Noise sensitivity of functionals of fractional Brownian motion driven stochastic differential equations: Results and perspectives

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Abstract We present an innovating sensitivity analysis for stochastic differential equations: We study the sensitivity, when the Hurst parameter H of the driving fractional Brownian motion tends to the pure Brownian value, of probability distributions of smooth functionals of the trajectories of the solutions $\{X_t^H\}_{t\in\mathbb{R}_+}$ and of the Laplace transform of the first passage time of X^H at a given threshold. We also present an improvement of already known Gaussian estimates on the density of X_t^H to estimates with constants which are uniform w.r.t. t in the whole half-line $\mathbb{R}_+ - \{0\}$ and w.r.t. H when H tends to $\frac{1}{2}$.

Key words: Fractional Brownian motion; First hitting time; Malliavin calculus.

1 Introduction

Recent statistical studies show memory effects in biological, financial, physical data: see e.g. [18] for a statistical evidence in climatology and [6] and citations therein for an evidence and important applications in finance. For such data the Markov structure of Lévy driven stochastic differential equations makes such models questionable. It seems worth proposing new models driven by noises with long-range memory such as fractional Brownian motions.

In practice the accurate estimation of the Hurst parameter H of the noise is difficult (see e.g. [4]) and therefore one needs to develop sensitivity analysis w.r.t. H of probability distributions of smooth and non smooth functionals of the solutions

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(X_t^H) to stochastic differential equations. Similar ideas were developed in [11] for symmetric integrals of the fractional Brownian motion.

Here we review and illustrate by numerical experiments our theoretical results obtained in [17] for two extreme situations in terms of Malliavin regularity: on the one hand, expectations of smooth functions of the solution at a fixed time; on the other hand, Laplace transforms of first passage times at prescribed thresholds. Our motivation to consider first passage times comes from their many use in various applications: default risk in mathematical finance or spike trains in neuroscience (spike trains are sequences of times at which the membrane potential of neurons reach limit thresholds and then are reset to a resting value, are essential to describe the neuronal activity), stochastic numerics (see e.g. [3, Sec.3]) and physics (see e.g. [13]). Long-range dependence leads to analytical and numerical difficulties: see e.g. [10].

In a Markovian setting the simplest partial differential equations characterizing the probability distributions of first hitting times are those satisfied by their Laplace transforms. In some circumstances they even have explicit solutions. It is thus natural to concentrate our study on Laplace transforms. We have a second motivation. Laplace transforms of first hitting times are expectations of singular functionals on the Wiener space. It seemed worth to us showing that a sensitivity analysis can be developed in such singular situations.

Our theoretical estimates and numerical results tend to show that the Markov Brownian model is a good proxy model as long as the Hurst parameter remains close to $\frac{1}{2}$. This robustness property, even for probability distributions of singular functionals (in the sense of Malliavin calculus) of the paths such as first hitting times, is an important information for modeling and simulation purposes: when statistical or calibration procedures lead to estimated values of H close to $\frac{1}{2}$, then it is reasonable to work with Brownian SDEs, which allows to analyze the model by means of PDE techniques and stochastic calculus for semimartingales, and to simulate it by means of standard stochastic simulation methods.

Our main results

The fractional Brownian motion $\{B_t^H\}_{t\in\mathbb{R}_+}$ with Hurst parameter $H\in(0,1)$ is the centred Gaussian process with covariance

$$R_H(s,t)=\frac{1}{2}\left(s^{2H}+t^{2H}-|t-s|^{2H}\right),\quad \forall s,t\in\mathbb{R}_+.$$

Given $H\in(\frac{1}{2},1)$, we consider the process $\{X_t^H\}_{t\in\mathbb{R}_+}$ solution to the following stochastic differential equation driven by $\{B_t^H\}_{t\in\mathbb{R}_+}$:

$$X_t^H=x_0+\int_0^t b(X_s^H)\,\mathrm{d}s+\int_0^t \sigma(X_s^H)\circ\mathrm{d}B_s^H, \tag{1}$$

where the last integral is a pathwise Stieltjes integral in the sense of [19]. For $H=\frac{1}{2}$ the process X solves the following SDE in the classical Stratonovich sense:

$$X_t = x_0 + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s) \circ \mathrm{d}B_s. \tag{2}$$

Below we use the following set of hypotheses:

(H1) There exists $\gamma \in (0,1)$ such that $b, \sigma \in \mathcal{C}^{1+\gamma}(\mathbb{R})$;
(H2) $b, \sigma \in \mathcal{C}^2(\mathbb{R})$;
(H3) The function σ satisfies a strong ellipticity condition: $\exists \sigma_0 > 0$ such that $|\sigma(x)| \geq \sigma_0, \forall x \in \mathbb{R}$.

Our first theorem is elementary. It describes the sensitivity w.r.t. H around the critical Brownian parameter $H = \frac{1}{2}$ of time marginal probability distributions of $\{X_t^H\}_{t\in\mathbb{R}_+}$.

Theorem 1. *Let $H \in (\frac{1}{2}, 1)$, and let X^H and X be as before. Suppose that b and σ satisfy (H1) and (H3), and φ is bounded and Hölder continuous of order $2+\beta$ for some $\beta > 0$. Then, for any $T > 0$ there exists $C_T > 0$ such that*

$$\forall H \in [\tfrac{1}{2}, 1), \quad \sup_{t\in[0,T]} \left| \mathbb{E}\varphi(X_t^H) - \mathbb{E}\varphi(X_t) \right| \leq C_T \ (H - \tfrac{1}{2}).$$

Our next theorem concerns the first passage time at threshold 1 of X^H issued from $x_0 < 1$: $\tau_H^X := \inf\{t \geq 0 : X_t^H = 1\}$. The probability distribution of the first passage time τ_H of a fractional Brownian motion is not explictly known. [14] obtained the asymptotic behaviour of its tail distribution function and [7] obtained an upper bound on the Laplace transform of τ_H^{2H}. The recent work of [8] proposes an asymptotic expansion (in terms of $H - \frac{1}{2}$) of the density of τ_H formally obtained by perturbation analysis techniques.

Theorem 2. *Suppose that b and σ satisfy Hypotheses (H2) and (H3) and let $x_0 < 1$. There exist constants $\lambda_0 \geq 1$, $\mu \geq 0$ (both depending on b and σ only), $\alpha > 0$ and $0 < \eta_0 < \frac{1-x_0}{2}$ such that: for all $\varepsilon \in (0, \frac{1}{4})$ and $0 < \eta \leq \eta_0$, there exists $C_{\varepsilon,\eta} > 0$ such that*

$$\forall \lambda \geq \lambda_0,\ \forall H \in [\tfrac{1}{2}, 1), \quad \left| \mathbb{E}\left(e^{-\lambda \tau_H^X}\right) - \mathbb{E}\left(e^{-\lambda \tau_{\frac{1}{2}}^X}\right) \right| \leq C_{\varepsilon,\eta} (H - \tfrac{1}{2})^{\frac{1}{2}-\varepsilon}\ e^{-\alpha S(1-x_0-2\eta)(\sqrt{2\lambda+\mu^2}-\mu)},$$

where $S(x) = x \wedge x^{\frac{1}{2H}}$. In the pure fBm case (where $b \equiv 0$ and $\sigma \equiv 1$) the result holds with $\lambda_0 = 1$ and $\mu = 0$.

