^3\#W).$$

The space complexity of storing $\mathrm{Wit}_\varpi(L, W)$ in the simplex-tree data structure is bounded by $O\left(\#\mathrm{Wit}_\varpi(L, W)\right)$. Hence the overall space complexity of the algorithm is

$$d\#W + (2^{O(m^2)} + d)\#L + O(m\#L \times \#W).$$

Note that the $2^{O(m^2)}\#L$ term bounds the space complexity of storing $N(p)$ and $S(p)$ for all $p \in L$ and $\mathrm{Wit}_\varpi(L, W)$, the $O(m\#L \times \#W)$ term comes from storing the ϖ-weighted $(m+1)$-nearest neighbors in L for each witness w in W, and finally the terms $d\#L$ and $d\#W$ come from storing the coordinates of the points in L and W respectively. □

[4] Note that $\mathrm{poly}(k)$ denotes a polynomial in k of degree $O(1)$.

[5] For a simplicial complex $\mathcal{K}$, the m-skeleton of $\mathcal{K}$ is set of simplices of $\mathcal{K}$ of dimension at most m.

5.3 Regarding the assumptions

We have assumed that we know the dimension of the manifold m, and the value of λ (having an upper bound would have been good enough) where L is a λ-net of W.

We will address the second question first. Given a point sample W, and beginning with an arbitrary point from W, it is simple to show that a furthest point sampling [Gon85] from W will generate a λ-net of W, for some $\lambda > 0$, and it is possible to keep track of the value of λ. For an analysis of this procedure, refer to [BGO09, Lem. 5.1].

Let $\mathsf{P} \subset \mathcal{M}$ be an (ν, ϵ)-net of $\mathcal{M}$. If $\frac{\nu}{\epsilon} = O(1)$ and if we know an upper bound on this quantity and if $\epsilon \leq \epsilon_0$, where ϵ_0 depends only on the reach and the dimension of $\mathcal{M}$, then we can learn the local dimension of the manifold at each sample point with time and space complexity $2^{O(m)}(\#\mathsf{P})^2$ and $2^{O(m)}\#\mathsf{P}$ respectively, see [CWW08, CC09, GW04]. Note that, in these papers, the dimension estimation is done locally around each sample point and therefore is exactly in the spirit of this paper. For a more detailed discussions on these things refer to Section 6.

6 Conclusion: only distances required

The algorithm we have outlined can be simply adapted to work in the setting where the input is just a *distance matrix* corresponding to a dense point sample on the submanifold $\mathcal{M}$. Rather than giving explicit coordinates of the points, we will be given a distance matrix $M = (a_{ij})$ where $a_{ij} = \|p_i - p_j\|$ and $p_i,\, p_j \in W$.

In our reconstruction algorithm, we have to compute things like local neighbors $N(p)$, candidate simplices $S(p)$, forbidden intervals $I_\varpi(\sigma, p)$, and the witness complex. We will end the section with a discussion on the sampling conditions, extension to noisy distance matrix and comparisons with other methods.

λ-net L of W. As already discussed in Section 5.2.2, the distance matrix can be used to generate a λ-net L of W by repeatedly inserting a farthest point.

Computing $N(p)$ and $S(p)$. Computing $N(p)$ is simple. For computing $S(p)$, we need to compute the altitude of simplices. This reduces to computing volume of simplices,

$$D(p, \sigma^j) = \frac{j\,\mathrm{vol}(\sigma^j)}{\mathrm{vol}(\sigma^j_p)},$$

which can be done from the knowledge of the lengths of its edges. To see this, observe that for a simplex $\sigma = [p_0, \ldots, p_k]$

$$\mathrm{vol}(\sigma) = \frac{1}{k!}\sqrt{|\det M(\sigma)|}$$

where $M(\sigma) \stackrel{\text{def}}{=} (b_{ij})_{1 \leq i,\, j \leq k}$ with

$$b_{ij} = \langle p_i - p_0,\, p_j - p_0 \rangle = \frac{\|p_i - p_0\|^2 + \|p_j - p_0\|^2 - \|p_i - p_j\|^2}{2}.$$

The above discussion shows that $N(p)$ and $S(p)$ can be computed directly from the distance matrix.

Computing forbidden intervals $I_\varpi(\sigma, p)$. Assume σ is a k-simplex. Recall that computing $I_\varpi(\sigma, p)$ will boil down to computing $D(p, \sigma)$, $d(p, N_\varpi(\sigma_p))$ and $R_\varpi(\sigma_p)$, see Equations (9), (10) and (11). We have already discussed how to compute $D(p, \sigma)$, but observe that $d(p, N_\varpi(\sigma_p))$ and $R_\varpi(\sigma_p)$ can be computed if we can find a distance preserving embedding of σ into an Euclidean space. Since we know the pairwise distance between vertices of the simplex, a distance preserving embedding of σ can be computed in $O(k^3)$, where σ is a k-simplex. See [Mat02, Mat13].

Dimension estimation from distance matrix. We will now show that using known algorithms, such as, for example, [GW04, CC09, CWW08], on dimension estimation of submanifolds, one can estimate the dimensions of submanifolds from distance matrices. We will be adapting Cheng, Wang and Wu's approach [CWW08]. Let $L \subset \mathcal{M}$ be a $(\lambda, 2\lambda)$-net of the manifold $\mathcal{M}$ and $p \in L$. For the time being we will assume that we have explicit coordinates of the points in the $\mathbb{R}^d$, the ambient space, and $\mathrm{rch}(\mathcal{M}) = 1$. We want to estimate the unknown dimension m of the manifold at p. Cheng et al. [CWW08] showed that if λ is less than some λ_0, where λ_0 depends only on m, then there exist an absolute constant C such that the *covariance matrix* [6] of the set of vectors $\{q - p \mid q \in X(p)\}$, where $X(p) = \{q \in L \mid \|p - q\| \le C\lambda\}$, has a sharp gap between the top m eigenvalues of the covariance matrix and the rest of the eigenvalues. More explicitly, if the $\beta_1 \ge \beta_2 \ge \cdots \ge \beta_d$ are the eigenvalues of the covariance matrix then

$$\frac{\beta_k}{\beta_i} = \Theta(1) \quad \text{and} \quad \frac{\beta_j}{\beta_i} = O(\lambda^2), \qquad \text{where } 1 \le i,\, k \le m \text{ and } m+1 \le j \le d \tag{13}$$

Using this gap Cheng et al. [CWW08] estimated the dimension of $\mathcal{M}$ at p. If the set $X(p)$ is given, then the running time of this algorithm will be $O(d2^{O(m)})$. First note that using the sparsity condition on L, as in Lemma 22, we can show that $\#X(p) = 2^{O(m)}$, and secondly the set $X(p)$ can be computed directly from the distance matrix. The part about covariance matrix can be done by first computing a distance preserving Euclidean embedding of the point set $X(p)$ and then checking the gap in the eigenvalues of the covariance matrix constructed using the coordinate of the embedded points. Note that the dimension of the Euclidean space where the points in $X(p)$ will be embedded is bounded by $\#X(p) = 2^{O(m)}$. Therefore given the set $X(p)$, the time complexity of the dimension procedure will be $2^{O(m)}$.

Computing witness complex. By its very definition, the witness complex can be built from an interpoint distance matrix. So, we can easily adapt our algorithm, without increasing its complexity, to the setting of interpoint distance matrices, which was not possible with the other reconstruction algorithms that explicitly need coordinates of the points [CDR05, BGO09, BG14].

Noisy distances and geodesic distances. The algorithm given in this paper is quite robust to noise and other distortions of the Euclidean distances. For example, as we discuss below, it could accommodate a distance matrix defined via *geodesic distances* on the manifold. Recalling the input to the Algorithm 1, let $W \subset \mathcal{M}$ be an ε-sample of $\mathcal{M}$, and $L \subset W$ be a λ-net of W with $\varepsilon \le \lambda$. This implies that L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$. Without loss of generality we will assume that $\mathrm{rch}(\mathcal{M}) = 1$.

[6] Covariance matrix $Cov(V)$ of the set of vectors $V = \{v_1, \ldots, v_l\}$ in $\mathbb{R}^d$ is defined as

$$Cov(V) = \sum_{i=1}^{l} (v_{i1}, v_{i2}, \ldots, v_{id})^T (v_{i1}, v_{i2}, \ldots, v_{id}),$$

where $v_i = (v_{i1}, v_{i2}, \ldots, v_{id})$. Observe that $Cov(V)$ is a $d \times d$ matrix.

We will have access to a *noisy distance matrix* $\tilde{d}(\cdot,\cdot)$ of W satisfying the following inequality:

$$\left|\frac{\tilde{d}(p,q)}{\|p-q\|}-1\right| \leq \gamma.$$

We will call γ the *error fraction* of the noisy distances. We will assume that $\gamma = O(\lambda^\rho)$ for some $\rho > 0$. This is a standard noise model [NSW11, CWW08, CFG+05, DG06], and we will show how the algorithm given in this paper can be adapted to handle this amount of noise, i.e., the case when the error fraction $\gamma = O(\lambda^\rho)$ for some constant $\rho > 0$.

One of the central objects in the paper is thickness, but for the discussion on noisy distances we will be using the notion of *fatness*. The fatness of a j-simplex σ is defined as

$$\Theta(\sigma) = \frac{\mathrm{vol}(\sigma)}{\Delta(\sigma)^j}.$$

Like in the case of thickness, using the definition of fatness one can characterize bad simplices.

Definition 25 (Θ_0-good simplices and Θ_0-slivers) Let Θ_0 be a positive real number smaller than one. A simplex σ is Θ_0-*good* if $\Theta(\sigma^j) \geq \Theta_0^j$ for all j-simplices $\sigma^j \leq \sigma$. A simplex is Θ_0-*bad* if it is not Θ_0-good. A Θ_0-*sliver* is a Θ_0-bad simplex in which all the proper faces are Θ_0-good.

Thickness and fatness are analogous concepts and the calculations done in this paper can easily be done using fatness. For more details on fatness refer to the manifold reconstruction paper of Boissonnat and Ghosh [BG14].

Let σ be a j-simplex with vertices from L, and let $\widetilde{\mathrm{vol}}(\sigma)$ and $\widetilde{\Delta}(\sigma)$ denote *noisy volume* and *noisy diameter* of σ obtained by using noisy distances. Expressions for $\widetilde{\mathrm{vol}}(\sigma)$ and $\widetilde{\Delta}(\sigma)$ for the simplex $\sigma = [p_0, \ldots, p_j]$ are the following:

$$\widetilde{\Delta}(\sigma) \stackrel{\mathrm{def}}{=} \max_{p_i, p_k \in \sigma} \tilde{d}(p_i, p_k),$$

and

$$\widetilde{\mathrm{vol}}(\sigma) \stackrel{\mathrm{def}}{=} \sqrt{\det(\widetilde{M}(\sigma))},$$

where $\widetilde{M}(\sigma) \stackrel{\mathrm{def}}{=} (a_{ik})_{1\leq i,\, k \leq j}$ with

$$a_{ik} = \frac{\tilde{d}(p_i,p_0)^2 + \tilde{d}(p_k,p_0)^2 - \tilde{d}(p_i,p_k)^2}{2}.$$

We came up with this definition of $\widetilde{M}(\sigma)$ to mimic the definition of $M(\sigma)$ given earlier in the section.

We will call $\widetilde{\Theta}(\sigma) := \frac{\widetilde{\mathrm{vol}}(\sigma)}{\widetilde{\Delta}(\sigma)^j}$ the *noisy fatness* of the j-simplex σ. Using [Gho12, Lem. 4.3.6] we can show that noisy fatness and actual fatness are closely related:

$$\left|\frac{\widetilde{\Theta}(\sigma)^2}{\Theta(\sigma)^2}-1\right| \leq O(j\lambda^\rho). \tag{14}$$

This shows that we can modify Algorithm 1 in terms of fatness and the candidate simplices can be detected using noisy fatness. Note that candidate simplices in the case of a noisy distance matrix will be defined using Θ_0-slivers (defined in terms of fatness) rather than Γ_0-slivers (defined in terms of thickness).

