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ERGODICITY OF INHOMOGENEOUS MARKOV CHAINS THROUGH ASYMPTOTIC PSEUDOTRAJECTORIES[1]

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In this work, we consider an inhomogeneous (discrete time) Markov chain and are interested in its long time behavior. We provide sufficient conditions to ensure that some of its asymptotic properties can be related to the ones of a homogeneous (continuous time) Markov process. Renowned examples such as a bandit algorithms, weighted random walks or decreasing step Euler schemes are included in our framework. Our results are related to functional limit theorems, but the approach differs from the standard "Tightness/Identification" argument; our method is unified and based on the notion of pseudotrajectories on the space of probability measures.

1. Introduction. In this paper, we consider an inhomogeneous Markov chain $(y_n)_{n \geq 0}$ on $\mathbb{R}^D$, and a nonincreasing sequence $(\gamma_n)_{n \geq 1}$ converging to 0, such that $\sum_{n=1}^{\infty} \gamma_n = +\infty$. For any smooth function f, we set

$$(1.1) \qquad \mathcal{L}_n f(y) := \frac{\mathbb{E}[f(y_{n+1}) - f(y_n) \mid y_n = y]}{\gamma_{n+1}}.$$

We shall establish general asymptotic results when $\mathcal{L}_n$ converges, in some sense explained below, toward some infinitesimal generator $\mathcal{L}$. We prove that, under reasonable hypotheses, one can deduce properties (trajectories, ergodicity, etc.) of $(y_n)_{n \geq 1}$ from the ones of a process generated by $\mathcal{L}$.

This work is mainly motivated by the study of the rescaling of stochastic approximation algorithms (see, e.g., [4, 33]). Classically, such rescaled algorithms converge to Normal distributions (or linear diffusion processes); see, for example, [15, 18, 30]. This central limit theorem is usually proved with the help of "Tightness/Identification" methods. With the same structure of proof, Lamberton and

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Pagès get a different limit in [32]; namely, they provide a convergence to the stationary measure of a nondiffusive Markov process. Closely related, the decreasing step Euler scheme (as developed in [31, 34]) behaves in the same way.

In contrast to this classical approach, we rely on the notion of asymptotic pseudotrajectories introduced in [5]. Therefore, we focus on the asymptotic behavior of $\mathcal{L}_n$ using Taylor expansions to deduce immediately the form of a limit generator $\mathcal{L}$. A natural way to understand the asymptotic behavior of $(y_n)_{n \geq 0}$ is to consider it as an approximation of a Markov process generated by $\mathcal{L}$. Then, provided that the limit Markov process is ergodic and that we can estimate its speed of convergence toward the stationary measure, it is natural to deduce convergence and explicit speeds of convergence of $(y_n)_{n \geq 0}$ toward equilibrium. Our point of view can be related to the Trotter–Kato theorem (see, e.g., [28]). The proof of our main theorem, Theorem 2.6 below, is related to Lindeberg's proof of the central limit theorem; namely, it is based on a telescopic sum and a Taylor expansion.

With the help of Theorem 2.6, the study of the long time behavior of $(y_n)_{n \geq 0}$ reduces to the one of a homogeneous-time Markov process. Their convergence has been widely studied in the literature, and we can differentiate several approaches. For instance, there are so-called "Meyn-and-Tweedie" methods (or Foster–Lyapunov criteria, see [12, 23, 24, 36]) which provide qualitative convergence under mild conditions; we can follow this approach to provide qualitative properties for our inhomogeneous Markov chain. However, the speed is usually not explicit or very poor. Another approach consists in the use of *ad hoc* coupling methods (see, e.g., [7, 16, 35]) either for a diffusion or a piecewise deterministic Markov process (PDMP). Those methods usually prove themselves to be efficient for providing explicit speeds of convergence, but rely on extremely particular strategies. Among other approaches, let us also mention functional inequalities or spectral gap methods (see, e.g., [1, 2, 11, 37]).

In this article, we develop a unified approach to study the long time behavior of inhomogeneous Markov chains, which may also provide speeds of convergence or functional convergence. To our knowledge, this method is original, and Theorems 2.6 and 2.8 have the advantage of being self-contained. The main goal of our illustrations, in Section 3, is to provide a simple framework to understand our approach. For these examples, proofs seem more simple and intuitive, and we are able to recover classical results as well as slight improvements.

This paper is organized as follows. In Section 2, we state the framework and the main assumptions that will be used throughout the paper. We recall the notion of asymptotic pseudotrajectory, and present our main result, Theorem 2.6, which describes the asymptotic behavior of a Markov chain. We also provide two consequences, Theorems 2.8 and 2.12, precising the geometric ergodicity of the chain or its functional convergence. In Section 3, we illustrate our results by showing how some renowned examples, including weighted random walks, bandit algorithms or decreasing step Euler schemes, can be easily studied with this unified approach. In

Section 4 and Appendix, we provide the proofs of our main theorems and of the technical parts left aside while dealing with the illustrations.

2. Main results.

2.1. *Framework.* We shall use the following notation in the sequel:

- A multi-index is a D-tuple $N = (N_1, \ldots, N_D) \in \mathbb{N}^D$; we define the order $N \leq \widetilde{N}$ if, for all $1 \leq i \leq D$, $N_i \leq \widetilde{N}_i$. We define $|N| = \sum_{i=1}^{D} N_i$ and and we identify an integer N with the multi-index $(N, \ldots, N)$.
- For some multi-index N, $\mathscr{C}^N$ is the set of functions $f : \mathbb{R}^D \to \mathbb{R}$ which are N_i times continuously differentiable in the direction i. For any $f \in \mathscr{C}^N(\mathbb{R}^D)$, we define

$$f^{(N)} = \frac{\partial^{|N|}}{\partial_{x_1}^{N_1} \cdots \partial_{x_D}^{N_D}} f, \qquad \|f^{(N)}\|_\infty = \sup_{x \in \mathbb{R}^D} |f^{(N)}(x)|.$$

- $\mathscr{C}_b^N$ is the set of $\mathscr{C}^N$ functions such that $\sum_{j \leq N} \|f^{(j)}\|_\infty < +\infty$. Also, $\mathscr{C}_c^N$ is the set of $\mathscr{C}^N$ functions with compact support, and $\mathscr{C}_0^N$ is the set of $\mathscr{C}^N$ functions such that $\lim_{\|x\| \to \infty} f(x) = 0$.
- $\mathscr{L}(X)$ is the law of a random variable X and $\mathrm{Supp}(\mathscr{L}(X))$ its support.
- $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$ for any $x, y \in \mathbb{R}$.
- For some multi-index N, $\chi_N(x) := \sum_{i=1}^{D} \sum_{k=0}^{N_i} |x_i|^k$ for $x \in \mathbb{R}^D$.

Let us recall some basics about Markov processes. Given a homogeneous Markov process $(X_t)_{t \geq 0}$ with càdlàg trajectories a.s., we define its Markov semigroup $(P_t)_{t \geq 0}$ by

$$P_t f(x) = \mathbb{E}[f(X_t) \mid X_0 = x].$$

It is said to be Feller if, for all $f \in \mathscr{C}_0^0$, $P_t f \in \mathscr{C}_0^0$ and $\lim_{t \to 0} \|P_t f - f\|_\infty = 0$. We can define its generator $\mathcal{L}$ acting on functions f satisfying $\lim_{t \to 0} \|t^{-1}(P_t f - f) - \mathcal{L}f\|_\infty = 0$. The set of such functions is denoted by $\mathcal{D}(\mathcal{L})$, and is dense in $\mathscr{C}_0^0$; see, for instance, [17]. The semigroup property of (P_t) ensures the existence of a semiflow:

(2.1) $$\Phi(\nu, t) := \nu P_t,$$

defined for any probability measure ν and $t \geq 0$; namely, for all $s, t > 0$, $\Phi(\nu, t + s) = \Phi(\Phi(\nu, t), s)$.

Let $(y_n)_{n \geq 0}$ be a (inhomogeneous) Markov chain and let $(\mathcal{L}_n)_{n \geq 0}$ be a sequence of operators satisfying, for $f \in \mathscr{C}_b^0$,

$$\mathcal{L}_n f(y_n) := \frac{\mathbb{E}[f(y_{n+1}) - f(y_n) \mid y_n]}{\gamma_{n+1}},$$

where $(\gamma_n)_{n\geq 1}$ is a decreasing sequence converging to 0, such that $\sum_{n=1}^{\infty} \gamma_n = +\infty$. Note that the sequence $(\mathcal{L}_n)$ exists thanks to Doob's lemma. Let (τ_n) be the sequence defined by $\tau_0 := 0$ and $\tau_n := \sum_{k=1}^{n} \gamma_k$, and let $m(t) := \sup\{n \geq 0 : t \geq \tau_n\}$ be the unique integer such that $\tau_{m(t)} \leq t < \tau_{m(t)+1}$. We denote by (Y_t) the process defined by $Y_t := y_n$ when $t \in [\tau_n, \tau_{n+1})$ and we set

(2.2) $$\mu_t := \mathcal{L}(Y_t).$$

Following [4, 5], we say that $(\mu_t)_{t\geq 0}$ is an asymptotic pseudotrajectory of Φ (with respect to a distance d over probability distributions) if, for any $T > 0$,

(2.3) $$\lim_{t\to\infty} \sup_{0\leq s\leq T} d(\mu_{t+s}, \Phi(\mu_t, s)) = 0.$$

Likewise, we say that $(\mu_t)_{t\geq 0}$ is a λ-pseudotrajectory of Φ (with respect to d) if there exists $\lambda > 0$ such that, for all $T > 0$,

(2.4) $$\limsup_{t\to+\infty} \frac{1}{t} \log\Big(\sup_{0\leq s\leq T} d(\mu_{t+s}, \Phi(\mu_t, s)) \Big) \leq -\lambda.$$

This definition of λ-pseudotrajectories is the same as in [4], up to the sign of λ.

In the sequel, we discuss asymptotic pseudotrajectories with distances of the form

$$d_{\mathscr{F}}(\mu, \nu) := \sup_{f\in\mathscr{F}} |\mu(f) - \nu(f)| = \sup_{f\in\mathscr{F}} \left| \int f \, d\mu - \int f \, d\nu \right|,$$

for a certain class of functions $\mathscr{F}$. In particular, this includes total variation, Fortet–Mourier and Wasserstein distances. In general, $d_{\mathscr{F}}$ is a pseudodistance. Nevertheless, it is a distance whenever $\mathscr{F}$ contains an algebra of bounded continuous functions that separates points (see [17], Theorem 4.5(a), Chapter 3). In all the cases considered here, $\mathscr{F}$ contains the algebra $\mathscr{C}_c^{\infty}$ and then convergence in $d_{\mathscr{F}}$ entails convergence in distribution (see Lemma A.1, whose proof is classical and is given in the Appendix for the sake of completeness).

2.2. *Assumptions and main theorem.* In the sequel, let d_1, N_1, N_2 be multi-indices, parameters of the model. We will assume, without loss of generality, that $N_1 \leq N_2$. Some key methods of how to check every assumption are provided in Section 3.

The first assumption we need is crucial. It defines the asymptotic homogeneous Markov process ruling the asymptotic behavior of (y_n).

ASSUMPTION 2.1 (Convergence of generators). There exist a nonincreasing sequence $(\epsilon_n)_{n\geq 1}$ converging to 0 and a constant M_1 [depending on $\mathscr{L}(y_0)$] such that, for all $f \in \mathcal{D}(\mathcal{L}) \cap \mathscr{C}_b^{N_1}$ and $n \in \mathbb{N}^{\star}$, and for any $y \in \mathrm{Supp}(\mathscr{L}(y_n))$,

$$\left| \mathcal{L}f(y) - \mathcal{L}_n f(y) \right| \leq M_1 \chi_{d_1}(y) \sum_{j\leq N_1} \|f^{(j)}\|_{\infty} \epsilon_n.$$

The following assumption is quite technical, but turns out to be true for most of the limit semigroups we deal with. Indeed, this is shown for large classes of PDMPs in Proposition 3.6 and for some diffusion processes in Lemma 3.12.

ASSUMPTION 2.2 (Regularity of the limit semigroup). For all $T > 0$, there exists a constant C_T such that, for every $t \leq T$, $j \leq N_1$ and $f \in \mathscr{C}_b^{N_2}$,

$$P_t f \in \mathscr{C}_b^{N_1}, \qquad \left\| (P_t f)^{(j)} \right\|_\infty \leq C_T \sum_{i \leq N_2} \left\| f^{(i)} \right\|_\infty.$$

The next assumption is a standard condition of uniform boundedness of the moments of the Markov chain. We also provide a very similar Lyapunov criterion to check this condition.

ASSUMPTION 2.3 (Uniform boundedness of moments). Assume that there exists a multi-index $d \geq d_1$ such that one of the following statements holds:

(i) There exists a constant M_2 [depending on $\mathscr{L}(y_0)$] such that

$$\sup_{n \geq 0} \mathbb{E}\big[\chi_d(y_n)\big] \leq M_2.$$

(ii) There exists $V : \mathbb{R}^D \to \mathbb{R}_+$ such that, for all $n \geq 0$, $\mathbb{E}[V(y_n)] < +\infty$. Moreover, there exist $n_0 \in \mathbb{N}^\star, a, \alpha, \beta > 0$, such that $V(y) \geq \chi_d(y)$ when $|y| > a$, such that, for $n \geq n_0$, and for any $y \in \mathrm{Supp}(\mathscr{L}(y_n))$

$$\mathcal{L}_n V(y) \leq -\alpha V(y) + \beta.$$

In this assumption, the function V is a so-called Lyapunov function. The multi-index d can be thought of as $d = d_1$ (which is sufficient for Theorem 2.6 to hold). However, in the setting of Assumption 2.11, it might be necessary to consider $d > d_1$. Of course, if Assumption 2.3 holds for $d' > d$, then it holds for d. Note that we usually can take $V(y) = \mathrm{e}^{\theta y}$, so that we can choose every component of d as large as needed.

REMARK 2.4 [(ii) $\Rightarrow$ (i)]. Computing $\mathbb{E}[\chi_d(y_n)]$ to check Assumption 2.3(i) can be involved, so we rather check a Lyapunov criterion. It is classic that (ii) entails (i). Indeed, denoting by $n_1 := n_0 \vee \min\{n \in \mathbb{N}^\star : \gamma_n < \alpha^{-1}\}$ and $v_n := \mathbb{E}[V(y_n)]$, it is clear that

$$v_{n+1} \leq v_n + \gamma_{n+1}(\beta - \alpha v_n).$$

From this inequality, it is easy to deduce that, for $n \geq n_1$, $v_{n+1} \leq \beta\alpha^{-1} \vee v_n$ and then by induction $v_n \leq \beta\alpha^{-1} \vee v_{n_1}$, which entails (i). Then

$$\mathbb{E}\big[\chi_d(y_n)\big] = \mathbb{P}\big(|y_n| \leq a\big)\mathbb{E}\big[\chi_d(y_n) \mid |y_n| \leq a\big] + \mathbb{P}\big(|y_n| > a\big)\mathbb{E}\big[\chi_d(y_n) \mid |y_n| > a\big]$$

$$\leq \chi_d(a) + \frac{\beta}{\alpha} \vee \Big(\sup_{k \leq n_1} v_k\Big).$$

Note that, with a classical approach, Assumption 2.3 would provide tightness and Assumption 2.1 would be used to identify the limit.

The previous three assumptions are crucial to provide a result on asymptotic pseudotrajectories (Theorem 2.6), but are not enough to quantify speeds of convergence. As it can be observed in the proof of Theorem 2.6, such speed relies deeply on the asymptotic behavior of $\gamma_{m(t)}$ and $\epsilon_{m(t)}$. To this end, we follow the guidelines of [4] to provide a condition in order to ensure such an exponential decay. For any nonincreasing sequences (γ_n), (ϵ_n) converging to 0, define

$$\lambda(\gamma, \epsilon) = -\limsup_{n \to \infty} \frac{\log(\gamma_n \vee \epsilon_n)}{\sum_{k=1}^{n} \gamma_k},$$

where γ and ϵ, respectively, stand for the sequences $(\gamma_n)_{n \geq 0}$ and $(\epsilon_n)_{n \geq 0}$.

REMARK 2.5 (Computation of $\lambda(\gamma, \epsilon)$).   With the notation of [4], Proposition 8.3, we have $\lambda(\gamma, \gamma) = -l(\gamma)$. It is easy to check that, if $\epsilon_n \leq \gamma_n$ for n large, $\lambda(\gamma, \epsilon) = \lambda(\gamma, \gamma)$ and, if $\epsilon_n = \gamma_n^\beta$ with $\beta \leq 1$, $\lambda(\gamma, \epsilon) = \beta \lambda(\gamma, \gamma)$. We can mimic [4], Remark 8.4, to provide sufficient conditions for $\lambda(\gamma, \epsilon)$ to be positive. Indeed, if $\gamma_n = f(n), \epsilon_n = g(n)$ with f, g two positive functions decreasing toward 0 such that $\int_1^{+\infty} f(s) \, ds = +\infty$, then

$$\lambda(\gamma, \epsilon) = -\limsup_{x \to \infty} \frac{\log(f(x) \vee g(x))}{\int_1^x f(s) \, ds}.$$

Typically, if

$$\gamma_n \sim \frac{A}{n^a \log(n)^b}, \quad \epsilon_n \sim \frac{B}{n^c \log(n)^d}$$

for $A, B, a, b, c, d \geq 0$, then:

- $\lambda(\gamma, \epsilon) = 0$ for $a < 1$.
- $\lambda(\gamma, \epsilon) = (c \wedge 1) A^{-1}$ for $a = 1$ and $b = 0$.
- $\lambda(\gamma, \epsilon) = +\infty$ for $a = 1$ and $0 < b \leq 1$.

Now, let us provide the main results of this paper.

THEOREM 2.6 (Asymptotic pseudotrajectories).   *Let $(y_n)_{n \geq 0}$ be an inhomogeneous Markov chain and let Φ and μ be defined as in (2.1) and (2.2). If Assumptions 2.1, 2.2, 2.3 hold, then $(\mu_t)_{t \geq 0}$ is an asymptotic pseudotrajectory of Φ with respect to $d_{\mathscr{F}}$, where*

$$\mathscr{F} = \left\{ f \in \mathcal{D}(\mathcal{L}) \cap \mathscr{C}_b^{N_2} : \mathcal{L}f \in \mathcal{D}(\mathcal{L}), \|\mathcal{L}f\|_\infty + \|\mathcal{L}\mathcal{L}f\|_\infty \right.$$

$$\left. + \sum_{j \leq N_2} \|f^{(j)}\|_\infty \leq 1 \right\}.$$

Moreover, if $\lambda(\gamma, \epsilon) > 0$, then $(\mu_t)_{t \geq 0}$ is a $\lambda(\gamma, \epsilon)$-pseudotrajectory of Φ with respect to $d_{\mathscr{F}}$.

2.3. *Consequences.* Theorem 2.6 relates the asymptotic behavior of the Markov chain (y_n) to the one of the Markov process generated by $\mathcal{L}$. However, to deduce convergence or speeds of convergence of the Markov chain, we need another assumption.