Remark 1. In [17] we extend the preceding result to the case $H < \frac{1}{2}$. The statement, the definition of the stochastic integrals, and technical arguments in the proofs are substantially different from the case $H > \frac{1}{2}$.

In addition to the preceding theorems, we provide accurate estimates on the density of X_t^H with constants which are uniform w.r.t. small and long times and w.r.t. H in $[\frac{1}{2}, 1)$. Our next theorem improves estimates in [2, 5]. Our contributions consists in getting constants which are uniform w.r.t. t in the whole half-line $\mathbb{R}_+ - \{0\}$ and H when H tends to $\frac{1}{2}$.

Theorem 3. *Assume that b and σ satisfy the conditions (H2) and (H3). Then for every $H \in [\frac{1}{2}, 1)$, the density of X^H satisfies: there exists $C(b,\sigma) \equiv C > 0$ such that, for all $t \in \mathbb{R}_+$ and $H \in [\frac{1}{2}, 1)$,*

$$\forall x \in \mathbb{R}, \;\; p_t^H(x) \leq \frac{e^{Ct}}{\sqrt{2\pi t^{2H}}} \, \exp\left(-\frac{(x-x_0)^2}{2Ct^{2H}}\right). \tag{3}$$

Theorems 1–2 are proved in [17]. We do not address the proof of Theorem 3 here.
We sketch the proofs of Theorems 1 and 2 in Section 2. In Section 3 we consider a case which was not tackled in [17], that is, the case $\lambda < 1$. Finally, in Section 4 we show numerical experiment results which illustrate Theorem 2 and suggest that the $(H-\frac{1}{2})^{\frac{1}{2}-}$ rate is sub-optimal.

2 Sketch of the proofs

Under Assumption (H3), the Lamperti transform F is a map such that $F(X^H)$ solves Eq. (1) with coefficients $\tilde{b} = \frac{b \circ F^{-1}}{\sigma \circ F^{-1}}$ and $\sigma(x) \equiv 1$. Since F is one-to-one, we may and do assume in the rest of this paper that $\sigma(x) \equiv 1$. See [17] for more details.

2.1 Reminders on Malliavin calculus

We denote by D and δ the classical derivative and Skorokhod operators of Malliavin calculus w.r.t. Brownian motion on the time interval $[0,T]$ (see e.g. [15]). In the fractional Brownian motion framework the Malliavin derivative D^H is defined as an operator on the smooth random variables with values in the Hilbert space $\mathcal{H}_H$ defined as the completion of the space of step functions on $[0,T]$ with the following scalar product:

$$\langle \varphi, \psi \rangle_{\mathcal{H}_H} := \alpha_H \int_0^T \int_0^T \varphi_s \; \psi_t \; |s-t|^{2H-2} \, \mathrm{d}s\mathrm{d}t < \infty,$$

where $\alpha_H = H(2H-1)$.
The domain of D^H in $L^p(\Omega)$ $(p>1)$ is denoted by $\mathbb{D}^{1,p}$ and is the closure of the space of smooth random variables with respect to the norm:

$$\|F\|_{1,p}^p = \mathbb{E}(|F|^p) + \mathbb{E}\left(\|D^H F\|_{\mathcal{H}_H}^p\right).$$

Equivalently, D^H and δ_H are defined as $D^H := (K_H^*)^{-1}D$ and $\delta_H(u) := \delta(K_H^* u)$ for $u \in (K_H^*)^{-1}(\mathrm{dom}\delta)$ (cf. [15, p.288]), where for any $H \in (\frac{1}{2},1)$ the operator K_H^* is defined as follows: for any φ with suitable integrability properties,

$$K_H^*\varphi(s) = (H-\tfrac{1}{2})c_H \int_s^T \left(\frac{\theta}{s}\right)^{H-\frac{1}{2}} (\theta-s)^{H-\frac{3}{2}}\ \varphi(\theta)\ \mathrm{d}\theta$$

with

$$c_H := \left(\frac{2H\,\Gamma(3/2-H)}{\Gamma(H+\frac{1}{2})\,\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

We denote by $\|\cdot\|_{\infty,[0,T]}$ the sup norm and $\|\cdot\|_\alpha$ the Hölder norm for functions on the interval $[0,T]$.

Let X^H be the solution to (1) with $\sigma(x) \equiv 1$. There exist modifications of the processes X^H and $D^H_\cdot X^H_\cdot$ such that for any $\alpha < H$ it a.s. holds that

$$\left\{\begin{array}{l} \|X^H\|_{\infty,[0,T]} \le C_T(1+|x_0|+\|B^H\|_{\infty,[0,T]}), \\ \|X^H\|_\alpha \le \|B^H\|_\alpha + C_T(1+|x_0|+\|B^H\|_{\infty,[0,T]}), \\ \|D^H_\cdot X^H_\cdot\|_{\infty,[0,T]^2} \le C_T\ , \\ \sup_{r\le t} \frac{|D^H_r X^H_t - 1|}{t-r} \le C_T\ , \forall t\in[0,T]\ . \end{array}\right. \tag{4}$$

These inequalities are simple consequences of the definition of X^H, assumptions (H1) and (H3), and the equality: $D^H_r X^H_t = \mathbf{1}_{\{r\le t\}}\left(1+\int_r^t D^H_r X^H_s b'(X^H_s)ds\right)$ (see Section 3 in [17] for more details).

2.2 Sketch of the proof of Theorem 1

Proving Theorem 1 is easy. A first technique consists in using pathwise estimates on $B^H - B^{1/2}$ with B^H and $B^{1/2}$ defined on the same probability space. A second technique, which we present here in order to introduce the reader to the method of proof for Theorem 2, consists in differentiating $u(t,X^H_t)$ where

$$u(s,x) := \mathbb{E}_x\left(\varphi(X_{t-s})\right),$$

which leads to

$$u(t,X^H_t) = u(0,x_0) + \int_0^t \left(\partial_s u(s,X^H_s) + \partial_x u(s,X^H_s) b(X^H_s)\right)\ \mathrm{d}s + \delta_H\left(\mathbf{1}_{[0,t]}\partial_x u(\cdot,X^H_\cdot)\right) \\ + \alpha_H \int_0^t\int_0^s |r-s|^{2H-2} D^H_r X^H_s\ \partial^2_{xx} u(s,X^H_s)\ \mathrm{d}r\mathrm{d}s.$$

As u solves a parabolic PDE driven by the generator of (X_t) and as the Skorokhod integral has zero mean we get

$$\begin{aligned}\mathbb{E}\varphi(X_t^H) - \mathbb{E}_{x_0}\varphi(X_t) &= \mathbb{E}u(t, X_t^H) - u(0, x_0)\\ &= \mathbb{E}\int_0^t \partial^2_{xx}u(s, X_s^H)\left(Hs^{2H-1} - \tfrac{1}{2}\right)\mathrm{d}s\\ &\quad + \alpha_H \mathbb{E}\int_0^t\int_0^s |r-s|^{2H-2}(D_r^H X_s^H - 1)\partial^2_{xx}u(s, X_s^H)\ \mathrm{d}r\mathrm{d}s.\end{aligned}$$

It then remains to use the estimates (4).

