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ABSTRACT. The set of all possible configurations of the Ehrenfest wind-tree model endowed with the Hausdorff topology is a compact metric space. For a typical configuration we show that the wind-tree dynamics has infinite ergodic index in almost every direction. In particular some ergodic theorems can be applied to show that if we start with a large number of initially parallel particles their directions decorrelate as the dynamics evolve answering the question posed by the Ehrenfests.

1. INTRODUCTION

In 1912 Paul and Tatiana Ehrenfest wrote a seminal article on the foundations of Statistical Mechanics in which the wind-tree model was introduced in order to interpret the work of Boltzmann and Maxwell on gas dynamics [EhEh]. In the wind-tree model a point particle moves without friction on the plane with infinitely many rigid obstacles removed, and collides elastically with the obstacles. The Ehrenfests' paper dates from times when the notions of probability theory where not yet rigorously defined. Thus they could not describe the distribution of the obstacles in a probabilistic way, they used the word "irregular" to describe it. However, they made precise what they did expected from the placement of the obstacles: obstacles are identical squares, all parallel to each other, the placement is irregular, every portion of the plane contains about the same number of obstacles, and the distances between the obstacles are large in comparison to the obstacle's size.

If we fix the direction of the particle, the billiard flow will take only four directions. The Ehrenfests asked the following question: start K particles in a given direction, will the number of particles in each of the four directions asymptotically equalize to about $K/4$? To answer this question we study the ergodic properties of the wind-tree model.

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Interestingly the birth of ergodic theory can be traced back to the Ehrenfests' article in which the word ergodic was used for the first time with a close mathematical meaning to the current one [GaBoGe]. We consider the set of all possible configurations and introduce a canonical topology which makes it a compact metric space. We show that for (Baire) generic configurations, for almost every direction the billiard flow has infinite ergodic index, i.e., all its powers are ergodic. In a finite measure space this would be equivalent to saying that the flow is weakly mixing. The asymptotic equalization of the directions of K particles in several senses then follows from various ergodic theorems (note that we are in the framework of infinite ergodic theory here, so the Birkhoff ergodic theorem is not directly applicable).

In two previous articles we have considered a subset of configurations which are small perturbations of lattice configurations, and we showed that the generic wind-tree is minimal and ergodic in almost every direction [MSTr1, MSTr2]. Although the space of configurations considered in those two articles is different from the one in this article, in the two cases, the topology considered is equivalent to our topology restricted to this smaller set of configurations. Furthermore the proofs of these two results hold mutatis mutandis in the more general setting which we consider here.

There have been a number of results on the wind-tree model [DeCoVB, Ga, HaCo, HaCo1, Tr, VBHa, WoLa], and on the wind-tree model with periodical distribution of obstacles of squares, rectangles and more recently other polygonal shapes [AvHu, BaKhMaPl, BiRo, De, DeHuLe, DeZo, FrHu, FrUl, HaWe, HuLeTr].

2. Definitions and main results

For sake of simplicity, a square whose sides are parallel to lines $y = \pm x$ will be referred to as *rhombus* in the rest of the article. The $\mathcal{L}_1$ distance in $\mathbb{R}^2$ will be denoted by d. Note that balls with respect to this distance are rhombii.

Fix $s > 0$. A *configuration* is an (at most) countable collection of rhombii with diameter s, whose interiors are pairwise disjoint. Since s is fixed it is enough to note the centers of the rhombii, thus a configuration g is an at most countable subset of $\mathbb{R}^2$ such that if $z_1, z_2 \in g$ then $d(z_1, z_2) \geq s$.

To define a topology on the set of configurations consider polar coordinates (r, θ) on the plane. Each point (r, θ) in the plane is the stereographic projection of a point in the sphere with spherical coordinates $(2\arctan(1/r), \theta)$. Apply the inverse of the stereographic projection to a configuration g to obtain a subset of the sphere. Let $\hat{g}$ denote the union of this set with the north pole of the sphere denoted by $\{\infty\}$. Then $\hat{g}$ is a closed subset of the sphere. Let ρ denote the geodesic distance on the sphere, i.e., the length of the shortest path from one point to another along the great circle passing through them. The topology we define on the set of configurations is then induced by the Hausdorff distance d_H given by

$$d_H(g_1, g_2) = \max(\sup_{z_1 \in \hat{g}_1} \inf_{z_2 \in \hat{g}_2} \rho(z_1, z_2), \sup_{z_2 \in \hat{g}_2} \inf_{z_1 \in \hat{g}_1} \rho(z_1, z_2)).$$

Let *Conf* be the set of all configurations.

Proposition 1. *($Conf, d_H$) is compact metric space, thus a Baire space.*

The proposition is proven in the appendix. Let $\mathcal{U}_\varepsilon(g)$ be the set of all configurations that are at most ε-close to g, i.e.,

$$\mathcal{U}_\varepsilon(g) := \{g' : d_H(g', g) < \varepsilon\}.$$

Next we show some topological properties of generic configurations.

Proposition 2. *There is a dense G_δ subset G of ($Conf, d_H$) such that for each $g \in G$*

(1) g is an infinite configuration,
(2) every pair of points $z_1, z_2 \in g$ satisfy $d(z_1, z_2) > s$.

