Autonomous Pseudomonoids

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This dissertation is the result of my own work and includes nothing which is the result of work done in collaboration.

This dissertation is not substantially the same as any that I have submitted for a degree or diploma or any other qualification at any other university.

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Summary

In this dissertation we generalise the basic theory of Hopf algebras to the context of autonomous pseudomonoids in monoidal bicategories.

Autonomous pseudomonoids were introduced in [13] as generalisations of both autonomous monoidal categories and Hopf algebras. Much of the theory of autonomous pseudomonoids developed in [13] was inspired by the example of autonomous (pro)monoidal enriched categories. The present thesis aims to further develop the theory with results inspired by Hopf algebra theory instead. We study three important results in Hopf algebra theory: the so-called *fundamental theorem of Hopf modules*, the *Drinfel'd* or *quantum double* and its relation with the centre of monoidal categories, and *Radford's formula*.

The basic result of this work is a general fundamental theorem of Hopf modules that establishes conditions equivalent to the existence of a left dualization. With this result as a base, we are able to construct the centre (defined in [83]) and the lax centre of an autonomous pseudomonoid as an Eilenberg-Moore construction for certain monad. As an application we show that the Drinfel'd double of a finite-dimensional Hopf algebra is equivalent to the centre of the associated pseudomonoid. The next piece of theory we develop is a general Radford's formula for autonomous map pseudomonoids; this yields an explicit formula in the case of a (coquasi) Hopf algebra. We also introduce *unimodular* autonomous pseudomonoids.

In the last part of the dissertation we apply the general theory to enriched categories with a (chosen) class of (co)limits, with emphasis in the case of finite (co)limits. We construct tensor products of such categories by means of pseudo-commutative enriched monads (a slight generalisation of the pseudo-commutative 2-monads of [37]), and showing that lax-idempotent 2-monads are pseudo-commutative. Finally we apply the general theory developed for pseudomonoids to deduce the main results of [27].

Contents

Chapter 1

Introduction

The present dissertation intends to give a *formal Hopf Algebra Theory*, by which we mean a theory general enough to cover the basic results on Hopf algebras and its generalisations, independent of any linear structure, and from which the results follow as easily as possible. By using concepts of Higher Dimensional Category Theory, our abstract perspective allows to cover at the same time, the cases of Hopf algebras and autonomous monoidal categories. The abstract categorical language forces us to work conceptually, in contrast to the complex calculations typical in Hopf Algebra Theory.

The relationship between Category Theory and Hopf Algebra Theory has been well studied during the last twenty years. Roughly speaking, each piece of structure that one adds to a coalgebra to obtain a Hopf algebra manifests as extra structure on the category of correpresentations of the coalgebra. Conversely, a series of results known as (Tannakian) reconstruction theorems allow us to construct a Hopf algebra structure on a coalgebra from certain extra structure on the category of correpresentations. To be a bit more explicit, if C is a coalgebra and $\mathscr{C} = \mathrm{Comod}_f(C)$ its category of finite-dimensional correpresentations or comodules: a bialgebra structure on C corresponds to a monoidal structure $\mathscr{C}$, and a Hopf algebra structure on C corresponds to a left autonomous (sometimes called left rigid) monoidal category. There are other corresponding structures, as for example (co)quasi-triangular elements and braidings, but we are not concerned with them here.

This correspondence between algebraic and categorical structures sometimes make us think of Hopf algebras as "the same" as left autonomous categories. This idea was formalised in [13] where Hopf algebras (and in fact the more general coquasi-Hopf algebras) and left autonomous monoidal categories were shown to be particular instances of an abstract concept of *left autonomous pseudomonoid*.

Pseudomonoids were introduced in [16], and some of the theory surrounding them has been developed by several authors. See [66, 17], and [52] where the more general pseudomonads are studied. Left autonomous pseudomonoids are pseudomonoids equipped with some extra structure called a *left dualization*. While most of the theory of pseudomonoids is a generalisation of constructions classically performed on (pro)monoidal categories, we propose to enlarge this theory with results inspired in Hopf Algebra Theory. The results about Hopf algebras that concern us are: the so-called *fundamental* or *structure theorem of Hopf modules*, the *Drinfel'd* or *quantum double* construction and *Radford's formula*.

Hopf algebras and generalisations.

Hopf algebras feature in many branches of modern Mathematics, from the more classical examples in Algebraic Geometry (rings of regular functions on an affine algebraic group), Lie Theory (universal enveloping algebras) and compact groups (algebras of representative functions), and Theoretical Physics (integrable systems and Yang-Baxter equation) to the most recent in Knot Theory, Combinatorics and Category Theory.

Hopf Algebra Theory is not only the study of Hopf algebras but also of a number of generalisations, such as (co)quasi-Hopf algebras [23], Hopf bialgebroids [86, 87, 58] and weak Hopf algebras [8]. Of these generalisations, the first lies in the scope of this work. This is because coquasi-Hopf algebras are left autonomous pseudomonoids in certain monoidal bicategory. All three results mentioned below have been proved in the context of (co)quasi-Hopf algebras, generalising the classical ones.

The three results about Hopf algebras that we generalise in this dissertation are at the heart of Hopf Algebra Theory, and are related to one another. The most basic of them is the *fundamental theorem of Hopf modules*, that translates the existence of an antipode for a bialgebra into the existence of certain equivalence of categories. One of the categories involved is the category of Hopf modules. This result is of pivotal importance in the theory of finite-dimensional Hopf algebras because it allow us to deduce the existence and uniqueness of *integrals*, the Hopf algebra analogue of Haar measures. We prove a general version of the fundamental theorem where we substitute Hopf algebras for left autonomous pseudomonoids and the category of Hopf modules for an Eilenberg-Moore construction for a special monad. Then we go on to study the internalisation of these constructions.

The second result, or rather construction, is the *Drinfel'd* or *quantum double* of a finite-dimensional Hopf algebra H. This is a new finite-dimensional Hopf algebra $D(H)$ constructed out of H (see [22] or [41]). In fact $D(H)$ supports more structure: it is quasi-triangular, also called almost-commutative or braided. From the categorical point of view, the Drinfel'd double is interesting because the category of $D(H)$-comodules $\mathrm{Comod}(D(H))$ is monoidally equivalent to the centre of the monoidal category $\mathrm{Comod}(H)$. The proof of this fact uses two intermediate monoidal categories: the category of Yetter-Drinfel'd modules and the category of two-sided Hopf modules. By using our general fundamental theorem of Hopf modules as a basis, we construct centres of finite coquasi-Hopf algebras (in the appropriate monoidal bicategory) and prove that $D(H)$ is the centre of H. Hence, $D(H)$ is not only related to the centre construction, but it *is* the centre.

The third result is Radford's formula for the fourth power of the antipode of a finite-dimensional Hopf algebra. Although many of the basic examples of Hopf algebras are involutive, *i.e.*, the antipode has order 2, there are many others where this is not true. Radford's formula (originally proven in [71], but see also [76] for another proof) tells us that although it does not have order 2, the antipode is not completely wild either: its fourth power has a very simple formula. In this formula intervene two special objects: the *modular element* and the *modular function*, which arise from the theory of integrals (therefore the connection with the fundamental theorem of Hopf modules). The first application of Radford's formula is the proof that the antipode of a finite-dimensional Hopf algebra has finite order. We show a Radford-like isomorphism for autonomous map pseudomonoids, that yields explicit formulas in the cases of finite quasi and coquasi-Hopf algebras.

A newer approach to the study of Hopf algebras has been taken in [28, 29, 27] where instead of working with algebraic structures in the classical sense, the authors manipulate categories directly. In a sense, this is a step further in this (informal) identification between Hopf algebras and autonomous categories, where one can forget about the algebra and work with a category that plays the role of the category of representations of the algebra. In [29, 27] the categories that abstract the properties of categories of representations of finite-dimensional quasi-Hopf algebras are called *finite tensor categories*. When these categories are moreover semisimple, they were called *fusion categories* [28]. In [27] "categorical" analogues of the fundamental theorem of Hopf modules and of Radford's formula were proved. We are able to deduce these results from the theory we developed for pseudomonoids (in contrast with the techniques used in the mentioned papers), by using a tensor product of categories with finite (co)limits.

Monoidal categories.

Since their introduction in [62], monoidal categories have become a basic tool in many areas of Mathematics, specially Representation Theory and Hopf Algebra Theory, but also Knot Theory, (topological) quantum field theories and others. An example of non trivial applications of monoidal categories to Algebraic Geometry can be found in [72, 20, 19]. The representations of most of the common algebraic structures, such as finite groups, Lie algebras and (rational representations of) algebraic groups, form a monoidal category. Many times, these monoidal categories come equipped with extra structure, such as braidings, duals, balanced structures, and others; see [39].

In many examples, the tensor product of a monoidal category preserves finite colimits in each variable (for example, whenever the category is monoidal closed), or even finite limits (as in the case of the category of vector spaces). As a tool to deal with these situations, [19] introduced a "tensor product" of abelian categories, commonly known as *Deligne's tensor product* of abelian categories. Given two abelian categories $\mathscr{A}, \mathscr{B}$, their tensor product is an abelian category $\mathscr{A} \boxtimes \mathscr{B}$ with a functor $\mathscr{A} \otimes \mathscr{B} \to \mathscr{A} \boxtimes \mathscr{B}$ right exact in each variable, universal in the sense that any other functor right exact in each variable into an abelian category $\mathscr{C}$ factors through $\mathscr{A} \boxtimes \mathscr{B}$ uniquely up to isomorphism. This construction is used in other works, for example in [59, 61, 60]. However, the definition of this tensor product as it stands is unsatisfactory, because there is no proof of its existence in general (at least none that I am aware of). In [19] the existence of the product in certain special class of abelian categories is shown.

We propose to drop the requirement that all the categories be abelian in Deligne's definition of the tensor product, asking only for the existence of finite colimits. In this way, we have at our disposal all the machinery of 2-monad theory to construct the tensor product. Notably, our new tensor product coincides with Deligne's on the class of abelian categories he works with in [19].

Organisation.

The dissertation is organised in seven chapters, the first being the present introduction. Chapters 2 to 4 constitute the theoretical core of this work and generalise to the context of pseudomonoids all three basic results on Hopf algebras mentioned above in this introduction. Chapter 6 also provides theory, although in another vein. The rest of the chapters are devoted to examples.

Chapter 2 sets the foundations which all the rest of the work rests upon: a generalised fundamental theorem of Hopf modules for map pseudomonoids, and its internalisation.

Chapter 3 constructs centres and lax centres of autonomous map pseudomonoids as an Eilenberg-Moore construction for certain a monad.

Chapter 4 proves a generalised Radford's formula for autonomous map pseudomonoids, and then goes on to study unimodular pseudomonoids.

Once the basic theory is developed, Chapter 5 interprets our results in the context of two important bicategories: the bicategory of $\mathscr{V}$-modules $\mathscr{V}$-**Mod** (also called profunctors, distributors or bimodules), and the bicategory of comodules in a monoidal braided or symmetric category **Comod**($\mathscr{V}$). Examples of left autonomous map pseudomonoids in the former are the left autonomous monoidal $\mathscr{V}$-categories, and in the latter coquasi-Hopf algebras. Applications to $\mathscr{V}$-**Mod** include the proof of existence of lax centres in $\mathscr{V}$-**Mod** of left autonomous monoidal $\mathscr{V}$-categories. In the case of **Comod**($\mathscr{V}$) we show that the classical fundamental theorem of Hopf modules is a particular case of our Chapter 2, that the Drinfel'd double of a finite coquasi-Hopf algebra *is* its centre, and deduce Radford-like formulas for quasi and coquasi-Hopf algebras.

Chapter 6 we construct monoidal structures on 2-categories of algebras and pseudomorphisms for a 2-monad, including the 2-categories of $\mathscr{V}$-categories with finite (co)limits. We use an extension of the pseudo-commutative 2-monads of Hyland-Power [37] to monads enriched in a monoidal 2-category. The connection with categories with a class of (co)limits is provided by the proof that every lax-idempotent (or KZ) 2-monad is pseudo-commutative.

Chapter 7 collects the consequences of the combination of the previous chapter with the first three chapters. We deduce the main results of [27] from the general theory; in particular we do not appeal to the Perron-Frobenius arguments used in [27]. We also relate fusion categories with bicategorical properties, such as the existence of certain adjoints.

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Chapter 2

Hopf modules and dualizations

In this chapter we make the first step towards our goal of extending the basic theory of Hopf algebras to the context of autonomous pseudomonoids in monoidal bicategories. We focus on the construction of *Hopf modules* and the *fundamental* or *structure theorem* for Hopf modules.

Left autonomous pseudomonoids, introduced in [13], generalise not only left autonomous (pro) monoidal enriched categories but also Hopf and (co)quasi-Hopf algebras. In fact, this is the conceptual reason underlying the well-known fact that the category of finite-dimensional (co)representations of a (co)quasi-Hopf algebra is left autonomous.

Hopf modules for a bialgebra H were introduced in [55] in connection with the integrals of H. In the cited paper, the authors prove the classical structure theorem of Hopf modules, stating that every Hopf module over a Hopf algebra is, in a specific way, free. This is the basic result that allows to prove the existence and uniqueness of the integrals in a finite-dimensional Hopf algebra. Integrals, the Hopf algebra analogue of Haar measures, are one of the more important tools in the theory of finite Hopf algebras, making the structure theorem of Hopf modules one of the fundamental results of this theory.

Generalisations of the above to the case of (co)quasi-Hopf algebras can be found in [34, 75]. A coquasibialgebra H, although not associative in **Vect**, is an associative algebra in the category $\mathbf{Comod}(H,H)$ of H-bicomodules and thus we can consider the category of left H-modules in $\mathbf{Comod}(H,H)$. This is by definition the category of H-Hopf modules. There is a monoidal functor from the category of right H-comodules to the category of H-Hopf modules sending M to the tensor product bicomodule $H \otimes M$, where M is considered as a trivial H-comodule on the left. It is shown in [75] that when H is a coquasi-Hopf algebra this functor is an equivalence, and in a dual fashion, that a finite-dimensional quasibialgebra is quasi-Hopf if and only if the module version of this functor is an equivalence.

We prove that an analogous result holds if we replace coquasibialgebras by map pseudomonoids (*i.e.,* pseudomonoids whose multiplication and unit have a right adjoint), Hopf modules by the Eilenberg-Moore construction for a certain monad and coquasi-Hopf algebras by left autonomous map pseudomonoids. So, a map pseudomonoid has a left dualization if and only if a generalised fundamental theorem of Hopf modules holds. This is also shown to be equivalent to the invertibility of certain special 2-cells. In our general setup no finiteness condition is necessary. We take this as an indication that the concept of dualization is more natural than the one of antipode.

When the monoidal bicategory involved is right closed, and in particular when it is right autonomous, our generalisation of the category of Hopf modules can be internalised. This internalisation, which we call a *Hopf module construction* for the map pseudomonoid A, is an Eilenberg-Moore construction for certain monad on the endo-hom object $[A, A]$. Naturally, this internalisation need not exist. We study its existence by embedding a Gray monoid $\mathscr{M}$ into a Gray monoid in which Hopf module constructions exist. This Gray monoid is the completion of the 2-category $\mathscr{M}$ under Eilenberg-Moore objects $\mathsf{EM}(\mathscr{M})$, described in [54]. We show that when $\mathscr{M}$ is a Gray monoid, right closed Gray monoid or right autonomous Gray monoid the 2-category $\mathsf{EM}(\mathscr{M})$ has the same structure. This is accomplished by extending the 2-functor EM on **2-Cat** to a homomorphism of tricategories on **Bicat**. Left autonomous pseudomonoids A always have a Hopf module construction, canonically equivalent to A itself.

We now describe the content of the chapter.

Section 2.1.1 provides the basic background on Gray monoids, pseudomonoids and Kleisli bicategories necessary to develop the rest of the paper.

In Section 2.2 we introduce the Hopf modules for a map pseudomonoid A in a monoidal bicategory $\mathscr{M}$ as the Eilenberg-Moore construction for a certain monad in $[\mathscr{M}^{\mathrm{op}}, \mathbf{Cat}]$, and explain what we mean by the theorem of Hopf modules.

Section 2.3 surveys some facts about lax actions and opmonoidal morphisms.

When the monad in the definition of Hopf modules is representable by a monad $t : [A, A] \to [A, A]$ in $\mathscr{M}$, we call an Eilenberg-Moore construction for it a *Hopf module construction* for A. This is introduced in Section 2.4 along with the proof that t is a opmonoidal monad.

Section 2.5 studies the existence of Hopf module constructions by extending the 2-functor $\mathsf{EM} : \mathbf{2\text{-}Cat} \to \mathbf{2\text{-}Cat}$ to a homomorphism of tricategories $\mathbf{Bicat} \to \mathbf{Bicat}$.

In Section 2.6 we prove our main result: a map pseudomonoid A is left autonomous if and only if the theorem of Hopf modules holds for A. Also, we use the results of the preceding section to show that a map pseudomonoid is left autonomous if and only if it has a Hopf module construction of a particular form, relating the problem of the existence of a dualization with a completeness problem.

2.1 Preliminaries

2.1.1 Monoidal bicategories and pseudomonoids

Recall that a Gray monoid [16] is a monoid in the monoidal category **Gray**. As a category, **Gray** is just the category of 2-categories and 2-functors. However, the monoidal structure we are interested in is not the one given by the cartesian product. Indeed, **Gray** has a monoidal closed structure with internal homs given by $\mathbf{Ps}(\mathscr{K}, \mathscr{L})$, the 2-category of 2-functors $\mathscr{K} \to \mathscr{L}$, *pseudonatural* transformations between them and modifications. The corresponding tensor product is called the *Gray tensor product* of 2-categories. This tensor product was introduced in [32, 33]; see also [31]. A monoid in **Gray**, also called a Gray monoid, is the same as a one-object **Gray**-category in the sense of enriched category theory, and therefore it can be thought of as a one-object tricategory, that is, a monoidal bicategory (see [31]). By the coherence theorem in [31], any monoidal bicategory is monoidally biequivalent (that is, triequivalent as a tricategory) to a Gray monoid. This allows us to develop the general theory using Gray monoids instead of general monoidal bicategories.

Our main examples of monoidal bicategories will be the bicategory of $\mathscr{V}$-modules $\mathscr{V}$-**Mod** and the bicategory of comodules $\mathbf{Comod}(\mathscr{V})$ in a monoidal category $\mathscr{V}$. See Examples 2.1 and 2.2.

We call 1-cells with right adjoints in a bicategory *maps*.

Let $\mathscr{M}$ be a Gray monoid and fix a map pseudomonoid (A, j, p) in $\mathscr{M}$, that is, a pseudomonoid whose unit $j : I \to A$ and multiplication $p : A \otimes A \to A$ are maps. Recall from [16] that a pseudomonoid, in addition to the unit and multiplication, is equipped with isomorphisms $\phi : p(p \otimes A) \Rightarrow p(A \otimes p)$, $p(j \otimes A) \Rightarrow 1_A$ and $p(A \otimes j) \Rightarrow 1_A$ satisfying three axioms. These axioms ensure, as shown in [52], that any 2-cell formed by pasting of tensor products of these isomorphisms, 1-cells and pseudonaturality constraints of the Gray monoid is uniquely determined by its domain and codomain 1-cells.

If (A, j, p) is a map pseudomonoid, then (A, j^*, p^*) is a *pseudocomonoid*, that is, a pseudomonoid in the opposite Gray monoid. By definition the unit isomorphism $(A \otimes j^*)p^* \cong 1_A$ of the pseudocomonoid (A, j^*, p^*) is the mate of the constraint $p(A \otimes j) \cong 1_A$, and thus the following equality holds.

$$\begin{tikzcd}
A \arrow[rr, equal] \arrow[dr, "p^*"'] & \arrow[d, phantom, "\cong"] & A \arrow[dr, "1\otimes j"] \arrow[d, phantom, "\Downarrow"] & \\
& A^2 \arrow[ur, "1\otimes j^*"] \arrow[rr, equal] & & A^2
\end{tikzcd}
=
\begin{tikzcd}
& A^2 \arrow[rr, equal] \arrow[dr, "p"] \arrow[d, phantom, "\cong"] & \arrow[d, phantom, "\Downarrow"] & A^2 \\
A \arrow[ur, "1\otimes j"] \arrow[rr, equal] & & A \arrow[ur, "p^*"'] &
\end{tikzcd}
\tag{2.1}$$

We mention this because it will be useful in Section 2.6.

Now we briefly mention the three main examples that concern us in this thesis.

Example 2.1 (The bicategory of $\mathscr{V}$-modules). One of our main examples of monoidal bicategory will be the bicategory of $\mathscr{V}$*-modules* $\mathscr{V}$-**Mod**. Some authors call $\mathscr{V}$-modules profunctors or distributors. Here $\mathscr{V}$ is a cocomplete monoidal closed category. The degree of completeness required from $\mathscr{V}$ varies between possible descriptions of $\mathscr{V}$-**Mod**. See [6, 78], [77, 81], [16, Section7]. We will use the presentation of $\mathscr{V}$-**Mod** where $\mathscr{V}$ is complete, objects are (small) enriched $\mathscr{V}$-categories and homs $\mathscr{V}\text{-}\mathbf{Mod}(\mathscr{A}, \mathscr{B})$ are the categories $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}, \mathscr{V}]_0$ of $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations. The monoidal structure on $\mathscr{V}$-**Mod** is induced by the tensor product of $\mathscr{V}$. A pseudomonoid in $\mathscr{V}$-**Mod** is a promonoidal category [11, 12].

Details can be found in Section 5.1.

Example 2.2 (The bicategory of comodules). Our second main example of monoidal bicategory will be the bicategory of comodules in a braided monoidal category $\mathscr{V}$, denoted by $\mathbf{Comod}(\mathscr{V})$. Objects are comonoids in $\mathscr{V}$, 1-cells are bicomodules and 2-cells bicomodule morphisms. This bicategory is dual to Example 2.1 in a sense that can be made precise. For details see Section 5.2, where it is also explained how (coquasi) bialgebras can be seen as pseudomonoids in $\mathbf{Comod}(\mathscr{V})$.

Example 2.3 (Algebras for a pseudo-commutative 2-monad). The relationship between monoidal closed categories and commutative monads has been long established in the series of papers [48, 49, 50]. A two-dimensional analogue of the notion of commutative monad was introduced in [37]: the notion of a *pseudo-commutative* 2-monad. For a pseudo-commutative 2-monad T on **Cat**, the 2-category T-Alg of T-algebras, pseudomorphisms of T-algebras and appropriate

2-cells is *pseudo-closed*. Pseudo-closedness is a higher dimensional analogue of the notion of closedness in a category. However, it is a "semi-strict" version of closedness, in the sense that it tries to be as strict as possible and only as "pseudo" or "weak" as it is necessary to cover the interesting examples. Under mild conditions on T (as such the existence of a *rank* for T), one can construct a (weak or pseudo) monoidal 2-category structure on T-Alg.

In Chapter 6 we extend these results to monads enriched in a (strict) monoidal 2-category, and apply the construction of the tensor product to the ($\mathscr{V}$-**Cat**)-monads whose algebras are categories with chosen (co)limits of certain class.

In Section 2.3 and subsequent sections we shall work with closed Gray monoids. A Gray monoid $\mathscr{M}$ is said to be *right closed* when equipped with a pseudofunctor $[-,-] : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$ and equivalences

$$\mathscr{M}(X \otimes Y, Z) \simeq \mathscr{M}(Y, [X, Z]) \tag{2.2}$$

pseudonatural in X, Y, Z, If one allows choice, this definition is equivalent to the one given in [16]: an object $[X, Z]$ for each pair of objects X, Y of $\mathscr{M}$ with equivalences (2.2) pseudonatural in Y.

2.1.2 Kleisli bicategories

In order to give a concise and conceptual definition of the Hopf modules in the next section, we shall use the Kleisli bicategory of a pseudocomonad. One can define a pseudomonad on the 2-category $\mathscr{K}$ as a pseudomonoid in the monoidal 2-category $\mathbf{Hom}(\mathscr{K}, \mathscr{K})$ of pseudofunctors, pseudonatural transformations and modifications, where tensor product is given by composition. A pseudocomonad is a pseudocomonoid in the same monoidal 2-category. As before, if T is a pseudomonad with unit $\eta : 1 \Rightarrow T$ and multiplication $\mu : T^2 \Rightarrow T$ which are maps, then T together with η^* and μ^* have a canonical structure of a pseudocomonad on $\mathscr{K}$.

A lax T-algebra is an arrow $a : TA \to A$ in $\mathscr{K}$ equipped with 2-cells $a(Ta) \Rightarrow a\mu_A : T^2A \to A$ and $1_A \to a\eta_A$ satisfying the axioms in [65, p. 39] and [52], but without the requirement of the invertibility of these 2-cells.

Let G be a pseudocomonad on the 2-category $\mathscr{K}$, and denote its comultiplication and counit by δ and ϵ, respectively. The Kleisli bicategory $\mathsf{Kl}(G)$ of $\mathscr{K}$ has the same objects as $\mathscr{K}$, and hom-categories $\mathsf{Kl}(G)(X, Y) = \mathscr{K}(GX, Y)$. We denote the 1-cells of $\mathsf{Kl}(G)$ by $f : X \nrightarrow Y$. The composition of this f with

$g : Y \nrightarrow Z$ is given by $g(Gf)\delta_X : GX \to Z$, while the identity of the object X is $\varepsilon_X : GX \to X$.

The following is a generalisation of part of [36, Prop. 4.6].

Lemma 2.4. *Let $T : \mathscr{K} \to \mathscr{K}$ be a pseudomonad whose unit η and multiplication μ are maps. There exists a bijection between the following structures on an arrow $a : TA \to A$ in $\mathscr{K}$: structures of a lax T-algebra and structures of a monad in $\mathsf{Kl}(T)$. Furthermore, there exists a bijection between the following structures on a 1-cell $h : TX \to A$: structures of a morphism of lax algebras from (TX, μ_X) to (A, a) and structures of an algebra $h : X \nrightarrow A$ for the monad $a : A \nrightarrow A$ in $\mathsf{Kl}(T)$.*

A structure of a monad in $\mathsf{Kl}(T)$ on $a : A \nrightarrow A$ is given by a pair of 2-cells $a(Ta)\mu_A^* \Rightarrow a$ and $\eta_A^* \Rightarrow a$ in $\mathscr{K}$. The bijection above is given by

$$\begin{tikzcd}
T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\swarrow"] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
 & T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\swarrow"] & TA \arrow[d, "a"] \\
TA \arrow[ur, "\mu_A^*"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow"'] & TA \arrow[r, "a"'] & A
\end{tikzcd}$$

2.2 The theorem of Hopf modules

If (A, j, p) is a map pseudomonoid in the Gray monoid $\mathscr{M}$, the 2-functor $A \otimes -$ has the structure of a pseudomonad with unit $j \otimes X : X \to A \otimes X$ and multiplication $p \otimes X : A \otimes A \otimes X \to A \otimes X$, and also the structure of a pseudocomonad with counit $j^* \otimes X$ and comultiplication $p^* \otimes X$. The associativity constraint $p(A \otimes p) \Rightarrow p(p \otimes A)$ endows $p : A \otimes A \to A$ with the structure of a lax $(A \otimes -)$-algebra, and hence by Lemma 2.4, with the structure of a monad $p : A \nrightarrow A$ in the Kleisli bicategory $\mathsf{Kl}(A \otimes -)$.

Definition 2.1. Consider the pseudofunctor $\mathsf{Kl}(A \otimes -) \to \mathbf{Hom}(\mathscr{M}^{\mathrm{op}}, \mathbf{Cat})$ induced by the identity on objects pseudofunctor $\mathscr{M} \to \mathsf{Kl}(A \otimes -)$. We will denote by θ the monad which is the image of the monad $p : A \nrightarrow A$ in $\mathsf{Kl}(A \otimes -)$. Hence, θ is a monad on the 2-functor $\mathscr{M}(A \otimes -, A)$ in the 2-category $\mathbf{Hom}(\mathscr{M}^{\mathrm{op}}, \mathbf{Cat})$ of pseudofunctors, pseudonatural transformations and modifications.

Explicitly, $\theta_X(f) = p(A \otimes f)(p^* \otimes X)$ and the multiplication and unit of the monad, depicted in (2.3) and (2.4), are induced by the counits of the adjunctions

$p \dashv p^*$ and $j \dashv j^*$ respectively.

$$
\begin{tikzcd}
A \otimes X \arrow[r, "p^* \otimes 1"] \arrow[dr, "p^* \otimes 1"'] & A^2 \otimes X \arrow[r, "1 \otimes p^* \otimes 1"] \arrow[d, phantom, "\phi^* \cong"] & A^3 \otimes X \arrow[r, "A^2 \otimes f"] \arrow[dr, "p \otimes 1 \otimes 1"] \arrow[d, phantom, "\Downarrow"] & A^3 \arrow[r, "1 \otimes p"] \arrow[dr, "p \otimes 1"] \arrow[d, phantom, "c \cong"] & A^2 \arrow[r, "p"] \arrow[d, phantom, "\cong"] & A \\
 & A^2 \otimes X \arrow[ur, "p^* \otimes 1 \otimes 1"] \arrow[rr, equal] & & A^2 \arrow[r, "1 \otimes f"'] & A^2 \arrow[ur, "p"'] &
\end{tikzcd}
\tag{2.3}
$$

$$
\begin{tikzcd}
 & & A \otimes X \arrow[r, "f"] \arrow[drr, "j \otimes 1 \otimes 1"] \arrow[d, phantom, "\Downarrow"] & A \arrow[dr, "j \otimes 1"] \arrow[drr, bend left, "1"] \arrow[d, phantom, "\cong c"] & \arrow[d, phantom, "\cong"] & \\
A \otimes X \arrow[urr, bend left, "1"] \arrow[r, "p^* \otimes 1"'] \arrow[ur, phantom, "\cong"] & A^2 \otimes X \arrow[ur, "j^* \otimes 1 \otimes 1"] \arrow[rr, equal] & & A^2 \otimes X \arrow[r, "1 \otimes f"'] & A^2 \arrow[r, "p"'] & A
\end{tikzcd}
\tag{2.4}
$$

Definition 2.2. Our generalisation of the category of Hopf modules is the Eilenberg-Moore construction $\upsilon : \mathscr{M}(A \otimes -, A)^\theta \to \mathscr{M}(A \otimes -, A)$ for the monad θ in $\mathbf{Hom}(\mathscr{M}^{\mathrm{op}}, \mathbf{Cat})$. We denote by φ the left adjoint of υ. Another way of viewing $\mathscr{M}(A \otimes -, A)^\theta$ is as the composition of the pseudofunctor

$$
\mathsf{Kl}(A \otimes -)(-, A)^{\mathsf{Kl}(A \otimes -)(-,p)} : \mathsf{Kl}(A \otimes -)^{\mathrm{op}} \to \mathbf{Cat}
$$

with the identity on objects pseudofunctor $\mathscr{M}^{\mathrm{op}} \to \mathsf{Kl}(A \otimes -)^{\mathrm{op}}$.

See Example 2.8 for an explanation of why this construction generalises the usual Hopf modules for a coquasibialgebra.

Observation 2.5. There is another equivalent way of defining Hopf modules. The category $\mathscr{M}(A, A)$ has a *convolution* monoidal structure, with tensor product $f * g = p(A \otimes g)(f \otimes A)p^*$ and unit jj^*. This monoidal category acts on the pseudofunctor $\mathscr{M}(A \otimes -, A) : \mathscr{M}^{\mathrm{op}} \to \mathbf{Cat}$ by sending $h : A \otimes X \to A$ to $p(A \otimes h)(p^* \otimes X)$, in the sense that this defines a monoidal functor from $\mathscr{M}(A, A)$ to $\mathbf{Hom}(\mathscr{M}^{\mathrm{op}}, \mathbf{Cat})(\mathscr{M}(A \otimes -, A), \mathscr{M}(A \otimes -, A))$. Now, $1_A : A \to A$ has a canonical structure of a monoid in $\mathscr{M}(A, A)$, with multiplication $pp^* \Rightarrow 1$ and $jj^* \Rightarrow 1$ the respective counits of the adjunctions. Hence 1_A defines via the action described above a monad on $\mathscr{M}(A \otimes -, A)$ in $\mathbf{Hom}(\mathscr{M}^{\mathrm{op}}, \mathbf{Cat})$. This monad is the monad θ of Definition 2.1.

Definition 2.3. We say that *the theorem of Hopf modules holds* for a map pseudomonoid A if the pseudonatural transformation λ given by

$$
\mathscr{M}(-, A) \xrightarrow{\mathscr{M}(j^* \otimes -, A)} \mathscr{M}(A \otimes -, A) \xrightarrow{\varphi} \mathscr{M}(A \otimes -, A)^\theta
$$

is an equivalence.

Observation 2.6. The composition $\upsilon_X \lambda_X = \theta_X \mathscr{M}(j^* \otimes X, A) : \mathscr{M}(X, A) \to \mathscr{M}(A \otimes X, A)$ is, up to isomorphism, the functor given by

$$(X \xrightarrow{f} A) \longmapsto (A \otimes X \xrightarrow{1 \otimes f} A \otimes A \xrightarrow{p} A).$$

Recall that a 1-cell in a bicategory is *fully faithful* if it is a map and the unit of the adjunction is an isomorphism.

Proposition 2.7. *The pseudonatural transformation λ is fully faithful.*

Proof. It is clear that λ has right adjoint $\mathscr{M}(j \otimes -, A)\upsilon$. By [38, Lemma 1.1.1], the unit of the adjunction is an isomorphism if and only if the composition $\mathscr{M}(j \otimes -, A)\upsilon\lambda$ is isomorphic to the identity pseudonatural transformation. This is clear from Observation 2.6, as we have isomorphisms $p(A \otimes f)(j \otimes X) \cong p(j \otimes A)f \cong f$, natural in $f : X \to A$. $\square$

Explicitly, the component corresponding to $f : X \to A$ of the unit of the adjunction $\lambda \dashv \mathscr{M}(j \otimes -, A)\upsilon$ is the pasting of 2-cells below (where the unlabelled 2-cells denote the obvious counits).

$$\begin{tikzcd}
X \arrow[rrrr, bend left, "1"] \arrow[d, "j\otimes 1"'] \arrow[r, phantom, "\Downarrow"] & {} & A \otimes X \arrow[rr, "j^*\otimes 1"'] \arrow[dr, "j\otimes 1\otimes 1"] \arrow[d, phantom, "\Downarrow"] & {} \arrow[d, phantom, "\cong"] & X \arrow[rr, "f"] \arrow[dr, "j\otimes 1"] & {} \arrow[d, phantom, "\cong"] & A \arrow[dr, "j\otimes 1"] \arrow[drr, bend left, "1"] & {} \arrow[d, phantom, "\cong"] & \\
A \otimes X \arrow[r, "p^*\otimes 1"'] \arrow[urr, bend left, "1"] \arrow[ur, phantom, "\cong"] & A^2 \otimes X \arrow[rr, equal] \arrow[ur, "j^*\otimes 1\otimes 1"] & {} & A^2 \otimes X \arrow[rr, "1\otimes j^*\otimes 1"'] & & A \otimes X \arrow[rr, "1\otimes f\otimes 1"'] & & A^2 \arrow[r, "p"'] & A
\end{tikzcd}$$

Example 2.8. We now explain why the Hopf modules for a map pseudomonoid generalise the usual Hopf modules for coquasi-Hopf algebras. As we mentioned in Example 2.2, a coquasibialgebra is a particular instance of a pseudomonoid in $\mathbf{Comon}(\mathscr{V})$ when $\mathscr{V}$ is the category of vector spaces. If C is a such a pseudomonoid with unit j and multiplication p, then (C, j_*, p_*) is a pseudomonoid in the bicategory of comodules. We claim that the category $\mathbf{Comod}(\mathscr{V})(C, C)^{\theta_I}$ is the category of Hopf modules considered in [74, 75].

The convolution monoidal structure on $\mathbf{Comod}(\mathscr{V})(C, C)$ is just the usual tensor product of bicomodules. The monad θ_I is given by the action of 1_C, which is simply the regular bicomodule (C, Δ^2). Therefore the $\mathbf{Comod}(\mathscr{V})(C, C)^{\theta_I}$ is the category of left modules for the monoid (C, Δ^2) within the monoidal category of C-bicomodules. This is exactly the definition of the category of Hopf modules given in [74] (actually, the formally dual definition) and [75].

The functor λ_I sends a right C-comodule M to the free Hopf module $C \otimes M$. The right adjoint to λ sends a Hopf module N to the right C-comodule of left coinvariants ${}^{\mathrm{co}C}N$. This is easy to see since by the definition of the composition in $\mathbf{Comod}(\mathscr{V})$, precomposing with $j_* : I \to C$ is exactly the same as taking left coinvariants. In [74, 75] the faithfulness of λ_I is argued using the fact that the functor $(C \otimes -)$ is exact when we work with vector spaces. We see that in fact the fully faithfulness of λ follows formally from the definitions.

The following observation will be of use in Section 2.6.

Observation 2.9. Consider the modification $\upsilon\varepsilon\varphi$, where ε is the counit of the adjunction $\lambda \dashv \mathscr{M}(j \otimes -, A)\upsilon$ as depicted below.

$$\begin{tikzcd}
\mathscr{M}(A\otimes -, A)^\theta \arrow[r, "\upsilon"] \arrow[drr, "1"'] & \mathscr{M}(A\otimes -, A) \arrow[r, "\mathscr{M}(j\otimes -,A)"] \arrow[dr, phantom, "\Downarrow\varepsilon"] & \mathscr{M}(-, A) \arrow[d, "\mathscr{M}(j^*\otimes -,A)"] \\
 & & \mathscr{M}(A\otimes -, A) \arrow[d, "\varphi"] \\
 & & \mathscr{M}(A\otimes -, A)^\theta
\end{tikzcd}$$

Observe that $\mathscr{M}(A\otimes X, A)^{\theta_X}$ is the closure under υ_X-split coequalizers of the full subcategory determined by the image of the functor φ_X, and these coequalizers are preserved by $\varphi_X\mathscr{M}(jj^* \otimes -, A)\upsilon_X$, since they become absolute coequalizers after applying υ_X. It follows that ε_X is an isomorphism if and only if $\varepsilon_X\varphi_X$ is an isomorphism. Using the fact that each υ_X is conservative, we deduce that ε is an isomorphism if and only if $\upsilon\varepsilon\varphi$ is so.

We finish the section by mentioning way of defining Hopf modules for a pseudomonoid whose unit and multiplication are not necessarily maps.

Observation 2.10. Consider the 2-category Lax-$(A\otimes -)$-Alg of lax algebras for the pseudomonad $(A \otimes -)$ on $\mathscr{M}$, and the 2-functor

$$\mathscr{M}^{\mathrm{op}} \xrightarrow{A\otimes -} \text{Lax-}(A\otimes -)\text{-Alg}^{\mathrm{op}} \xrightarrow{\text{Lax-}(A\otimes -)\text{-Alg}(-,A)} \mathbf{Cat}. \tag{2.5}$$

This 2-functor is exactly the 2-functor $\mathscr{M}(A\otimes -, A)^\theta$ in Definition 2.2. This is so because each Eilenberg-Moore category $\mathsf{Kl}(A\otimes -)(X, A)^{\mathsf{Kl}(A\otimes -)(X,p)}$ is isomorphic to Lax-$(A \otimes -)$-Alg$(A \otimes X, A)$, by Lemma 2.4. Hence, a Hopf module for A is a morphism of lax $(A \otimes -)$-algebras $h : A \otimes X \to A$. This means h is equipped with a 2-cell $\bar{h} : p(A \otimes h) \Rightarrow h(p \otimes X) : A \otimes A \otimes X \to A$ satisfying coherence conditions.

The functor $\lambda_X : \mathscr{M}(X, A) \to \text{Lax-}(A \otimes -)\text{-Alg}(A \otimes X, A)$ sends a 1-cell f to $p(A \otimes f)$ with the 2-cell $p(A \otimes p)(A \otimes A \otimes f) \cong p(p \otimes A)(A \otimes A \otimes f) \cong p(A \otimes f)(p \otimes X)$ induced by the pseudomonoid structure of A. In particular $\lambda_X(f)$ is a pseudomorphism between the pseudo $(A \otimes -)$-algebras $A \otimes X$ and A. Conversely, any such pseudomorphism is in the image of λ_X: if $(h, \bar{h})$ is a pseudomorphism, we have $(h, \bar{h}) \cong \lambda_X(h(j \otimes X))$. It follows that, when A is a map pseudomonoid, the theorem of Hopf modules hold for A if and only if every lax morphism $A \otimes X \to A$ is a pseudomorphism. This latter condition can be taken as the definition of theorem of Hopf modules for arbitrary pseudomonoids.

2.3 Opmonoidal morphisms and oplax actions

In this section we spell out the relation between opmonoidal morphisms and right oplax actions in a right closed Gray monoid. Everything in this section is well-known, though we have not found the present formulation in the literature. The case when the monoidal 2-category is strict and has certain completeness conditions is studied in [47].

Let A be a pseudomonoid in $\mathscr{M}$. Briefly, a *right oplax action* of A on an object B is an oplax algebra for the pseudomonad $- \otimes A$ on $\mathscr{M}$. This amounts to a 1-cell $h : B \otimes A \to B$ together with 2-cells

$$\begin{tikzcd} B \otimes A \otimes A \arrow[r, "h\otimes 1"] \arrow[d, "1\otimes p"'] \arrow[dr, phantom, "h_2\Uparrow"] & B \otimes A \arrow[d, "h"] \\ B \otimes A \arrow[r, "h"'] & A \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "1\otimes j"'] \arrow[dr, "1"] \arrow[d, phantom, "h_0\Uparrow"] & \\ B \otimes A \arrow[rr, "h"'] & {} & A \end{tikzcd}$$

satisfying axioms dual to those in [65, p. 39] or [52] but without the invertibility requirement on the 2-cells. A *morphism of right oplax actions* on B from h to $k : B \otimes A \to B$ is a 2-cell $\tau : h \Rightarrow k$ compatible with h_2, k_2 and h_0, k_0 in the obvious sense. Right oplax actions of A on B and their morphisms form a category $\mathbf{Opact}_A(B)$ which comes equipped with a canonical forgetful functor to $\mathscr{M}(B \otimes A, B)$.