2.3 Sketch of the proof of Theorem 2

We now sketch the proof of Theorem 2. We will soon limit ourselves to the pure fBm case ($b(x) \equiv 0$ and $\sigma \equiv 1$) in order to show the main ideas used in the proof and avoid too heavy technicalities. Recall that, after having used the Lamperti transform, we are reduced to the case $\sigma(x) \equiv 1$.

Our Laplace transforms sensititivity analysis is based on a PDE representation of first hitting time Laplace transforms in the case $H = \frac{1}{2}$.

For $\lambda > 0$ it is well known that

$$\forall x_0 \in (-\infty, 1],\ \mathbb{E}_{x_0}\left(e^{-\lambda \tau_{\frac{1}{2}}}\right) = u_\lambda(x_0),$$

where the function u_λ is the classical solution with bounded continuous first and second derivatives to

$$\begin{cases} 2b(x)u'_\lambda(x) + u''_\lambda(x) = 2\lambda u_\lambda(x),\ x < 1,\\ u_\lambda(1) = 1,\\ \lim_{x\to-\infty} u_\lambda(x) = 0.\end{cases} \tag{5}$$

For any $t \in [0,T]$ the process $\mathbf{1}_{[0,t]}u'_\lambda(B^H_\bullet)\,e^{-\lambda\bullet}$ is in dom $\delta_H^{(T)}$. One thus can apply Itô's formula to $e^{-\lambda t}u_\lambda(X_t^H)$ (see [17, Section 2] and [15]). As u_λ satisfies (5), for any $t \leq T \wedge \tau_H$ we get

$$\begin{aligned} e^{-\lambda t}u_\lambda(X_t^H) = u_\lambda(x_0) &+ \int_0^t e^{-\lambda s}\left(u'_\lambda(X_s^H)b(X_s^H) - \lambda u_\lambda(X_s^H)\right)\ \mathrm{d}s + \delta_H^{(T)}\left(\mathbf{1}_{[0,t]}(\bullet)e^{-\lambda\bullet}u'_\lambda(X_\bullet^H)\right)\\ &+ \alpha_H\int_0^t\int_0^t D_v^H\left(e^{-\lambda s}u'_\lambda(X_s^H)\right)|s-v|^{2H-2}\ \mathrm{d}v\mathrm{d}s\ ,\end{aligned}$$

where the last term corresponds to the Itô term. Using $D_v^H X_s^H = \mathbf{1}_{[0,s]}(v)\left(1 + \int_0^s b'(X_\theta^H)\ D_v^H X_\theta^H\ \mathrm{d}\theta\right)$ and the ODE (5) satisfied by u_λ, we get

$$\begin{aligned} e^{-\lambda t}u_\lambda(X_t^H) = u_\lambda(x_0) &+ \int_0^t \left(\alpha_H \int_0^s |s-v|^{2H-2}\mathrm{d}v - \frac{1}{2}\right) e^{-\lambda s} u_\lambda''(X_s^H)\ \mathrm{d}s \\ &+ \delta_H^{(T)}\left(\mathbf{1}_{[0,t]}(\cdot) e^{-\lambda \cdot} u_\lambda'(X_\cdot^H)\right) \\ &+ \alpha_H \int_0^t \int_0^s e^{-\lambda s} w_\lambda''(X_s^H)\ I(v,s)\ |s-v|^{2H-2}\ \mathrm{d}v\mathrm{d}s, \end{aligned}$$

where $I(v,s) = \mathbf{1}_{\{v\le s\}} \int_v^s b'(X_\theta^H)\ D_v^H X_\theta^H\ \mathrm{d}\theta$. Observe that the last term vanishes for H close to $\frac{1}{2}$, since $\alpha_H |s-v|^{2H-2}$ is an approximation of the identity and $I(v,s)$ converges to 0 as $|v-s| \to 0$. This argument is made rigorous in [17].

We now limit ourselves to the pure fBm case ($b(x) \equiv 0$ **and** $\sigma \equiv 1$**)** to make the rest of the computations more understandable, although the differences will be essentially technical. Given that now, $u_\lambda'(x) = \sqrt{2\lambda}\, u_\lambda(x)$, the previous equality becomes

$$u_\lambda(B_t^H)\ e^{-\lambda t} = u_\lambda(x_0) + \sqrt{2\lambda}\,\delta_H^{(T)}\left(\mathbf{1}_{[0,t]} u_\lambda(B_\cdot^H)\ e^{-\lambda \cdot}\right) + 2\lambda \int_0^t \left(H s^{2H-1} - \tfrac{1}{2}\right)\ u_\lambda(B_s^H)\ e^{-\lambda s}\ \mathrm{d}s.$$

Evaluate the previous equation at $T \wedge \tau_H$, take expectations and let T tend to infinity. For any $\lambda \ge 0$ it comes:

$$\begin{aligned} \mathbb{E}\left(e^{-\lambda \tau_H}\right) - \mathbb{E}\left(e^{-\lambda \tau_{\frac{1}{2}}}\right) &= \mathbb{E}\left[2\lambda \int_0^{\tau_H} (H s^{2H-1} - \tfrac{1}{2}) u_\lambda(B_s^H)\ e^{-\lambda s}\ \mathrm{d}s\right] \qquad (6) \\ &\quad + \sqrt{2\lambda} \lim_{T\to\infty} \mathbb{E}\left[\delta_H^{(T)}\left(\mathbf{1}_{[0,t]} u_\lambda(B_\cdot^H)\ e^{-\lambda \cdot}\right)\Big|_{t=\tau_H \wedge T}\right] \\ &=: I_1(\lambda) + I_2(\lambda). \qquad (7) \end{aligned}$$

Proposition 1. *Let T be the function of $\lambda \in \mathbb{R}_+$ defined by $T(\lambda) = (2\lambda)^{1-\frac{1}{4H}}$ if $\lambda \le 1$ and $T(\lambda) = \sqrt{2\lambda}$ if $\lambda > 1$. There exists a constant $C > 0$ such that*

$$|I_1(\lambda)| \le C\ (H - \tfrac{1}{2})\ e^{-\frac{1}{4} S(1-x_0) T(\lambda)},$$

where S is the function defined in Theorem 2.

Proof (Sketch of proof). From Fubini's theorem, we get

$$I_1(\lambda) = 2\lambda \int_0^{+\infty} (H s^{2H-1} - \tfrac{1}{2}) \mathbb{E}\left[\mathbf{1}_{\{\tau_H \ge s\}} u_\lambda(B_s^H)\right]\ e^{-\lambda s}\ \mathrm{d}s.$$

The inequalities

$$\forall H \in (\tfrac{1}{2}, 1),\ \forall s \in (0,\infty),\ |H s^{2H-1} - \tfrac{1}{2}| \le (H - \tfrac{1}{2})\ (1 \vee s^{2H-1})(1 + 2H|\log s|)$$

and

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_H \ge s\}} u_\lambda(B_s^H)\right] \le \int_{-\infty}^1 u_\lambda(x) \frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \mathrm{d}x = \int_{-\infty}^1 e^{-(1-x)\sqrt{2\lambda}} \frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \mathrm{d}x$$

lead to the desired result.

The above calculation can be extended to diffusions but then accurate estimates on the density of X^H are needed: They are provided by our Theorem 3.