Remark. *Point (2) means that the obstacles centered at z_1 and z_2 do not intersect.*

Proof. As seen in Proposition 1, ($Conf, d_H$) is compact metric space, thus a Hausdorff space. Consider the countable dense set $\{g_i\}$ of finite configurations with centers of the obstacles at rational coordinates of $\mathbb{R}^2$ such that the obstacles are pairwise disjoint. From this set we create a new countable set $\{g_{i,j}\}$ of configurations as follows. Consider the smallest N_i such that the Euclidean ball $B(0, N_i)$ of radius N_i centered at the origin contains all the obstacles of g_i. Fix $s' > s$. Then for each $j \geq 1$ define the configuration $g_{i,j}$ to have the obstacles of g_i and additionally obstacles centered at all the points $\{(N_i + s'k, 0) : k > j\}$. Note that this choice of placement of the additional obstacles is arbitrary, any other choice would work as well as long as the distance between any pair of obstacles is uniformly bounded away from 0. Clearly $g_{i,j}$ is an infinite configuration, the obstacles are pairwise disjoint, and since $g_{i,j} \to g_i$ as $j \to \infty$ we see that the set $\{g_{i,j}\}$ is dense.

We choose a total order on the set $\{g_{i,j}\}$, abusing notation we call it $\{g_n\}$. Let $B_\varepsilon(g_n) := \{z \in g_n : \rho(z, \infty) \geq \varepsilon\}$. Define $\varepsilon(g_n)$ to be be the infinimum of $\varepsilon > 0$ such that

i) $\mathrm{card}(\mathrm{B}_\varepsilon(\mathrm{g_n})) \geq \mathrm{n}$, and
ii) for any $h \in U_{\varepsilon(g_n)}(g_n)$ we have

$\min\{d(z_1, z_2) : z_1, z_2 \in h$ corresponding to points of g_n in $B_\varepsilon(g_n)\} \geq s + \varepsilon$.

Notice that by construction the obstacles of g_n are not only pairwise disjoint, but $\inf\{d(z, z') : z, z' \in g_n, z \neq z'\} > s$, thus $\varepsilon(g_n) > 0$.

Clearly

$$G := \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} U_{\varepsilon(g_n)}(g_n)$$

is a dense G_δ set. If $g \in G$, then g is in $U_{\varepsilon(g_n)}(g_n)$ for an arbitrarily large n and thus g is an infinite configuration. Point (2) follows directly from (ii). □

Fix $g \in$ *Conf*. The *wind-tree table* $\mathcal{B}^g$ is the plane $\mathbb{R}^2$ with the interiors of the union of the trees removed. Fix $\theta \in \mathbb{S}^1$. The *billiard flow* $\phi_t^{g,\theta}$ *in the direction* θ or simply ϕ_t^θ is the free motion in the interior of $\mathcal{B}^g$ with elastic collision from the boundary of $\mathcal{B}^g$ (the boundary of the union of the trees). Once launched in the direction θ, the billiard direction can only achieve four directions

$$[\theta] := \{\pm\theta, \pm(\pi - \theta)\};$$

thus the *phase space* $X^{g,\theta}$ of the billiard flow in the direction θ is a subset of the cartesian product of $\mathcal{B}^g$ with these four directions. We agree that if a billiard orbit hits a corner of a tree, the outcome of the collision is not defined, and the billiard orbit stops there, its future is not defined from this time on. Note that in this notation ϕ_t^θ, $\phi_t^{-\theta}$, $\phi_t^{\pi-\theta}$ and $\phi_t^{\theta-\pi}$ denote all the same flow.

A flow ψ_t preserving a Borel measure m is called *ergodic* if for each Borel measurable set $A, m(\psi_t(A) \triangle A) = 0\ \forall t \in \mathbb{R}$ implies that $m(A) = 0$ or $m(A^c) = 0$. The flow ψ_t is said to have *infinite ergodic index* if for each integer $K \geq 1$ the K-fold product flow $\psi_t \times \cdots \times \psi_t$ is ergodic with respect to the K-fold product measure $m \times \cdots \times m$. It is a well known fact that in the finite measure case the notion of infinite ergodic index is equivalent to weak-mixing. However we are working in the context of an infinite measure preserving flow.

For each direction θ, the billiard flow ϕ_t^θ preserves the area measure μ on $\mathcal{B}^g$ times a discrete measure on $[\theta]$, we will also call this measure μ. Note that μ is an infinite measure. The billiard flow on the full phase space preserves the volume measure $\mu \times \lambda$ with λ the length measure on $\mathbb{S}^1$. Let $K \geq 1$, and let $\vec{\theta} = (\theta_1, \ldots, \theta_K)$ be a vector of directions. When $\theta_1 = \cdots = \theta_K = \theta$, we will denote this vector of identical directions by $\bar{\theta}$. We note the *product billiard flow* $\phi_t^{\vec{\theta}} := \phi_t^{\theta_1} \times \cdots \times \phi_t^{\theta_K}$. This flow preserves the measure $\mu^K := \mu \times \cdots \times \mu$.

Now we can state our main result.

Theorem 3. *For any $s > 0$ there is a dense G_δ subset G of Conf such that*

(1) there exists a dense G_δ set of full measure of directions $\mathcal{H}$ such that the flow ϕ_t^θ has infinite ergodic index for every $\theta \in \mathcal{H}$ and each $g \in G$, and

(2) for every integer $K \geq 1$ there is a dense G_δ set $\mathcal{H}(K)$ of full measure of K-tuples of directions such that the flow $\phi_t^{\vec{\theta}}$ is ergodic for every $\vec{\theta} \in \mathcal{H}(K)$ for each $g \in G$.