ASSUMPTION 2.7 (Ergodicity). Assume that there exist a probability distribution π, constants v, $M_3 > 0$ [M_3 depending on $\mathcal{L}(y_0)$], and a class of functions $\mathcal{G}$ such that one of the following conditions holds:

(i) $\mathcal{G} \subseteq \mathcal{F}$ and, for any probability measure v, for all $t > 0$,

$$d_{\mathcal{G}}(\Phi(v, t), \pi) \leq d_{\mathcal{G}}(v, \pi) M_3 \mathrm{e}^{-vt}.$$

(ii) There exists r, $M_4 > 0$ such that, for all $s, t > 0$

$$d_{\mathcal{G}}(\Phi(\mu_s, t), \pi) \leq M_3 \mathrm{e}^{-vt}$$

and, for all $T > 0$, with C_T defined in Assumption 2.2,

$$T C_T \leq M_4 \mathrm{e}^{rT}.$$

(iii) There exist functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ and $W \in \mathscr{C}^0$ such that

$$\lim_{t \to \infty} \psi(t) = 0, \qquad \lim_{\|x\| \to \infty} W(x) = +\infty, \qquad \sup_{n \geq 0} \mathbb{E}[W(y_n)] < \infty$$

and, for any probability measure v, for all $t \geq 0$,

$$d_{\mathcal{G}}(\Phi(v, t), \pi) \leq v(W)\psi(t).$$

Since standard proofs of geometric ergodicity rely on the use of Grönwall's lemma, Assumption 2.7(i) and (ii) are quite classic. In particular, using Foster–Lyapunov methods entail such inequalities (see, e.g., [23, 36]). However, in a weaker setting (sub-geometric ergodicity for instance) Assumption 2.7(iii) might still hold; see, for example, [26], Theorem 3.6, [14], Theorem 3.2z, or [20], Theorem 4.1. Note that, if $W = \chi_d$, then $\sup_{n \geq 0} \mathbb{E}[W(y_n)] < \infty$ automatically from Assumption 2.3. Note that, in classical settings where $T C_T \leq M_4 \mathrm{e}^{rT}$, we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

THEOREM 2.8 (Speed of convergence toward equilibrium). *Assume that Assumptions* 2.1, 2.2, 2.3 *hold and let* $\mathcal{F}$ *be as in Theorem* 2.6:

(i) *If Assumption* 2.7(i) *holds and* $\lambda(\gamma, \epsilon) > 0$ *then, for any* $u < \lambda(\gamma, \epsilon) \wedge v$, *there exists a constant* M_5 *such that, for all* $t > t_0 := (v - u)^{-1} \log(1 \wedge M_3)$,

$$d_{\mathcal{G}}(\mu_t, \pi) \leq (M_5 + d_{\mathcal{G}}(\mu_0, \pi)) \mathrm{e}^{-ut}.$$

(ii) *If Assumption* 2.7(ii) *holds and* $\lambda(\gamma, \epsilon) > 0$ *then, for any* $u < v\lambda(\gamma, \epsilon)(r + v + \lambda(\gamma, \epsilon))^{-1}$, *there exists a constant* M_5 *such that, for all* $t > 0$,

$$d_{\mathcal{F} \cap \mathcal{G}}(\mu_t, \pi) \leq M_5 \mathrm{e}^{-ut}.$$

(iii) *If Assumption* 2.7(iii) *holds and convergence in $d_{\mathscr{G}}$ implies weak convergence, then μ_t converges weakly toward π when $t \to \infty$.*

The first part of this theorem is similar to [4], Lemma 8.7, but provides sharp bounds for the constants. In particular, M_5 and t_0 do not depend on μ_0 [in Theorem 2.8(i) only]; see the proof for an explicit expression of M_5. The second part, however, does not require $\mathscr{G}$ to be a subset of $\mathscr{F}$, which can be rather involved to check, given the expression of $\mathscr{F}$ given in Theorem 2.6. The third part is a direct consequence of [4], Theorem 6.10; we did not meet this case in our main examples, but we discuss the convergence toward sub-geometrically ergodic limit processes in Remark 3.14.

REMARK 2.9 (Rate of convergence in the initial scale).   Theorem 2.8(i) and (ii) provide a bound of the form

$$d_{\mathscr{H}}\big(\mathcal{L}(Y_t), \pi\big) \le Ce^{-ut},$$

for some $\mathscr{H}, C, u$ and all $t \ge 0$. This easily entails, for another constant C and all $n \ge 0$,

$$d_{\mathscr{H}}\big(\mathcal{L}(y_n), \pi\big) \le Ce^{-u\tau_n}.$$

Let us detail this bound for three examples where $\epsilon \le \gamma$:

- If $\gamma_n = An^{-1/2}$, then $d_{\mathscr{H}}(\mathcal{L}(y_n), \pi) \le Ce^{-2Au\sqrt{n}}$.
- If $\gamma_n = An^{-1}$, then $d_{\mathscr{H}}(\mathcal{L}(y_n), \pi) \le Cn^{-Au}$.
- If $\gamma_n = A(n\log(n))^{-1}$, then $d_{\mathscr{H}}(\mathcal{L}(y_n), \pi) \le C\log(n)^{-Au}$.

In a nutshell, if γ_n is large, the speed of convergence is good but $\lambda(\gamma, \gamma)$ is small. In particular, even if $\gamma_n = n^{-1/2}$ provides the better speed, Theorem 2.8 does not apply. Remark that the parameter u is more important at the discrete time scale than it is at the continuous time scale.

REMARK 2.10 (Convergence of unbounded functionals).   Theorem 2.8 provides convergence in distribution of (μ_t) toward π, that is, for every $f \in \mathscr{C}_b^0(\mathbb{R}^D)$,

$$\lim_{t \to \infty} \mu_t(f) = \pi(f).$$

Nonetheless, Assumption 2.3 enables us to extend this convergence to unbounded functionals f. Recall that, if a sequence $(X_n)_{n \ge 0}$ converges weakly to X and

$$M := \mathbb{E}\big[V(X)\big] + \sup_{n \ge 0} \mathbb{E}\big[V(X_n)\big] < +\infty$$

for some positive function V, then $\mathbb{E}[f(X_n)]$ converges to $\mathbb{E}[f(X)]$ for every function $|f| < V^\theta$, with $\theta < 1$. Indeed, let $(\kappa_m)_{m \ge 0}$ be a sequence of $\mathscr{C}_c^\infty$ functions

such that $\forall x \in \mathbb{R}^D$, $\lim_{m \to \infty} \kappa_m(x) = 1$ and $0 \le \kappa_m \le 1$. We have, for $m \in \mathbb{N}$,

$$
\begin{aligned}
\left| \mathbb{E}[f(X_n) - f(X)] \right| &\le \left| \mathbb{E}[(1 - \kappa_m(X_n))f(X_n)] \right| + \left| \mathbb{E}[(1 - \kappa_m(X))f(X)] \right| \\
&\quad + \left| \mathbb{E}[f(X_n)\kappa_m(X_n) - f(X)\kappa_m(X)] \right| \\
&\le \mathbb{E}[|f(X_n)|^{\frac{1}{\theta}}]^\theta \mathbb{E}[(1 - \kappa_m(X_n))^{\frac{1}{1-\theta}}]^{1-\theta} \\
&\quad + \mathbb{E}[|f(X)|^{\frac{1}{\theta}}]^\theta \mathbb{E}[(1 - \kappa_m(X))^{\frac{1}{1-\theta}}]^{1-\theta} \\
&\quad + \left| \mathbb{E}[f(X_n)\kappa_m(X_n) - f(X)\kappa_m(X)] \right| \\
&\le M^\theta \mathbb{E}[(1 - \kappa_m(X_n))^{\frac{1}{1-\theta}}]^{1-\theta} \\
&\quad + M^\theta \mathbb{E}[(1 - \kappa_m(X))^{\frac{1}{1-\theta}}]^{1-\theta} \\
&\quad + \left| \mathbb{E}[f(X_n)\kappa_m(X_n) - f(X)\kappa_m(X)] \right|,
\end{aligned}
$$

so that, for all $m \in \mathbb{N}$,

$$
\limsup_{n \to \infty} \mathbb{E}[f(X_n) - f(X)] \le 2M^\theta \mathbb{E}[(1 - \kappa_m(X))^{\frac{1}{1-\theta}}]^{1-\theta}.
$$

Using the dominated convergence theorem, $\lim_{n \to \infty} \mathbb{E}[f(X_n) - f(X)] = 0$ since the right-hand side converges to 0. Note that the condition $|f| \le V^\theta$ can be slightly weakened using the generalized Hölder's inequality on Orlicz spaces (see, e.g., [9]). Although, note that $\mathbb{E}[V(X_n)]$ may not converge to $\mathbb{E}[V(X)]$.

The following assumption is purely technical but is easy to verify in all of our examples, and will be used to prove functional convergence.

ASSUMPTION 2.11 (Control of the variance). Define the following operator:

$$
\Gamma_n f = \mathcal{L}_n f^2 - \gamma_{n+1}(\mathcal{L}_n f)^2 - 2f\mathcal{L}_n f.
$$

Assume that there exists a multi-index d_2 and $M_6 > 0$ such that, if φ_i is the projection on the ith coordinate,

$$
\mathcal{L}_n \varphi_i(y) \le M_6 \chi_{d_2}(y), \qquad \Gamma_n \varphi_i(y) \le M_6 \chi_{d_2}(y),
$$

and

$$
\mathcal{L}_n \chi_{d_2}(y) \le M_6 \chi_{d_2}(y), \qquad \Gamma_n \chi_{d_2}(y) \le M_6 \chi_d(y),
$$

where d is defined in Assumption 2.3.

THEOREM 2.12 (Functional convergence). *Assume that Assumptions 2.1, 2.2, 2.3, 2.7 hold and let π be as in Assumption 2.7. Let $Y_s^{(t)} := Y_{t+s}$ and X^π be the*

process generated by $\mathcal{L}$ *such that* $\mathscr{L}(X_0^\pi) = \pi$. *Then, for any* $m \in \mathbb{N}^\star$, *let* $0 < s_1 < \cdots < s_m$,

$$\left(Y_{s_1}^{(t)}, \ldots, Y_{s_m}^{(t)}\right) \xrightarrow{\mathscr{L}} \left(X_{s_1}^\pi, \ldots, X_{s_m}^\pi\right).$$

Further, if Assumption 2.11 *holds, then the sequence of processes* $(Y_s^{(t)})_{s \geq 0}$ *converges in distribution, as* $t \to +\infty$, *toward* $(X_s^\pi)_{s \geq 0}$ *in the Skorokhod space.*

For reminders about the Skorokhod space, the reader may consult [6, 25, 27]. Note that the operator Γ_n we introduced in Assumption 2.11 is very similar to the *carré du champ* operator in the continuous-time case, up to a term $\gamma_{n+1}(\mathcal{L}_n f)^2$ vanishing as $n \to +\infty$ (see, e.g., [1, 2, 25]). Moreover, if we denote by (K_n) the transition kernels of the Markov chain (y_n), then it is clear that

$$\forall n \in \mathbb{N}, \qquad \gamma_{n+1} \Gamma_n f = K_n f^2 - (K_n f)^2.$$

3. Illustrations.

3.1. *Weighted random walks.*   In this section, we apply Theorems 2.6 and 2.8 to weighted random walks (WRWs) on $\mathbb{R}^D$. Let (ω_n) be a positive sequence, and $\gamma_n := \omega_n (\sum_{k=1}^n \omega_k)^{-1}$. Then, set

$$x_n := \frac{\sum_{k=1}^n \omega_k E_k}{\sum_{k=1}^n \omega_k}, \qquad x_{n+1} := x_n + \gamma_{n+1}(E_{n+1} - x_n).$$

Here, x_n is the weighted mean of $E_1, \ldots, E_n$, where (E_n) is a sequence of centered independent random variables. Under standard assumptions on the moments of E_n, the strong law of large numbers holds and (x_n) converges to 0 a.s. Thus, it is natural to apply the general setting of Section 2 to $y_n := x_n \gamma_n^{-1/2}$ and to define μ_t as in (2.2). As we shall see, computations lead to the convergence of $\mathcal{L}_n$, as defined in (1.1), toward

$$\mathcal{L}f(y) := -ylf'(y) + \frac{\sigma^2}{2} f''(y),$$

where l and σ are defined below. Hence, the properly normalized process asymptotically behaves like the Ornstein–Uhlenbeck process; see Figure 1. This process is the solution of the following stochastic differential equation (SDE):

$$dX_t = -lX_t \, dt + \sigma \, dW_t,$$

see [2] for instance. In the sequel, define $\mathscr{F}$ as in Theorem 2.6 with $N_2 = 3$, and φ_i the projection on the ith coordinate.
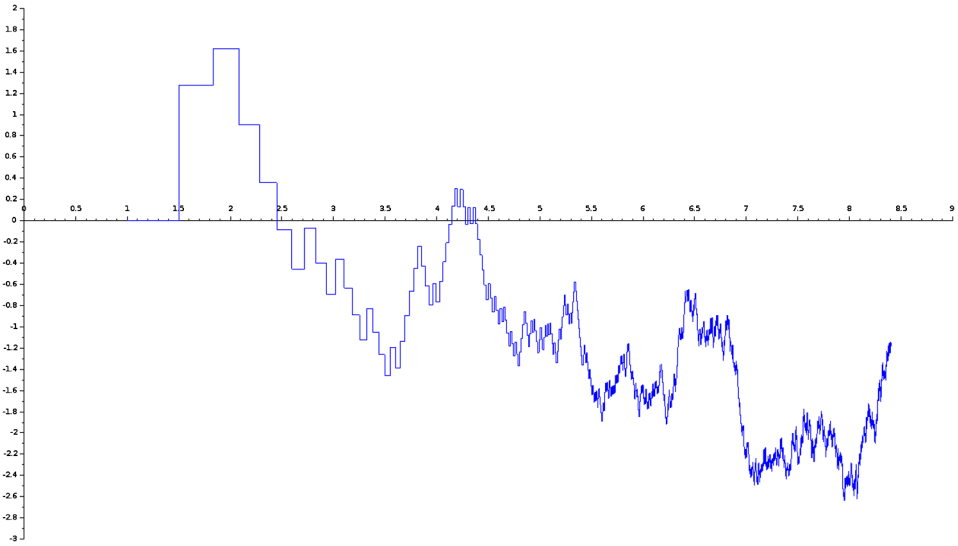
FIG. 1. *Trajectory of the interpolated process for the normalized mean of the WRW with $\omega_n = 1$ and $\mathcal{L}(E_n) = (\delta_{-1} + \delta_1)/2$.*

PROPOSITION 3.1 (Results for the WRW). *Assume that*

$$\mathbb{E}\left[\sum_{i=1}^{D} \varphi_i(E_{n+1})^2\right] = \sigma^2, \qquad \sup_{n \geq 1} \gamma_n^2 \omega_n^4 \mathbb{E}\big[\|E_n\|^4\big] < +\infty,$$

$$\sup_n \gamma_n \sum_{i=1}^{n} \omega_i^2 < +\infty,$$

and that there exist $l > 0$ and $\beta > 1$ such that

$$(3.1) \qquad \sqrt{\frac{\gamma_n}{\gamma_{n+1}}} - 1 - \sqrt{\gamma_n \gamma_{n+1}} = -\gamma_n l + \mathcal{O}(\gamma_n^\beta).$$

Then (μ_t) is an asymptotic pseudotrajectory of Φ, with respect to $d_{\mathscr{F}}$.

Moreover, if $\lambda(\gamma, \gamma^{(\beta-1)\wedge\frac{1}{2}}) > 0$ then, for any $u < l\lambda(\gamma, \gamma^{(\beta-1)\wedge\frac{1}{2}})(l + \lambda(\gamma, \gamma^{(\beta-1)\wedge\frac{1}{2}}))^{-1}$, there exists a constant C such that, for all $t > 0$,

$$(3.2) \qquad d_{\mathscr{F}}(\mu_t, \pi) \leq C e^{-ut},$$

where π is the Gaussian distribution $\mathcal{N}(0, \sigma^2/(2l))$.

Moreover, the sequence of processes $(Y_s^{(t)})_{s \geq 0}$ converges in distribution, as $t \to +\infty$, toward $(X_s^\pi)_{s \geq 0}$ in the Skorokhod space.

It is possible to recover the functional convergence using classical results: for instance, one can apply [30], Theorem 2.1, Chapter 10, with a slightly stronger assumption on (γ_n). Yet, to our knowledge, the rate of convergence (3.2) is original.

REMARK 3.2 (Powers of n).   Typically, if $\gamma_n \sim An^{-\alpha}$, then we can easily check that:

- if $\alpha = 1$, then (3.1) holds with $l = 1 - \frac{1}{2A}$ and $\beta = 2$;
- if $0 < \alpha < 1$, then (3.1) holds with $l = 1$ and $\beta = \frac{1+\alpha}{\alpha} > 2$.

Observe that, if $\omega_n = n^a$ for any $a > -1$, then $\gamma_n \sim \frac{1+a}{n}$ and (3.1) hold with $l = \frac{1+2a}{2+2a}$ and $\beta = 2$.

We will see during the proof that checking Assumptions 2.1, 2.2, 2.3 and 2.7 is quite direct.

PROOF OF PROPOSITION 3.1.   For the sake of simplicity, we do the computations for $D = 1$. We have

$$y_{n+1} = \sqrt{\frac{\gamma_n}{\gamma_{n+1}}} y_n + \sqrt{\gamma_{n+1}}(E_{n+1} - \sqrt{\gamma_n} y_n),$$

so

$$\mathcal{L}_n f(y) = \gamma_{n+1}^{-1} \mathbb{E}[f(y_{n+1}) - f(y_n) \mid y_n = y] = \gamma_{n+1}^{-1} \mathbb{E}[f(y + I_n(y)) - f(y)],$$

with $I_n(y) := (\sqrt{\frac{\gamma_n}{\gamma_{n+1}}} - 1 - \sqrt{\gamma_n \gamma_{n+1}})y + \sqrt{\gamma_{n+1}} E_{n+1}$. Simple Taylor expansions provide the following equalities (where $\mathcal{O}$ is the Landau notation, deterministic and uniform over y and f, and $\tilde{\beta} := \beta \wedge \frac{3}{2}$):

$$I_n(y) = (-\gamma_n l + \mathcal{O}(\gamma_n^{\tilde{\beta}}))y + \sqrt{\gamma_{n+1}} E_{n+1},$$

$$I_n^2(y) = \gamma_{n+1} E_{n+1}^2 + \chi_2(y)(1 + E_{n+1})\mathcal{O}(\gamma_{n+1}^{\tilde{\beta}}),$$

$$I_n^3(y) = \chi_3(y)(1 + E_{n+1} + E_{n+1}^2 + E_{n+1}^3)\mathcal{O}(\gamma_{n+1}^{\tilde{\beta}}).$$

In the setting of Remark 3.2, note that $\tilde{\beta} = \frac{3}{2}$. Now, Taylor formula provides a random variable ξ_n^y such that

$$f(y + I_n(y)) - f(y) = I_n(y)f'(y) + \frac{I_n^2(y)}{2}f''(y) + \frac{I_n^3(y)}{6}f^{(3)}(\xi_n^y).$$

Then it follows that

$$\mathcal{L}_n f(y) = \gamma_{n+1}^{-1} \mathbb{E}\left[I_n(y)f'(y) + \frac{I_n^2(y)}{2}f''(y) + \frac{I_n^3(y)}{6}f^{(3)}(\xi_n^y) \,\Big|\, y_n = y \right]$$

$$= \gamma_{n+1}^{-1}[(-\gamma_n l + \mathcal{O}(\gamma_n^{3/2}))y + \sqrt{\gamma_{n+1}}\mathbb{E}[E_{n+1}]]f'(y)$$

(3.3) $$\qquad + \frac{1}{2\gamma_{n+1}}\gamma_{n+1}\mathbb{E}[E_{n+1}^2 + \chi_2(y)\mathcal{O}(\gamma_{n+1}^{\tilde{\beta}})]f''(y)$$

$$+ \gamma_{n+1}^{-1} \chi_3(y) \mathbb{E}[1 + E_{n+1} + E_{n+1}^2 + E_{n+1}^3] \|f^{(3)}\|_\infty \mathcal{O}(\gamma_{n+1}^\beta)$$

$$= -ylf'(y) + \chi_1(y)\|f'\|_\infty \mathcal{O}(\gamma_n^{\beta-1}) + \frac{\sigma^2}{2} f''(y)$$

$$+ \chi_2(y)\|f''\|_\infty \mathcal{O}(\gamma_n^{\beta-1}) + \chi_3(y)\|f^{(3)}\|_\infty \mathcal{O}(\gamma_n^{\beta-1}).$$

From (3.3), we can conclude that

$$|\mathcal{L}_n f(y) - \mathcal{L}f(y)| = \chi_3(y)(\|f'\|_\infty + \|f''\|_\infty + \|f^{(3)}\|_\infty)\mathcal{O}(\gamma_n^{\beta-1}).$$

As a consequence, the WRW satisfies Assumptions 2.1 with $d_1 = 3$, $N_1 = 3$ and $\epsilon_n = \gamma_n^{\beta-1}$. Note that (see Remark 2.5) $\lambda(\gamma, \epsilon) = \beta - 1$ if $\gamma_n = n^{-1}$.