The calculation for computing forbidden intervals in Section B can be easily extended to the case of noisy distances. Also the dimension estimation algorithm of Cheng et al. [CWW08] works for noisy point samples and also can be directly extended to the case of noisy distance matrix. This shows that the algorithm given in this paper can be adapted to handle this amount of noise, i.e., the case when the error fraction $\gamma = O(\lambda^\rho)$ for some constant $\rho > 0$.

Another interesting problem to consider is the case when the entries to the distance matrix are *geodesic distances*, i.e., the entries in the distance matrix are geodesic distances $d_{\mathcal{M}}(p, q)$ between the points p and q on the manifold $\mathcal{M}$. For the case of submanifolds, Niyogi, Smale and Weinberger [NSW08, Prop. 6.3] proved the following result connecting geodesic distance and Euclidean distances in a small neighborhood of the submanifold $\mathcal{M}$.

Lemma 26 *Let p and q be points on the submanifold $\mathcal{M}$ of $\mathbb{R}^d$ with $\mathrm{rch}(\mathcal{M}) = 1$ and $\|p - q\| \leq \frac{1}{2}$. Then*

$$\left| \frac{d_{\mathcal{M}}(p,q)}{\|p-q\|} - 1 \right| \leq O\left(\|p-q\|\right).$$

Therefore for the case of geodesic distances we are again back to the noisy distance framework with the error fraction $\gamma = O(\lambda)$.

Sampling conditions. The sampling condition, i.e., the bound on λ in Theorem 19 is quite pessimistic with respect to the reach of the manifold. Naturally this makes the results in this paper to be of more theoretical interest and less relevant for applications. But we feel that the techniques introduced in this paper could be used to get better reconstruction algorithms, and or design new and better reconstruction heuristics. Also, note that the only assumption in this paper was that $\mathcal{M}$ is a smooth submanifold with positive reach. If one restrict to a more narrow class of manifolds such as compact flat manifolds then one could easily improve on the sampling conditions. For more details on triangulating closed Euclidean Orbifolds refer to a recent paper of Caroli and Teillaud [CT16].

Comparison with previous works. It is a natural question to ask how other manifold reconstruction algorithms [CDR05, BGO09, BG14] will fair when given only a distance matrix to work with. We will assume that input is a distance matrix corresponding to a dense point sample P of $\mathcal{M}$. Since all the previous algorithms work with explicit coordinates of the points in the point sample one needs to start by first getting a distance preserving embedding into an Euclidean space. Note that this can be done in time $O((\#\mathsf{P})^3)$ via *Cholesky factorization* of positive definite matrix. For more details on Cholesky factorization refer to [GVL13, TB97], and for details on distance preserving embedding into Euclidean space see [Mat02, Mat13]. Once we have the embedding then we can apply the previous manifold reconstruction algorithms [CDR05, BGO09, BG14].

We will now outline some of the issues with this approach:

1. **Time complexity of getting a distance preserving embedding.** The distance preserving embedding will be computed via Cholesky factorization with the time complexity $O((\#\mathsf{P})^3)$.

2. **Noisy distance matrix and geodesic distances.** These approaches won't work if the distance matrix given is noisy. Since the entries in the noisy distance matrix may not be Euclidean distance like for example when the entries are geodesic distance on the manifold $\mathcal{M}$. Since in these cases the entries to the noisy distance matrix are not Euclidean distances, the approach via distance preserving Euclidean embedding won't work.

3. **Large dimension of the embedding space.** Observe that the dimension d of the Euclidean space we can get via the above distance preserving embedding can be as large as $d = \Omega(\#\mathsf{P})$. This will make time complexity of the manifold reconstruction algorithms given in [CDR05, BGO09] exponential in $\#\mathsf{P}$, since both these approaches compute Voronoi diagrams in the ambient space using the whole point sample or a subset of the sample.

4. **Problems with tangential Delaunay complex.** Once we have the distance preserving embedding of the point sample P we can use the tangential Delaunay complex (TDC) for reconstruction [BG14]. As already mentioned, the dimension of the Euclidean space obtained from a distance preserving embedding of P can be as large as $\Omega(\#\mathsf{P})$. Note that TDC construction will compute the following structures:

   (a) The approximate tangent space at each point of the point sample. This if the dimension of the embedding space is $\Omega(\#\mathsf{P})$ will incur an $\Omega(\#\mathsf{P})$ time complexity.

   (b) The TDC construction will need to compute Delaunay triangulations restricted to the approximate tangent spaces. So the problem of using complicated and highly structured data structure still stays.

   (c) In the TDC construction one needs to compute $(m+1)$-dimensional simplices corresponding to each *inconsistent simplex*. See the definition of inconsistent simplex from [BG14]. The time complexity for constructing each of these simplices will be $\Omega(\#\mathsf{P})$ if the dimension of the Euclidean space obtained from a distance preserving embedding is $\Omega(\#\mathsf{P})$.

   As with the other reconstruction algorithms [CDR05, BGO09], the TDC won't be able to handle noisy distance matrix as input or when the entries to the distance matrix are geodesic distances on the manifold $\mathcal{M}$.

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A Proof of Theorem 19

This section is devoted to the proof of Theorem 19. When we talk about properties P1, P2 and P3 in this section, we are actually referring to properties introduced in Section 4.

Before we go into the detailed calculations, we want to make this small and obvious observation that directly follows from triangle inequality.

Observation 27 If W is an ε-sample of $\mathcal{M}$ and L a λ-net of W with $\varepsilon \leq \lambda$, then L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$.

A.1 Proof of properties P1, P2 and P3

We will use the following structural result from [BGO09].

Lemma 28 *Let $\theta \in [0, \frac{\pi}{2})$, and $\mathsf{P} \subset \mathcal{M}$ be an ϵ-sample of $\mathcal{M}$ with $\epsilon < \frac{1}{9}(1-\sin\theta)^2 \mathrm{rch}(\mathcal{M})$. For any weight assignment $\varpi : L \to [0, \infty)$ with $\tilde{\varpi} < \frac{1}{2}$, for any $p \in \mathsf{P}$ and $x \in \mathrm{Vor}_\varpi(p) \cap \mathrm{K}^\theta(p)$, we have*

$$\|p - x\| \leq \frac{3\epsilon}{1 - \sin\theta} .$$

Following lemma is a direct consequence of Lemma 28.

Lemma 29 *Let L be an ϵ-sample of $\mathcal{M}$ with $\epsilon < \frac{1}{9}(1-\sin\theta_0)^2 \mathrm{rch}(\mathcal{M})$, and let $\varpi : L \to [0,\infty)$ be a weight assignment with $\tilde{\varpi} < \frac{1}{2}$. Let $\sigma \in \mathrm{K}_\varpi(L)$.*

1. *Let p be a vertex of σ with $\mathrm{Vor}_\varpi(p) \cap \mathrm{K}^{\theta_0}(p) \neq \emptyset$. For all vertices q of σ and $x \in \mathrm{Vor}_\varpi(p) \cap \mathrm{K}^{\theta_0}(p)$, we have $\|x - q\| < 4\epsilon$. This implies for all the vertices q of σ, $\|q - C_\varpi(\sigma)\| < 4\epsilon$.*

2. *$\Delta(\sigma) < 8\epsilon$.*

Proof 1. Observe that

$$\begin{aligned}
\|q - x\| &= \sqrt{\|p-x\|^2 - \varpi(p)^2 + \varpi(q)^2} \\
&\leq \sqrt{\left(\frac{3\epsilon}{1 - \sin\theta_0}\right) + \epsilon^2} && \text{from Lemma 28, Lemma 8 (1) and } \tilde{\varpi} < \tfrac{1}{2} \\
&< 4\epsilon
\end{aligned}$$

The bound on $\|q - C_\varpi(\sigma)\|$ follows from the fact that $\|q - x\| \geq \|q - C_\varpi(\sigma)\|$.

2. The bound on $\Delta(\sigma)$ follows from part (1) and triangle inequality. □

The following corollary about witness complex is from [dS08, Cor. 7.6].

Corollary 30 *For any subsets $W, L \subseteq \mathbb{R}^d$ with L finite, for any $\varpi : L \to [0,\infty)$, we have $\mathrm{Wit}_\varpi(L, W) \subseteq \mathrm{Del}_\varpi(L)$. Moreover, for any simplex σ of $\mathrm{Wit}_\varpi(L,W)$, the weighted Voronoi face of σ intersects the convex hull of the ϖ-witnesses (among the points of W) of σ and of its subsimplices.*

Following result is a direct consequence of Lemma 8 (2).

Lemma 31 *Let $W \subseteq \mathcal{M}$ be an ε-sample of $\mathcal{M}$, $L \subseteq W$ be a λ-net of W with $\lambda + \varepsilon < \frac{\mathrm{rch}(\mathcal{M})}{4}$, and $\varpi : L \to [0,\infty)$ be a weight assignment with $\tilde{\varpi} < \frac{1}{2}$.*

1. *For all $pq \in \mathrm{Wit}_\varpi(L,W)$, $\|p - q\| \leq (4 + 10\tilde{\varpi})(\lambda + \varepsilon)$.*

2. *Let $\sigma \in \mathrm{Wit}_\varpi(L,W)$. The distance between any vertex p of σ and any witness w of $\tau \leq \sigma$ is at most $(5 + 12\tilde{\varpi})(\lambda + \varepsilon)$.*

Proof 1. First observe that L is a $(\lambda + \varepsilon)$-sample of $\mathcal{M}$ via triangle inequality. Let $w \in W$ be a ϖ-witness for the edge $pq \in \mathrm{Wit}_\varpi(L,W)$, and without loss of generality assume that p is closest neighbor of w in terms of the weighted distance. Then from Lemma 8 (2), we have $\|w - p\| \leq (1 + 2\tilde{\varpi})(\lambda + \varepsilon)$ and $\|w - q\| \leq (3 + 8\tilde{\varpi})(\lambda + \varepsilon)$. Therefore again from triangle inequality we get $\|p - q\| \leq (4 + 10\tilde{\varpi})(\lambda + \varepsilon)$.

2. Let p be a vertex of σ, and let $q \in \tau$ is the vertex closest to w in terms of the weighted distance. Then from part (1), we have $\|p - q\| \leq (4 + 10\tilde{\varpi})(\lambda + \varepsilon)$. From Lemma 8 (2), we have $\|w - q\| \leq (1 + 2\tilde{\varpi})(\lambda + \varepsilon)$. From triangle inequality, we have $\|p - w\| \leq (5 + 12\tilde{\varpi})(\lambda + \varepsilon)$. □

We will now give the proof of Properties **P1**, **P2**, and **P3**.

Lemma 32 *Let L be an ϵ-sample of $\mathcal{M}$ with $\epsilon < \frac{1}{9}(1-\sin\theta_0)^2 \mathrm{rch}(\mathcal{M})$, and let $\varpi : L \to [0,\infty)$ be a weight assignment with $\tilde{\varpi} < \frac{1}{2}$. Let $\sigma \in \mathrm{K}_\varpi(L)$.*

1. **(Property P1)** *Assume $\dim\sigma = k \le m$ and $\Upsilon(\sigma) \ge \Gamma_0^k$. Additionally, if $\epsilon < \frac{\mathrm{rch}(\mathcal{M})}{8}$ then for all vertices p of σ we have*

$$\sin\angle(\mathrm{aff}(\sigma), T_p\mathcal{M}) \le \frac{8\epsilon}{\Gamma_0^m \mathrm{rch}(\mathcal{M})}.$$

2. **(Property P2)** *If $\mathrm{K}_\varpi(L)$ does not contain any Γ_0-slivers of dimension $\le m+1$ and*

$$\epsilon < \frac{\Gamma_0^{2m+1}\mathrm{rch}(\mathcal{M})}{12},$$

then dimension of maximal simplices in $\mathrm{K}_\varpi(L)$ is at most m.