Remark. *If $K > 1$, we do not know that the set $\mathcal{H}(K)$ has product structure, thus (1) does not follow from (2).*

2.1. **The precise question posed by the Ehrenfests.** Consider a large but finite number K of initial points in the wind-tree model in a given direction θ. The Ehrenfests asked whether the particles directions asymptotically equalize under the wind-tree dynamics, i.e., whether there are approximately $K/4$ particles in each direction after a large time.

This question is the motivation for our study. Let $(\vec{z}, \vec{\theta})$ denote the initial positions and velocities of these particles, and $f_i((\vec{z}, \vec{\theta}))$ denote the number of particles pointing in the direction $i \in \{\pm\theta, \pm(\pi - \theta)\}$. If the functions f_i were integrable, then we could give a nice answer to this question using Theorem 3, but unfortunately this is not the case. We give three partial answers.

First a finite measure version. Let $A \subset \mathcal{B}^g$ be a positive but finite measure subset of the wind-tree table, and let f_i^A denote the function f_i restricted to the set $A \times \cdots \times A$. This function is integrable, thus applying the Hopf ergodic theorem to the wind-tree flow yields the following corollary (here G and $\mathcal{H}$ are the dense G_δ sets from Theorem 3 which depends on K and s).

Corollary 4. *For each $s > 0$, for each K positive integer, for each $g \in G$, for each $A \subset \mathcal{B}^g$ of positive measure, for each $\theta \in \mathcal{H}$, for each i, j the following limit holds almost surely as $T \to \infty$:*

$$\frac{\int_0^T f_i^A\big(\phi_t^\theta \times \cdots \times \phi_t^\theta(\vec{z}, \bar{\theta})\big)\, dt}{\int_0^T f_j^A\big(\phi_t^\theta \times \cdots \times \phi_t^\theta(\vec{z}, \bar{\theta})\big)\, dt} \to 1.$$

This means that if we only count when all the particles are in the set A then the average over times of the number going in each direction is asymptotically the same.

If we replace the flow $\phi_t^{\bar{\theta}} = \phi_t^\theta \times \cdots \times \phi_t^\theta$ by its first return flow $\psi_t^{A,\bar{\theta}}$ to the region $A \times \cdots \times A$, then we can apply the Birkhoff ergodic theorem.

Corollary 5. *For each $s > 0$, for each K positive integer, for each $g \in G$, for each $A \subset \mathcal{B}^g$ of positive measure, for each i, the following limit holds almost surely as $T \to \infty$:*

$$\frac{1}{T} \int_0^T f_i^A\big(\psi_t^{A,\bar{\theta}}(\vec{z}, \bar{\theta})\big)\, dt \to \int_A f_i^A \, d\mu \times \cdots \times d\mu = \frac{K}{4} \cdot area(A).$$

This means that the average over time of the direction converges to $K/4$, but for the first return flow.

Finally we can replace the f_i by integrable functions which somehow measure a similar phenomenon. For example the sum of the cubes of the reciprocal of the distance of the particles from the origin: $\hat{f}_i((\vec{z},\vec{\theta})) = \sum_{\{k:\vec{\theta}_k=i\}} \frac{1}{\min(1,|z_k|^3)}$. These functions are positive and integrable, thus we can apply the Hopf ergodic theorem to conclude:

Corollary 6. *For each $s > 0$, for each K positive integer, for each $g \in G$, for each $\theta \in \mathcal{H}$, for each i, j the following limit holds almost surely as $T \to \infty$:*

$$\frac{\int_0^T \hat{f}_i\big(\phi_t^{\bar{\theta}}(\vec{z},\bar{\theta})\big)\, dt}{\int_0^T \hat{f}_j\big(\phi_t^{\bar{\theta}}(\vec{z},\bar{\theta})\big)\, dt} \to 1.$$

This means that then the average over time of the weighted number going in each direction is asymptotically the same.

For all three results we can replace $\bar{\theta}$ whose components are all equal to a single $\theta \in \mathcal{H}$ by a vector $\vec{\theta} \in \mathcal{H}(G)$ of directions that are maybe no longer identical, we can define $\psi^{A,\vec{\theta}}$ analogously to $\psi^{A,\bar{\theta}}$ and state a similar result for the functions $f_i, f_i^A, \hat{f}_i$ which counts the number of particles with direction in the ith quadrant ($i \in \{1,2,3,4\}$).

Corollary 7. *For each $s > 0$, for each K positive integer, for each $g \in G$, for each $A \subset \mathcal{B}^g$ of positive measure, for each $\vec{\theta} \in \mathcal{H}(K)$, for each i, j the following limits hold almost surely as $T \to \infty$:*

$$\frac{\int_0^T f_i^A\big(\phi_t^{\vec{\theta}}(\vec{z},\vec{\theta})\big)\, dt}{\int_0^T f_j^A\big(\phi_t^{\vec{\theta}}(\vec{z},\vec{\theta})\big)\, dt} \to 1$$

$$\frac{1}{T}\int_0^T f_i^A\big(\psi_t^{A,\vec{\theta}}(\vec{z},\vec{\theta})\big)\, dt \to \int_A f_i^A \, d\mu \times \cdots \times d\mu = \frac{K}{4} \cdot area(A)$$

$$\frac{\int_0^T \hat{f}_i\big(\phi_t^{\vec{\theta}}(\vec{z},\vec{\theta})\big)\, dt}{\int_0^T \hat{f}_j\big(\phi_t^{\vec{\theta}}(\vec{z},\vec{\theta})\big)\, dt} \to 1.$$

We interpret these results in the following way; if K particles are launched in arbitrary generic directions, then the average over time of the number of particles in the different quadrants are asymptotically the same in the three senses mentioned above.