For each Gray monoid $\mathscr{M}$ we have a 2-category $\mathbf{Mon}(\mathscr{M})$ whose objects, 1-cells and 2-cells are respectively pseudomonoids in $\mathscr{M}$, lax monoidal morphisms and monoidal 2-cells. See for example [66] and references therein. Define $\mathbf{Opmon}(\mathscr{M}) = \mathbf{Mon}(\mathscr{M}^{\mathrm{co}})^{\mathrm{co}}$. The objects of $\mathbf{Opmon}(\mathscr{M})$ may be identified with the pseudomonoids, the 1-cells, called *opmonoidal morphisms*, are 1-cells

$f : A \to B$ of $\mathscr{M}$ equipped with 2-cells

$$\begin{tikzcd} A \otimes A \arrow[r, "(B\otimes f)(f\otimes A)"] \arrow[d, "p"'] \arrow[dr, phantom, "f_2\Uparrow"] & B \otimes B \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "j"'] \arrow[dr, "j"] \arrow[d, phantom, "f_0\Uparrow"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

satisfying the obvious equations, and the 2-cells $f \Rightarrow g$ are the 2-cells of $\mathscr{M}$ satisfying compatibility conditions with f_2, g_2 and f_0, g_0.

Now suppose that $\mathscr{M}$ is a right closed Gray monoid in the sense of [16], that is, there is a pseudofunctor $[-,-] : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$ and a pseudonatural equivalence

$$\mathscr{M}(X \otimes Y, Z) \simeq \mathscr{M}(X, [Y, Z]). \tag{2.6}$$

Equivalently, for each pair of objects Y, Z of $\mathscr{M}$ there is another one denoted by $[Y, Z]$ and an evaluation 1-cell $\mathrm{ev}_{Y,Z} : Y \otimes [Y, Z] \to Z$ inducing (2.6). For any object X of $\mathscr{M}$, the internal hom $[X, X]$ has a canonical structure of a pseudomonoid; namely, there are composition and identity 1-cells $\mathrm{comp} : [X, X] \otimes [X, X] \to [X, X]$ and $\mathrm{id} : I \to [X, X]$ corresponding respectively to

$$X \otimes [X, X] \otimes [X, X] \xrightarrow{\mathrm{ev}\otimes 1} X \otimes [X, X] \xrightarrow{\mathrm{ev}} X \quad \text{and} \quad X \xrightarrow{1_X} X.$$

Example 2.11.

Proposition 2.12. *For any pseudomonoid A and any object B, the closedness equivalence $\mathscr{M}(B \otimes A, B) \simeq \mathscr{M}(A, [B, B])$ lifts to an equivalence*

$$\mathbf{Opact}_A(B) \simeq \mathbf{Opmon}(\mathscr{M})(A, [B, B]).$$

Moreover, under this equivalence pseudoactions correspond to pseudomonoidal morphisms.

Using Proposition 2.12 one can easily establish the following facts.

Proposition 2.13.

1. *For any map $f : X \to Y$ the 1-cell $[f^*, f]$ from $[X, X]$ to $[Y, Y]$ has a canonical structure of an opmonoidal morphism. If $\tau : f \Rightarrow g$ is an invertible 2-cell then $[(\tau^{-1})^*, \tau] : [f^*, f] \Rightarrow [g^*, g]$ is an invertible monoidal 2-cell.*

2. *For any pair of objects X, Y of $\mathscr{M}$, the 1-cell $i_X^Y : [X, X] \to [Y \otimes X, Y \otimes X]$ corresponding to $Y \otimes \mathrm{ev} : Y \otimes X \otimes [X, X] \to Y \otimes X$ has a canonical structure*

of a strong monoidal morphism. Moreover, there are canonical monoidal isomorphisms $(i^W_{Y\otimes X})(i^Y_X) \cong i^{W\otimes Y}_X$.

3. *For any map* $f : X \to Z$ *and any object* Y *there exists a canonical monoidal isomorphism*

$$\begin{tikzcd}
{[X,X]} \arrow[r, "i^Y_X"] \arrow[d, "{[f^*,f]}"'] \arrow[dr, phantom, "\cong"] & {[Y\otimes X, Y\otimes X]} \arrow[d, "{[1\otimes f^*, 1\otimes f]}"] \\
{[Z,Z]} \arrow[r, "i^Y_Z"'] & {[Y\otimes Z, Y\otimes Z]}
\end{tikzcd} \tag{2.7}$$

4. *Given a map* $f : Y \to Z$ *and an object* X*, the counit of* $f \dashv f^*$ *induces a monoidal 2-cell*

$$\begin{tikzcd}
{[X,X]} \arrow[r, "i^Y_X"] \arrow[dr, "i^Z_X"'] & {[Y\otimes X, Y\otimes X]} \arrow[d, "{[f^*\otimes 1, f\otimes 1]}"] \arrow[dl, phantom, "\Downarrow"] \\
& {[Z\otimes X, Z\otimes X]}
\end{tikzcd} \tag{2.8}$$

Proof. (1) It is not hard to show that the 2-cells (2.9) and (2.10) equip

$$Y \otimes [X,X] \xrightarrow{f^*\otimes 1} X \otimes [X,X] \xrightarrow{\mathrm{ev}} X \xrightarrow{f} Y$$

with a structure of right oplax action of $[X,X]$ on Y, and that

$$\begin{tikzcd}
Y\otimes[X,X] \arrow[r, bend left, "f^*\otimes 1"] \arrow[r, bend right, "g^*\otimes 1"'] \arrow[r, phantom, "\Downarrow (\tau^{-1})^*\otimes 1"] & X\otimes[X,X] \arrow[r, "\mathrm{ev}"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\tau"] & Y
\end{tikzcd}$$

is a morphism of right oplax actions on Y.

(2) The evaluation $\mathrm{ev} : X \otimes [X,X] \to X$ has a canonical structure of right oplax action (in fact, pseudoaction) and it is obvious that any 2-functor $Y \otimes -$ preserves right oplax actions. This shows that i^Y_X has a canonical opmonoidal structure. The existence of the isomorphism $(i^W_{Y\otimes X})(i^Y_X) \cong i^{W\otimes Y}_X$ follows from the fact that both 1-cells correspond to the right pseudoaction $W \otimes Y \otimes \mathrm{ev} : W \otimes Y \otimes X \otimes [X,X] \to W \otimes Y \otimes X$.

(3) The two legs of the rectangle (2.7) correspond, up to isomorphism, to the 1-cell

$$Y \otimes Z \otimes [X,X] \xrightarrow{1\otimes f^*\otimes 1} Y \otimes X \otimes [X,X] \xrightarrow{1\otimes \mathrm{ev}} Y \otimes X \xrightarrow{1\otimes f} Y \otimes Z$$

$$\begin{tikzcd}
Y \otimes [X,X]^2 \arrow[r, "f^* \otimes 1 \otimes 1"] \arrow[ddd, "1\otimes\mathrm{comp}"'] \arrow[dddr, phantom, "\cong"] & X \otimes [X,X]^2 \arrow[r, "\mathrm{ev}\otimes 1"] \arrow[ddd, "1\otimes\mathrm{comp}"'] \arrow[dddrr, phantom, "\cong"] & X \otimes [X,X] \arrow[r, "f\otimes 1"] \arrow[dr, equal] \arrow[r, phantom, shift right=3, "\Uparrow"] & Y \otimes [X,X] \arrow[d, "f^*\otimes 1"] \\
 & & & X \otimes [X,X] \arrow[d, "\mathrm{ev}"] \\
 & & & X \arrow[d, "f"] \\
Y \otimes [X,X] \arrow[r, "f^*\otimes 1"'] & X \otimes [X,X] \arrow[r, "\mathrm{ev}"'] & X \arrow[r, "f"'] \arrow[ur, equal] & Y
\end{tikzcd} \tag{2.9}$$

$$\begin{tikzcd}
 & Y \arrow[d, "f^*"'] \arrow[ddl, "1\otimes\mathrm{id}"'] \arrow[ddrr, equal] & & \\
 & X \arrow[d, "1\otimes\mathrm{id}"'] \arrow[dr, equal] \arrow[r, phantom, "\Uparrow"] \arrow[dl, phantom, "\cong"] & & \\
Y \otimes [X,X] \arrow[r, "f^*\otimes 1"'] & X \otimes [X,X] \arrow[r, "\mathrm{ev}"'] & X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{2.10}$$

Figure 2.1:

and therefore there exists an isomorphism as claimed. Moreover, this isomorphism is monoidal by Proposition 2.12.

(4) The 2-cell (2.8) corresponds under the closedness equivalence to

$$\begin{tikzcd}
Z \otimes X \otimes [X,X] \arrow[r, "1\otimes\mathrm{ev}"] & Z \otimes X \arrow[r, "f^*\otimes 1"] \arrow[rr, bend right, "1"'] & Y \otimes X \arrow[r, "f\otimes 1"] \arrow[d, phantom, "\Downarrow \varepsilon\otimes 1"] & Z \otimes X. \\
 & & &
\end{tikzcd}$$

This 2-cell is readily shown to be a morphism of right $[X,X]$-actions on $Z\otimes X$. $\square$

2.4 The object of Hopf modules

In this section we shall assume that A is a map pseudomonoid in a closed Gray monoid $\mathscr{M}$ (see Section 2.3). Under these assumptions the monad θ on $\mathscr{M}(A\otimes -, A)$ is representable by a monad $t : [A,A] \to [A,A]$; that is, there is an isomorphism

$$
\begin{tikzcd}
\mathscr{M}(A \otimes X, A) \arrow[r, "\theta_X"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\cong"] & \mathscr{M}(A \otimes X, A) \arrow[d, "\simeq"] \\
\mathscr{M}(X, [A, A]) \arrow[r, "\mathscr{M}(X,t)"'] & \mathscr{M}(X, [A, A])
\end{tikzcd}
$$

pseudonatural in X. More explicitly, t is the 1-cell

$$
[A, A] \xrightarrow{i_A^A} [A \otimes A, A \otimes A] \xrightarrow{[p^*,p]} [A, A] \tag{2.11}
$$

where i_A^A was defined in Proposition 2.13. The multiplication and unit of t are respectively

$$
\begin{tikzcd}
{[A,A]} \arrow[r, "i_A^A"] \arrow[dr, "i_A^{A^2}"'] \arrow[ddr, bend right, "i_A^A"'] & {[A^2,A^2]} \arrow[r, "{[p^*,p]}"] \arrow[d, "i_{A^2}^A"'] \arrow[dl, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & {[A,A]} \arrow[d, "i_A^A"] \\
& {[A^3,A^3]} \arrow[r, "{[1\otimes p^*,1\otimes p]}"] \arrow[d, "{[p^*\otimes 1,p\otimes 1]}"'] \arrow[l, phantom, "\Swarrow"] \arrow[dr, phantom, "\cong"] & {[A^2,A^2]} \arrow[d, "{[p^*,p]}"] \\
& {[A^2,A^2]} \arrow[r, "{[p^*,p]}"'] & {[A,A]}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& {[A,A]} \arrow[dr, "{[j^*\otimes 1,j\otimes 1]}"] \arrow[drr, bend left, "1"] \arrow[d, phantom, "\Downarrow"] \arrow[drr, phantom, "\cong"] & & \\
{[A,A]} \arrow[ur, "1=i_A^I"] \arrow[rr, "i_A^A"'] & & {[A^2,A^2]} \arrow[r, "{[p^*,p]}"'] & {[A,A]}
\end{tikzcd}
$$

where the unlabelled 2-cells are the ones defined in Proposition 2.13.4. Recall that an *opmonoidal monad* is a monad in $\mathbf{Opmon}(\mathscr{M})$ (see Section 2.3).

Proposition 2.14. *The monad $t : [A, A] \to [A, A]$ is opmonoidal.*

Proof. It is a consequence of the description of the multiplication and unit of t above and Proposition 2.13 applied to the closed Gray monoid $\mathscr{M}$. $\square$

Recall that a (bicategorical) Eilenberg-Moore construction for a monad $s : B \to B$ in a bicategory $\mathscr{B}$ is a birepresentation of the pseudofunctor

$$
\mathscr{B}(-, B)^{\mathscr{B}(-,s)} : \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}
$$

or equivalently, the unit $u : B^s \to B$ of that birepresentation. Opmonoidal monads $s : B \to B$ have the property that if they have an Eilenberg-Moore construction $u : B^s \to B$ in $\mathscr{M}$, then this construction lifts to $\mathbf{Opmon}(\mathscr{M})$; in other words, the forgetful 2-functor $\mathbf{Opmon}(\mathscr{M}) \to \mathscr{M}$ creates Eilenberg-Moore objects. Moreover, $u : B^s \to B$ is strong monoidal and an arrow $g : C \to B^s$ is opmonoidal (strong monoidal) if and only if ug is so. The case of $\mathscr{B} = \mathbf{Cat}$ can be found in [67], while the general case is in [17, Lemma 3.2].

Definition 2.4. Suppose that the monad t has an Eilenberg-Moore construction $u : [A, A]^t \to [A, A]$, with $f \dashv u$. So, $[A, A]^t$ has a unique (up to isomorphism) structure of a pseudomonoid such that u is strong monoidal. An Eilenberg-Moore construction $u : [A, A]^t \to [A, A]$ is called *a Hopf module construction* for the map pseudomonoid A.

The Hopf module construction, of course, need not exist in general. However, it does exist when the theorem of Hopf modules holds, as we shall show in subsequent sections.

Observation 2.15. When A has a Hopf module construction the pseudonatural transformation λ in Definition 2.3 is representable by

$$\ell : A \xrightarrow{[j^*,1]} [A, A] \xrightarrow{f} [A, A]^t. \tag{2.12}$$

There exist isomorphisms as depicted below, where w is the 1-cell corresponding to 1_{A^2} under the closedness equivalence $\mathscr{M}(A, [A, A^2]) \simeq \mathscr{M}(A^2, A^2)$.

$$\begin{tikzcd} A \arrow[r, "{[j^*,1]}"] \arrow[d, "w"'] \arrow[dr, phantom, "\cong"] & {[A, A]} \arrow[r, "f"] \arrow[d, "i_A^A"'] \arrow[dr, phantom, "\cong"] & {[A, A]^t} \arrow[d, "u"] \\ {[A, A^2]} \arrow[r, "{[1\otimes j^*,1]}"] \arrow[rr, bend right, "{[1,p]}"'] & {[A^2, A^2]} \arrow[r, "{[p^*,p]}"] \arrow[d, phantom, "\cong"] & {[A, A]} \\ & & \end{tikzcd} \tag{2.13}$$

The isomorphism on the right hand side of (2.13) is the isomorphism of t-algebras $uf \cong t$ induced by the universal property of u. We consider $[A, p]w$ as equipped with the unique t-algebra structure such that (2.13) is a morphism of t-algebras.

Corollary 2.16. *The theorem of Hopf modules holds for A if and only if the 1-cell*

$$A \xrightarrow{w} [A, A^2] \xrightarrow{[A,p]} [A, A] \tag{2.14}$$

provides a Hopf module construction for A.

Proof. The pseudonatural transformation λ in Definition 2.3 is an equivalence if and only if the composition $\upsilon\lambda : \mathscr{M}(-, A) \to \mathscr{M}(A \otimes -, A)^\theta \to \mathscr{M}(A \otimes -, A)$

is an Eilenberg-Moore construction for the monad θ in $[\mathscr{M}^{\mathrm{op}}, \mathbf{Cat}]$. But $\upsilon\lambda$ is represented by the 1-cell (2.14) and θ is represented by t, and the result follows. $\square$

Observe that in the corollary above we do not assume *a priori* the existence of a Hopf module construction for A.

Proposition 2.17. *Suppose that A has a Hopf module construction. The 1-cell ℓ in* (2.12) *is fully faithful and strong monoidal. Moreover, ℓ is an equivalence if and only if the theorem of Hopf modules holds for A (see Definition 2.3).*

Proof. The first and last assertions follow trivially from Proposition 2.7 and Definition 2.3, so we only have to prove that ℓ is strong monoidal, or equivalently, that $u\ell \cong t[j^*, A]$ is strong monoidal. This 1-cell is isomorphic to $[A, p]w$ as in Observation 2.15.

The 1-cell $[A, p]w : A \to [A, A]$ corresponds up to isomorphism under

$$\mathscr{M}(A, [A, A]) \simeq \mathscr{M}(A \otimes A, A)$$

to $p : A \otimes A \to A$, which is obviously a right pseudoaction of A on A, and hence $[A, p]w$ is strong monoidal by Proposition 2.12. This endows $u\ell$ with the structure of a strong monoidal morphism, by transport of structure. $\square$

Corollary 2.18.

1. *Suppose that the monad t has an Eilenberg-Moore construction $f \dashv u : [A, A]^t \to [A, A]$. If the theorem of Hopf modules holds for A then f is a Kleisli construction for t.*
2. *Suppose that the monad t has a Kleisli construction $k : [A, A] \to [A, A]_t$. If the theorem of Hopf modules holds for A then k^* is an Eilenberg-Moore construction for t.*

Proof. Let $\mathscr{C} \subset \mathscr{M}(A \otimes X, A)^{\theta_X}$ be the full image of the free θ_X-algebra functor $\varphi_X : \mathscr{M}(A \otimes X, A) \to \mathscr{M}(A \otimes X, A)^{\theta_X}$. When thought of as with codomain $\mathscr{C}$, φ_X provides a Kleisli construction for θ_X. The theorem of Hopf modules holds if and only if $\lambda_X = \varphi_X \mathscr{M}(j^* \otimes X, A)$ is an essentially surjective on objects, since it is always fully faithful by Proposition 2.7. Hence, the theorem of Hopf modules holds if and only if the inclusion of $\mathscr{C}$ into $\mathscr{M}(A \otimes X, A)^{\theta_X}$ is an equivalence, which is equivalent to saying that φ_X is a (bicategorical) Kleisli construction for θ. This proves (1) since t and f represent θ and φ respectively. To show (2), since $\varphi_X : \mathscr{M}(A \otimes X, A) \to \mathscr{C}$ is a Kleisli construction for θ_X, the 1-cell k^*

is an Eilenberg-Moore construction for t if and only if the right adjoint of φ_X, $\mathscr{C} \hookrightarrow \mathscr{M}(A \otimes X, A)^{\theta_X} \to \mathscr{M}(A \otimes X, A)$, is an Eilenberg-Moore construction for θ_X and this happens only if the inclusion $\mathscr{C} \hookrightarrow \mathscr{M}(A \otimes X, A)^{\theta_X}$ is an equivalence. $\square$

2.5 On the existence of Hopf modules

In this section we study the existence of the Hopf module construction for an arbitrary map pseudomonoid. Since this construction is an Eilenberg-Moore construction for a certain monad, it is natural to embed $\mathscr{M}$ into a 2-category where this exists, and the obvious choice is the completion of $\mathscr{M}$ under (**Cat**-enriched) Eilenberg-Moore objects. This is a 2-category $\mathsf{EM}(\mathscr{M})$ with a fully faithful universal 2-functor $E : \mathscr{M} \to \mathsf{EM}(\mathscr{M})$. However, in order to speak of the Hopf module construction for a map pseudomonoid B in $\mathsf{EM}(\mathscr{M})$ we need $\mathsf{EM}(\mathscr{M})$ to be a monoidal 2-category and the pseudofunctor $B \otimes -$ to have right biadjoint.

We prove that when $\mathscr{M}$ is a Gray monoid there exists a model of its completion under Eilenberg-Moore objects which is also a Gray monoid and such that the 2-functor $E : \mathscr{M} \to \mathsf{EM}(\mathscr{M})$ is strict monoidal; this model is the 2-category explicitly described in [54]. In fact, we prove this by extending the assignment $\mathscr{M} \mapsto \mathsf{EM}(\mathscr{M})$ to a monoidal functor on the monoidal category **Gray**, which turns out to be a **Gray**-functor. In order to show that if $A \otimes - : \mathscr{M} \to \mathscr{M}$ has right biadjoint then the same is true for $E(A)$ in $\mathsf{EM}(\mathscr{M})$ we have to move from **Gray**, where the 1-cells are 2-functors, to **Bicat**, where 1-cells are pseudofunctors. For this we extend EM to a homomorphism of tricategories on **Bicat**.

So far we have only considered bicategorical Eilenberg-Moore constructions. However, in this section we will use the completion of a 2-category under **Cat**-enriched Eilenberg-Moore objects. Recall that a **Cat**-enriched Eilenberg-Moore construction on a monad $s : Y \to Y$ in a 2-category $\mathscr{K}$ is a representation of the 2-functor $\mathscr{K}(-, Y)^{\mathscr{K}(-,t)} : \mathscr{K}^{\mathrm{op}} \to \mathbf{Cat}$. Any 2-categorical Eilenberg-Moore construction is also a bicategorical one because 2-natural isomorphisms are pseudonatural equivalences.

From [54] we know that $\mathsf{EM}(\mathscr{K})$, the completion under Eilenberg-Moore objects of the 2-category $\mathscr{K}$, may be described as the 2-category with objects the monads in $\mathscr{K}$, 1-cells from (X, r) to (Y, s) monad morphisms, *i.e.*, a 1-cells

$f : X \to Y$ equipped with a 2-cell $\psi : sf \Rightarrow ft$ satisfying

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[rr, bend left=50, "t"] \arrow[rr, phantom, bend left=25, "\mu\Uparrow"] \arrow[d, "f"'] & X \arrow[r, "t"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "s"'] \arrow[ur, phantom, "\psi\Nearrow"] & Y \arrow[r, "s"'] \arrow[ur, phantom, "\psi\Nearrow"] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[rr, "t"] \arrow[d, "f"'] & & X \arrow[d, "f"] \\ Y \arrow[rr, "s"] \arrow[dr, "s"'] \arrow[urr, phantom, "\psi\Nearrow"] & & Y \\ & Y \arrow[ur, "s"'] \arrow[u, phantom, "\mu\Uparrow"] & \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "s"] \arrow[r, bend right=60, "1"'] \arrow[r, phantom, bend right=30, "\eta\Uparrow"] \arrow[ur, phantom, "\psi\Nearrow"] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "t"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\eta\Uparrow"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

and 2-cells $(f, \psi) \Rightarrow (g, \chi)$ 2-cells $\rho : sf \Rightarrow gt$ in $\mathscr{K}$ such that

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[rr, bend left=50, "t"] \arrow[rr, phantom, bend left=25, "\mu\Uparrow"] \arrow[d, "f"'] & X \arrow[r, "t"] \arrow[d, "g"] & X \arrow[d, "g"] \\ Y \arrow[r, "s"'] \arrow[ur, phantom, "\rho\Nearrow"] & Y \arrow[r, "s"'] \arrow[ur, phantom, "\chi\Nearrow"] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[rr, "t"] \arrow[d, "f"'] & & X \arrow[d, "g"] \\ Y \arrow[rr, "s"] \arrow[dr, "s"'] \arrow[urr, phantom, "\rho\Nearrow"] & & Y \\ & Y \arrow[ur, "s"'] \arrow[u, phantom, "\mu\Uparrow"] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[rr, bend left=50, "t"] \arrow[rr, phantom, bend left=25, "\mu\Uparrow"] \arrow[d, "f"'] & X \arrow[r, "t"] \arrow[d, "f"] & X \arrow[d, "g"] \\ Y \arrow[r, "s"'] \arrow[ur, phantom, "\psi\Nearrow"] & Y \arrow[r, "s"'] \arrow[ur, phantom, "\rho\Nearrow"] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[rr, "t"] \arrow[d, "f"'] & & X \arrow[d, "g"] \\ Y \arrow[rr, "s"] \arrow[dr, "s"'] \arrow[urr, phantom, "\rho\Nearrow"] & & Y \\ & Y \arrow[ur, "s"'] \arrow[u, phantom, "\mu\Uparrow"] & \end{tikzcd}$$

This is called the *unreduced form* of the 2-cells in [54].

The completion comes equipped with a fully faithful 2-functor $E : \mathscr{K} \to \mathsf{EM}(\mathscr{K})$ given on objects by $X \mapsto (X, 1_X)$. This 2-functor has a universal property: for any 2-category with Eilenberg-Moore objects $\mathscr{L}$, E induces an isomorphism of categories $[\mathsf{EM}(\mathscr{K}), \mathscr{L}]_{\text{EM}} \to [\mathscr{K}, \mathscr{L}]$, where $[\mathsf{EM}(\mathscr{K}), \mathscr{L}]_{\text{EM}} \subset [\mathsf{EM}(\mathscr{K}), \mathscr{L}]$ is the full sub 2-category of Eilenberg-Moore object-preserving 2-functors. Moreover, any object of $\mathsf{EM}(\mathscr{K})$ is the Eilenberg-Moore construction on some monad in the image of E.

Denote by **Hom** the category whose objects are 2-categories and whose arrows are pseudofunctors. This category is monoidal under the cartesian product.

Proposition 2.19. *Completion under Eilenberg-Moore objects defines a strong monoidal functor* $\mathsf{EM} : \mathbf{Hom} \to \mathbf{Hom}$.

Proof. We use the explicit description of the Eilenberg-Moore completion given in [54]. Define EM on a pseudofunctor $F : \mathscr{K} \to \mathscr{L}$ as sending an object (X, r) to the monad (FX, Fr) in $\mathscr{L}$, a 1-cell (f, ψ) to $(Ff, F\psi)$ and a 2-cell ρ to $F\rho$. The comparison 2-cell $(\mathsf{EM}F(g, \chi))(\mathsf{EM}F(f, \psi)) \to \mathsf{EM}F((g, \chi)(f, \psi))$ is defined to be

$$(Fr)(Fg)(Ff) \xrightarrow{\cong} F(rgf) \xrightarrow{F((g\psi)\cdot(\chi f))} F(gft) \xrightarrow{\cong} F(gf)(Ft)$$

or what is the same thing

$$\begin{aligned}(Fr)(Fg)(Ff) \xrightarrow{\cong} F(rg)(Ff) \xrightarrow{(F\chi)(Ff)} F(sg)(Ff) \xrightarrow{\cong} (Fg)F(sf) \longrightarrow \\ \xrightarrow{(Fg)(F\psi)} (Fg)F(ft) \xrightarrow{\cong} F(gf)(Ft) \quad (2.15)\end{aligned}$$

where the unlabelled isomorphisms are (the unique possible) compositions of the structural constraints of the pseudofunctor F. The axioms of a 2-cell in $\mathsf{EM}(\mathscr{L})$ follow from the fact that $(Fg)(Ff)$ and $F(gf)$ are monad morphisms. Similarly, the identity constraint of $1_{\mathsf{EM}F(X)} \to (\mathsf{EM}F)(1_X)$ is defined as

$$\big((Ft)1_{FX} \xrightarrow{(Ft)F_0} (Ft)(F1_x) \xrightarrow{\cong} (F1_x)(Ft)\big) = \big(1_{FX}(Ft) \xrightarrow{F_0(Ft)} (F1_X)(Ft)\big)$$

where F_0 is the identity constraint of F.

It is clear that this defines a functor EM. It is also clear that it is strong monoidal, with constraints the evident isomorphisms $\mathsf{EM}(\mathscr{K}) \times \mathsf{EM}(\mathscr{L}) \cong \mathsf{EM}(\mathscr{K} \times \mathscr{L})$ and $E_1 : 1 \cong \mathsf{EM}(1)$. ☐

Observation 2.20. If $F : \mathscr{K} \to \mathscr{L}$ is a biequivalence between 2-categories, then $\mathsf{EM}F$ is a biequivalence too. This is straightforward from the definition of EM on pseudofunctors in the proof of Proposition 2.19 above.

Recall from Section 2.1.1 the notion of cubical functor.

Corollary 2.21. *The pseudofunctor below is a cubical functor whenever* $F : \mathscr{K} \times \mathscr{L} \to \mathscr{J}$ *is one.*

$$\mathsf{EM}(\mathscr{K}) \times \mathsf{EM}(\mathscr{L}) \xrightarrow{\cong} \mathsf{EM}(\mathscr{K} \times \mathscr{L}) \xrightarrow{\mathsf{EM}F} \mathsf{EM}(\mathscr{J})$$

Proof. Consider 1-cells in $\mathsf{EM}(\mathscr{K}) \times \mathsf{EM}(\mathscr{L})$

$$((X',t'),(X,t)) \xrightarrow{((f',\psi'),(f,\psi))} ((Y',s'),(Y,s)) \xrightarrow{((g',\chi'),(g,\chi))} ((Z',r'),(Z,r)).$$

If $(X,t) = (Y,s)$ and (f,ψ) is the identity 1-cell of (X,t), that is $(f,\psi) = (1_X, 1_t)$, then the constraint defined in (2.15) above is

$$\begin{aligned} F(r',r)F(g',g)F(f',1) = F(r',r)F(g'f',g) &\xrightarrow{\cong} F(r'g'f',rg) \\ &\xrightarrow{F((g'\psi')\cdot(\chi'f'),\chi)} F(g'f't',gt) \xrightarrow{\cong} F(g'f',g)F(t',t') \end{aligned}$$

which is exactly the identity 2-cell of the 1-cell $\mathsf{EM}F((g',\chi')(f',\psi'),(g\chi))$ in the 2-category $\mathsf{EM}(\mathscr{J})$. The rest of the proof is similar. $\square$

Recall from Section 2.1.1 the Gray tensor product of 2-categories. If $\mathscr{K},\mathscr{L}$ are 2-categories, its Gray tensor product $\mathscr{K}\square\mathscr{L}$ is a 2-category classifying cubical functors out of $\mathscr{K}\times\mathscr{L}$.

Corollary 2.22. *Completion under Eilenberg-Moore objects induces a monoidal functor* EM *from* **Gray** *to itself. Furthermore, the 2-functors* $E_{\mathscr{K}} : \mathscr{K} \to \mathsf{EM}(\mathscr{K})$ *are the components of a monoidal natural transformation.*

Proof. Define the structural arrow $\mathsf{EM}(\mathscr{K})\square\mathsf{EM}(\mathscr{L}) \to \mathsf{EM}(\mathscr{K}\square\mathscr{L})$ as corresponding to $\mathsf{EM}(\mathscr{K})\times\mathsf{EM}(\mathscr{L}) \cong \mathsf{EM}(\mathscr{K}\times\mathscr{L}) \to \mathsf{EM}(\mathscr{K}\square\mathscr{L})$, which is a cubical functor by Corollary 2.21, and the arrow $1 \to \mathsf{EM}(1)$ as the universal E_1. Here the symbol $\square$ denotes the Gray tensor product. The axioms of lax monoidal functor follow from the fact that EM is monoidal with respect to the cartesian product.

The naturality of the arrows $E_{\mathscr{K}}$ follows from the universal property of the completion under Eilenberg-Moore objects. We only have to prove that the resulting natural transformation is monoidal. Consider the diagram

$$\begin{tikzcd}
\mathsf{EM}(\mathscr{K})\square\mathsf{EM}(\mathscr{L}) \arrow[rrr, bend left] & & & \\
& \mathsf{EM}(\mathscr{K})\times\mathsf{EM}(\mathscr{L}) \arrow[ul] \arrow[r, "\cong"] & \mathsf{EM}(\mathscr{K}\times\mathscr{L}) \arrow[r] & \mathsf{EM}(\mathscr{K}\square\mathscr{L}) \\
& \mathscr{K}\times\mathscr{L} \arrow[u, "E_{\mathscr{K}}\times E_{\mathscr{L}}"] \arrow[ur, "E_{\mathscr{K}\times\mathscr{L}}"'] \arrow[dl] & & \\
\mathscr{K}\square\mathscr{L} \arrow[uuu, "E_{\mathscr{K}}\square E_{\mathscr{L}}"] \arrow[uurrr, "E_{\mathscr{K}\square\mathscr{L}}"'] & & &
\end{tikzcd}$$

One of the two axioms we have to check is the commutativity of the exterior diagram. This commutativity can be proven by observing that each one of the four internal diagrams commute and then applying the universal property of $\mathscr{K} \times \mathscr{L} \to \mathscr{K} \square \mathscr{L}$. The other axiom, involving $E_1 : 1 \to \mathsf{EM}(1)$ is trivial, since E_1 itself is the unit constraint. $\square$

Corollary 2.23. $\mathsf{EM}(\mathscr{M})$ *is a Gray monoid whenever* $\mathscr{M}$ *is a Gray monoid. Moreover, the 2-functor* $E_{\mathscr{M}} : \mathscr{M} \to \mathsf{EM}(\mathscr{M})$ *is strict monoidal, so that* $\mathscr{M}$ *can be identified with a full monoidal sub 2-category of* $\mathsf{EM}(\mathscr{M})$*.*

Proof. We know that EM is a monoidal functor, and as such it preserves monoids. Moreover, $E_{\mathscr{M}}$ is strict monoidal, that is, a morphism of monoids in **Gray**, since E is a monoidal natural transformation (see Corollary 2.22). $\square$

The tensor product in $\mathsf{EM}(\mathscr{M})$ is induced by the one of $\mathscr{M}$; for instance, the tensor product of (X, r) with (Y, s), denoted by $(X, r) \circledcirc (Y, s)$, is $(X \otimes Y, r \otimes s)$.

In order to show that EM is in fact a **Gray**-functor we state the following easy result.

Lemma 2.24. *Let* $\mathscr{V}$ *be a symmetric monoidal closed category and* $F : \mathscr{V} \to \mathscr{V}$ *be a lax monoidal functor. Then, any monoidal natural transformation* $\eta : 1_{\mathscr{V}} \Rightarrow F$ *induces on* F *a structure of a* $\mathscr{V}$*-functor.*

Proof. Define F on enriched homs as

$$F_{X,Y} : [X, Y] \xrightarrow{\eta_{[X,Y]}} F([X, Y]) \xrightarrow{\vartheta_{X,Y}} [FX, FY]$$

where $\vartheta_{X,Y}$ is the arrow corresponding to $F[X, Y] \otimes FX \longrightarrow F([X, Y] \otimes X) \xrightarrow{F\text{ev}} FY$. $\square$

Corollary 2.25. EM : **Gray** $\to$ **Gray** *has a canonical structure of* **Gray***-functor.*

Proof. Let $\mathscr{V}$ in the lemma above be **Gray** and η be the transformation defined by the inclusions $E_{\mathscr{K}} : \mathscr{K} \to \mathsf{EM}(\mathscr{K})$, which is easily shown to be a monoidal transformation. Now apply the lemma. $\square$

Let $\mathbf{Ps}(\mathscr{K}, \mathscr{L})$ denote the 2-category of pseudofunctors from $\mathscr{K}$ to $\mathscr{L}$, pseudonatural transformations between them and modifications between these.

Observation 2.26. In the case of EM, the transformation $\vartheta_{\mathscr{K},\mathscr{L}}$ is defined by the commutativity of the following diagram

$$\begin{tikzcd}
\mathbf{Ps}(\mathsf{EM}(\mathscr{K}),\mathsf{EM}(\mathscr{L}))\otimes\mathsf{EM}(\mathscr{K}) \arrow[rr, "\mathrm{ev}"] & & \mathsf{EM}(\mathscr{L}) \\
\mathsf{EM}\mathbf{Ps}(\mathscr{K},\mathscr{L})\otimes\mathsf{EM}(\mathscr{K}) \arrow[u, "\vartheta_{\mathscr{K},\mathscr{L}}\otimes 1"] \arrow[rr] & & \mathsf{EM}(\mathbf{Ps}(\mathscr{K},\mathscr{L})\otimes\mathscr{L}) \arrow[u, "\mathsf{EM}\mathrm{ev}"']
\end{tikzcd}$$

that is,

$$\begin{aligned}
(\vartheta_{\mathscr{K},\mathscr{L}}(F,\tau))(X,t) &= (\mathsf{EM}\mathrm{ev})((F,\tau),(X,t)) \\
&= (\mathrm{ev}(F,X),\mathrm{ev}(\tau,t)) \\
&= (FX,(Ft)\tau_X),
\end{aligned}$$

and then EM is defined on homs by the 2-functor

$$\vartheta_{\mathscr{K},\mathscr{L}}E_{\mathscr{K},\mathscr{L}} : \mathbf{Ps}(\mathscr{K},\mathscr{L}) \to \mathbf{Ps}(\mathsf{EM}(\mathscr{K}),\mathsf{EM}(\mathscr{L}))$$

whose value on a 2-functor F is the 2-functor sending a monad (X,t) to (FX,Ft). Then we see that our **Gray**-functor has as underlying ordinary functor just the restriction to **Gray** of the functor in Proposition 2.19.

Denote by **Bicat** the tricategory of bicategories, pseudofunctors, pseudonatural transformations and modifications as defined in [31, 5.6]. (There is another canonical choice for a tricategory structure on **Bicat**, as explained in that paper.) We shall describe an extension of the **Gray**-functor EM to a homomorphism of tricategories $\widetilde{\mathsf{EM}} : \mathbf{Bicat} \to \mathbf{Bicat}$. In order to do this we will use the construction of a homomorphism of tricategories $\mathbf{Bicat} \to \mathbf{Gray}$ given in [31], of which we recall some aspects. For each bicategory $\mathscr{B}$ there is a 2-category $\mathbf{st}\mathscr{B}$ and a pseudofunctor $\xi_{\mathscr{B}} : \mathscr{B} \to \mathbf{st}\mathscr{B}$ inducing for each 2-category $\mathscr{K}$ an *isomorphism* of 2-categories $\mathbf{Bicat}(\mathscr{B},\mathscr{K}) \cong \mathbf{Ps}(\mathbf{st}\mathscr{B},\mathscr{K})$. Moreover, $\xi_{\mathscr{B}}$ is a biequivalence of bicategories. As usual, we get a pseudofunctor

$$\mathbf{st}_{\mathscr{A},\mathscr{B}} : \mathbf{Bicat}(\mathscr{A},\mathscr{B}) \to \mathbf{Ps}(\mathbf{st}\mathscr{A},\mathbf{st}\mathscr{B})$$

which turns out to be an biequivalence. Finally, the object part of the homomorphism of tricategories $\mathbf{Bicat} \to \mathbf{Gray}$ is given by $\mathscr{B} \mapsto \mathbf{st}\mathscr{B}$ while on hom-bicategories it is given by the biequivalence $\mathbf{st}_{\mathscr{A},\mathscr{B}}$.

Define a homomorphism of tricategories $\widetilde{\mathsf{EM}}$ by

$$\begin{tikzcd} \mathbf{Bicat} \arrow[r, "\sim"] \arrow[d, dotted, "\widetilde{\mathsf{EM}}"'] & \mathbf{Gray} \arrow[d, "\mathsf{EM}"] \\ \mathbf{Bicat} & \mathbf{Gray} \arrow[l, hook'] \end{tikzcd}$$

It is given on objects by $\mathscr{B} \mapsto \mathsf{EM}(\mathbf{st}\mathscr{B})$ and on homs by

$$\mathbf{Bicat}(\mathscr{A}, \mathscr{B}) \xrightarrow{\mathbf{st}} \mathbf{Ps}(\mathbf{st}\mathscr{A}, \mathbf{st}\mathscr{B}) \xrightarrow{E} \mathsf{EM}\mathbf{Ps}(\mathbf{st}\mathscr{A}, \mathbf{st}\mathscr{B}) \xrightarrow{\vartheta} \mathbf{Ps}(\mathsf{EM}\mathbf{st}\mathscr{A}, \mathsf{EM}\mathbf{st}\mathscr{B}),$$

which by the Observation 2.26 sends a pseudofunctor $F : \mathscr{A} \to \mathscr{B}$ to the 2-functor $\mathsf{EM}(\mathbf{st}F)$ defined in Proposition 2.19.