3. **(Property P3)** *Assume hypothesis in part (4) of this lemma. Additionally, if L is $\frac{\epsilon}{2}$-sparse and*

$$\epsilon \le \frac{\Gamma_0^m \sin\theta_0}{2^9}\mathrm{rch}(\mathcal{M}),$$

then for all m-simplex $\sigma^m \in \mathrm{Del}_\varpi(L, T\mathcal{M})$ and $\forall\, p \in \sigma$, we have $\mathrm{Vor}_\varpi(\sigma^m) \cap T_p\mathcal{M} \neq \emptyset$.

Proof 1. Follows directly from Corollary 9 and Lemma 32 (2).

2. Using part 1 and exactly the proof idea used in the proof of [BG14, Lem. 4.9], we can show that all $m+1$-simplices in $\mathrm{K}_\varpi(L)$ are either Γ_0-bad or have thickness $\frac{12\epsilon}{\Gamma_0^m \mathrm{rch}(\mathcal{M})}$. Using the bound on ϵ, we can complete the proof of part 4.

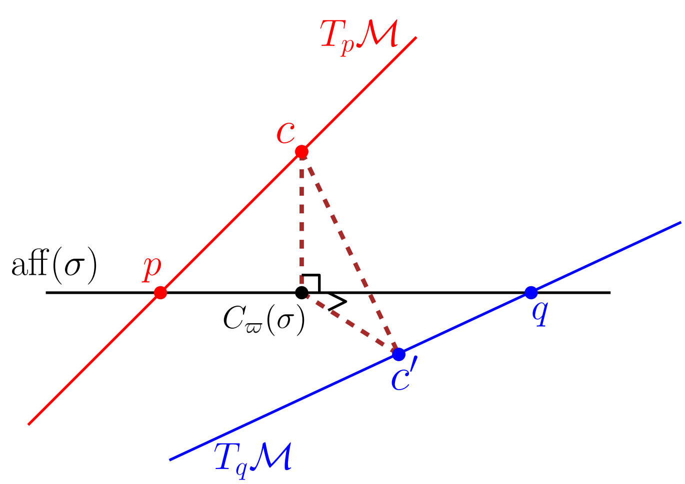

Figure 12: Diagram for the Lemma 32 (3).

3. Using the facts that L is $\frac{\epsilon}{2}$-sparse and $\tilde{\varpi} < \frac{1}{2}$, we can show $\forall\, p \in \sigma$,

$$d(p, N_\varpi(\sigma^m)) \ge \frac{3\epsilon}{16}. \tag{15}$$

To reach a contradiction let p be a vertex of σ^m such that $\mathrm{Vor}_\varpi(\sigma) \cap T_p\mathcal{M} \neq \emptyset$ and q be a vertex of σ^m with $\mathrm{Vor}_\varpi(\sigma) \cap T_q\mathcal{M} = \emptyset$. Using the bound on ϵ and part 1 of this lemma, we can show, for all $x \in \sigma^m$, that

$$\sin(\mathrm{aff}\,\sigma^m, T_x\mathcal{M}) \leq \sin\theta \stackrel{\mathrm{def}}{=} \frac{8\epsilon}{\Gamma_0^m \mathrm{rch}(\mathcal{M})} < \frac{1}{2}$$

and

$$\#(N_\varpi(\sigma) \cap T_x\mathcal{M}) = 1.$$

Observe that $C_\varpi(\sigma^m)$ is orthogonal projection of c onto $\mathrm{aff}(\sigma^m)$. Let c' denotes the intersection of $N_\varpi(\sigma^m)$ and $T_q\mathcal{M}$. Note that, by construction, the line segment $cc' \in N_\varpi(\sigma^m)$.

From part (1) of this lemma, we have for all $x \in \sigma^m$,

$$\|x - C_\varpi(\sigma^m)\| \leq \|x - c\| \leq 4\epsilon. \tag{16}$$

Therefore

$$\begin{aligned}\|c - c'\| &\leq \|c - C_\varpi(\sigma^m)\| + \|C_\varpi(\sigma^m) - c'\| \\ &\leq \|p - c\|\sin\theta + \|q - C_\varpi(\sigma^m)\|\tan\theta \\ &\leq 4\epsilon\sin\theta\left(1 + \frac{1}{\cos\theta}\right) \\ &< 12\epsilon\sin\theta\end{aligned}$$

Using the facts that $\epsilon \leq \frac{\Gamma_0^m \sin\theta_0}{2^9}\mathrm{rch}(\mathcal{M})$ and $d(p, N_\varpi(\sigma^m)) \geq \frac{3\epsilon}{16}$, see Equation (15), we can show that the line segment $cc' \in \mathrm{K}(p)$. So, $\mathrm{Vor}(\sigma^m) \cap T_p\mathcal{M} = \emptyset$ implies there exists a $\sigma^{m+1} \in \mathrm{K}_\varpi(L)$ with $\sigma^m < \sigma^{m+1}$. We have reached a contradiction via part 2 of this lemma. □

A.2 Proof of Theorem 19

Homemorphism and geometric guarantees given in Theorem 19 is a direct consequence of the following lemma[7] from [BG14].

Lemma 33 *Let $L \subset \mathcal{M}$ be an $(\lambda, 2\lambda)$-net of $\mathcal{M}$, and $\varpi : L \to [0, \infty)$ be a weight assignment satisfying the following properties:*

1. $\tilde{\varpi} \leq \alpha_0$.
2. *Dimension of maximal simplices in* $\mathrm{Del}_\varpi(L, T\mathcal{M})$ *is equal to* m
3. *All the simplices in* $\mathrm{Del}_\varpi(L, T\mathcal{M})$ *are* Γ_0*-good.*
4. *For all* $\sigma = [p_0, \ldots, p_k] \in \mathrm{Del}_\varpi(L, T\mathcal{M})$ *and* $\forall\, i \in \{0, \ldots, k\}$, $\mathrm{Vor}_\varpi(p_i) \cap T_{p_i}\mathcal{M} \neq \emptyset$.

There exists $\lambda_0 > 0$ that depends only on α_0, Γ_0 and m such that for $\lambda \leq \lambda_0$, $\mathrm{Del}_\varpi(\mathsf{P}, T\mathcal{M})$ is homeomorphic to and a close geometric approximation of $\mathcal{M}$.

Rest of this section is devoted to the proof of

$$\mathrm{Wit}_\varpi(L, W) = \mathrm{Del}_\varpi(L, T\mathcal{M}).$$

The proof goes through multiple stages.

[7] Note that this lemma is a special case of the result proved in [BG14].

Stage (1). We will first show, in Lemma 34, that no Γ_0-slivers of dimension $\leq m+1$ in $\mathrm{K}_\varpi(L)$ implies $\mathrm{Wit}_\varpi(L,W) \subseteq \mathrm{Del}_\varpi(L,T\mathcal{M})$.

Stage (2). In Lemma 35 we prove that if $\varpi : L \to [0,\infty)$ is a stable weight assignment then for all m-simplices $\sigma \in \mathrm{Del}_\varpi(L,T\mathcal{M})$ and $p \in \sigma$, σ will be δ^2-power protected on $T_p\mathcal{M}$.

Stage (3). Let $\mathrm{Del}_\varpi(L,T\mathcal{M})$ does not contain any Γ_0-slivers and for all m-simplex $\sigma^m \in \mathrm{Del}_\varpi(L,T\mathcal{M})$ and $\forall\, p \in \sigma^m$, σ^m is δ^2-power protected on $T_p\mathcal{M}$. Then for all $\sigma \in \mathrm{Del}_\varpi(L,T\mathcal{M})$, σ will be $\Omega\left(\frac{\delta^2}{m+1}\right)$-power protected on $\mathcal{M}$. See, Lemma 38.

Stage (4). Finally in Lemma 39, we will show that conditions in **Stage (3)** implies

$$\mathrm{Del}_\varpi(L,T\mathcal{M}) \subseteq \mathrm{Wit}_\varpi(L,W).$$

To see that this will complete the proof, observe that ϖ being stable weight assignment implies $K_\varpi(L)$ contains no Γ_0-slivers of dimension $\leq m+1$. This would imply dimension of maximal simplices in $K_\varpi(L)$ is at most m. Once we have this, simply going through **Stage (1)** till **Stage (4)** will give us the result.

Lemma 34 *Let $W \subseteq \mathcal{M}$ be a ε-sample of $\mathcal{M}$, $L \subset W$ be a λ-net of W with $\varepsilon \leq \lambda$, and $\varpi : L \to [0,\infty)$ be a weight assignment with $\tilde{\varpi} < \frac{1}{2}$ and $\mathrm{K}_\varpi(L)$ does not contain any Γ_0-sliver of dimension $\leq m+1$. If*

$$\lambda < \min\left\{\frac{3\sin\theta_0}{2^{11}(1+m)}, \frac{\Gamma_0^{2m+1}}{24}\right\} \mathrm{rch}(\mathcal{M})$$

then

$$\mathrm{Wit}_\varpi(L,W) \subseteq \mathrm{Del}_\varpi(L,T\mathcal{M}).$$

Proof Note that L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$.

To reach a contradiction, let σ^k be a k-simplex in $\mathrm{Wit}_\varpi(L,W)$ and p be a vertex of σ such that $\mathrm{Vor}_\varpi(\sigma^k) \cap T_p\mathcal{M} = \emptyset$.

Let $w \in W$ be a ϖ-witness of a subface of σ^k. From Lemma 31, and the facts that $\tilde{\varpi} < \frac{1}{2}$ and $\varepsilon \leq \lambda$, we have

$$\|p-w\| \leq (5+12\tilde{\varpi})(\lambda+\varepsilon) < 22\lambda.$$

From Lemma 7, we have

$$d(w,T_p\mathcal{M}) \leq \frac{2\times 11^2\lambda^2}{\mathrm{rch}(\mathcal{M})}.$$

From Corollary 30, there exist $c_k \in \mathrm{Vor}_\varpi(\sigma^k)$ that lies in the convex hull of the ϖ-witness of σ (in W) and its subfaces. This implies,

$$\rho \stackrel{\text{def}}{=} d(c_k, T_p\mathcal{M}) \leq \frac{2\times 11^2\lambda^2}{\mathrm{rch}(\mathcal{M})}.$$

Note that since L is λ-sparse and $\tilde{\varpi} < \frac{1}{2}$,

$$\|p-c_k\| \geq d(p, C_\varpi(\sigma^k)) \geq \frac{3\lambda}{8}.^{8}$$

[8] Let p and q be distinct vertices of σ^k, and let x be an orthogonal projection of $C_\varpi(\sigma^k)$ on the line through p and

Note that λ is sufficiently small such that

$$\frac{\rho}{\frac{3\lambda}{8} - 2m\rho} \leq \sin\theta_0 \tag{18}$$

and

$$\sin\theta \stackrel{\text{def}}{=} \frac{16\lambda}{\Gamma_0^m \mathrm{rch}(\mathcal{M})} < \frac{\sqrt{3}}{2}. \tag{19}$$

We will now generate sequence of simplices

$$\sigma^k < \sigma^{k+1} < \cdots < \sigma^m < \sigma^{m+1}$$

and points

$$c_k,\, c_{k+1},\, \ldots,\, c_m,\, c_{m+1}$$

by walking on $\mathrm{Vor}_\varpi(\sigma^k)$ satisfying the following properties:

Prop-1. For all σ^{k+i}, there exists $c_{k+i} \in \mathrm{Vor}_\varpi(\sigma^{k+i})$ such that

$$d(c_{k+i}, T_p\mathcal{M}) \leq \rho$$

and

$$\|p - c_{k+i}\| \geq \|p - c_k\| - 2i\rho.$$

From Eq. (18), this implies $\sigma^{k+i} \in \mathrm{K}_\varpi^{\theta_0}(L)$.

Prop-2. For all σ^{k+i}, we have

$$\mathrm{Vor}_\varpi(\sigma^{k+i}) \cap T_p\mathcal{M} = \emptyset.$$

Note that once we have shown that such sequence of simplices exists, then we would have reached a contradiction from Lemma 32 (2).

We will now show how to generate the above sequence of simplices.

Base case. From Eq. (18), it is easy to see that σ^k and c_k satisfy Prop-1 and Prop-2.