Now, let us show that $P_t f$ admits bounded derivatives for $f \in \mathscr{F}$. Here, the expressions of the semigroup and its derivatives are explicit and the computations are simple (see [1, 2]). Indeed, $P_t f(x) = \mathbb{E}[f(xe^{-lt} + \sqrt{1 - e^{-2lt}}G)]$ and $(P_t f)^{(j)}(y) = e^{-jlt} P_t f^{(j)}(y)$, where $\mathscr{L}(G) = \mathscr{N}(0, 1)$. Then it is clear that

$$\|(P_t f)^{(j)}\|_\infty = e^{-jlt} \|P_t f^{(j)}\|_\infty \le \|f^{(j)}\|_\infty.$$

Hence, Assumption 2.2 holds with $N_2 = 3$ and $C_T = 1$. Without loss of generality (in order to use Theorem 2.12 later), we set $d = 4$.

Now, we check that the moments of order 4 of y_n are uniformly bounded. Applying Cauchy–Schwarz's inequality,

$$\mathbb{E}\left[\left\|\sum_{i=1}^n \omega_i E_i\right\|^4\right] = \mathbb{E}\left[\sum_{i=1}^n \omega_i^4 \|E_i\|^4 + 6 \sum_{i<j} \omega_i^2 \|E_i^2\| \omega_j^2 \|E_j\|^2\right]$$

$$\le C\left(\sum_{i=1}^n \omega_i^2\right)^2,$$

for some explicit constant C. Then, since

$$\mathbb{E}[\|y_n\|^4] = \gamma_n^2 \mathbb{E}\left[\left\|\sum_{i=1}^n \omega_i E_i\right\|^4\right] \le C \sup_{n\ge 1}\left(\gamma_n \sum_{i=1}^n \omega_i^2\right)^2,$$

the sequence $(y_n)_{n\ge 0}$ satisfies Assumption 2.3.

It is classic, using coupling methods with the same Brownian motion for instance, that, for any probability measure ν,

$$d_\mathscr{G}(\Phi(\nu, t), \pi) \le d_\mathscr{G}(\nu, \pi)e^{-lt},$$

where $\pi = \mathscr{N}(0, \sigma^2/(2l)I_D)$ and $d_\mathscr{G}$ is the Wasserstein distance ($\mathscr{G}$ is the set of 1-Lipschitz functions, see [10]). We have, for $s, t > 0$,

$$d_\mathscr{G}(\Phi(\mu_s, t), \pi) \le d_\mathscr{G}(\mu_s, \pi)e^{-lt} \le (M_2 + \pi(\chi_1))e^{-lt}.$$

In other words, Assumption 2.7(ii) holds for the WRW model with $M_3 = M_2 + \pi(\chi_1)$, $M_4 = 1$, $v = l$, $r = 0$ and $\mathscr{F} \subseteq \mathscr{G}$.
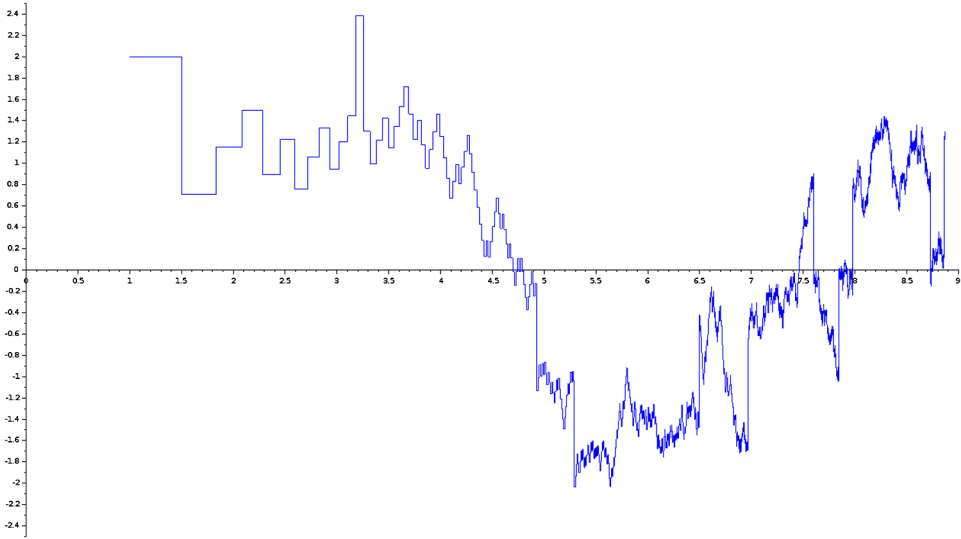
FIG. 2. *Trajectory of the interpolated process for the toy model of Remark 3.3 with* $\mathscr{L}(F_n) = \mathscr{L}(G_n) = (\delta_{-1} + \delta_1)/2.$

Finally, it is easy to check Assumption 2.11 in the case of the WRW, with $d_2 = 2$, and then $\Gamma_n \chi_2 \le M_6 \chi_4$ (that is why we set $d = 4$ above).

Then Theorems 2.6, 2.8 and 2.12 achieve the proof of Proposition 3.1.   □

REMARK 3.3 (Building a limit process with jumps).   In this paper, we mainly provide examples of Markov chains converging (in the sense of Theorem 2.6) toward diffusion processes (see Section 3.1) or jump processes (see Section 3.2). However, it is not hard to adapt the previous model to obtain an example converging toward a diffusion process with jumps (see Figure 2): this illustrates how every component (drift, jump and noise) appears in the limit generator. The intuition is that the jump terms appear when larger and larger jumps of the Markov chain occur with smaller and smaller probability. For an example when $D = 1$, take

$$\omega_n := 1, \qquad E_n := \begin{cases} F_n & \text{if } U_n \ge \sqrt{\gamma_n}, \\ \gamma_n^{-1/2} G_n & \text{if } U_n < \sqrt{\gamma_n}, \end{cases} \qquad y_n := \frac{1}{\sqrt{\gamma_n}} \sum_{k=1}^{n} E_k,$$

where $(F_n)_{n \ge 1}, (G_n)_{n \ge 1}$ and $(U_n)_{n \ge 1}$ are three sequences of i.i.d. random variables, such that $\mathbb{E}[F_1] = 0, \mathbb{E}[F_1^2] = \sigma^2, \mathscr{L}(G_1) = Q, \mathscr{L}(U_1)$ is the uniform distribution on $[0, 1]$. In this case, $\gamma_n = 1/n$ and it is easy to show that $\mathcal{L}_n$ as defined in (1.1) converges toward the following infinitesimal generator:

$$\mathcal{L}f(y) := -\frac{1}{2}yf'(y) + \frac{\sigma^2}{2}f''(y) + \int_{\mathbb{R}} [f(y+z) - f(y)]Q(dz),$$

so that Assumption 2.1 holds with $d_1 = 3, N_1 = 3, \epsilon_n = n^{-1/2}$.

3.2. *Penalized bandit algorithm.* In this section, we slightly generalize the penalized bandit algorithm (PBA) model introduced by Lamberton and Pagès, and we recover [32], Theorem 4. Such algorithms aim at optimizing the gain in a game with two choices, A and B, with respective unknown gain probabilities p_A and p_B. Originally, A and B are the two arms of a slot machine, or *bandit*. Throughout this section, we assume $0 \le p_B < p_A \le 1$.

Let $s : [0, 1] \to [0, 1]$ be a function, which can be understood as a player's strategy, such that $s(0) = 0, s(1) = 1$. Let $x_n \in [0, 1]$ be a measure of her trust level in A at time n. She chooses A with probability $s(x_n)$ independently from the past, and updates x_n as follows:

x_{n+1}	Choice	Result
$x_n + \gamma_{n+1}(1 - x_n)$	A	Gain
$x_n - \gamma_{n+1} x_n$	B	Gain
$x_n + \gamma_{n+1}^2(1 - x_n)$	B	Loss
$x_n - \gamma_{n+1}^2 x_n$	A	Loss

Then (x_n) satisfies the following stochastic approximation algorithm:

$$x_{n+1} := x_n + \gamma_{n+1}(X_{n+1} - x_n) + \gamma_{n+1}^2(\widetilde{X}_{n+1} - x_n),$$

where

$$(3.4) \qquad (X_{n+1}, \widetilde{X}_{n+1}) := \begin{cases} (1, x_n) & \text{with probability } p_1(x_n), \\ (0, x_n) & \text{w.p. } p_0(x_n), \\ (x_n, 1) & \text{w.p. } \widetilde{p}_1(x_n), \\ (x_n, 0) & \text{w.p. } \widetilde{p}_0(x_n), \end{cases}$$

with

$$(3.5) \qquad \begin{aligned} p_1(x) &= s(x)p_A, & p_0(x) &= (1 - s(x))p_B, \\ \widetilde{p}_1(x) &= (1 - s(x))(1 - p_B), & \widetilde{p}_0(x) &= s(x)(1 - p_A). \end{aligned}$$

Note that the PBA of [32] is recovered by setting $s(x) = x$ in (3.5).

From now on, we consider the algorithm (3.4) where $p_1, p_0, \widetilde{p}_1, \widetilde{p}_0$ are nonnecessarily given by (3.5), but are general nonnegative functions whose sum is 1. Let $\mathscr{F}$ be as in Theorem 2.6 with $N_2 = 2$, and $y_n := \gamma_n^{-1}(1 - x_n)$ the rescaled algorithm. Let $\mathcal{L}_n$ be defined as in (1.1),

$$(3.6) \qquad \mathcal{L}f(y) := [\widetilde{p}_0(1) - yp_1(1)]f'(y) - yp_0'(1)[f(y + 1) - f(y)],$$

and π the invariant distribution for $\mathcal{L}$ (which exists and is unique, see Remark 3.7).

Under the assumptions of Proposition 3.4, it is straightforward to mimic the results [32] and ensure that our generalized algorithm $(x_n)_{n \ge 0}$ satisfies the ODE

lemma (see, e.g., [30], Theorem 2.1, Chapter 5), and converges toward 1 almost surely.

PROPOSITION 3.4 (Results for the PBA). *Assume that $\gamma_n = n^{-1/2}$, that $p_1, \widetilde{p}_1, \widetilde{p}_0 \in C_b^1$, $p_0 \in C_b^2$, and that*

$$p_0(1) = \widetilde{p}_1(1) = 0, \qquad p_0'(1) \leq 0, \qquad p_1(1) + p_0'(1) > 0, \qquad \widetilde{p}_1(0) > 0.$$

If, for $0 < x < 1$, $(1 - x)p_1(x) > xp_0(x)$, then (μ_t) is an asymptotic pseudotrajectory of Φ, with respect to $d_{\mathscr{F}}$.

Moreover, (μ_t) converges to π and the sequence of processes $(Y_s^{(t)})_{s \geq 0}$ converges in distribution, as $t \to +\infty$, toward $(X_s^\pi)_{s \geq 0}$ in the Skorokhod space.

The proof is given at the end of the section; before that, let us give some interpretation and heuristic explanation of the algorithm. The random sequence (y_n) satisfies

$$y_{n+1} = y_n + \left(\frac{\gamma_n}{\gamma_{n+1}} - 1\right)y_n - (X_{n+1} - x_n) - \gamma_{n+1}(\widetilde{X}_{n+1} - x_n),$$

thus, defining $\mathcal{L}_n$ as in (1.1),

$$\mathcal{L}_n f(y) = \gamma_{n+1}^{-1}\mathbb{E}[f(y_{n+1}) - f(y_n) \mid y_n = y]$$
$$= \gamma_{n+1}^{-1}\mathbb{E}[f(y + I_n(y)) - f(y) \mid y_n = y],$$

where

$$(3.7) \qquad I_n(y) := \begin{cases} I_n^1(y) := \left(\dfrac{\gamma_n}{\gamma_{n+1}} - 1 - \gamma_n\right)y \\ \qquad \text{w.p. } p_1(1 - \gamma_n y), \\ I_n^0(y) := 1 + \left(\dfrac{\gamma_n}{\gamma_{n+1}} - 1 - \gamma_n\right)y \\ \qquad \text{w.p. } p_0(1 - \gamma_n y), \\ \widetilde{I}_n^1(y) := \left(\dfrac{\gamma_n}{\gamma_{n+1}} - 1 - \gamma_n \gamma_{n+1}\right)y \\ \qquad \text{w.p. } \widetilde{p}_1(1 - \gamma_n y), \\ \widetilde{I}_n^0(y) := \gamma_{n+1} + \left(\dfrac{\gamma_n}{\gamma_{n+1}} - 1 - \gamma_n \gamma_{n+1}\right)y \\ \qquad \text{w.p. } \widetilde{p}_0(1 - \gamma_n y). \end{cases}$$

Taylor expansions provide the convergence of $\mathcal{L}_n$ toward $\mathcal{L}$. As a consequence, the properly renormalized interpolated process will asymptotically behave like a PDMP (see Figure 3). Classically, one can read the dynamics of the limit process through its generator (see, e.g., [13]): the PDMP generated by (3.6) has upward jumps of height 1 and follows the flow given by the ODE $y' = \widetilde{p}_0(1) - yp_1(1)$, which means it converges exponentially fast toward $\widetilde{p}_0(1)/p_1(1)$.
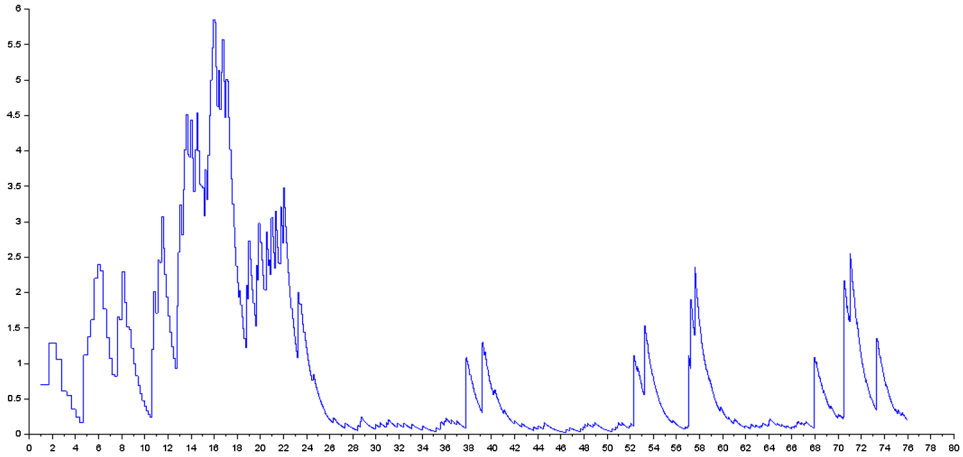
FIG. 3. *Trajectory of the interpolated process for the rescaled PBA, setting $s(x) = x$ in* (3.5).

REMARK 3.5 (Interpretation). Consider the case (3.5). Here, Proposition 3.4 states that the rescaled algorithm (y_n) behaves asymptotically like the process generated by

$$\mathcal{L}f(x) = (1 - p_A - x p_A)f'(x) + p_B s'(1) x [f(x+1) - f(x)].$$

Intuitively, it is more and more likely to play the arm A (the one with the greatest gain probability). Its successes and failures appear within the drift term of the limit infinitesimal generator, whereas playing the arm B with success will provoke a jump. Finally, playing the arm B with failure does not affect the limit dynamics of the process (as $\tilde{p}_1$ does not appear within the limit generator). To carry out the computations in this section, where we establish the speed of convergence of $(\mathcal{L}_n)$ toward $\mathcal{L}$, the main idea is to condition $\mathbb{E}[y_{n+1}]$ given typical events on the one hand, and rare events on the other hand. Typical events generally construct the drift term of $\mathcal{L}$ and rare events are responsible of the jump term of $\mathcal{L}$ (see also Remark 3.3).

Note that one can tune the frequency of jumps with the parameter $s'(1)$. The more concave s is in a neighborhood of 1, the better the convergence is. In particular, if $s'(1) = 0$, the limit process is deterministic. Also, note that choosing a function s nonsymmetric with respect to $(1/2, 1/2)$ introduces an a priori bias; see Figure 4.

Let us start the analysis of the rescaled PBA with a global result about a large class of PDMPs, whose proof is postponed to the Appendix. This lemma provides the necessary arguments to check Assumption 2.2.

PROPOSITION 3.6 (Assumption 2.2 for PDMPs). *Let X be a PDMP with infinitesimal generator*

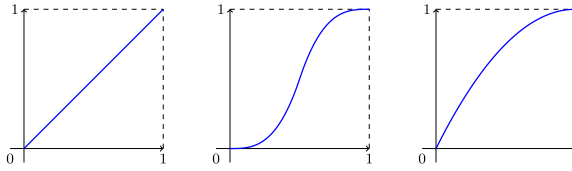$$\mathcal{L}f(x) = (a - bx)f'(x) + (c + dx)[f(x+1) - f(x)],$$

FIG. 4.   *Various strategies for* $s(x) = x$, s *concave*, s *with a bias.*

such that $a, b, c, d \geq 0$. Assume that either $b > 0$, or $b = 0$ and $a \neq 0$. If $f \in \mathscr{C}_b^N$, then for all $0 \leq t \leq T$, $P_t f \in \mathscr{C}_b^N$. Moreover, for all $n \leq N$,

$$\left\| (P_t f)^{(n)} \right\|_\infty \leq \begin{cases} \displaystyle\sum_{k=0}^n \left(\frac{2|d|}{b} \right)^{n-k} \left\| f^{(k)} \right\|_\infty & \text{if } b > 0, \\[2em] \displaystyle\sum_{k=0}^n \frac{n!}{k!} (2|d|T)^{n-k} \left\| f^{(k)} \right\|_\infty & \text{if } b = 0. \end{cases}$$

Note that a very similar result is obtained in [3], but for PDMPs with a diffusive component.

REMARK 3.7 (The stationary probability distribution).   Let $(X_t)_{t \geq 0}$ be the PDMP generated by $\mathcal{L}$ defined in Proposition 3.6. By using the same tools as in [32], Theorem 6, it is possible to prove existence and uniqueness of a stationary distribution π on $\mathbb{R}_+$. Applying Dynkin's formula with $f(x) = x$, we get

$$\partial_t \mathbb{E}[X_t] = a + c - (b - d)\mathbb{E}[X_t].$$

If one uses the same technique with $f(x) = x^n$, it is possible to deduce the nth moment of the invariant measure π, and Dynkin's formula applied to $f(x) = \exp(\lambda x)$ provides exponential moments of π (see [8], Remark 2.2, for the detail).

In the setting of (3.6), one can use the reasoning above to show that, by denoting by $m_n = \int_0^\infty x^n \pi(dx)$ for $n \geq 0$,

$$m_n = \frac{-p_0'(1)}{n(p_1(1) + p_0'(1))} \sum_{k=1}^{n-2} \binom{n}{k-1} m_k + \frac{2\tilde{p}_0(1) + (n-1)p_0'(1)}{2(p_1(1) + p_0'(1))} m_{n-1},$$

with the convention $\sum_{k=i+1}^i = 0$.