Proposition 2.27. *Every biadjunction between pseudofunctors $F \dashv_{\mathrm{b}} G : \mathscr{L} \to \mathscr{K}$, where $\mathscr{K}$ and $\mathscr{L}$ are 2-categories, induces a biadjunction $\mathsf{EM}F \dashv_{\mathrm{b}} \mathsf{EM}G$.*

Proof. Since $\widetilde{\mathsf{EM}}$ is a homomorphism of tricategories on **Bicat**, $\widetilde{\mathsf{EM}}F = \mathsf{EM}(\mathbf{st}F)$ is left biadjoint to $\widetilde{\mathsf{EM}}G = \mathsf{EM}(\mathbf{st}G)$. The 2-functor $\mathbf{st}F$ is defined as the unique 2-functor such that $(\mathbf{st}F)\xi_{\mathscr{K}} = \xi_{\mathscr{L}}F$, and similarly for G. It follows, by functoriality of EM with respect to pseudofunctors (Proposition 2.19), that

$$\mathsf{EM}(\mathbf{st}F)\mathsf{EM}\xi_{\mathscr{K}} = \mathsf{EM}\xi_{\mathscr{L}}\mathsf{EM}F \quad \text{and} \quad \mathsf{EM}(\mathbf{st}G)\mathsf{EM}\xi_{\mathscr{L}} = \mathsf{EM}\xi_{\mathscr{K}}\mathsf{EM}G.$$

Since each component of ξ is a biequivalence and these are preserved by EM (see Observation 2.20), we have

$$\mathsf{EM}F \simeq (\mathsf{EM}\xi_{\mathscr{L}})^*\mathsf{EM}(\mathbf{st}F)\mathsf{EM}\xi_{\mathscr{K}} \dashv_{\mathrm{b}} (\mathsf{EM}\xi_{\mathscr{K}})^*\mathsf{EM}(\mathbf{st}G)\mathsf{EM}\xi_{\mathscr{L}} \simeq \mathsf{EM}G$$

□

Corollary 2.28. *If X is an object in a Gray monoid $\mathscr{M}$ such that $X \otimes -$ has right biadjoint $[X, -]$, then $(EX \circledcirc -) : \mathsf{EM}(\mathscr{M}) \to \mathsf{EM}(\mathscr{M})$ has right biadjoint $\langle EX, -\rangle$ given by $\langle EX, (Y, s)\rangle = ([X, Y], [X, s])$.*

Proof. The 2-functor $(EX \circledcirc -)$ is just $\mathsf{EM}(X \otimes -)$, and then by the proposition above it has right biadjoint $\mathsf{EM}([X, -])$. This is given by the stated formula as a consequence of the description of the effect of EM on pseudofunctors in the proof of Proposition 2.19. □

Theorem 2.29. *For any closed Gray monoid $\mathscr{M}$ there exists another Gray monoid $\mathscr{N}$ and a fully faithful strict monoidal 2-functor $\mathscr{M} \to \mathscr{N}$ such that* *any map pseudomonoid in $\mathscr{M}$ has a Hopf module construction in $\mathscr{N}$. Moreover, $\mathscr{N}$ can be taken to be $\mathsf{EM}(\mathscr{M})$.*

Proof. The proof is only a matter of putting Corollaries 2.23 and 2.28 together with the definition of the object of Hopf modules. □

Proposition 2.30. *Let A be a map pseudomonoid in a Gray monoid $\mathscr{M}$ such that $A \otimes -$ has right biadjoint. Suppose that the theorem of Hopf modules holds for $E(A) \in \mathrm{ob}\mathsf{EM}(\mathscr{M})$; then it also holds for A. Moreover, in this case A has a Hopf module construction provided by*

$$A \xrightarrow{w} [A, A \otimes A] \xrightarrow{[A,p]} [A, A] \tag{2.16}$$

as in Corollary 2.16.

Proof. Consider the image of the monad t under the 2-functor $E : \mathscr{M} \to \mathsf{EM}(\mathscr{M})$. Denote by $\hat{\theta}$ the monad $\mathsf{EM}(\mathscr{M})(-, Et)$ on $\mathsf{EM}(\mathscr{M})(-, E[A, A])$ and $\hat{\varphi} \dashv \hat{\upsilon}$ the adjunction arising from its Eilenberg-Moore construction in $\mathbf{Hom}(\mathsf{EM}(\mathscr{M})^{\mathrm{op}}, \mathbf{Cat})$. Observe that by the fully faithfulness of E, the monad $\hat{\theta}_{EXMP}$ can be identified with the monad θ_X of Definition 2.1, and the adjunction $\hat{\varphi}_{EXMP} \dashv \hat{\upsilon}_{EXMP}$ with the adjunction $\varphi_X \dashv \upsilon_X$ corresponding to θ.

If the theorem of Hopf modules holds for $E(A)$ then in particular for each object X of $\mathscr{M}$ the functor

$$\begin{aligned}\mathsf{EM}(\mathscr{M})(E(X), E(A)) \xrightarrow{\mathsf{EM}(\mathscr{M})(1,E([j^*,A]))} \mathsf{EM}(\mathscr{M})(E(X), E[A, A]) \longrightarrow \\ \xrightarrow{\hat{\varphi}_{E(X)}} \mathsf{EM}(\mathscr{M})(E(X), E[A, A])^{\hat{\theta}_{E(X)}}\end{aligned} \tag{2.17}$$

is an equivalence (Definition 2.3). But by the fully faithfulness of the 2-functor E this is, up to composing with suitable isomorphisms, just the functor λ_X in Definition 2.3 and then the theorem of Hopf modules holds for A.

The last assertion follows directly from Corollary 2.16. □

2.6 Left autonomous pseudomonoids and the theorem of Hopf modules

In this section we specialise to the kind of pseudomonoid central to our work, namely the autonomous pseudomonoids. We begin by recalling the necessary background.

2.6.1 Background on dualizations

A *bidual pair* in a Gray monoid $\mathscr{M}$ is a pseudoadjunction (see for example [52]) in the one-object Gray-category $\mathscr{M}$. Explicitly, it consists of a pair of 1-cells $\mathrm{e} : X \otimes Y \to I$ and $\mathrm{n} : I \to Y \otimes X$ together with invertible 2-cells

$$1_Y \Rightarrow (Y \otimes \mathrm{e})(\mathrm{n} \otimes Y) : Y \to Y \qquad (\mathrm{e} \otimes X)(X \otimes \mathrm{n}) \Rightarrow 1_X : X \to X$$

satisfying the following two axioms.

$$\begin{tikzcd}
& & f \otimes u \arrow[dr, "\mathrm{e}"] & \\
f \otimes u \arrow[r, "1\otimes \mathrm{n} \otimes 1"] \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] & f \otimes u \otimes f \otimes u \arrow[ur, "1\otimes 1\otimes \mathrm{e}"] \arrow[dr, "\mathrm{e}\otimes 1\otimes 1"'] \arrow[u, phantom, "\Downarrow 1\otimes\eta"] \arrow[d, phantom, "\Downarrow \varepsilon\otimes 1"] \arrow[rr, phantom, "\Downarrow c_{\mathrm{e},\mathrm{e}}^{-1}"] & & 1 \\
& & f \otimes u \arrow[ur, "\mathrm{e}"'] &
\end{tikzcd} \tag{2.18}$$

$$\begin{tikzcd}
& u \otimes f \arrow[dr, "\mathrm{n}\otimes 1\otimes 1"] \arrow[drrr, bend left, "1"] & & & \\
1 \arrow[ur, "\mathrm{n}"] \arrow[dr, "\mathrm{n}"'] \arrow[rr, phantom, "\Downarrow c_{\mathrm{n},\mathrm{n}}^{-1}"] & & u \otimes f \otimes u \otimes f \arrow[rr, "1\otimes \mathrm{e} \otimes 1"] \arrow[u, phantom, "\Downarrow \eta\otimes 1"] \arrow[d, phantom, "\Downarrow 1\otimes\varepsilon"] & & u \otimes f \\
& u \otimes f \arrow[ur, "1\otimes 1\otimes \mathrm{n}"'] \arrow[urrr, bend right, "1"'] & & &
\end{tikzcd} \tag{2.19}$$

The object X is called a *right bidual* of Y, denoted by Y°, and Y is called a *left bidual* of X, denoted by $X^\vee$. A Gray monoid in which every object has a right (left) bidual is called right (left) autonomous.

If X has a right bidual X°, then the 2-functor $X \otimes -$ has a right biadjoint $X^\circ \otimes -$, and $- \otimes X$ has a left biadjoint $- \otimes X^\circ$, and dually for left biadjoints. In particular, any right (left) autonomous Gray monoid is right (left) closed with internal hom $[X, Y] = X^\circ \otimes Y$ ($[X, Y] = Y \otimes X^\vee$). If both X and Y have a bidual and $f : X \to Y$ is a 1-cell, the bidual of f is the 1-cell $f^\circ = (X^\circ \otimes \mathrm{e})(X^\circ \otimes f \otimes Y^\circ)(\mathrm{n} \otimes Y^\circ) : Y^\circ \to X^\circ$. Similarly with 2-cells. If $\mathscr{N}$ is the full sub-2-category of $\mathscr{M}$ whose objects are the objects with right bidual, we have a monoidal pseudofunctor $(-)^\circ : (\mathscr{N}^{\mathrm{op}})^{\mathrm{rev}} \to \mathscr{M}$, where the superscript rev indicates the reverse monoidal structure. The structural constraints are given by the canonical equivalences $I \simeq I^\circ$ and $Y^\circ \otimes X^\circ \simeq (X \otimes Y)^\circ$.

Recall from [13] that a *left dualization* for a pseudomonoid (A, j, p) in $\mathscr{M}$ is a 1-cell $d : A^\circ \to A$ equipped with two 2-cells $\alpha : p(d \otimes A)\mathrm{n} \Rightarrow j$ and $\beta : j\mathrm{e} \Rightarrow p(A \otimes d)$ satisfying two axioms. Let us write $f \bullet g$ for the composition $p(f \otimes A)(X \otimes g) : X \otimes Y \to A$, for a pair of arrows $f : X \to A$, $g : Y \to A$. The 2-cells α, β are *extraordinary 2-cells* in the sense of [80], that we write $\alpha : d \bullet 1_A \to j$ and $\beta : j \to A \bullet d$. The axioms of a left dualization state that $\alpha, beta$ satisfy the usual triangular equalities of an adjunction

$$1 = \big(1_A \xrightarrow{\cong} j \bullet 1_A \xrightarrow{\beta \bullet 1_A} (1_A \bullet d) \bullet 1_A \xrightarrow{\cong} 1_A \bullet (d \bullet 1_A) \xrightarrow{1_A \bullet \alpha} 1_A \bullet j \xrightarrow{1_A} 1_A\big)$$
$$1 = \big(d \xrightarrow{\cong} d \bullet j \xrightarrow{d \bullet \beta} d \bullet (1_A \bullet d) \xrightarrow{\cong} (d \bullet 1_A) \bullet d \xrightarrow{\alpha \bullet d} j \bullet d \xrightarrow{\cong} d\big)$$

Left dualization structures on $d : A^\circ \to A$ are in bijection with adjunctions

$$p \dashv (p \otimes A)(A \otimes d \otimes A)(A \otimes \mathrm{n}) \tag{2.20}$$

satisfying the following condition. Consider the pseudomonad $(A \otimes -)$, and the free pseudo-$(A \otimes -)$-algebras A and $A \otimes A$. The 1-cell p has the canonical structure of a pseudomorphism of pseudo-$(A \otimes -)$-algebras, given by the associativity constraint. Also, the three 1-cells composed in the right hand side of (2.20) are clearly pseudomorphisms; we consider $(p \otimes A)(A \otimes d \otimes A)(A \otimes \mathrm{n})$ with the composition pseudomorphism structure. The required condition is that the adjunction (2.20) must be an adjunction in the 2-category of pseudoalgebras Ps-$(A \otimes -)$-Alg. This condition is missing in [13] and will appear in [57]. Similarly, left dualization structures on d are in bijection with adjunctions $p(d \otimes A) \dashv (A^\circ \otimes p)(\mathrm{n} \otimes A)$ in Ps-$(- \otimes A)$-Alg. For example, given α and β the counit of the corresponding adjunction (2.20) is

$$\begin{tikzcd}
A \otimes A^\circ \otimes A \arrow[r, "1\otimes d\otimes 1"] \arrow[dr, phantom, "\Downarrow 1\otimes\alpha"] & A^3 \arrow[r, "p\otimes 1"] \arrow[d, "1\otimes p"] \arrow[dr, phantom, "\cong"] & A^2 \arrow[d, "p"] \\
A \arrow[u, "1\otimes \mathrm{n}"] \arrow[r, "1\otimes j"] \arrow[rr, bend right, "1"'] & A^2 \arrow[r, "p"] \arrow[d, phantom, "\Downarrow\cong"] & A \\
& &
\end{tikzcd}$$

(To be precise, in [13] the authors define left dualization in a right autonomous Gray monoid, *i.e.*, a Gray monoid where any object has a right bidual, but the only really necessary condition is that the pseudomonoid itself have a right bidual).

A pseudomonoid equipped with a left dualization is called *left autonomous.*

If a left dualization exists, then it is isomorphic to

$$d \cong (A \otimes \mathrm{e})(p^* \otimes A^\circ)(j \otimes A^\circ) : A^\circ \to A \tag{2.21}$$

by [13, Proposition 1.2]. Furthermore, when j is a map, a left dualization d has always a right adjoint given by

$$d^* \cong (A^\circ \otimes j^* p)(\mathrm{n} \otimes A) : A \to A^\circ. \tag{2.22}$$

Example 2.31. The bicategory of $\mathscr{V}$-modules is left and right autonomous. The bidual of a $\mathscr{V}$-category $\mathscr{A}$ is the opposite $\mathscr{V}$-category $\mathscr{A}^{\mathrm{op}}$ (if $\mathscr{V}$ is braided non symmetric, we have different left and right opposites, providing left and right biduals). The pseudonatural equivalence $\mathscr{V}$-$\mathbf{Mod}(\mathscr{A} \otimes \mathscr{B}, \mathscr{C}) \simeq \mathscr{V}$-$\mathbf{Mod}(\mathscr{B}, \mathscr{A}^{\mathrm{op}} \otimes \mathscr{C})$ can be taken as the obvious isomorphism $[(\mathscr{A} \otimes \mathscr{B})^{\mathrm{op}} \otimes \mathscr{C}, \mathscr{V}]_0 \cong [\mathscr{B}^{\mathrm{op}} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{C}, \mathscr{V}]_0$. The $\mathscr{V}$-modules n and e are given by $\mathrm{n}(a, a') = \mathscr{A}(a, a')$ and $\mathrm{e}(a, a') = \mathscr{A}(a', a)$. (Note that the $\mathscr{V}$-modules e and n do not induce the isomorphism above, but only equivalences.)

An example of a left autonomous pseudomonoid in $\mathscr{V}$-$\mathbf{Mod}$ is a monoidal $\mathscr{V}$-category with left duals. More precisely, if $\mathscr{A}$ is a monoidal category regarded as a pseudomonoid in $\mathscr{V}$-$\mathbf{Mod}$ and $D : \mathscr{A}^{\mathrm{op}} \to \mathscr{A}$ is a $\mathscr{V}$-functor, then D_* is a left dualization for $\mathscr{A}$ if and only if $\mathscr{A}$ has left duals and D is isomorphic to the functor ${}^*(-)$ defined by a choice of left duals in $\mathscr{A}$. This was shown in [13].

Example 2.32. A right bidual of an object C in the monoidal bicategory of comodules $\mathbf{Comod}(\mathscr{V})$ of Example 2.2, this is, a comonoid in the braided category $\mathscr{V}$, is the opposite comonoid, which we will denote by C°. The comultiplication of C° is equal to the one of C composed with the braiding of $\mathscr{V}$. The braiding induces functors $\mathbf{Comod}(\mathscr{V})(C \otimes D, E) \to \mathbf{Comod}(\mathscr{V})(D, C^\circ \otimes E)$ which are isomorphisms of categories.

In [13] it is noted that a coquasibialgebra H has a structure of a left autonomous pseudomonoid in $\mathbf{Comod}(\mathscr{V})$ with dualization s_*, represented by a comonoid morphism $s : H^\circ \to H$ if and only if s_* is a left dualization for H.

2.6.2 The main result

Given a left autonomous pseudomonoid A define the following important 2-cell.

$$
\gamma := \begin{tikzcd}
{} & A^3 \arrow[r, "p\otimes 1"] \arrow[dr, "1\otimes p"'] \arrow[d, phantom, "1\otimes\varepsilon \;\Downarrow"] & A^2 \arrow[rr, equal] \arrow[dr, "p"] \arrow[d, phantom, "\cong \phi"] & \arrow[d, phantom, "\Downarrow\eta"] & A^2 \\
A^2 \arrow[ur, "1\otimes p^*"] \arrow[rr, equal] & {} & A^2 \arrow[r, "p"'] & A \arrow[ur, "p^*"'] &
\end{tikzcd}
\tag{2.23}
$$

In the lemma below we show that this 2-cell γ is invertible, and in fact this property will turn out to be equivalent to the existence of a left dualization.

Lemma 2.33. *For a left autonomous pseudomonoid A the following equality holds.*

$$
\gamma = \begin{tikzcd}
A^2 \arrow[r, "A^2\otimes \mathrm{n}"] \arrow[d, "p"'] \arrow[rrr, bend left, "1\otimes p^*"] \arrow[dr, phantom, "\cong"] & A^2\otimes A^\circ\otimes A \arrow[r, "A^2\otimes d\otimes 1"] \arrow[d, "p\otimes 1\otimes 1"] \arrow[dr, phantom, "\cong"] & A^4 \arrow[r, "1\otimes p\otimes 1"] \arrow[d, "p\otimes A^2"] \arrow[dr, phantom, "\cong"] & A^3 \arrow[d, "p\otimes 1"] \\
A \arrow[r, "1\otimes \mathrm{n}"'] \arrow[rrr, bend right, "p^*"'] & A\otimes A^\circ\otimes A \arrow[r, "1\otimes d\otimes 1"'] & A^3 \arrow[r, "p\otimes 1"'] & A^2
\end{tikzcd}
\tag{2.24}
$$

In particular, γ is invertible.

Proof. A short proof of this result is possible using the missing condition in [13, Proposition 1.1] discussed in page 26. However, since this condition will only appear in [57], we prefer to give a slightly longer version here.

The 2-cell on the right hand of (2.24) pasted with the counit of the adjunction (2.20) gives the following 2-cell

$$
\begin{tikzcd}
A^2 \arrow[r, "A^2\otimes \mathrm{n}"] \arrow[d, "p"'] \arrow[dr, phantom, "\cong"] & A^2\otimes A^\circ\otimes A \arrow[r, "A^2\otimes d\otimes 1"] \arrow[d, "p\otimes 1\otimes 1"] \arrow[dr, phantom, "\cong"] & A^4 \arrow[r, "1\otimes p\otimes 1"] \arrow[d, "p\otimes 1\otimes 1"] \arrow[dr, phantom, "\cong"] & A^3 \arrow[d, "p\otimes 1"] \\
A \arrow[r, "1\otimes \mathrm{n}"] \arrow[drr, "1\otimes j"] \arrow[drrr, bend right, "1"'] & A\otimes A^\circ\otimes A \arrow[r, "1\otimes d\otimes 1"'] \arrow[dr, phantom, "\Downarrow 1\otimes\alpha"] & A^3 \arrow[r, "p\otimes 1"'] \arrow[d, "1\otimes p"] \arrow[dr, phantom, "\cong"] & A^2 \arrow[d, "p"] \\
& & A^2 \arrow[r, "p"'] \arrow[d, phantom, "\cong"] & A \\
& & {} &
\end{tikzcd}
$$

which itself is equal to

$$\begin{tikzcd}
A^2 \arrow[r, "A^2\otimes\mathrm{n}"] \arrow[dd, "p"'] \arrow[drr, bend right=10, "A^2\otimes j"'] & A^2\otimes A^\circ\otimes A \arrow[r, "A^2\otimes d\otimes 1"] \arrow[d, phantom, "\Downarrow A^2\otimes\alpha"] & A^4 \arrow[r, "1\otimes p\otimes 1"] \arrow[d, "A^2\otimes p"] \arrow[dr, "p\otimes A^2"] & A^3 \arrow[dr, "p\otimes 1"] \arrow[d, phantom, "\cong"] & \\
& {} \arrow[d, phantom, "\cong"] & A^3 \arrow[d, "p\otimes 1"'] \arrow[r, phantom, "\cong"] & A^3 \arrow[r, "p\otimes 1"'] \arrow[dl, "1\otimes p"] \arrow[d, phantom, "\cong"] & A^2 \arrow[dl, "p"] \\
A \arrow[rr, "1\otimes j"'] \arrow[rrr, bend right=40, "1"'] & {} \arrow[r, phantom, "\cong"] & A^2 \arrow[r, "p"] & A &
\end{tikzcd}$$

$$= \begin{tikzcd}
A^2 \arrow[r, "A^2\otimes\mathrm{n}"] \arrow[dd, "p\otimes 1"'] \arrow[drr, bend right=10, "A^2\otimes j"'] & A^2\otimes A^\circ\otimes A \arrow[r, "A^{\otimes}d\otimes 1"] \arrow[d, phantom, "\Downarrow A^2\otimes\alpha"] & A^4 \arrow[r, "1\otimes p\otimes 1"] \arrow[d, "A^2\otimes p"] \arrow[dr, phantom, "\cong"] & A^3 \arrow[dr, "p\otimes 1"] \arrow[d, "1\otimes p"] & \\
& {} \arrow[d, phantom, "\cong"] & A^3 \arrow[d, "p\otimes 1"'] \arrow[r, "1\otimes p"'] \arrow[dr, phantom, "\cong"] & A^2 \arrow[d, "p"] \arrow[r, phantom, "\cong"] & A^2 \arrow[dl, "p"] \\
A \arrow[rr, "1\otimes j"'] \arrow[rrr, bend right=40, "1"'] & {} \arrow[r, phantom, "\cong"] & A^2 \arrow[r, "p"] & A &
\end{tikzcd}$$

$$= \begin{tikzcd}
A^2 \arrow[r, "A^2\otimes\mathrm{n}"] \arrow[drr, bend right=10, "A^2\otimes j"] \arrow[drrr, bend right=30, "1"'] & A^2\otimes A^\circ\otimes A \arrow[r, "A^2\otimes d\otimes 1"] \arrow[d, phantom, "\Downarrow A^2\otimes\alpha"] & A^4 \arrow[r, "1\otimes p\otimes 1"] \arrow[d, "A^2\otimes p"] \arrow[dr, phantom, "\cong"] & A^3 \arrow[dr, "p\otimes 1"] \arrow[d, "1\otimes p"] & \\
& {} \arrow[r, phantom, "\cong"] & A^3 \arrow[r, "1\otimes p"'] & A^2 \arrow[d, "p"'] \arrow[r, phantom, "\cong"] & A^2 \arrow[dl, "p"] \\
& & & A &
\end{tikzcd}$$

$$= \begin{tikzcd}
A^2 \arrow[r, "1\otimes p^*"] \arrow[dr, "1"'] \arrow[r, phantom, shift right=15, "\Swarrow 1\otimes\varepsilon"] & A^3 \arrow[d, "1\otimes p"] \arrow[dr, "p\otimes 1"] & \\
& A^2 \arrow[d, "p"'] \arrow[r, phantom, "\cong"] & A^2 \arrow[dl, "p"] \\
& A &
\end{tikzcd}$$

The result follows. $\square$

Define the 2-cell ω as

$$\begin{tikzcd}
A \arrow[r, "1\otimes j"] \arrow[dr, bend right, "1"'] \arrow[dr, phantom, "\cong"] & A^2 \arrow[r, "1\otimes p^*"] \arrow[d, "p"] \arrow[dr, phantom, "\Downarrow\gamma"] & A^3 \arrow[d, "p\otimes 1"] \\
& A \arrow[r, "p^*"'] & A^2
\end{tikzcd} \tag{2.25}$$

Now we state the basic result of this work.

Theorem 2.34. *Let (A, j, p) be a map pseudomonoid in a Gray monoid $\mathscr{M}$ and suppose that A has a right bidual. Then, the following assertions are equivalent.*

1. *A is left autonomous.*
2. *The 2-cell γ in (2.23) is invertible.*
3. *The 2-cell ω in (2.25) is invertible.*
4. *The theorem of Hopf modules holds for A.*
5. *The functor*

$$\lambda_{A^\circ} : \mathscr{M}(A^\circ, A) \xrightarrow{\mathscr{M}(j^*\otimes 1,1)} \mathscr{M}(A\otimes A^\circ, A) \xrightarrow{\varphi_{A^\circ}} \mathscr{M}(A\otimes A^\circ, A)^{\theta_{A^\circ}}$$

is an equivalence.

Proof. (1) implies (2) by Lemma 2.33, and (3) follows trivially from (2) as (5) does from (4). By Observation 2.9, to prove that (3) implies (4) it is enough to show that for each object X the natural transformation $\upsilon_X\varepsilon_X\varphi_X$ is an isomorphism. For $g \in \mathscr{M}(A\otimes X, A)^{\theta_X}$, the component $\upsilon_X(\varepsilon_X)_g$ is the pasting

$$\begin{tikzcd}
A\otimes X \arrow[r, "p^*\otimes 1"] \arrow[rrrrr, bend right, "g"'] \arrow[rrrrr, phantom, bend right=20, "\Downarrow\nu"] & A^2\otimes X \arrow[r, "1\otimes j^*\otimes 1"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\Downarrow"] & A\otimes X \arrow[r, "1\otimes j\otimes 1"] & A^2\otimes X \arrow[r, "1\otimes g"] & A^2 \arrow[r, "p"] & A
\end{tikzcd}$$

where ν is the action of θ_X on g and the unlabelled arrow is induced by the counit of $j \dashv j^*$. This 2-cell pasted with $1_{A\otimes X} \cong (A\otimes j^*\otimes X)(p^*\otimes X)$ gives, by the equality (2.1),

$$\begin{tikzcd}
A\otimes X \arrow[r, "1\otimes j\otimes 1"] \arrow[drr, bend right, "1"'] & A^2\otimes X \arrow[rr, equal] \arrow[dr, "p\otimes 1"'] & \arrow[d, phantom, "\Downarrow\eta\otimes 1"] & A^2\otimes X \arrow[r, "1\otimes g"] & A^2 \arrow[r, "p"] & A \\
& \arrow[u, phantom, "\Downarrow\cong"] & A\otimes X \arrow[ur, "p^*\otimes 1"'] \arrow[urrr, bend right, "g"'] & & \arrow[u, phantom, "\Downarrow\nu"] &
\end{tikzcd} \tag{2.26}$$

When $g=\varphi_X(h)$ for some $h\in\mathscr{M}(A\otimes X,A)$, that is $g=\theta_X(h)=p(A\otimes h)(p^*\otimes X)$ and ν is equal to

$$\begin{tikzcd}
A\otimes X \arrow[r, "p^*\otimes 1"] \arrow[dr, "p^*\otimes 1"'] & A^2\otimes X \arrow[r, "1\otimes p^*\otimes 1"] \arrow[d, phantom, "\cong\phi^*\otimes 1"] & A^3\otimes X \arrow[r, "1\otimes 1\otimes h"] \arrow[dr, "p\otimes 1\otimes 1"] \arrow[d, phantom, "\Downarrow"] & A^3 \arrow[r, "1\otimes p"] \arrow[drr, "p\otimes 1"'] \arrow[d, phantom, "\cong c"] & A^2 \arrow[r, "p"] \arrow[dr, phantom, "\cong\phi^{-1}"] & A \\
& A^2\otimes X \arrow[ur, "p^*\otimes 1\otimes 1"] \arrow[rr, equal] & & A^2\otimes X \arrow[rr, "1\otimes h"'] & & A^2 \arrow[u, "p"']
\end{tikzcd}$$

then (2.26) is equal to the pasting of $\phi^{-1}:p(A\otimes p)\Rightarrow p(p\otimes A)$ with the following 2-cell

$$\begin{tikzcd}
A^2\otimes X \arrow[r, "p\otimes 1"] \arrow[rr, bend left, "1"] & A\otimes X \arrow[r, "p^*\otimes 1"'] \arrow[dr, "p^*\otimes 1"'] \arrow[u, phantom, "\Downarrow"] & A^2\otimes X \arrow[r, "1\otimes p^*\otimes 1"] & A^3\otimes X \arrow[r, "A^2\otimes h"] \arrow[dr, "p\otimes 1\otimes 1"] & A^3 \arrow[dr, "p\otimes 1"] \arrow[d, phantom, "\cong"] & \\
A\otimes X \arrow[u, "1\otimes j\otimes 1"] \arrow[ur] \arrow[u, phantom, "\Downarrow\cong"'] & & A^2\otimes X \arrow[ur, "p^*\otimes 1\otimes 1"] \arrow[u, phantom, "\cong\phi^*\otimes 1"] \arrow[rr, equal] & \arrow[u, phantom, "\Downarrow"] & A^2\otimes X \arrow[r, "1\otimes h"'] & A^2
\end{tikzcd}$$

which is nothing but $\omega\otimes X$ pasted on the right with an isomorphism, and so it is itself an isomorphism.

Now we show that (5) implies (1). Recall from Observation 2.10 that a Hopf module structure on a 1-cell $A\otimes X\to A$ is the same as a structure of a lax morphism between the free pseudo-$(A\otimes -)$-algebras $A\otimes X$ and A. We want to prove that $(p\otimes A)(A\otimes d\otimes A)(A\otimes \mathrm{n})$ is a right adjoint to p in Lax-$(A\otimes -)$-Alg, for some d; or equivalently, that the former pseudomorphism is isomorphic to p^* equipped with the lax morphism structure given by the 2-cell γ in (2.23).

Suppose that λ_{A° is an equivalence. Define a 1-cell $b=(A\otimes \mathrm{e})(p^*\otimes A^\circ):A\otimes A^\circ\to A$. Since γ is a lax morphism structure for $p^*:A\to A\otimes A$, we obtain a lax morphism structure on b by simply composing with $A\otimes\mathrm{e}$. This lax morphism structure translates into a Hopf module structure, and then, $b\cong\lambda_{A^\circ}(d)=p(A\otimes d)$ as Hopf modules, for some $d:A^\circ\to A$. Using Observation 2.10 again, we have that b and $p(A\otimes d)$ are isomorphic in Lax-$(A\otimes -)$-Alg, and one easily deduce that

p^* equipped with the lax morphism structure γ is isomorphic to $p(A\otimes d\otimes A)(A\otimes \mathrm{n})$. ☐

If A has a right bidual the 2-functor $A \otimes -$ has right biadjoint given by $[A, -] = A^\circ \otimes -$ (see the discussion on biduals at the beginning of the section). In this case, the monad t of (2.11) can be expressed as

$$t : A^\circ \otimes A \xrightarrow{1\otimes \mathrm{n}\otimes 1} A^\circ \otimes A^\circ \otimes A \otimes A \xrightarrow{(p^*)^\circ\otimes 1\otimes 1} A^\circ \otimes A \otimes A \xrightarrow{1\otimes p} A^\circ \otimes A \quad (2.27)$$

or

$$A^\circ \otimes A \xrightarrow{\mathrm{n}\otimes 1\otimes 1} A^\circ \otimes A \otimes A^\circ \otimes A \xrightarrow{1\otimes p^*\otimes 1\otimes 1} A^\circ \otimes A \otimes A \otimes A^\circ \otimes A \to$$
$$\xrightarrow{1\otimes 1\otimes \mathrm{e}\otimes 1} A^\circ \otimes A \otimes A \xrightarrow{1\otimes p} A^\circ \otimes A \quad (2.28)$$

(we omitted the canonical equivalence $A^\circ \otimes A^\circ \simeq (A \otimes A)^\circ$), and the 1-cell ℓ in (2.12) can be expressed as

$$A \xrightarrow{(j^*)^\circ\otimes 1} A^\circ \otimes A \xrightarrow{f} (A^\circ \otimes A)^t.$$

The 1-cell (2.14) can be expressed as $(A^\circ \otimes p)(\mathrm{n} \otimes A) : A \to A^\circ \otimes A \otimes A \to A \otimes A$. Recall that this 1-cell has a canonical t-algebra structure, described in Observation 2.15.

Theorem 2.35. *For any map pseudomonoid A with right bidual the following are equivalent.*

1. *A is left autonomous.*
2. *A has a Hopf module construction provided by*

$$A \xrightarrow{\mathrm{n}\otimes 1} A^\circ \otimes A \otimes A \xrightarrow{1\otimes p} A^\circ \otimes A. \quad (2.29)$$

Moreover, in this case the dualization is given by $A^\circ \xrightarrow{1\otimes j} A^\circ \otimes A \xrightarrow{f} A$, where f is left adjoint to (2.29).

Proof. By Corollary 2.16, (2.29) is a Hopf module construction for A if and only if the theorem of Hopf modules holds for A, and this is equivalent to the existence of a left dualization by Theorem 2.34. The last assertion follows from the existence of an adjunction $p(d \otimes A) \dashv (A^\circ \otimes p)(\mathrm{n} \otimes A)$ whenever d is a left dualization (see [13, Prop. 1.1]). ☐

Corollary 2.36. *For a left autonomous map pseudomonoid A the adjunction $p(d \otimes A) \dashv (A^\circ \otimes p)(\mathrm{n} \otimes A)$ induces the monad t. Moreover, this adjunction is monadic.*

Proof. By Corollary 2.16 we know that $(A^\circ \otimes p)(\mathrm{n} \otimes A) : A \to A^\circ \otimes A$ provides an Eilenberg-Moore construction for t. $\square$

By definition [13], a *right dualization* $d' : A^\vee \to A$ for a pseudomonoid A in $\mathscr{M}$ is a left dualization for A in $\mathscr{M}^{\mathrm{rev}}$, $\mathscr{M}$ with the reverse tensor product. In particular, $A^\vee$ is a left bidual for A. A pseudomonoid equipped with a right dualization is called *right autonomous* and a left and right autonomous pseudomonoid is simply called *autonomous*. A left autonomous *map* pseudomonoid with dualization d is autonomous if and only if d is an equivalence [13, Propositions 1.4 and 1.5].

Corollary 2.37. *Suppose that A is an autonomous map pseudomonoid. Then there exists an equivalence of monads*

$$\begin{tikzcd} A^\circ \otimes A \arrow[r, "t"] \arrow[d, "d\otimes 1"'] \arrow[dr, phantom, "\cong"] & A^\circ \otimes A \arrow[d, "d\otimes 1"] \\ A \otimes A \arrow[r, "p^* p"'] & A \otimes A \end{tikzcd}$$

and, moreover, $p^* : A \to A \otimes A$ is monadic.

Proof. The first assertion is clear since d is an equivalence and t is induced by $p(d \otimes A) \dashv (d^* \otimes A)p^*$; see Proposition 2.35. By the same theorem, $(d^* \otimes A)p^*$ is monadic, and then so is p^* since d is an equivalence. $\square$

Proposition 2.38. *Any left dualization $d : A^\circ \to A$ has the structure of a strong monoidal morphism from $(A^\circ, (j^*)^\circ, (p^*)^\circ)$ to (A, j, p).*

Proof. It is enough to show that

$$A^\circ \xrightarrow{d} A \xrightarrow{\mathrm{n}\otimes 1} A^\circ \otimes A \otimes A \xrightarrow{1\otimes p} A^\circ \otimes A \qquad (2.30)$$

is strong monoidal, since $(A^\circ \otimes p)(\mathrm{n} \otimes A)$ is an Eilenberg-Moore object in the 2-category $\mathbf{Opmon}(\mathscr{M})$. In the proof of Theorem 2.34 we saw that $p(A \otimes d) \cong (A \otimes \mathrm{e})(p^* \otimes A^\circ)$, so we have to show that $(A^\circ \otimes A \otimes \mathrm{e})(A^\circ \otimes p^* \otimes A^\circ)(\mathrm{n} \otimes A^\circ)$ is a strong monoidal morphism, or equivalently, by Proposition 2.12, that $(A \otimes \mathrm{e})(p^* \otimes A^\circ) : A \otimes A^\circ \to A$ is a right pseudoaction of A° on A (*i.e.*, a $(- \otimes A^\circ)$-pseudoalgebra structure on A). This itself turns to be equivalent to saying that $p^* : A \to A \otimes A$ is a right pseudocoaction of A on A (*i.e.*, a $(- \otimes A)$-pseudocoalgebra structure on A), which is obviously true. $\square$

2.7 Preservation of dualizations

This short section contains some comments on autonomous monoidal lax functors. The notion of *right autonomous monoidal lax functor* was introduced in [13], and it consists of a monoidal lax functor equipped with the structure necessary to ensure that it preserves, in lax sense, right biduals. More explicitly, if F is a lax monoidal lax functor with monoidal structure $\chi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$, $\iota : I \to F(I)$, a right autonomous structure for F is pseudonatural transformation $\kappa_X : (FX)^\circ \to F(X^\circ)$ with modifications

$$\xi_X : \iota \mathrm{e} \Rightarrow (F\mathrm{e})\chi_{X,X^\circ}(F(X) \otimes \kappa_X) \qquad \zeta_X : \chi_{X^\circ,X}(\kappa_X \otimes F(X))\mathrm{n} \Rightarrow (F\mathrm{n})\iota$$

satisfying two axioms.

What is proved in [13] is that if $F : \mathscr{M} \to \mathscr{N}$ is a right autonomous monoidal special lax functor and A is a left autonomous pseudomonoid in $\mathscr{M}$ with left dualization d, then $F(A)$ is left autonomous with left dualization $F(d)\kappa_A : F(A)^\circ \to F(A)$. The term *special* means that F is normal (in the sense that the constraint $1_{FX} \to F1_X$ is an isomorphism for all X) and the constraints $(Fg)(Ff) \Rightarrow F(gf)$ are isomorphisms whenever f is a map. Special lax functors have the property of preserving adjunctions.

If we restrict ourselves to map pseudomonoids, as application of Theorem 2.34, we can deduce the following result.

Proposition 2.39. *Let $F : \mathscr{M} \to \mathscr{N}$ be a monoidal special lax functor between right autonomous Gray monoids and A be a left autonomous map pseudomonoid in $\mathscr{M}$. Assume F has the following two properties: the monoidal constraints $\iota : I \to FI$ and $\chi_{A,A} : F(A) \otimes F(A) \to F(A \otimes A)$ are maps, and the 2-cell below is invertible.*

$$\begin{tikzcd}
 & F(A)^3 \arrow[r, "Fp\chi\otimes 1"] \arrow[dr, "1\otimes Fp\chi"] \arrow[d, phantom, "\Downarrow 1\otimes\varepsilon"] & F(A)^2 \arrow[rr, equal] \arrow[drr, "Fp\chi"'] \arrow[d, phantom, "\cong"] & & F(A)^2 \arrow[d, phantom, "\Downarrow\eta"] \\
F(A)^2 \arrow[ur, "1\otimes(Fp\chi)^*"] \arrow[rr, equal] & & F(A)^2 \arrow[rr, "Fp\chi"'] & & F(A) \arrow[u, "(Fp\chi)^*"']
\end{tikzcd}$$

Then, the map pseudomonoid $F(A)$ is left autonomous with left dualization

$$F(A)^\circ \xrightarrow{(Fj)\iota\otimes 1} F(A)\otimes F(A)^\circ \xrightarrow{\chi^*(Fp^*)\otimes 1} F(A)^2\otimes F(A)^\circ \xrightarrow{1\otimes \mathrm{e}} F(A). \qquad (2.31)$$

Proof. Recall that $F(A)$ has multiplication $F(p)\chi : F(A)\otimes F(A) \to F(A)$ and unit $F(j)\iota : I \to F(A)$, so that it is a map pseudomonoid. Using the conditions above plus the fact that (2.23) is invertible, it can be shown that the corresponding 2-cell (2.23) for $F(A)$ is invertible, and hence $F(A)$ is left autonomous. The formula for the left dualization is just the general expression of any left dualization in terms of the product, unit and evaluation. □

If F is strong monoidal (sometimes called weak monoidal) in the sense that ι and χ are equivalences, then F preserves biduals; more explicitly, there exists $\kappa : F(A)^\circ \to F(A^\circ)$, unique up to isomorphism, such that

$$(I \xrightarrow{\mathrm{n}} (FA)^\circ\otimes FA \xrightarrow{\kappa\otimes 1} F(A^\circ)\otimes FA) = (I \xrightarrow{\iota} FI \xrightarrow{F\mathrm{n}} F(A^\circ\otimes A) \xrightarrow{\chi^*} F(A^\circ)\otimes FA) \qquad (2.32)$$

and κ is *a fortiori* an equivalence.

Proposition 2.40. *Suppose $F : \mathscr{M} \to \mathscr{N}$ is a strong monoidal special lax functor between Gray monoids and A is a left autonomous map pseudomonoid in $\mathscr{M}$ with left dualization d. Then FA is a left autonomous map pseudomonoid too, with left dualization $(Fd)\kappa : (FA)^\circ \to F(A^\circ) \to FA$.*

Proof. The fact that (2.23) is invertible and that $\chi : F(A)\otimes F(A) \to F(A\otimes A)$ is an equivalence ensures that the hypotheses of Proposition 2.39 are satisfied, and hence $F(A)$ is left autonomous. The formula for the dualization follows from (2.31) using (2.32), the fact that χ is an equivalence and the canonical isomorphism $(d\otimes A)\mathrm{n} \cong p^*j$. □

Note that although losing some generality, we gain in simplicity by restricting to the case of left autonomous *map* pseudomonoids, in that our proofs are not based on big diagrams but on the theory of Hopf modules.

We finish the section with an application in the case of a braided Gray monoid. In [16] a braided Gray monoid is defined as a Gray monoid $\mathscr{M}$ equipped with pseudonatural equivalences $c_{X,Y} : X\otimes Y \to Y\otimes X$ and certain invertible 2-cells satisfying axioms. These axioms ensure that the pseudofunctor $\otimes : \mathscr{M}\times\mathscr{M} \to \mathscr{M}$ equipped with constraints $X\otimes c_{Y,X'}\otimes Y' : X\otimes X'\otimes Y\otimes Y' \to X\otimes Y\otimes X'\otimes Y'$ and $1 : I\otimes I \to I$ is strong monoidal. See Section 3.1 for more explanation.

Corollary 2.41. *If A and B are left autonomous map pseudomonoids, with left dualizations d_A and d_B respectively, in a braided Gray monoid $\mathscr{M}$, then $A \otimes B$ is a left autonomous map pseudomonoid too, with left dualization*

$$B^\circ \otimes A^\circ \xrightarrow{c_{B^\circ,A^\circ}} A^\circ \otimes B^\circ \xrightarrow{(d_A \otimes 1)(1 \otimes d_B)} A \otimes B.$$

Proof. The objects (A°, B°) and $B^\circ \otimes A^\circ$ can be taken as left bidual of $(A, B) \in \mathscr{M} \times \mathscr{M}$ and $A \otimes B \in \mathscr{M}$ respectively. With these choices, the corresponding 1-cell κ is just c_{B°,A°. □

2.8 Opposite pseudomonoids

We give another application of the theorem of Hopf modules that we will revisit in Chapter 3.