Inductive step. Wlog lets assume that we have generated till σ^{k+i}, satisfying properties **Prop-1** and **Prop-2**, and we also assume $k + i \leq m$. Since $\sigma^{k+i} \in \mathrm{K}_\varpi(L)$, we can show, using Lemma 32 (1), that

$$\sin\angle(N_p\mathcal{M}, N_\varpi(\sigma^{k+i})) \leq \sin\theta.$$

From **Prop-1**, we have $\|p - c_{k+i}\| \geq \|p - c_k\| - 2i\rho$ and $d(c_{k+i}, T_p\mathcal{M}) \leq \rho$. Therefore, from Eq. 19, there exists $\tilde{c}_{k+i} \in T_p\mathcal{M} \cap N_\varpi(\sigma^{k+i})$ such that

$$\|c_{k+i} - \tilde{c}_{k+i}\| \leq \frac{\rho}{\cos\theta} \leq 2\rho.$$

As $\mathrm{Vor}_\varpi(\sigma^{k+i}) \cap T_p\mathcal{M} = \emptyset$ hence there exists $c_{k+i+1} \in [c_{k+i}, \tilde{c}_{k+i})$ such that $c_{k+i+1} \in \mathrm{Vor}_\varpi(\sigma^{k+i+1})$ with $\sigma^{k+i} < \sigma^{k+i+1}$. Note that, as in the base case, we can show that

$$d(c_{k+i+1}, T_p\mathcal{M}) \leq \rho$$

and

$$\|p - c_{k+i+1}\| \leq \|p - c_k\| - 2(i+1)\rho\,.$$

---

q. Since $\tilde{\varpi} < \frac{1}{2}$, $x \in pq$. Using the facts that $d(x, p^\varpi) = d(x, q^\varpi)$, $\|p - q\| \geq \lambda$ and $\varpi(q) < \frac{\|p-q\|}{2}$ (as $\tilde{\varpi} < 1/2$), we get

$$\|p - x\| \;=\; \frac{\|p-q\|}{2}\left(1 + \frac{\varpi(p)^2 - \varpi(q)^2}{\|p-q\|^2}\right) \;\geq\; \frac{\|p-q\|}{2}\left(1 - \frac{\varpi(q)^2}{\|p-q\|^2}\right) \;\geq\; \frac{3\lambda}{8} \tag{17}$$

The bound on $d(p, C_{\varpi(\sigma^k)})$ follows from the fact that $d(p, x) \geq d(p, C_\varpi(\sigma^k))$.

$\square$

The following lemma connects power protection of m-dimensional simplices in $\mathrm{Del}_{\varpi}(T\mathcal{M})$ with stability of ϖ.

Lemma 35 *Let $L \subset \mathcal{M}$ be a $(\lambda, 2\lambda)$-net of $\mathcal{M}$ with*

$$\lambda < \min\left\{\frac{\Gamma_0^m \sin\theta_0}{2^{10}}, \frac{\Gamma_0^{2m+1}}{24}\right\} \mathrm{rch}(\mathcal{M}),$$

and let $\varpi : L \to [0, \infty)$, with $\tilde{\varpi} \leq \tilde{\alpha}_0$, be a stable weight assignment. Then all the m-simplices $\sigma \in \mathrm{Del}_{\varpi}(L, T\mathcal{M})$ are δ^2-power protected on $T_p\mathcal{M}$ for all $p \in \sigma$, where $\delta = \delta_0\lambda$.

Proof Note that L is $(\lambda, 2\lambda)$-net of $\mathcal{M}$. For all m-simplices σ in $\mathrm{Del}_{\varpi}(L, T\mathcal{M})$, we have $\mathrm{Vor}_{\varpi}(\sigma) \cap T_p\mathcal{M} \neq \emptyset$ for all $p \in \sigma$.

To reach a contradiction, lets assume that $\sigma \in \mathrm{Del}_{\varpi}(L, T\mathcal{M})$ to be not δ^2-power protected on $T_p\mathcal{M}$ for some $p \in \sigma$. Let $c \in \mathrm{Vor}_{\varpi}(\sigma) \cap T_p\mathcal{M}$ and $q \in L \setminus \sigma$ such that for all $x \in \sigma$

$$\|q - c\|^2 - \varpi(q)^2 - \delta^2 \leq \|x - c\|^2 - \varpi(x)^2.$$

Let $\beta^2 = \|q - c\|^2 - \|p - c\|^2 - (\varpi(q)^2 - \varpi(p)^2)$ where $p \in \sigma$. Note that $\beta \leq \delta$.

Let $\xi : L \to [0, \infty)$

$$\xi(x) = \begin{cases} \varpi(x) & \text{if } x \neq q \\ \sqrt{\varpi(q)^2 + \beta^2} & \text{if } x = q \end{cases}$$

Since L is λ-sparse, $\delta = \delta_0\lambda$ and $\tilde{\alpha}_0^2 + \delta_0^2 \leq \alpha_0^2$, we have $\tilde{\xi} \leq \alpha_0$. It is easy to see ξ is an ewp of ϖ, and the $(m+1)$-dimensional simplex $\tau = q * \sigma \in \mathrm{K}_{\xi}(L)$. As ϖ is stable weight assignment and ξ is an ewp of ϖ, we get a contradiction from Lemma 32 (2) and the fact that $\lambda < \min\left\{\frac{\Gamma_0^m \sin\theta_0}{2^{10}}, \frac{\Gamma_0^{2m+1}}{24}\right\} \mathrm{rch}(\mathcal{M})$. $\square$

We will need the following result due to Boissonnat and Ghosh [BGO09, Lem 2.2].

Lemma 36 *Let $L \subset \mathbb{R}^d$ be a point set, $\varpi : L \to [0, \infty)$ be a weight distribution, and $H \subseteq \mathbb{R}^d$ be a k-dimensional flat. Also, let L' denotes the projection of the point set L onto H, and p' denotes the projection of $p \in L$ onto H. For all $p \in L$, we have*

$$\mathrm{Vor}_{\varpi}(p) \cap H = \mathrm{Vor}_{\xi}(p')$$

where $\xi : L' \to [0, \infty)$ with

$$\xi(q')^2 = \varpi(q)^2 - \|q - q'\|^2 + \max_{x \in \mathsf{P}} \|x - x'\|^2$$

and $\mathrm{Vor}_{\xi}(p')$ denotes the Voronoi diagram of p' in H and not in $\mathbb{R}^d$.

From Lemma 36, we have get the following corollary.

Corollary 37 *Let $L \subset \mathbb{R}^d$ be a finite set, $\varpi : L \to [0, \infty)$, and let $H \subseteq \mathbb{R}^d$ be k-flat. For a point $p \in L$, if $\mathrm{Vor}_{\varpi}(p) \cap H$ is bounded then the dimension of maximal simplices incident to p in $\mathrm{Del}_{\varpi}(L, H) \stackrel{\text{def}}{=} \{\sigma : \mathrm{Vor}_{\varpi}(\sigma) \cap H \neq \emptyset\}$ is greater than k.*

Following lemma connects power protection of m-simplices on the tangent space to that on the manifold.

Lemma 38 *Let $L \subset \mathcal{M}$ be a $(\lambda, 2\lambda)$-net of $\mathcal{M}$, and $\delta = \delta_0\lambda$ with $\delta_0 < 1$. Let $\varpi : L \to [0, \infty)$ be a weight assignment with $\widetilde{\varpi} < \frac{1}{2}$ and satisfying the following properties:*

1. $\mathrm{Del}_{\varpi}(\mathsf{P}, T\mathcal{M})$ *does not contain any Γ_0-sliver, and*
2. $\forall\ \sigma^m \in \mathrm{Del}_{\varpi}(\mathsf{P}, T\mathcal{M})$, *$\sigma^m$ is δ^2-power protected on $T_p\mathcal{M}$ for all $p \in \sigma^m$.*

If

$$\lambda \leq \frac{\Gamma_0^m \mathrm{rch}(\mathcal{M})}{2^{11}},$$

then all $\sigma \in \mathrm{Del}_{\varpi}(L, T\mathcal{M})$ are δ_1^2-power protected on $\mathcal{M}$ where

$$\delta_1^2 = \frac{\delta^2}{m+1} - \frac{B\lambda^3}{\mathrm{rch}(\mathcal{M})}$$

and $B \stackrel{\mathrm{def}}{=} 2^{15}$.

Proof Let p be a point in L, and L' denotes the projection of the point sample L onto $T_p\mathcal{M}$. For a point $x \in L$, x' is the projection of x onto $T_p\mathcal{M}$ and vise versa, and similarly, let $\sigma = [p_0, \ldots, p_k]$ be a simplex with p_i's in L then σ' denotes the simplex $[p_0', \ldots, p_k']$ and vise versa. Note that $p' = p$.

The weight assignment $\xi : L' \to [0, \infty)$ is defined in the following way:

$$\xi(x')^2 = \varpi(x)^2 - \|x - x'\|^2 + \max_{y \in L} \|y - y'\|^2 .$$

For $\sigma' \subseteq L'$, $\mathrm{Vor}_{\xi}(\sigma')$ denotes the Voronoi cell in $T_p\mathcal{M}$ and not in $\mathbb{R}^d$.

From Lemmas 36 and 29 (1) we have:

Prop. (a) For $\sigma \subseteq L$, $\mathrm{Vor}_{\varpi}(\sigma) \cap T_p\mathcal{M} = \mathrm{Vor}_{\xi}(\sigma')$.

Prop. (b) $\mathrm{Vor}_{\varpi}(p) \cap T_p\mathcal{M} = \mathrm{Vor}_{\xi}(p) \subset B(p, 8\lambda) \cap T_p\mathcal{M}$.

From Prop. (a) and the definition of tangential complex, if $\sigma' \in \mathrm{st}(p; \mathrm{Del}_{\xi_p}(L'))$ then $\sigma \in \mathrm{Del}_{\varpi}(L, T\mathcal{M})$. Since all the m-simplices of $\mathrm{Del}_{\varpi}(L, T\mathcal{M})$ are δ^2-power protected on the tangent space of the vertices (Hyp. 4), therefore, from the definition of $\xi :\to [0, \infty)$, all the m-simplices σ' incident to p in $\mathrm{Del}_{\xi}(L')$ are also δ^2-power protected on $T_p\mathcal{M}$, i.e., there exists $x \in \mathrm{Vor}_{\xi_p}(\sigma')$ such that for all $q' \in \sigma'$ and $r' \in L' \setminus \sigma'$

$$\|r' - x\|^2 - \xi(r')^2 > \|q' - x\|^2 - \xi(q')^2 + \delta^2 .$$

Following properties are a direct consequence of Prop. (b), and Lemmas 12 (1) and 11

Prop. (c) Dimension of maximal simplices incident to p in $\mathrm{Del}_{\xi}(L')$ is equal to m.

Prop. (d) Let σ' be a m-simplex incident to p in $\mathrm{Del}_{\xi}(L')$. Then $\mathrm{Vor}_{\xi}(\sigma') = C_{\xi}(\sigma')$.

Prop. (e) Let $\sigma' \in \mathrm{Del}_{\xi}(L')$ be a j-simplex incident, with $p \in \sigma'$, then σ' is $\frac{\delta^2}{m-j+1}$-power protected.

Note that Prop. (c) and the definition of tangential complex implies the following

Prop. (f) Dimension of maximal simplices in $\mathrm{Del}_{\varpi}(L, T\mathcal{M})$ is equal to m.

We will now prove the power protection of simplices in $\mathrm{Del}_{\varpi}(L, T\mathcal{M})$ on the manifold $\mathcal{M}$. Let $\sigma \in \mathrm{Del}_{\varpi}(L, T\mathcal{M})$ be a k-simplex, with $k \leq m$, incident to p. From Prop. (e), $\exists\, c' \in \mathrm{Vor}_{\xi}(\sigma')$ such that $\forall\, x' \in \sigma'$ and $\forall\, y' \in L' \setminus \sigma'$

$$\|y' - c'\|^2 - \xi(y')^2 > \|x' - c'\|^2 - \xi(x')^2 + \frac{\delta^2}{m+1}.$$

Which, from the definition of ξ and Prop. (a), implies $\forall\, x \in \sigma$ and $\forall\, y \in L \setminus \sigma$

$$\|y - c'\|^2 - \varpi(y)^2 > \|x - c'\|^2 - \varpi(x)^2 + \frac{\delta^2}{m+1},$$

and $c' \in \mathrm{Vor}_{\varpi}(\sigma)$.