PROOF OF PROPOSITION 3.4.   First, let us specify the announced convergence of $\mathcal{L}_n$ toward $\mathcal{L}$; recall that $\gamma_n = n^{-1/2}$ and $\chi_d(y) = \sum_{k=0}^d |y|^k$, so that $I_n(y)$ in

(3.7) rewrites

$$I_n(y) = \begin{cases} \dfrac{\sqrt{n+1} - \sqrt{n} - 1}{\sqrt{n}} y & \text{w.p. } p_1(1 - \gamma_n y), \\[3mm] 1 + \dfrac{\sqrt{n+1} - \sqrt{n} - 1}{\sqrt{n}} y & \text{w.p. } p_0(1 - \gamma_n y), \\[3mm] \dfrac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n+1}} y & \text{w.p. } \widetilde{p}_1(1 - \gamma_n y), \\[3mm] \dfrac{1}{\sqrt{n+1}} + \dfrac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n+1}} y & \text{w.p. } \widetilde{p}_0(1 - \gamma_n y), \end{cases}$$

and the infinitesimal generator rewrites

$$
\begin{aligned}
\mathcal{L}_n f(y) = {} & \frac{p_1(1 - \gamma_n y)}{\gamma_{n+1}} \big[f(y + I_n^1(y)) - f(y) \big] \\[2mm]
& + \frac{p_0(1 - \gamma_n y)}{\gamma_{n+1}} \big[f(y + I_n^0(y)) - f(y) \big] \\[2mm]
& + \frac{\widetilde{p}_1(1 - \gamma_n y)}{\gamma_{n+1}} \big[f(y + \widetilde{I}_n^1(y)) - f(y) \big] \\[2mm]
& + \frac{\widetilde{p}_0(1 - \gamma_n y)}{\gamma_{n+1}} \big[f(y + \widetilde{I}_n^0(y)) - f(y) \big].
\end{aligned}
$$

(3.8)

In the sequel, the Landau notation $\mathcal{O}$ will be deterministic and uniform over both y and f.

First, we consider the first term of (3.8) and observe that

$$p_1(1 - \gamma_n y) = p_1(1) + y\mathcal{O}(\gamma_n),$$

and that

$$I_n^1(y) = \left(\frac{\gamma_n}{\gamma_{n+1}} - 1 - \gamma_n \right) y = \left(\frac{1}{2n} + o(n^{-1}) - \frac{1}{\sqrt{n}} \right) y = -y\gamma_n(1 + \mathcal{O}(\gamma_n)),$$

so that $I_n^1(y)^2 = y^2 \mathcal{O}(\gamma_n^2)$. Since $\gamma_n \sim \gamma_{n+1}$, and since the Taylor formula gives a random variable ξ_n^y such that

$$f(y + I_n^1(y)) - f(y) = I_n^1(y) f'(y) + \frac{I_n^1(y)^2}{2} f''(\xi_n^y),$$

we have

$$\gamma_{n+1}^{-1}\big[f(y + I_n^1(y)) - f(y) \big] = -y f'(y) + \chi_2(y)(\|f'\|_\infty + \|f''\|_\infty)\mathcal{O}(\gamma_n).$$

Then easy computations show that

(3.9)
$$
\begin{aligned}
& \frac{p_1(1 - \gamma_n y)}{\gamma_{n+1}} \big[f(y + I_n^1(y)) - f(y) \big] \\[2mm]
& = -p_1(1) y f'(y) + \chi_3(y)(\|f'\|_\infty + \|f''\|_\infty)\mathcal{O}(\gamma_n).
\end{aligned}
$$

The third term in (3.8) is expanded similarly and writes

$$(3.10) \quad \frac{\widetilde{p}_1(1 - \gamma_n y)}{\gamma_{n+1}} [f(y + \widetilde{I}_n^1(y)) - f(y)] = \chi_3(y)(\|f'\|_\infty + \|f''\|_\infty)\mathcal{O}(\gamma_n),$$

while the fourth term becomes

$$(3.11) \quad \begin{aligned} &\frac{\widetilde{p}_0(1 - \gamma_n y)}{\gamma_{n+1}} [f(y + \widetilde{I}_n^0(y)) - f(y)] \\ &\qquad = \widetilde{p}_0(1) f'(y) + \chi_3(y)(\|f'\|_\infty + \|f''\|_\infty)\mathcal{O}(\gamma_n). \end{aligned}$$

Note the slight difference with the expansion of the second term, since we have, on the one hand,

$$\frac{p_0(1 - \gamma_n y)}{\gamma_{n+1}} = -\frac{\gamma_n}{\gamma_{n+1}} y p_0'(1) + \frac{\gamma_n^2}{\gamma_{n+1}} y^2 p''(\xi_n^y) = -y p_0'(1) + \chi_2(y)\mathcal{O}(\gamma_n),$$

where ξ_n^y is a random variable, while, on the other hand,

$$f(y + I_n^0(y)) - f(y) = f(y + 1) - f(y) + \chi_1(y)\|f'\|_\infty \mathcal{O}(\gamma_n).$$

Then

$$(3.12) \quad \begin{aligned} &\frac{p_0(1 - \gamma_n y)}{\gamma_{n+1}} [f(y + I_n^0(y)) - f(y)] \\ &\qquad = -y p_0'(1)[f(y + 1) - f(y)] + \chi_3(y)(\|f\|_\infty + \|f'\|_\infty)\mathcal{O}(\gamma_n). \end{aligned}$$

Finally, combining (3.9), (3.10), (3.11) and (3.12), we obtain the following speed of convergence for the infinitesimal generators:

$$(3.13) \quad |\mathcal{L}_n f(y) - \mathcal{L}f(y)| = \chi_3(y)(\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty)\mathcal{O}(\gamma_n),$$

establishing that the rescaled PBA satisfies Assumption 2.1 with $d_1 = 3$, $N_1 = 2$ and $\epsilon_n = \gamma_n$. Assumption 2.2 follows from Proposition 3.6 with $N_2 = 2$.

In order to apply Theorem 2.6, it would remain to check Assumption 2.3, that is to prove that the moments of order 3 of (y_n) are uniformly bounded. This happens to be very difficult and we do not even know whether it is true. As an illustration of this difficulty, the reader may refer to [19], Remark 4.4, where uniform bounds for the first moment are provided using rather technical lemmas, and only for an overpenalized version of the algorithm.

In order to overcome this technical difficulty, we introduce a truncated Markov chain coupled with (y_n), which does satisfy a Lyapunov criterion. For $l \in \mathbb{N}^\star$ and $\delta \in (0, 1]$, we define $(y_n^{(l,\delta)})_{n \geq 0}$ as follows:

$$y_n^{(l,\delta)} := \begin{cases} y_n & \text{for } n \leq l, \\ (y_{n-1}^{(l,\delta)} + I_{n-1}(y_{n-1}^{(l,\delta)})) \wedge \delta\gamma_n^{-1} & \text{for } n > l. \end{cases}$$

In the sequel, we denote with an exposant (l, δ) the equivalents of $\mathcal{L}_n, Y_t, \mu_t$ for $(y_n^{(l,\delta)})_{n \geq 0}$. We prove that $(\mathcal{L}_n^{(l,\delta)})_{n \geq 0}$ satisfies our main assumptions, and consequently $(\mu_t^{(l,\delta)})_{t \geq 0}$ is an asymptotic pseudotrajectory of Φ (at least for δ small enough and l large enough), which is the result of the combination of Lemma 3.8 and Theorem 2.6.

LEMMA 3.8 (Behavior of the truncated chain). *For δ small enough and l large enough, the inhomogeneous Markov chain $(y_n^{(l,\delta)})_{n \geq 0}$ satisfies Assumptions* 2.1, 2.2, 2.3 *and* 2.11.

Now, we shall prove that $(\mu_t)_{t \geq 0}$ is an asymptotic pseudotrajectory of Φ as well. Indeed, let $\varepsilon > 0$ and l be large enough such that $\mathbb{P}(\forall n \geq l, \gamma_n y_n \leq \delta) \geq 1 - \varepsilon$ (it is possible since $\gamma_n y_n = 1 - x_n$ converges to 0 in probability). Then, for $T > 0, f \in \mathscr{F}, s \in [0, T]$

$$
\begin{aligned}
\left|\mu_{t+s}(f) - \Phi(\mu_t, s)(f)\right| &\leq \left|\mu_{t+s}(f) - \mu_{t+s}^{(l,\delta)}(f)\right| \\
&\quad + \left|\Phi(\mu_t^{(l,\delta)}, s)(f) - \Phi(\mu_t, s)(f)\right| \\
&\quad + \left|\mu_{t+s}^{(l,\delta)}(f) - \Phi(\mu_t^{(l,\delta)}, s)(f)\right| \\
&\leq \left(2\|f\|_\infty + 2\|f\|_\infty\right)\left(1 - \mathbb{P}(\forall n \geq l, \gamma_n y_n \leq \delta)\right) \\
&\quad + \left|\mu_{t+s}^{(l,\delta)}(f) - \Phi(\mu_t^{(l,\delta)}, s)(f)\right| \\
&\leq 4\varepsilon + \left|\mu_{t+s}^{(l,\delta)}(f) - \Phi(\mu_t^{(l,\delta)}, s)(f)\right|,
\end{aligned}
$$

since $\|f\|_\infty \leq 1$. Taking the suprema over $[0, T]$ and $\mathscr{F}$ yields

(3.14)
$$
\begin{aligned}
&\limsup_{t \to \infty} \sup_{s \in [0,T]} d_{\mathscr{F}}\left(\mu_{t+s}, \Phi(\mu_t, s)\right) \\
&\qquad \leq 4\varepsilon + \limsup_{t \to \infty} \sup_{s \in [0,T]} d_{\mathscr{F}}\left(\mu_{t+s}^{(l,\delta)}, \Phi(\mu_t^{(l,\delta)}, s)\right).
\end{aligned}
$$

Using Lemma 3.8, Theorem 2.6 holds for $(\mu_t^{(l,\delta)})_{t \geq 0}$ and (3.14) rewrites

$$
\limsup_{t \to \infty} \sup_{s \in [0,T]} d_{\mathscr{F}}\left(\mu_{t+s}, \Phi(\mu_t, s)\right) \leq 4\varepsilon,
$$

so that $(\mu_t)_{t \geq 0}$ is an asymptotic pseudotrajectory of Φ.

Finally, for $t > 0, T > 0, f \in \mathscr{C}_b^0, s \in [0, T]$, set $\nu_t := \mathscr{L}((Y_s^{(t)})_{0 \leq T})$ and $\nu := \mathscr{L}((X_s^\pi)_{0 \leq T})$. We have

(3.15)
$$
\begin{aligned}
\left|\nu_t(f) - \nu(f)\right| &\leq \left|\nu_t(f) - \nu_t^{(l,\delta)}(f)\right| + \left|\nu_t^{(l,\delta)}(f) - \nu(f)\right| \\
&\leq 2\|f\|_\infty\left(1 - \mathbb{P}(\forall n \geq l, \gamma_n y_n \leq \delta)\right) + \left|\nu_t^{(l,\delta)}(f) - \nu(f)\right| \\
&\leq 2\varepsilon + \left|\nu_t^{(l,\delta)}(f) - \nu(f)\right|.
\end{aligned}
$$

Since $(y_n^{(l,\delta)})_{n\geq 0}$ satisfies Assumption 2.11, we can apply Theorem 2.12 so that the right-hand side of (3.15) converges to 0, which concludes the proof.   □

REMARK 3.9 (Rate of convergence toward the stationary measure).   For such PDMPs, exponential convergence in Wasserstein distance has already been obtained (see [8], Proposition 2.1, or [19], Theorem 3.4). However, we are not in the setting of Theorem 2.8, since $\gamma_n = n^{-1/2}$. Thus, $\lambda(\gamma, \epsilon) = 0$, and there is no exponential convergence. This highlights the fact that the rescaled algorithm converges too slowly toward the limit PDMP.

REMARK 3.10 (The overpenalized bandit algorithm).   Even though we do not consider the overpenalized bandit algorithm introduced in [19], the tools are the same. The behavior of this algorithm is the same as the PBAs, except from a possible (random) penalization of an arm in case of a success; it writes

$$x_{n+1} = x_n + \gamma_{n+1}(X_{n+1} - x_n) + \gamma_{n+1}^2(\widetilde{X}_{n+1} - x_n),$$

where

$$(X_{n+1}, \widetilde{X}_{n+1}) = \begin{cases} (1, x_n) & \text{with probability } p_A x_n \sigma, \\ (0, x_n) & \text{with probability } p_B(1 - x_n)\sigma, \\ (1, 0) & \text{with probability } p_A x_n(1 - \sigma), \\ (0, 1) & \text{with probability } p_B(1 - x_n)(1 - \sigma), \\ (x_n, 1) & \text{with probability } (1 - p_B)(1 - x_n), \\ (x_n, 0) & \text{with probability } (1 - p_A)x_n. \end{cases}$$

Setting $y_n = \gamma_n^{-1}(1 - x_n)$, and following our previous computations, it is easy to show that the rescaled overpenalized algorithm converges, in the sense of Assumption 2.1, toward

$$\mathcal{L}f(y) = [1 - \sigma p_A - p_A y]f'(y) + p_B y[f(y + 1) - f(y)].$$

3.3. *Decreasing step Euler scheme.*   In this section, we turn to the study of the so-called decreasing step Euler scheme (DSES). This classical stochastic procedure is designed to approximate the stationary measure of a diffusion process of the form

$$(3.16) \qquad X_t^x = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$$

with a discrete Markov chain

$$(3.17) \qquad y_{n+1} := y_n + \gamma_{n+1}b(y_n) + \sqrt{\gamma_{n+1}}\sigma(y_n)E_{n+1},$$

for any nonincreasing sequence $(\gamma_n)_{n\geq 1}$ converging toward 0 such that $\sum_{n=1}^{\infty}\gamma_n = +\infty$ and (E_n) a suitable sequence of random variables. In the sequel, we shall

recover the convergence of the DSES toward the diffusion process at equilibrium, as defined by (3.16). If $\gamma_n = \gamma$ in (3.17), this model would be a constant step Euler scheme as studied by [39, 40], which approaches the diffusion process at time t when γ tends to 0. By letting $t \to +\infty$ in (3.16), it converges to the equilibrium of the diffusion process. We can concatenate those steps by choosing γ_n vanishing but such that $\sum_n \gamma_n$ diverges. The DSES has already been studied in the literature; see, for instance, [31, 34].

It is simple, following the computations of Sections 3.1 and 3.2, to check that $\mathcal{L}_n$ converges (in the sense of Assumption 2.1) toward

$$\mathcal{L}f(y) := b(y)f'(y) + \frac{\sigma^2(y)}{2}f''(y).$$

In the sequel, define $\mathscr{F}$ as in Theorem 2.6 with $N_2 = 3$.

PROPOSITION 3.11 (Results for the DSES). *Assume that (E_n) is a sequence of sub-Gaussian random variables (i.e., there exists $\kappa > 0$ such that $\forall \theta \in \mathbb{R}, \mathbb{E}[\exp(\theta E_1)] \leq \exp(\kappa \theta^2 / 2)$), and $\mathbb{E}[E_1] = 0$ and $\mathbb{E}[E_1^2] = 1$. Moreover, assume that $b, \sigma \in \mathscr{C}^\infty$ whose derivatives of any order are bounded, and that σ is bounded. Eventually, assume that there exist constants $0 < b_1 \leq b_2$ and $0 < \sigma_1$ such that, for $|y| > A$,*

$$(3.18) \qquad -b_2 y^2 \leq b(y)y \leq -b_1 y^2, \qquad \sigma_1 \leq \sigma(y).$$

If $\gamma_n = 1/n$, then (μ_t) is a $\frac{1}{2}$-pseudotrajectory of Φ, with respect to $d_\mathscr{F}$.

Moreover, there exists a probability distribution π and $C, u > 0$ such that, for all $t > 0$,

$$d_\mathscr{F}(\mu_t, \pi) \leq Ce^{-ut}.$$

Further, the sequence of processes $(Y_s^{(t)})_{s \geq 0}$ converges in distribution, as $t \to +\infty$, toward $(X_s^\pi)_{s \geq 0}$ in the Skorokhod space.

Note that one could choose a more general (γ_n), provided that $\lambda(\gamma, \gamma) > 0$. In contrast to classical results, Proposition 3.11 provides functional convergence. Moreover, we obtain a rate of convergence in a more general setting than [34], Theorem IV.1, see also [31]. Indeed, let us detail the difference between those settings with the example of the Kolmogorov–Langevin equation:

$$dX_t = \nabla V(X_t)\,dt + \sigma\,dB_t.$$

A rate of convergence may be obtained in [34] only for V uniformly convex; although, we only need V to be convex outside some compact set. Let us recall that the uniform convexity is a strong assumption ensuring log-Sobolev inequality, Wasserstein contraction; see, for instance, [1, 2].

PROOF OF PROPOSITION 3.11.   Recalling (y_n) in (3.17) and $\mathcal{L}_n$ in (1.1), we have

$$\mathcal{L}_n(y) = \gamma_{n+1}^{-1} \mathbb{E}\big[f(y + \gamma_{n+1}b(y) + \sqrt{\gamma_{n+1}}\sigma(y)E_{n+1}) - f(y) \mid y_n = y\big].$$

Easy computations show that Assumption 2.1 holds with $\epsilon_n = \sqrt{\gamma_n}$, $N_1 = 3$, $d_1 = 3$.

We aim at proving Assumption 2.2, that is, for $f \in \mathcal{F}$, $j \le 3$ and $t \le T$, that $(P_t f)^{(j)}$ exists and

$$\big\|(P_t f)^{(j)}\big\|_\infty \le C_T \sum_{k=0}^{3} \big\|f^{(k)}\big\|_\infty.$$

It is straightforward for $j = 0$, but computations are more involved for $j \ge 1$. Let us denote by $(X_t^x)_{t \ge 0}$ the solution of (3.16) starting at x. Since b and σ are smooth with bounded derivatives, it is standard that $x \mapsto X_t^x$ is $\mathscr{C}^4$ (see, for instance, [29], Chapter II, Theorem 3.3). Moreover, $\partial_x X_t^x$ satisfies the following SDE:

$$\partial_x X_t^x = 1 + \int_0^t b'(X_s^x)\partial_x X_s^x \, ds + \int_0^t \sigma'(X_s^x)\partial_x X_s^x \, dW_s.$$

For our purpose, we need the following lemma, which provides a constant for Assumption 2.2 of the form $C_T = C_1 e^{C_2 T}$. It is of course possible to explicit the constants C_1 and C_2 for the second and third derivatives in its proof. However, here we are only interested in checking Assumption 2.7(ii).

LEMMA 3.12 (Estimates for the derivatives of the diffusion).   *Under the assumptions of Proposition* 3.11, *for $p \ge 2$ and $t \le T$,*

$$\mathbb{E}\big[|\partial_x X_t^x|^p\big] \le \exp\bigg(\bigg(p\|b'\|_\infty + \frac{p(p-1)}{2}\|\sigma'\|_\infty^2\bigg)T\bigg)$$

and

$$\mathbb{E}\big[|\partial_x X_t^x|\big] \le \exp\bigg(\bigg(\|b'\|_\infty + \frac{1}{2}\|\sigma'\|_\infty^2\bigg)T\bigg).$$

For any $p \in \mathbb{N}^\star$, there exist positive constants C_1, C_2 not depending on x, such that
$$\mathbb{E}\big[|\partial_x^2 X_t^x|^p\big] \le C_1 e^{C_2 T}, \qquad \mathbb{E}\big[|\partial_x^3 X_t^x|^p\big] \le C_1 e^{C_2 T}.$$

The proof of the lemma is postponed to the Appendix. Using Lemma 3.12, and since f and its derivatives are bounded, it is clear that $x \mapsto P_t f(x)$ is three times differentiable, with

$$(P_t f)'(x) = \mathbb{E}\big[f'(X_t^x)\partial_x X_t^x\big],$$

$$(P_t f)''(x) = \mathbb{E}\big[f''(X_t^x)(\partial_x X_t^x)^2 + f'(X_t^x)(\partial_x^2 X_t^x)\big],$$

$$(P_t f)^{(3)}(x) = \mathbb{E}\big[f^{(3)}(X_t^x)(\partial_x X_t^x)^3 + 3f''(X_t^x)(\partial_x X_t^x)(\partial_x^2 X_t^x)$$
$$+ f'(X_t^x)(\partial_x^3 X_t^x)\big].$$

As a consequence, Assumption 2.2 holds, with $C_T = 3C_1^3 e^{3C_2 T}$ and $N_2 = 3$.