Compared to the proof of Theorem 1, an important difficulty appears when estimating $|I_2(\lambda)|$: as the optional stopping theorem does not hold for Skorokhod integrals of the fBm one has to carefully estimate expectations of stopped Skorokhod integrals and obtain estimates which decrease infinitely fast when λ goes to infinity. We obtained the following result.

Proposition 2.

$$\forall\lambda > 1,\ |I_2(\lambda)| \leq C(H-\tfrac{1}{2})^{\frac{1}{2}-\varepsilon} e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}}. \tag{8}$$

Proof. Proposition 13 of [16] shows that

$$\forall T>0, \quad \mathbb{E}\left(\delta^{(T)}(\mathbf{1}_{[0,t]}(\cdot)u_\lambda(B^H_\cdot)e^{-\lambda\cdot})\Big|_{t=T\wedge\tau_H}\right)=0.$$

Thus $I_2(\lambda)$ satisfies

$$\begin{aligned}
|I_2(\lambda)| &= \sqrt{2\lambda}\,\left|\lim_{N\to\infty}\mathbb{E}\left[\delta_H^{(N)}\left(\mathbf{1}_{[0,t]}(\cdot)u_\lambda(B^H_\cdot)e^{-\lambda\cdot}\right)\Big|_{t=\tau_H\wedge N} - \delta^{(N)}\left(\mathbf{1}_{[0,t]}(\cdot)u_\lambda(B^H_\cdot)e^{-\lambda\cdot}\right)\Big|_{t=\tau_H\wedge N}\right]\right| \\
&= \sqrt{2\lambda}\left|\lim_{N\to\infty}\mathbb{E}\left[\delta^{(N)}\left(\{K_H^*-\mathrm{Id}\}(\mathbf{1}_{[0,t]}(\cdot)u_\lambda(B^H_\cdot)e^{-\lambda\cdot})\right)\Big|_{t=\tau_H\wedge N}\right]\right| \\
&\leq \sqrt{2\lambda}\lim_{N\to\infty}\mathbb{E}\sup_{t\in[0,\tau_H\wedge N]}|\delta^{(N)}\left(\{K_H^*-\mathrm{Id}\}(\mathbf{1}_{[0,t]}(\cdot)u_\lambda(B^H_\cdot)e^{-\lambda\cdot})\right)| \\
&\leq \sqrt{2\lambda}\lim_{N\to\infty}\mathbb{E}\sup_{t\in[0,N]}\left[\mathbf{1}_{\{\tau_H\geq t\}}|\delta^{(N)}\left(\{K_H^*-\mathrm{Id}\}(\mathbf{1}_{[0,t]}(\cdot)u_\lambda(B^H_\cdot)e^{-\lambda\cdot})\right)|\right].
\end{aligned}$$

Define the field $\{U_t(v), t\in[0,N], v\geq 0\}$ and the process $\{\Upsilon_t, t\in[0,N]\}$ by

$$\forall t\in[0,N],\ U_t(v)=\{K_H^*-\mathrm{Id}\}\left(\mathbf{1}_{[0,t]}(\cdot)\ u_\lambda(B^H_\cdot)\ e^{-\lambda\cdot}\right)(v),$$

and

$$\Upsilon_t=\delta^{(N)}(U_t(\cdot)).$$

For any real-valued function f with $f(0)=0$ one has

$$\begin{aligned}
\mathbf{1}_{\{\tau_H\geq t\}}|f(t)| &\leq \mathbf{1}_{\{\tau_H\geq t\}}\sum_{n=0}^{[t]}\sup_{s\in[n,n+1]}\mathbf{1}_{\{\tau_H\geq s\}}|f(s)-f(n)| \\
&\leq \sum_{n=0}^{[t]}\sup_{s\in[n,n+1]}\mathbf{1}_{\{\tau_H\geq s\}}|f(s)-f(n)|.
\end{aligned}$$

Therefore

$$
\begin{aligned}
|I_2(\lambda)| &\leq \sqrt{2\lambda} \lim_{N\to\infty} \mathbb{E} \sup_{t\in[0,N]} \left[\mathbf{1}_{\{\tau_H \geq t\}} |\Upsilon_t|\right] \\
&\leq \sqrt{2\lambda} \lim_{N\to\infty} \sum_{n=0}^{N-1} \mathbb{E} \sup_{t\in[n,n+1]} \left[\mathbf{1}_{\{\tau_H \geq t\}} |\Upsilon_t - \Upsilon_n|\right].
\end{aligned} \tag{9}
$$

Suppose for a while that we have proven: there exists $\eta_0 \in (0, \frac{1-x_0}{2})$ such that for all $\eta \in (0,\eta_0]$ and all $\varepsilon \in (0, \frac{1}{4})$, there exist constants $C, \alpha > 0$ such that

$$
\mathbb{E} \sup_{t\in[n,n+1]} \left[\mathbf{1}_{\{\tau_H \geq t\}} |\Upsilon_t - \Upsilon_n|\right] \leq C\,(H - \tfrac{1}{2})^{\frac{1}{2}-\varepsilon}\, e^{-\frac{1}{3(2+4\varepsilon)}\lambda n} e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}}. \tag{10}
$$

We would then get:

$$
\begin{aligned}
|I_2(\lambda)| &\leq C\,\sqrt{2\lambda} \sum_{n=0}^{\infty} e^{-\frac{\lambda n}{3(2+4\varepsilon)}} (H - \tfrac{1}{2})^{\frac{1}{4}-\varepsilon} e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}} \\
&\leq C\,(H - \tfrac{1}{2})^{\frac{1}{2}-\varepsilon} e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}},
\end{aligned}
$$

which is the desired result (8).

In order to estimate the left-hand side of Inequality (10) we aim to apply Garsia-Rodemich-Rumsey's lemma (see below). However, it seems hard to get the desired estimate by estimating moments of increments of $\mathbf{1}_{\{\tau_H \geq t\}} |\Upsilon_t - \Upsilon_n|$, in particular because $\mathbf{1}_{\{\tau_H \geq t\}}$ is not smooth in the Malliavin sense. We thus proceed by localization and construct a continuous process $\bar{\Upsilon}_t$ which is smooth on the event $\{\tau_H \geq t\}$ and is close to 0 on the complementary event. To this end we introduce the following new notations.

For some small $\eta > 0$ to be fixed set

$$
\forall t \in [0,N],\ \bar{U}_t(v) = \{K_H^* - \mathrm{Id}\} \left(\mathbf{1}_{[0,t]}(\boldsymbol{\cdot})\, u_\lambda(B^H_{\boldsymbol{\cdot}}) \phi_\eta(B^H_{\boldsymbol{\cdot}})\, e^{-\lambda \boldsymbol{\cdot}}\right)(v)
$$

and

$$
\bar{\Upsilon}_t = \delta^{(N)}(\bar{U}_t),
$$

where ϕ_η is a smooth function taking values in $[0,1]$ such that $\phi_\eta(x) = 1,\ \forall x \leq 1$, and $\phi_\eta(x) = 0,\ \forall x > 1+\eta$.