Let $\hat{c}$ be the point closest to c' on $\mathcal{M}$ and c denotes the point closest to c' in $\mathcal{M} \cap N_{\varpi}(\sigma)$.

Using the facts that $\|p - c'\| \leq 8\lambda$ (from Lemma 29 (1)) and

$$\|c' - \hat{c}\| \leq \frac{2^7\lambda^2}{\mathrm{rch}(\mathcal{M})} \leq \frac{\lambda}{16} < \frac{\mathrm{rch}(\mathcal{M})}{25} \tag{20}$$

from Lemma 7 (3) and $\lambda \leq \frac{\Gamma_0^m \mathrm{rch}(\mathcal{M})}{2^{11}} \leq \frac{\mathrm{rch}(\mathcal{M})}{2^{11}}$, we get

$$\|p - \hat{c}\| \leq \|p - c'\| + \|c' - \hat{c}\| \leq \left(8 + \frac{1}{16}\right)\lambda < \frac{\mathrm{rch}(\mathcal{M})}{4}. \tag{21}$$

Therefore, using $\sin\angle(\mathrm{aff}(\sigma), T_p\mathcal{M}) \leq \frac{16\lambda}{\Gamma_0^m \mathrm{rch}(\mathcal{M})}$ (from Lemma 32 (1)) and $\sin\angle(T_p\mathcal{M}, T_{\hat{c}}\mathcal{M}) < \frac{6\|p-\hat{c}\|}{\mathrm{rch}(\mathcal{M})}$ (from Lemma 7 (3) and $\|p - \hat{c}\| < \frac{\mathrm{rch}(\mathcal{M})}{4}$), we have

$$\begin{aligned}
\sin\angle(\mathrm{aff}\,\sigma, T_{\hat{c}}\mathcal{M}) &\leq \sin\angle(\mathrm{aff}\,\sigma, T_p\mathcal{M}) + \sin\angle(T_p\mathcal{M}, T_{\hat{c}}\mathcal{M}) \\
&\leq \frac{16\lambda}{\Gamma_0^m \mathrm{rch}(\mathcal{M})} + \frac{6\|p - \hat{c}\|}{\mathrm{rch}(\mathcal{M})} \\
&\leq \frac{16\lambda}{\Gamma_0^m \mathrm{rch}(\mathcal{M})} + \frac{387\lambda}{8\,\mathrm{rch}(\mathcal{M})} && \text{as } \|p - \hat{c}\| \leq \tfrac{129\lambda}{16} \\
&\leq \frac{1}{4} && \text{as } \lambda \leq \tfrac{\Gamma_0^m \mathrm{rch}(\mathcal{M})}{2^{11}}
\end{aligned}$$

Using the above bound on $\sin\angle(\mathrm{aff}\,\sigma, T_{\hat{c}}\mathcal{M})$, the fact that $\|c' - \hat{c}\| \leq \frac{2^7\lambda^2}{\mathrm{rch}(\mathcal{M})} < \frac{\mathrm{rch}(\mathcal{M})}{25}$ (Eq. (20)) and Lemma 48, we get

$$\|c' - c\| \leq 4\|c' - \hat{c}\| \leq \frac{2^9\lambda^2}{\mathrm{rch}(\mathcal{M})} \stackrel{\text{def}}{=} \frac{C\lambda^2}{\mathrm{rch}(\mathcal{M})} \leq \frac{\lambda}{4}.$$

Let $q \in L \setminus \sigma$ and $p \in \sigma$. We will consider the following two cases:

Case-1. $\|q - c\|^2 > \|p - c\|^2 + 2(2\lambda)^2$. Using the facts that $\varpi(q) \leq 4\alpha_0\lambda$ (from part 2(a) of Lemma 8) and $\alpha_0 < \frac{1}{2}$, we have

$$\begin{aligned}
\|q - c\|^2 - \varpi(q)^2 - (\|p - c\|^2 - \varpi(p)^2) &> 8\lambda^2 - \varpi(q)^2 + \varpi(p)^2 \\
&> 8\lambda^2 - \varpi(q)^2 \\
&> 4\lambda^2
\end{aligned}$$

Case-2. $\|q-c\|^2 \leq \|p-c\|^2 + 8\lambda^2$. This implies $\|q-c\| < \|p-c\| + 3\lambda$.

Using the facts that $\|p-c'\| \leq 8\lambda$ (from Lemma 29 (1)), $\|c-c'\| \leq \frac{C\lambda^2}{\mathrm{rch}(\mathcal{M})} \leq \frac{\lambda}{4}$,

$$\begin{aligned} \|p-c\| &\leq \|p-c'\| + \|c-c'\|\| \leq \left(8+\frac{1}{4}\right)\lambda, \text{ and} \\ \|q-c'\| &\leq \|q-c\| + \|c-c'\| \leq \left(8+\frac{13}{4}\right)\lambda, \end{aligned}$$

we get

$$\begin{aligned} \|q-c\|^2 - \varpi(q)^2 &\geq (\|q-c'\| - \|c-c'\|)^2 - \varpi(q)^2 \\ &\geq \|q-c'\|^2 - \varpi(q)^2 - 2\|c-c'\|\|q-c'\| \\ &> \|p-c'\|^2 - \varpi(p)^2 + \frac{\delta^2}{m+1} - 2\|c-c'\|\|q-c'\| \\ &\geq (\|p-c\| - \|c-c'\|)^2 - \varpi(p)^2 + \frac{\delta^2}{m+1} - 2\|c-c'\|\|q-c'\| \\ &\geq \|p-c\|^2 - \varpi(p)^2 + \frac{\delta^2}{m+1} - 2\|c-c'\|(\|q-c'\| + \|p-c\|) \\ &> \|p-c\|^2 - \varpi(p)^2 + \frac{\delta^2}{m+1} - \frac{B\lambda^3}{\mathrm{rch}(\mathcal{M})} \end{aligned}$$

where $B = 2^{15}$.

From **Case-1** and **2**, we get

$$\|q-c\|^2 - \varpi(q)^2 > \|p-c\|^2 - \varpi(p)^2 + \frac{\delta^2}{m+1} - \frac{B\lambda^3}{\mathrm{rch}(\mathcal{M})}.$$

$\square$

Lemma 39 *Let $W \subseteq \mathcal{M}$ be an ε-sample of $\mathcal{M}$, $L \subseteq W$ be a λ-net of W with $\varepsilon \leq \lambda$, and $\delta = \delta_0\lambda$. Also, let $\varpi : L \to [0,\infty)$ be a weight assignment with $\widetilde{\varpi} < \frac{1}{2}$ and satisfying conditions (1) to (2) of Lemma 38. If $\delta = \delta_0\lambda$,*

$$\lambda < \min\left\{\frac{\Gamma_0^m}{2^{11}}, \frac{\delta_0^2}{B(m+1)}\right\} \mathrm{rch}(\mathcal{M})$$

and

$$\varepsilon < \frac{\lambda}{24}\left(\frac{\delta_0^2}{m+1} - \frac{B\lambda}{\mathrm{rch}(\mathcal{M})}\right),$$

then

$$\mathrm{Del}_\varpi(L, T\mathcal{M}) \subseteq \mathrm{Wit}_\varpi(L, W).$$

Proof Note that, as $\varepsilon \leq \lambda$, L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$.

Let $\sigma^k \in \mathrm{Del}_\varpi(L, \mathcal{M})$. From Lemma 38, there exists $c \in \mathrm{Vor}_\varpi(\sigma^k) \cap \mathcal{M}$ such that σ^k is δ_1^2-protected at c, where

$$\delta_1^2 = \frac{\delta^2}{m+1} - \frac{B\lambda^3}{\mathrm{rch}(\mathcal{M})}.$$

From Lemma 8 (2) as $c \in \mathrm{Vor}_\varpi(\sigma^k) \cap \mathcal{M}$, we have for all $p \in \sigma^k$, $\|p-c\| \leq 4\lambda$.

Let $w \in W$ be such that $\|c - w\| \leq \varepsilon$. For all $q \in L \setminus \sigma^k$ and $p \in \sigma^k$ we have

$$\begin{aligned}
\|p - w\|^2 - \varpi(p)^2 &\leq (\|p - c\| + \|c - w\|)^2 - \varpi(p)^2 \\
&= \|p - c\|^2 - \varpi(p)^2 + \|c - w\| \left(\|c - w\| + 2\|p - c\|\right) \\
&\leq \|p - c\|^2 - \varpi(p)^2 + 9\varepsilon\lambda \\
&< \|q - c\|^2 - \varpi(q)^2 - (\delta_1^2 - 9\varepsilon\lambda) \\
&\leq \|q - w\|^2 - \varpi(q)^2 + \beta - (\delta_1^2 - 9\varepsilon\lambda) \qquad (22)
\end{aligned}$$

Where $\beta = \|w - c\| \left(\|w - c\| + 2\|q - w\|\right)$.

We have to consider the following two case:

1. If $\|q - w\|^2 > \|p - w\|^2 + 4\lambda^2$. Using the fact that $\varpi(q) < 2\lambda$, from Lemma 8 (1), we get

$$\begin{aligned}
\|q - w\|^2 - \varpi(q)^2 &> \|p - w\|^2 + 4\lambda^2 - \varpi(q)^2 \\
&> \|p - w\|^2 \\
&\geq \|p - w\|^2 - \varpi(p)^2
\end{aligned}$$

2. If $\|q - w\|^2 \leq \|p - w\|^2 + 4\lambda^2$. This implies

$$\begin{aligned}
\|q - w\| &\leq \|p - w\| + 2\lambda \\
&\leq \|p - c\| + \|c - w\| + 2\lambda \\
&\leq 7\lambda.
\end{aligned}$$

Now, using Eq. (22) and the facts that $\|q - w\| = 7\lambda$ and $\|c - w\| \leq \varepsilon \leq \lambda$, we get

$$\begin{aligned}
\|p - w\|^2 - \varpi(p)^2 &\leq \|q - w\|^2 - \varpi(q)^2 + \beta - (\delta_1^2 - 9\varepsilon\lambda) \\
&\leq \|q - w\|^2 - \varpi(q)^2 - (\delta_1^2 - 24\varepsilon\lambda) \qquad \text{as } \beta \leq 15\varepsilon\lambda \\
&< \|q - w\|^2 - \varpi(q)^2
\end{aligned}$$

The last inequality follows from the fact that

$$\lambda < \frac{\delta_0^2 \mathrm{rch}(\mathcal{M})}{B\, m} \text{ and } \varepsilon < \frac{\lambda}{24}\left(\frac{\delta_0^2}{m} - \frac{B\lambda}{\mathrm{rch}(\mathcal{M})}\right).$$

This implis w is a witness of σ^k.

As this is true for all $\sigma^k \in \mathrm{Del}_\varpi(L, T\mathcal{M})$, we get $\mathrm{Del}_\varpi(L, T\mathcal{M}) \subseteq \mathrm{Wit}_\varpi(L, W)$. □

B Proof of Lemma 20

B.1 Outline of the proof

We will use a variant of Pumping equation, Lemma 41, from [CDE+00] and bound on the height of slivers, Lemma 42, from [BDG14]. Let $\varpi : L \to [0, \infty)$ be a weight assignment with $\tilde{\varpi} \leq \tilde{\alpha}_0$, and $\sigma \subset L$ be a Γ_0-sliver incident to the point $p \in L$. As in Lemma 20, ϖ_1 is an ewp of ϖ such that $\sigma \in \mathrm{K}_\varpi(L)$. To prove Lemma 20, we distinguish the following two cases depending on the point whose weight is changed when replacing ϖ by ϖ_1:

Case 1. The point whose weight is changed is p. Lemma 43 takes care of this case and states that

$$\varpi(p)^2 \in J_\varpi(p,\sigma) = \left[F_\varpi(p,\sigma) - \frac{\eta_1}{2} - \delta_0^2\lambda^2,\ F_\varpi(p,\sigma) + \frac{\eta_1}{2}\right],$$

for some $\eta_1 \le \eta - 2\delta_0^2\lambda^2$.