3. Proof of wind-tree results

Fix a wind-tree configuration $g \in Conf$. Fix K and $n \in \mathbb{N}$, and let $\mathcal{B}_n^g := \big(\mathcal{B}^g \cap \{(x,y) : |x| + |y| \le ns\}\big)^K$. Note that K does not appear in this notation, as well as certain other notations in this section, since it is fixed throughout much of the proof.

We will define a series of notations depending on a vector of directions $\vec{\theta}$. When all the components of the vector are identical, recall that we agreed to note this vector by $\bar{\theta}$ - the notations defined will apply automatically to that case as well.

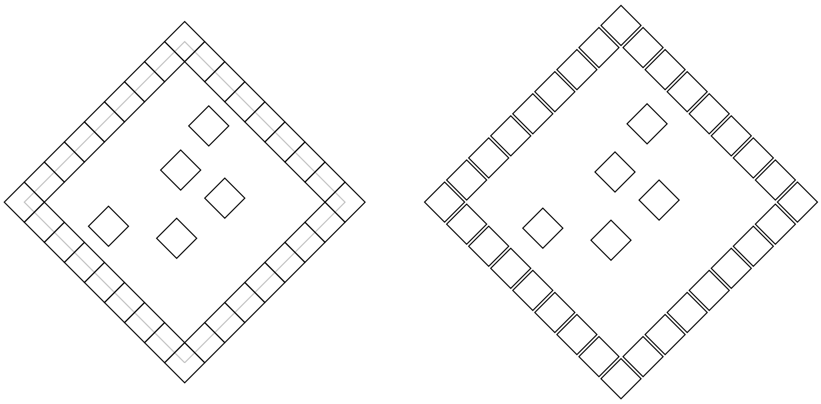

Figure 1. An 8-ringed configuration and a configuration close to it.

Recall that for each θ, $[\theta]$ is the set of all possible directions under the billiard flow starting in direction θ, i.e., $[\theta] = \{\pm\theta, \pm(\pi - \theta)\}$. We denote the components of the vectors $\vec{\theta}$ and $\vec{\psi}$ by θ_i and ψ_i. Let $[\vec{\theta}] := \{\vec{\psi} : \psi_i \in [\theta_i] \text{ for all } i\}$ and

$$X_n^{g,\vec{\theta}} := \{(\vec{z}, \vec{\psi}) : \vec{z} \in \mathcal{B}_n^g, \vec{\psi} \in [\vec{\theta}]\}, X^{g,\vec{\theta}} := \{(\vec{z}, \vec{\psi}) : \vec{z} \in \mathcal{B}^g, \vec{\psi} \in [\vec{\theta}]\}.$$

We note the product billiard flow on $X^{g,\vec{\theta}}$ by $\phi^{g,\vec{\theta}}$. For each $n \geq 1$ we consider the first return flow of $\phi^{g,\vec{\theta}}$,

$$\phi_t^{g,\vec{\theta},n} : X_n^{g,\vec{\theta}} \to X_n^{g,\vec{\theta}}.$$

In the future, when it is clear from the context, we will drop the superscript g from this notation. For each $\theta \in \mathbb{S}^1$, the flow $\phi_t^{g,\vec{\theta},n}$ preserves the measure μ^K. For sake of simplicity, we will denote μ^K by μ.

Proof of Theorem 3. We prove both statements with the same strategy: we choose a dense set $\{f_i\}$ of configurations which satisfy the goal dynamical property of K-fold ergodicity on certain compact sets. Then we will show that wind-tree tables which are sufficiently well approximated by this dense set will satisfy the dynamical property on the whole phase space. The proof for $K = 1$ is simpler, and we will mention the simplification in the proof even though this is not formally necessary for the proof.

A configuration f is called *n-ringed* if the boundary of the rhombus $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq ns\}$ is completely covered by trees as in Figure 1 left (i.e., the obstacles covering the boundary intersect with each other on a whole side or do not intersect at all). The union of the trees which cover the boundary of the rhombus is called the associated n-ring.

For the proof of ergodicity, i.e., $K = 1$, let $\{f_i\}$ be a dense set of parameters such that each f_i is an n_i-ringed configuration, is not an n-ringed configuration for all $n < n_i$, and n_i is increasing with i. Then by [KeMaSm] the billiard flow $\psi_t^{f_i,\theta}$ is ergodic in almost every direction inside the n_i-ring. So, the return flow $\phi_t^{f_i,\theta,n}$ is ergodic for all n such that $1 \le n \le n_i$ and for almost every direction θ. Since in the rest of the proof we will consider all the cases $K \ge 1$ simultaneously we will denote $\phi_t^{f_i,\theta,n}$ by $\phi_t^{f_i,\vec{\theta},n}$ or $\phi_t^{\vec{\theta},n}$ where $\vec{\theta}$ is the vector (θ).