The crucial property of $\bar{\Upsilon}_t$ is the following: For all $n \in \mathbb{N}$ and $n \leq r \leq t < n+1$, $\mathbf{1}_{\{\tau_H \geq t\}} \Upsilon_r = \mathbf{1}_{\{\tau_H \geq t\}} \bar{\Upsilon}_r$ a.s. This is a consequence of the local property of δ ([15, p.47]). Therefore, for any $n \leq N-1$,

$$
\mathbb{E}\left(\sup_{t\in[n,n+1]} \mathbf{1}_{\{\tau_H \geq t\}} |\Upsilon_t - \Upsilon_n|\right) = \mathbb{E}\left(\sup_{t\in[n,n+1]} \mathbf{1}_{\{\tau_H \geq t\}} |\bar{\Upsilon}_t - \bar{\Upsilon}_n|\right) \leq \mathbb{E}\left(\sup_{t\in[n,n+1]} |\bar{\Upsilon}_t - \bar{\Upsilon}_n|\right). \tag{11}
$$

Recall the Garsia-Rodemich-Rumsey lemma: if X is a continuous process, then for $p \geq 1$ and $q > 0$ such that $pq > 2$, one has

$$\mathbb{E}\left(\sup_{t\in[a,b]}|X_t-X_a|\right)\leq C\frac{pq}{pq-2}(b-a)^{q-\frac{2}{p}}\,\mathbb{E}\left[\left(\int_a^b\int_a^b\frac{|X_s-X_t|^p}{|t-s|^{pq}}\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{p}}\right]$$
$$\leq C\frac{pq}{pq-2}(b-a)^{q-\frac{2}{p}}\left(\int_a^b\int_a^b\frac{\mathbb{E}\left(|X_s-X_t|^p\right)}{|t-s|^{pq}}\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{p}} \quad (12)$$

provided the right-hand side in each line is finite. In order to apply (12), we thus need to estimate moments of $\bar{\Upsilon}_t-\bar{\Upsilon}_s$. Lemmas 1 and Lemmas 2 below give bounds on the moments of $\bar{\Upsilon}_t-\bar{\Upsilon}_s$ in terms of a power of $|t-s|$. Thus Kolmogorov's continuity criterion implies that $\bar{\Upsilon}$ has a continuous modification, which justifies to apply the GRR lemma to $\bar{\Upsilon}$.

In addition, we can easily obtain bounds on the norm $\left\|\bar{\Upsilon}_t-\bar{\Upsilon}_s\right\|_{L^2(\Omega)}$ in terms of $(H-\frac{1}{2})$. This observation leads us to notice that

$$\mathbb{E}\left(|\bar{\Upsilon}_s-\bar{\Upsilon}_t|^{2+4\varepsilon}\right)\leq\left\|\bar{\Upsilon}_t-\bar{\Upsilon}_s\right\|_{L^2(\Omega)}\times\mathbb{E}\left(|\bar{\Upsilon}_t-\bar{\Upsilon}_s|^{2+8\varepsilon}\right)^{\frac{1}{2}}.$$

We then combine Lemmas 1 and 2 below to obtain: For every $[n\leq s\leq t\leq n+1]$,

$$\begin{aligned}\mathbb{E}\left(|\bar{\Upsilon}_s-\bar{\Upsilon}_t|^{2+4\varepsilon}\right)&\leq C\,(H-\tfrac{1}{2})(t-s)^{\frac{1}{2}-\varepsilon}\;e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}}\\&\quad\times(t-s)^{\frac{1}{2}+2\varepsilon}\;e^{-\frac{1}{3}\lambda s}e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}}\\&\leq C\,(H-\tfrac{1}{2})\;(t-s)^{1+\varepsilon}\;e^{-\frac{1}{3}\lambda s}e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}}.\end{aligned}$$

Choosing $p=2+4\varepsilon$ and $q=\frac{2+\varepsilon/2}{2+4\varepsilon}$ we thus get

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[n,n+1]}\mathbf{1}_{\{\tau_H\geq t\}}|\Upsilon_t-\Upsilon_n|\right)&\leq C\,(H-\tfrac{1}{2})^{\frac{1}{2+4\varepsilon}}\;e^{-\frac{\alpha}{2+4\varepsilon}S(1-x_0-2\eta)\sqrt{2\lambda}}\\&\qquad\left(\int_n^{n+1}\int_s^{n+1}e^{-\frac{1}{3}\lambda s}(t-s)^{\frac{\varepsilon}{2}-1}\,\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{2+4\varepsilon}}\\&\leq C\,(H-\tfrac{1}{2})^{\frac{1}{2+4\varepsilon}}\;e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}}e^{-\frac{1}{3(2+4\varepsilon)}\lambda n},\end{aligned}$$

from which Inequality (10) follows.

It now remains to prove the above estimates on $\left\|\bar{\Upsilon}_t-\bar{\Upsilon}_s\right\|_{L^2(\Omega)}$ and $\mathbb{E}\left(|\bar{\Upsilon}_t-\bar{\Upsilon}_s|^{2+8\varepsilon}\right)^{\frac{1}{2}}$: These estimates are provided by Lemmas 1 and 2 below whose proofs are very technical.

Lemma 1. *There exists $\eta_0\in(0,\frac{1-x_0}{2})$ such that: for all $0<\eta\leq\eta_0$, for all $H\in[\frac{1}{2},1)$ and for all $0<\varepsilon<\frac{1}{4}$, there exist $C,\alpha>0$ such that*

$$\forall\lambda\geq1,\;\forall0\leq n\leq s\leq t\leq n+1\leq N,$$
$$\mathbb{E}\left(|\bar{\Upsilon}_t-\bar{\Upsilon}_s|^{2+8\varepsilon}\right)^{\frac{1}{2}}\leq C\,(t-s)^{\frac{1}{2}+2\varepsilon}\;e^{-\frac{1}{3}\lambda s}e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}}\;,$$

where the function S is defined as in Theorem 2.

Lemma 2. *There exists* $\eta_0 \in (0, \frac{1-x_0}{2})$ *such that: For all* $0 < \eta \leq \eta_0$ *and* $0 < \varepsilon < \frac{1}{4}$*, there exist* $C, \alpha > 0$ *such that*

$$\forall n \in [0,N],\ \forall H \in [\tfrac{1}{2},1),\ \forall n \leq s \leq t \leq n+1,\ \forall \lambda \geq 1,$$
$$\left\|\bar{\Upsilon}_t - \bar{\Upsilon}_s\right\|_{L^2(\Omega)} \leq C\,(H-\tfrac{1}{2})(t-s)^{\frac{1}{2}-\varepsilon}\ e^{-\alpha S(1-x_0-2\eta)\sqrt{2\lambda}}.$$

3 Discussion on the fBm case with $\lambda < 1$

We believe that Theorem 2 also holds true for $\lambda \in (0,1]$. One of the main issues consists in getting accurate enough bounds on the right-hand side of Inequality (9).