Now, we shall prove that Assumption 2.3(ii) holds with $V(y) = \exp(\theta y)$, for some (small) $\theta > 0$. Thanks to (3.18), we easily check that, for $\widetilde{V}(y) = 1 + y^2$,

$$(3.19) \qquad \mathcal{L}\widetilde{V}(y) \leq -\widetilde{\alpha}\widetilde{V}(y) + \widetilde{\beta},$$

with $\widetilde{\alpha} = 2b_1$ and

$$\widetilde{\beta} = (2b_1 + S) \vee \left(A \sup_{[-A, A]} b + \frac{S^2}{2} + 2b_1(1 + A^2) \right).$$

Then [34], Proposition III.1, entails Assumption 2.3(ii). Finally, Theorem 2.6 applies and we recover [30], Theorem 2.1, Chapter 10.

Then Theorem 2.6 provides the asymptotic behavior of the Markov chain $(y_n)_{n \geq 0}$ (in the sense of asymptotic pseudotrajectories). If we further want speeds of convergence, we shall use Theorem 2.8 and prove the ergodicity of the limit process; to that end, combine (3.19) with [36], Theorem 6.1 (which provides exponential ergodicity for the diffusion toward some stationary measure π), as well as Lemma 3.12, to ensure Assumption 2.7(ii) with $\mathscr{G} = \{g \in \mathscr{C}^0(\mathbb{R}) : |g(y)| \leq 1 + y^2\}$ (v and r are not explicitly given). Note that we used the fact that σ is lower-bounded, which implies that the compact sets are small sets. Moreover, the choice $\gamma_n = n^{-1}$ implies $\lambda(\gamma, \epsilon) = 1/2$. Then the assumptions of Theorem 2.8 are satisfied, with $u_0 = v(1 + 2v + 2r)^{-1}$.

Finally, we can easily check Assumption 2.11 for some $d \in \mathbb{N}$, since y_n admits uniformly bounded exponential moments. Then using Theorem 2.12 completes the proof. □

3.4. *Lazier and lazier random walk.* We consider the lazier and lazier random walk (LLRW) $(y_n)_{n \geq 0}$ defined as follows:

$$(3.20) \qquad y_{n+1} := \begin{cases} y_n + Z_{n+1} & \text{with probability } \gamma_{n+1}, \\ y_n & \text{with probability } 1 - \gamma_{n+1}, \end{cases}$$

where (Z_n) is such that $\mathscr{L}(Z_{n+1} \mid y_0, \ldots, y_n) = \mathscr{L}(Z_{n+1} \mid y_n)$; we denote the conditional distribution $Q(y_n, \cdot) := \mathscr{L}(Z_{n+1} \mid y_n)$. In the sequel, define $\mathscr{F} := \{f \in \mathscr{C}_b^0 : 7\|f\|_\infty \leq 1\}$ and $\mathcal{L}f(y) = \int_{\mathbb{R}} f(y + z)Q(y, dz) - f(y)$, which is the generator of a pure-jump Markov process (constant between two jumps).

This example is very simple and could be studied without using our main results; however, we still develop it in order to check the sharpness of our rates of convergence (see Remak 3.15).

PROPOSITION 3.13 (Results for the LLRW model). *The sequence (μ_t) is an asymptotic pseudotrajectory of Φ, with respect to $d_{\mathscr{F}}$.*

Moreover, if $\lambda(\gamma, \gamma) > 0$, then (μ_t) is a $\lambda(\gamma, \gamma)$-pseudotrajectory of Φ.

Further, if $\mathcal{L}$ satisfies Assumption 2.7(i) for some $v > 0$ then, for any $u < v \wedge \lambda(\gamma, \gamma)$, there exists a constant C such that, for all $t > 0$,

$$d_{\mathscr{F}}(\mu_t, \pi) \leq Ce^{-ut}.$$

Remark that the distance $d_{\mathscr{F}}$ in Proposition 3.13 is the total variation distance up to a constant.

PROOF OF PROPOSITION 3.13.   It is easy to check that (1.1) entails

$$\mathcal{L}_n f(y) = \int_{\mathbb{R}} f(y+z) Q(y, dz) - f(y) = \mathcal{L} f(y).$$

It is clear that the LLRW satisfies Assumption 2.1 with $d_1 = 0$, $N_1 = 0$, $\epsilon_n = 0$, and Assumption 2.2 with $C_T = 1$, $N_2 = 0$. Since $d = d_1 = 0$, Assumption 2.3 is also clearly satisfied. Eventually, note that $\lambda(\gamma, \epsilon) = \lambda(\gamma, \gamma)$. Then Theorem 2.6 holds. Finally, if $\mathcal{L}$ satisfies Assumption 2.7(i), it is clear that Theorem 2.8 applies. □

The assumption on $\mathcal{L}$ satisfying Assumption 2.7(i) (which strongly depends on the choice of Q), can be checked with the help of a Foster–Lyapunov criterion; see [36] for instance.

REMARK 3.14 (Constructing limit processes with a slow speed of convergence).   The framework of the LLRW provides a large pool of toy examples. Let R be some Markov transition kernel on $\mathbb{R}$, and define $Q(y, A) = R(y, y + A)$, for any $y \in \mathbb{R}$ and A Borelian set, where $y + A = \{z \in \mathbb{R} : z - y \in A\}$. Let $(y_n)_{n \geq 0}$ be the LLRW defined in (3.20). Proposition 3.13 holds, and the limit process generated by $\mathcal{L} f(y) = \int_{\mathbb{R}} f(y+z) Q(y, dz) - f(y)$ is just a Markov chain generated by R indexed by a Poisson process. Precisely, if N_t is a Poisson process of intensity 1,

$$\Phi(\nu, t) = \mathbb{E}[\nu R^{N_t}].$$

This construction allows us to build a variety of limit processes for the LLRW, with a slow speed of convergence if needed. Indeed, choose R to be the Markov kernel of a sub-geometrically ergodic Markov chain converging to a stationary measure π at polynomial speed (for instance the kernels introduced in [26]); the limit process will inherit the slow speed of convergence. More precisely, there exist $\beta \geq 1$, a class of functions $\mathscr{G}$ and a function W such that

$$d_{\mathscr{G}}(\nu R^n, \pi) \leq \frac{\nu(W)}{(1+n)^\beta}.$$

Then

$$d_{\mathscr{G}}(\Phi(\nu, t), \pi) \leq \mathbb{E}\left[\frac{\nu(W)}{(1+N_t)^\beta}\right]$$

which goes to 0 at polynomial speed. Then, if $\sup_n \mathbb{E}[W(y_n)] < +\infty$, which could be proven via troncature arguments as in Section 3.2, we can use Theorem 2.8(iii) to conclude that (y_n) converges weakly toward π.

Note that another example of sub-geometrically ergodic process is provided in [14], Theorem 5.4. The elliptic diffusions mentioned in this article converge

slowly toward equilibrium, and could be approximated by a Euler scheme as in Section 3.3. In this example again, the use of troncature arguments to check Assumption 2.3 could be enough for Theorem 2.8(iii) to hold.

REMARK 3.15 (Speed of convergence under Doeblin condition). Assume there exists a measure ψ and $\varepsilon > 0$ such that for every y and measurable set A, we have

$$\int \mathbf{1}_{y+z \in A} Q(y, dz) \geq \varepsilon \psi(A).$$

It is the classical Doeblin condition, which ensures exponential uniform ergodicity in total variation distance. It is classic to prove that under this condition there exists an invariant distribution π, such that, for every μ and $t \geq 0$,

$$d_{\mathcal{F}}(\mu P_t, \pi) \leq e^{-t\varepsilon} d_{\mathcal{F}}(\mu, \pi) \leq e^{-t\varepsilon}.$$

Indeed, one can couple two trajectories as follows: choose the same jump times and, using the Doeblin condition, at each jumps, couple them with probability ε. The coupling time then follows an exponential distribution with parameter ε. Then the conclusion of Proposition 3.13 holds with $v = \varepsilon^{-1}$.

However, one can use the Doeblin argument directly with the inhomogeneous chain. Let us denote by (K_n) its sequence of transition kernels. From the Doeblin condition, we have, for every μ, v and $n \geq 0$,

$$d_{\mathcal{F}}(\mu K_n, v K_n) \leq (1 - \gamma_{n+1}\varepsilon) d_{\mathcal{F}}(\mu, v)$$

and as π is invariant for K_n (it is straighforward because π is invariant for Q) then

$$d_{\mathcal{F}}(\mu K_n, \pi) \leq (1 - \gamma_{n+1}\varepsilon) d_{\mathcal{F}}(\mu, \pi).$$

A recursion argument then gives

$$d_{\mathcal{F}}(\mathcal{L}(y_n), \pi) \leq \prod_{k=0}^{n} (1 - \gamma_{k+1}\varepsilon) d_{\mathcal{F}}(\mathcal{L}(y_0), \pi).$$

But

$$\prod_{k=0}^{n} (1 - \gamma_{k+1}\varepsilon) = \exp\left(\sum_{k=0}^{n} \ln(1 - \gamma_{k+1}\varepsilon)\right) \leq \exp\left(\sum_{k=0}^{n} \ln(1 - \gamma_{k+1}\varepsilon)\right)$$

$$\leq e^{-\varepsilon \sum_{k=0}^{n} \gamma_{k+1}}.$$

As a conclusion, Proposition 3.13 and the direct approach provide the same rate of convergence for the LLRW under Doeblin condition.

REMARK 3.16 (Nonconvergence in total variation).   Assume that $y_n \in \mathbb{R}_+$ and $Z_n = -y_n/2$. We then have that

$$y_n = \prod_{i=1}^{n} \widetilde{\Theta}_i \, y_0, \qquad \widetilde{\Theta}_i = \begin{cases} 1 & \text{with probability } 1 - \gamma_i, \\ \dfrac{1}{2} & \text{with probability } \gamma_i, \end{cases}$$

where $\widetilde{\Theta}_i$ are independent random variables. Borel–Cantelli's lemma entails that $(y_n)_{n \geq 0}$ converges to 0 almost surely and, here,

$$\mathcal{L} f(y) = f\left(\frac{y}{2}\right) - f(y).$$

A process with such a generator never hits 0 whenever it starts with a positive value and, then does not converge in total variation distance. Nevertheless, it is easy to prove that for any y and $t \geq 0$,

$$d_{\mathcal{G}}(\delta_y P_t, \delta_0) \leq \mathbb{E}\left[\frac{1}{2^{N_t}}\right] y \leq e^{-t/2} y,$$

where $\mathcal{G}$ is any class of functions included in $\{f \in \mathscr{C}_b^1 : \|f'\|_\infty \leq 1\}$, and (N_t) a Poisson process. In particular, Assumption 2.7(ii) holds and there is convergence of our chain to zero in distribution, as well as a rate of convergence in the Fortet–Mourier distance.

4. Proofs of theorems.

In the sequel, we consider the following classes of functions:

$$\mathscr{F}_1 := \{f \in \mathcal{D}(\mathcal{L}) : \mathcal{L} f \in \mathcal{D}(\mathcal{L}), \|f\|_\infty + \|\mathcal{L} f\|_\infty + \|\mathcal{L}\mathcal{L} f\|_\infty \leq 1\},$$

$$\mathscr{F}_2 := \left\{f \in \mathcal{D}(\mathcal{L}) \cap \mathscr{C}_b^{N_2} : \sum_{j \leq N_2} \|f^{(j)}\|_\infty \leq 1\right\},$$

$$\mathscr{F} := \mathscr{F}_1 \cap \mathscr{F}_2.$$

The class $\mathscr{F}_1$ is particularly useful to control $P_t f$ (see Lemma 4.1), and the class $\mathscr{F}_2$ enables us to deal with smooth and bounded functions (for the second part of the proof of Theorem 2.6). Note that an important feature of $\mathscr{F}$ is that Lemma A.1 holds for $\mathscr{F}_1 \cap \mathscr{F}_2$, so that $\mathscr{F}$ contains $\mathscr{C}_c^\infty$ "up to a constant."

Let us begin with preliminary remarks on the properties of the semigroup (P_t).

LEMMA 4.1 (Expansion of $P_t f$).   *Let $f \in \mathscr{F}_1$. Then, for all $t > 0$, $P_t f \in \mathscr{F}_1$ and*

$$\sup_{f \in \mathscr{F}_1} \|P_t f - f - t\mathcal{L} f\|_\infty \leq \frac{t^2}{2}.$$

PROOF. It is clear that $P_t f \in \mathcal{F}_1$, since for all $g \in \mathcal{D}(\mathcal{L})$, $P_t \mathcal{L} g = \mathcal{L} P_t g$ and $\|P_t g\|_\infty \leq \|g\|_\infty$. Now, if $f \in \mathcal{F}_1$, then

$$P_t f = f + \int_0^t P_s \mathcal{L} f \, ds = f + t \mathcal{L} f + K(f, t),$$

where $K(f, t) = P_t f - f - t \mathcal{L} f$. Using the mean value inequality, we have, for $x \in \mathbb{R}^D$,

$$|K(f, t)(x)| = \left| \int_0^t P_s \mathcal{L} f(x) \, ds - \mathcal{L} f(x) \right| \leq \int_0^t |P_s \mathcal{L} f(x) - \mathcal{L} f(x)| \, ds$$

$$\leq \int_0^t s \|\mathcal{L} \mathcal{L} f\|_\infty \, ds \leq \frac{t^2}{2},$$

which concludes the proof. □

PROOF OF THEOREM 2.6. For every $t \geq 0$, set $K(f, t) := P_t f - f - t \mathcal{L} f$ and recall that $m(t) = \sup\{n \geq 0 : t \geq \tau_n\}$. Then we have $Y_{\tau_{m(t)}} = Y_t$ and $\tau_{m(t)} \leq t < \tau_{m(t)+1}$. Let $0 < s < T$. Using the following telescoping sum, we have

$$d_{\mathcal{F}}\big(\mu_{t+s}, \Phi(\mu_t, s)\big)$$

$$= d_{\mathcal{F}}\big(\mu_{\tau_{m(t+s)}}, \Phi(\mu_{\tau_{m(t)}}, s)\big)$$

$$\leq d_{\mathcal{F}}\big(\Phi(\mu_{\tau_{m(t)}}, \tau_{m(t+s)} - \tau_{m(t)}), \Phi(\mu_{\tau_{m(t)}}, s)\big)$$

(4.1)

$$\quad + d_{\mathcal{F}}\big(\mu_{\tau_{m(t+s)}}, \Phi(\mu_{\tau_{m(t)}}, \tau_{m(t+s)} - \tau_{m(t)})\big)$$

$$\leq d_{\mathcal{F}}\big(\Phi(\mu_{\tau_{m(t)}}, \tau_{m(t+s)} - \tau_{m(t)}), \Phi(\mu_{\tau_{m(t)}}, s)\big)$$

$$\quad + \sum_{k=m(t)}^{m(t+s)-1} d_{\mathcal{F}}\left(\Phi\left(\mu_{\tau_{k+1}}, \sum_{j=k+2}^{m(t+s)} \gamma_j\right), \Phi\left(\mu_{\tau_k}, \sum_{j=k+1}^{m(t+s)} \gamma_j\right)\right),$$

with the convention $\sum_{k=i+1}^{i} = 0$. Our aim is now to bound each term of this sum. The first one is the simplest: indeed, we have $s \leq \tau_{m(t+s)+1} - \tau_{m(t)}$, so $s - \gamma_{m(t+s)+1} \leq \tau_{m(t+s)} - \tau_{m(t)}$ and $\tau_{m(t+s)} - \tau_{m(t)} \leq s + \gamma_{m(t)+1}$. Denoting by $u = s \wedge (\tau_{m(t+s)} - \tau_{m(t)})$ and $h = |\tau_{m(t+s)} - \tau_{m(t)} - s|$ we have, by the semigroup property,

$$d_{\mathcal{F}}\big(\Phi(\mu_t, \tau_{m(t+s)} - \tau_{m(t)}), \Phi(\mu_t, s)\big) = d_{\mathcal{F}}\big(\Phi(\Phi(\mu_t, u), h), \Phi(\mu_t, u)\big).$$

From Lemma 4.1, we know that for every $f \in \mathcal{F}_1$ and every probability measure ν,

$$\big|\Phi(\nu, h)(f) - \nu(f)\big| = |\nu(P_h f - f)| \leq h + \frac{h^2}{2} \leq \frac{3}{2} h,$$

for $h \leq 1$. It is then straightforward that

(4.2) $$d_{\mathcal{F}}\big(\Phi(\mu_t, \tau_{m(t+s)} - \tau_{m(t)}), \Phi(\mu_t, s)\big) \leq \frac{3}{2} h \leq \frac{3}{2} \gamma_{m(t)+1}.$$

Now, we provide bounds for the generic term of the telescoping sum in (4.1). Let $f \in \mathscr{F}_1$ and $m(t) \leq k \leq m(t+s) - 1$. On the one hand, using Lemma 4.1,

$$\Phi\left(\mu_{\tau_k}, \sum_{j=k+1}^{m(t+s)} \gamma_j\right)(f) = \mu_{\tau_k} P_{\sum_{j=k+1}^{m(t+s)} \gamma_j}(f)$$

$$= \mu_{\tau_k}(P_{\tau_{m(t+s)} - \tau_{k+1}} f)$$

$$+ \int_0^{\gamma_{k+1}} \mu_{\tau_k}(\mathcal{L} P_{\tau_{m(t+s)} - \tau_{k+1} + u} f) \, du$$

$$= \mu_{\tau_k}(P_{\tau_{m(t+s)} - \tau_{k+1}} f) + \gamma_{k+1} \mu_{\tau_k}(\mathcal{L} P_{\tau_{m(t+s)} - \tau_{k+1}} f)$$

$$+ K(P_{\tau_{m(t+s)} - \tau_{k+1}} f, \gamma_{k+1}).$$

On the other hand,

$$\mu_{\tau_{k+1}}(f) = \mu_{\tau_k}(f) + \gamma_{k+1} \mu_{\tau_k}(\mathcal{L}_k f)$$

so that

$$\Phi\left(\mu_{\tau_{k+1}}, \sum_{j=k+2}^{m(t+s)} \gamma_j\right)(f) = \mu_{\tau_{k+1}}(P_{\tau_{m(t+s)} - \tau_{k+1}} f)$$

$$= \mu_{\tau_k}(P_{\tau_{m(t+s)} - \tau_{k+1}} f) + \gamma_{k+1} \mu_{\tau_k}(\mathcal{L}_k P_{\tau_{m(t+s)} - \tau_{k+1}} f).$$

Henceforth,

$$\Phi\left(\mu_{\tau_{k+1}}, \sum_{j=k+2}^{m(t+s)} \gamma_j\right)(f) - \Phi\left(\mu_{\tau_k}, \sum_{j=k+1}^{m(t+s)} \gamma_j\right)(f)$$

$$\leq \gamma_{k+1} \mu_{\tau_k}\left((\mathcal{L}_k - \mathcal{L}) P_{\tau_{m(t+s)} - \tau_{k+1}} f\right) + K(P_{\tau_{m(t+s)} - \tau_{k+1}} f, \gamma_{k+1}).$$

Now, we bound the previous term using Assumption 2.1, Assumption 2.2 and Assumption 2.3. Let $m(t) \leq k \leq m(t+s) - 1$. Recall that, since $s < T$, $\tau_{m(t+s)} - \tau_{k+1} \leq \tau_{m(t+s)} - \tau_{m(t)+1} \leq (t+s) - t \leq T$. Then, for all $f \in \mathscr{F}_2$,

$$\left|\mu_{\tau_k}\left((\mathcal{L}_k - \mathcal{L}) P_{\tau_{m(t+s)} - \tau_{k+1}} f\right)\right| \leq \mu_{\tau_k}\left(\left|(\mathcal{L}_k - \mathcal{L}) P_{\tau_{m(t+s)} - \tau_{k+1}} f\right|\right)$$

$$\leq \mu_{\tau_k}\left(M_1 \chi_{d_1} \sum_{j \leq N_1} \left\|(P_{\tau_{m(t+s)} - \tau_{k+1}} f)^{(j)}\right\|_\infty \epsilon_k\right)$$

$$\leq \mu_{\tau_k}\left(M_1(N_1 + 1) C_T \chi_d \sum_{j \leq N_2} \left\|f^{(j)}\right\|_\infty \epsilon_k\right)$$

$$\leq M_1(N_1 + 1) C_T \mathbb{E}\left[\chi_d(y_k)\right] \sum_{j \leq N_2} \left\|f^{(j)}\right\|_\infty \epsilon_k$$

$$\leq M_1 M_2 (N_1 + 1) C_T \sum_{j \leq N_2} \| f^{(j)} \|_\infty \epsilon_k$$

$$\leq M_1 M_2 (N_1 + 1) C_T \epsilon_k.$$

Gathering the previous bounds entails

$$\sum_{k=m(t)}^{m(t+s)-1} d_{\mathscr{F}} \left(\Phi \left(\mu_{\tau_{k+1}}, \sum_{j=k+2}^{m(t+s)} \gamma_j \right), \Phi \left(\mu_{\tau_k}, \sum_{j=k+1}^{m(t+s)} \gamma_j \right) \right)$$

$$(4.3) \qquad \leq \sum_{k=m(t)}^{m(t+s)-1} \left(M_1 M_2 (N_1 + 1) C_T \gamma_{k+1} \epsilon_k + \frac{\gamma_{k+1}^2}{2} \right)$$

$$\leq (T + 1) \left(M_1 M_2 (N_1 + 1) C_T + \frac{1}{2} \right) (\gamma_{m(t)} \vee \epsilon_{m(t)}).$$

Thus, combining (4.1), (4.2) and (4.3) yields

$$(4.4) \qquad \sup_{s \leq T} d_{\mathscr{F}} \left(\mu_{t+s}, \Phi(\mu_t, s) \right) \leq C_T' (\gamma_{m(t)} \vee \epsilon_{m(t)}),$$

with $C_T' = \frac{3}{2} + (T + 1)(M_1 M_2 (N_1 + 1) C_T + \frac{1}{2})$. Then $(\mu_t)_{t \geq 0}$ is an asymptotic pseudotrajectory of Φ (with respect to $d_{\mathscr{F}}$).