Case 2. The point whose weight is changed is *not* p. Lemma 20 takes care of this case and states that

$$\varpi(p)^2 \in I_\varpi(\sigma,p) = \left[F_\varpi(p,\sigma) - \frac{\eta}{2},\ F_\varpi(p,\sigma) + \frac{\eta}{2}\right].$$

Since $J_\varpi(\sigma,p) \subset I_\varpi(\sigma,p)$, Lemma 20 is proved.

The proof of **Case 1** is in the same vein as the proofs of [CDR05, Lem. 10] and [BG14, Lem. 4.14].

The main technical ingredient in completing the proof of **Case 2** is in showing that

$$|F_\varpi(p,\sigma) - F_{\varpi_1}(p,\sigma)| = O\left(\frac{\delta_0^2\lambda^2}{\Gamma_0^m}\right)$$

One way to proving this is by proving

$$\max\left\{ \left|R_\varpi(\sigma_p)^2 - R_{\varpi_1}(p,\sigma)^2\right|,\ \left|d(p, N_\varpi(\sigma_p))^2 - d(p, N_{\varpi_1}(p,\sigma))^2\right| \right\} = O\left(\frac{\delta_0^2\lambda^2}{\Gamma_0^m}\right),$$

and this will be done in Lemma 46 using Lemma 44 and Corollary 45,

B.2 Details of the proof

For the rest of this section we will assume the following hypothesis

Hypothesis 40 $L \subset \mathcal{M}$ is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$ with

$$\lambda < \frac{1}{18}(1-\sin\theta_0)^2\,\mathrm{rch}(\mathcal{M}).$$

For a simplex σ and a vertex $p \in \sigma$, *excentricity* $H_\varpi(p,\sigma)$ of σ with respect to p is the signed distance of $C_\varpi(\sigma)$ from $\mathrm{aff}(\sigma_p)$, i.e., $H_\varpi(p,\sigma)$ is positive if $C_\varpi(\sigma)$ and p lie on the same side of $\mathrm{aff}(\sigma_p)$ and negative if they lie on different sides of $\mathrm{aff}(\sigma_p)$.

The following lemma is a variant of the pumping equation from [CDE$^+$00, BG14, CDR05].

Lemma 41 (Pumping equation, see Figure 13) *We will assume that the weight of p is varying and the weight of the other vertices of σ are fixed. Then*

$$2D(p,\sigma)\,H_\varpi(p,\sigma) = F_\varpi(p,\sigma) - \varpi(p)^2.$$

The above "pumping equation" will be used to bound the length of the forbidden intervals.

The following result is from [BDG14].

Lemma 42 (Sliver altitude bound) *If a $(k+1)$-simplex τ is a Γ_0-sliver, then for any vertex p of σ we have*

$$D(p,\sigma) < \frac{2\Gamma_0\Delta(\sigma)^2}{L(\sigma)}.$$

A variant of the following result can be found in [CDR05, Lem. 10] and [BG14, Lem. 4.14]. We have included the proof for completeness.

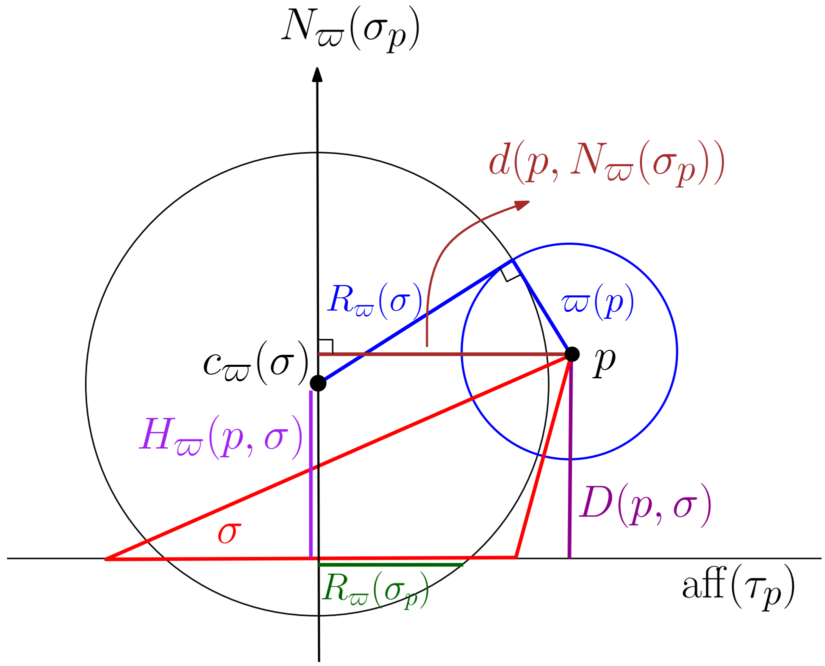

Figure 13: Diagram for the Lemma 41.

Lemma 43 (Case 1) *Let $\varpi : L \to [0,\infty]$ be a weight assignment with $\widetilde{\varpi} \leq \tilde{\alpha}_0$, and $\sigma \subset L$ be a Γ_0-sliver incident to the point $p \in L$. Let ϖ_1 be a ewp of ϖ satisfying the following conditions*

$$\varpi(q) = \varpi_1(q),\ \forall\ q \in \sigma \setminus p,\ \textit{and}\ \sigma \in \mathrm{K}_{\varpi_1}(L).$$

If λ is sufficiently small then

$$\varpi(p)^2 \in \left[F_\varpi(p,\sigma) - \frac{\eta_1}{2} - \delta_0^2\lambda^2,\ F_\varpi(p,\sigma) - \frac{\eta_1}{2}\right]$$

where $\eta_1 \stackrel{\mathrm{def}}{=} 2^{14}\Gamma_0\lambda^2$.

Proof Since $|H_{\varpi_1}(p,\sigma)| \leq \|C_{\varpi_1}(\sigma) - p\|$, we have from Lemma 32 (1)

$$|H_{\varpi_1}(p,\sigma)| \leq \|C_{\varpi_1}(\sigma) - p\| < 8\lambda.$$

Since L is λ-sparse, we have from Lemma 32 (2)

$$\lambda \leq L(\sigma) \leq \Delta(\sigma) < 16\lambda$$

From Lemma 42, we have

$$D(p,\sigma) < \frac{2\Gamma_0\Delta(\sigma)^2}{L(\sigma)} < 2^9\Gamma_0\lambda.$$

Therefore, using Lemma 41, we have

$$\begin{aligned} F_{\varpi_1}(p,\sigma) - 2D(p,\sigma)|H_{\varpi_1}(p,\sigma)| \leq\ & \varpi_1(p)^2 && \leq F_{\varpi_1}(p,\sigma) + 2D(p,\sigma)|H_{\varpi_1}(p,\sigma)| \\ F_{\varpi_1}(p,\sigma) - 2^{13}\Gamma_0\lambda^2 \leq\ & \varpi_1(p)^2 && \leq F_{\varpi_1}(p,\sigma) + 2^{13}\Gamma_0\lambda^2 \end{aligned}$$

The result now follows from the facts that

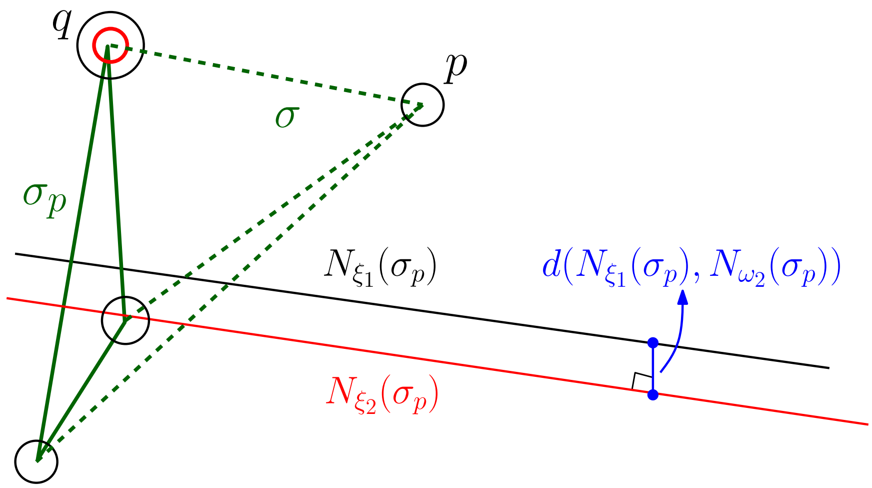

Figure 14: Diagram for the Lemma 44.

- $F_{\varpi_1}(p, \sigma) = F_{\varpi}(p, \sigma)$ as, from the definition, $F_{\varpi_1}(p, \sigma)$ (and $F_{\varpi}(p, \sigma)$) depends only on the weights of the vertices in σ_p and for all $q \in \sigma \setminus p$, $\varpi(q) = \varpi_1(q)$.
- $\varpi_1(p)^2 \in [\varpi(p)^2, \varpi(p)^2 + \delta_0^2\lambda^2]$.

$\square$

The following lemma show the stability of weighted centers of well shaped simplices under small perturbations of weight assignments. The proof is in the same vein as the proof of [BDG13b, Lem. 4.1], and will use singular values of matrices associated with the simplices.

Lemma 44 *Let σ be a simplex with $L(\sigma) \geq \lambda$ and $\Upsilon(\sigma) > 0$, and $\xi_i : \sigma \to [0, \infty)$, with $i \in \{1, 2\}$, be weights assignments, with $\tilde{\xi_i} \leq \alpha_0$, satisfy the following properties: $\exists\, p \in \sigma$ such that*

1. *$\forall\, q \in \sigma \setminus p$, $\xi_1(q) = \xi_2(q)$, and*
2. *$|\xi_1(p)^2 - \xi_2(p)^2| \leq \delta_0^2\lambda^2$.*

Then

$$\|C_{\xi_1}(\sigma) - C_{\xi_2}(\sigma)\| \leq \frac{\delta_0^2\lambda}{2\Upsilon(\sigma)},$$

and for $r \notin \sigma$, we have

$$\left|d(r, N_{\xi_1}(\sigma)) - d(r, N_{\xi_2}(\sigma))\right| \leq \frac{\delta_0^2\lambda}{2\Upsilon(\sigma)}$$

Proof Let $\sigma = [p_0 \ldots p_k]$, and wlog let $p \neq p_0$ and $\sigma \subset \mathbb{R}^k$. The ortho-radius of σ satisfy the following system of k-linear equations:

$$(p_j - p_0)^{\mathrm{T}} C_{\xi_i}(\sigma) = \frac{1}{2}(\|p_j\|^2 - \xi_i(p_j)^2 - \|p_0\|^2 + \xi_i(p_0)^2)$$

Rewriting the above system of equation we get

$$\begin{aligned}(p_j - p_0)^{\mathrm{T}}(C_{\xi_2}(\sigma) - C_{\xi_1}(\sigma)) &= \frac{1}{2}(\xi_1(p_j)^2 - \xi_2(p_j)^2 + \xi_2(p_0)^2 - \xi_1(p_0)^2) \\ &= \frac{1}{2}(\xi_1(p_j)^2 - \xi_2(p_j)^2) \;\text{ as } p \neq p_0\end{aligned}$$

Letting P be a $k \times k$ matrix whose j^{th} column is $(p_j - p_0)$, we have

$$P^T(C_{\xi_2}(\sigma) - C_{\xi_1}(\sigma)) = \frac{x_\xi}{2}$$

where

$$x_\xi = (\xi_1(p_1)^2 - \xi_2(p_1)^2, \ldots, \xi_1(p_k)^2 - \xi_2(p_k)^2)^{\mathrm{T}}.$$