For $a_\lambda = \lambda^{-\frac{1}{2H}}$ and $b_\lambda = \frac{-\log\sqrt{\lambda}}{\lambda}$ $(\lambda < 1)$ we have

$$\begin{aligned}
|I_2(\lambda)| \leq & \sqrt{2\lambda}\,\mathbb{E}\left[\sup_{t\in[0,a_\lambda]} \mathbf{1}_{\{\tau_H\geq t\}}\left|\delta\left(\{K_H^*-\mathrm{Id}\}(\mathbf{1}_{[0,t]}u_\lambda(B_\bullet^H)e^{-\lambda\bullet})\right)\right|\right]\\
&+\sqrt{2\lambda}\,\mathbb{E}\left[\sup_{t\in[a_\lambda,b_\lambda]} \mathbf{1}_{\{\tau_H\geq t\}}\left|\delta\left(\{K_H^*-\mathrm{Id}\}(\mathbf{1}_{[a_\lambda,t]}u_\lambda(B_\bullet^H)e^{-\lambda\bullet})\right)\right|\right]\\
&+\sqrt{2\lambda}\lim_{N\to+\infty}\mathbb{E}\left[\sup_{t\in[b_\lambda,N]} \mathbf{1}_{\{\tau_H\geq t\}}\left|\delta\left(\{K_H^*-\mathrm{Id}\}(\mathbf{1}_{[b_\lambda,t]}u_\lambda(B_\bullet^H)e^{-\lambda\bullet})\right)\right|\right].
\end{aligned}$$

We here limit ourselves to examine the second summand on the r.h.s and we denote it by $I_2^{(2)}(\lambda)$. The two other terms (corresponding to $t < a_\lambda$ and $t > b_\lambda$) are easier to study.

Compared to Subsection 2.3 we localize the Skorokhod integral in a slightly different manner by using $\phi_\eta(S_t^H)$ instead of $\phi_\eta(B_t^H)$, where S_t^H denotes the running supremum of the fBm up to time t. Hence

$$\begin{aligned}
&\mathbf{1}_{\{\tau_H\geq t\}}\delta\left(\{K_H^*-\mathrm{Id}\}\left(\mathbf{1}_{[0,t]}u_\lambda(B_\bullet^H)e^{-\lambda\bullet}\right)\right)\\
&\qquad= \mathbf{1}_{\{\tau_H\geq t\}}\delta\left(\{K_H^*-\mathrm{Id}\}\left(\mathbf{1}_{[0,t]}u_\lambda(B_\bullet^H)\phi_\eta(S_\bullet^H)e^{-\lambda\bullet}\right)\right) \text{ a.s.}
\end{aligned}$$

Set $\bar{V}_\lambda(s) := u_\lambda(B_s^H)\phi_\eta(S_s^H)$ and

$$\tilde{\Upsilon}_t := \delta\left(\{K_H^*-\mathrm{Id}\}\left(\mathbf{1}_{[0,t]}\bar{V}_\lambda(\cdot)e^{-\lambda\bullet}\right)\right).$$

Proceeding as from Eq.(11) to Eq.(12) we get for some $p > 1$ and $m > 0$ (chosen later):

$$\mathbb{E}\left(\sup_{t\in[a_\lambda,b_\lambda]}\mathbf{1}_{\{\tau_H\geq t\}}|\delta_H\left(\mathbf{1}_{[0,t]}u_\lambda(B^H_\bullet)e^{-\lambda\bullet}\right)|\right)\leq\mathbb{P}(\tau_H\geq a_\lambda)^{\frac{p-1}{p}}C(b_\lambda-a_\lambda)^{\frac{m}{p}}$$
$$\times\left(\int_{a_\lambda}^{b_\lambda}\int_{a_\lambda}^{b_\lambda}\frac{\mathbb{E}\left(|\tilde{\Upsilon}_t-\tilde{\Upsilon}_s|^p\right)}{|t-s|^{m+2}}\,\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}. \tag{13}$$

We then use the proposition 3.2.1 in [15] to bound $\mathbb{E}|\tilde{\Upsilon}_t-\tilde{\Upsilon}_s|^p$:

$$\mathbb{E}|\tilde{\Upsilon}_t-\tilde{\Upsilon}_s|^p\leq C(t-s)^{\frac{p}{2}-1}\int_s^t|\mathbb{E}\left(\bar{V}_\lambda(r)e^{-\lambda r}\right)|^p + \mathbb{E}\left[\left(\int_0^{b_\lambda}|D_\theta\bar{V}_\lambda(r)e^{-\lambda r}|^2\,\mathrm{d}\theta\right)^{\frac{p}{2}}\right]\mathrm{d}r. \tag{14}$$

The Malliavin derivative of the supremum of the fBm is obtained for example in [7]. Denoting by ϑ_r the first time at which B^H reaches S^H_r on the interval $[0,r]$ we have $D^H_\theta S^H_r=\mathbf{1}_{\{\vartheta_r>\theta\}}$. It follows that $D_\theta S^H_r=K_H(\vartheta_r,\theta)$. Since $D_\theta\bar{V}_\lambda(r)=\phi_\eta(S^H_r)D_\theta u_\lambda(B^H_r)+u_\lambda(B^H_r)D_\theta\phi_\eta(S^H_r)$, we are led to study the three following terms (for $p>2$):

(i) $\mathbb{E}\left(\bar{V}_\lambda(r)e^{-\lambda r}\right)\leq\mathbb{E}\left(\phi_\eta(S^H_r)\right)\leq\mathbb{P}(S^H_r\leq 1+\eta)$.

(ii) $e^{-p\lambda r}\mathbb{E}\left[\left(\int_0^{b_\lambda}|\phi_\eta(S^H_r)D_\theta u_\lambda(B^H_r)|^2\,\mathrm{d}\theta\right)^{\frac{p}{2}}\right]$
$\leq\mathbb{E}\left[\mathbf{1}_{\{S^H_r\leq 1+\eta\}}\left(\int_0^r|\sqrt{2\lambda}K_H(r,\theta)u_\lambda(B^H_r)|^2\,\mathrm{d}\theta\right)^{\frac{p}{2}}\right]$
$=(\sqrt{2\lambda})^p\ r^{pH}\ \mathbb{E}(\mathbf{1}_{\{S^H_r\leq 1+\eta\}}u_\lambda(B^H_r)^p)$.

(iii) $e^{-p\lambda r}\mathbb{E}\left[\left(\int_0^{b_\lambda}|u_\lambda(B^H_r)D_\theta\phi_\eta(S^H_r)|^2\,\mathrm{d}\theta\right)^{\frac{p}{2}}\right]$
$\leq\mathbb{E}\left[\phi'_\eta(S^H_r)^p\ \vartheta_r^{Hp}\right]\leq\|\phi'_\eta\|^p_\infty\mathbb{E}\left[\mathbf{1}_{\{S^H_r\leq 1+\eta\}}\vartheta_r^{Hp}\right]$.

We do not know any accurate estimate on the joint law of either $(S^H_\bullet,B^H_\bullet)$ or $(S^H_\bullet,\vartheta_\bullet)$. We thus can only use the rough bounds $\mathbf{1}_{\{S^H_r\leq 1+\eta\}}u_\lambda(B^H_r)\leq C\mathbf{1}_{\{S^H_r\leq 1+\eta\}}$ for (ii) and $\vartheta_r\leq r$ for (iii). Then one is in a position to use the following refinement of Molchan's asymptotic [14] obtained by Aurzada [1]: $\mathbb{P}(\tau_H\geq t)\leq t^{-(1-H)}(\log t)^c$ for some constant $c>0$. However, when plugged into (14) and then into (13), these bounds lead us to an upper bound for $|I_2^{(2)}(\lambda)|$ which diverges when $\lambda\to 0$.