Now, we turn to the study of the case $\lambda(\gamma, \epsilon) > 0$. For any $\lambda < \lambda(\gamma, \epsilon)$, we have (for n large enough) $\gamma_n \vee \epsilon_n \leq \exp(-\lambda \tau_n)$. Then, for any t large enough,

$$\gamma_{m(t)} \vee \epsilon_{m(t)} \leq e^{-\lambda \tau_{m(t)}} \leq e^{\lambda(t - \tau_{m(t)})} e^{-\lambda t} \leq e^{\lambda(\gamma, \epsilon)} e^{-\lambda t}.$$

Now, plugging this upper bound in (4.4), we get, for $\lambda < \lambda(\gamma, \epsilon)$,

$$(4.5) \qquad \sup_{s \leq T} d_{\mathscr{F}} \left(\mu_{t+s}, \Phi(\mu_t, s) \right) \leq e^{\lambda(\gamma, \epsilon)} C_T' e^{-\lambda t}.$$

Finally, we can deduce that

$$\limsup_{t \to +\infty} \frac{1}{t} \log \left(\sup_{0 \leq s \leq T} d \left(\mu_{t+s}, \Phi(\mu_t, s) \right) \right) \leq -\lambda$$

for any $\lambda < \lambda(\gamma, \epsilon)$, which concludes the proof of Theorem 2.6. $\quad \square$

PROOF OF THEOREM 2.8. The first part of the proof is an adaptation of [4]. Assume Assumption 2.7(i) and, without loss of generality, assume $M_3 > 1$. If $v > \lambda(\gamma, \epsilon)$, fix $\varepsilon > v - \lambda(\gamma, \epsilon)$, otherwise let $\varepsilon > 0$, and set $u := v - \varepsilon$, $T_\varepsilon := \varepsilon^{-1} \log M_3$. Since $u < \lambda(\gamma, \epsilon)$, and using (4.5), the following sequence of inequalities holds, for any $T \in [T_\varepsilon, 2T_\varepsilon]$ and $n \in \mathbb{N}$:

$$d_{\mathscr{G}}(\mu_{(n+1)T}, \pi) \leq d_{\mathscr{G}} \left(\mu_{(n+1)T}, \Phi(\mu_{nT}, T) \right) + d_{\mathscr{G}} \left(\Phi(\mu_{nT}, T), \pi \right)$$

$$\leq e^{\lambda(\gamma, \epsilon)} C_T' e^{-unT} + M_3 d_{\mathscr{G}}(\mu_{nT}, \pi) e^{-vT}$$

$$\leq e^{\lambda(\gamma, \epsilon)} C_T' e^{-unT} + d_{\mathscr{G}}(\mu_{nT}, \pi) e^{-uT},$$

with $C_T' = \frac{3}{2} + (T+1)(M_1 M_2 (N_1 + 1)C_T + \frac{1}{2})$. Denoting by $\delta_n := d_{\mathscr{G}}(\mu_{nT}, \pi)$ and $\rho := e^{-uT}$, the previous inequality turns into $\delta_{n+1} \leq e^{\lambda(\gamma,\epsilon)} C_T' \rho^n + \rho \delta_n$, from which we derive

$$\delta_n \leq n \rho^{n-1} C_T' e^{\lambda(\gamma,\epsilon)} + \rho^n \delta_0.$$

Hence, for every $n \geq 0$ and $T \in [T_\varepsilon, 2T_\varepsilon]$, we have

$$d_{\mathscr{G}}(\mu_{nT}, \pi) \leq e^{-(u-\varepsilon)nT}\left(M_5 + d_{\mathscr{G}}(\mu_0, \pi)\right)$$

with

$$M_5 = e^{\lambda(\gamma,\epsilon)}\left(\sup_{n \geq 0} n e^{-\varepsilon nT}\right)\left(\sup_{T \in [T_\varepsilon, 2T_\varepsilon]} C_T'\right).$$

Then, for any $t > T_\varepsilon$, let $n = \lfloor t T_\varepsilon^{-1} \rfloor$ and $T = t n^{-1}$. Then $T \in [T_\varepsilon, 2T_\varepsilon]$ and the following upper bound holds:

$$d_{\mathscr{G}}(\mu_t, \pi) \leq \left(M_5 + d_{\mathscr{G}}(\mu_0, \pi)\right)e^{-(u-\varepsilon)t}.$$

Now, assume Assumption 2.7(ii). For any (small) $\varepsilon > 0$, there exists $e^{\lambda(\gamma,\epsilon)}$ such that $\gamma_{m(t)} \vee \epsilon_{m(t)} \leq e^{\lambda(\gamma,\epsilon)} \exp(-(\lambda(\gamma,\epsilon)-\varepsilon)t)$. For any $\alpha \in (0,1)$, we have

$$d_{\mathscr{F} \cap \mathscr{G}}(\mu_t, \pi) \leq d_{\mathscr{F} \cap \mathscr{G}}\left(\mu_t, \Phi(\mu_{\alpha t}, (1-\alpha)t)\right) + d_{\mathscr{F} \cap \mathscr{G}}\left(\Phi(\mu_{\alpha t}, (1-\alpha)t), \pi\right)$$

$$(4.6) \qquad \leq C_{(1-\alpha)t}'\left(\gamma_{m(\alpha t)} \vee \epsilon_{m(\alpha t)}\right) + M_3 e^{-v(1-\alpha)t}$$

$$\leq M_4 e^{r(1-\alpha)t} e^{\lambda(\gamma,\epsilon)} e^{-(\lambda(\gamma,\epsilon)-\varepsilon)\alpha t} + M_3 e^{-v(1-\alpha)t}.$$

Optimizing (4.6) by taking $\alpha = (r+v)(r+v+\lambda(\gamma,\epsilon)-\varepsilon)^{-1}$, we get

$$d_{\mathscr{F} \cap \mathscr{G}}(\mu_t, \pi) \leq M_5 \exp\left(-\frac{v(\lambda(\gamma,\epsilon)-\varepsilon)}{r+v+\lambda(\gamma,\epsilon)-\varepsilon}t\right),$$

with $M_5 = M_4 e^{\lambda(\gamma,\epsilon)} + M_3$, which depends on ε only through M_3.

Lastly, assume Assumption 2.1(iii). Denote by $\mathcal{K}$ the set of probability measures ν such that

$$\nu(W) < M = \sup_{n \geq 0} \mathbb{E}[W(y_n)].$$

Let $\varepsilon > 0$ and $K = \{x \in \mathbb{R}^D : W(x) \leq M/\varepsilon\}$. For every $\nu \in \mathcal{K}$, using Markov's inequality, it is clear that

$$\nu(K^C) \leq \frac{\varepsilon}{M}\nu(W) \leq \varepsilon.$$

Then $\mathcal{K}$ is a relatively compact set (by Prokhorov's theorem). The measure π is an attractor in the sense of [4], which means that $\lim d_{\mathscr{G}}(\Phi(\nu, t), \pi) = 0$ uniformly in $\nu \in \mathcal{K}$. Then, since for any $t > 0$, $\mu_t \in \mathcal{K}$, we can apply [4], Theorem 6.10, to achieve the proof. $\square$

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PROOF OF THEOREM 2.12. We shall prove the convergence of the sequence of processes $(Y_s^{(t)})_{0 \le s \le T}$, as $t \to +\infty$, toward $(X_s^\pi)_{0 \le s \le T}$ in the Skorokhod space $D([0, T])$, for any $T > 0$. Then, using [6], Theorem 16.7, this convergence entails Theorem 2.12, that is, convergence of the sequence $(Y^{(t)})$ in $D([0, \infty))$.

Let $T > 0$. The proof of functional convergence classically relies on proving the convergence of finite-dimensional distributions, on the one hand, and tightness, on the other hand. First, we prove the former, which is the first part of Theorem 2.12. We choose to prove the convergence of the finite-dimensional distributions in the case $m = 2$. The proof for the general case is similar but with a laborious notation. Denote by $T_{u,v}g(y) := \mathbb{E}[g(Y_v) \mid Y_u = y]$. With this notation, (4.4) becomes

$$\sup_{s \le T} \sup_{g \in \mathscr{F}} (\mu_t T_{t,t+s} g - \mu_t P_s g) \le C_T'(\gamma_{m(t)} \vee \epsilon_{m(t)}).$$

This upper bound does not depend on μ_t, so, for any probability distribution ν, we have

$$\sup_{s \le T} \sup_{g \in \mathscr{F}} (\nu T_{t,t+s} g - \nu P_s g) \le C_T'(\gamma_{m(t)} \vee \epsilon_{m(t)}).$$

This inequality implies that, for any ν,

$$(4.7) \qquad \sup_{s_1 \le s_2 \le T} \sup_{g \in \mathscr{F}} (\nu T_{t+s_1,t+s_2} g - \nu P_{s_2-s_1} g) \le C_T'(\gamma_{m(t)} \vee \epsilon_{m(t)}),$$

which converges toward 0 as $t \to +\infty$. From now on, we denote, for any function f, $\widehat{f}_x(y) := f(x, y)$. If f is a smooth function (say in $\mathscr{C}_c^\infty$ with enough derivatives bounded), $\widehat{f}_.(\cdot) \in \mathscr{F}$. On the one hand, for $0, s_1 < s_2 < T$,

$$\mathbb{E}[f(X_{s_1}^\pi, X_{s_2}^\pi)] = \int P_{s_2-s_1} \widehat{f}_y(y) \pi(dy) = \pi P_{s_2-s_1} \widehat{f}_.(\cdot).$$

On the other hand, we have

$$\mathbb{E}[f(Y_{s_1}^{(t)}, Y_{s_2}^{(t)})] = \mathbb{E}[\mathbb{E}[f(Y_{s_1}^{(t)}, Y_{s_2}^{(t)}) \mid Y_{s_1}^{(t)}]] = \mathbb{E}[T_{t+s_1,t+s_2} \widehat{f}_{Y_{t+s_1}}(Y_{t+s_1})]$$

$$= T_{0,t+s_1}(T_{t+s_1,t+s_2} \widehat{f}_.(\cdot)).$$

We have the following triangle inequality:

$$|\mathbb{E}[f(Y_{s_1}^{(t)}, Y_{s_2}^{(t)})] - \mathbb{E}[f(X_{s_1}^\pi, X_{s_2}^\pi)]| = |T_{0,t+s_1}(T_{t+s_1,t+s_2} \widehat{f}_.(\cdot)) - \pi P_{s_2-s_1} \widehat{f}_.(\cdot)|$$

$$(4.8) \qquad\qquad \le |T_{0,t+s_1}(T_{t+s_1,t+s_2} \widehat{f}_.(\cdot) - P_{s_2-s_1} \widehat{f}_.(\cdot))|$$

$$+ |T_{0,t+s_1}(P_{s_2-s_1} \widehat{f}_.(\cdot)) - \pi P_{s_2-s_1} \widehat{f}_.(\cdot)|.$$

First, using (4.7), and if $\widehat{f}_.(\cdot) \in \mathscr{F}$,

$$\lim_{t \to \infty} T_{0,t+s_1}(T_{t+s_1,t+s_2} \widehat{f}_.(\cdot) - P_{s_2-s_1} \widehat{f}_.(\cdot))$$

$$= \lim_{t \to \infty} \mu_{t+s_1}(T_{t+s_1,t+s_2} \widehat{f}_.(\cdot) - P_{s_2-s_1} \widehat{f}_.(\cdot)) = 0.$$

Second, $P_{s_2-s_1}f.(\cdot) \in \mathscr{C}_b^0$ and, using Theorem 2.8,

$$\lim_{t\to\infty} T_{0,t+s_1}\big(P_{s_2-s_1}\widehat{f}.(\cdot)\big) - \pi\, P_{s_2-s_1}\widehat{f}.(\cdot) = 0.$$

From (4.8), it is straightforward that, for a smooth f,

$$\lim_{t\to\infty}\big|\mathbb{E}\big[f\big(Y_{s_1}^{(t)}, Y_{s_2}^{(t)}\big)\big] - \mathbb{E}\big[f\big(X_{s_1}^\pi, X_{s_2}^\pi\big)\big]\big| = 0,$$

and applying Lemma A.1 achieves the proof of finite dimensional convergence for $m = 2$.

To prove tightness, which is the second part of Theorem 2.12, we need the following lemma, whose proof is postponed to the Appendix.

LEMMA 4.2 (Martingale properties).   *Let f be a continuous and bounded function. The process $(\widehat{M}_n^f)_{n\geq 0}$, defined for every $n \geq 0$ by*

$$\widehat{M}_n^f = f(y_n) - f(y_0) - \sum_{k=0}^{n-1} \gamma_{k+1}\mathcal{L}_k f(y_k),$$

is a martingale, with

$$\langle \widehat{M}^f \rangle_n = \sum_{k=0}^{n-1} \gamma_{k+1}\Gamma_k f(y_k).$$

Moreover, under Assumption 2.11, if $d \geq d_2$ then for every $N \geq 0$, there exist a constant $M_7 > 0$ (depending on N and y_0) such that

$$\mathbb{E}\Big[\sup_{n\leq N} \chi_{d_1}(y_n)\Big] \leq M_7.$$

Now, define

$$M_s^{(t,i)} = \widehat{M}_{m(t+s)}^{\varphi_i} - \widehat{M}_{m(t)}^{\varphi_i},$$

$$A_s^{(t,i)} = \varphi_i(Y_t) + \int_{\tau_{m(t)}}^{\tau_{m(t+s)}} \mathcal{L}_{m(u)}\varphi_i(Y_u)\,du = \varphi_i(y_{m(t)})$$

$$+ \sum_{k=m(t)}^{m(t+s)-1} \gamma_{k+1}\mathcal{L}_k\varphi_i(y_k)$$

and

$$Y_s^{(t,i)} = \varphi_i\big(Y_s^{(t)}\big).$$

With this notation and Lemma 4.2 we have

$$Y_s^{(t,i)} = A_s^{(t,i)} + M_s^{(t,i)}$$

and $(M_s^{(t,i)})_{s \geq 0}$ is a martingale with quadratic variation

$$\langle M^{(t,i)} \rangle_s = \int_{\tau_{m(t)}}^{\tau_{m(t+s)}} \Gamma_{m(u)} \varphi_i(Y_u) \, du,$$

where Γ_n is as in Assumption 2.11. From the convergence of finite-dimensional distributions, for every $s \in [0, T]$, the sequence $(Y_s^{(t)})_{t \geq 0}$ is tight. It is then enough, from the Aldous–Rebolledo criterion (see Theorems 2.2.2 and 2.3.2 in [27]) and Lemma 4.2 to show that: for every $S \geq 0$, $\varepsilon, \eta > 0$, there exists a $\delta > 0$ and $t_0 > 0$ with the property that whatever the family of stopping times $(\sigma^{(t)})_{t \geq 0}$, with $\sigma^{(t)} \leq S$, for every $i \in \{1, \dots D\}$,

$$(4.9) \qquad \sup_{t \geq t_0} \sup_{\theta \leq \delta} \mathbb{P}\big(\big| \langle M^{(t,i)} \rangle_{\sigma^{(t)}} - \langle M^{(t,i)} \rangle_{\sigma^{(t)} + \theta} \big| \geq \eta \big) \leq \varepsilon$$

and

$$(4.10) \qquad \sup_{t \geq t_0} \sup_{\theta \leq \delta} \mathbb{P}\big(\big| A_{\sigma^{(t)}}^{(t,i)} - A_{\sigma^{(t)} + \theta}^{(t,i)} \big| \geq \eta \big) \leq \varepsilon.$$

We have, using Assumption 2.11,

$$A_{\sigma^{(t)} + \theta}^{(t,i)} - A_{\sigma^{(t)}}^{(t,i)} = \int_{\tau_{m(t+\sigma^{(t)})}}^{\tau_{m(t+\sigma^{(t)}+\theta)}} \mathcal{L}_{m(u)} \varphi_i(Y_u) \, du \leq \int_{\tau_{m(t+\sigma^{(t)})}}^{\tau_{m(t+\sigma^{(t)}+\theta)}} M_6 \chi_{d_2}(Y_u) \, du$$

$$\leq M_6 |\tau_{m(t+\sigma^{(t)}+\theta)} - \tau_{m(t+\sigma^{(t)})}| \sup_{r \leq T} \chi_{d_2}(Y_r).$$

From the definition of τ_n,

$$|\tau_{m(t+\sigma^{(t)}+\theta)} - \tau_{m(t+\sigma^{(t)})}| \leq \theta + \gamma_{m(t)+1},$$

and then, using Lemma 4.2 and Markov's inequality,

$$\mathbb{P}\big(\big| A_{\sigma^{(t)}}^{(t,i)} - A_{\sigma^{(t)} + \theta}^{(t,i)} \big| \geq \eta \big) \leq \frac{M_6 (\theta + \gamma_{m(t_0)+1})}{\eta} \mathbb{E}\Big[\sup_{s \leq T} \chi_{d_2}(Y_r) \Big]$$

$$\leq M_6 M_7 \frac{(\delta + \gamma_{m(t_0)+1})}{\eta}.$$

Proving inequality (4.9) is done in a similar way and achieves the proof. $\square$

APPENDIX

A.1. General appendix.