Consider now the K-fold case. We apply Theorem 1 of [MSTr3], we state this theorem in the special context of our setting.

Theorem 8. *Fix $n \ge 2$. Consider the set Z of n-ringed configurations with a fixed number of non-intersecting obstacles inside the ring with the topology given by the positions of centers of the inner obstacles. The set Z contains a dense G_δ set of tables Y such that for each configuration in Y there exists a full measure G_δ-dense set Θ of directions such that the billiard flow on table given by this the configuration is weakly mixing inside the n ring in every direction $\theta \in \Theta$.*

Note that the topology in Theorem 8 is different from the topology in this article, none this less in our context it yields a countable dense set of configurations $\{f_i\}$ such that each f_i is n_i-ringed, not n ringed for all $n < n_i$, and the flow is weakly mixing for all $\theta \in \Theta$ inside the ring. To see this note that if f is n-ringed then there is a maximum number of obstacles that can be contained inside the ring, call this number $m(n)$. For each $0 \le m \le m(n)$ consider now a countable dense set $\{f_j^{n.m}\}$ of n-ringed configurations, with m obstacles inside the ring, where density is in the sense of the topology of Theorem 8 . Since the obstacles in Theorem 8 are non-intersecting, the $f_j^{m,n}$ are not n'-ringed for all $n' < n$. Then the set $\{f_j^{n.m}\}_{m,n,j}$ is the required countable dense set, we linearly enumerate this set, calling it $\{f_i\}$ so that f_i is n_i-ringed and n_i is increasing with i.

Fix $K \ge 1$. Suppose that δ_i are strictly positive numbers. Then the set

$$\mathcal{G}_K := \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} \mathcal{U}_{\delta_i}(f_i)$$

is a dense G_δ set. We will show that the δ_i can be chosen in such a way that all the configurations in $\mathcal{G}$ are K-fold ergodic for all $\vec{\theta} \in \Theta^K$. Note that the set Θ does not depend on K. This Θ will be the G_δ set $\mathcal{H}$ of full measure that has to be found in the proof. Taking the intersection $\cap_K \mathcal{G}_K$ will finish the proof, thus for sake of simplicity, we will fix K from here on and drop it from the notations when convenient.

For any $\vec{\theta}$ and $g \in Conf$, let $\{h_j\}_{j\ge1}$ be a countable dense collection of continuous functions in $\mathcal{L}^1(X^{g,\vec{\theta}}, \mu^K)$. For any $\vec{\theta}$ such that $\theta_i \notin \{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}$ for all i, the sets $X^{g,\vec{\theta}}$ can be homeomorphically identified, and thus functions $\{h_j\}$ can be considered to not depend on θ. Finally by restriction, we think of this collection as a collection in $\mathcal{L}^1(X_n^{g,\vec{\theta}}, \mu_n^K)$.

The flow $\phi^{g,\vec{\theta}}$ is ergodic if and only if the flow $\phi^{g,\vec{\theta},n}$ is ergodic for all n. Consider the Cesaro average

$$S^g_{n,\ell} h_j(\vec{z},\vec{\theta}) := \frac{1}{\ell}\int_0^\ell h_j\big(\phi_t^{g,\vec{\theta},n}(\vec{z},\vec{\theta})\big)\,dt.$$

By the Birkhoff ergodic theorem, the flow $\phi^{g,\vec{\theta},n}$ is ergodic for all n if and only if for all n we have

$$S^g_{n,\ell} h_j(\vec{z},\vec{\theta}) \to \int_{X_n^{\vec{\theta}}} h_j(y)\,d\mu(y) \tag{1}$$

as ℓ goes to infinity for all $j \geq 1$.

Now fix i. The billiard flow $\phi_t^{f_i,\theta}$ is weakly-mixing inside the ring for each $\theta \in \Theta$, thus $\phi_t^{f_i,\vec{\theta}}$ inside the ring is ergodic for every $\vec{\theta}$ in Θ^K. Thus the first return flows $\phi_t^{f_i,\vec{\theta},n}$ are ergodic for every $\vec{\theta}$ in Θ^K, for all $1 \leq n \leq n_i$. Note that in particular, $\bar{\theta} \in \Theta^K$ for all $\theta \in \Theta$.

Consider a large time ℓ_i so that the Cesaro average is $1/i$ close to its limit in Equation (1) for a large set of points (many points $\vec{z}$ for many directions $\vec{\theta}$). More precisely we can find positive integers $\ell_i \geq n_i$, open sets $H_i \subset \mathbb{S}^1$ and sets $C_n^{f_i,\vec{\theta}} \subset X_n^{f_i,\vec{\theta}}$ so that $\mu(C_n^{f_i,\vec{\theta}}) > \mu(X_n^{f_i,\vec{\theta}}) - \frac{1}{i}$, $\lambda\left(H_i\right) > 1 - \frac{1}{i}$ and

$$\left| S^{f_i}_{n,\ell_i} h_j(\vec{z},\vec{\theta}) - \int_{X_n^{f_i,\vec{\theta}}} h_j(y)\,d\mu(y) \right| < \frac{1}{i} \tag{2}$$

for all $\vec{z} \in C_n^{f_i,\vec{\theta}}$, $\vec{\theta} \in (H_i)^K$, $1 \leq j \leq i$, and $1 \leq n \leq n_i$.