Hence the preceding rough bounds on (ii) and (iii) must be improved. In the Brownian motion case, the joint laws of $(B_r,S_r^{\frac{1}{2}})$ and $(\vartheta_r,S_r^{\frac{1}{2}})$ are known (see e.g. [12, p.96–102]). In particular, for $p\in(2,3)$ the term (iii) leads to

$$\forall r\geq 0,\quad \mathbb{E}\left[\mathbf{1}_{\{S_r^{1/2}\leq 1+\eta\}}\vartheta_r^{\frac{p}{2}}\right]\leq C \tag{15}$$

instead of the bound $r^{\frac{p}{2}-\frac{1}{2}}(\log t)^c$ when one uses the previous rough method.

From numerical simulations and an incomplete mathematical analysis using arguments developed by [14] and [1] we believe that Inequality (15) remains true for $H > \frac{1}{2}$. If so, the bound on $|I_2^{(2)}(\lambda)|$ would become

$$|I_2^{(2)}(\lambda)| \le C\sqrt{2\lambda}\, a_\lambda^{-(1-H)\frac{p-1}{p}}\ (b_\lambda - a_\lambda)^{\frac{1}{2}},$$

which, in view of $a_\lambda = \lambda^{-\frac{1}{2H}}$ and $b_\lambda = \frac{-\log\sqrt{\lambda}}{\lambda}$, can now be bounded as $\lambda \to 0$.

4 Optimal rate of convergence in Theorem 2: Comparison with numerical results

In this section, we numerically approximate the quantity $\mathcal{L}(H,\lambda) = \mathbb{E}\left[e^{-\lambda\tau_H}\right]$, where τ_H is the first time a fractional Brownian motion started from 0 hits 1.
As already recalled this Laplace transform is explictely known in the Brownian case: $\mathcal{L}(\frac{1}{2},\lambda) = e^{-\sqrt{2\lambda}}$, $\forall\lambda \ge 0$. Our simulations suggest that the convergence of $\mathcal{L}(H,\lambda)$ towards $\mathcal{L}(\frac{1}{2},\lambda)$ is faster than what we were able to prove. We also show numerical experiments which concern the convergence of hitting time densities.

Although several numerical schemes permit to decrease the weak error when estimating $\tau_{\frac{1}{2}}$, none seem to be available in the fractional Brownian motion case. We thus propose a heuristic extension of the bridge correction of Gobet [9] (valid in the Markov case) and compare this procedure to the standard Euler scheme.

Convergence of $\mathbb{E}\left[e^{-\lambda\tau_H}\right]$ to $\mathbb{E}\left[e^{-\lambda\tau_{\frac{1}{2}}}\right]$.

Let us fix a time horizon T and N points on each trajectory. Let $\delta = \frac{T}{N}$ be the time step. Denote by M the number of Monte-Carlo samples. For each $m \in \{1,\dots,M\}$, we simulate $\{B^{H,N}_{n\delta}(m)\}_{1\le n\le N}$, from which we obtain

$$\tau_H^{\delta,T}(m) = \inf\{n\delta : B^{H,N}_{n\delta}(m) > 1\}.$$

We then approximate $\mathcal{L}(H,\lambda)$ as follows:

$$\mathcal{L}(H,\lambda) \approx \frac{1}{M}\sum_{m=1}^{M} e^{-\lambda\tau_H^{\delta,T}(m)} =: \mathcal{L}^{\delta,T,M}(H,\lambda)\ .$$

The bias $\tau_H^{\delta,T}(m) \ge \tau_H(m)$ due to the time discretization implies $\lim_{M\to\infty}\mathcal{L}^{\delta,T,M}(H,\lambda) \le \mathcal{L}(H,\lambda)$.

In view of Theorem 2 we have

$$\log\left|\mathcal{L}(H,\lambda) - \mathcal{L}(\tfrac{1}{2},\lambda)\right| \le C_\lambda + \beta\ \log(H - \frac{1}{2})\ ,$$

with $\beta = (\frac{1}{4} - \varepsilon)$. We approximate $\log\left|\mathcal{L}(H,\lambda) - \mathcal{L}(\frac{1}{2},\lambda)\right|$ by $\log\left|\mathcal{L}^{\delta,T,M}(H,\lambda) - \mathcal{L}(\frac{1}{2},\lambda)\right|$ for several values of H close to $\frac{1}{2}$ and then perform a linear regression analysis around $\log(H-\frac{1}{2})$. The slope of the regression line provides a hint on the optimal value of β.

The global error $|\mathcal{L}(H,1)-\mathcal{L}^{\delta,T,M}(H,1)|$ results from the discretization error $\mathrm{err}(\delta)$ and the statistical error $\mathrm{err}(M)$. For $M=2^{13}$ and $\delta=3.10^{-4}$ the estimator of the standard deviation of $\mathcal{L}^{\delta,T,M}(H,\lambda)$ is 0.259. This allows to decrease the number of simulations to 100,000 to have a statistical error of order 0.0016.

The numerical results are presented in Table 1 for several values of $\lambda(=1,2,3,4)$ and of the parameter $H\in\{0,5;0,51;0,52;0,54;0,6\}$. These results suggest that $|\mathcal{L}^{\delta,T,M}(\frac{1}{2},\lambda)-\mathcal{L}^{\delta,T,M}(H,\lambda)|$ is linear w.r.t. $(H-\frac{1}{2})$. For each λ we thus perform a linear regression on these quantities (without the above log transformation). The regression line is plotted in Fig. 1.

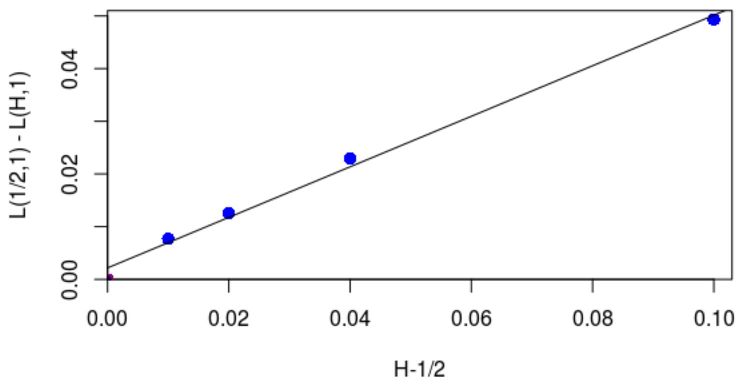

Fig. 1 Regression of $\mathcal{L}(\frac{1}{2},1)-\mathcal{L}(H,1)$ against $H-\frac{1}{2}$ using the values from Table 1.

Our numerical results suggest that Theorem 2 is not optimal but the optimal convergence rate seems hard to get. An even more difficult result to obtain concerns the convergence rate of the density of the first hitting time of fBm to the density of the first hitting time of Brownian motion. We analyze it numerically: See Fig. 2.