LEMMA A.1 (Weak convergence and $d_{\mathscr{F}}$). *Assume that $\mathscr{F}$ is a star domain with respect to 0 (i.e., if $f \in \mathscr{F}$ then $\lambda f \in \mathscr{F}$ for $\lambda \in [0, 1]$). Let (μ_n), μ be probability measures. If $\lim_{n \to \infty} d_{\mathscr{F}}(\mu_n, \mu) = 0$ and, for every $g \in \mathscr{C}_c^\infty$, there exists $\lambda > 0$ such that $\lambda g \in \mathscr{F}$, then (μ_n) converges weakly toward μ. If $\mathscr{F} \subseteq \mathscr{C}_b^1$, then $d_{\mathscr{F}}$ metrizes the weak convergence.*

PROOF. Let $f \in \mathscr{C}_b^0, g \in \mathscr{C}_c^\infty$. Note that $fg \in \mathscr{C}_c^0$ and, using Weierstrass' theorem, it is well known that, for all $\varepsilon > 0$, there exists $\varphi \in \mathscr{C}_c^\infty$ such that $\|fg - \varphi\|_\infty \le \varepsilon$. By hypothesis, and since $\mathscr{F}$ is a star domain, there exists $\lambda > 0$ such that $\lambda g, \lambda \varphi \in \mathscr{F}$. Then

$$|\mu_n(fg) - \mu(fg)| \le |\mu_n(fg) - \mu_n(\varphi)| + \frac{1}{\lambda}|\mu_n(\lambda\varphi) - \mu(\lambda\varphi)| + |\mu(fg) - \mu(\varphi)|;$$

thus, $\limsup_{n\to\infty} |\mu_n(fg) - \mu(fg)| \le 2\varepsilon$. Now,

$$\begin{aligned}
|\mu_n(f) - \mu(f)| &\le |\mu_n(f - fg) - \mu(f - fg)| + |\mu_n(fg) - \mu(fg)| \\
&\le \|f\|_\infty |\mu_n(1 - g) - \mu(1 - g)| + |\mu_n(fg) - \mu(fg)| \\
&\le \frac{\|f\|_\infty}{\lambda} |\mu_n(\lambda g) - \mu(\lambda g)| + |\mu_n(fg) - \mu(fg)|
\end{aligned}$$

so that $\limsup_{n\to\infty} |\mu_n(f) - \mu(f)| \le 2\varepsilon$, for any $\varepsilon > 0$, which concludes the proof.

Now, assuming $\mathscr{F} \subseteq \mathscr{C}_b^1$, use [10], Theorem 5.6. Then convergence with respect to $d_{\mathscr{F}}$ is equivalent to weak convergence. Indeed, $d_{\mathscr{C}_b^1}$ is the well-known Fortet–Mourier distance, which metrizes the weak topology. It is also the Wasserstein distance W_δ, with respect to the distance δ such that

$$\forall x, y \in \mathbb{R}^D, \qquad \delta(x, y) = \sup_{f \in \mathscr{C}_b^1} |f(x) - f(y)| = |x - y| \wedge 2.$$

See also [38], Theorem 4.4.2. $\square$

PROOF OF LEMMA 4.2. Let $\mathscr{F}_n = \sigma(y_0, \ldots, y_n)$ be the natural filtration. Classically, we have

$$\begin{aligned}
\mathbb{E}[\widehat{M}_{n+1}^f \mid \mathscr{F}_n] &= \mathbb{E}\left[f(y_{n+1}) - f(y_0) - \sum_{k=0}^n \gamma_{k+1}\mathcal{L}_k f(y_k) \,\middle|\, \mathscr{F}_n\right] \\
&= f(y_n) + \gamma_{n+1}\mathcal{L}_n f(y_n) - f(y_0) - \sum_{k=0}^n \gamma_{k+1}\mathcal{L}_k f(y_k) \\
&= \widehat{M}_n^f.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathbb{E}[(\widehat{M}_{n+1}^f)^2 \mid \mathscr{F}_n] &= \mathbb{E}\left[f(y_{n+1})^2 + f(y_0)^2 + \left(\sum_{k=0}^n \gamma_{k+1}\mathcal{L}_k f(y_k)\right)^2 \,\middle|\, \mathscr{F}_n\right] \\
&\quad - \mathbb{E}\left[2f(y_{n+1})\left(f(y_0) + \sum_{k=0}^n \gamma_{k+1}\mathcal{L}_k f(y_k)\right) \,\middle|\, \mathscr{F}_n\right]
\end{aligned}$$

$$+ \mathbb{E}\left[2f(y_0)\left(\sum_{k=0}^{n} \gamma_{k+1}\mathcal{L}_k f(y_k)\right) \Big| \mathscr{F}_n \right]$$

$$= f(y_n)^2 + \gamma_{n+1}\mathcal{L}_n f^2(y_n) + f(y_0)^2 + \left(\sum_{k=0}^{n} \gamma_{k+1}\mathcal{L}_k f(y_k)\right)^2$$

$$- 2(f(y_n) + \gamma_{n+1}\mathcal{L}_n f(y_n))\left(f(y_0) + \sum_{k=0}^{n} \gamma_{k+1}\mathcal{L}_k f(y_k)\right)$$

$$+ 2f(y_0)\left(\sum_{k=0}^{n} \gamma_{k+1}\mathcal{L}_k f(y_k)\right).$$

Henceforth,

$$\mathbb{E}\big[(\widehat{M}_{n+1}^f)^2 \mid \mathscr{F}_n\big] = \gamma_{n+1}\mathcal{L}_n f^2(y_n) + 2\gamma_{n+1}\mathcal{L}_n f(y_n)\left(\sum_{k=0}^{n-1} \gamma_{k+1}\mathcal{L}_k f(y_k)\right)$$

$$+ (\gamma_{n+1}\mathcal{L}_n f(y_n))^2 - 2f(y_n)\gamma_{n+1}\mathcal{L}_n f(y_n)$$

$$- 2\gamma_{n+1}\mathcal{L}_n f(y_n)\left(f(y_0) + \sum_{k=0}^{n} \gamma_{k+1}\mathcal{L}_k f(y_k)\right)$$

$$+ 2f(y_0)\gamma_{n+1}\mathcal{L}_n f(y_n) + (m_n^f)^2$$

$$= (\widehat{M}_n^f)^2 + \gamma_{n+1}\mathcal{L}_n f^2(y_n) - (\gamma_{n+1}\mathcal{L}_n f(y_n))^2$$

$$- 2f(y_n)\gamma_{n+1}\mathcal{L}_n f(y_n)$$

$$= (\widehat{M}_n^f)^2 + \gamma_{n+1}\Gamma_n f.$$

Now, on the first hand, using Assumption 2.11,

$$\mathbb{E}\big[\langle \widehat{M}^{\chi_{d_2}}\rangle_N\big] = \mathbb{E}\left[\sum_{k=0}^{N-1} \gamma_{k+1}\Gamma_{k+1}\chi_{d_2}(y_k)\right] \le M_6 \sum_{k=0}^{N-1} \gamma_{k+1}\mathbb{E}\big[\chi_d(y_k)\big]$$

$$\le M_2 M_6 \sum_{k=0}^{N-1} \gamma_{k+1},$$

and then Doob's inequality gives

$$\mathbb{E}\Big[\Big(\sup_{n \le N} \widehat{M}_n^{\chi_{d_2}}\Big)^2\Big]^{1/2} \le 2\mathbb{E}\big[\langle \widehat{M}^{\chi_{d_2}}\rangle_N\big]^{1/2} \le C,$$

for some constant C which only depends on N. On the other hand, from Lemma 4.2 and Assumption 2.11,

$$\sup_{n \le N} \chi_{d_2}(y_n) \le \chi_{d_2}(y_0) + M_6 \sum_{k=0}^{N-1} \gamma_{k+1}\sup_{n \le k}\chi_{d_2}(y_n) + \sup_{n \le N}\widehat{M}_n^{\chi_{d_2}}.$$

Using the triangle inequality, we then have

$$\mathbb{E}\bigg[\Big(\sup_{n\leq N}\chi_{d_2}(y_n)\Big)^2\bigg]^{1/2}$$

$$\leq \mathbb{E}\big[(\chi_{d_2}(y_0))^2\big]^{1/2}$$

$$+ M_6 \sum_{k=0}^{N-1}\gamma_{k+1}\mathbb{E}\bigg[\Big(\sup_{n\leq k}\chi_{d_2}(y_n)\Big)^2\bigg]^{1/2} + \mathbb{E}\bigg[\Big(\sup_{n\leq N}\widehat{M}_n^{\chi_{d_2}}\Big)^2\bigg]^{1/2}.$$

Then, using (discrete) Grönwall's lemma as well as Cauchy–Schwarz's inequality completes the proof. □

A.2. Appendix for the penalized bandit algorithm.

PROOF OF PROPOSITION 3.6.   The unique solution of the ordinary differential equation $y'(t) = a - by(t)$ with initial condition x is given by

$$\Psi(x,t) = \begin{cases} \Big(x - \dfrac{a}{b}\Big)e^{-bt} + \dfrac{a}{b} & \text{if } b > 0, \\ x + at & \text{if } b = 0. \end{cases}$$

First, assume that $b > 0$ and let $t \in [0, T]$. We have, for $x > 0$,

$$P_t f(x) = \mathbb{E}_x\big[f(X_t)\big]$$

$$= f\big(\Psi(x,t)\big)\mathbb{P}_x(T > t) + \mathbb{E}_x\big[f(X_t) \mid T \leq t\big]\mathbb{P}_x(T \leq t)$$

(A.1)
$$= f\big(\Psi(x,t)\big)\exp\Big(-\int_0^t (c + d\Psi(x,s))\,ds\Big)$$

$$+ \int_0^t P_{t-u}f\big(\Psi(x,u) + 1\big)(c + d\Psi(x,u))$$

$$\times \exp\Big(-\int_0^u (c + d\Psi(x,s))\,ds\Big)\,du.$$

At this stage, the smoothness of the right-hand side of (A.1) with respect to x is not clear. Let $0 < \varepsilon < \min(a/b, 1/2)$. If $0 \leq x \leq a/b - \varepsilon$, use the substitution

$$v = \Psi(x,u), \qquad u = \varphi(x,v) = \frac{1}{b}\log\Big(\frac{x - \frac{a}{b}}{v - \frac{a}{b}}\Big),$$

to get

$$P_t f(x) = f\big(\Psi(x,t)\big)\exp\Big(-\int_0^t (c + d\Psi(x,s))\,ds\Big)$$

$$+ \int_x^{\Psi(x,t)} P_{t-\varphi(x,v)}f(v + 1)\exp\Big(-\int_0^{\varphi(x,v)} (c + d\Psi(x,s))\,ds\Big)$$

$$\times \frac{c + dv}{a - bv}\,dv.$$

Note that $\Psi(x, t) \leq \Psi(a/b - \varepsilon, t) < a/b$, so that $a - bv \neq 0$. Since $s \mapsto P_s f(x)$, Ψ, φ and f are smooth, $x \mapsto P_t f(x) \in \mathscr{C}^N([o, a/b - \varepsilon])$. The reasoning holds with the same substitution for $x \geq a/b + \varepsilon$, so that $P_t f \in \mathscr{C}^N(\mathbb{R}_+ \setminus \{a/b\})$. Now, if $x > a/b - \varepsilon$, for any $u > 0$,

$$\Psi(x, u) + 1 \geq a/b + 1 - \varepsilon \geq a/b + \varepsilon,$$

so $x \mapsto P_{t-u} f(\Psi(x, u) + 1)$ is smooth. Thus, the right-hand side of (A.1) is smooth as well and $P_t f \in \mathscr{C}^N(\mathbb{R}_+)$.

Now, let us show that the semigroup generated by $\mathcal{L}$ has bounded derivatives. Note that it is possible to mimic this proof for the example of the WRW treated in Section 3.1 when the derivatives of $P_t f$ are not explicit. Let $\mathcal{A}_n f = f^{(n)}$, $\mathcal{J} f(x) = f(x + 1) - f(x)$ and $\psi_n(s) = P_{t-s} \mathcal{A}_n P_s f$ for $0 \leq n \leq N$. So, $\psi_n'(s) = P_{t-s}(\mathcal{A}_n \mathcal{L} - \mathcal{L} \mathcal{A}_n) P_s f$. It is clear that $\mathcal{A}_{n+1} = \mathcal{A}_1 \mathcal{A}_n$, that $\mathcal{A}_n \mathcal{J} = \mathcal{J} \mathcal{A}_n$ and that

$$\mathcal{L}g(x) = (a - bx)\mathcal{A}_1 g(x) + (c + dx)\mathcal{J} g(x).$$

It is straightforward by induction that

$$\mathcal{A}_n \mathcal{L}g = \mathcal{L}\mathcal{A}_n g - nb\mathcal{A}_n g + nd\mathcal{J}\mathcal{A}_{n-1}g,$$

so the following inequality holds:

$$(\mathcal{A}_n \mathcal{L} - \mathcal{L}\mathcal{A}_n)g \leq -nb\mathcal{A}_n g + 2|d|n\|\mathcal{A}_{n-1}g\|_\infty.$$

Hence,

$$\psi_n'(s) \leq -nb\psi_n(s) + 2|d|n\|\mathcal{A}_{n-1} P_s f\|_\infty.$$

In particular, $\psi_1'(s) \leq -b\psi_1(s) + 2d\|f\|_\infty$, so, by Grönwall's inequality,

$$\psi_1(s) \leq \left(\psi_1(0) - \frac{2|d|}{b}\|f\|_\infty\right)e^{-bs} + \frac{2|d|}{b}\|f\|_\infty \leq \|f'\|_\infty + \frac{2d}{b}\|f\|_\infty.$$

Let us show by induction that

$$(A.2) \qquad \psi_n(s) \leq \sum_{k=0}^n \left(\frac{2|d|}{b}\right)^{n-k} \|f^{(k)}\|_\infty.$$

If (A.2) is true for some $n \geq 1$ (we denote by K_n its right-hand side), then for all $t < T$, $\psi_n(t) \leq K_n$ and, since $\mathcal{A}_n P_t(-f) = -\mathcal{A}_n P_t f$, $|\psi_n(t)| \leq K_n$, so $\|\mathcal{A}_n P_s f\|_\infty \leq K_n$. Then we deduce that $\psi_{n+1}'(s) \leq -(n + 1)b\psi_{n+1}(s) + 2(n + 1)dK_n$. Use Grönwall's inequality once more to have $\psi_{n+1}(s) \leq K_{n+1}$ and achieve the proof by induction. In particular, taking $s = t$ in (A.2) provides $\mathcal{A}_n P_t f \leq K_n$ and, since $\mathcal{A}_n P_t(-f) = -\mathcal{A}_n P_t f$, $\mathcal{A}_n P_t f \leq K_n$. As a conclusion, for $n \in \{0, \ldots, N\}$,

$$\|(P_t f)^{(n)}\|_\infty \leq \sum_{k=0}^n \left(\frac{2|d|}{b}\right)^{n-k} \|f^{(k)}\|_\infty,$$

which concludes the proof when $b > 0$.

The case $b = 0$ is dealt with in a similar way. We use the substitution $\varphi(x, v) = (v - x)/a$ in (A.1), which is enough to prove smoothness [this time, $\Psi(x, \cdot)$ is a diffeomorphism for any $x \geq 0$[, and it is easy to mimic the proof to obtain the following estimates, for $s \leq t$:

$$|\psi_n(s)| \leq \sum_{k=0}^{n} \frac{n!}{k!} (2|d|T)^{n-k} \|f^{(k)}\|_\infty.$$

$\square$

PROOF OF LEMMA 3.8.   First, we shall prove that Assumption 2.1 holds; let

$$y \in \text{Supp}(\mathscr{L}(y_n^{(l,\delta)})) = [0, \delta\sqrt{n}].$$

Note that $\tilde{I}_n^0(y), I_n^0(y) \leq 1$ and $\tilde{I}_n^1(y), I_n^1(y) \leq 0$, so if $y_n^{(l,\delta)} \leq \delta\gamma_{n+1}^{-1} - 1$, then $y_{n+1}^{(l,\delta)} \leq \delta\gamma_{n+1}^{-1}$. For $f \in \mathscr{F}$,

$$\begin{aligned}
|\mathcal{L}_n^{(l,\delta)} f(y) - \mathcal{L}_n f(y)| &\leq \gamma_{n+1}^{-1} \mathbb{E}[f(y_{n+1}^{(l,\delta)}) - f(y_{n+1}) \mid y_n = y_n^{(l,\delta)} = y] \\
&\leq \frac{\mathbf{1}_{y \geq \delta\gamma_{n+1}^{-1} - 1}}{\gamma_{n+1}} \left(p_0(1 - \gamma_n y) | f(\delta\gamma_{n+1}^{-1}) - f(y + I_n^0(y))| \right. \\
&\quad \left. + \tilde{p}_0(1 - \gamma_n y) | f(\delta\gamma_{n+1}^{-1}) - f(y + \tilde{I}_n^0(y))| \right) \\
&\leq \frac{\|f'\|_\infty \mathbf{1}_{y \geq \delta\gamma_{n+1}^{-1} - 1}}{\gamma_{n+1}} \left(p_0(1 - \gamma_n y) + \tilde{p}_0(1 - \gamma_n y) \right) \\
&\leq \frac{y + 1}{\delta} \|f'\|_\infty \mathbf{1}_{y \geq \delta\gamma_{n+1}^{-1} - 1} \\
&\leq \frac{(y + 1)^2}{\delta^2} \|f'\|_\infty \gamma_{n+1}.
\end{aligned}$$

Using this inequality and (3.13), we can bound the difference between $\mathcal{L}_n^{(l,\delta)}$ and $\mathcal{L}$ defined in (3.6):

(A.3)
$$\begin{aligned}
|\mathcal{L}_n^{(l,\delta)} f(y) - \mathcal{L}f(y)| &\leq |\mathcal{L}_n^{(l,\delta)} f(y) - \mathcal{L}_n f(y)| + |\mathcal{L}_n f(y) - \mathcal{L}f(y)| \\
&= \chi_3(y)(\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty)\mathcal{O}(\gamma_n).
\end{aligned}$$

Note that the notation $\mathcal{O}$ depends here on l and δ, but is uniform over y and f.