If (A, j, p) is a map pseudomonoid we call $(A^\circ, j^{*\circ}, p^{*\circ})$ its *bidual pseudomonoid.*

Proposition 2.42. *A 1-cell $d : A^\circ \to A$ is a left dualization for the map pseudomonoid (A, j, p) if and only if $d^{*\circ} : A^{\circ\circ} \to A^\circ$ is a left dualization for the bidual pseudomonoid.*

Proof. First we use Theorem 2.34.2 to prove that (A, j, p) is left autonomous if and only if $(A^\circ, j^{*\circ}, p^{*\circ})$ is left autonomous. The bidual pseudomonoid of A is left autonomous if and only if the 2-cell (2.33) in Figure 2.2 is an isomorphism. This is equivalent to saying that (2.34) is an isomorphism, because taking biduals is a locally fully faithful pseudofunctor. It is easy to see that (2.34) is the 2-cell γ in (2.23), which is invertible if and only if A is left autonomous.

All that remains to do is to express the left dualization of the bidual pseudomonoid of A in terms of the left dualization d of A. By [13, Proposition 1.2] or (2.21), the left dualization of A° is isomorphic to the first 1-cell in the following chain of isomorphisms.

$$\begin{aligned}
&(A^\circ \otimes \mathrm{e}_{A^\circ})(((p^{*\circ})^* j^{*\circ}) \otimes A^{\circ\circ}) \cong (A^\circ \otimes \mathrm{e}_{A^\circ})((p^\circ j^{*\circ}) \otimes A^{\circ\circ}) \\
&\cong (A^\circ \otimes \mathrm{e}_{A^\circ})(((j^* p)^\circ) \otimes A^{\circ\circ}) \cong (A^\circ \otimes \mathrm{e}_{A^\circ})(A^\circ \otimes ((A^\circ \otimes j^* p)(\mathrm{n}_A \otimes A)))(\mathrm{n}_A \otimes A^{\circ\circ}) \\
&\qquad \cong (A^\circ \otimes \mathrm{e}_{A^\circ})(A^\circ \otimes d^* \otimes A^{\circ\circ})(\mathrm{n}_A \otimes A)
\end{aligned}$$

The last isomorphism is induced by the isomorphism (2.22) of [13, Proposition 1.2]. □

$$\begin{tikzcd}
& (A^\circ)^3 \arrow[r, "p^{*\circ}\otimes 1"] \arrow[dr, "1\otimes p^{*\circ}" description] & (A^\circ)^2 \arrow[rr, equal] \arrow[dr, "p^{*\circ}" description] \arrow[d, phantom, "\cong"] & & (A^\circ)^2 \\
(A^\circ)^2 \arrow[ur, "1\otimes p^\circ"] \arrow[rr, equal] \arrow[r, phantom, "\Downarrow"] & & (A^\circ)^2 \arrow[r, "p^{*\circ}"'] & A^\circ \arrow[ur, "p^\circ"'] \arrow[u, phantom, "\Downarrow"] &
\end{tikzcd} \tag{2.33}$$

$$\begin{tikzcd}
& A^3 \arrow[dl, "p^*\otimes 1"'] & A^2 \arrow[l, "1\otimes p^*"'] \arrow[rr, equal] \arrow[d, phantom, "\cong"] & & A^2 \arrow[dl, "p"] \\
A^2 \arrow[rr, equal] \arrow[r, phantom, "\Downarrow"] & & A^2 \arrow[ul, "p^*\otimes 1" description] & A \arrow[l, "p^*"] \arrow[ul, "p^*" description] \arrow[u, phantom, "\Downarrow"] &
\end{tikzcd} \tag{2.34}$$

Figure 2.2: Diagrams of the proof of Proposition 2.42.

2.9 Frobenius and autonomous map pseudomonoids

In this section we study the relationship between autonomous pseudomonoids, the condition (2) in Theorem 2.34 and Frobenius pseudomonoids. In [17] it is shown that any autonomous pseudomonoid is Frobenius, and we showed in Theorem 2.34.2 that autonomy is equivalent to the invertibility of the 2-cell γ in (2.23) and its dual, *i.e.*, the corresponding 2-cell γ' in $\mathscr{M}^{\mathrm{rev}}$. We show a converse in the absence of biduals, namely: if γ and γ' are invertible, then A is Frobenius, and as such it has right and left bidual, and moreover A is autonomous.

A *Frobenius structure* for a pseudomonoid A is defined in [82] as a 1-cell $\varepsilon : A \to I$ such that $\varepsilon p : A \otimes A \to I$ is the evaluation of a bidual pair.

Lemma 2.43. *Let A be a pseudomonoid whose multiplication p is a map, and call γ and γ', respectively, the following 2-cells.*

$$\begin{tikzcd}
& A^3 \arrow[r, "p\otimes 1"] \arrow[dr, "1\otimes p" description] & A^2 \arrow[rr, equal] \arrow[dr, "p" description] \arrow[d, phantom, "\cong\ \phi"] & & A^2 \\
A^2 \arrow[ur, "1\otimes p^*"] \arrow[rr, equal] \arrow[r, phantom, "1\otimes\varepsilon\Downarrow"] & & A^2 \arrow[r, "p"'] & A \arrow[ur, "p^*"'] \arrow[u, phantom, "\Downarrow\eta"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A^3 \arrow[r, "1\otimes p"] \arrow[dr, "p\otimes 1" description] & A^2 \arrow[rr, equal] \arrow[dr, "p" description] \arrow[d, phantom, "\cong\ \phi^{-1}"] & & A^2 \\
A^2 \arrow[ur, "p^*\otimes 1"] \arrow[rr, equal] \arrow[r, phantom, "\varepsilon\otimes 1\Downarrow"] & & A^2 \arrow[r, "p"'] & A \arrow[ur, "p^*"'] \arrow[u, phantom, "\Downarrow\eta"] &
\end{tikzcd}$$

Then the following equalities in Figure 2.3 hold.

Proof. The proof is a standard calculation involving mates and the axioms of a pseudomonoid. □

Proposition 2.44. *Suppose A is a map pseudomonoid and that the 2-cells γ and γ' in Lemma 2.43 are invertible. Then $j^*p : A \otimes A \to I$ and $p^*j : I \to A \otimes A$*

$$
\begin{tikzcd}
A^4 \arrow[r, "1\otimes p\otimes 1"] \arrow[dr, phantom, "1\otimes\gamma\ \Downarrow"] & A^3 & \\
A^3 \arrow[u, "A^2\otimes p"] \arrow[r, "1\otimes p"'] \arrow[dr, phantom, "\Downarrow\gamma'"] & A^2 \arrow[u, "1\otimes p^*"'] \arrow[r, phantom, "\cong"] & A^2 \arrow[ul, "p^*\otimes 1"'] \\
A^2 \arrow[u, "p^*\otimes 1"] \arrow[r, "p"'] & A \arrow[u, "p^*"] \arrow[ur, "p^*"'] &
\end{tikzcd}
=
\begin{tikzcd}
& A^4 \arrow[r, "1\otimes p\otimes 1"] \arrow[dr, phantom, "\Downarrow\gamma'\otimes 1"] & A^3 \\
A^3 \arrow[ur, "A^2\otimes p^*"] \arrow[r, phantom, "\cong"] & A^3 \arrow[u, "p^*\otimes A^2"'] \arrow[r, "p\otimes 1"] \arrow[dr, phantom, "\Downarrow\gamma"] & A^2 \arrow[u, "p^*\otimes 1"'] \\
& A^2 \arrow[ul, "p^*\otimes 1"] \arrow[u, "1\otimes p^*"'] \arrow[r, "p"'] & A \arrow[u, "p^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
A^4 \arrow[r, "A^2\otimes p"] \arrow[dr, phantom, "1\otimes\gamma'\ \Downarrow"] & A^3 \arrow[r, "p\otimes 1"] \arrow[dr, phantom, "\Downarrow\gamma"] & A^2 \\
A^3 \arrow[u, "1\otimes p^*\otimes 1"] \arrow[r, "1\otimes p"] \arrow[dr, "p\otimes 1"'] & A^2 \arrow[u, "1\otimes p^*"'] \arrow[r, "p"] \arrow[d, phantom, "\cong"] & A \arrow[u, "p^*"'] \\
& A^2 \arrow[ur, "p"'] &
\end{tikzcd}
=
\begin{tikzcd}
& A^3 \arrow[dr, "p\otimes 1"] \arrow[d, phantom, "\cong"] & \\
A^4 \arrow[ur, "A^2\otimes p"] \arrow[r, "p\otimes A^2"] \arrow[dr, phantom, "\gamma\otimes 1\ \Downarrow"] & A^3 \arrow[r, "1\otimes p"] \arrow[dr, phantom, "\Downarrow\gamma'"] & A^2 \\
A^3 \arrow[u, "1\otimes p^*\otimes 1"] \arrow[r, "p\otimes 1"'] & A^2 \arrow[u, "p^*\otimes 1"'] \arrow[r, "p"'] & A \arrow[u, "p^*"']
\end{tikzcd}
$$

Figure 2.3:

have the structure of a bidual pair. In particular, A is a Frobenius pseudomonoid and given a choice of right and left biduals, A is autonomous.

Proof. The 2-cells

$$(j^* \otimes A)(p \otimes A)(A \otimes p^*)(A \otimes j) \xrightarrow{(j^*\otimes A)\gamma(A\otimes j)} (j^* \otimes A)p^*p(A \otimes j) \cong 1_A$$

$$(A \otimes j^*)(A \otimes p)(p^* \otimes A)(j \otimes A) \xrightarrow{(A\otimes j^*)\gamma'(j\otimes A)} (A \otimes j^*)p^*p(j \otimes A) \cong 1_A$$

endow j^*p and p^*j with the structure of a bidual pair. The axioms of a bidual pair follow from Lemma 2.43. $\square$

Observation 2.45. In the hypothesis of the proposition above, different choices of a bidual for A give rise to different dualizations. For example, when we take the bidual pair j^*p, p^*j, so that A is right and left bidual of itself, the resulting left and right dualizations are just the identity 1_A. Slightly more generally, given any equivalence $f : B \to A$, B has a canonical structure of right bidual of A such that the corresponding left dualization is (isomorphic) to f. To see this just consider the evaluation $j^*p(A\otimes f) : A\otimes B \to I$ and the coevaluation $(f^*\otimes A)p^*j : I \to B \otimes A$.

Chapter 3

Centres of autonomous pseudomonoids

In this second chapter we continue extending Hopf algebra theory to the context of autonomous pseudomonoids in monoidal bicategories. We use the results in Chapter 2 to study centres and lax centres of autonomous map pseudomonoids, and their relationship with the Drinfel'd double.

A classical notion of centre of an algebraic structure is the centre of a monoid. If M is a monoid, its centre is the set of elements of M with the *property* of commuting with every element of M. We can slightly change our point of view and say that the centre of M is the set whose elements are pairs $(x, (x \cdot -) = (- \cdot x))$, *i.e.*, elements of $x \in M$ equipped with the extra *structure* of an equality between the multiplication with x on the left and on the right. The centre of a monoidal category, defined in [39], follows the spirit of the latter point of view: from the algebraic structure of a monoidal category $\mathscr{C}$ one forms a new algebraic structure $Z\mathscr{C}$, called the centre of $\mathscr{C}$. What we actually have is a functor $Z\mathscr{C} \to \mathscr{C}$, and $Z\mathscr{C}$ has a monoidal structure such that this functor is strong monoidal. Moreover, $Z\mathscr{C}$ has a canonical braiding. The objects of $Z\mathscr{C}$ are pairs (x, γ_x) where $\gamma_x : (- \otimes x) \Rightarrow (x \otimes -)$ is an invertible natural transformation. In this context one can also consider the *lax centre* $Z_\ell\mathscr{C}$ of $\mathscr{C}$, simply by dropping the requirement of the invertibility of γ_x. See Example 3.1. The functor $Z\mathscr{C} \to \mathscr{C}$ is the universal one satisfying certain commutation properties.

Another classically considered centre-like object is the Drinfel'd double of a finite-dimensional Hopf algebra, or, more recently, of a (co)quasi-Hopf algebra. See [64, 74]. Here the concept is not the one of the object classifying maps with certain commutation properties, but it is a representational one. Roughly speaking, the Drinfel'd double of a finite dimensional Hopf algebra H is a Hopf algebra $D(H)$ such that the category of representations of $D(H)$ is monoidally equivalent to the centre of the category of representations of H.

We study lax centres $Z_\ell A$ of a map pseudomonoid A in a braided Gray monoid $\mathscr{M}$ from two points of view. Firstly we would like to have canonical equivalences $\mathscr{M}(I, Z_\ell A) \simeq Z_\ell(\mathscr{M}(I, A))$. The simple minded choice is to take the object on right hand side of the equivalence as the lax centre of the monoidal category $\mathscr{M}(I, A)$. However, this turns out to be insufficient to obtain an equivalence. We are led to consider $\mathscr{M}(I, A)$ as a $\mathscr{M}(I, I)$-enriched category and its lax centre in $\mathscr{M}(I, I)$-**Cat**. This context provides an enriched equivalence as above, at the price of requiring certain mild conditions on $\mathscr{M}$. We apply these constructions to (pro)monoidal enriched categories.

Secondly, we construct lax centres of autonomous map pseudomonoids. By means of the Hopf module construction of Chapter 2, we construct the lax centre as an internal analogue of the category of two sided Hopf modules. This generalises the fact that for a Hopf algebra the category of two sided Hopf modules is monoidally equivalent to the centre of the category of representations of the Hopf algebra (and to the category of representations of the Drinfel'd double of the Hopf algebra). Later, in Chapter 5, Section 5.2.4, we show that the (lax)centre of a finite dimensional coquasi-Hopf algebra H always exists within the bicategory of comodules. Moreover, the construction of this centre is explicit, can be taking to be finite dimensional and it is isomorphic as a coalgebra and equivalent as a coquasibialgebra to the Drinfel'd double of H.

Now we describe the organisation of the present chapter.

Section 3.1 recalls the notion of a braided Gray monoid.

In Section 3.2 we introduce lax centres of pseudomonoids and give the first examples.

Section 3.3 studies the relationship between $\mathscr{M}(I, Z_\ell A)$ and the centre of the monoidal category $\mathscr{M}(I, A)$. We show that the universal $Z_\ell A \to A$ induces an equivalence between the categories above, when we consider them as $\mathscr{M}(I, I)$-enriched categories.

Section 3.4 exhibits lax centres of left autonomous map pseudomonoids as Eilenberg-Moore constructions for certain monad. When the pseudomonoid is also right autonomous, the lax centre coincides with the centre.

3.1 Braided Gray monoids

As in this chapter we study centres and lax centres of pseudomonoids, we shall need extra structure on the Gray monoids where the pseudomonoids lie. For example, if one looks at the definition of the centre of a monoidal category given in [39] (see also Introduction above), one realises that the symmetry of the cartesian product in **Cat** is used. We require similar structure on our Gray monoids, but a symmetry is too strict a structure.

The definition of a braided Gray monoid was first introduced by Kapranov and Veovodsky in [40] and modified by Baez and Neuchl in [1]. Here we use the equivalent definition given by Day and Street [16].

Let $\mathscr{M}$ be a Gray monoid and denote by $\mathrm{sw} : \mathscr{M} \times \mathscr{M} \to \mathscr{M} \times \mathscr{M}$ the obvious symmetry. A *braiding* for $\mathscr{M}$ is a pseudonatural transformation $c : \otimes \mathrm{sw} \Rightarrow \otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ with invertible 2-cells

$$\begin{tikzcd}
W \otimes X \otimes Y \otimes Z \arrow[r, "1\otimes 1\otimes c_{Y,Z}"] \arrow[d, "c_{W,X}\otimes 1\otimes 1"'] \arrow[dr, phantom, "\Uparrow \varpi_{XYZW}"] & W \otimes X \otimes Z \otimes Y \arrow[d, "c_{W,X\otimes Z}\otimes 1"] \\
X \otimes W \otimes Y \otimes Z \arrow[r, "1\otimes c_{W\otimes Y,Z}"'] & X \otimes Y \otimes Z \otimes W
\end{tikzcd}$$

satisfying three axioms. These axioms ensure that $\otimes : \mathscr{M}\times\mathscr{M} \to \mathscr{M}$ is a monoidal pseudofunctor when equipped with $1\otimes c_{Y,Z}\otimes 1 : X\otimes Y\otimes Z\otimes W \to X\otimes Z\otimes Y\otimes W$ and $1 : I \to I \otimes I$, and the obvious 2-cells.

3.2 Centres and lax centres

We shall work in a *braided Gray monoid*. See Section 3.1 above. The centre of a pseudomonoid was defined in [83]. Here we will be interested in the lax version of the centre, called the *lax centre* of a pseudomonoid. The definition is exactly the same as that of the centre but for the fact that we drop the requirement of the invertibility of certain 2-cells.

Definition 3.1. Given a pseudomonoid in a braided Gray monoid $\mathscr{M}$ we define for each object X a category $CP_\ell(X, A)$. The objects, called *lax centre pieces*, are pairs (f, γ) where $f : X \to A$ is a 1-cell and γ is a 2-cell

$$\begin{tikzcd} A \otimes X \arrow[d, "1\otimes f"'] & & X \otimes A \arrow[ll, "c_{X,A}"'] \arrow[d, "f\otimes 1"] \\ A \otimes A \arrow[dr, "p"'] \arrow[rr, phantom, "\Leftarrow_{\gamma}"] & & A \otimes A \arrow[dl, "p"] \\ & A & \end{tikzcd} \tag{3.1}$$

satisfying axioms (3.2) and (3.3) in Figure 3.1. The arrows $(f, \gamma) \to (f', \gamma')$ are the 2-cells $f \Rightarrow g$ which are compatible with γ and γ' in the obvious sense.

This is the object part of a pseudofunctor $CP_\ell(-, A) : \mathscr{M}^{\mathrm{op}} \to \mathbf{Cat}$, that is defined on 1-cells and 2-cells just by precomposition. When CP_ℓ is birepresentable we call a birepresentation $z_\ell : Z_\ell A \to A$ a *lax centre* of the pseudomonoid A.

A *centre piece* is a lax centre piece (f, γ) such that γ is invertible. The full subcategories $CP(X, A) \subset CP_\ell(X, A)$ with objects the centre pieces define a pseudofunctor $CP(-, A) : \mathscr{M}^{\mathrm{op}} \to \mathbf{Cat}$, and we call a birepresentation of it a *centre* of A, denoted by $z : ZA \to A$.

Definition 3.2. The inclusion $CP(-, A) \hookrightarrow CP_\ell(-, A)$ induces a 1-cell $z_c : ZA \to Z_\ell A$, unique up to isomorphism such that $z_\ell z_c \cong z$ as centre pieces. When z_c is an equivalence we will say that the centre of A coincides with the lax centre.

Example 3.1. The centre of a pseudomonoid in **Cat**, that is, of a monoidal category, is the usual centre defined in [39]. In fact, lax centres and centres of pseudomonoids in $\mathscr{V}$-**Cat** exist and are given by the constructions in [15]. Lax centres or (ordinary) monoidal categories were also considered in [74] under the name of 'weak centers'. If $\mathscr{A}$ is a monoidal $\mathscr{V}$-category, its lax centre $Z_\ell \mathscr{C}$ has objects pairs (x, γ) where x is an object of $\mathscr{C}$ and $\gamma : (- \otimes x) \Rightarrow (x \otimes -)$ is a $\mathscr{V}$-natural transformation. The $\mathscr{V}$-enriched hom $Z_\ell \mathscr{C}((x, \gamma), (y, \delta))$ is the equalizer of the pair of arrows

$$\begin{tikzcd} \mathscr{C}(x, y) \arrow[r] \arrow[d] & {[\mathscr{C}, \mathscr{C}](- \otimes x, - \otimes y)} \arrow[d, "{[\mathscr{C},\mathscr{C}](\gamma,1)}"] \\ {[\mathscr{C}, \mathscr{C}](x \otimes -, y \otimes -)} \arrow[r, "{[\mathscr{C},\mathscr{C}](1,\delta)}"'] & {[\mathscr{C}, \mathscr{C}](x \otimes -, - \otimes y)} \end{tikzcd}$$

Observation 3.2. By [83], in a monoidal closed Gray monoid with finite limits, *every* pseudomonoid has a centre.

$$
\begin{gathered}
\begin{tikzcd}
A\otimes A\otimes X \arrow[d, "1\otimes1\otimes f"'] \arrow[drr, "p\otimes1"] & & & \cong & & & X\otimes A\otimes A \arrow[llllll, "c_{X,A\otimes A}"'] \arrow[d, "f\otimes1\otimes1"] \arrow[dll, "1\otimes p"'] \\
A\otimes A\otimes A \arrow[d, "1\otimes p"'] \arrow[drr, "p\otimes1"'] \arrow[rr, phantom, "\cong"] & & A\otimes X \arrow[d, "1\otimes f"] & & X\otimes A \arrow[ll, "c_{X,A}"'] \arrow[d, "f\otimes1"'] \arrow[rr, phantom, "\cong"] & & A\otimes A\otimes A \arrow[d, "p\otimes1"] \arrow[dll, "1\otimes p"] \\
A\otimes A \arrow[drrr, "p"'] \arrow[rr, phantom, "\cong"] & & A\otimes A \arrow[dr, "p"] \arrow[rr, phantom, "\Leftarrow\gamma"] & & A\otimes A \arrow[dl, "p"'] \arrow[rr, phantom, "\cong"] & & A\otimes A \arrow[dlll, "p"] \\
& & & A & & &
\end{tikzcd}
\\
\|
\\
\begin{tikzcd}
A\otimes A\otimes X \arrow[d, "1\otimes1\otimes f"'] & & A\otimes X\otimes A \arrow[ll, "1\otimes c_{X,A}"'] \arrow[d, "1\otimes f\otimes1"] & & X\otimes A\otimes A \arrow[ll, "c_{X,A}\otimes1"'] \arrow[llll, bend right, "c_{X,A\otimes A}"', "\cong"] \arrow[d, "f\otimes1\otimes1"] \\
A\otimes A\otimes A \arrow[dr, "1\otimes p"'] \arrow[rr, phantom, "\Leftarrow 1\otimes\gamma"] & & A\otimes A\otimes A \arrow[dl, "1\otimes p"] \arrow[dr, "p\otimes1"'] \arrow[rr, phantom, "\Leftarrow \gamma\otimes1"] & & A\otimes A\otimes A \arrow[dl, "p\otimes1"] \\
& A\otimes A \arrow[dr, "p"'] \arrow[rr, phantom, "\cong"] & & A\otimes A \arrow[dl, "p"] & \\
& & A & &
\end{tikzcd}
\end{gathered}
\tag{3.2}
$$

$$
\begin{tikzcd}
X \arrow[rrrr, equal, "\cong"'] \arrow[dr, "j\otimes1"] \arrow[dd, "f"'] & & & & X \arrow[dl, "1\otimes j"'] \arrow[dd, "f"] \\
\cong & A\otimes X \arrow[d, "1\otimes f"'] & & X\otimes A \arrow[ll, "c_{X,A}"'] \arrow[d, "f\otimes1"] & \cong \\
A \arrow[r, "j\otimes1"] \arrow[drr, bend right, "1"', "\cong"] & A\otimes A \arrow[dr, "p"] \arrow[rr, phantom, "\Leftarrow\gamma"] & & A\otimes A \arrow[dl, "p"'] & A \arrow[l, "1\otimes j"'] \arrow[dll, bend left, "1", "\cong"'] \\
& & A & &
\end{tikzcd}
= 1_f
\tag{3.3}
$$

Figure 3.1: Lax centre piece axioms

3.3 Lax centres of convolution monoidal categories

For any pseudomonoid (A, j, p) in a Gray monoid $\mathscr{M}$ we know from [16] that the category $\mathscr{M}(I, A)$ has a canonical *convolution* monoidal structure. The tensor product is given by $f * g = p(f \otimes A)g$ with unit j. We would like to exhibit an equivalence $\mathscr{M}(I, Z_\ell A) \simeq Z_\ell(\mathscr{M}(I, A))$. Our leading example is the bicategory $\mathscr{V}$-**Mod** of $\mathscr{V}$-categories and $\mathscr{V}$-modules. In this example the tensor product just described is just Day's convolution tensor product introduced in [11]. For details about this bicategory see Section 5.1. Henceforth, we shall assume our Gray monoid $\mathscr{M}$ satisfies additional properties, which we explain below.

Recall that a 2-cell

$$\begin{tikzcd} & Y \arrow[dl, "{}^{f}g"'] \arrow[d, "g"] \arrow[dl, phantom, shift right=4, "\Leftarrow\lambda"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in a bicategory $\mathscr{B}$ is said to *exhibit* ${}^{f}g$ *as the right lifting of* g *through* f if it induces a bijection $\mathscr{B}(Y, X)(k, {}^{f}g) \cong \mathscr{B}(Y, Z)(fk, g)$, natural in k. Clearly, right liftings are unique up to compatible isomorphisms. See [84].

We shall assume that our braided Gray monoid $\mathscr{M}$ is closed (see Section 3.1 and references therein) and has *right liftings* of arrows out of I through arrows out of I. As explained in [16], this endows each $\mathscr{M}(X, Y)$ with the structure of a $\mathscr{V}$-category where $\mathscr{V} = \mathscr{M}(I, I)$ is a symmetric monoidal closed category whose tensor product is given by composition. The $\mathscr{V}$-enriched hom $\mathscr{M}(X, Y)(f, g)$ is ${}^{\hat{f}}\hat{g}$, the right lifting of $\hat{g} : I \to [X, Y]$ through $\hat{f} : I \to [X, Y]$, where these two arrows correspond to f and g under the closedness biadjunction. Both $\hat{f}$ and $\hat{g}$ are determined up to isomorphism, and then so is $\mathscr{M}(X, Y)(f, g)$. The compositions $\mathscr{M}(X, Y)(g, h) \otimes \mathscr{M}(X, Y)(f, g) \to \mathscr{M}(X, Y)(f, h)$ and units $1_I \to \mathscr{M}(X, Y)(f, f)$, along with the $\mathscr{V}$-category axioms, are easily deduced from the universal property of the right liftings. Observe that the underlying category of the $\mathscr{V}$-category $\mathscr{M}(X, Y)$ is the hom-category $\mathscr{M}(X, Y)$. For, $\mathscr{V}(1_I, \mathscr{M}(X, Y)(f, g)) = \mathscr{V}(1_I, {}^{\hat{f}}\hat{g}) \cong \mathscr{M}(I, [X, Y])(\hat{f}, \hat{g}) \cong \mathscr{M}(X, Y)(f, g)$.

One can define *composition* $\mathscr{V}$*-functors* $\mathscr{M}(Y, Z) \otimes \mathscr{M}(X, Y) \to \mathscr{M}(X, Z)$ on objects just by composition in $\mathscr{M}$ and on $\mathscr{V}$-enriched homs in the following way. Given $f, h : Y \to Z$ and $g, k : X \to Y$, define an arrow $\mathscr{M}(I, [Y, Z])(\hat{f}, \hat{h}) \otimes \mathscr{M}(I, [X, Y])(\hat{g}, \hat{k}) \to \mathscr{M}(I, [X, Z])(\widehat{fg}, \widehat{hk})$ as the 2-cell in $\mathscr{M}$ corresponding to the following pasting.

$$
\begin{tikzcd}
 & & I \arrow[dl, swap, "\hat{g}\hat{k}"] \arrow[d, "\hat{k}"] \arrow[ddr, "\widehat{hk}"] \arrow[dl, phantom, shift right=4, "\Downarrow"] & \\
 & I \arrow[r, "\hat{g}"] \arrow[d, "\hat{h}"] \arrow[dl, swap, "\hat{f}\hat{h}"] \arrow[dr, phantom, "\cong"] & {[X,Y]} \arrow[d, "\hat{h}\otimes 1"] \arrow[dr, phantom, "\cong"] & \\
I \arrow[r, "\hat{f}"] \arrow[ur, phantom, shift left=4, "\Downarrow"] \arrow[rrr, bend right, swap, "\widehat{fg}"] & {[Y,Z]} \arrow[r, "1\otimes\hat{g}"] \arrow[r, phantom, shift right=3, "\cong"] & {[Y,Z]\otimes[X,Y]} \arrow[r, "\mathrm{comp}"] & {[X,Z]}
\end{tikzcd}
$$

There are also *identity $\mathscr{V}$-functors* from the trivial $\mathscr{V}$-category to $\mathscr{M}(X,X)$. On objects they just pick the identity 1-cells 1_X and homs they are given by the arrows $1_I \to {}^{(\hat{1_X})}\hat{1_X}$ corresponding to the identity 2-cells $\hat{1_X} \Rightarrow \hat{1_X}$. These composition and identity $\mathscr{V}$-functors endow $\mathscr{M}$ with the structure of a category weakly enriched in $\mathscr{V}$-**Cat**, in the sense that the category axioms hold only up to specified $\mathscr{V}$-natural isomorphisms (*e.g.* when $\mathscr{V}$ is the category of sets, we get a (locally small) bicategory.

Now we shall further suppose that the category $\mathscr{V} = \mathscr{M}(I,I)$ is complete. This allows us to consider functor $\mathscr{V}$-categories. In this situation, the composition $\mathscr{V}$-functors induce $\mathscr{V}$-functors $\mathscr{M}(X,-)_{Y,Z} : \mathscr{M}(Y,Z) \to [\mathscr{M}(X,Y), \mathscr{M}(X,Z)]$ making the pseudofunctor $\mathscr{M}(X,-) : \mathscr{M} \to \mathscr{V}$-**Cat** locally a $\mathscr{V}$-functor.

Lemma 3.3. *In the hypothesis above, if A is a pseudomonoid in $\mathscr{M}$, $CP_\ell(I,A)$ has a canonical structure of a $\mathscr{V}$-category such that the forgetful functor*

$$CP_\ell(I,A) \longrightarrow \mathscr{M}(I,A)$$

is the underlying functor of a $\mathscr{V}$-functor. Moreover, $CP(I,A)$ is a full sub-$\mathscr{V}$-category of $CP_\ell(I,A)$.

Proof. We give only a sketch of a proof; the details are an exercise in the universal property of right liftings. Given two lax centre pieces (f,α) and (g,β), define the $\mathscr{V}$-enriched hom $CP_\ell(I,A)((f,\alpha),(g,\beta))$ as the equalizer in $\mathscr{V}$ of the pair

$$
\begin{tikzcd}
\mathscr{M}(I,A)(f,g) \arrow[r] \arrow[d] & \mathscr{M}(A,A)(p(A\otimes f), p(A\otimes g)) \arrow[d, "\mathscr{M}(A,A)(\alpha,1)"] \\
\mathscr{M}(A,A)(p(f\otimes A), p(g\otimes A)) \arrow[r, swap, "\mathscr{M}(A,A)(1,\beta)"] & \mathscr{M}(A,A)(p(f\otimes A), p(A\otimes g))
\end{tikzcd}
\tag{3.4}
$$

where the unlabelled arrows are induced by the universal property of right liftings under postcomposition with the arrows $A \to [A, A]$ corresponding to p and $pc_{A,A}$. With this definition, an arrow $1_I \to CP_\ell(I, A)((f, \alpha), (g, \beta))$ in $\mathscr{V} = \mathscr{M}(I, I)$ corresponds to an arrow $(f, \alpha) \to (g, \beta)$ in the ordinary category $CP_\ell(I, A)$. The composition

$$CP_\ell(I, A)((g, \beta), (h, \gamma)) \otimes CP_\ell(I, A)((f, \alpha), (g, \beta)) \to CP_\ell(I, A)((f, \alpha), (h, \gamma))$$

is induced by the composition

$$\mathscr{M}(I, A)(g, h) \otimes \mathscr{M}(I, A)(f, g) \to \mathscr{M}(I, A)(f, h)$$

and the universal property of the equalizers, and likewise for the identities. $\square$

Proposition 3.4. *Assume the lax centre of A exists, with universal centre piece (z_ℓ, γ). In the hypothesis above, (z_ℓ, γ) induces a $\mathscr{V}$-enriched equivalence U making the following diagram commute.*

$$\begin{tikzcd}
\mathscr{M}(I, Z_\ell A) \arrow[rr, "U"] \arrow[dr, "\mathscr{M}(I,z_\ell)"'] & & CP_\ell(I, A) \arrow[dl] \\
& \mathscr{M}(I, A) &
\end{tikzcd}$$

Moreover, the same holds if the centre of A exists and we use $CP(I, A)$ instead of $CP_\ell(I, A)$.

Proof. On objects, U is equal to the usual functor, that is, it sends $f : I \to Z_\ell A$ to the lax centre piece $(z_\ell f, \gamma(f \otimes A))$. Next we describe U on $\mathscr{V}$-enriched homs. Define ϱ by the following equality, where π exhibits ${}^h k$ as a right lifting of k through h and ϖ exhibits ${}^{(z_\ell h)}(z_\ell k)$ as a right lifting of $z_\ell k$ through $z_\ell h$.

$$\begin{tikzcd}
& I \arrow[dl, "{}^h k"'] \arrow[d, "k"] \arrow[dl, phantom, shift left=3, "\Downarrow\pi"] & \\
I \arrow[r, "h"'] & Z_\ell A \arrow[dr, "z_\ell"'] & \\
& & A
\end{tikzcd}
=
\begin{tikzcd}
& & I \arrow[d, "k"] \arrow[ddll, bend right, "{}^h k"'] \arrow[ddll, "{}^{(z_\ell h)}(z_\ell k)"] \arrow[ddll, phantom, shift right=4, "\Downarrow\varrho"] \\
& & Z_\ell A \arrow[d, "z_\ell"] \arrow[dl, phantom, "\Downarrow\varpi"] \\
I \arrow[r, "h"'] & Z_\ell A \arrow[r, "z_\ell"'] & A
\end{tikzcd}
\tag{3.5}$$

This pasted composite is trivially a morphism of lax centre pieces $U(h({}^h k)) \to$

$$\begin{tikzcd}
 & & A \arrow[ddll, bend right, "v\otimes 1"'] \arrow[ddll, "{}^{(z_\ell h)}(z_\ell k)\otimes 1" description] \arrow[d, "k\otimes 1"] \arrow[drr, "1\otimes k"] \arrow[dl, phantom, "\Downarrow\nu\otimes 1"] \arrow[dr, phantom, "\cong"] & & \\
 & & Z_\ell A\otimes A \arrow[rr, "\simeq"] \arrow[d, "z_\ell\otimes 1"] \arrow[l, phantom, "\Downarrow\varpi\otimes 1"] & & A\otimes Z_\ell A \arrow[d, "1\otimes z_\ell"] \\
A \arrow[r, "h\otimes 1"'] & Z_\ell A\otimes A \arrow[r, "z_\ell\otimes 1"'] & A^2 \arrow[dr, "p"'] \arrow[rr, phantom, "\overset{\gamma}{\Rightarrow}"] & & A^2 \arrow[dl, "p"] \\
 & & & A &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & & A \arrow[ddll, bend right, "1\otimes v"'] \arrow[ddll, "1\otimes{}^{(z_\ell h)}(z_\ell k)" description] \arrow[d, "1\otimes k"] \arrow[dl, phantom, "\Downarrow 1\otimes\nu"] & \\
 & & A\otimes Z_\ell A \arrow[d, "1\otimes z_\ell"] \arrow[l, phantom, "\Downarrow 1\otimes\varpi"] & \\
A \arrow[r, "1\otimes h"] \arrow[dr, "h\otimes 1"'] \arrow[dr, phantom, "\cong"] & A\otimes Z_\ell A \arrow[r, "1\otimes z_\ell"] & A^2 \arrow[r, "p"] \arrow[d, phantom, "\gamma\Uparrow"] & A \\
 & Z_\ell A\otimes A \arrow[u, "\simeq"] \arrow[r, "z_\ell\otimes 1"'] & A^2 \arrow[ur, "p"'] &
\end{tikzcd}$$

Figure 3.2:

$U(k)$, and this means exactly that ϱ factors through the equalizer

$$CP_\ell(I,A)(U(h),U(k)) \rightarrowtail {}^{(z_\ell h)}(z_\ell k) = \mathscr{M}(I,A)(z_\ell h, z_\ell k);$$

in (3.4) defining $CP_\ell(I,A)(U(h),U(k))$ on $\mathscr{V}$-enriched homs. Denote by $\tilde{\varrho} : {}^h k = \mathscr{M}(I,A)(h,k) \to CP_\ell(I,A)(U(h),U(k))$ the resulting arrow in $\mathscr{V}$. This is by definition the effect of U on enriched homs.

Observe that the underlying ordinary functor of U is the usual equivalence given by the universal property of the lax centre. Hence, U is essentially surjective on objects as a $\mathscr{V}$-functor. It is sufficient, then, to show that U is fully faithful, or, in other words, that $\tilde{\varrho}$ is invertible. To do this, we shall show that ϱ has the universal property of the equalizer defining $CP_\ell(I,A)(U(h),U(k))$.

Suppose $\nu : v \to {}^{(z_\ell h)}(z_\ell k)$ is an arrow in $\mathscr{V}$ equalising the pair of arrows ${}^{(z_\ell h)}(z_\ell k) \to \mathscr{M}(A,A)(p(z_\ell h\otimes A), p(A\otimes z_\ell k))$ analogues to (3.4). If one unravels this condition, one gets the equality in Figure 3.2. This means that the 2-cell $\varpi(z_\ell h\nu)$ is an arrow in the ordinary category $CP_\ell(I,A)$ from $U(hv) =$

$(z_\ell hv, \gamma((hv)\otimes A))$ to $U(k) = (z_\ell k, \gamma(k \otimes A))$, and therefore there exists a unique 2-cell $\tau : hv \Rightarrow k : I \to Z_\ell A$ such that $z_\ell \tau = \varpi(z_\ell h\nu)$. From the universal property of right liftings, we deduce the existence of a unique $\tau' : v \Rightarrow {}^h k$ such that $\pi(h\tau') = \tau$. In order to show that $\varrho : {}^h k \Rightarrow {}^{(z_\ell h)}(z_\ell k)$ has the universal property of the equalizer as explained above, we have to show that $\varrho\tau' = \nu$. But the pasting of $\varrho\tau'$ with ϖ, $\varpi(z_\ell h(\varrho\tau'))$, is equal, by definition of ϱ, to $z_\ell(\pi(h\tau')) = z_\ell\tau = \varpi(z_\ell h\nu)$. It follows that $\varrho\tau' = \nu$.

The case of the centre is completely analogous to that of the lax centre. The $\mathscr{V}$-functor U is defined on objects by sending $f : I \to ZA$ to the centre piece $(zf, \gamma(f \otimes A))$, where (z, γ) is the universal centre piece. The definition of U on $\mathscr{V}$-enriched homs is the same as in the case of the lax centre above. $\square$

In order to exhibit the desired equivalence $\mathscr{M}(I, Z_\ell A) \simeq Z_\ell(\mathscr{M}(I, A))$, we shall require of our closed braided Gray monoid $\mathscr{M}$ two further properties.

Firstly, the pseudofunctor $\mathscr{M}(I, -) : \mathscr{M} \to \mathscr{V}$-**Cat** must be locally faithful. In other words, for every pair of 1-cells f, g, the following must be a monic arrow in $\mathscr{V}$:

$$\mathscr{M}(X,Y)(f,g) \to [\mathscr{M}(I,X), \mathscr{M}(I,Y)](\mathscr{M}(I,f), \mathscr{M}(I,g)). \qquad (3.6)$$

Secondly, for any $f, g : X \to Y$, the image of the arrow (3.6) under $\mathscr{V}(I, -) : \mathscr{V} \to$ **Set** must be surjective. This condition is saying that every $\mathscr{V}$-natural transformation $\mathscr{M}(I,f) \Rightarrow \mathscr{M}(I,g)$ is induced by a 2-cell $f \Rightarrow g$; this 2-cell is unique by the condition in the previous paragraph.

All these properties are satisfied by our main example of $\mathscr{V}$-**Mod**, as we shall see later.

Theorem 3.5. *In the hypothesis above, if A has a lax centre then there exists a $\mathscr{V}$-enriched equivalence making the following diagram commute up to a canonical isomorphism.*

$$\begin{tikzcd} \mathscr{M}(I, Z_\ell A) \arrow[rr, "\simeq"] \arrow[dr, "\mathscr{M}(I,z_\ell)"'] & & Z_\ell(\mathscr{M}(I,A)) \arrow[dl, "V"] \\ & \mathscr{M}(I,A) & \end{tikzcd}$$

*Here the $\mathscr{V}$-category on the right hand side is a lax centre in $\mathscr{V}$-**Cat** and V is the forgetful $\mathscr{V}$-functor. Furthermore, the result remains true if we write centres in place of lax centres.*

Proof. By Proposition 3.4 it is enough to exhibit a $\mathscr{V}$-enriched equivalence between $CP_\ell(I,A)$ and $Z_\ell(\mathscr{M}(I,A))$ commuting with the forgetful functors.

Define a $\mathscr{V}$-functor $\Phi : CP_\ell(I,A) \to Z_\ell(\mathscr{M}(I,A))$ as follows. On objects $\Phi(f,\alpha) = (f, \Phi_1(\alpha))$ where

$$\Phi_1(\alpha)_h : h * f \cong p(A \otimes f)h \xrightarrow{\alpha h} p(f \otimes A)h \cong f * h.$$

Recall that the $\mathscr{V}$-enriched hom $CP_\ell(I,A)((f,\alpha),(g,\beta))$ is the equalizer of (3.4) and $Z_\ell(\mathscr{M}(I,A))(\Phi(f,\alpha),\Phi(g,\beta))$ is the equalizer of the diagram in Example 3.1, where $\mathscr{C} = \mathscr{M}(I,A)$, $x = f$, $y = g$, $\gamma = \Phi_1(\alpha)$ and $\delta = \Phi_1(\beta)$. We can draw a diagram

$$\begin{tikzcd}
\mathscr{M}(I,A)(f,g) \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & \mathscr{M}(A,A)(p(f \otimes A), p(A \otimes g)) \arrow[d, "\mathscr{M}(I,-)"] \\
& {[\mathscr{M}(I,A), \mathscr{M}(I,A)](f * -, - * g)}
\end{tikzcd}$$

where $CP_\ell(I,A)((f,\alpha),(g,\beta))$ is the equalizer of the pair of arrows in the top row and $Z_\ell(\mathscr{M}(I,A))(\Phi(f,\alpha),\Phi(g,\beta))$ is the equalizer of the other diagonal pair of arrows. Moreover, the diagram serially commutes. The vertical arrow is induced by the effect of the pseudofunctor $\mathscr{M}(I,-) : \mathscr{M} \to \mathscr{V}$-**Cat** on $\mathscr{V}$-enriched homs, and hence monic by hypothesis. It follows that there exists an isomorphism $CP_\ell(I,A)((f,\alpha),(g,\beta)) \to Z_\ell(\mathscr{M}(I,A))(\Phi(f,\alpha),\Phi(g,\beta))$. One can check that these isomorphisms are part of a $\mathscr{V}$-functor Φ, which, obviously, is fully faithful.