Brownian bridge correction. We apply the following rule (which is only heuristic when $H>\frac{1}{2}$): at each time step, if the threshold has not yet been hit and if $B^{H,N}_{(n-1)\delta}(m)<1$ and $B^{H,N}_{n\delta}(m)<1$, we sample a uniform random variable U on $[0,1]$ and compare it to

$$p_H=\exp\left\{-2\frac{\left(1-B^{H,N}_{(n-1)\delta}(m)\right)\left(1-B^{H,N}_{n\delta}(m)\right)}{\delta^{2H}}\right\}.$$

If $U<p_H$ then decide $\tau^{\delta,T}_H(m)=n\delta$. Otherwise let the algorithm continue. In the sequel we denote by $\widetilde{\mathcal{L}}^{\delta,T,M}(H,\lambda)$ the corresponding Laplace transform. This algorithm is an adaptation to our non-Markovian framework of the algorithm of [9]

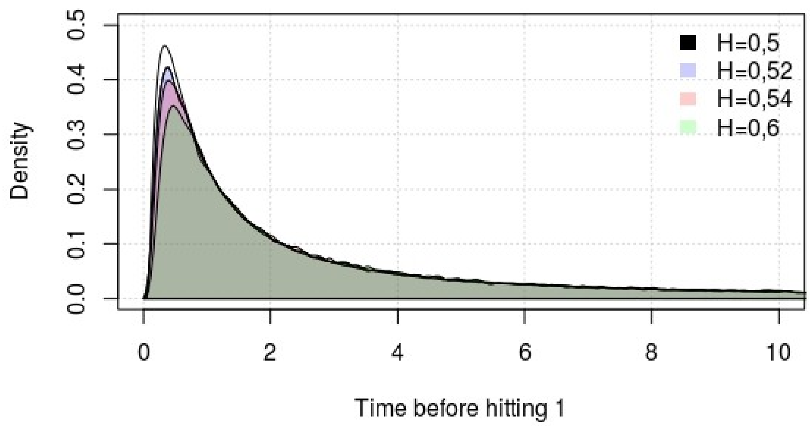

Fig. 2 Density of τ_H for several values of H

which is fully justified when $H = \frac{1}{2}$. In particular $p_{\frac{1}{2}}$ is the exact probability that a Brownian motion conditioned by its values at time $(n-1)\delta$ and $n\delta$ crosses 1 in the time interval $[(n-1)\delta, n\delta]$. Here, we approximate the unknown value of p_H by a heuristic value which coincides with $p_{\frac{1}{2}}$ when $H = \frac{1}{2}$.

Table 2 shows the corresponding results for the simple estimator $\mathcal{L}^{\delta_0,T,M}(\frac{1}{2},\lambda)$ and the Brownian Bridge estimator $\widetilde{\mathcal{L}}^{\delta_1,T,M}(\frac{1}{2},\lambda)$ with $\delta_0 < \delta_1$ in the Brownian case (we kept $M = 10^5$). Consistently with theoretical results, Table 2 shows that the estimator $\widetilde{\mathcal{L}}^{\delta,T,M}(H,\lambda)$ allows to substantially reduce the number of discretization steps (thus the computational time) to get a desired accuracy. The figure also shows a reasonable choice of δ_1 which we actually keep when tackling the fractional Brownian motion case.

The exact value $\mathcal{L}(H,\lambda)$ is unknown. Our reference value is the lower bound $\mathcal{L}^{\delta_0,T,M}(H,\lambda)$. The parameter δ_1 used in Table 3 allows to conjecture that the Brownian bridge correction is useful even in the non-Markovian case. Although the approximation errors of the estimators $\mathcal{L}^{\delta_1,T,M}$ and $\widetilde{\mathcal{L}}^{\delta_1,T,M}$ are similar when compared to $\mathcal{L}^{\delta_0,T,M}(H,\lambda)$, we recommend to use the latter because we have $\mathcal{L}^{\delta_1,T,M}(H,\lambda) \leq \mathcal{L}^{\delta_0,T,M}(H,\lambda) \leq \mathcal{L}(H,\lambda)$ whereas $\mathcal{L}^{\delta_0,T,M}(H,\lambda) \leq \widetilde{\mathcal{L}}^{\delta_1,T,M}(H,\lambda)$.

Appendix: tables

Table 1 Values of $\Delta_H = \mathbb{E}\left[e^{-\lambda\tau_{\frac{1}{2}}}\right] - \mathbb{E}\left[e^{-\lambda\tau_H}\right]$ when $H \to \frac{1}{2}$.
Set of parameters: $T = 20,\ N = 2^{16}\ (\delta \approx 3.10^{-4}),\ M = 10^5$

H	$\lambda = 1$ $\mathcal{L}^{\delta,T,M}(H,\lambda)$	Δ_H	$\lambda = 2$ $\mathcal{L}^{\delta,T,M}(H,\lambda)$	Δ_H	$\lambda = 3$ $\mathcal{L}^{\delta,T,M}(H,\lambda)$	Δ_H	$\lambda = 4$ $\mathcal{L}^{\delta,T,M}(H,\lambda)$	Δ_H
0,50	0,2400	–	0,1329	–	0,0846	–	0,0578	–
0,51	0,2323	0,0077	0,1271	0,0059	0,0800	0,0046	0,0542	0,0037
0,52	0,2275	0,0125	0,1232	0,0098	0,0769	0,0077	0,0517	0,0061
0,54	0,2171	0,0229	0,1149	0,0180	0,0703	0,0143	0,0464	0,0114
0,60	0,1907	0,0493	0,0958	0,0372	0,0560	0,0286	0,0354	0,0224

Table 2 Test case: Error estimation of our procedure in the Brownian case ($H = \frac{1}{2}$)
Set of parameters: $T = 20,\ N = 2^{16}\ (\delta_0 \approx 3.10^{-4}),\ M = 10^5$ for the simple estimator
$T = 20,\ N = 2^{15}\ (\delta_1 \approx 6.10^{-4}),\ M = 10^5$ for the Bridge estimator

λ	$\mathcal{L}(\frac{1}{2},\lambda)$	$\mathcal{L}^{\delta,T,M}(\frac{1}{2},\lambda)$	Error (%)	$\widetilde{\mathcal{L}}^{\delta,T,M}(\frac{1}{2},\lambda)$	Error (%)
1	0,2431	0,2400	1,3	0,2438	0,3
2	0,1353	0,1329	1,7	0,1358	0,4
3	0,0863	0,0846	2,0	0,0867	0,5
4	0,0591	0,0578	2,2	0,0594	0,5

Table 3 Comparison of estimators in the fractional case ($H = 0,54$)
Set of parameters: $T = 20,\ N = 2^{16}\ (\delta_0 \approx 3.10^{-4}),\ M = 10^5$ for the simple estimator
$T = 20,\ N = 2^{15}\ (\delta_1 \approx 6.10^{-4}),\ M = 10^5$ for the simple estimator
$T = 20,\ N = 2^{15}\ (\delta_1 \approx 6.10^{-4}),\ M = 10^5$ for the Bridge estimator

λ	$\mathcal{L}^{\delta_0,T,M}(H,\lambda)$	$\mathcal{L}^{\delta_1,T,M}(H,\lambda)$	Error (%)	$\widetilde{\mathcal{L}}^{\delta_1,T,M}(H,\lambda)$	Error (%)
1	0,2171	0,2147	1,1	0,2186	0,7
2	0,1149	0,1131	1,6	0,1165	1,4
3	0,07003	0,0689	2,0	0,0717	1,9
4	0,0464	0,0453	2,3	0,0476	2,5

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