Now we would like to extend these estimates to the neighborhood $\mathcal{U}_{\delta_i}(f_i)$ for a sufficiently small strictly positive δ_i (see Figure 1 right). For any n such that $1 \leq n \leq n_i$ let $\bar{B}_n^i$ be the intersection of $\mathcal{B}_n^g$ for all g in the δ_i-neighbourhood $\mathcal{U}_{\delta_i}(f_i)$. Let

$$\bar{X}_n^{i,\vec{\theta}} := \bar{B}_n^i \times [\vec{\theta}].$$

For every $1 \leq n \leq n_i$ we define $\vec{\psi} = \vec{\psi}^{g,f_i}$ a piecewise continuous map from $\mathcal{B}_n^g$ to $\mathcal{B}_n^{f_i}$. When convenient we will write $\vec{\psi}(\vec{z},\vec{\theta})$ instead of $(\vec{\psi}(\vec{z}),\vec{\theta})$. The behavior of $\vec{\psi}$ will be defined coordinate by coordinate, more precisely $\vec{\psi}(\vec{z}) = (\psi(z_1)\ldots,\psi(z_K))$ where ψ will be defined right now. For z outside the obstacles of f_i, we define $\psi(z) = z$. For each obstacle O_1 of g inside the ring, we consider the corresponding obstacle O_2 of f_i and $C_{12} = O_2 \setminus O_1$. We define a direction ξ that points from a corner of O_1 to a corner of O_2 in such a way that the segment along this direction between the two corners is completely included in C_{12} (as in Figure 2). Then for any $z \in C_{12}$, the image $\psi(z)$ of z is the closest point in the direction ξ in the table $\mathcal{B}_n^{f_i}$ (Figure 2). The difference between the Lebesgue measure of $\bar{B}_n^i$ and the measure of $\mathcal{B}_n^g$ can be made arbitrarily small by an adequate choice of δ_i, simultaneously for all g in $\mathcal{U}_{\delta_i}(f_i)$. From now on, we make the choice of δ_i such that $\mu(\bar{X}_n^{i,\vec{\theta}}) = \mu(\bar{B}_n^i) > \mu(\mathcal{B}_n^g) - \frac{1}{i}$ for all $g \in \mathcal{U}_{\delta_i}(f_i)$ and all i.

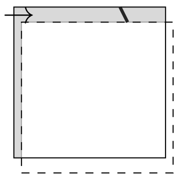

FIGURE 2. The dashed obstacle is O_1 and the solid obstacle is the associated obstacle O_2. The map ψ maps all the points on in the gray region C_{12} to the boundary of the dashed obstacle, for example it maps the bold segment in the direction ξ to a point in its top endpoint.

By the triangular inequality we have:

$$\begin{aligned}\left|S^g_{n,\ell_i} h_j(\vec{z},\vec{\theta}) - \textstyle\int_{X_n^{g,\vec{\theta}}} h_j(y)\,d\mu(y)\right| \le \; & \left|S^g_{n,\ell_i} h_j(\vec{z},\vec{\theta}) - S^{f_i}_{n,\ell_i} h_j(\vec{\psi}(\vec{z}),\vec{\theta})\right| + \\ & \left|S^{f_i}_{n,\ell_i} h_j(\vec{\psi}(\vec{z}),\vec{\theta}) - \textstyle\int_{X_n^{f_i,\vec{\theta}}} (h_j(y))d\mu(y)\right| + \\ & \left|\textstyle\int_{X_n^{f_i,\vec{\theta}}} h_j(y)\,d\mu(y) - \int_{X_n^{g,\vec{\theta}}} h_j(y)\,d\mu(y)\right|.\end{aligned}$$

Futhermore we choose δ_i so small that

$$\left|\int_{X_n^{f_i,\vec{\theta}} \setminus \bar{X}_n^{i,\vec{\theta}}} h_j(y)\,d\mu(y)\right| < \frac{1}{i}$$

and

$$\left|\int_{X_n^{g,\vec{\theta}} \setminus \bar{X}_n^{i,\vec{\theta}}} h_j(y)\,d\mu(y)\right| < \frac{1}{i}$$

thus by the triangular inequality

$$\left|\int_{X_n^{f_i,\vec{\theta}}} h_j(y)\,d\mu(y) - \int_{X_n^{g,\vec{\theta}}} h_j(y)\,d\mu(y)\right| < \frac{2}{i}. \tag{3}$$