It only rests to prove that Φ is essentially surjective on objects. An object (f,γ) of $Z_\ell(\mathscr{M}(I,A))$ gives rise to a $\mathscr{V}$-natural transformation

$$\gamma'_h : p(A \otimes f)h \cong h * f \xrightarrow{\gamma_h} f * h \cong p(f \otimes A)h.$$

By the hypothesis on $\mathscr{M}(I,-)$ introduced in the paragraph previous to this theorem, γ' is induced by a unique $\alpha : p(A \otimes f) \Rightarrow p(f \otimes A)$. The equalities (3.2) and (3.3) for the 2-cell α follow from the fact that (f,γ) is an object in the lax centre of $\mathscr{M}(I,A)$ and the fact that $\mathscr{M}(A^2,A) \to [\mathscr{M}(I,A^2), \mathscr{M}(I,A)]$ is fully faithful. Now observe that $\Phi(f,\alpha) = (f,\gamma)$. This shows that Φ is essentially surjective on objects. Finally, α is invertible if and only if γ is invertible, so that proof also applies to centres. $\square$

Recall from [13] that for a right autonomous pseudomonoid A, with right dualization $\bar{d} : A^\vee \to A$, every map $f : I \to A$ has a right dual in the monoidal $\mathscr{V}$-category $\mathscr{M}(I,A)$. A right dual of f is given by $\bar{d}(f^*)^\vee$, where f^* is a right adjoint to f. Then the full subcategory Map$\mathscr{M}(I,A)$ of $\mathscr{M}(I,A)$ is right autonomous (in the classical sense that it has right duals).

Theorem 3.6. *In addition to the hypothesis above, assume the following: $\mathscr{V}$ is a complete and cocomplete monoidal closed category, $\mathscr{M}$ has* all *right liftings, $\mathscr{M}(I,A)$ has a dense sub $\mathscr{V}$-category included in* $\mathrm{Map}\mathscr{M}(I,A)$ *and $\mathscr{M}(I,-)$: $\mathscr{M} \to$* **Cat** *reflects equivalences. If A is left autonomous, then the centre of A coincides with the lax centre whenever both exist.*

Proof. By Theorem 3.5, there exists an isomorphism as depicted below.

$$\begin{tikzcd} \mathscr{M}(I,ZA) \arrow[r, "\simeq"] \arrow[d, "\mathscr{M}(I,z_c)"'] \arrow[dr, phantom, "\cong"] & Z(\mathscr{M}(I,A)) \arrow[d, hook] \\ \mathscr{M}(I,Z_\ell A) \arrow[r, "\simeq"] & Z_\ell(\mathscr{M}(I,A)) \end{tikzcd}$$

A straightforward modification of [16, Prop. 6] (using the property of the right liftings with respect to composition dual to [84, Prop. 1]) shows that the monoidal $\mathscr{V}$-category $\mathscr{M}(I,A)$ is closed as a $\mathscr{V}$-category. It follows that the $\mathscr{V}$-functors $(f * -) = p(f \otimes A)- : \mathscr{M}(I,A) \to \mathscr{M}(I,A)$ given by tensoring with an object f are cocontinuous. As $\mathscr{M}(I,A)$ has a dense sub monoidal $\mathscr{V}$-category with right duals, the hypotheses of [15, Theorem 3.4] are satisfied, and we deduce that the inclusion $Z(\mathscr{M}(I,A)) \hookrightarrow Z_\ell(\mathscr{M}(I,A))$ is the identity. It follows that $\mathscr{M}(I,z_c)$ is an equivalence and hence z_c is an equivalence. □

The theorem above applies to the case of promonoidal enriched categories. See Section 5.1.

3.4 Lax centres of autonomous pseudomonoids

In this section we exhibit the lax centre of a left autonomous map pseudomonoid as an Eilenberg-Moore construction for a certain monad.

The lax centre of a pseudomonoid was defined as a birepresentation of the pseudofunctor $CP_\ell(-,A)$. An object of the category $CP_\ell(X,A)$, *i.e.*, a lax centre piece, is a 2-cell $p(f \otimes A) \Rightarrow p(A \otimes f)c_{X,A}$. We observe that the same notion of lax centre can be defined by using c^* instead of c. In an entirely analogous way to Definition 3.1, one defines a category $CP^*_\ell(X,A)$ as follows. It has objects (f,γ) where $f : X \to A$ and $\gamma : p(f \otimes A)c^*_{X,A} \Rightarrow p(A \otimes f)$, and arrows $(f,\gamma) \to (g,\delta)$ those 2-cells $f \Rightarrow g$ which are compatible with γ and δ. Pasting with the structural isomorphism $c_{X,A}c^*_{X,A} \cong 1_{X\otimes A}$ induces pseudonatural equivalences

$CP_\ell(X,A) \to CP^*_\ell(X,A)$. This is the reason why the c^* appears in the following definition.

Definition 3.3. Given a map pseudomonoid A in a braided Gray monoid $\mathscr{M}$ define a pseudonatural transformation $\sigma : \mathscr{M}(A\otimes -, A) \Rightarrow \mathscr{M}(A \otimes -, A)$ with components

$$\sigma_X(g) = \big(A\otimes X \xrightarrow{p^*\otimes 1} A^2 \otimes X \xrightarrow{1\otimes c^*_{X,A}} A\otimes X\otimes A \xrightarrow{g\otimes 1} A^2 \xrightarrow{p} A\big).$$

Lemma 3.7. *The pseudonatural transformation σ has a canonical structure of a monad.*

Proof. Just note that σ is isomorphic to the monad θ of Section 2.2 for the map pseudomonoid $(A, j, pc^*_{A,A})$. $\square$

Explicitly, the multiplication of σ is given by components

$$\begin{tikzcd}
A\otimes X\otimes A \arrow[r, "p^*\otimes 1\otimes 1"] & A^2\otimes X\otimes A \arrow[dr, bend left, "1\otimes c^*_{X,A}\otimes 1"] & & & & \\
A^2\otimes X \arrow[u, "1\otimes c^*_{X,A}"] \arrow[r, "p^*\otimes 1\otimes 1"] \arrow[ur, phantom, "\cong"] & A^3\otimes X \arrow[u, "A^2\otimes c^*_{X,A}"] \arrow[r, "1\otimes c^*_{X,A^2}"] \arrow[dr, "1\otimes p\otimes 1"] \arrow[u, phantom, shift left=8, "\cong"] & A\otimes X\otimes A^2 \arrow[r, "g\otimes A^2"] \arrow[dr, "1\otimes 1\otimes p"] & A^3 \arrow[r, "p\otimes 1"] \arrow[dr, "1\otimes p"] & A^2 \arrow[dr, "p"] & \\
A\otimes X \arrow[u, "p^*\otimes 1"] \arrow[r, "p^*\otimes 1"'] \arrow[ur, phantom, "\cong"] & A^2\otimes X \arrow[u, "1\otimes p^*\otimes 1"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow"] & A^2\otimes X \arrow[r, "1\otimes c^*_{X,A}"'] \arrow[ur, phantom, "\cong"] & A\otimes X\otimes A \arrow[r, "g\otimes 1"'] \arrow[ur, phantom, "\cong"] & A^2 \arrow[r, "p"'] \arrow[ur, phantom, "\cong"] & A
\end{tikzcd} \tag{3.7}$$

and the unit by

$$\begin{tikzcd}
& A\otimes X \arrow[r, equal] & A\otimes X \arrow[r, "g"] & A \arrow[dr, "1\otimes j"] \arrow[drr, bend left, "1"] & & \\
A\otimes X \arrow[ur] \arrow[r, "p^*\otimes 1"'] \arrow[ur, phantom, shift right=4, "\cong"] & A^2\otimes X \arrow[u, "1\otimes j^*\otimes 1"] \arrow[r, "1\otimes c^*_{X,A}"'] \arrow[ur, phantom, "\cong"] & A\otimes X\otimes A \arrow[u, "1\otimes 1\otimes j^*"] \arrow[r, "g\otimes 1"'] \arrow[ur, phantom, "\cong"] & A^2 \arrow[u, "1\otimes j^*"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow"] & A^2 \arrow[r, "p"'] \arrow[u, phantom, "\cong"] & A
\end{tikzcd} \tag{3.8}$$

Now we assume that the braided Gray monoid $\mathscr{M}$ is also closed. In this situation the monads θ and σ are represented by monads t and $s : [A,A] \to [A,A]$. The monad s is

$$[A,A] \xrightarrow{i^A_A} [A\otimes A, A\otimes A] \xrightarrow{[c_{A,A}, c^*_{A,A}]} [A\otimes A, A\otimes A] \xrightarrow{[p^*,p]} [A,A], \tag{3.9}$$

which is the monad t for the opposite pseudomonoid of A with respect to c^*, in other words, $(A, j, pc^*_{A,A})$. Alternatively, t and s can be taken respectively as

$$[A,A] \xrightarrow{\mathrm{id}\otimes 1} [A,A]\otimes[A,A] \longrightarrow [A\otimes A, A\otimes A] \xrightarrow{[p^*,p]} [A,A] \tag{3.10}$$

$$[A,A] \xrightarrow{1\otimes\mathrm{id}} [A,A]\otimes[A,A] \longrightarrow [A\otimes A, A\otimes A] \xrightarrow{[p^*,p]} [A,A] \tag{3.11}$$

where $\mathrm{id} : I \to [A,A]$ is the 1-cell corresponding to 1_A under the equivalence $\mathscr{M}(A,A) \simeq \mathscr{M}(I,[A,A])$.

Observation 3.8. At this point we should remark that for a map pseudomonoid A, $[A,A]$ has two pseudomonoid structures. The one we have considered so far is the *composition* pseudomonoid structure, but we also have the *convolution* pseudomonoid structure.

If (C,e,b) is a pseudocomonoid in the closed braided Gray monoid $\mathscr{M}$, $[C,-]$ is lax monoidal in the standard way. The unit constraint $I \to [C,I]$ corresponds under the closedness equivalence to the counit $e : C \to I$ and the 1-cells $[C,X]\otimes[C,Y] \to [C,X\otimes Y]$ correspond

$$\begin{aligned} C\otimes[C,X]\otimes[C,Y] \xrightarrow{b\otimes 1\otimes 1} C^2\otimes[C,X]\otimes[C,Y] \xrightarrow{1\otimes c\otimes 1} (C\otimes[C,X])^2 \\ \xrightarrow{(\mathrm{ev}\otimes 1)(1\otimes 1\otimes \mathrm{ev})} X\otimes Y. \end{aligned}$$

In particular, for a pseudomonoid A, $[C,A]$ has a canonical *convolution* pseudomonoid structure. This structure corresponds to the usual convolution tensor product in $\mathscr{M}(C,A)$ given by $f * g = p(A\otimes g)(f\otimes C)b$ with unit je. As we remarked before, for a map pseudomonoid A the identity 1_A has a canonical structure of a monoid in the convolution monoidal category $\mathscr{M}(A,A)$. It follows that the corresponding 1-cell $\mathrm{id} : I \to [A,A]$ is a monoid in $\mathscr{M}(I,[A,A])$.

Observation 3.9. Let B be a pseudomonoid in $\mathscr{M}$ and consider $\mathscr{M}(I,B)$ and $\mathscr{M}(B,B)$ as monoidal categories with the convolution and the composition tensor product respectively. We have monoidal functors $L,R : \mathscr{M}(I,B) \to \mathscr{M}(B,B)$ given by $L(f) = p(f\otimes B)$ and $R(f) = p(B\otimes f)$. The associativity constraint of B induces isomorphisms $L(f)R(g) \cong R(g)L(f)$, natural in f and g. If m and n are monoids in $\mathscr{M}(I,B)$, then these isomorphisms form an invertible distributive law between the monads $L(m)$ and $R(n)$.

The monoidal functors L,R are compatible with monoidal pseudofunctors: if

$F : \mathscr{M} \to \mathscr{N}$ is a monoidal pseudofunctor, then there are monoidal isomorphisms

$$\begin{tikzcd}
\mathscr{M}(I,B) \arrow[r, "{L,R}"] \arrow[d, "F_{I,B}"'] & \mathscr{M}(B,B) \arrow[dd, "F_{B,B}"] \\
\mathscr{N}(FI,FB) \arrow[d, "\simeq"'] \arrow[r, phantom, "\cong"] & \\
\mathscr{N}(I,FB) \arrow[r, "{L,R}"] & \mathscr{N}(FB,FB)
\end{tikzcd}$$

In particular, if m is a monoid in $M(I, B)$, we have an isomorphisms $F(L(m)) \cong L(Fm)$ and $F(R(m)) \cong R(Fm)$ of monoids in $\mathscr{M}(B, B)$.

Proposition 3.10. *There exists an invertible distributive law between the monads t and s, and hence between the monads θ and σ.*

Proof. Apply Observation 3.9 above to the *convolution* pseudomonoid $B = [A, A]$ and the monoid $m = n = \mathrm{id} : I \to [A, A]$, noting that $t = L(\mathrm{id})$ and $s = R(\mathrm{id})$. The 1-cell id is a monoid with the structure given by Observation 3.8. $\square$

If t has an Eilenberg-Moore construction $u : [A, A]^t \to [A, A]$ the monad $\hat{\sigma}$ is represented by some $\hat{s} : [A, A]^t \to [A, A]^t$.

Proposition 3.11. *The monads s and $\hat{s}$ are opmonoidal monads.*

Proof. As we noted above, s is the monad t corresponding to the pseudomonoid $(A, j, pc^*_{A,A})$. It can also be regarded as the corresponding monad t for the pseudomonoid (A, j, p) in $\mathscr{M}^{\mathrm{rev}}$, and thus it is opmonoidal in $\mathscr{M}^{\mathrm{rev}}$; hence it is opmonoidal in $\mathscr{M}$. The monad $\hat{s}$ is opmonoidal since $[A, A]^t$ is an Eilenberg-Moore construction in $\mathbf{Opmon}(\mathscr{M})$. $\square$

Denote by $\hat{\sigma}$ the monad on $\mathscr{M}(A \otimes -, A)^\theta$ induced by σ. There exists an isomorphism $(\mathscr{M}(A \otimes -, A)^\theta)^{\hat{\sigma}} \simeq \mathscr{M}(A \otimes -, A)^{\sigma\theta}$.

Suppose that there exists a pseudonatural transformation $\tilde{\sigma} : \mathscr{M}(-, A) \to \mathscr{M}(-, A)$ such that $\lambda\tilde{\sigma} \cong \hat{\sigma}\lambda$; since λ is fully faithful (see Proposition 2.7), this is equivalent to saying that for each X the monad $\hat{\sigma}_X$ restricts to a monad on the replete image of λ_X in $\mathscr{M}(A \otimes X, A)^{\theta_X}$, and in this case $\tilde{\sigma} = \lambda^*\hat{\sigma}\lambda$. Moreover, $\tilde{\sigma}$ carries the structure of a monad induced by the one of $\hat{\sigma}$, making λ together with the isomorphism $\lambda\tilde{\sigma} \cong \hat{\sigma}\lambda$ a monad morphism. Such a monad $\tilde{\sigma}$ clearly exists if the theorem of Hopf modules holds for A, *i.e.*, if λ is an equivalence.

Theorem 3.12. *There exists an equivalence in the 2-category $[\mathscr{M}^{\mathrm{op}}, \mathbf{Cat}]$ between $\mathscr{M}(-, A)^{\tilde{\sigma}}$ and $CP_\ell(-, A)$ whenever the monad $\tilde{\sigma}$ exists. Moreover, this* *equivalence commutes with the corresponding forgetful pseudonatural transformations.*

Proof. Instead of $\tilde{\sigma}_X$, we shall consider the restriction of $\hat{\sigma}_X$ to the replete image of λ_X. Take $f : X \to A$ and assume that $\lambda_X(f : X \to A)$ has a structure ν of $\hat{\sigma}$-algebra. This means that the action ν is a 2-cell

$$\begin{tikzcd}
A \otimes A \otimes X \arrow[rr, "1\otimes c^*_{X,A}"] & & A \otimes X \otimes A \arrow[d, "1\otimes f\otimes 1"] \\
A \otimes X \arrow[u, "p^*\otimes 1"] \arrow[d, "1\otimes f"'] & \Leftarrow_{\nu} & A \otimes A \otimes A \arrow[d, "p\otimes 1"] \\
A \otimes A \arrow[dr, "p"'] & & A \otimes A \arrow[dl, "p"] \\
& A &
\end{tikzcd} \tag{3.12}$$

which is a morphism of θ_X-algebras from $\hat{\sigma}_X\lambda_X(f)$ to $\lambda_X(f)$. Furthermore, the pasting

$$\begin{tikzcd}
& A^2 \otimes X \otimes A \arrow[rr, "1\otimes c^*\otimes 1"] & & A \otimes X \otimes A^2 \arrow[d, "1\otimes f\otimes A^2"] \\
A^2 \otimes X \arrow[r, "1\otimes c^*"] & A \otimes X \otimes A \arrow[u, "p^*\otimes 1\otimes 1"] \arrow[d, "1\otimes f\otimes 1"'] & \Leftarrow_{\nu\otimes 1} & A^4 \arrow[d, "p\otimes A^2"] \\
A \otimes X \arrow[u, "p^*\otimes 1"] \arrow[d, "1\otimes f"'] & A^3 \arrow[dr, "p\otimes 1"'] & & A^3 \arrow[dl, "p\otimes 1"] \\
A^2 \arrow[dr, "p"'] & \Leftarrow_{\nu} & A^2 \arrow[dl, "p"] & \\
& A & &
\end{tikzcd}$$

should be equal to the composition $\sigma_X\sigma_X\lambda_X(f) \to \sigma_X\lambda_X(f) \xrightarrow{\nu} \lambda_X(f)$ of the multiplication of σ_X (3.7) and ν, and the composition $\lambda_X(f) \to \sigma_X\lambda_X(f) \xrightarrow{\nu} \lambda_X(f)$ of the unit of σ (3.8) and ν is the identity. The 2-cells (3.12) correspond, under pasting with $\phi^{-1} : p(A\otimes p) \cong p(p\otimes A)$, to 2-cells $p(A\otimes(p(f\otimes A)c^*_{X,A}))(p^*\otimes X) \Rightarrow p(A\otimes f)$, and then to 2-cells $p(A\otimes(p(f\otimes A)c^*_{X,A})) \Rightarrow p(A\otimes f)(p\otimes A) \cong p(A\otimes p)(A\otimes A\otimes f)$. Since λ_X is fully faithful, and $\hat{\sigma}$ restricts to its replete image, it follows that the 2-cells ν correspond to the 2-cells γ (3.1). The axiom of associativity for the action ν translates into the axiom (3.2) for γ and the axiom of unit for ν into the axiom (3.3) for γ. This shows that the composition of the forgetful functor $V : CP_\ell(X, A) \to \mathscr{M}(X, A)$ with λ_X factors as a pseudonatural transformation G followed by $\hat{U}$, as depicted below.

$$\begin{tikzcd}
CP_\ell(X,A) \arrow[rr, dashed, "H_X"] \arrow[d, "V_X"'] \arrow[drr, "G_X"' near start] & & \mathscr{M}(X,A)^{\tilde{\sigma}_X} \arrow[d, "\tilde{\lambda}_X"] \arrow[dll, "\tilde{U}_X" near start] \\
\mathscr{M}(X,A) \arrow[dr, "\lambda_X"'] & & (\mathscr{M}(A\otimes X,A)^{\theta_X})^{\hat{\sigma}_X} \arrow[dl, "\hat{U}_X"] \\
& \mathscr{M}(A\otimes X,A)^{\theta_X} &
\end{tikzcd}$$

Moreover, G_X factors through the image of $\tilde{\lambda}_X$, since $\hat{U}_X G_X$ factors through λ_X, and in fact G_X is an equivalence into the image of $\tilde{\lambda}_X$. Here $\tilde{\lambda}_X$ is the functor induced on Eilenberg-Moore constructions by λ_X; in particular, $\tilde{\lambda}_X$ is fully faithful since λ_X is fully faithful. Therefore we have an equivalence H_X as in the diagram, such that $\tilde{\lambda}_X H_X = G_X$. Hence, $\lambda_X \tilde{U}_X H_X = \hat{U}_X \tilde{\lambda}_X H_X = \hat{U}_X G_X = \lambda_X V_X$, and $\tilde{U}_X H_X = V_X$. The equivalences H_X are clearly pseudonatural in X.

□

Corollary 3.13. *If the theorem of Hopf modules holds for a map pseudomonoid A then there exists an equivalence $CP_\ell(-, A) \simeq \mathscr{M}(A \otimes -, A)^{\sigma\theta}$.*

Proof. λ_X is an equivalence and then the monad $\tilde{\sigma}$ exists and

$$\mathscr{M}(-, A)^{\tilde{\sigma}} \simeq (\mathscr{M}(A \otimes -, A)^{\theta})^{\hat{\sigma}} \simeq \mathscr{M}(A \otimes -, A)^{\sigma\theta}.$$

□

Theorem 3.14. *Suppose that the theorem of Hopf modules holds for the map pseudomonoid A and that it has a Hopf module construction. Then the lax centre of A is the Eilenberg-Moore construction for the opmonoidal monad*

$$\tilde{s} := \ell^* \hat{s} \ell = A \to A$$

one of them existing if the other does. Moreover,

$$\tilde{s} \cong \big(A \xrightarrow{j\otimes 1} A \otimes A \xrightarrow{p^*\otimes 1} A \otimes A \otimes A \xrightarrow{1\otimes c^*_{A,A}} A \otimes A \otimes A \xrightarrow{p\otimes A} A \otimes A \xrightarrow{p} A\big).$$

Proof. The monad $\hat{s}$ exists and is opmonoidal since $t : [A, A] \to [A, A]$ has an

Eilenberg-Moore construction in **Opmon**($\mathscr{M}$). Hence, $\tilde{s}$ has a canonical opmonoidal monad structure induced by the one of $\hat{s}$. Theorem 3.12 implies that the lax centre of A exists, that is, $CP(-,A)$ is birepresentable, if and only if the monad $\tilde{s}$ has an Eilenberg-Moore construction.

To obtain an expression for the 1-cell $\tilde{s}$ recall that, by definition, $\mathscr{M}(-,\tilde{s})$ is isomorphic to $\lambda^*\hat{\sigma}\lambda$. It is easy to show that

$$\begin{aligned}\lambda^*_X\hat{\sigma}_X\lambda_X(f : X \to A) &= p(p\otimes A)(A\otimes f\otimes A)(A\otimes c^*_{X,A})(p^*\otimes X)(j\otimes X)\\ &\cong p(p\otimes A)(A\otimes c^*_{A,A})(p^*\otimes A)(j\otimes A)f;\end{aligned}$$

see the definition of λ in Section 2.2 and Definition 3.3. It follows that the expression for $\tilde{s}$ of the statement holds. $\square$

Observation 3.15. The thesis of Theorem 3.14 above holds under the sole hypothesis of A being left autonomous. This is so because every left autonomous map pseudomonoid has a Hopf module construction (see Theorem 2.35).

Theorem 3.16. *For a (left and right) autonomous map pseudomonoid the centre equals the lax centre, either existing if the other does.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
(\mathscr{M}(A\otimes X,A)^{\theta_X})^{\hat{\sigma}_X} \arrow[r] \arrow[d] & \mathscr{M}(A\otimes X,A)^{\theta_X} \arrow[r, "\hat{\sigma}"] \arrow[d] & \mathscr{M}(A\otimes X,A)^{\theta_X} \arrow[d, "\upsilon_X"] \\
\mathscr{M}(A\otimes X,A)^{\sigma_X} \arrow[r] & \mathscr{M}(A\otimes X,A) \arrow[r, "\sigma"'] & \mathscr{M}(A\otimes X,A)
\end{tikzcd}$$

In Theorem 3.12 we proved that any lax centre piece arises as

$$\begin{tikzcd}
A\otimes A\otimes X \arrow[r, equal] \arrow[dr, "p\otimes 1"'] \arrow[r, phantom, shift right=3, "\Downarrow"] & A\otimes A\otimes X \arrow[rr, "1\otimes c_{A,X}"] & & A\otimes X\otimes A \arrow[d, "h\otimes 1"] \\
& A\otimes X \arrow[u, "p^*\otimes 1"'] \arrow[r, phantom, "\Leftarrow\nu"] \arrow[dr, "h"'] & \; & A\otimes A \arrow[dl, "p"] \\
& & A &
\end{tikzcd} \tag{3.13}$$

for some $\hat{\sigma}_X$-algebra $\nu : \hat{\sigma}_X(h) \to h$, so we have to prove that (3.13) is invertible. Consider the canonical split coequalizer $\hat{\sigma}^2_X(h) \rightrightarrows \hat{\sigma}_X(h) \twoheadrightarrow h$ in $\mathscr{M}(A\otimes X,A)^{\theta_X}$, and its image $\nu : \sigma_X(h) \to h$ in $\mathscr{M}(A\otimes X,A)$. The arrow ν is a morphism of σ_X-

algebras. This implies that the lower rectangle in the diagram below commutes.

$$\begin{tikzcd}
p(\sigma_X(h)\otimes A)(A\otimes c_{A,X}) \arrow[rr] \arrow[d, "p(\sigma_X(h)\otimes A)(A\otimes c_{A,X})(\eta\otimes X)"'] & & p(h\otimes A)(A\otimes c_{A,X}) \arrow[d, "p(h\otimes A)(A\otimes c_{A,X})(\eta\otimes X)"] \\
p(\sigma_X(h)\otimes A)(A\otimes c_{A,X})(p^*p\otimes X) \arrow[d, equal] & & p(h\otimes A)(A\otimes c_{A,X})(p^*p\otimes X) \arrow[d, equal] \\
\sigma_X^2(h)(p\otimes X) \arrow[rr] \arrow[d, "(\mu_X)_h(p\otimes X)"'] & & \sigma_X(h)(p\otimes X) \arrow[d, "\nu(p\otimes X)"] \\
\sigma_X(h)(p\otimes X) \arrow[rr] & & h(p\otimes X)
\end{tikzcd}$$

The upper rectangle commutes by naturality of composition. Here η denotes the unit of the adjunction $p \dashv p^*$ and μ the multiplication of the monad σ. Observe that the rows are coequalizers and the right-hand column is just (3.13). Then, to show that this last arrow is invertible it suffices to show that the left-hand side column, which is the pasting of η with the multiplication of σ (3.7), is so. But this 2-cell is invertible because A is right autonomous and hence by the dual of Theorem 2.34.2 the 2-cell below is invertible. This completes the proof.

$$\begin{tikzcd}
A^2 \arrow[rr, equal] \arrow[dr, "p"'] & \arrow[d, phantom, "\Downarrow"] & A^2 \arrow[r, "p^*\otimes 1"] \arrow[d, phantom, "\cong"] & A^3 \arrow[dr, "1\otimes p"] \arrow[d, phantom, "\Downarrow"] & \\
& A \arrow[r, "p^*"'] \arrow[ur, "p^*"'] & A^2 \arrow[rr, equal] \arrow[ur, "1\otimes p^*"'] & & A^2
\end{tikzcd}
=
\begin{tikzcd}
& A^3 \arrow[r, "1\otimes p"] \arrow[dr, "p\otimes 1"] \arrow[d, phantom, "\Downarrow"] & A^2 \arrow[rr, equal] \arrow[dr, "p"] \arrow[d, phantom, "\cong"] & \arrow[d, phantom, "\Downarrow"] & A^2 \\
A^2 \arrow[rr, equal] \arrow[ur, "p^*\otimes 1"] & & A^2 \arrow[r, "p"'] & A \arrow[ur, "p^*"'] &
\end{tikzcd}$$

□

Finally, putting together the results above we obtain:

Corollary 3.17. *Any autonomous map pseudomonoid in a braided monoidal bicategory with Eilenberg-Moore objects has both a centre and a lax centre, and the two coincide.*

Finally, we state and prove the following easy preservation result.

Corollary 3.18. *Suppose $F : \mathscr{M} \to \mathscr{N}$ is a pseudofunctor between Gray monoids with the following properties: F preserves Eilenberg-Moore objects, is braided and strong monoidal. Then, F preserves lax centres of left autonomous map pseudomonoids.*

Proof. Let A be a left autonomous map pseudomonoid in $\mathscr{M}$. By Observation 3.15, the lax centre of A is the Eilenberg-Moore construction for the opmonoidal monad $\tilde{s} : A \to A$, one existing if the other does. On the other hand, FA is also a left autonomous map pseudomonoid by Proposition 2.40. Therefore, it is enough to show that F preserves the monad $\tilde{s}$, in the sense that $F\tilde{s}$ is isomorphic to the corresponding monad $\tilde{s}$ for FA.

Since $\tilde{s}$ is the lifting of the monad s on $A^\circ \otimes A$ to the Eilenberg-Moore construction $(A^\circ \otimes p)(\mathrm{n} \otimes A) : A \to A^\circ \otimes A$ of the monad t (see Theorem 2.35) it suffices to prove that F preserves the monads t and s. We only work with t, the proof for the monad s being completely analogous. Now, we know from the proof of Proposition 3.10 that $t = L(\mathrm{n})$ and $s = R(\mathrm{n})$, where $L, R : \mathscr{M}(I, A^\circ \otimes A) \to \mathscr{M}(A^\circ \otimes A, A^\circ \otimes A)$ are the functors defined in Observation 3.9. Therefore, $Ft = F(L(\mathrm{n})) \cong L(I \xrightarrow{\cong} FI \xrightarrow{F\mathrm{n}_A} F(A^\circ \otimes A)) \cong L(\mathrm{n}_{FA})$, which is the monad t corresponding to the pseudomonoid FA.

$\square$

Chapter 4

Radford's formula for autonomous pseudomonoids

In this Chapter we give a generalisation of Radford's formula for finite-dimensional Hopf algebras to the context of autonomous pseudomonoids. We also define and study *unimodular* autonomous map pseudomonoids.

Radford's formula, originally proven in [71] but also see [76], states that for a finite-dimensional Hopf algebra H with antipode S the following equality holds

$$S^4(x) = a(\alpha^{-1} \rightharpoonup x \leftharpoonup \alpha)a. \tag{4.1}$$

Here $\rightharpoonup$: $H^* \otimes H \to H$, $\leftharpoonup$: $H \otimes H^* \to H$ are actions of H^* on H given by $(f \rightharpoonup x) = \sum x_1 f(x_2) = (H \otimes f)\Delta(x)$, $(x \leftharpoonup f) = \sum f(x_1)x_2 = (f \otimes H)\Delta(x)$.

An element $t \in H$ is called a *right integral* if $tx = \varepsilon(x)t$ for all $x \in H$ (ε is the counit of H). Dually, a right cointegral is a integral in the dual of H. In (4.1), a is the *modular element* of H and α is the *modular function* of H, defined by the properties that $xt = \alpha(x)t$ and $\phi * \alpha = \phi(a)\alpha$ for all $x \in H$, $\phi \in H^*$. The existence and uniqueness of a, α are consequences of the fundamental theorem of Hopf modules and the finiteness of H. See for example [85, 76]. It also follows that a is a group-like element and $\alpha : H \to k$ is a morphism of algebras. It is worth mention that in fact, in all the proofs of Radford's formula we are aware of, what one actually deduces is

$$S^2(x) = a(\alpha^{-1} \rightharpoonup S^{-2}(x) \leftharpoonup \alpha)a \tag{4.2}$$

and then apply S^2 to get (4.1). The formula (4.2) has the same form as the formulas for finite dimensional quasi and coquasi Hopf algebras we will deduce in Section 5.2.5 from the results of this chapter. Moreover, the passage from (4.2) to (4.1) depends of special properties of Hopf algebras, and does not behave well for more general algebraic structures.

Radford's formula was first proved in [71], and plays an important role in the theory of finite-dimensional Hopf algebras. Other techniques are used in the proof in [76]. The formula has been generalised to various contexts, such as quasi-Hopf algebras [34], coquasi-Hopf algebras [30], weak Hopf algebras [69], bi-Frobenius algebras [21, 14], co-Frobenius Hopf algebras [3], Hopf algebras in a braided category [5] and finite tensor categories [27].

The main results of the present chapter, Theorem 4.15 and Corollary 4.16, generalise Radford's formula for Hopf algebras, coquasi-Hopf algebras and finite tensor categories. See Sections 5.2.5 and 7.5.

The role of a Hopf algebra and its antipode is played by an autonomous map pseudomonoid A and its left dualization d. The role of the inverse of the antipode (that automatically exists for finite-dimensional Hopf algebras) is played by the right dualization $\bar{d}$ of A. The finiteness hypothesis on the Hopf algebra is replaced by the assumption that the counit $\mathrm{n} : I \to A^\circ \otimes A$ of the pseudoadjunction has a right adjoint. The modular element a takes the form of an invertible element w in the monoidal category $\mathscr{M}(A, I)$ and the modular function α appears as an isomorphism. Moreover, this isomorphism is monoidal, generalising the fact that the modular function is a morphism of algebras.

4.1 Duals in convolution hom-categories

Recall closed Gray monoids and braided Gray monoids from Sections 2.1.1 and 3.1 respectively.

Suppose that $\mathscr{M}$ is a closed braided Gray-monoid [16] with internal hom $[-,-]$ and braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$. In this situation, the pseudofunctor $[-,-] : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$ is weak monoidal (see [16, Corollary 9]). If (A, j, p) is a map pseudomonoid in $\mathscr{M}$, the object $[A, A]$ has the structure of a pseudomonoid with multiplication

$$[A, A] \otimes [A, A] \longrightarrow [A \otimes A, A \otimes A] \xrightarrow{[p^*, p]} [A, A] \tag{4.3}$$

and unit

$$I \xrightarrow{\simeq} [I, I] \xrightarrow{[j^*, j]} [A, A] \tag{4.4}$$

where the unlabelled arrows are part of the monoidal structure of $[-,-]$. These 1-cells correspond under the closedness biadjunction respectively to

$$A\otimes[A,A]\otimes[A,A]\xrightarrow{p^*\otimes 1\otimes 1}A^2\otimes[A,A]\otimes[A,A]\xrightarrow{1\otimes c\otimes 1}A\otimes[A,A]\otimes A\otimes[A,A]$$
$$\xrightarrow{\mathrm{ev}\otimes 1\otimes 1}A^2\otimes[A,A]\xrightarrow{1\otimes\mathrm{ev}}A\otimes A\xrightarrow{p}A \quad (4.5)$$

and $A\xrightarrow{j^*}I\xrightarrow{j}A$.

Suppose further that the object A has right bidual A° with evaluation and coevaluation $\mathrm{e}:A\otimes A^\circ\to I$ and $\mathrm{n}:I\to A^\circ\otimes A$. The 1-cells

$$\bar{\mathrm{e}}:A^\circ\otimes A\xrightarrow{c_{A^\circ,A}}A\otimes A^\circ\xrightarrow{\mathrm{e}}I \qquad \text{and} \qquad \bar{\mathrm{n}}:I\xrightarrow{\mathrm{n}}A^\circ\otimes A\xrightarrow{c^*_{A,A^\circ}}A\otimes A^\circ \quad (4.6)$$

makes of A° a left bidual for A. This is so because the braiding c gives rise to a strong monoidal 2-functor $S:\mathscr{M}^{\mathrm{rev}}\to\mathscr{M}$ with underlying 2-functor the identity 2-functor of $\mathscr{M}$, and $\bar{\mathrm{e}}=S(\mathrm{e})$, $\bar{\mathrm{n}}=S(\mathrm{n})$. There is a canonical equivalence $A^\circ\otimes A\simeq[A,A]$, and the pseudomonoid structure on $[A,A]$ transports to one on $A^\circ\otimes A$, described in the following Lemma.

Lemma 4.1. *When A has right bidual the map pseudomonoid structure on $A^\circ\otimes A$ described in* (4.3) *and* (4.4) *is given by the multiplication*

$$A^\circ\otimes A\otimes A^\circ\otimes A\xrightarrow{1\otimes c^*\otimes 1}A^\circ\otimes A^\circ\otimes A\otimes A\xrightarrow{(pc)^{*\circ}\otimes 1\otimes 1}A^\circ\otimes A\otimes A\xrightarrow{1\otimes p}A^\circ\otimes A \quad (4.7)$$

and unit

$$I\xrightarrow{j^{*\circ}}A^\circ\xrightarrow{1\otimes j}A^\circ\otimes A. \quad (4.8)$$

Proof. Simple exercise. □

4.1.1 Opposite and bidual autonomous pseudomonoids

If (A,j,p) is a pseudomonoid in the braided Gray monoid $\mathscr{M}$, then (A,j,pc) is a pseudomonoid too, called the *opposite pseudomonoid.* This is the image of the pseudomonoid (A,j,p) in $\mathscr{M}^{\mathrm{rev}}$ under the monoidal functor identity $S:\mathscr{M}^{\mathrm{rev}}\to\mathscr{M}$ (see comments above).

Through this subsection we will equip A with the left bidual with evaluation and coevaluation $\bar{\mathrm{e}}=\mathrm{e}c_{A,A^\circ}$ and $\bar{\mathrm{n}}=c^*_{A,A^\circ}\mathrm{n}$ respectively. With this choice of left biduals A° is a right and a left bidual of A.

Proposition 4.2. *Suppose the pseudomonoid A has a left dualization $d : A^\circ \to A$. Then d is a right dualization for the opposite pseudomonoid of A.*

Proof. As already mentioned, the braiding of $\mathscr{M}$ induces a strong monoidal structure on the identity 2-functor, yielding a strong monoidal 2-functor $S : \mathscr{M}^{\mathrm{rev}} \to \mathscr{M}$. Then S preserves biduals and left dualizations. This proves the proposition, observing that the image under S of the right bidual pair n, e in $\mathscr{M}$ is the left bidual pair $\bar{\mathrm{n}}, \bar{\mathrm{e}}$. □

Example 4.3. A particular instance of Proposition 4.2 is the obvious fact that left duals in a monoidal category $\mathscr{C}$ are right duals in the reverse monoidal category $\mathscr{C}^{\mathrm{rev}}$; that is, the category $\mathscr{C}$ equipped with the tensor product $X \otimes^{\mathrm{rev}} Y = Y \otimes X$. To see this, we consider $\mathscr{C}$ as a map pseudomonoid in $\mathscr{V}$-**Mod** as in Chapter 5.1 or [13, Prop. 1.6]; the opposite pseudomonoid is given just by $\mathscr{C}^{\mathrm{rev}}$, while the left and right dualizations are given by the left and right-dual functors respectively.

Another example of Proposition 4.2 is the fact that if $s : H \to H$ is the antipode of a coquasi-Hopf algebra H, then s^{-1} is an antipode for the coquasibialgebra with the same comultiplication as H but with the opposite multiplication. See Section 5.2.

Recall from Section 2.8 that if (A, j, p) is a map pseudomonoid, its opposite pseudomonoid is $(A^\circ, j^{*\circ}, p^{*\circ})$. In Proposition 2.42 we proved that A° is left autonomous if and only if A is so, and expressed the left dualization of the latter in terms of the one of the former.