Assumption 2.2 holds, since it takes into account only the limit process generated by $\mathcal{L}$, and it is a consequence of Proposition 3.6: for $n \leq 3$,

$$\|(P_t f)^{(n)}\|_\infty \leq \sum_{k=0}^{n} \left(\frac{2|p_0'(1)|}{p_1(1)} \right)^{n-k} \|f^{(k)}\|_\infty.$$

Now, we shall check a Lyapunov criterion for the chain $(y_n^{(l,\delta)})_{n \geq 0}$, in order to ensure Assumption 2.3. Taking $V(y) = e^{\theta y}$, where (small) $\theta > 0$ will be chosen

afterwards, we have, for $n \geq l$ and $y \leq \delta\gamma_n^{-1}$,

$$\mathcal{L}_n^{(l,\delta)}V(y) \leq \gamma_{n+1}^{-1}\mathbb{E}\big[V\big((y+I_n(y))\wedge\delta\sqrt{n}\big) - V(y)\big]$$

$$\leq \gamma_{n+1}^{-1}\mathbb{E}\big[V(y+I_n(y)) - V(y)\big]$$

$$\leq V(y)\sqrt{n+1}\big(\mathbb{E}[e^{\theta I_n(y)}] - 1\big).$$

Let $\varepsilon > 0$; we are going to decompose $I_n(y)$. The first term is

$$\sqrt{n+1}\Big(\exp\Big(\frac{\sqrt{n+1}-\sqrt{n}-1}{\sqrt{n}}\theta y\Big) - 1\Big)p_1(1-\gamma_n y)$$

$$\leq \sqrt{n+1}\Big(\frac{\sqrt{n+1}-\sqrt{n}-1}{\sqrt{n}}\theta y + \frac{1}{2}\Big(\frac{\sqrt{n+1}-\sqrt{n}-1}{\sqrt{n}}\theta y\Big)^2\Big)$$

$$\times\, p_1(1-\gamma_n y)$$

$$\leq \Big(-\alpha_n\theta y + \frac{\alpha_n^2}{2\sqrt{n+1}}\theta^2 y^2\Big)p_1(1-\gamma_n y) \leq \theta y\Big(-\alpha_n + \frac{\alpha_n^2}{2}\theta\delta\Big)$$

$$\times\, p_1(1-\gamma_n y)$$

$$\leq \Big(\varepsilon + \Big(-1 + \frac{\theta\delta}{2}\Big)\Big)\theta y \qquad \text{for } n \text{ large,}$$

where $\alpha_n = (1-\sqrt{n+1}+\sqrt{n})\gamma_n\gamma_{n+1}^{-1}$. There exists $\xi^{(\delta)}$, such that $1-\delta \leq \xi^{(\delta)} \leq 1$ and the second term writes

$$\sqrt{n+1}\Big(\exp\Big(\theta + \frac{\sqrt{n+1}-\sqrt{n}-1}{\sqrt{n}}\theta y\Big) - 1\Big)p_0(1-\gamma_n y)$$

$$\leq \sqrt{n+1}\,p_0(1-\gamma_n y)(e^\theta - 1)$$

$$\leq -\sqrt{n+1}\gamma_n y p_0'(\xi^{(\delta)})(e^\theta - 1) \leq (\varepsilon - (e^\theta - 1)p_0'(1))y \qquad \text{for } n \text{ large.}$$

The third term is negative, and the fourth term writes

$$\sqrt{n+1}\Big(\exp\Big(\frac{\theta}{\sqrt{n+1}} + \frac{n-\sqrt{n(n+1)}}{\sqrt{n(n+1)}}\theta y\Big) - 1\Big)\widetilde{p}_0(1-\gamma_n y)$$

$$\leq \sqrt{n+1}\Big(\exp\Big(\frac{\theta}{\sqrt{n+1}}\Big) - 1\Big) \leq \theta + \varepsilon \qquad \text{for } n \text{ large.}$$

Hence, there exists some (deterministic) $n_0 \geq l$ such that, for $n \geq n_0$,

$$\mathcal{L}_n^{(l,\delta)}V(y) \leq V(y)\Big[\theta + \varepsilon - y\Big(p_0'(1)(e^\theta - 1) - \Big(\theta + \frac{\theta\delta}{2}\Big)p_1(1) + \epsilon(1+\theta)\Big)\Big].$$

Then, for $\varepsilon, \delta, \theta$ small enough, there exists $\widetilde{\alpha} > 0$ such that, for $n \geq n_0$ and for any $M \geq (\theta + \epsilon)\widetilde{\alpha}^{-1}$,

$$\mathcal{L}_n^{(l,\delta)}V(y) \leq V(y)(\theta + \varepsilon - \widetilde{\alpha}y) \leq -(\widetilde{\alpha}M - \theta - \varepsilon)V(y) + \widetilde{\alpha}MV(M).$$

Then Assumption 2.3(iii) holds with

$$
\begin{cases}
\alpha = \left(p_0'(1)(e^\theta - 1) - \left(\theta + \dfrac{\theta\delta}{2}\right) p_1(1) + \epsilon(1+\theta) \right) M - \theta - \varepsilon, \\
\beta = \widetilde{\alpha} M V(M).
\end{cases}
$$

Finally, checking Assumption 2.11 is easy [using (A.3) for instance] with $d_2 = 3$, which forces us to set $d = 6$ (since $\Gamma_n \chi_3 \le M_6 \chi_6$). The chain $(y_n^{(l,\delta)})_{n \ge 0}$ satisfying a Lyapunov criterion with $V(y) = e^{\theta y}$, its moments of order 6 are also uniformly bounded. $\square$

A.3. Appendix for the decreasing step Euler scheme.

PROOF OF LEMMA 3.12.   Applying Itô's formula with $x \mapsto |x|^p$, we get

$$
|\partial_x X_t^x|^p = 1 + \int_0^t p \left(b'(X_s^x) |\partial_x X_s^x|^p + \frac{p-1}{2} (\sigma'(X_s^x))^2 |\partial_x X_s^x|^p \right) ds
$$

(A.4)
$$
+ \int_0^t p \sigma'(X_s^x) |\partial_x X_s^x|^p \, dW_s
$$

$$
\le 1 + C \int_0^t |\partial_x X_s^x|^p \, ds + \int_0^t p \sigma'(X_s^x) |\partial_x X_s^x|^p \, dW_s,
$$

where $C = p\|b'\|_\infty + \frac{p(p-1)}{2}\|\sigma'\|_\infty^2$. Let us show that $\int_0^t p\sigma'(X_s^x)|\partial_x X_s^x|^p \, dW_s$ is a martingale. To that end, since $|\partial_x X_t^x|^p$ is nonnegative and $(x + y + z)^2 \le 2(x^2 + y^2 + z^2)$, we use the Burkholder–Davis–Gundy's inequality so there exists a constant C' such that

$$
|\partial_x X_t^x|^p \le 1 + C \int_0^t \sup_{u \in [0,s]} |\partial_x X_u^x|^p \, ds + \int_0^t p\sigma'(X_s^x) |\partial_x X_s^x|^p \, dW_s,
$$

$$
\sup_{u \in [0,t]} |\partial_x X_u^x|^p \le 1 + C \int_0^t \sup_{u \in [0,s]} |\partial_x X_u^x|^p \, ds
$$

$$
+ \sup_{u \in [0,t]} \int_0^u p\sigma'(X_s^x) |\partial_x X_s^x|^p \, dW_s,
$$

$$
\mathbb{E}\left[\sup_{u \in [0,t]} |\partial_x X_u^x|^{2p} \right] \le 2 + 2C^2 T \int_0^t \mathbb{E}\left[\sup_{u \in [0,s]} |\partial_x X_u^x|^{2p} \right] ds
$$

$$
+ 2\mathbb{E}\left[\left(\sup_{u \in [0,t]} \int_0^u p\sigma'(X_s^x) |\partial_x X_s^x|^p \, dW_s \right)^2 \right]
$$

$$
\le 2 + 2C^2 T \int_0^t \mathbb{E}\left[\sup_{u \in [0,s]} |\partial_x X_u^x|^{2p} \right] ds
$$

$$
+ 2C' \int_0^t \mathbb{E}[\sigma'(X_s^x)^2 |\partial_x X_s^x|^{2p}] \, ds
$$

$$\leq 2 + 2C^2 T \int_0^t \mathbb{E}\Big[\sup_{u \in [0,s]} |\partial_x X_u^x|^{2p}\Big] ds$$

$$+ 2C' \|\sigma'\|_\infty^2 \int_0^t \mathbb{E}\Big[\sup_{u \in [0,s]} |\partial_x X_u^x|^{2p}\Big] ds$$

$$\leq 2 \exp((C^2 T + C' \|\sigma'\|_\infty^2) T)$$

by Grönwall's lemma. Hence, $\int_0^t p\sigma'(X_s^x)|\partial_x X_s^x|^p dW_s$ is a martingale and, taking the expected values in (A.4) and applying Grönwall's lemma once again, we have

$$\mathbb{E}[|\partial_x X_t^x|^p] \leq \exp\Big(\Big(p\|b'\|_\infty + \frac{p(p-1)}{2}\|\sigma'\|_\infty^2\Big) T\Big).$$

Using Hölder's inequality for $p = 2$ completes the case of the first derivative.

Since the following computations are more and more tedious, we choose to treat only the case of the second derivative. Note that $\partial_x^2 X_t^x$ exists and satisfies the following SDE:

$$\partial_x^2 X_t^x = \int_0^t \big(b'(X_s^x)\partial_x^2 X_s^x + b''(X_s^x)(\partial_x X_s^x)^2\big) ds$$

$$+ \int_0^t \big(\sigma'(X_s^x)\partial_x^2 X_s^x + \sigma''(X_s^x)(\partial_x X_s^x)^2\big) dW_s.$$

Itô's formula provides the following inequality:

$$|\partial_x^2 X_t^x|^p \leq C_1 \int_0^t |\partial_x^2 X_s^x|^p ds + C_2 \int_0^t |\partial_x^2 X_s^x|^{p-1} |\partial_x X_s^x|^2 ds$$

$$+ C_3 \int_0^t |\partial_x^2 X_s^x|^{p-2} |\partial_x X_s^x|^4 ds$$

$$+ \int_0^t p\big(|\partial_x^2 X_s^x|^p \sigma'(X_s^x)$$

$$+ |\partial_x^2 X_s^x|^{p-1} \operatorname{sgn}(\partial_x^2 X_s^x)\sigma''(X_s^x)|\partial_x X_s^x|^2\big) dW_s,$$

with constants C_i depending on $p, \|b'\|_\infty, \|b''\|_\infty, \|\sigma'\|_\infty, \|\sigma''\|_\infty$. The last term proves to be a martingale, with similar arguments as above. We take the expected values, and apply Hölder's inequality twice to find, for $p > 2$,

$$\mathbb{E}[|\partial_x^2 X_t^x|^p] \leq C_1 \int_0^t \mathbb{E}[|\partial_x^2 X_s^x|^p] ds + C_2 \int_0^t \mathbb{E}[|\partial_x^2 X_s^x|^{p-1} |\partial_x X_s^x|^2] ds$$

$$+ C_3 \int_0^t \mathbb{E}[|\partial_x^2 X_s^x|^{p-2} |\partial_x X_s^x|^4] ds$$

$$\leq C_1 \int_0^t \mathbb{E}[|\partial_x^2 X_s^x|^p] ds + C_2 \int_0^t \mathbb{E}[|\partial_x^2 X_s^x|^p]^{\frac{p-1}{p}} \mathbb{E}[|\partial_x X_s^x|^{2p}]^{\frac{1}{p}} ds$$

$$+ C_3 \int_0^t \mathbb{E}\big[|\partial_x^2 X_s^x|^p\big]^{\frac{p-2}{p}} \mathbb{E}\big[|\partial_x X_s^x|^{2p}\big]^{\frac{2}{p}} \, ds$$

$$\leq C_3 \mathrm{e}^{C_4 T} + C_1 \int_0^t \mathbb{E}\big[|\partial_x^2 X_s^x|^p\big] \, ds$$

$$+ (C_2 + C_3)\mathrm{e}^{C_4 T} \int_0^t \mathbb{E}\big[|\partial_x^2 X_s^x|^p\big]^{\frac{p-1}{p}} \, ds,$$

with $C_4 = 4\|b'\|_\infty + 2(p-1)\|\sigma'\|_\infty^2$. The case $p = 2$ is deduced straightforwardly:

$$\mathbb{E}\big[|\partial_x^2 X_t^x|^2\big] \leq C_3 \mathrm{e}^{C_4 T} + C_1 \int_0^t \mathbb{E}\big[|\partial_x^2 X_s^x|^2\big] \, ds + C_3 \mathrm{e}^{C_4 T} \int_0^t \mathbb{E}\big[|\partial_x^2 X_s^x|^2\big]^{\frac{1}{2}} \, ds.$$

Regardless, since the unique solution of $u = Au + Bu^\alpha$ is

$$u(t) = \left(\left(u(0)^{1-\alpha} + \frac{B}{A} \right) \exp(A(1-\alpha)t) - \frac{B}{A} \right)^{\frac{1}{1-\alpha}},$$

for $A, B > 0, \alpha \in (0,1), u(0) > 0$, we have

$$\mathbb{E}\big[|\partial_x^2 X_t^x|^2\big] \leq \left(\left(C_2^{\frac{1}{p}} \mathrm{e}^{\frac{C_4}{p} T} + \frac{C_2 + C_3}{C_1} \mathrm{e}^{C_4 T} \right) \mathrm{e}^{\frac{C_1}{p} T} - \frac{C_2 + C_3}{C_1} \mathrm{e}^{C_4 T} \right)^p$$

$$\leq \left(C_2^{\frac{1}{p}} \mathrm{e}^{\frac{C_4}{p} T} + \frac{C_2 + C_3}{C_1} \mathrm{e}^{C_4 T} \right)^p \mathrm{e}^{C_1 T}.$$

The same reasoning for the third derivative achieves the proof. $\quad\square$

REMARK A.2 (Regularity of general diffusion processes).   The quality of approximation of a diffusion process is not completely unrelated to its regularity; see, for instance, [22], Theorem 1.3. In higher dimension, smoothness is generally checked under Hörmander conditions (see, e.g., [21, 22]).

REFERENCES

[1] ANÉ, C., BLACHÈRE, S., CHAFAÏ, D., FOUGÈRES, P., GENTIL, I., MALRIEU, F., ROBERTO, C. and SCHEFFER, G. (2000). *Sur les Inégalités de Sobolev Logarithmiques. Panoramas et Synthèses* [*Panoramas and Syntheses*] **10**. Société Mathématique de France, Paris. MR1845806

[2] BAKRY, D. (1994). L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lectures on Probability Theory* (*Saint-Flour*, 1992). *Lecture Notes in Math.* **1581** 1–114. Springer, Berlin. MR1307413

[3] BALLY, V. and RABIET, V. (2015). Asymptotic behavior for multi-scale PDMP's. Preprint, HAL.

[4] BENAÏM, M. (1999). Dynamics of stochastic approximation algorithms. In *Séminaire de Probabilités*, *XXXIII*. *Lecture Notes in Math.* **1709** 1–68. Springer, Berlin. MR1767993

[5] BENAÏM, M. and HIRSCH, M. W. (1996). Asymptotic pseudotrajectories and chain recurrent flows, with applications. *J. Dynam. Differential Equations* **8** 141–176. MR1388167

[6] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[7] BOUGUET, F. (2015). Quantitative speeds of convergence for exposure to food contaminants. *ESAIM Probab. Stat.* **19** 482–501. MR3423303

[8] BOUGUET, F., MALRIEU, F., PANLOUP, F., POQUET, C. and REYGNER, J. (2015). Long time behavior of Markov processes and beyond. In *Modélisation Aléatoire et Statistique—Journées MAS* 2014. *ESAIM Proc. Surveys* **51** 193–211. EDP Sci., Les Ulis. MR3440798

[9] CHAFAÏ, D., GUÉDON, O., LECUÉ, G. and PAJOR, A. (2012). *Interactions Between Compressed Sensing Random Matrices and High Dimensional Geometry*. Panoramas et Synthèses [*Panoramas and Syntheses*] **37**. Société Mathématique de France, Paris.

[10] CHEN, M.-F. (2004). *From Markov Chains to Non-equilibrium Particle Systems*, 2nd ed. World Scientific Co., Inc., River Edge, NJ. MR2091955

[11] CLOEZ, B. (2012). Wasserstein decay of one dimensional jump-diffusions. ArXiv e-prints.

[12] CLOEZ, B. and HAIRER, M. (2015). Exponential ergodicity for Markov processes with random switching. *Bernoulli* **21** 505–536. MR3322329

[13] DAVIS, M. H. A. (1993). *Markov Models and Optimization*. Monographs on Statistics and Applied Probability **49**. Chapman & Hall, London. MR1283589

[14] DOUC, R., FORT, G. and GUILLIN, A. (2009). Subgeometric rates of convergence of f-ergodic strong Markov processes. *Stochastic Process. Appl.* **119** 897–923. MR2499863

[15] DUFLO, M. (1996). *Algorithmes Stochastiques*. Mathématiques & Applications (*Berlin*) [*Mathematics & Applications*] **23**. Springer, Berlin. MR1612815

[16] EBERLE, A. (2011). Reflection coupling and Wasserstein contractivity without convexity. *C. R. Math. Acad. Sci. Paris* **349** 1101–1104. MR2843007

[17] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. Wiley, New York. MR0838085

[18] FORT, G. (2015). Central limit theorems for stochastic approximation with controlled Markov chain dynamics. *ESAIM Probab. Stat.* **19** 60–80. MR3374869

[19] GADAT, S., PANLOUP, F. and SAADANE, S. (2015). Regret bounds for Narendra–Shapiro bandit algorithms. ArXiv e-prints.

[20] HAIRER, M. (2010). Convergence of Markov processes. Available at http://www.hairer.org/notes/Convergence.pdf.

[21] HAIRER, M. (2011). On Malliavin's proof of Hörmander's theorem. *Bull. Sci. Math.* **135** 650–666. MR2838095

[22] HAIRER, M., HUTZENTHALER, M. and JENTZEN, A. (2015). Loss of regularity for Kolmogorov equations. *Ann. Probab.* **43** 468–527. MR3305998

[23] HAIRER, M. and MATTINGLY, J. C. (2011). Yet another look at Harris' ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis*, *Random Fields and Applications VI*. Progress in Probability **63** 109–117. Birkhäuser/Springer Basel AG, Basel. MR2857021

[24] HAIRER, M., MATTINGLY, J. C. and SCHEUTZOW, M. (2011). Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations. *Probab. Theory Related Fields* **149** 223–259. MR2773030

[25] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. Grundlehren der Mathematischen Wissenschaften [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[26] JARNER, S. F. and ROBERTS, G. O. (2002). Polynomial convergence rates of Markov chains. *Ann. Appl. Probab.* **12** 224–247. MR1890063

[27] JOFFE, A. and MÉTIVIER, M. (1986). Weak convergence of sequences of semimartingales with applications to multitype branching processes. *Adv. in Appl. Probab.* **18** 20–65. MR0827331

[28] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York. MR1876169

[29] KUNITA, H. (1984). Stochastic differential equations and stochastic flows of diffeomorphisms. In *École D'été de Probabilités de Saint-Flour, XII*—1982. *Lecture Notes in Math.* **1097** 143–303. Springer, Berlin. MR0876080

[30] KUSHNER, H. J. and YIN, G. G. (2003). *Stochastic Approximation and Recursive Algorithms and Applications*, 2nd ed. *Stochastic Modelling and Applied Probability* **35**. Springer, New York. MR1993642

[31] LAMBERTON, D. and PAGÈS, G. (2002). Recursive computation of the invariant distribution of a diffusion. *Bernoulli* **8** 367–405. MR1913112

[32] LAMBERTON, D. and PAGÈS, G. (2008). A penalized bandit algorithm. *Electron. J. Probab.* **13** 341–373. MR2386736

[33] LARUELLE, S. and PAGÈS, G. (2013). Randomized urn models revisited using stochastic approximation. *Ann. Appl. Probab.* **23** 1409–1436. MR3098437

[34] LEMAIRE, V. (2005). Estimation récursive de la mesure invariante d'un processus de diffusion. Theses, Univ. de Marne la Vallée.

[35] LINDVALL, T. (1992). *Lectures on the Coupling Method*. Wiley, New York. MR1180522

[36] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes. III. Foster–Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.* **25** 518–548. MR1234295

[37] MONMARCHÉ, P. (2014). Hypocoercive relaxation to equilibrium for some kinetic models. *Kinet. Relat. Models* **7** 341–360. MR3195078

[38] RACHEV, S. T., KLEBANOV, L. B., STOYANOV, S. V. and FABOZZI, F. J. (2013). *The Methods of Distances in the Theory of Probability and Statistics*. Springer, New York. MR3024835

[39] TALAY, D. (1984). Efficient numerical schemes for the approximation of expectations of functionals of the solution of a SDE and applications. In *Filtering and Control of Random Processes* (*Paris*, 1983). *Lect. Notes Control Inf. Sci.* **61** 294–313. Springer, Berlin. MR0874837

[40] TALAY, D. and TUBARO, L. (1990). Expansion of the global error for numerical schemes solving stochastic differential equations. *Stoch. Anal. Appl.* **8** 483–509. MR1091544

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