Now the proof bifurcates a bit according to the different cases stated in the theorem. Consider part (1) of the theorem. Note that ψ is not continuous, not invertible, and not onto. However it is not far from being continuous: $||\vec{z} - \vec{\psi}(\vec{z})||_{\mathcal{L}^\infty} < \delta_i$ for any $\vec{z} \in \mathcal{B}_n^g$. By our convention the billiard flow stops at corners, thus any point $(\vec{z},\bar{\theta})$ for which the flow is defined up to time ℓ_i is a point of continuity for $\phi_{\ell_i}^{f_i,\bar{\theta},n}$. Consider such a point, then the point $\vec{\psi}(\phi_{\ell_i}^{g,\bar{\theta},n}(\vec{z},\bar{\theta}))$ stays δ_i-close to $\phi_{\ell_i}^{f_i,\bar{\theta},n}(\vec{z},\bar{\theta})$ for g in a small enough neighborhood of f_i; thus we can find $\delta_i > 0$, an open set $\hat{H}_i \subset H_i$ and a set $\hat{C}_n^{i,\bar{\theta}} \subset \bar{X}_n^{i,\bar{\theta}} \cap C_n^{f_i,\bar{\theta}}$ so that if $g \in \mathcal{U}(f_i,\delta_i)$, then

$$\left|S^g_{n,\ell_i} h_j(\vec{z},\bar{\theta}) - S^{f_i}_{n,\ell_i} h_j(\vec{\psi}(\vec{z}),\bar{\theta})\right| < \frac{2}{i} \tag{4}$$

for all $z \in \hat{C}_n^{i,\bar{\theta}}$, $\theta \in \hat{H}_i$, $1 \leq j \leq i$, $1 \leq n \leq n_i$; and $\mu(\hat{C}_n^{i,\bar{\theta}}) > \mu(\mathcal{B}_n^g) - \frac{2}{i}$ and $\hat{H}_i$ is of measure larger than $1 - \frac{2}{i}$.

Since $\lambda(\hat{H}_i) > 1 - 2/i$, the set $\mathcal{H} = \cap_{M=1}^{\infty} \cup_{i=M}^{\infty} \hat{H}_i$ has full measure. Fix $g \in \mathcal{G}$ and $\theta \in \mathcal{H}$, then there is an infinite sequence i_k such that $g \in \mathcal{U}_{\delta_{i_k}}(f_{i_k})$ and $\theta \in \hat{H}_{i_k}$. Fix $n \geq 1$ and consider $\mathcal{C}_n^{g,\bar{\theta}} := \cap_{M=1}^{\infty} \cup_{k=M}^{\infty} \widehat{C}_n^{i_k,\bar{\theta}}$. Recall that we made the choice of δ_i such that $\mu(\bar{X}_n^{i_k,\bar{\theta}}) > \mu(\mathcal{B}_n^g) - \frac{1}{i_k}$. Since $\mu(\widehat{C}_n^{i_k,\bar{\theta}}) > \mu(\bar{X}_n^{i_k,\bar{\theta}}) - \frac{1}{i_k}$, it follows that $\mu(\mathcal{C}_n^{g,\bar{\theta}}) = \mu(\mathcal{B}_n^g)$.

Suppose $g \in \mathcal{G}$. Thus for $\theta \in \mathcal{H}$, for each $n \geq 1$ the three inequalities (2), (3), (4) imply that

$$|S_{n,\ell_{i_k}}^g(h_j^\theta) - \int_{X_n^\theta} h_j(\vec{z},\bar{\theta})d\mu| < \frac{5}{i}$$

for all $z \in \hat{C}_n^{i,\bar{\theta}}$, $\theta \in \hat{H}_i$, $1 \leq j \leq i$, $1 \leq n \leq n_i$ and thus

$$\lim_{k\to\infty} S_{n,\ell_{i_k}}^g(h_j^\theta) \to \int_{X_n^\theta} h_j(\vec{z},\bar{\theta})d\mu \tag{5}$$

for all $(\vec{z},\bar{\theta})$ in $\mathcal{C}_n^{g,\bar{\theta}}$, for each $j \geq 1$. The h_j^θ are dense in $\mathcal{L}^1(X_n^\theta,\mu)$ and $\lim_{k\to\infty} \ell_{i_k} = \infty$, thus Equation (5) together with the Birkhoff ergodic theorem imply that for each $n \geq 1$, the first return flow $\phi_t^{g,\bar{\theta},n}$ is ergodic for all $\theta \in \mathcal{H}$. This implies the ergodicity of the billiard flow $\phi_t^{g,\bar{\theta}}$ in every direction in $\mathcal{H}$.

For part (2) of the theorem we have to slightly modify the previous arguments. The only difference being that the set of directions we construct depends on K. For any point $(\vec{z},\vec{\theta})$ of continuity of $\phi_{\ell_i}^{f_i,\vec{\theta},n}$, the point $\phi_{\ell_i}^{g,\vec{\theta},n}(\vec{z},\vec{\theta})$ varies continuously with g in a small neighborhood of f_i; thus we can find $\delta_i > 0$, an open set $\hat{H}_i(K) \subset (H_i)^K$ and a set $\hat{C}_n^{i,\vec{\theta}} \subset \bar{X}_n^{i,\vec{\theta}} \cap C_n^{f_i,\vec{\theta}}$ so that that if $g \in \mathcal{U}(f_i,\delta_i)$, then

$$\left|S_{n,\ell_i}^g h_j(\vec{z},\vec{\theta}) - S_{n,\ell_i}^{f_i} h_j(\vec{\psi}(\vec{z}),\vec{\theta})\right| < \frac{2}{i}$$

for all $z \in \hat{C}_n^{i,\vec{\theta}}$, $\vec{\theta} \in \hat{H}_i(K)$, $1 \leq n \leq n_i$, $1 \leq j \leq i$; and $\mu(\hat{C}_n^{i,\vec{\theta}}) > \mu(\mathcal{B}_n^g) - \frac{2}{i}$ and $\hat{H}_i(K)$ is of measure larger than $1 - \frac{2}{i}$.