Corollary 4.4. *If $d : A^\circ \to A$ is a left dualization for (A, j, p) then $d^* : A^{\circ\circ} \to A^\circ$ is a right dualization for the pseudomonoid $(A^\circ, j^{*\circ}, (pc_{A,A})^{*\circ})$.*

Proof. Combine Propositions 2.42 and 4.2. □

Corollary 4.5. *Let A be an autonomous map pseudomonoid in a braided Gray monoid $\mathscr{M}$, and denote by d and $\bar{d}$ the left and right dualizations. Then, $A^\circ \otimes A$ with the pseudomonoid structure described in Lemma 4.1 is autonomous. Moreover, the left dualization is given by*

$$A^\circ \otimes A^{\circ\circ} \xrightarrow{c^*} A^{\circ\circ} \otimes A^\circ \xrightarrow{\bar{d}^{*\circ}\otimes 1} A^\circ \otimes A^\circ \xrightarrow{1\otimes d} A^\circ \otimes A.$$

Proof. The pseudomonoid structure of $A^\circ \otimes A$ in Lemma 4.1 can be obtained as the tensor product of the pseudomonoids $(A^\circ, j^{*\circ}, (pc)^{*\circ})$ and (A, j, p) in the Gray monoid $\mathscr{M}$ with braiding c^*. Note that $\bar{d}^{*\circ}$ is a left dualization for $(A^\circ, j^{*\circ}, (pc)^{*\circ})$ by Corollary 4.4. The result now follows from Corollary 2.41. □

4.1.2 Duals

Let A be an autonomous map pseudomonoid in the (not necessarily braided) Gray-monoid $\mathscr{M}$. In particular, we suppose that A has left bidual $A^\vee$ and right bidual A°. Denote the left and right dualizations by $d : A^\circ \to A$ and $\bar{d} : A^\vee \to A$. Duals in the convolution category $\mathscr{M}(I, A)$ were studied in [16]. If $f : I \to A$ has right adjoint f^* then it has right dual $f^\triangleright$ and left dual $f^\triangleleft$ in $\mathscr{M}(I, A)$ given by

$$f^\triangleright : I \xrightarrow{(f^*)^\vee} A^\vee \xrightarrow{\bar{d}} A \qquad \text{and} \qquad f^\triangleleft : I \xrightarrow{(f^*)^\circ} A^\circ \xrightarrow{d} A. \tag{4.9}$$

Dually, if $f : A \to I$ has left adjoint f^ℓ, then it has right and left dual in $\mathscr{M}(A, I)$ given by

$$f^\triangleright : A \xrightarrow{d^*} A^\circ \xrightarrow{(f^\ell)^\circ} I \qquad \text{and} \qquad f^\triangleleft : A \xrightarrow{\bar{d}^*} A^\vee \xrightarrow{(f^\ell)^\vee} I. \tag{4.10}$$

The evaluation and coevaluation for the left dual $f^\triangleleft$ in $\mathscr{M}(I, A)$ are

$$f^\triangleleft * f \cong p(d \otimes A)(A^\circ \otimes f)(A^\circ \otimes f^*)\mathrm{n} \xrightarrow{p(d\otimes A)(A^\circ\otimes\varepsilon)\mathrm{n}} p(d \otimes A)\mathrm{n} \xrightarrow{\alpha} j$$

and

$$j \xrightarrow{j\eta} jf^*f \cong jf^*(\mathrm{e} \otimes A)(A \otimes \mathrm{n})f \cong j\mathrm{e}(A \otimes A^\circ \otimes f^*)(A \otimes n)f \to$$
$$\xrightarrow{\beta(A\otimes A^\circ\otimes f^*)(A\otimes n)} p(A \otimes d)(A \otimes A^\circ \otimes f^*)(A \otimes \mathrm{n}) \cong f * f^\triangleleft$$

where α, β are the 2-cells giving to d the structure of a left dualization (see Section 2.6), and η, ε are the unit and counit of the adjunction $f \dashv f^*$.

The formulas (4.10) for the duals in $\mathscr{M}(A, I)$ can be deduced from (4.9) applied to the pseudomonoid A° together with the monoidal equivalence $\mathscr{M}(A, I) \simeq \mathscr{M}(I, A^\circ)^{\mathrm{op}}$.

In the case of *map* pseudomonoids it is not only true that maps have duals in $\mathscr{M}(I, A)$ but the converse also holds. We leave to the reader the various dualizations of the following result.

Proposition 4.6. *Let A be a left autonomous map pseudomonoid in $\mathscr{M}$. Then, a 1-cell $f \in \mathscr{M}(I, A)$ has a left dual if and only if it has a right adjoint in $\mathscr{M}$.*

Proof. We only have to prove the direct implication. We know from Proposition 2.17 that the functor $\lambda_I : \mathscr{M}(I, A) \to \mathscr{M}(A, A)^{\theta_I}$ in Definition 2.3 is strong monoidal; therefore the composition of λ_I with the forgetful functor $\mathscr{M}(A, A)^{\theta_I} \to$

$\mathscr{M}(A,A)$ is strong monoidal. Explicitly, there are coherent natural transformations $\lambda_I(g)\lambda_I(f) \cong \lambda_I(f*g)$ and $1_A \cong \lambda_I(j)$. Then we have an adjunction $\lambda_I(f) \dashv \lambda_I(f^{\triangleleft}) : A \to A$, and composing with j we obtain $f \cong (\lambda_I(f))j \dashv j^*\lambda_I(f^{\triangleleft})$. □

Example 4.7. In Chapter 4 we will interpret Proposition 4.6 in terms of comodules for a (coquasi) Hopf algebra H, recovering the following well-known fact. In this case the monoidal bicategory is the bicategory of comodules **Comod**($\mathscr{V}$) and H is a left autonomous pseudomonoid in it (*e.g.*, $\mathscr{V} = \mathbf{Vect}$). Hence a right H-comodule M has a left dual if and only if $M \in \mathscr{V}$ has a dual (*e.g.*, M has finite dimension), by Proposition 5.14.

We shall write $\mathscr{M}_r(I,A)$ for the full subcategory of $\mathscr{M}(I,A)$ determined by the 1-cells with right adjoint (the maps). This is clearly a monoidal subcategory.

Corollary 4.8. *If A is a left autonomous map pseudomonoid in $\mathscr{M}$, then the category $\mathscr{M}_r(I,A)$ is left autonomous. If A is also right autonomous, $\mathscr{M}_r(I,A)$ is autonomous.*

Proof. We know that the left dual of a map f is given by $d(f^{*\circ})$, which is again a map, with right dual given by $f^\circ d^*$. □

Observation 4.9. We will compute the right and left double duals of a 1-cell $f \in \mathscr{M}(A,I)$ with left adjoint, where A is an autonomous map pseudomonoid.

$$f^{\triangleleft\triangleleft} = (f^{\ell\vee}\bar{d}^*)^{\triangleleft} = (f^{\ell\vee}\bar{d}^*)^{\ell\vee}\bar{d}^* \cong (\bar{d}f^\vee)^\vee\bar{d}^* \cong f^{\vee\vee}\bar{d}^\vee\bar{d}^* \tag{4.11}$$

$$f^{\triangleright\triangleright} = (f^{\ell\circ}d^*)^{\triangleright} = (f^{\ell\circ}d^*)^{\ell\circ}d^* \cong (df^\circ)^\circ d^* \cong f^{\circ\circ}d^\circ d^*. \tag{4.12}$$

These isomorphisms are the components of monoidal transformations with respect to the convolution tensor product. In fact (4.11) is just (4.12) in the reverse Gray monoid, so we will only show that the first of them is monoidal.

Consider the following diagram of monoidal categories and monoidal functors.

$$\begin{tikzcd}
\mathscr{M}_\ell(A,I) \arrow[r, "(-)^\ell"] \arrow[dr, "(-)^\vee"'] & \mathscr{M}_r(I,A) \arrow[r, "(-)^\vee"] & \mathscr{M}_\ell(A^\vee,I)^{\mathrm{rev}} \arrow[r, "\bar{d}^*"] \arrow[d, "(-)^\ell"] & \mathscr{M}_\ell(A,I)^{\mathrm{rev}} \arrow[r, "(-)^\ell"] & \mathscr{M}_r(I,A)^{\mathrm{rev}} \arrow[d, "(-)^\vee"'] \\
& \mathscr{M}_r(I,A^\vee)^{\mathrm{rev}} \arrow[r, "1"'] \arrow[ur, "(-)^*"] & \mathscr{M}_r(I,A^\vee)^{\mathrm{rev}} \arrow[r, "(-)^*"'] \arrow[urr, "\bar{d}"] & \mathscr{M}_\ell(A^{\vee\vee},I)^{\mathrm{rev}} \arrow[r, "\bar{d}^\vee"'] & \mathscr{M}_\ell(A^\vee,I)
\end{tikzcd}$$

It is easy to see that this diagram commutes up to canonical monoidal isomorphisms. The resulting monoidal isomorphism postcomposed with the monoidal functor $\mathscr{M}_\ell(\bar{d}^*,I)$ has components (4.11).

Proposition 4.10. *Assume $\mathscr{M}$ is braided with braiding c. If the coevaluation $\mathrm{n} : I \to A^\circ \otimes A$ has right adjoint n^* then any map $f : A \to A$ has a left and a right dual in the convolution monoidal category $\mathscr{M}(A, A)$. Moreover, if we take the left bidual of A as A° with evaluation and coevaluation as in (4.6), the left dual is given by*

$$A \xrightarrow{\bar{d}^*} A^\circ \xrightarrow{1\otimes \mathrm{n}} A^\circ \otimes A^\circ \otimes A \xrightarrow{c\otimes 1} A^\circ \otimes A^\circ \otimes A \xrightarrow{1\otimes 1\otimes f^*} A^\circ \otimes A^\circ \otimes A \xrightarrow{1\otimes \mathrm{n}^*} A^\circ \xrightarrow{d} A. \tag{4.13}$$

Proof. If we prove the existence of a left dual, the existence of a right dual follows automatically by considering the pseudomonoid A in the reverse Gray monoid $\mathscr{M}^{\mathrm{rev}}$.

The equivalence $\mathscr{M}(A, A) \simeq \mathscr{M}(I, A^\circ \otimes A)$ becomes monoidal when we equip $A^\circ \otimes A$ with the pseudomonoid structure of Lemma 4.1. Then, an arrow $f \in \mathscr{M}(A, A)$ has left dual if $\hat{f} = (A^\circ \otimes f)\mathrm{n} \in \mathscr{M}(I, A^\circ \otimes A)$ does so, which is the case if f and n are maps. A left dual of $\hat{f}$ is

$$I \xrightarrow{\hat{f}^{*\circ}} A^\circ \otimes A^{\circ\circ} \xrightarrow{c^*} A^{\circ\circ} \otimes A^\circ \xrightarrow{\bar{d}^{*\circ}\otimes 1} A^\circ \otimes A^\circ \xrightarrow{1\otimes d} A^\circ \otimes A$$

by Corollary 4.5. This 1-cell corresponds to (4.13). $\square$

4.2 Radford's formula

Throughout this section we shall assume that A is an autonomous map pseudomonoid in a braided Gray monoid $\mathscr{M}$ and that the coevaluation $\mathrm{n} : I \to A^\circ \otimes A$ has a right adjoint. Under these conditions, Proposition 4.10 ensures that the identity 1-cell of A has a left and right adjoints in the convolution category $\mathscr{M}(A, A)$. On the other hand, in Section 2.2 we showed that the identity 1-cell has the structure of a monoid in this monoidal category, and then its dual $1^\triangleleft$ has a canonical structure of a $(- * 1)$-algebra. Here $(- * 1)$ is the monad on $\mathscr{M}(A, A)$ given by tensoring with 1 on the right; the $(- * 1)$-algebra structure on $1^\triangleleft$ is

$$1^\triangleleft * 1 \xrightarrow{1^\triangleleft * 1 * \mathrm{coev}} 1^\triangleleft * 1 * 1 * 1^\triangleleft \xrightarrow{1^\triangleleft * \varepsilon * 1^\triangleleft} 1^\triangleleft * 1 * 1^\triangleleft \xrightarrow{\mathrm{ev} * 1^\triangleleft} 1^\triangleleft$$

where $\varepsilon : 1 * 1 = pp^* \to 1$ is the counit of the adjunction $p \dashv p^*$.

By Theorem 2.34, or rather this applied to the autonomous map pseudomonoid $(A^\circ, j^{*\circ}, p^{*\circ})$, there exists a 1-cell $\mathsf{w} : A \to I$, unique up to isomorphism, and an isomorphism of $(-*1)$-algebras

$$1^{\lhd} \cong (j\mathsf{w}) * 1; \tag{4.14}$$

recall that the arrow on the right hand side is isomorphic to $(\mathsf{w} \otimes A)p^*$.

Observation 4.11. The 1-cell w is isomorphic to $(j^*)1^{\lhd}$; this follows from the last paragraph above. Therefore, Proposition 4.10 implies that w is isomorphic to

$$A \xrightarrow{d^*} A^{\circ} \xrightarrow{1\otimes j} A^{\circ} \otimes A \xrightarrow{\mathrm{n}^*} I.$$

Proposition 4.12. *If $\mathsf{w} : A \to I$ has a left adjoint, then it is an invertible object in $\mathscr{M}(A, I)$. Equivalently, $1^{\lhd} : A \to A$ is an equivalence.*

Proof. We use Theorem 2.34 repeatedly. We know from Proposition 4.8 that w has a left and a right dual in $\mathscr{M}(A, I)$ if it has a left adjoint in $\mathscr{M}$. The right dual $1^{\rhd}$ of 1 is a $(1*-)$-algebra, and hence of the form $1*(j\mathsf{v})$ for a unique up to isomorphism $\mathsf{v} : A \to I$. Then, $1 \cong (1^{\lhd})^{\rhd} \cong 1^{\rhd} * j\mathsf{w}^{\rhd} \cong 1 * j(\mathsf{v} * \mathsf{w}^{\rhd})$. It follows that $j^* \cong \mathsf{v} * \mathsf{w}^{\rhd}$ and hence $j^* \cong \mathsf{w} * \mathsf{v}^{\lhd}$. Similarly, $1 \cong (1^{\rhd})^{\lhd} \cong j\mathsf{v}^{\lhd} * 1^{\lhd} \cong j(\mathsf{v}^{\lhd} * \mathsf{w}) * 1$ and $\mathsf{v}^{\lhd} * \mathsf{w} \cong j^*$. ☐

Consider the isomorphism $p^* \cong (p \otimes A)(A \otimes d \otimes A)(A \otimes \mathrm{n})$ that gives to d the structure of a left dualization. It induces an isomorphism between the functors $\mathscr{M}(A, I) \to \mathscr{M}(A, A)^{(1*-)}$ given by

$$f \mapsto (A \otimes f)p^* \tag{4.15}$$

and

$$f \mapsto (A \otimes f)(p \otimes A)(A \otimes d \otimes A)(A \otimes \mathrm{n}) = p(A \otimes df^{\circ}). \tag{4.16}$$

The first functor is the composition of $\mathscr{M}(A, j) : \mathscr{M}(A, I) \to \mathscr{M}(A, A)$ with the free $(1*-)$-algebra functor $\mathscr{M}(A, A) \to \mathscr{M}(A, A)^{(1*-)}$, and hence strong monoidal by a dual of Proposition 2.17. Explicitly, the monoidal structure is induced by the isomorphism $(A \otimes p^*)p^* \cong (p^* \otimes A)p^*$:

$$\begin{aligned}(((A \otimes g)p^*) \otimes A)p^* &\cong (A \otimes g)(A \otimes A \otimes f)(p^* \otimes A)p^* \\ &\cong (A \otimes g)(A \otimes A \otimes f)(A \otimes p^*)p^* \\ &\cong (A \otimes (g * f))p^*\end{aligned}$$

and the isomorphism $(A \otimes j^*)p^* \cong 1$. The second functor is the composition of

$\mathscr{M}(I,d)(-)^\circ : \mathscr{M}(A,I) \to \mathscr{M}(I,A)$ with the monoidal functor $\lambda_I : \mathscr{M}(I,A) \to \mathscr{M}(A,A)^{(1*-)}$ of Definition 2.3 (that is strong monoidal by Proposition 2.17).

Lemma 4.13. *The isomorphism between the functors* (4.15) *and* (4.16) *described above is monoidal.*

Proof. An equivalent formulation of the lemma is that the isomorphism $(A \otimes \mathrm{e})(p^* \otimes A^\circ) \cong p(A \otimes d)$ induces a monoidal transformation between the respective functors $\mathscr{M}(I, A^\circ) \to \mathscr{M}(A,A)^{(1*-)}$, or in other words, that the isomorphism

$$(A^\circ \otimes A \otimes \mathrm{e})(A^\circ \otimes p^* \otimes A^\circ)(\mathrm{n} \otimes A^\circ) \cong (A^\circ \otimes p)(\mathrm{n} \otimes A)d \tag{4.17}$$

induces a monoidal transformation between the respective functors $\mathscr{M}(I, A^\circ) \to \mathscr{M}(I, A^\circ \otimes A)^{(\mathrm{n}*-)}$, where $A^\circ \otimes A$ has the monoidal structure of Lemma 4.1. In the proof of Proposition 2.38 we saw that the monoidal structure of the 1-cell on the left hand side is a consequence of the pseudocomonoid structure of $(A^\circ, j^{*\circ}, p^{*\circ})$, and we equipped the 1-cell on the right hand side with the unique monoidal structure such that (4.17) is a monoidal isomorphism. Therefore the lemma is proved. $\square$

In the course of the proof of Radford's formula will need the following easy result, which we state as a Lemma.

Lemma 4.14. *Let M be a monoid in a monoidal category $\mathscr{V}$ with left duals. Equip the left dual $M^\triangleleft$ of M with its canonical structure of $(-\otimes M)$-algebra and $W \otimes M \otimes W^\triangleleft$ with the obvious monoid structure given by the the one of M together with the evaluation and coevaluation of W.*

1. *The functor $- \otimes W^\triangleleft : \mathscr{V} \to \mathscr{V}$ lifts to a functor between Eilenberg-Moore categories $\mathscr{V}^{(-\otimes M)} \to \mathscr{V}^{(-\otimes W \otimes M \otimes W^\triangleleft)}$. These functors are equivalences when W is invertible.*

2. *If $\xi : M^\triangleleft \to W \otimes M$ is an isomorphism of $(-\otimes M)$-algebras, where the codomain is a free algebra, then the arrow*

$$M^{\triangleleft\triangleleft} \xrightarrow{(\xi^{-1})^\triangleleft} (W \otimes M)^\triangleleft \xrightarrow{\cong} M^\triangleleft \otimes W^\triangleleft \xrightarrow{\xi \otimes W^\triangleleft} W \otimes M \otimes W^\triangleleft$$

is a morphism of monoids.

Proof. (1) The functor $- \otimes W^\triangleleft$ together with the transformation

$$- \otimes W^\triangleleft \otimes W \otimes M \otimes W^\triangleleft \xrightarrow{-\otimes \mathrm{ev} \otimes 1 \otimes 1} - \otimes M \otimes W^\triangleleft. \tag{4.18}$$

is a lax morphism of monads from $(- \otimes M)$ to $(- \otimes W \otimes M \otimes W^{\triangleleft})$. Therefore, $(-\otimes W^{\triangleleft})$ lifts to a functor between the respective categories of algebras. When W is invertible, the functor $(-\otimes W^{\triangleleft})$ is an equivalence and (4.18) is an isomorphism; in other words, $(- \otimes W^{\triangleleft})$ is an equivalence of monads, and hence the induced functor between the categories of algebras is an equivalence.

(2) is a routine exercise. □

Applying Lemma 4.13 to $\mathscr{M}^{\mathrm{rev}}$ we get a monoidal isomorphism between strong monoidal functors $\mathscr{M}(A, I) \to \mathscr{M}(A, A)^{(-*1)}$ with components

$$\rho_f : (jf) * 1 \cong (\bar{d} f^{\vee} j^*) * 1. \tag{4.19}$$

Now we state the main result of this chapter. The idea of the proof has certain similarities to [27, Theorem 3.3].

Theorem 4.15. *Let A be an autonomous map pseudomonoid in a braided Gray monoid and suppose that $1 \in \mathscr{M}(A, A)$ has both a left and a right duals (or equivalently, that $\mathrm{n} : I \to A^{\circ} \otimes A$ has a right adjoint) and the 1-cell $\mathsf{w} : A \to I$ in* (4.14) *has a left adjoint. Then there exists a monoidal isomorphism between strong monoidal endo-functors on $\mathscr{M}(A, I)$ with components*

$$\zeta_f : \mathsf{w}^{\triangleleft} * f^{\triangleleft\triangleleft} * \mathsf{w} \cong f^{\triangleright\triangleright}.$$

Proof. The identity $1 : A \to A$ has a left and a right dual in $\mathscr{M}(A, A)$ by Proposition 4.10; hence w exists, and it is invertible by Proposition 4.12.

There exists a monoidal isomorphism of strong monoidal functors $\mathscr{M}_{\ell}(A, I) \to \mathscr{M}(A, I)$ with components

$$(\bar{d} f^{\vee})^{\triangleleft\triangleleft} \cong \bar{d}(f^{\triangleright\triangleright})^{\vee}; \tag{4.20}$$

this follows from the fact that $(-)^{\vee} : \mathscr{M}(A, I)^{\mathrm{rev}} \to \mathscr{M}(I, A^{\vee})$ is a strong monoidal functor and $\bar{d} : A^{\vee} \to A$ is a strong monoidal 1-cell. From (4.14) we obtain an isomorphism

$$1^{\triangleleft\triangleleft} \cong 1^{\triangleleft} * (j\mathsf{w}^{\triangleleft}) \cong (j\mathsf{w}) * 1 * (j\mathsf{w}^{\triangleleft}); \tag{4.21}$$

moreover, by Lemma 4.14.2, this is an isomorphism of monoids when we equip $(j\mathsf{w}) * 1 * (j\mathsf{w}^{\triangleleft})$ with the monoidal structure induced by that 1_A.

Now consider the monoidal isomorphisms between strong monoidal functors from $\mathscr{M}_\ell(A,I)$ to $\mathscr{M}(A,A)^{(-*1^{\lhd\lhd})}$

$$\rho_f^{\lhd\lhd} : (jf^{\lhd\lhd}) * 1^{\lhd\lhd} \cong (\bar{d}f^\vee j^*)^{\lhd\lhd} * 1^{\lhd\lhd},$$

The isomorphism of monoids (4.21) induces a natural isomorphism of free $(- * (j\mathsf{w}) * 1 * (j\mathsf{w}^\lhd))$-algebras depicted on the bottom of the following diagram,

$$\begin{tikzcd}
(jf^{\lhd\lhd}) * 1^{\lhd\lhd} \arrow[r, "\rho_f^{\lhd\lhd}"] \arrow[d, "\cong"'] & (\bar{d}f^\vee j^*)^{\lhd\lhd} * 1^{\lhd\lhd} \arrow[d, "\cong"] \\
(jf^{\lhd\lhd}) * (j\mathsf{w}) * 1 * (j\mathsf{w}^\lhd) \arrow[r, "\tau_f"] & (\bar{d}f^\vee j^*)^{\lhd\lhd} * (j\mathsf{w}) * 1 * (j\mathsf{w}^\lhd)
\end{tikzcd}$$

and by Lemma 4.14.1 τ_f is of the form $\hat{\tau}_f * (j\mathsf{w}^\lhd)$ for a unique $(- * 1)$-algebra isomorphism

$$\hat{\tau}_f : (jf^{\lhd\lhd}) * (j\mathsf{w}) * 1 \cong (\bar{d}f^\vee j^*)^{\lhd\lhd} * (j\mathsf{w}) * 1 \tag{4.22}$$

natural in f. We can form, then, an isomorphism

$$j(\mathsf{w}^\lhd * f^{\lhd\lhd} * \mathsf{w}) * 1 \cong jf^{\rhd\rhd} * 1 \tag{4.23}$$

between functors $\mathscr{M}_\ell(A,I) \to \mathscr{M}(A,A)^{(-*1)}$ given by the composition

$$\begin{aligned} j\mathsf{w}^\lhd * (jf^{\lhd\lhd}) * j\mathsf{w} * 1 &\xrightarrow{j\mathsf{w}^\lhd * \hat{\tau}_f} j\mathsf{w}^\lhd * ((\bar{d}f)^{\vee\lhd\lhd} j^*) * j\mathsf{w} * 1 \xrightarrow{\cong} j\mathsf{w}^\lhd * j\mathsf{w} * ((\bar{d}f)^{\vee\lhd\lhd} j^*) * 1 \\ &\xrightarrow{\mathrm{ev}*(4.20)*1} (\bar{d}(f^{\rhd\rhd})^\vee j^*) * 1 \xrightarrow{\rho^{-1}_{f^{\rhd\rhd}}} jf^{\rhd\rhd} * 1. \end{aligned} \tag{4.24}$$

The unnamed isomorphism is the obvious one induced by $((\bar{d}f^\vee)^{\lhd\lhd} j^*) * j\mathsf{w} \cong ((\bar{d}f^\vee)^{\lhd\lhd} j^*)\mathsf{w} \cong j\mathsf{w} * ((\bar{d}f^\vee)^{\lhd\lhd} j^*)$. Finally, by Theorem 2.34, (4.23) is the image under $\mathscr{M}(A,I) \to \mathscr{M}(A,A)^{(-*1)}$ of a unique isomorphism $\zeta_f : \mathsf{w}^\lhd * f^{\lhd\lhd} * \mathsf{w} \cong f^{\rhd\rhd}$.

Now we shall prove that the isomorphism ζ_f monoidal. We have to prove that the following diagrams commute.

$$\begin{tikzcd}
\mathsf{w}^\lhd * f^{\lhd\lhd} * \mathsf{w} * \mathsf{w}^\lhd * g^{\lhd\lhd} * \mathsf{w} \arrow[r, "\cong"] \arrow[d, "\zeta_f * \zeta_g"'] & \mathsf{w}^\lhd * f^{\lhd\lhd} * g^{\lhd\lhd} * \mathsf{w} \arrow[r, "\cong"] & \mathsf{w}^\lhd * (f*g)^{\lhd\lhd} * \mathsf{w} \arrow[d, "\zeta_{f*g}"] \\
f^{\rhd\rhd} * g^{\rhd\rhd} \arrow[rr, "\cong"] & & (f*g)^{\rhd\rhd}
\end{tikzcd} \tag{4.29}$$

$$
\begin{tikzcd}[column sep=small]
j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}j\mathsf{w}^{\triangleleft}{*}jg^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}1 \arrow[r] \arrow[d, "j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}j\mathsf{w}^{\triangleleft}{*}\hat{\tau}_g"] & j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}jg^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}1 \arrow[r] \arrow[d, "j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}\hat{\tau}_g"] & j\mathsf{w}^{\triangleleft}{*}j(f{*}g)^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}1 \arrow[dd, "j\mathsf{w}^{\triangleleft}{*}\hat{\tau}_{f*g}"] \\
j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}j\mathsf{w}^{\triangleleft}{*}(\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}{*}1 \arrow[r, "\cong"] \arrow[d, "\cong"] & j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}(\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}{*}1 \arrow[d, "\cong"] & \\
j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}j\mathsf{w}^{\triangleleft}{*}j\mathsf{w}{*}(\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}1 \arrow[d, "\cong"] & j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}(\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}1 \arrow[dl, equal] \arrow[d, "\cong"] \arrow[r, phantom, "(4.27)"] & j\mathsf{w}^{\triangleleft}{*}(\bar{d}(f{*}g)^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}{*}1 \arrow[dd, "\cong"] \\
j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}(\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}1 \arrow[dd, "j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}\rho^{-1}_{g^{\triangleright\triangleright}}"] & (\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}^{\triangleleft}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}1 \arrow[d, "(\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}^{\triangleleft}\hat{\tau}_f"] & \\
 & (\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}^{\triangleleft}{*}(\bar{d}f^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}{*}1 \arrow[d, "\cong"] & (\bar{d}(f{*}g)^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}^{\triangleleft}{*}j\mathsf{w}{*}1 \arrow[dd, "\cong"] \\
j\mathsf{w}^{\triangleright}{*}jf^{\triangleleft\triangleleft}{*}j\mathsf{w}{*}jg^{\triangleright\triangleright}{*}1 \arrow[dddd, "\zeta_f{*}jg^{\triangleright\triangleright}{*}1"] \arrow[r, phantom, "(+)"] & (\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}(\bar{d}f^{\vee\triangleleft\triangleleft}j^*){*}j\mathsf{w}^{\triangleleft}{*}j\mathsf{w}{*}1 \arrow[d, "\cong"] \arrow[ur, "\cong"] & \\
 & (\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}(\bar{d}f^{\vee\triangleleft\triangleleft}j^*){*}1 \arrow[r, "\cong"] \arrow[d, "(\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}\rho^{-1}_{f^{\triangleright\triangleright}}"] & (\bar{d}(f{*}g)^{\vee\triangleleft\triangleleft}j^*){*}1 \arrow[ddd, "\rho^{-1}_{f*g}"] \\
 & (\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}jf^{\triangleright\triangleright}{*}1 \arrow[d, "\cong"] & \\
 & jf^{\triangleright\triangleright}{*}(\bar{d}g^{\vee\triangleleft\triangleleft}j^*){*}1 \arrow[dl, "jf^{\triangleright\triangleright}{*}\rho^{-1}_{g^{\triangleright\triangleright}}"] \arrow[r, phantom, "(4.25)"] & \\
jf^{\triangleright\triangleright}{*}jg^{\triangleright\triangleright}{*}1 \arrow[rr, "\cong"'] & & j(f{*}g)^{\triangleright\triangleright}{*}1
\end{tikzcd}
$$

Figure 4.1: Proof of the monoidality of ζ.

$$
\begin{tikzcd}
(jf*1)(jg*1) \arrow[r, "\cong"] \arrow[ddd, "\rho_f\rho_g"'] & jg*jf*1 \arrow[r, "\cong"] \arrow[d, "jg*\rho_f"] & j(g*f)*1 \arrow[ddd, "\rho_{g*f}"] \\
& jg*(\bar{d}f^{\vee}j^*)*1 \arrow[d, "\cong"] & \\
& (\bar{d}f^{\vee}j^*)*jg*1 \arrow[d, "(\bar{d}f^{\vee}j^*)*\rho_g"] & \\
(\bar{d}f^{\vee}j^**1)(\bar{d}g^{\vee}j^**1) \arrow[r, "\cong"] & (\bar{d}f^{\vee}j^*)*(\bar{d}g^{\vee}j^**1) \arrow[r, "\cong"] & \bar{d}(g*f)^{\vee}j^**1
\end{tikzcd}
\tag{4.25}
$$

$$
\begin{tikzcd}
jg^{\lhd\lhd}*jf^{\lhd\lhd}*1^{\lhd\lhd} \arrow[r, "\cong"] \arrow[d, "jg^{\lhd\lhd}*\rho_f^{\lhd\lhd}"'] & j(g*f)^{\lhd\lhd}*1^{\lhd\lhd} \arrow[ddd, "\rho_{g*f}^{\lhd\lhd}"] \\
jg^{\lhd\lhd}*\bar{d}f^{\vee\lhd\lhd}j^**1^{\lhd\lhd} \arrow[d, "\cong*1^{\lhd\lhd}"'] & \\
\bar{d}f^{\vee\lhd\lhd}j^**jg^{\lhd\lhd}*1^{\lhd\lhd} \arrow[d, "\bar{d}f^{\vee\lhd\lhd}j^**\rho_g^{\lhd\lhd}"'] & \\
\bar{d}f^{\vee\lhd\lhd}j^**\bar{d}g^{\vee\lhd\lhd}j^**1^{\lhd\lhd} \arrow[r, "\cong"] & \bar{d}(g*f)^{\vee\lhd\lhd}j^**1^{\lhd\lhd}
\end{tikzcd}
\tag{4.26}
$$

$$
\begin{tikzcd}
jg^{\lhd\lhd}*jf^{\lhd\lhd}*j\mathsf{w}*1 \arrow[r, "\cong*j\mathsf{w}*1"] \arrow[d, "jg^{\lhd\lhd}*\hat{\tau}_f"'] & j(g*f)^{\lhd\lhd}*j\mathsf{w}*1 \arrow[ddd, "\hat{\tau}_{g*f}"] \\
jg^{\lhd\lhd}*\bar{d}f^{\vee\lhd\lhd}j^**j\mathsf{w}*1 \arrow[d, "\cong*j\mathsf{w}*1"'] & \\
\bar{d}f^{\vee\lhd\lhd}j^**jg^{\lhd\lhd}*j\mathsf{w}*1 \arrow[d, "df^{\vee\lhd\lhd}j^**\hat{\tau}_g"'] & \\
df^{\vee\lhd\lhd}j^**dg^{\vee\lhd\lhd}j^**j\mathsf{w}*1 \arrow[r, "\cong*j\mathsf{w}*1"] & \bar{d}(g*f)^{\lhd\lhd\vee}*j\mathsf{w}*1
\end{tikzcd}
\tag{4.27}
$$

Figure 4.2: Proof of the monoidality of ζ.

$$
\begin{tikzcd}
 & & j\mathsf{w}^{\lhd} * jj^{*\lhd\lhd} * j\mathsf{w} * 1 \arrow[d, "j\mathsf{w}^{\lhd} * \hat{\tau}_{j^*}"] \\
 & j\mathsf{w}^{\lhd} * jj^{*} * j\mathsf{w} * 1 \arrow[ur, "j\mathsf{w}^{\lhd} * j\iota_{\ell} * j\mathsf{w} * 1"] \arrow[r, "\cong"] & j\mathsf{w}^{\lhd} * (\bar{d}j^{*\vee\lhd\lhd} j^{*}) * j\mathsf{w} * 1 \arrow[d, "\cong"] \\
jj^{*} * 1 \arrow[ur, "\cong"] \arrow[drr, "(\bar{d}\iota_r^{\vee} j^{*} * 1)\rho_{j^*}"] \arrow[ddrr, "j\iota_r * 1"'] & & (\bar{d}j^{*\rhd\rhd\vee} j^{*}) * j\mathsf{w}^{\lhd} * j\mathsf{w} * 1 \arrow[d, "\cong"] \\
 & & (\bar{d}j^{*\rhd\rhd\vee} j^{*}) * 1 \arrow[d, "\rho^{-1}_{j^{*\rhd\rhd}}"] \\
 & & jj^{*\rhd\rhd} * 1
\end{tikzcd}
\tag{4.28}
$$

Figure 4.3: Proof of the monoidality of ζ.

$$
\begin{tikzcd}
 & & j^{*} \arrow[dl, "\cong"'] \arrow[ddr, "\iota_r"] & \\
 & \mathsf{w}^{\lhd} * j^{*} * \mathsf{w} \arrow[dl, "\mathsf{w}^{\lhd} * \iota_{\ell} * \mathsf{w}"'] & & \\
\mathsf{w}^{\lhd} * j^{*\lhd\lhd} * \mathsf{w} \arrow[rrr, "\zeta_{j^*}"'] & & & j^{*\rhd\rhd}
\end{tikzcd}
\tag{4.30}
$$

Here ι_ℓ and ι_r denote the canonical isomorphisms between j^* (which is the unit of the convolution tensor product in $\mathscr{M}(A, I)$) and its double left dual and double right dual, respectively. We will deal with (4.29) first. The image of this diagram under the equivalence $\mathscr{M}(A, I) \to \mathscr{M}(A, A)^{(-*1)}$ is, by definition of ζ, the diagram in Figure 4.1. In it, the diagrams left blank commute trivially and the one marked with (+) does by definition of ζ_f; so the diagram in Figure 4.1 commutes if (4.25) and (4.27) in Figure 4.2 do so. The exterior rectangle in (4.25) in Figure 4.2 commutes because ρ is monoidal, while the square on the left hand side does by direct verification; hence the square on the right hand side also commutes. Finally, the commutativity of (4.27) follows easily from (4.26).

We now turn our attention to (4.30). Its image under

$$\mathscr{M}(A, I) \to \mathscr{M}(A, A)^{(-*1)}$$

is the exterior diagram (4.28) in Figure 4.3, where the internal pentagon commutes trivially, the lower triangle commutes by naturality of ρ, and the upper triangle by naturality of ι_ℓ and definition of $\hat{\tau}_{j^*}$ in terms of $\rho^{\lhd\lhd}_{j^*}$. $\square$

Our next result is a special case of Theorem 4.15 that will be useful when we consider our main applications.

We remind the reader that we are considering the braided Gray monoid $\mathscr{M}$ as autonomous with left biduals induced by the right biduals via the braiding.

Corollary 4.16. *Suppose $\mathscr{M}$ is an autonomous braided Gray monoid and A is an autonomous map pseudomonoid in $\mathscr{M}$. Suppose further that the coevaluation* $\mathrm{n} : I \longrightarrow A^{\circ} \otimes A$ *is a map.*

1. *If the unique up to isomorphism $\mathsf{w} : A \longrightarrow I$ such that $j\mathsf{w} * 1 \cong 1^{\triangleleft}$ has a left adjoint, then there exists a monoidal isomorphism of strong monoidal endo-functors of $\mathscr{M}_{\ell}(A, I)$ with components*
$$(A \xrightarrow{p^*} A \otimes A \xrightarrow{\mathsf{w}^{\triangleleft} \otimes 1} A \xrightarrow{p^*} A \otimes A \xrightarrow{1 \otimes \mathsf{w}} A \xrightarrow{d\bar{d}^*} A \xrightarrow{f} I) \cong (A \xrightarrow{\bar{d}d^*} A \xrightarrow{f} I). \tag{4.31}$$

2. *If the unique up to isomorphism $\mathsf{w} : A \longrightarrow I$ such that $1 * j\mathsf{w} \cong 1^{\triangleright}$ has left adjoint, then there exists a monoidal isomorphism of strong monoidal endo-functors of $\mathscr{M}_{\ell}(A, I)$ with components*
$$(A \xrightarrow{p^*} A \otimes A \xrightarrow{1 \otimes \mathsf{w}^{\triangleright}} A \xrightarrow{p^*} A \otimes A \xrightarrow{\mathsf{w} \otimes 1} A \xrightarrow{\bar{d}d^*} A \xrightarrow{f} I) \cong (A \xrightarrow{d\bar{d}^*} A \xrightarrow{f} I). \tag{4.32}$$

3. *If the unique up to isomorphism $\mathsf{w} : I \longrightarrow A$ such that $1 * \mathsf{w}j^* \cong 1^{\triangleright}$ has a right adjoint, then there exists a monoidal isomorphism of strong monoidal endo-functors of $\mathscr{M}_r(I, A)$ with components*
$$(I \xrightarrow{f} A \xrightarrow{\bar{d}d^*} A \xrightarrow{\mathsf{w} \otimes 1} A \otimes A \xrightarrow{p} A \xrightarrow{1 \otimes \mathsf{w}^{\triangleright}} A \otimes A \xrightarrow{p} A) \cong (I \xrightarrow{f} A \xrightarrow{d\bar{d}^*} A). \tag{4.33}$$

4. *If the unique up to isomorphism $\mathsf{w} : I \longrightarrow A$ such that $\mathsf{w}j^* * 1 \cong 1^{\triangleleft}$ has a right adjoint, then there exists a monoidal isomorphism of strong monoidal endo-functors of $\mathscr{M}_r(I, A)$ with components*
$$(I \xrightarrow{f} A \xrightarrow{d\bar{d}^*} A \xrightarrow{1 \otimes \mathsf{w}} A \otimes A \xrightarrow{p} A \xrightarrow{\mathsf{w}^{\triangleleft} \otimes 1} A \otimes A \xrightarrow{p} A) \cong (I \xrightarrow{f} A \xrightarrow{\bar{d}d^*} A). \tag{4.34}$$

Proof. First of all, the assumption that n is a map ensures the existence of $1^{\triangleleft}$ and $1^{\triangleright}$ in $\mathscr{M}(A, A)$.

Part 1 is simply a restatement of Theorem 4.15 using $\bar{d} \cong d^{\vee}$ and $d \cong \bar{d}^{\circ}$, while 2 is just 1 applied to the autonomous pseudomonoid A in $\mathscr{M}^{\mathrm{rev}}$ (so that d and $\bar{d}$ are interchanged).

To prove 3, we first write part 1 for the case of the autonomous map pseudomonoid $(A^\circ, j^{*\circ}, p^{*\circ})$. Applying $(-)^\circ$ to the hypothesis $1 * \mathsf{w}j^* \cong 1^\triangleright$ we get $j^{*\circ}\mathsf{w}^\circ * 1_{A^\circ} \cong 1^\triangleleft_{A^\circ}$. Recalling from Proposition 2.42 that A° has both left and right dualizations $d^{*\circ}$ and $\bar{d}^{*\circ}$ respectively, we obtain from part 1 monoidal isomorphisms

$$gd^{*\circ}\bar{d}^\circ(A^\circ\mathsf{w}^\circ)p^\circ(\mathsf{w}^{\circ\triangleleft} \otimes A^\circ)p^\circ \cong g\bar{d}^{*\circ}d^\circ$$

with $g \in \mathscr{M}_\ell(A^\circ, I)$; putting $g = f^\circ$, with $f \in \mathscr{M}_r(I, A)$, and using that $(-)^\circ$ is a monoidal biequivalence, we obtain (4.33).

Finally, 4 is obtained from 3 by considering the reverse Gray monoid. $\square$

Example 4.17. The results above when applied to a coquasi Hopf algebra give a formula that generalises the classical Radford's formula for finite-dimensional Hopf algebras. This is explained in Section 5.2.5.

4.3 Unimodular pseudomonoids

Recall from the comments at the beginning of the chapter that a right cointegral for a bialgebra H is a functional $\phi : H \to k$ such that the convolution product $\phi * \psi = \psi(1)\phi$ for all $\psi : H \to k$. If H is a finite-dimensional Hopf algebra, there exists a unique up to scalars *modular element* $a \in H$ such that $\psi * \phi = \psi(a)\psi$ for all $\psi : H \to k$. The Hopf algebra H is *unimodular* if $a = 1$. This is equivalent to saying that the right cointegrals coincide with the left cointegrals (functionals χ such that $\psi * \chi = \psi(1)\chi$ for all $\psi : H \to k$).

If one takes the point of view that autonomous pseudomonoids are generalised Hopf algebras, there must be a corresponding notion of unimodularity for them. In this section we introduce unimodular autonomous map pseudomonoids and deduce the first consequences of the definition. One classical result on Hopf algebras establishes that semisimple Hopf algebras are unimodular. We address the relationship between semisimplicity and unimodularity in Section 7.6.

Throughout this section A will be an autonomous map pseudomonoid, for which the coevaluation $\mathrm{n} : I \to A^\circ \otimes A$ is a map.

As we mentioned before, and will explain in Section 5.2.5, the 1-cell $\mathsf{w} : A \to I$ of (4.14) plays the role of the modular element. This 1-cell is defined by the property that $1^\triangleleft \cong j\mathsf{w} * 1$ in $\mathscr{M}(A, A)^{(-*1)}$. This leads us to the following definition.

Definition 4.1. A *unimodularity isomorphism* on A is an isomorphism between the 1-cell w and the unit of the convolution product, j^*. We say that A is *unimodular* when it is equipped with a unimodularity isomorphism.

Note that unimodularity isomorphisms are in bijection with isomorphisms of $(- * 1)$-algebras $1^{\triangleleft} \cong 1$.