Therefore

$$\begin{aligned}\|C_{\xi_2}(\sigma) - C_{\xi_1}(\sigma)\| &= \frac{1}{2}\|P^{-\mathrm{T}}x_\xi\| \\ &\leq \frac{1}{2}\|P^{-1}\|\,\|x_\xi\| \\ &\leq s_1(P^{-1}) \times \frac{\delta_0^2\lambda^2}{2} && \text{as } s_1(P^{-1}) = \|P^{-1}\| \text{ and } \|x_\xi\| \leq \delta_0^2\lambda^2 \\ &= s_k(P)^{-1} \times \frac{\delta_0^2\lambda^2}{2} && \text{as } s_1(P^{-1}) = s_k(P)^{-1}, \text{ Lemma 2} \\ &\leq \frac{\delta_0^2\lambda^2}{2\sqrt{k}\Upsilon(\sigma)\Delta(\sigma)} && \text{as } s_k(P) \geq \sqrt{k}\Upsilon(\sigma)\Delta(\sigma), \text{ Lemma 3} \\ &\leq \frac{\delta_0^2\lambda}{2\Upsilon(\sigma)} && \text{as } \tfrac{\Delta(\sigma)}{\lambda} \geq 1\end{aligned}$$

The bound on $\left|d(r, N_{\xi_1}(\sigma)) - d(r, N_{\xi_2}(\sigma))\right|$ follows directly from the part 1 of the lemma and the fact that

$$\left|d(r, N_{\xi_1}(\sigma)) - d(r, N_{\xi_2}(\sigma))\right| \leq \|C_{\xi_1}(\sigma) - C_{\xi_2}(\sigma)\|.$$

□

Corollary 45 *Let $\varpi : L \to [0, \infty]$ be a weight assignment with $\tilde{\varpi} \leq \tilde{\alpha}_0$, and $\sigma \subset L$ be a j-dimensional Γ_0-sliver with $p \in \sigma$ and $j \leq m+1$. In addition, we assume*

$$\frac{\delta_0^2}{\Gamma_0^m} \leq 2.$$

If ϖ_1 be an ewp of ϖ satisfying the following: $\exists\, q \in \sigma_p$ such that $\forall\, x \in L \setminus \{q\}$, $\varpi(x) = \varpi_1(x)$ and $\sigma \in \mathrm{K}_{\varpi_1}(L, \mathcal{M})$. Then

$$\left|d(p, N_\varpi(\sigma_p))^2 - d(p, N_{\varpi_1}(\sigma_p))^2\right|, \left|R_\varpi(\sigma_p)^2 - R_{\varpi_1}(\sigma_p)^2\right| \leq \frac{49\delta_0^2\lambda^2}{2\Gamma_0^m}.$$

Proof Using the fact that L is an $(\lambda, 2\lambda)$-net of $\mathcal{M}$, and from Lemmas 32 (1) and (2) we have

$$\begin{aligned}d(p, N_{\varpi_1}(\sigma_p)) &\leq \|C_{\varpi_1}(\sigma_p) - q\| + \|p - q\| \\ &\leq 24\lambda\end{aligned}$$

From Lemma 44 we have

$$\begin{aligned} d(p, N_\varpi(\sigma_p)) + d(p, N_{\varpi_1}(\sigma_p)) &\le 2d(p, N_{\varpi_1}(\sigma_p)) + \frac{\delta_0^2\lambda}{2\Upsilon(\sigma_p)} \\ &\le 2d(p, N_{\varpi_1}(\sigma_p)) + \frac{\delta_0^2\lambda}{2\Gamma_0^m} \qquad \text{as } \sigma \text{ is a } \Gamma_0\text{-sliver} \\ &\le 49\lambda \end{aligned}$$

From Lemma 44 and the fact that $d(p, N_\varpi(\sigma_p)) + d(p, N_{\varpi_1}(\sigma_p)) \le 49\lambda$, we have

$$\left| d(p, N_\varpi(\sigma_p))^2 - d(p, N_{\varpi_1}(\sigma_p))^2 \right| \le \frac{49\delta_0^2\lambda^2}{2\Gamma_0^m}$$

Since σ is a Γ_0-sliver, $j \ge 2$ (see Remark 5). As $j \ge 2$, there exists $r \in \sigma_p \setminus q$. This implies $\varpi(r) = \varpi_1(r)$.

Using the facts that $\varpi_1(r) = \varpi(r)$, $\|C_{\varpi_1}(\sigma_p) - r\| \le 8\lambda$ (from Lemma 32 (1)) and

$$\|C_\varpi(\sigma_p) - C_{\varpi_1}(\sigma_p)\| \le \frac{\delta_0^2\lambda}{2\Gamma_0^m}$$

(from Lemma 44 and the fact that $\Upsilon(\sigma_p) \ge \Gamma_0^{j-1} \ge \Gamma_0^m$), we get

$$\begin{aligned} R_\varpi(\sigma_p)^2 &= \|C_\varpi(\sigma_p) - r\|^2 - \varpi(r)^2 \\ &\le (\|C_{\varpi_1}(\sigma_p) - r\| + \|C_\varpi(\sigma_p) - C_{\varpi_1}(\sigma_p)\|)^2 - \varpi_1(r)^2 \\ &\le R_{\varpi_1}(\sigma_p)^2 + \Big(2\|C_{\varpi_1}(\sigma_p) - r\| \\ &\qquad\qquad + \|C_\varpi(\sigma_p) - C_{\varpi_1}(\sigma_p)\|\Big) \|C_\varpi(\sigma_p) - C_{\varpi_1}(\sigma_p)\| \\ &\le R_{\varpi_1}(\sigma_p)^2 + \left(16\lambda + \frac{\delta_0^2\lambda}{2\Gamma_0^m}\right) \frac{\delta_0^2\lambda}{2\Gamma_0^m} \\ &\le R_{\varpi_1}(\sigma_p)^2 + \frac{17\delta_0^2\lambda^2}{2\Gamma_0^m} \end{aligned}$$

and

$$\begin{aligned} R_\varpi(\sigma_p)^2 &= \|C_\varpi(\sigma_p) - r\|^2 - \varpi(r)^2 \\ &\ge (\|C_{\varpi_1}(\sigma_p) - r\| - \|C_\varpi(\sigma_p) - C_{\varpi_1}(\sigma_p)\|)^2 - \varpi_1(r)^2 \\ &\ge R_{\varpi_1}(\sigma_p)^2 - \Big(2\|C_{\varpi_1}(\sigma_p) - r\| \\ &\qquad\qquad - \|C_\varpi(\sigma_p) - C_{\varpi_1}(\sigma_p)\|\Big) \|C_\varpi(\sigma_p) - C_{\varpi_1}(\sigma_p)\| \\ &\ge R_{\varpi_1}(\sigma_p)^2 - 2\|C_{\varpi_1}(\sigma_p) - r\| \, \|C_\varpi(\sigma_p) - C_{\varpi_1}(\sigma_p)\| \\ &\ge R_{\varpi_1}(\sigma_p)^2 - \frac{8\delta_0^2}{\Gamma_0^m}\lambda^2 \end{aligned}$$

□

Lemma 46 (Case 2) *Assuming the same conditions on ϖ, ϖ_1, p, q and σ as in Corollary 45, we get*

$$\varpi(p)^2 \in \left[F_\varpi(p,\sigma) - \frac{\eta}{2},\, F_\varpi(p,\sigma) + \frac{\eta}{2}\right].$$

Proof As in the proof of Lemma 43, we can show that $|2D(p,\sigma)H_{\varpi_1}(p,\sigma)| \leq 2^{13}\Gamma_0\lambda$.
From Corollary 45, we have

$$\left|d(p, N_{\varpi}(\sigma_p))^2 - d(p, N_{\varpi_1}(\sigma_p))^2\right|, \left|R_{\varpi}(\sigma_p)^2 - R_{\varpi_1}(\sigma_p)^2\right| \leq \frac{49\delta_0^2}{2\Gamma_0^m}\lambda^2.$$

This implies, from the definition of $F_{\varpi_1}(p,\sigma)$,

$$\left|F_{\varpi_1}(p,\sigma) - F_{\varpi}(p,\sigma)\right| \leq \frac{49\delta_0^2}{\Gamma_0^m}\lambda^2.$$

From Lemma 41, and the above bounds we get

$$\varpi_1(p)^2 \in \left[F_{\varpi}(p,\sigma) - \frac{\eta}{2}, F_{\varpi}(p,\sigma) + \frac{\eta}{2}\right].$$

The result now follows from the fact that $\varpi(p) = \varpi_1(p)$. □

Remark 47 Note that $\eta \geq \eta_1 + 2\delta_0^2\lambda^2$.

Combining Lemmas 43 and 46, completes the proof of Lemma 20.

C Almost normal flats intersecting submanifolds

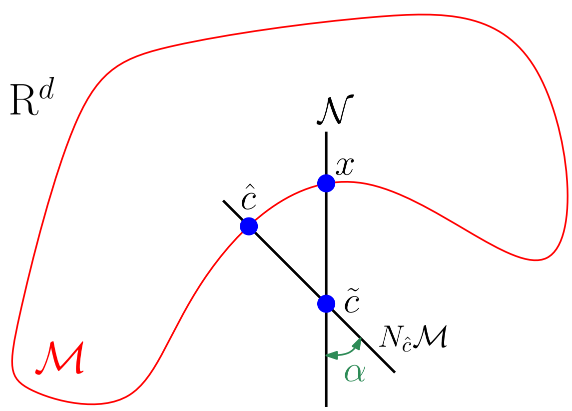

Figure 15: Almost normal flat $\mathcal{N}$ intersecting the manifold $\mathcal{M}$.

The following technical lemma, which asserts that, for $j \leq m = \dim \mathcal{M}$, if a $(d-j)$-flat, $\mathcal{N}$, passes through a point $\tilde{c}$ that is close to $\mathcal{M}$, and the normal space at the point on $\mathcal{M}$ closest to $\tilde{c}$ makes a small angle with $\mathcal{N}$, then $\mathcal{N}$ must intersect $\mathcal{M}$ in that vicinity. The technical difficulty stems from the fact that the codimension may be greater than one.

Lemma 48 *Let $\tilde{c} \in \mathbb{R}^d$ be such that it has a unique closest point $\hat{c}$ on $\mathcal{M}$ and $\|\tilde{c} - \hat{c}\| \leq \rho \leq \frac{\mathrm{rch}(\mathcal{M})}{25}$. Let $j \leq m = \dim \mathcal{M}$, and let $\mathcal{N}$ be a $(d-j)$-dimensional affine flat passing through $\tilde{c}$ such that $\angle(N_{\hat{c}}\mathcal{M}, \mathcal{N}) \leq \alpha$ with $\sin\alpha \leq \frac{1}{4}$. Then there exists an $x \in \mathcal{N} \cap \mathcal{M}$ such that $\|\tilde{c} - x\| \leq 4\rho$.*

The idea of the proof is to consider the m-dimensional affine space $\widetilde{T}_{\hat{c}}\mathcal{M}$ that passes through $\hat{c}$ and is orthogonal to a $(d-m)$-dimensional affine subspace of $\mathcal{N}$. We show that the orthogonal projection onto $\widetilde{T}_{\hat{c}}\mathcal{M}$ induces, in some neighbourhood V of $\hat{c}$, a diffeomorphism between $\mathcal{M} \cap V$, and $\widetilde{T}_{\hat{c}}\mathcal{M} \cap V$ (Lemma 51). We use $T_{\hat{c}}\mathcal{M}$ as an intermediary in this calculation (Lemma 50). Then, since $\mathcal{N}$ intersects $T_{\hat{c}}\mathcal{M}$ near $\hat{c}$ (Lemma 49), we can argue that it must also intersect $\mathcal{M}$ because the established diffeomorphisms make a correspondence between points along segments parallel to $\mathcal{N}$.