Since $\lambda(\hat{H}_i) > 1 - 2/i$, the G_δ set $\mathcal{H}(K) = \cap_{M=1}^{\infty} \cup_{i=M}^{\infty} \hat{H}_i(K)$ has full measure. The rest of the proof of part (2) is identical to that of part (1). □

3.1. **Generalization.** If we consider a subset C of $(Conf, d_H)$ which is itself a Baire set such that the set $\{h : h \text{ is N}-\text{ringed for } N \geq N_0\}$ is dense in C for each $N_0 \geq 1$ then Theorem 3 holds in (C, d_H) as well. In particular the set of configurations considered in the articles [MSTr1],[MSTr2] is a Baire subset of $(Conf, d_H)$ thus Theorem 3 holds in that context as well.

For convenience we recall the setup, to interpret in our framework we need to rotate the lattice by 45 degree. In these articles we considered the plane $\mathbb{R}^2$ tiled by one by one closed square *cells* with corners on the lattice $\mathbb{Z}^2$. Then fix $r \in [1/4, 1/2)$ and consider the set of $2r$ by $2r$ squares, with vertical and horizontal sides, centered at (a, b) contained in the unit cell $[0, 1]^2$, this set is naturally parametrized by

$$\mathcal{A} := \{t = (a, b) : r \le a \le 1 - r,\ r \le b \le 1 - r\}$$

with the usual topology inherited from $\mathbb{R}^2$. The parameter space we considered was $\mathcal{A}^{\mathbb{Z}^2}$ with the product topology. It is a Baire space. Each parameter $g = (a_{i,j}, b_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \mathcal{A}^{\mathbb{Z}^2}$ corresponds to a wind-tree table in the plane in the following manner: the tree inside the cell corresponding to the lattice point $(i, j) \in \mathbb{Z}^2$ is a $2r$ by $2r$ square with center at position $(a_{i,j}, b_{i,j}) + (i, j)$. The wind-tree table B^g is the plane $\mathbb{R}^2$ with the interiors of the union of these trees removed. Note that trees can intersect only at the boundary of cells.

4. Appendix

Proof of Proposition 1. Remember that $\mathcal{U}_\varepsilon(g) := \{g' : d_H(g', g) < \varepsilon\}$.

Let $(g_i)_{i\in\mathbb{N}}$ be a sequence of configurations. Consider $\varepsilon_n = 1/n$. Let $B_\varepsilon := \{z : \rho(z, \infty) > \varepsilon)\}$. Let k_j be the cardinality of $g_j \cap B_{\varepsilon_1}$. The sequence $\{k_j\}$ is uniformly bounded above since pairs of points have by assumption distance at least s, thus the sequence k_i only take a finite number of values. Thus we can choose subsequence $(g_j)_{j\in J_0}$ such that the sequence $(k_j : j \in J_0)$ is constant, call this constant c_1.

If $c_1 = 0$ then for each $j \in J_0$ let g_j^1 be the empty configuration. Otherwise for each $j \in J_0$ let $g_j^1 := \{z_{1,j}^1, \dots, z_{c_1,j}^1\} = g_j \cap B_{\varepsilon_1}$. For each j we think of g_j^1 as a finite configuration in *Conf*, but also as a point in $\mathbb{S}^{c_1}$. By compactness of $\mathbb{S}^{c_1}$ we can find a subsequence $J_1 \subset J_0$ such that the $(g_j^1 : j \in J_1)$ converge to a point $g^1 := (z_1^1, \dots, z_{c_1}^1) \in \mathbb{S}^{c_1}$. Note that $d(z_i^1, z_j^1) \ge s$ for all $i \ne j$, thus $g^1 \in$ *Conf*. Furthermore we have $g_j^1 \in \mathcal{U}_{\varepsilon_1}(g^1)$ for all sufficiently large $j \in J_1$. Repeat this argument for $n = 2$ to produce a subsequence $J_2 \subset J_1$ which converge to a point $g^2 \in \mathbb{S}^{c_2}$. Again we have $g^2 \in$ *Conf* and $g_j^2 \in \mathcal{U}_{\varepsilon_2}(g^2)$ for all sufficiently large $j \in J_2$. Note that $c_2 \ge c_1 \ge 0$ and for $j = 1, \dots, c_1$ we have $z_j^2 = z_j^1$. Repeat this construction for each n. Finally we define g to be the collection of points such that every $z \in g$ is in g^k for all sufficiently large k. By construction for any $z_1, z_2 \in g$ we have $d(z_1, z_2) \ge s$, thus $g \in$ *Conf*. Note that g can be an infinite, finite, or even the empty configuration.

We claim that g is an accumulation point of the sequence $(g_j)_{j\in\mathbb{N}}$. Fix a neighborhood $\mathcal{U}$ of g. Since $\varepsilon_n \to 0$ as $n \to \infty$ we can choose n so large that $\mathcal{U}_{\varepsilon_n}(g) \subset \mathcal{U}$. By construction of g and g^n we have

$\mathcal{U}_{\varepsilon_n}(g^n) = \mathcal{U}_{\varepsilon_n}(g)$. The result follows since $g_j \in \mathcal{U}_{\varepsilon_n}(g^n)$ for all sufficiently large $j \in J_n \subset \mathbb{N}$. □

Remark. *If we remove the empty and finite configurations from Conf the space is not even locally compact.*

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