We show below that this condition is related to the notion of Frobenius monad. A *Frobenius monad* in a bicategory $\mathscr{K}$ is a monad (r, η, μ) in $\mathscr{K}$ equipped with a 2-cell $\sigma : r \Rightarrow 1$ such that $\sigma\mu : rr \Rightarrow 1$ is the counit of an adjunction $r \dashv r$. In [82] several conditions equivalent to this definition are given. The Frobenius structure reflects on Eilenberg-Moore constructions in the following way. If r has an Eilenberg-Moore construction $f \dashv u : X^r \to X$, then there exists a bijection between Frobenius structures on r and adjunctions $u \dashv f$.

In the special case when r has left adjoint r^ℓ, the composition

$$r^\ell r \xrightarrow{r^\ell r\eta} r^\ell r r r^\ell \xrightarrow{r^\ell \mu r^\ell} r^\ell r r^\ell \xrightarrow{\varepsilon r^\ell} r^\ell$$

equips r^ℓ with the structure of a r-algebra in $\mathscr{K}^{\mathrm{op}}$.

The proof of the following lemma is a standard calculation

Lemma 4.18. *Suppose (r, η, μ) is a monad in $\mathscr{K}$ and the 1-cell r has left adjoint r^ℓ. Then, there is a bijection between the Frobenius structures on r and isomorphisms of r-algebras in $\mathscr{K}^{\mathrm{op}}$ between r^ℓ and r.*

A *Frobenius monoid* in a monoidal category is just a Frobenius monad in the respective one-object bicategory.

Theorem 4.19. *There exist a bijection between the following structures.*

1. *Unimodularity isomorphisms on A.*
2. *Frobenius structures on the monoid $1 : A \to A$ in the convolution monoidal category $\mathscr{M}(A, A)$.*

Proof. By Definition 4.1, we have to establish a bijection between isomorphisms of $(- * 1)$-algebras $1^{\triangleleft} \cong 1$ and Frobenius structures on $1 \in \mathscr{M}(A, A)$. This is exactly what Lemma 4.18 does. $\square$

Proposition 4.20. 1. *If A is unimodular and $j : I \to A$ has a left adjoint, then $j^* \dashv j$.*

2. *If $j^* \dashv j$, then A is unimodular.*

Proof. 1. Assume $f \dashv j$. By Observation 4.11, there is an isomorphism $\mathrm{n}^*(A^\circ \otimes j)d^* \cong j^*$. Taking left adjoints, we get an isomorphism $j \cong d(A^\circ \otimes f)\mathrm{n}$. Together with the isomorphism $(j^*p \otimes A)(A \otimes d \otimes A)(A \otimes \mathrm{n}) \cong 1_A$ that can be easily deduced from [13, Proposition 1.2], we get

$$\begin{aligned} f &\cong f((j^*p) \otimes A)(A \otimes d \otimes A)(A \otimes \mathrm{n}) \\ &\cong j^*p(A \otimes ((A \otimes f)(d \otimes A)\mathrm{n})) \\ &\cong j^*p(A \otimes j) \\ &\cong j^*. \end{aligned}$$

(2) If $j^* \dashv j$, by Observation 4.11, w is right adjoint to $(A \otimes j^*)(d \otimes A)\mathrm{n}$. This 1-cell is isomorphic to $(A \otimes j^*)p^*j \cong j$, and hence $\mathsf{w} \cong j^*$. □

Before our next result, we recall the monads t, s on $A^\circ \otimes A$. In Observation 3.9 we saw that there are two strong monoidal functors $L, R : \mathscr{M}(I, A^\circ A) \to \mathscr{M}(A^\circ A, A^\circ A)$, where the domain has the convolution monoidal structure and the codomain the composition monoidal structure, such that $L(\mathrm{n}) = t$ and $R(\mathrm{n}) = s$. In this way we showed the existence of an invertible distributive law between t and s. When A is left autonomous, a lifting $\tilde{s} : A \to A$ of s to the Eilenberg-Moore object of t (which can be taken to be A), is a monad with Eilenberg-Moore construction the lax centre of A (Theorem 3.14).

Proposition 4.21. *For a unimodular autonomous map pseudomonoid, the corresponding monads t, s are Frobenius. Moreover, the Frobenius structure on s lifts to $\tilde{s}$.*

Proof. The strong monoidal functors L, R of the paragraph above preserve Frobenius monoids, and n is Frobenius by Theorem 4.19. Since everything in the image of the functor R commutes with the action of $t = L(\mathrm{n})$, the Frobenius structure of s lifts to $\tilde{s}$. □

Corollary 4.22. *If A is a unimodular autonomous pseudomonoid, then there exist an adjunction $p^* \dashv p$.*

Proof. The monad t is Frobenius by the proposition above, and as such, the left adjoint to its Eilenberg-Moore construction is also its right adjoint. By Corollary 2.37, $p^* : A \to A \otimes A$ is an Eilenberg-Moore construction for t, and hence Frobenius structures on t are in bijection with adjunctions $p^* \dashv p$. □

Corollary 4.23. *If A is unimodular and the (lax)centre of A exists, then the universal $Z(A) \to A$ is not only monadic, but the generated monad is Frobenius.*

Proof. Since A is a left and right autonomous pseudomonoid, if a lax centre of A exists, it is also a centre for A (Theorem 3.16). Now, if $Z(A) \to A$ is a (lax) centre, then it is an Eilenberg-Moore construction for the monad $\tilde{s}$ on A, by Theorem 3.14. But $\tilde{s}$ is a Frobenius monad by Proposition 4.21. $\square$

Chapter 5

Monoidal categories and coquasi-Hopf algebras

The present chapter comprises the first two main applications of the theory of autonomous pseudomonoids developed so far. The first application is to (pro)monoidal enriched categories and the second to coquasi-Hopf algebras.

In section 5.1, after interpreting the fundamental theorem of Hopf modules (Theorem 2.34) and Hopf module constructions in the case of the bicategory of $\mathscr{V}$-modules $\mathscr{V}$-**Mod**, we study centres and lax centres of (pro)monoidal categories. One consequence of our results is that if a promonoidal $\mathscr{V}$-category $\mathscr{A}$ has a lax centre $Z_\ell\mathscr{A}$, then there is a canonical equivalence $[Z_\ell\mathscr{A},\mathscr{V}] \simeq Z_\ell[\mathscr{A},\mathscr{V}]$, where the $\mathscr{V}$-category on the right hand side is the lax centre of $[\mathscr{A},\mathscr{V}]$ in $\mathscr{V}$-**Cat**. We also show that if $\mathscr{A}$ is a left autonomous map pseudomonoid (*e.g.*, a left autonomous monoidal $\mathscr{V}$-category), $Z_\ell\mathscr{A}$ does exists and can be given explicitly.

Section 5.2 deals with the examples more directly related to Hopf algebra theory by means of the monoidal bicategory of comodules **Comod**($\mathscr{V}$). We explain why Theorem 2.34 generalises the fundamental theorem of Hopf modules for (coquasi) Hopf algebras, and describe the Hopf module construction of a coquasi-Hopf algebra as a bicomodule. Then we relate the centre construction for pseudomonoids with the Drinfel'd or quantum double of a finite dimensional (coquasi) Hopf algebra. In fact, we show that the Drinfel'd double is exactly the centre construction of the autonomous map pseudomonoid in **Comod**($\mathscr{V}$) associated to the coquasi Hopf algebra.

5.1 $\mathscr{V}$-categories and $\mathscr{V}$-modules

The concept of $\mathscr{V}$-module, also called bimodule, distributor or profunctor, arose in connection with enriched categories [56], [4]. The bicategory of $\mathscr{V}$-categories and $\mathscr{V}$-modules can be introduced in several different ways. See for example [6, 78, 16]. It can be viewed as an extension of the 2-category $\mathscr{V}$-**Cat** with good properties, and experience indicates that it is the right environment to study enriched categories and related structures.

We take a short definition of the bicategory of $\mathscr{V}$-modules, but at the same time we must ask for some properties (such as completeness) on the base monoidal category $\mathscr{V}$ that are not necessary in other approaches. However, in our examples $\mathscr{V}$ always has these properties, necessary to develop the usual theory of enriched categories as in [42].

Let $\mathscr{V}$ be a complete and cocomplete closed symmetric monoidal category. There is a bicategory $\mathscr{V}$-**Mod** whose objects are the small $\mathscr{V}$-categories and hom-categories $\mathscr{V}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B}) = [\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}, \mathscr{V}]_0$, the category of $\mathscr{V}$-functors from the tensor product of the $\mathscr{V}$-categories $\mathscr{A}^{\mathrm{op}}$ and $\mathscr{B}$ to $\mathscr{V}$, and $\mathscr{V}$-natural transformations between them. Objects of this category are called $\mathscr{V}$-modules and arrows morphisms of $\mathscr{V}$-modules. The composition of two $\mathscr{V}$-modules $M : \mathscr{A} \to \mathscr{B}$ and $N : \mathscr{B} \to \mathscr{C}$ is given by $(NM)(a,c) = \int^x N(x,c) \otimes M(a,x)$. The identity module $1_{\mathscr{A}}$ is given by $1_{\mathscr{A}}(a,a') = \mathscr{A}(a,a')$. Our convention is that a $\mathscr{V}$-module from $\mathscr{A}$ to $\mathscr{B}$ as a $\mathscr{V}$-functor $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathscr{V}$.

There is a pseudofunctor $(-)_* : \mathscr{V}\text{-}\mathbf{Cat}^{\mathrm{co}} \to \mathscr{V}\text{-}\mathbf{Mod}$ which is the identity on objects and on hom-categories $[\mathscr{A},\mathscr{B}]_0^{\mathrm{op}} \to [\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}, \mathscr{V}]_0$ sends a $\mathscr{V}$-functor F to the $\mathscr{V}$-functor $F_*(a,b) = \mathscr{B}(F(a),b)$. Moreover, the $\mathscr{V}$-module F_* has right adjoint F^* given by $F^*(b,a) = \mathscr{B}(b,F(a))$. The pseudofunctor $(-)_*$ is easily shown to be strong monoidal and symmetry-preserving.

The tensor product of $\mathscr{V}$-categories induces a structure of a monoidal bicategory on $\mathscr{V}$-**Mod**. Moreover, the usual symmetry of $\mathscr{V}$-**Cat** together with the symmetry of $\mathscr{V}$ induce a structure of symmetric monoidal bicategory on $\mathscr{V}$-**Mod**, or rather, induce a symmetry in the sense of [16] in any Gray monoid monoidally equivalent to $\mathscr{V}$-**Mod**.

Example 5.1 (Promonoidal enriched categories). A pseudomonoid in $\mathscr{V}$-**Mod** is a promonodial $\mathscr{V}$-category [11]. The pseudomonoid structure amounts to a multiplication and a unit $\mathscr{V}$-functors $P : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{V}$ and $J : \mathscr{A} \to \mathscr{V}$ together with associativity and unit $\mathscr{V}$-natural constraints satisfying axioms. Any monoidal $\mathscr{V}$-category can be thought of as a promonoidal $\mathscr{V}$-category, in fact a map pseudomonoid, by using the monoidal pseudofunctor $(-)_* : \mathscr{V}\text{-}\mathbf{Cat}^{\mathrm{co}} \to \mathscr{V}\text{-}\mathbf{Mod}$; explicitly, if $\mathscr{A}$ is a monoidal $\mathscr{V}$-category, then the induced promonoidal structure is given by $P(a,b;c) = \mathscr{A}(b \otimes a, c)$ and $J(a) = \mathscr{A}(I,a)$.

One of the many pleasant properties of $\mathscr{V}$-$\mathbf{Mod}$ is that it has right liftings. If $M : \mathscr{B} \to \mathscr{C}$ and $N : \mathscr{A} \to \mathscr{C}$ are $\mathscr{V}$-modules, a right lifting of N through M is given by the formula ${}^{M}N(a,b) = \int_{c\in\mathscr{C}} [M(b,c), N(a,c)]$. As explained in Section 3.3, the existence of right liftings endows each hom-category $\mathscr{V}$-$\mathbf{Mod}(I,\mathscr{A})$ with a canonical structure of a $\mathscr{V}$-$\mathbf{Mod}(I,I)$-category, where I is the trivial $\mathscr{V}$-category. Therefore, each $\mathscr{V}$-$\mathbf{Mod}(I,\mathscr{A})$ is canonically a $\mathscr{V}$-category via the monoidal isomorphism $\mathscr{V}\text{-}\mathbf{Mod}(I,I) \cong \mathscr{V}$. This is exactly the usual $\mathscr{V}$-category structure of $[\mathscr{A},\mathscr{V}]$. In fact, each hom-category $\mathscr{V}$-$\mathbf{Mod}(\mathscr{A},\mathscr{B})$ is canonically a $\mathscr{V}$-category, in a way such that the equivalence $\mathscr{V}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B}) \simeq \mathscr{V}\text{-}\mathbf{Mod}(I, \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B})$ is a $\mathscr{V}$-functor.

Another feature of $\mathscr{V}$-$\mathbf{Mod}$ we will need is the existence of Kleisli and Eilenberg-Moore constructions for monads. The existence of the former was shown in [77]. Here we recall the explicit construction for later use. If (M,η,μ) is a monad in $\mathscr{V}$-$\mathbf{Mod}$ on $\mathscr{A}$, $\mathsf{Kl}(M)$ has the same objects as $\mathscr{A}$ and homs $\mathsf{Kl}(M)(a,b) = M(a,b)$. Composition is given by

$$M(b,c) \otimes M(a,b) \to \int^{b\in\mathscr{A}} M(b,c) \otimes M(a,b) \xrightarrow{\mu_{a,c}} M(a,c)$$

and the units by $I \xrightarrow{\mathrm{id}} \mathscr{A}(a,a) \xrightarrow{\eta_{a,a}} M(a,a)$. One can verify that the $\mathscr{V}$-module K_* induced by the $\mathscr{V}$-functor $K : \mathscr{A} \to \mathsf{Kl}(M)$ given by the identity on objects and by $\eta_{a,b} : \mathscr{A}(a,b) \to M(a,b)$ on homs has the universal property of the Kleisli construction. It is not hard to see that K^* is an Eilenberg-Moore construction for M.

5.1.1 Hopf modules for autonomous (pro)monoidal enriched categories

We already established our notations and conventions regarding the bicategory of $\mathscr{V}$-modules in Examples 2.1 and 5.1. Next we show how the results on Hopf modules specialise to the bicategory of $\mathscr{V}$-modules, and give explicit descriptions of the main constructions. Although these descriptions carry over to arbitrary left autonomous map pseudomonoids, here we will concentrate on the simpler case of the left autonomous monoidal $\mathscr{V}$-categories $\mathscr{A}$.

The opmonoidal monad $T : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ defined in Section 2.4 is given as a $\mathscr{V}$-module by $T(a, b; c, d) = \int^x \mathscr{A}(b \otimes x, d) \otimes \mathscr{A}(c, a \otimes x)$. The multiplication is has components

$$\begin{aligned} T^2(a, b; c, d) &= \textstyle\int^{u,v} \int^x \mathscr{A}(v \otimes x, d) \otimes \mathscr{A}(c, u \otimes x) \otimes \int^y \mathscr{A}(b \otimes y, v) \otimes \mathscr{A}(u, a \otimes y) \\ &\cong \textstyle\int^{x,y} \mathscr{A}((b \otimes y) \otimes x, d) \otimes \mathscr{A}(c, (a \otimes y) \otimes x) \\ &\cong \textstyle\int^{x,y} \mathscr{A}(b \otimes (y \otimes x), d) \otimes \mathscr{A}(c, a \otimes (y \otimes x)) \longrightarrow T(a, b; c, d) \end{aligned}$$

where the last arrow is induced by the obvious arrows $\mathscr{A}(b \otimes (y \otimes x), d) \otimes \mathscr{A}(c, a \otimes (y \otimes x)) \to \int^x \mathscr{A}(b \otimes x, d) \otimes \mathscr{A}(c, a \otimes x)$. The unit has components

$$(\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})(a, b; c, d) = \mathscr{A}(b, d) \otimes \mathscr{A}(c, a) \xrightarrow{\eta} \int^x \mathscr{A}(b \otimes x, d) \otimes \mathscr{A}(c, a \otimes x),$$

the component corresponding to $I \in \mathrm{ob}\mathscr{A}$.

The existence of Eilenberg-Moore constructions in $\mathscr{V}$-**Mod** implies the following.

Proposition 5.2. *Any map pseudomonoid in $\mathscr{V}$-**Mod** has a Hopf module construction.*

Following the remarks on Eilenberg-Moore constructions above, one can give an explicit description of the Hopf module construction for a map pseudomonoid $\mathscr{A}$. The $\mathscr{V}$-category $(\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})^T = (\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})_T$ has the same objects as $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$, homs $(\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})(a, b; c, d) = T(a, b; c, d)$ and composition and identities induced by the multiplication and unit of T. The unit of the monad T defines a $\mathscr{V}$-functor $\eta : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to (\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})^T$; the Kleisli construction for T is just the module η_* and the Eilenberg-Moore construction is η^*. The module $L : \mathscr{A} \to (\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})^T$ in (2.12), which is an equivalence if and only if $\mathscr{A}$ has a left dualization, equals

$$L = \big(\mathscr{A} \xrightarrow{(J^*)^\circ \otimes \mathscr{A}} \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \xrightarrow{\eta_*} (\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})^T\big) \tag{5.1}$$

When the promonoidal structure is induced by a monoidal structure on $\mathscr{A}$, *i.e.*, $P(a, b; c) = \mathscr{A}(b \otimes a, c)$ and $J(a) = \mathscr{A}(I, a)$, we can compute L more explicitly. Firstly note that for any $\mathscr{V}$-functor $F : \mathscr{B} \to \mathscr{C}$ there exists a canonical isomorphism of $\mathscr{V}$-modules $(F^*)^\circ \cong (F^{\mathrm{op}})_* : \mathscr{B}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$, where $F^{\mathrm{op}} : \mathscr{B}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$

is the usual opposite functor. Then

$$L \cong \eta_*((J^{\mathrm{op}})_* \otimes \mathscr{A}) \cong (\eta(J^{\mathrm{op}} \otimes \mathscr{A}))_*.$$

In components,

$$L(a; b, c) \cong (\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})^T(\eta(I, a), (b, c)) = T(I, a; b, c) \cong \mathscr{A}(a \otimes b, c)$$

with right $\mathscr{A}$-action and left $(\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})^T$-action. The latter is given by the composition of $(\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A})^T$, while the $\mathscr{A}$-action can be shown to be given as

$$\mathscr{A}(a \otimes b, c) \otimes \mathscr{A}(a', a) \xrightarrow{1 \otimes (- \otimes b)} \mathscr{A}(a \otimes b, c) \otimes \mathscr{A}(a' \otimes b, a \otimes b) \xrightarrow{\mathrm{comp}} \mathscr{A}(a' \otimes b, c).$$

The fact that L is a fully faithful $\mathscr{V}$-module (Proposition 2.17) means exactly that the $\mathscr{V}$-functor $\eta(J^{\mathrm{op}} \otimes \mathscr{A})$ is fully faithful. This can be also verified directly, for the effect of this $\mathscr{V}$-functor on homs is

$$\mathscr{A}(b, d) \xrightarrow{1 \otimes 1_I} \mathscr{A}(b, d) \otimes \mathscr{A}(I, I) \xrightarrow{\eta} \int^x \mathscr{A}(b \otimes x, d) \otimes \mathscr{A}(I, I \otimes x) \cong \mathscr{A}(b, d)$$

sending an arrow f to $\big(b \xrightarrow{\cong} b \otimes I \xrightarrow{f \otimes 1_I} d \otimes I \xrightarrow{\cong} d\big)$.

We finish the section by characterising monoidal categories which are left autonomous as pseudomonoids in the bicategory of **Set**-modules, sometimes called profunctors or distributors. We will denote this bicategory simply by **Mod**, and use the conventions of Example 2.2.

If $\mathscr{A}$ is a monoidal category, consider the arrows $\mathscr{A}(b, d \otimes x) \times \mathscr{A}(x \otimes a, c) \to \mathscr{A}(b \otimes a, d \otimes c)$ sending (f, g) to the composition

$$b \otimes a \xrightarrow{f \otimes 1} (d \otimes x) \otimes a \xrightarrow{\cong} d \otimes (x \otimes a) \xrightarrow{1 \otimes g} d \otimes c$$

where the isomorphism is the associativity constraint of the monoidal category $\mathscr{A}$. These arrows are dinatural in x, inducing arrows

$$\int^x \mathscr{A}(b, d \otimes x) \times \mathscr{A}(x \otimes a, c) \to \mathscr{A}(b \otimes a, d \otimes c). \tag{5.2}$$

When $b = I$, the neutral object of $\mathscr{A}$, we get arrows

$$\int^x \mathscr{A}(I, d \otimes x) \times \mathscr{A}(x \otimes a, c) \to \mathscr{A}(a, d \otimes c). \tag{5.3}$$

Proposition 5.3. *A monoidal category $\mathscr{A}$ has a structure of a left autonomous pseudomonoid in* **Mod** *if and only if the arrows* (5.2) *are isomorphisms, if and only if the arrows* (5.3) *are isomorphisms, for all objects a, b, c, d in $\mathscr{A}$.*

Proof. The result follows from Theorem 2.34 since the arrows (5.2) and (5.3) are the components of the natural transformations γ and ω defined in (2.23) and (2.25) respectively. □

When idempotents in $\mathscr{A}$ split, the conditions in the proposition above imply that $\mathscr{A}$ has left duals, by classical arguments. Indeed, if (5.3) is an isomorphism, by taking $c = I$ and $a = d$, we deduce that $1_a = (1_a \otimes g)(f \otimes 1_a)$ for some $f : I \to a \otimes x$, $g : x \otimes a \to I$. By an argument due to Paré, the splitting of the idempotent $(g \otimes 1_x)(1_x \otimes f) : x \to x$ provides a left dual for a. There is another way of seeing this. The splitting of idempotents in $\mathscr{A}$ means that $\mathscr{A}$ is Cauchy complete, and therefore the dualization $\mathscr{A}^{\mathrm{op}} \to \mathscr{A}$ in **Mod** is represented by a functor. This functor assigns a left dual to each object of $\mathscr{A}$ (see Example 2.31).

5.1.2 Lax centres in $\mathscr{V}$-Mod

In this section we study the centre and lax centre of pseudomonoid in the monoidal bicategory of $\mathscr{V}$-modules by means of the theory developed in previous sections. Along the way, we compare our work with [15, 18].

First we consider lax centres of arbitrary pseudomonoids. We shall show that the results in Section 3.3 apply to $\mathscr{V}$-**Mod**. To realise this aim, we have to verify all the hypothesis required in that section.

We already saw at the beginning of Section 5.1 that liftings exist. In order to show $\mathscr{V}$-**Mod** satisfies the other two hypotheses required in Section 3.3 it is enough to prove that the arrow (3.6) is an isomorphism for $\mathscr{M}$ the bicategory of $\mathscr{V}$-modules. In this case (3.6) becomes

$$[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}, \mathscr{V}](M, N) \to [[\mathscr{A}, \mathscr{V}], [\mathscr{B}, \mathscr{V}]]((M \circ -), (N \circ -)), \tag{5.4}$$

where $(M \circ -)$ is the $\mathscr{V}$-functor given by composition with the $\mathscr{V}$-module M. To show that (5.4) is an isomorphism, recall that the $\mathscr{V}$-functor

$$[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}, \mathscr{V}] \cong [\mathscr{B}, [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]] \to \mathrm{Cocts}[[\mathscr{B}^{\mathrm{op}}, \mathscr{V}], [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]] \tag{5.5}$$

into the sub-$\mathscr{V}$-category of cocontinuous $\mathscr{V}$-functors is an equivalence by [42, Theorem 4.51]. This $\mathscr{V}$-functor sends $R : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{V}$ to the left extension of the corresponding $R' : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ along the Yoneda embedding $\mathsf{y} : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$, which is exactly $(R \circ -)$.

Theorem 3.5 gives:

Corollary 5.4. *Suppose the lax centre of the promonoidal $\mathscr{V}$-category $\mathscr{A}$ exists. Then there exits an equivalence of $\mathscr{V}$-categories $[Z_\ell \mathscr{A}, \mathscr{V}] \simeq Z_\ell[\mathscr{A}, \mathscr{V}]$, where on the left hand side appears the lax centre in $\mathscr{V}$-**Mod** and on the right hand side the lax centre in $\mathscr{V}$-**Cat**. The composition of this equivalence with the forgetful $\mathscr{V}$-functor $Z_\ell[\mathscr{A}, \mathscr{V}] \to [\mathscr{A}, \mathscr{V}]$ is canonically isomorphic to the $\mathscr{V}$-functor given by composing with the universal $\mathscr{V}$-module $Z_\ell \mathscr{A} \to \mathscr{A}$. If the centre of $\mathscr{A}$, rather than the lax centre, exists, then the above holds substituting lax centres by centres throughout.*

Now we turn our attention to autonomous pseudomonoids. The existence of Eilenberg-Moore constructions in $\mathscr{V}$-**Mod** together with Theorem 3.14 and Theorem 3.16 imply:

Proposition 5.5. *Any left autonomous map pseudomonoid in $\mathscr{V}$-**Mod** has a lax centre. Moreover, if the pseudomonoid is also right autonomous then the lax centre is the centre.*

Proposition 5.6. *If a left autonomous pseudomonoid $\mathscr{A}$ in $\mathscr{V}$-**Mod** has a centre construction, then its lax centre and its centre coincide.*

Proof. We saw that the lax centre of a $\mathscr{A}$ exists. The result, then, follows from Theorem 3.6. The category $\mathscr{V}$-$\mathbf{Mod}(I, \mathscr{A})$ has a dense small sub $\mathscr{V}$-category, namely the one determined by the representable $\mathscr{V}$-functors; and representables are maps in the bicategory of $\mathscr{V}$-modules. The rest of the hypotheses on $\mathscr{M}$ are easily verified. □

We shall describe the lax centre explicitly. In order to simplify the description, we will suppose $\mathscr{A}$ is a left autonomous monoidal $\mathscr{V}$-category, and not merely a promonoidal one. However, all the following description carries over to the case of map pseudomonoids.

By Theorem 3.14, the lax centre of $\mathscr{A}$ in $\mathscr{V}$-**Mod** is the Eilenberg-Moore construction for the monad $\tilde{S}$ given by

$$\mathscr{A} \xrightarrow{J \otimes 1} \mathscr{A} \otimes \mathscr{A} \xrightarrow{P^* \otimes 1} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{1 \otimes c_*} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{P \otimes 1} \mathscr{A} \otimes \mathscr{A} \xrightarrow{P} \mathscr{A} \quad (5.6)$$

where c denotes the usual symmetry in $\mathscr{V}$-**Cat**. Explicitly,

$$\tilde{S}(a;b) \cong \int^{x,y} \mathscr{A}(y \otimes (a \otimes x), b) \otimes \mathscr{A}(I, y \otimes x) \cong \int^{y} \mathscr{A}(y \otimes (a \otimes y^{\vee}), b),$$

where $y^{\vee}$ denotes the left dual of y in $\mathscr{A}$. The multiplication of this monad is given by

$$\begin{aligned}
&\tilde{S}^2(a;b) \cong \int^{u,y,z} \mathscr{A}\big(y \otimes (u \otimes y^{\vee}), b\big) \otimes \mathscr{A}\big(z \otimes (a \otimes z^{\vee}), u\big) \\
&\cong \int^{y,z} \mathscr{A}\big(y \otimes (z \otimes (a \otimes z^{\vee})) \otimes y^{\vee}, b\big) \cong \int^{y,z} \mathscr{A}\big((y \otimes z) \otimes (a \otimes (y \otimes z)^{\vee}), b\big) \to \\
&\qquad\qquad\qquad\qquad \longrightarrow \int^{x} \mathscr{A}\big(x \otimes (a \otimes x^{\vee}), b\big) \cong \tilde{S}(a;b)
\end{aligned}$$

where the last arrow is induced by the components $\zeta^{a,b}_{y \otimes z} : \mathscr{A}\big((y \otimes z) \otimes (a \otimes (y \otimes z)^{\vee}), b\big) \to \int^x \mathscr{A}\big(x \otimes (a \otimes x^{\vee}), b\big)$ of the universal dinatural transformation defining the latter coend in the codomain above. The unit of $\tilde{S}$ is given by components $\zeta^{a,b}_I : \mathscr{A}(a,b) \to \int^x \mathscr{A}\big(x \otimes (a \otimes x^{\vee}), b\big)$ of the same dinatural transformation corresponding to $x = I$. Now we have all the ingredients to describe the lax centre $Z_\ell(\mathscr{A})$, that is, a Kleisli construction for $\tilde{S}$. It has the same objects as $\mathscr{A}$, enriched homs $Z_\ell(\mathscr{A})(a,b) = \tilde{S}(a,b)$, composition given by the multiplication and unit given by

$$I \to \mathscr{A}(a,a) \xrightarrow{\zeta^{a,a}_I} \tilde{S}(a,a),$$

where the first arrow is the identity of a in $\mathscr{A}$. The arrows $\zeta^{a,b}_I : \mathscr{A}(a,b) \to \tilde{S}(a,b)$ define a $\mathscr{V}$-functor, which we also call ζ, and the universal $Z_\ell(\mathscr{A}) \to \mathscr{A}$ is none other than ζ^*.

Observation 5.7. The monad $\tilde{S}$ is closely related to the monad $\check{M}$ in [18, Section 5]. There the authors show that for a general small promonoidal $\mathscr{V}$-category $\mathscr{C}$ there exists a monad $\check{M}$ on $\mathscr{C}$ in $\mathscr{V}$-**Mod** with the following property. Whenever $[\mathscr{C}, \mathscr{V}]$ has a small dense sub-$\mathscr{V}$-category of objects with left duals (it is *right-dual controlled*, in the terminology of [18]), the forgetful $\mathscr{V}$-functor $Z_\ell[\mathscr{C}, \mathscr{V}] \to [\mathscr{C}, \mathscr{V}]$ is a Eilenberg-Moore construction for the monad M on $[\mathscr{C}, \mathscr{V}]$ in $\mathscr{V}$-**Cat** given by composition with $\check{M}$. The module $\check{M}$ is given by

$$\check{M}(a,b) = \int^{x,y} P(P \otimes \mathscr{C})(y,a,x,b) \otimes x^{\wedge}(y),$$

where $x^{\wedge}$ is the internal hom $[\![\mathscr{C}(x,-), J]\!] \in [\mathscr{C}, \mathscr{V}]$ (J is the unit of the promonoidal structure).

When $\mathscr{C}$ is equipped with a left dualization $D : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$, each $\mathscr{V}$-module $I \to \mathscr{C}$ with right adjoint in $\mathscr{V}$-**Mod** has a left dual in the monoidal $\mathscr{V}$-category $\mathscr{V}\text{-}\mathbf{Mod}(I, \mathscr{C}) = [\mathscr{C}, \mathscr{V}]$. This was first shown in [13]. In particular, $\mathscr{C}(x, -)$, which is the $\mathscr{V}$-module induced by the $\mathscr{V}$-functor $I \to \mathscr{C}$ constant on x, has left dual. It follows that $[\mathscr{C}, \mathscr{V}]$ has a small dense sub-$\mathscr{V}$-category with left duals, and the results of [18] mentioned above apply.

In this situation, if we assume J is a map, so that $\tilde{S}$ exists, we claim that the monads $\check{M}$ and $\tilde{S}$ are isomorphic, or more precisely, that both are isomorphic as monoids in the monoidal $\mathscr{V}$-category $\mathscr{V}\text{-}\mathbf{Mod}(\mathscr{C}, \mathscr{C}) = [\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}, \mathscr{V}]$. To show this, it is enough to prove that the monads $(\check{M} \circ -)$ and $(\tilde{S} \circ -)$ on $\mathscr{V}\text{-}\mathbf{Mod}(I, \mathscr{C}) = [\mathscr{C}, \mathscr{V}]$ given by composition with $\check{M}$ and $\tilde{S}$ respectively are isomorphic. Now, the monad $(\tilde{S} \circ -)$ is $\mathscr{V}\text{-}\mathbf{Mod}(I, \tilde{S})$, and then it has the forgetful $\mathscr{V}$-functor $Z_\ell[\mathscr{C}, \mathscr{V}] \to [\mathscr{C}, \mathscr{V}]$ as a (bicategorical) Eilenberg-Moore construction by Corollary 5.4 and Proposition 5.5. Then, $(\tilde{S} \circ -)$ and $M = (\check{M} \circ -)$ have the same Eilenberg-Moore construction in $\mathscr{V}$-**Cat** and it follows that both monads are isomorphic as required.

Example 5.8. Let $\mathscr{G}$ be a groupoid. Write $\Delta : \mathscr{G} \to \mathscr{G} \times \mathscr{G}$ for the diagonal functor and $E : \mathscr{G} \to 1$ the only possible functor. These give $\mathscr{G}$ a structure of comonoid in **Cat** and thus $P = \Delta^*$ and $J = E^*$ is a promonoidal structure on $\mathscr{G}$. Explicitly, $P(a, b; c) = \mathscr{G}(a, c) \times \mathscr{G}(b, c)$ and $J(a) = 1$; the monoidal structure induced in $[\mathscr{G}, \mathbf{Set}]$ is given by the point-wise cartesian product. Define a functor $D : \mathscr{G}^{\mathrm{op}} \to \mathscr{G}$ as the identity on objects an $D(f) = f^{-1}$ on arrows. In [16, Example 10] it was essentially shown that D is a left and right dualization for the map pseudomonoid $(\mathscr{G}, J, P)$ in $\mathbf{Set}\text{-}\mathbf{Mod}^{\mathrm{co}}$. Then, by Corollary 3.17, $\mathscr{G}$ has centre and lax centre in $\mathbf{Set}\text{-}\mathbf{Mod}^{\mathrm{co}}$ and both coincide. On the other hand, there is a category $\mathscr{G}^{\mathbf{Z}}$ described in [16, Example 10] with the property that $[\mathscr{G}^{\mathbf{Z}}, \mathbf{Set}] \simeq Z[\mathscr{G}, \mathbf{Set}]$. In [15] $\mathscr{G}^{\mathbf{Z}}$ is shown to be equivalent to the *centre of the promonoidal category* $\mathscr{G}$ in the sense of that article. As the centre in **Set-Mod** satisfies $[Z\mathscr{G}, \mathbf{Set}] \simeq Z[\mathscr{G}, \mathbf{Set}]$, we have that $\mathscr{G}^{\mathbf{Z}}$ is Morita equivalent to $Z\mathscr{G}$.

5.2 Hopf algebras and comodules

5.2.1 Hopf algebras

This section pretends to be a very short introduction to Hopf algebras and some related structures. Readers familiar with the concepts of Hopf algebras and (co)quasi-Hopf algebras can skip to the next section.

Hopf algebras originally appeared as cohomology rings of topological groups or H-spaces, to later find applications to many branches of Mathematics. Basic examples of Hopf algebras are the group algebras of finite groups, the algebra of regular functions of an affine algebraic group, the universal enveloping algebra of a Lie algebra. Hopf algebra theory is of interest to mathematical physicists because of its connection to quantum theories, via for example the q-enveloping algebras of a Lie algebra, and to category theorists because of its relationship with braided monoidal categories.

Let $\mathscr{V}$ be a braided monoidal category. The braiding makes the category of comonoids and comonoid morphisms in $\mathscr{V}$ a monoidal category. A bimonoid in $\mathscr{V}$ is a monoid in this category of comonoids. Therefore, a bimonoid is an object H of $\mathscr{V}$ with a comonoid structure with comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to I$, and a monoid structure with multiplication $p : H \otimes H \to H$ and unit $j : I \to H$. These structures are compatible in the sense that Δ, ε are monoid morphisms, or equivalently, p, j are comonoid morphisms. We use Greek letters for the comonoid structure and Roman ones for the monoid structure because, although completely dual to each other in the case of a bimonoid, these structures will play different roles when we move to the coquasibialgebras.

There is a *convolution product* on $\mathscr{V}(H, H)$ given by $f * g = p(f \otimes g)\Delta$ with unit $j\varepsilon$. An *antipode* for the bialgebra H is an inverse S to $1 : H \to H$ under the convolution product. Hence an antipode, if it exists, is unique, and moreover it can be shown to be an anti-monoid and anti-comonoid morphism. A bimonoid equipped with an antipode is called a *Hopf monoid.*

In the case when $\mathscr{V}$ is the category of modules over a commutative ring, these structures are usually called coalgebras, algebras, bialgebras and Hopf algebras.

Sometimes we will use *Sweedler's notation* which we briefly recall. If (H, ε, Δ) is a coalgebra, we write $\Delta(x) = \sum x_1 \otimes x_2$. These expressions are subject to the coassociativity rule $\sum (x_1)_1 \otimes (x_1)_2 \otimes x_2 = \sum x_1 \otimes (x_2)_1 \otimes (x_2)_2$, and we write this element $\sum x_1 \otimes x_2 \otimes x_2$. The counit condition is written $\sum \varepsilon(x_1)x_2 = x = \sum x_1 \varepsilon(x_2)$.

Example 5.9 (Group algebras)*.* If G is a finite group and k a commutative ring, the algebra $k[G]$ is a Hopf algebra with algebra structure induced by the monoid structure of G (the free vector space functor is monoidal, and thus it sends monoids in **Set** to monoids in **Vect**), comultiplication and counit given by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = \delta_{g,e}$ respectively, where $e \in G$ is the unit. The antipode is given by

$S(g) = g^{-1}$.

Example 5.10 (Universal enveloping algebras). If $\mathfrak{g}$ is a Lie algebra, the universal enveloping algebra $U(\mathfrak{g})$ has a canonical structure of a Hopf algebra. Recall that $U(\mathfrak{g})$ can be constructed as a quotient of the tensor algebra $T(\mathfrak{g})$ over the vector space $\mathfrak{g}$ by the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$. For $a \in T(\mathfrak{g})$ we write $\bar{a}$ for the corresponding element of $U(\mathfrak{g})$. Then the comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is defined by $\Delta(\bar{x}) = \overline{x \otimes 1} + \overline{1 \otimes x}$ for $x \in T(\mathfrak{g})$ of degree 1; this formula extends to the whole of $U(\mathfrak{g})$. The counit $\varepsilon : U(\mathfrak{g}) \to k$ is given by $\varepsilon(\bar{x}) = 0$ and the antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ by $S(\bar{x}) = -\bar{x}$ for all $x \in T(\mathfrak{g})$ of degree 1.

Example 5.11 (Taft's algebras). Let k be a field and $\xi \in k$ a primitive N^{th} root of the unity. Define a Hopf algebra H in the following way. As an algebra, H is generated by two elements g, x with relations

$$g^N = 1 \qquad x^N = 1 \qquad xg = \xi gx.$$

The comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to k$ and antipode $S : H^{\text{op}} \to H$ are defined on the generators by

$$\begin{gathered} \Delta(g) = g \otimes g \qquad \Delta(x) = 1 \otimes x + x \otimes g \\ \varepsilon(g) = 1 \qquad \varepsilon(x) = 1 \\ S(g) = g^{-1} \qquad S(x) = -xg^{-1}. \end{gathered}$$

This Hopf algebra is called a Taft's Hopf algebra, and it has dimension N^2 with basis $\{g^i x^j \ : \ 0 \leq i, j \leq N_1\}$. Observe that H is not commutative nor cocommutative, and the antipode does not satisfy $S^2 = \text{id}$, as $S^2(g) = gxg^{-1}$. In fact, it can be shown that S has order $2N$

In [23] Drinfel'd introduced a weaken version of Hopf algebras called *quasi-Hopf algebras*. A quasibialgebra is a k-algebra H equipped with a counit $\varepsilon : H \to k$ which is a morphism of algebras and a comultiplication $\Delta : H \to H \otimes H$ that is not coassociative. Instead, Δ is coassociative up to conjugation with an invertible 3-cycle. We give the definition of the dual concept of *coquasi-Hopf algebra* that serves best to our purposes.

A *coquasi bialgebra* structure on the k-coalgebra (C, Δ, ε) is a triple (p, j, ϕ) where $p : C \otimes C \to C$ (the product) and $j : k \to C$ (the unit) are coalgebra morphisms, and $\phi : C \otimes C \otimes C \to k$ (the associator) is a convolution-invertible functional, satisfying the following axioms, where we write $p(x \otimes y)$ as xy.

$$p(j \otimes \mathrm{id}) = \mathrm{id} = p(\mathrm{id} \otimes j)$$

$$\sum \phi(x_1 \otimes y_1 \otimes z_1)(x_2 \cdot y_2) \cdot z_2 = \sum x_1 \cdot (y_1 \cdot z_1)\phi(x_2 \otimes y_2 \otimes z_2)$$

$$(\epsilon \otimes \phi) * \phi(1 \otimes p \otimes 1) * (\phi \otimes \epsilon) = \phi(1 \otimes 1 \otimes p) * \phi(p \otimes 1 \otimes 1)$$

$$\phi(x \otimes j \otimes y) = \varepsilon(x)\varepsilon(y)$$

where $*$ denotes the convolution product in the dual of $C \otimes C \otimes C \otimes C$.

Of course, these axioms can be written as string diagrams in any braided monoidal category $\mathscr{V}$, giving rise to a structure we call a coquasibialgebra or coquasibimonoid in $\mathscr{V}$. Coquasibialgebras were first considered under another name in [10]; see also [64]. Observe that when $\phi = \varepsilon \otimes \varepsilon \otimes \varepsilon$, a coquasibialgebra is just a bialgebra.

If one consider the two-dimensional aspects of reconstruction theorems, the algebraic structures corresponding to monoidal categories are coquasibialgebras. The reason for this is that both are pseudomonoids in different monoidal bicategories; see Example 5.12. This approach is taken in [66].