The final bounds are established in Lemma 52, from which Lemma 48 follows by a direct calculation, together with the following observations: If $\dim \mathcal{N} = \dim N_{\hat{c}}\mathcal{M}$, then $\angle(N_{\hat{c}}\mathcal{M}, \mathcal{N}) = \angle(\mathcal{N}, N_{\hat{c}}\mathcal{M})$, and if $\dim \mathcal{N} \geq \dim N_{\hat{c}}\mathcal{M}$, then there is an affine subspace $\widetilde{\mathcal{N}} \subset \mathcal{N}$, such that $\dim \widetilde{\mathcal{N}} = \dim N_{\hat{c}}\mathcal{M}$, and $\angle(N_{\hat{c}}\mathcal{M}, \widetilde{\mathcal{N}}) = \angle(N_{\hat{c}}\mathcal{M}, \mathcal{N})$. Indeed, we may take $\widetilde{\mathcal{N}}$ to be the orthogonal projection of $N_{\hat{c}}\mathcal{M}$ into $\mathcal{N}$.

We now bound distances to the intersection of $\mathcal{N}$ and $T_{\hat{c}}\mathcal{M}$.

Lemma 49 *Let $\tilde{c}$, $\hat{c}$ be points in $\mathbb{R}^d$ such that the projection of $\tilde{c}$ onto $\mathcal{M}$ is $\hat{c}$ and $\|\tilde{c} - \hat{c}\| \leq \rho$. Let $\mathcal{N}$ be a $d - m$ dimensional affine flat passing through $\tilde{c}$ such that $\angle(\mathcal{N}, N_{\hat{c}}\mathcal{M}) \leq \alpha$. For all $x \in \mathcal{N} \cap T_{\hat{c}}\mathcal{M}$, we have*

1. $\|\tilde{c} - x\| \leq \frac{\rho}{\cos\alpha}$
2. $\|\hat{c} - x\| \leq \left(1 + \frac{1}{\cos\alpha}\right)\rho$

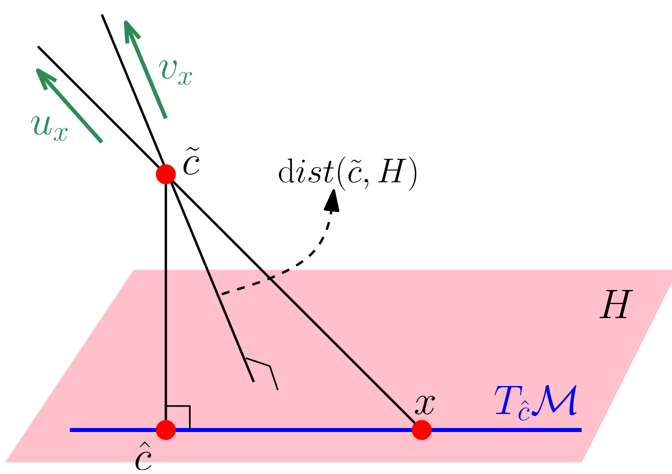

Figure 16: Diagram for the Lemma 49.

Proof For a point $x \in \mathcal{N} \cap T_{\hat{c}}\mathcal{M}$, let u_x denote the unit vector from $\tilde{c}$ to x, and let $v_x \in N_{\hat{c}}\mathcal{M}$ be the unit vector that makes the smallest angle with u_x. Let H denote the hyperplane passing through $\hat{c}$ and orthogonal to v_x. Since $\|\tilde{c} - \hat{c}\| \leq \rho$, $\mathrm{dist}(\tilde{c}, H) \leq \rho$. Therefore,

$$\|\tilde{c} - x\| \leq \frac{\mathrm{dist}(\tilde{c}, H)}{\cos\alpha}$$

and

$$\|\hat{c} - x\| \leq \|\hat{c} - \tilde{c}\| + \|\tilde{c} - x\| \leq \left(1 + \frac{1}{\cos\alpha}\right)\rho$$

$\square$

The following lemma is a direct consequence of the definition of the angle between two affine spaces.

Lemma 50 *Let p be a point in $\mathcal{M}$ and let $\widetilde{T}_p\mathcal{M}$ denote a m-dimensional flat passing through p with $\angle(T_p\mathcal{M}, \widetilde{T}_p\mathcal{M}) \leq \alpha < \frac{\pi}{2}$. If f_p^α denote the orthogonal projection of $T_p\mathcal{M}$ onto $\widetilde{T}_p\mathcal{M}$, then*

1. *The map f_p^α is bijective.*
2. *For $r > 0$, $f_p^\alpha(B_p(r)) \supseteq \widetilde{B}_p(r\cos\alpha)$ where $B_p(r) = B(p,r) \cap T_p\mathcal{M}$ and $\widetilde{B}_p(r) = B(p,r) \cap \widetilde{T}_p\mathcal{M}$.*

Lemma 51 *Let p be a point in $\mathcal{M}$, and let $\widetilde{T}_p\mathcal{M}$ be a m-dimensional affine flat passing through p with $\angle(T_p\mathcal{M}, \widetilde{T}_p\mathcal{M}) \leq \alpha$. There exists an $r(\alpha)$ satisfying :*

$$\frac{7\,r(\alpha)}{\mathrm{rch}(\mathcal{M})} + \sin\alpha \,<\, 1 \quad \textit{and} \quad r(\alpha) \leq \frac{\mathrm{rch}(\mathcal{M})}{10}$$

such that the orthogonal projection map, g_p^α, of $B_{\mathcal{M}}(p, r(\alpha)) = B(p, r(\alpha)) \cap \mathcal{M}$ into $\widetilde{T}_p\mathcal{M}$ satisfy the following conditions:

1. *g_p^α is a diffeomorphism.*
2. *$g_p^\alpha(B_{\mathcal{M}}(p, r(\alpha))) \supseteq \widetilde{B}_p(r(\alpha)\cos\alpha_1)$ where $\sin\alpha_1 = \frac{r(\alpha)}{2\mathrm{rch}(\mathcal{M})} + \sin\alpha$.*
3. *Let $x \in g_p^\alpha(B_{\mathcal{M}}(p, r(\alpha)))$, then $\|x - (g_p^\alpha)^{-1}(x)\| \leq \|p - x\| \tan\alpha_1$.*

Proof 1. Let $\pi_{\widetilde{T}_p\mathcal{M}}$ denote the orthogonal projection of $\mathbb{R}^d$ onto $\widetilde{T}_p\mathcal{M}$. The derivative of this map, $D\pi_{\widetilde{T}_p\mathcal{M}}$, has a kernel of dimension $(d-m)$ that is parallel to the orthogonal complement of $\widetilde{T}_p\mathcal{M}$ in $\mathbb{R}^d$.

We will first show that Dg_p^α is nonsingular for all $x \in B_{\mathcal{M}}(p, r(\alpha))$. From Lemma 7 (3) and the fact that $\angle(T_p\mathcal{M}, \widetilde{T}_p\mathcal{M}) \leq \alpha$, we have

$$\begin{aligned}
\sin\angle(\widetilde{T}_p\mathcal{M}, T_x\mathcal{M}) &\leq \sin\angle(T_x\mathcal{M}, T_p\mathcal{M}) + \sin\angle(T_p\mathcal{M}, \widetilde{T}_p\mathcal{M}) \\
&\leq \frac{6r(\alpha)}{\mathrm{rch}(\mathcal{M})} + \sin\alpha \,<\, 1
\end{aligned}$$

Since g_p^α is the restriction of $\pi_{\widetilde{T}_p\mathcal{M}}$ to $B_{\mathcal{M}}(p, r(\alpha))$, the above inequality implies that Dg_p^α is nonsingular. Therefore, g_p^α is a local diffeomorphism.

Let x, $y \in B_{\mathcal{M}}(p, r(\alpha))$. From Lemma 7 part (1) and (3), we have

$$\begin{aligned}
\sin\angle([x,y], \widetilde{T}_p\mathcal{M}) &\leq \sin\angle([x,y], T_x\mathcal{M}) + \sin\angle(T_x\mathcal{M}, T_p\mathcal{M}) + \sin\angle(\widetilde{T}_p, T_p\mathcal{M}) \\
&\leq \frac{\|x-y\|}{2\mathrm{rch}(\mathcal{M})} + \frac{6\|p-x\|}{\mathrm{rch}(\mathcal{M})} + \sin\alpha \\
&\leq \frac{7r(\alpha)}{\mathrm{rch}(\mathcal{M})} + \sin\alpha \,<\, 1
\end{aligned}$$

This implies $g_p^\alpha(x) \neq g_p^\alpha(y)$.

Since g_p^α is nonsingular and injective on $B_{\mathcal{M}}(p, r(\alpha))$, it is a diffeomorphism onto its image.

2. Notice that, for $x \in B_{\mathcal{M}}(p, r(\alpha))$, the angle α_1 is a bound on the angle between px and $\tilde{T}_p\mathcal{M}$. The inclusion $g_p^\alpha(B_{\mathcal{M}}(p, r(\alpha))) \supseteq \widetilde{B}_p(r(\alpha)\cos\alpha_1)$ follows since $xg_P^\alpha(x)$ is orthogonal to $\tilde{T}_p\mathcal{M}$.

3. Follows similarly. $\square$

Lemma 52 *Let $\tilde{c}$, $\hat{c}$ be points in $\mathbb{R}^d$ such that the projection of $\tilde{c}$ onto $\mathcal{M}$ is $\hat{c}$ and $\|\tilde{c}-\hat{c}\| \leq \rho$. Let $\mathcal{N}$ be a $d-m$ dimensional affine flat passing through $\tilde{c}$ such that $\angle(\mathcal{N}, N_{\hat{c}}\mathcal{M}) \leq \alpha$. If*

$$\rho \leq \frac{r(\alpha)\cos\alpha\cos\alpha_1}{1+\cos\alpha}$$

Then there exists an $x \in \mathcal{N} \cap \mathcal{M}$ such that

$$\|\tilde{c}-x\| \leq \left(\frac{1}{\cos\alpha} + \left(1+\frac{1}{\cos\alpha}\right)(\sin\alpha + \sin\alpha_1)\right)\rho\,.$$

Proof Let $\tilde{T}_{\hat{c}}\mathcal{M}$ denote the orthogonal complement of $\mathcal{N}$ in $\mathbb{R}^d$ passing through $\hat{c}$. Note that $\angle(T_{\hat{c}}\mathcal{M}, \widetilde{T}_{\hat{c}}\mathcal{M}) = \angle(\mathcal{N}, N_{\hat{c}}\mathcal{M})$.

Let $\hat{x} \in \mathcal{N} \cap T_{\hat{c}}\mathcal{M}$ and $\tilde{x} = f^{\alpha}_{\hat{c}}(\hat{x})$. Then from Lemma 49, we have

$$\|\tilde{x}-\hat{c}\| \leq \|\hat{x}-\hat{c}\| \leq \left(1+\frac{1}{\cos\alpha}\right)\rho$$

and

$$\|\hat{x}-\tilde{x}\| \leq \|\hat{x}-\hat{c}\|\sin\alpha \leq (\sin\alpha + \tan\alpha)\ \rho\,.$$

Using the fact that $\rho \leq \frac{r(\alpha)\cos\alpha\cos\alpha_1}{1+\cos\alpha}$, we have

$$\|\tilde{x}-\hat{c}\| \leq \left(1+\frac{1}{\cos\alpha}\right)\rho \leq r(\alpha)\cos\alpha_1.$$

Therefore, from Lemma 49, there exists an $x \in B_{\mathcal{M}}(p, r(\alpha))$ such that $g^{\alpha}_{p}(x) = \tilde{x}$ and

$$\|\tilde{x}-x\| \leq \|\tilde{x}-\hat{c}\|\tan\alpha_1 \leq \left(1+\frac{1}{\cos\alpha}\right)\tan\alpha_1\,\rho.$$

Therefore

$$\begin{aligned}
\|\tilde{c}-x\| &\leq \|\tilde{c}-\hat{x}\| + \|\hat{x}-\tilde{x}\| + \|\tilde{x}-x\| \\
&\leq \frac{\rho}{\cos\alpha} + \left(1+\frac{1}{\cos\alpha}\right)(\sin\alpha + \tan\alpha_1)\,\rho \\
&= \left(\frac{1}{\cos\alpha} + \left(1+\frac{1}{\cos\alpha}\right)(\sin\alpha + \tan\alpha_1)\right)\rho.
\end{aligned}$$

Note that the line segment $\tilde{c}x \in \mathcal{N}$. □

This completes the proof of Lemma 48.

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