An *antipode* for the coquasi bialgebra H is a triple (S, α, β) where $S : H^{\mathrm{cop}} \to H$ is a coalgebra morphism from the opposite coalgebra of H to H, and the functionals $\alpha, \beta : H \to k$ satisfy the following equations, where we write 1 for the element of H corresponding to the unit $j : k \to H$.

$$\sum S(x_1)\alpha(x_2)x_3 = \alpha(x)j \qquad \sum x_1\beta(x_2)S(x_3) = \beta(x)j \tag{5.7}$$

$$\sum \phi(x_1 \otimes Sx_3 \otimes x_5)\beta(x_2)\alpha(x_4) = \varepsilon(x) \tag{5.8}$$

$$\sum \phi^{-1}(Sx_1 \otimes x_3 \otimes Sx_5)\alpha(x_2)\beta(x_4) = \varepsilon(x) \tag{5.9}$$

The definition of coquasibialgebra and coquasi-Hopf algebra are designed in such a way that the following holds. If B is a coquasibialgebra, the category of (right) comodules $\mathrm{Comod}(B)$ is monoidal and the forgetful functor U into **Vect** is multiplicative in the sense that $U(M) \otimes U(N) \cong U(M \otimes N)$, but not monoidal. This is because the associativity constraint of $\mathrm{Comod}(B)$ is the composition

$$M \otimes N \otimes L \to M \otimes B \otimes N \otimes B \otimes L \otimes B \to M \otimes N \otimes L \otimes B^{\otimes 3} \xrightarrow{M \otimes N \otimes L \otimes \phi} M \otimes N \otimes L$$

where the first arrow is the tensor product of the coactions, the second is induced by the symmetry of **Vect**, and we omit the associativity constraint of **Vect**.

In the case of a coquasi-Hopf algebra H with antipode S, the category of finite-dimensional comodules $\mathrm{Comod}_f(H)$ is left autonomous. A left dual ${}^*M$ for a comodule M with coaction $\chi : M \to M \otimes H$ is given by the dual $M^\vee$ of the vector space M with coaction

$$M^\vee \xrightarrow{1\otimes c} M^\vee \otimes M \otimes M^\vee \xrightarrow{1\otimes\chi\otimes 1} M^\vee \otimes M \otimes H \otimes M^\vee \xrightarrow{e\otimes c_{H,M^\vee}} M^\vee \otimes H$$

where c, e are the coevaluation and evaluation of vector spaces and c is the symmetry. The evaluation ${}^*M \otimes M \to k$ and the coevaluation $k \to M \otimes {}^*M$ are the morphisms of comodules below.

$$M^\vee \otimes M \xrightarrow{1\otimes\chi} M^\vee \otimes M \otimes H \xrightarrow{e\otimes\alpha} k$$

$$k \xrightarrow{c} M \otimes M^\vee \xrightarrow{\chi\otimes 1} M \otimes H \otimes M^\vee \xrightarrow{1\otimes\beta\otimes 1} M \otimes M^\vee$$

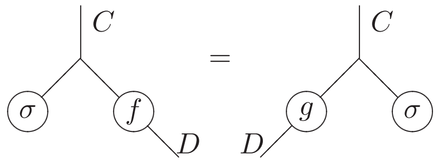
When H is a bialgebra (*i.e.*, ϕ is trivial) and $\alpha = \beta = \varepsilon$, we recover Hopf algebras.

5.2.2 The bicategory of comodules

In this section we apply the general theory we have developed for autonomous pseudomonoids to Hopf algebras and some of their generalisations. After reviewing the definition and basic properties of the bicategory of comodules, we interpret the material of the three previous chapters in three sections. In the first of them, Section 5.2.3, we explain why Theorem 2.34 generalises the fundamental theorem of Hopf modules for (coquasi) Hopf algebras. In Section 5.2.4 we describe the centre in the monoidal bicategory of comodules of (the pseudomonoid induced by) a finite dimensional coquasi-Hopf algebra. We show that this centre is equivalent to the *Drinfel'd* or *quantum double* of the coquasi-Hopf algebra. Finally, in Section 5.2.5 we deduce from the isomorphism we called Radford's formula for autonomous map pseudomonoids in Section 4.2 formulas for the cases of quasi-Hopf algebras (Theorem 5.34) and coquasi-Hopf algebras (Theorem 5.36).

Given a monoidal category $\mathscr{V}$, there is a monoidal 2-category $\mathbf{Comon}(\mathscr{V})$ called the *2-category of comonoids.* Its objects are comonoids in $\mathscr{V}$, its 1-cells comonoid morphisms and 2-cells $\sigma : f \Rightarrow g : C \to D$ are arrows $\sigma : C \to I$ in $\mathscr{V}$

such that



The vertical composition of a pair of 2-cells $\sigma : f \Rightarrow g$ and $\tau : g \Rightarrow h$ is the usual convolution product: $\tau * \sigma = (\tau \otimes \sigma)\Delta$, where Δ denotes the comultiplication. The horizontal compositions

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \arrow[r, "h"] & C \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} D \arrow[r, "k"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd}$$

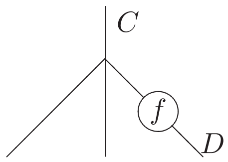
are $A \xrightarrow{\sigma} I$ and $D \xrightarrow{k} A \xrightarrow{\sigma} I$ respectively.

Now suppose further that $\mathscr{V}$ has equalizers of reflexive pairs and each functor $X \otimes -$ preserves them. Then we can construct the *bicategory of comodules* over $\mathscr{V}$, denoted by $\mathbf{Comod}(\mathscr{V})$. It has comonoids in $\mathscr{V}$ as objects and homs $\mathbf{Comod}(\mathscr{V})(C, D)$ the category of C-D-bicomodules; this is the category of Eilenberg-Moore algebras for the comonad $C \otimes - \otimes D$ on $\mathscr{V}$. The composition of two comodules $M : C \to D$ and $N : D \to E$ is given by the equalizer of the following reflexive pair

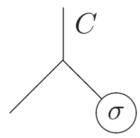
$$\begin{tikzcd} M\square_D N \arrow[r] & M \otimes N \arrow[r, shift left, "\chi_r^M \otimes N"] \arrow[r, shift right, "M \otimes \chi_\ell^N"'] & M \otimes D \otimes N \end{tikzcd}$$

where the various χ denote the obvious coactions. This equalizer is denoted by $M\square_D N$, and has a C-E-comodule structure induced by the structures of M and N. The comodule $M\square_D N$ is sometimes called the *cotensor product* of M and N over D. The identity 1-cell corresponding to a comonoid C is the *regular comodule* C, *i.e.* it is C with coaction $\Delta^2 = (\Delta \otimes 1)\Delta : C \to C \otimes C \otimes C$.

There is a pseudofunctor $(-)_* : \mathbf{Comod}(\mathscr{V}) \to \mathbf{Comod}(\mathscr{V})$ acting as the identity on objects, sending a comonoid morphism $f : C \to D$ to the comodule, denoted by $f_* : C \to D$, with underlying object C and coaction

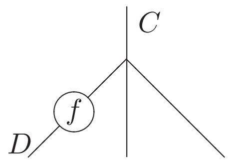


and sending a 2-cell $\sigma : f \Rightarrow g$ to the comodule morphism $\sigma_* : f_* \Rightarrow g_*$ given by

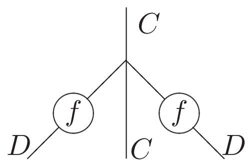


The axioms of coaction and of comodule morphism follow from the ones of comodule morphism and 2-cell in **Comon**($\mathscr{V}$) respectively. It is easy to show that the pseudofunctor $(-)_*$ is locally fully faithful (in fact, locally it can be viewed as a $\mathscr{V}^{\mathrm{op}}$-enriched Yoneda embedding).

An important property of $(-)_*$ is that it sends any 1-cell in **Comon**($\mathscr{V}$) to a map in **Comod**($\mathscr{V}$). For, if $f : C \to D$ is a comonoid morphism, then f_* has a right adjoint, denoted by f^*, with underlying object C and coaction



The composition $f_* f^*$ is the comodule with object C and coaction



and the counit of the adjunction is just the arrow $f : C \to D$, which turns out to be a comodule morphism; the unit is the unique map such that

$$\begin{tikzcd}
f^* f_* = f_* \Box_D f^* \arrow[r] & C \otimes C \\
C \arrow[u, dotted, "\eta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

where the horizontal arrow is the defining equalizer of $f^* f_*$.

When $\mathscr{V}$ is braided, **Comon**($\mathscr{V}$) and **Comod**($\mathscr{V}$) have the structure of monoidal bicategories with tensor product given by the tensor product of $\mathscr{V}$; note that the braiding is used in defining the comultiplication and coactions on the tensor product of comonoids and comodules. The pseudofunctor $(-)_*$ is strong monoidal. Through $(-)_*$ we can think of **Comon**($\mathscr{V}$) as a monoidal sub bicategory of **Comod**($\mathscr{V}$).

Example 5.12 (Coquasibialgebras). A pseudomonoid (C, j, p) in the bicategory of comonoids $\mathbf{Comon}(\mathscr{V})$ amounts to a comonoid C with two comonoid morphisms $j : I \to C$ and $p : C \otimes C \to C$ and the invertible 2-cells $\phi : p(p \otimes C) \Rightarrow p(C \otimes p)$, $\lambda : p(j \otimes C) \Rightarrow 1$ and $\rho : p(C \otimes j) \Rightarrow 1$ satisfying axioms. These 2-cells are convolution-invertible arrows $\phi : C \otimes C \otimes C \to I$ and $\lambda, \rho : C \to I$.

Normal pseudomonoids, that is, pseudomonoids whose unit constraints λ, ρ are identities, in the monoidal bicategory $\mathbf{Comon}(\mathbf{Vect})$ are *coquasibialgebras.* The dual of this algebraic structure, called *quasibialgebra,* was first defined in [23] where also were defined the quasi-Hopf algebras. Coquasibialgebras and coquasi Hopf algebras can be found for example in [63, 10, 73]. See Section 5.2.1.

The bicategory $\mathbf{Comod}(\mathscr{V})$ is not just monoidal but it is also left and right autonomous. The right bidual of a comonoid C is the opposite comonoid C°. The braiding provides pseudonatural equivalences

$$\mathbf{Comod}(\mathscr{V})(C \otimes D, E) \simeq \mathbf{Comod}(\mathscr{V})(D, C^{\circ} \otimes E).$$

The coevaluation $\mathrm{n} : I \to C^{\circ} \otimes C$ and evaluation $\mathrm{e} : C \otimes C^{\circ} \to I$ comodules are the object C with coaction

The left bidual is defined by using the inverse of the braiding.

Example 5.13 (Coquasi-Hopf algebras). As shown in [13], Coquasi-Hopf algebras are exactly the left autonomous normal pseudomonoids in $\mathbf{Comod}(\mathbf{Vect})$ whose unit, multiplication and dualization are representable by coalgebra morphisms.

Regard a coquasibialgebra H as a pseudomonoid (H, j_*, p_*) in $\mathbf{Comod}(\mathbf{Vect})$, and assume H has a left dualization (s_*, α, β) where $s : H^{\circ} \to H$ is a comonoid morphism. We write Δ^2 for the arrow $(\Delta \otimes H)\Delta = (H \otimes \Delta)\Delta$ as it is custom in Hopf algebra theory. The 2-cell α is a comodule morphism from $p_*(s_* \otimes H)\mathrm{n}$ to j_*. Then $\alpha : H \to k$ is a functional satisfying $p(s \otimes \alpha \otimes H)\Delta^2 = j\alpha$, or in Sweedler's notation: $\sum \alpha(x_2)s(x_1) \cdot x_3 = \alpha(x)j$. This is one of the equations in (5.7). Taking mates, the 2-cell $\beta : j^*\mathrm{e} \Rightarrow p_*(H \otimes s_*)$ corresponds to a 2-cell $\bar{\beta} : \mathrm{e}(H \otimes s^*)p^* \Rightarrow j^*$. This comodule morphism is an arrow $\bar{\beta} : H \to k$ satisfying $p(H \otimes \bar{\beta} \otimes H)\Delta^2 = j\bar{\beta}$, or in Sweedler's notation $\sum \bar{\beta}(x_3)x_1 \cdot s(x_3) = \bar{\beta}(x)j$,

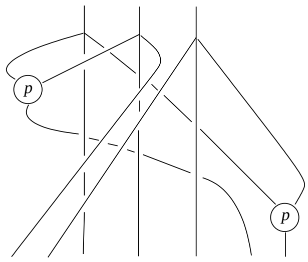


Figure 5.1: Comodule structure of the monad t.

that is the second equation in (5.7). The bijection between β and $\bar{\beta}$ is given by $\beta = (H \otimes \bar{\beta} \otimes H)\Delta^2$, $\bar{\beta} = (\varepsilon \otimes \varepsilon)\beta$. Now it is not hard to check that the axioms of a left dualization translate into the equations (5.8) and (5.9).

5.2.3 Hopf modules

From now on $\mathscr{V}$ will not only have equalizers of reflexive pairs, but all equalizers. Equalizers are necessary as the proof of the proposition below uses the Adjoint Triangle Theorem [24]. In any case, these properties are certainly satisfied in our main example of the category of vector spaces.

Proposition 5.14 ([13]). *A comodule $M : C \to D$ has a right adjoint if and only if its composition with $\varepsilon_* : D \to I$ has a right adjoint.*

Now we shall describe for a pseudomonoid C in **Comon**($\mathscr{V}$) the underlying comodule of the monad t on $C^\circ \otimes C$ representing θ. Recall from (2.28) that

$$t \cong (C^\circ \otimes p_*)(C^\circ \otimes C \otimes \mathrm{e} \otimes C)(C^\circ \otimes p^* \otimes C^* \otimes C)(\mathrm{n} \otimes C^* \otimes C)$$

and so it has underlying object $C \otimes C \otimes C$ with coaction depicted in Figure 5.1.

The Hopf module construction for a map pseudomonoid in **Comod**($\mathscr{V}$) may not exist, as this bicategory does not have Eilenberg-Moore objects for monads. However, it does have Eilenberg-Moore constructions for comonads.

Observation 5.15. The bicategory **Comod**($\mathscr{V}$) has Eilenberg-Moore objects for comonads. If G is a comonad on the comonoid C with comultiplication $\delta : G \to G\Box_C G$ and counit $\epsilon : G \to C$, its Eilenberg-Moore object admits the following description (which is dual to the description of Kleisli objects for monads in $\mathscr{V}$-**Cat** in [77]). As a comonoid, it is G equipped with comultiplication and counit

$$G \xrightarrow{\delta} G\Box_C G \rightarrowtail G \otimes G \qquad \text{and} \qquad G \xrightarrow{\epsilon} C \xrightarrow{\varepsilon} I.$$

Note that the arrow $\epsilon : G \to C$ in $\mathscr{V}$ becomes a morphism of comonoids. The universal 1-cell is just the comodule $\epsilon_* : G \to C$.

Proposition 5.16. *Given a map pseudomonoid C in $\mathbf{Comod}(\mathscr{V})$, if the monad $t : C^\circ \otimes C \to C^\circ \otimes C$ has right adjoint, then C has a Hopf module construction. In particular, this holds if $C \in \mathrm{ob}\mathscr{V}$ has a dual.*

Proof. The 1-cell t^* has a canonical structure of a right adjoint comonad to the monad t. It is well-known that the Eilenberg-Moore construction for the comonad t^* is an Eilenberg-Moore construction for the monad t. To finish, we show that if C has a dual in $\mathscr{V}$ then $t \cong ((p^*)^\circ \otimes p)(C^\circ \otimes \mathrm{n} \otimes C)$ has a right adjoint, and for that it suffices to prove that n does. But by Proposition 5.14, n is a map if and only if C has a dual. $\square$

When $\mathscr{V}$ is the category of vector spaces and C is a coquasi-bialgebra, the assertion that the functor λ_I from $\mathbf{Comod}(\mathscr{V})(I, C)$ to the category of Hopf modules is an equivalence is what Schauenburg [75] calls the theorem of Hopf modules. See Example 2.8. We shall show that when C has a Hopf module construction both notions are equivalent.

Let $\mathscr{W}$ be a braided monoidal replete full subcategory of $\mathscr{V}$ closed under equalizers of reflexive pairs. There is an inclusion monoidal pseudofunctor

$$\mathbf{Comod}(\mathscr{W}) \to \mathbf{Comod}(\mathscr{V}).$$

This inclusion, being monoidal, preserves biduals.

Corollary 5.17. *Let $\mathscr{W}$ and $\mathscr{V}$ be as above. Suppose C is a map pseudomonoid in $\mathbf{Comod}(\mathscr{W})$ such that C has a dual in $\mathscr{W}$. Then, the theorem of Hopf modules holds for C in $\mathbf{Comod}(\mathscr{W})$ if and only if it holds for C in $\mathbf{Comod}(\mathscr{V})$.*

Proof. We begin by observing that since C has a dual in $\mathscr{W}$, and hence in $\mathscr{V}$, by Proposition 5.16, C has a Hopf module construction both in $\mathbf{Comod}(\mathscr{W})$ and in $\mathbf{Comod}(\mathscr{V})$. Moreover, the two coincide. To see this, observe that the monad t is given by (2.28) and each of the 1-cells in the composition lies in $\mathbf{Comod}(\mathscr{W})$. Since C has a dual, t has a right adjoint comonad, whose Eilenberg-Moore construction, described in Observation 5.15, is the Hopf module construction for C.

By the description of this Eilenberg-Moore construction, one sees that it lies in $\mathbf{Comod}(\mathscr{W})$.

Hence, we have to prove that the 1-cell $\ell : C \to (C^\circ \otimes C)^t$ (see Proposition 2.17) is an equivalence in $\mathbf{Comod}(\mathscr{W})$ if and only if it is one in $\mathbf{Comod}(\mathscr{V})$. One direction is trivial, so we shall suppose ℓ is an equivalence in $\mathbf{Comod}(\mathscr{V})$. We have, then, an adjoint equivalence $\ell \dashv \ell^*$; as ℓ is always a map (by Proposition 2.17), this adjoint equivalence lifts to $\mathbf{Comod}(\mathscr{W})$. □

Corollary 5.18. *Suppose that C is a map pseudomonoid in $\mathbf{Comod}(\mathbf{Vect})$. If C is finite-dimensional, the theorem of Hopf modules holds for C if and only if the functor*

$$\lambda_I : \mathbf{Comod}(\mathbf{Vect})(I, C) \to \mathbf{Comod}(\mathbf{Vect})(C, C)^{\theta_I}$$

(see Definition 2.3) is an equivalence.

Proof. Only the converse is non trivial. Write $\mathscr{V}$ for **Vect** and $\mathscr{V}_f$ for the full subcategory of finite-dimensional vector spaces. By Proposition 5.17, it is enough to show that the theorem of Hopf modules holds for C in $\mathbf{Comod}(\mathscr{V}_f)$.

The functor λ_I is represented by the 1-cell $\ell : C \to (C^\circ \otimes C)^t$. We have that the functor $\mathbf{Comod}(\mathscr{V}_f)(I, \ell)$ is an equivalence, and the result follows from the fact that the functor $\mathbf{Comod}(\mathscr{V}_f)(I, -)$ reflects equivalences. □

We obtain the following generalisation of [75, Thm. 3.1].

Corollary 5.19. *Let C be a map pseudomonoid in $\mathbf{Comod}(\mathbf{Vect})$ whose underlying space is finite-dimensional. Then C has a left dualization if and only if the functor $\lambda_I : \mathbf{Comod}(\mathbf{Vect})(I, C) \to \mathbf{Comod}(\mathbf{Vect})(C, C)^{\theta_I}$ is an equivalence.*

Proof. By the corollary above, the theorem of Hopf modules holds for C; hence, C has a left dualization by Theorem 2.34. □

Corollary 5.20. *For any finite-dimensional coquasi-bialgebra C there exists a map pseudomonoid D in $\mathbf{Comod}(\mathbf{Vect})$ such that the category of Hopf modules for C (as defined in [75]) is monoidally equivalent to the category of right D-comodules $\mathbf{Comod}(\mathbf{Vect})(I, D)$. Moreover, D can be taken to be the Hopf module construction for C, and in particular, finite-dimensional.*

By Observation 5.15, the Hopf module construction $(C^\circ \otimes C)^t \to C^\circ \otimes C$ can be taken to be of the form ϵ_*, where $\epsilon : (C^\circ \otimes C)^t \to C^\circ \otimes C$ is a comonoid morphism.

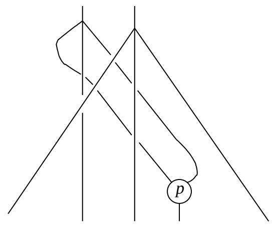


Figure 5.2: Hopf module construction for a coquasibialgebra with a left dualization.

Note that, in general, the forgetful functor $\mathbf{Comod}(\mathbf{Vect})(I, D) \to \mathbf{Vect}$ is not monoidal.

Now suppose that C is a left autonomous map pseudomonoid in $\mathbf{Comod}(\mathscr{V})$. The existence of a left dualization forces the multiplication to be a map [13, Prop. 1.2]. On the other hand, the unit of C is a map because its underlying object $I \in \mathscr{V}$ has a (right) dual by Proposition 5.14. It follows that any left autonomous pseudomonoid in $\mathbf{Comod}(\mathscr{V})$ is a map pseudomonoid. A Hopf module construction for C is provided by $(C^\circ \otimes p)(\mathrm{n} \otimes C) \cong (p(d \otimes C))^* : C \to C^\circ \otimes C$. In the case when C is a coquasibialgebra, the comodule $(C^\circ \otimes p_*)(\mathrm{n} \otimes C)$ is $C \otimes C$ with coaction depicted in Figure 5.2.

5.2.4 Centres and Drinfel'd double

We now consider the results of Section 3.4 on the lax centre in the context of comodules. We suppose the underlying monoidal category $\mathscr{V}$ is symmetric, and thus $\mathbf{Comon}(\mathscr{V})$ is a symmetric monoidal $\mathbf{Cat}$-enriched category. Via the monoidal pseudofunctor $(-)_*$ we obtain comodules $c_{M,N} : M \otimes N \to N \otimes M$ making the usual diagrams commute up to canonical isomorphisms in $\mathbf{Comod}(\mathscr{V})$.

Proposition 5.21. *Any left autonomous pseudomonoid in* $\mathbf{Comod}(\mathscr{V})$ *whose underlying object in* $\mathscr{V}$ *has a dual has a lax centre. If the pseudomonoid is also right autonomous then the lax centre equals the centre.*

Proof. We have already mention that any left autonomous pseudomonoid C in $\mathbf{Comod}(\mathscr{V})$ is a map pseudomonoid. By Theorem 3.14 we have to show that the monad $\tilde{s} : A \to A$ has an Eilenberg-Moore construction, and for that it is enough to show that it has a right adjoint, since $\mathbf{Comod}(\mathscr{V})$ has Eilenberg-Moore objects for comonads. By Theorem 3.14, we have $\tilde{s} \cong p(p \otimes C)(C \otimes c_{C,C})(p^* \otimes C)(j \otimes C)$ and therefore $\tilde{s}$ has a right adjoint if $p^* j : I \to C \otimes C$ has one; but C being left autonomous, this 1-cell is isomorphic to $(d \otimes C)\mathrm{n}$ which is a composition of maps: d by [13, Prop. 1.2] and n by [13, Prop. 5.1]. □

Example 5.22. The proposition above implies that any finite-dimensional coquasi-Hopf algebra H has a lax centre in **Comod**(**Vect**). Moreover, the antipode of a finite-dimensional coquasi-Hopf algebra is always invertible by [9, 75]. This means that the dualization of the induced map pseudomonoid is an equivalence, and hence we have a left and right autonomous pseudomonoid (see [13, Prop. 1.5]). It follows that H has a centre and it coincides with the lax centre.

Observation 5.23. In the proposition above, suppose that the full subcategory $\mathscr{V}_f$ of objects with a dual in $\mathscr{V}$ is closed under equalizers of reflexive pairs. Then the lax centre $Z_\ell(C) \to C$ lies in $\mathbf{Comod}(\mathscr{V}_f)$, and it is a lax centre in it.

To prove this observe that $t : C^\circ \otimes C \to C^\circ \otimes C$ and its Eilenberg-Moore construction $C \to C^\circ \otimes C$ lie in $\mathbf{Comod}(\mathscr{V}_f)$, and the monad s and the distributive law between t and s do so too; see the description of Eilenberg-Moore constructions for comonads in Observation 5.15. It follows that the induced monad $\tilde{s}$ on C lies in $\mathbf{Comod}(\mathscr{V}_f)$, and it has right adjoint in this bicategory, as shown in the proof above, and it is necessarily the same as in $\mathbf{Comod}(\mathscr{V})$. It follows from the description of Eilenberg-Moore objects mentioned above that $\tilde{s}^*$ has an Eilenberg-Moore construction in $\mathbf{Comod}(\mathscr{V}_f)$ and coincides with the respective construction in $\mathbf{Comod}(\mathscr{V})$. Moreover, this construction is given by $\epsilon_* : C^{\tilde{s}^*} \to C$, where ϵ is the comonoid morphism induced by the counit of the comonad $\tilde{s}^*$. Therefore, the lax centre of C in $\mathbf{Comod}(\mathscr{V}_f)$ is the lax centre of C in $\mathbf{Comod}(\mathscr{V})$.

The *Drinfel'd double* or *quantum double* of a finite-dimensional Hopf algebra is a finite-dimensional braided (also called quasitriangular) Hopf algebra $D(H)$ with underlying vector space $H^* \otimes H$ (one can also take $H \otimes H$) and suitably defined structure. It is a classical result that the category of left $D(H)$-modules is monoidally equivalent to the category of (two-sided) H-Hopf modules and to the centre of the category of H-modules. The Drinfel'd double of a finite-dimensional *quasi-Hopf algebra* was defined in [64] using a reconstruction theorem, and explicit constructions were given in [35, 74]. This last paper shows that the category of $D(H)$-modules is monoidally equivalent to the centre of the category of H-modules, via a generalisation of the Yetter-Drinfel'd modules. The quantum double of a coquasi-Hopf algebra was described in [10]. Alternatively, it can be described by dualising the explicit constructions for the quasi-Hopf case. Then the Drinfel'd or quantum double $D(H)$ of a finite-dimensional coquasi-Hopf H

algebra is finite-dimensional and has the property that the category of $D(H)$-comodules $\mathrm{Comod}(D(H))$ is monoidally equivalent to the centre of $\mathrm{Comod}(H)$, and the equivalence commutes with the forgetful functors.

Given a finite-dimensional coquasi-Hopf algebra H, we would like to study the relationship between the centre $Z(H)$ in $\mathbf{Comod}(\mathbf{Vect})$ and the Drinfel'd double $D(H)$. To this aim we will need some of the machinery of *Tannakian reconstruction*, of which we give the most basic aspects following [68].

Let $\mathscr{V}$ be a monoidal category and $\mathscr{V}_f$ the full sub-monoidal category with objects with left duals. We denote by $\mathscr{V}_f$-Act the 2-category of pseudoalgebras for the pseudomonad $(\mathscr{V}_f \times -)$ on $\mathbf{Cat}$. Objects of this 2-category are pseudoactions of $\mathscr{V}_f$ and 1-cells are pseudomorphisms of pseudoactions. Observe that $\mathscr{V}_f$ has a canonical $\mathscr{V}_f$-pseudoaction given by the tensor product. We form the 2-category $\mathscr{V}_f$-Alg$/\mathscr{V}_f$ with objects 1-cells $\sigma : \mathscr{A} \to \mathscr{V}_f$ in $\mathscr{V}_f$-Act. The 1-cells are pairs $(F, \phi) : \sigma \to \sigma'$ where $F : \mathscr{A} \to \mathscr{A}'$ is a 1-cell in $\mathscr{V}_f$-Act and $\phi : \sigma' F \cong \sigma$ is a 2-cell in $\mathscr{V}_f$-Act. 2- cells $(F, \phi) \Rightarrow (F', \phi')$ are just 2-cells $F \Rightarrow F'$ in $\mathscr{V}_f$-Act. There is a 2-functor $\mathrm{Comod}_f : \mathbf{Comon}(\mathscr{V}) \to \mathscr{V}_f\text{-Act}/\mathscr{V}_f$ sending a comonoid C to the forgetful functor $\omega_C : \mathrm{Comod}_f(C) \to \mathscr{V}_f$; here $\mathrm{Comod}_f(C)$ is the category of right coactions of C with underlying object in $\mathscr{V}_f$. This category has a canonical $\mathscr{V}_f$-pseudoaction such that ω is an object of $\mathscr{V}_f\text{-Act}/\mathscr{V}_f$. The definition of Comod_f on 1-cells and 2-cells should be more or less obvious; see [68].

Under certain hypothesis on $\mathscr{V}$, the 2-functor Comod_f is bi-fully faithful. Here is the case we will need: the 2-functor

$$\mathrm{Comod}_f : \mathbf{Comon}(\mathbf{Vect}) \to \mathbf{Vect}_f\text{-Act}/\mathbf{Vect}_f$$

is bi-fully faithful. We refer the reader to [68] for a proof of this result.

Theorem 5.24. *For any finite-dimensional coquasi-Hopf algebra H, the coalgebras $H^{\tilde{s}^*}$ and $D(H)$ are equivalent coquasibialgebras. Moreover, they are isomorphic as coalgebras.*

Proof. By Observation 5.23, $H^{\tilde{s}^*}$ is a centre for the pseudomonoid H in the monoidal bicategory $\mathbf{Comod}(\mathbf{Vect}_f)$. Hence we have an equivalence in the 2-category $\mathbf{Vect}_f$-Act$/\mathbf{Vect}_f$ from the forgetful functor $\mathrm{Comod}_f(H^{\tilde{s}^*}) \to \mathbf{Vect}_f$ to the forgetful functor $Z(\mathrm{Comod}_f(H)) \to \mathbf{Vect}_f$. On the other hand, there is an equivalence from the latter to $\mathrm{Comod}_f(D(H)) \to \mathbf{Vect}_f$. In this way we get an equivalence from $\mathrm{Comod}_f(H^{\tilde{s}^*})$ to $\mathrm{Comod}_f(D(H))$ in $\mathbf{Vect}_f$-Act$/\mathbf{Vect}_f$. By the result mentioned above this theorem, we have an equivalence $f : H^{\tilde{s}^*} \to D(H)$

in **Comon**(**Vect**). That is, both coquasibialgebras are equivalent. As every equivalence in **Comon**($\mathscr{V}$) has an invertible underlying arrow in $\mathscr{V}$, we deduce that f is an isomorphism of coalgebras. $\square$

5.2.5 Radford's formula

In this section we interpret the results on Radford's formula obtained in Chapter 4 in the case of the bicategory of comodules. We will assume that $\mathscr{V}$ is a symmetric monoidal category with equalizers and whose tensor product preserves equalizers of reflexive pairs in each variable, so that **Comod**($\mathscr{V}$) exists. Since the base monoidal category $\mathscr{V}$ is symmetric, the canonical right and left bidual pseudofunctors we have chosen on **Comod**($\mathscr{V}$) are equal. Therefore we shall write both right and left biduals by $(-)^{\circ}$.

For the sake of simplicity, and with view to the applications to Hopf algebra theory, from now on we will suppose that (H, j, p, s) is a coquasi-Hopf algebra in the symmetric monoidal category $\mathscr{V}$. We consider the map pseudomonoid (H, j_*, p_*) in **Comod**($\mathscr{V}$) with left dualization $d = s_*$.

Lemma 5.25. *The arrows sj and $j : I \to H$ are equal in $\mathscr{V}$.*

Proof. Write j^{op} for the arrow j regarded as a comonoid morphism $I \to H^{\circ}$. It is easy to see that $j^{*\circ} \cong j_*^{\mathrm{op}} : I \to H^{\circ}$. By Proposition 2.38, the left dualization s_* is a strong monoidal morphism. In particular, there exists an isomorphism $(sj^{\mathrm{op}})_* \cong s_* j^{*\circ} \cong j_*$, and then an isomorphism $sj^{\mathrm{op}} \cong j$ in **Comon**($\mathscr{V}$). This amounts to an invertible arrow $\gamma : I \to I$ such that $\gamma \otimes sj = j \otimes \gamma$; hence $sj = j$. $\square$

Lemma 5.26. *Let M be a left H-comodule.*

1. *M is invertible in the monoidal category* **Comod**($\mathscr{V}$)(H, I) *if and only if the object $M \in \mathscr{V}$ is invertible.*

2. *If M is invertible, M is isomorphic to $f^* \otimes (M\varepsilon^*)$ where $f : I \to H$ is the unique morphism such that $f \otimes M$ is the coaction of M.*

3. *If M is invertible and $f, g : I \to H$ are the comonoid morphisms corresponding to M and its left dual respectively, then $p(f \otimes g) = j = p(g \otimes f)$.*

Proof. 1. Clearly, $\varepsilon : H \to I$ is a strict monoidal 1-cell from the pseudomonoid (H, j, p) to I. Hence, the monadic functor

$$\mathbf{Comod}(\mathscr{V})(\varepsilon^*, I) : \mathbf{Comod}(\mathscr{V})(H, I) \to \mathbf{Comod}(\mathscr{V})(I, I) \simeq \mathscr{V} \tag{5.10}$$

is strong monoidal. Therefore, $M\varepsilon^*$ is invertible whenever M is so.

Conversely, by [13, Prop. 5.1] (see Proposition 5.14), $M\varepsilon^*$ has a dual if and only if M has left adjoint, and this happens if and only if M has left a dual in $\mathbf{Comod}(\mathscr{V})(H, I)$, by Proposition 4.6. Moreover, the coevaluation and evaluation for $M\varepsilon^*$ are the image under (5.10) of the ones of M. Then, the invertibility of M follows, because (5.10) is conservative.

2. If $M \in \mathscr{V}$ is invertible, tensoring the coaction $M \to H \otimes M$ with $M^\vee$ on the right we obtain an arrow $f : I \to H$. The coassociativity of the coaction ensures that f is a comonoid morphism. Clearly $f^* \otimes M\varepsilon^* \cong M$.

3. Denote a left dual of M by $M^\triangleleft$. The underlying arrow in $\mathscr{V}$ of the evaluation $\mathrm{ev} : M^\triangleleft \otimes M \to I$ is an isomorphism between $(g^* \otimes M^\triangleleft\varepsilon^*) \otimes (f^* \otimes M\varepsilon^*) = (p(g \otimes f)^* \otimes (M^\triangleleft \otimes M)\varepsilon^*)$ and j^*. If one writes this explicitly as diagrams in $\mathscr{V}$, one immediately sees that it implies $p(g \otimes f) = j : I \to H$. An analogous reasoning proves that $p(f \otimes g) = j$. □

When s is invertible, denote by $\bar{s} : H^\circ \to H$ the morphism of comonoids whose underlying arrow in $\mathscr{V}$ is the inverse of s.

Lemma 5.27. *With the notation above, H is also right autonomous if and only if $s : H^\circ \to H$ is invertible in $\mathbf{Comon}(\mathscr{V})$, or equivalently, in $\mathscr{V}$. Moreover, in this case the right dualization is given by $\bar{s}_*$.*

Proof. From [13, Propositions 1.4 and 1.5] we know that a left autonomous map pseudomonoid is right autonomous if and only if the left dualization is an equivalence. Then, H is right autonomous if and only if $s_* \dashv s^*$ is an equivalence. In particular, the counit, that is given by $s : H \to H$ (see beginning of Section 5.2.2), is an isomorphism. The converse is clear.

The right dualization is given by $(s_*)^\circ$, by [13, Prop. 1.4] (recall that in our case left and right bidual coincide); so it is the comodule with underlying object H and left and right coactions

$$(H \xrightarrow{\Delta} H \otimes H \xrightarrow{c} H \otimes H \xrightarrow{1\otimes s} H \otimes H \xrightarrow{c} H \otimes H) = (H \xrightarrow{\Delta} H \otimes H \xrightarrow{s\otimes 1} H \otimes H)$$

and

$$(H \xrightarrow{\Delta} H \otimes H \xrightarrow{c} H \otimes H \xrightarrow{c} H \otimes H) = \Delta$$

respectively. It is clear that the arrow $s : H \to H$ is an isomorphism of comodules from $(s_*)^\circ$ to $\bar{s}_*$. □

From now on we will suppose that the antipode of H is invertible. The right dualization $\bar{d}$ is just $\bar{s}_*$, where $\bar{s} : H^\circ \to H$ is the inverse of s.

If H has dual in $\mathscr{V}$, so that $\mathrm{n} : I \to H^\circ \otimes H$ has a right adjoint, then there exists a comodule $\mathsf{W} : H \to I$ such that $(j_*\mathsf{W}) * 1_H \cong 1_H^{\triangleleft}$ has $(- * 1_H)$-algebras (see (4.14)). Here, 1_H is the regular bicomodule H, $j_*\mathsf{W}$ is just W with trivial right coaction induced by the unit j; then, $(j_*\mathsf{W}) * 1_H$ can be taken as the comodule with underlying object $\mathsf{W} \otimes H$, left and right coactions

$$\mathsf{W} \otimes H \xrightarrow{\chi \otimes \Delta} H \otimes \mathsf{W} \otimes H^{\otimes 2} \xrightarrow{1 \otimes c \otimes 1} H^{\otimes 2} \otimes \mathsf{W} \otimes H \xrightarrow{p \otimes 1 \otimes 1} H \otimes W \otimes H$$
$$\mathsf{W} \otimes H \xrightarrow{1 \otimes \Delta} \mathsf{W} \otimes H \otimes H.$$

The right action of 1_H on $j_*\mathsf{W} * 1_H$ is just the morphism $\mathsf{W} \otimes p : \mathsf{W} \otimes H \otimes H \to \mathsf{W} \otimes H$. Using Proposition 4.10, the comodule $1_H^{\triangleleft}$ can be taken to be the left dual $H^\vee$ of H with coaction

$$H^\vee \xrightarrow{1 \otimes \mathrm{coev}} H^\vee \otimes H \otimes H^\vee \xrightarrow{1 \otimes (1 \otimes \Delta)\Delta \otimes 1} H^\vee \otimes H^{\otimes 3} \otimes H^\vee \longrightarrow$$
$$\xrightarrow{c_{H^\vee, H} \otimes 1 \otimes c_{H, H^\vee}} H \otimes H^\vee \otimes H \otimes H^\vee \otimes H \xrightarrow{1 \otimes \mathrm{ev} 1 \otimes 1} H \otimes H^\vee \otimes H \xrightarrow{s^{-1} \otimes 1 \otimes s} H \otimes H^\vee \otimes H$$

where c denotes the symmetry of $\mathscr{V}$. Analogously, the right action of 1_H on $1_H^{\triangleleft}$ is the morphism

$$H^\vee \otimes H \xrightarrow{1 \otimes 1 \otimes \mathrm{coev}} H^\vee \otimes H^{\otimes 2} \otimes H^\vee \xrightarrow{1 \otimes p \otimes 1} H^\vee \otimes H \otimes H^\vee \xrightarrow{\mathrm{ev} \otimes 1} H^\vee.$$

Observation 5.28. The comodule $\mathsf{W} : H \to I$ is isomorphic to the composition of $1_H^{\triangleleft}$ with $j^* : H \to I$; in other words, W is the equalizer of $(H^\vee \otimes s)(c_{H,H^\vee})(\mathrm{ev} \otimes H \otimes H^\vee)(H^\vee \otimes \Delta \otimes H^\vee)(H^\vee \otimes \mathrm{coev})$ and $H^\vee \otimes j$, or, composing with the symmetry and s^{-1} and using Lemma 5.25, the equalizer of

$$H^\vee \xrightarrow{1 \otimes \mathrm{coev}} H^\vee \otimes H \otimes H^\vee \xrightarrow{1 \otimes \Delta \otimes 1} H^\vee \otimes H^{\otimes 2} \otimes H^\vee \xrightarrow{\mathrm{ev} \otimes 1 \otimes 1} H \otimes H^\vee$$

and $j \otimes H^\vee$. We call W the *object of right cointegrals* of H. When the base monoidal category is **Vect**, W is the usual space of right cointegrals

$$\{\phi \in H^\vee \mid \sum \phi(x_1)x_2 = \phi(x)j\}.$$

Definition 5.1. When the comodule $\mathsf{W} : H \to I$ is invertible, define $b : I \to H$ as the invertible comonoid morphism given by Lemma 5.26. Define the *modular* *element* of H as $a = sb : I \to H$.

Example 5.29. When the base monoidal category is **Vect**, the element $a \in H$ defined above is the usual modular element. In order to show this we compare two expressions for the coaction of W. The diagram on the left hand side below, in which the vertical arrows are the coactions, commutes by definition of W.

$$\begin{tikzcd} \mathsf{W} \arrow[r, hook] \arrow[d] & H^\vee \arrow[d] & \mathsf{W} \otimes H \arrow[r] \arrow[d] & H^\vee \otimes H \arrow[d] \\ H \otimes \mathsf{W} \arrow[r, hook] & H \otimes H^\vee & H \otimes \mathsf{W} \otimes H \arrow[r] & H \end{tikzcd}$$

Taking mates under duals, we get the commutative square on the right hand side, *i.e.*, for any $\phi \in \mathsf{W}$ and $x \in H$, $\phi(x)b = \sum \phi(x_2)s^{-1}(x_1)$; equivalently,

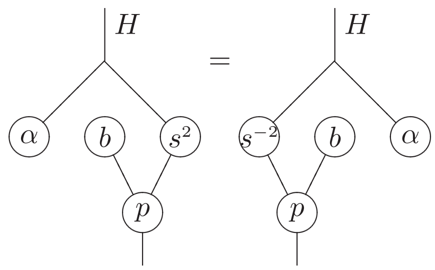
$$\phi(x)a = \sum \phi(x_2)x_1.$$

This means that a is the modular element, since $(\psi * \phi)(x) = \sum \psi(x_1)\phi(x_2) = \psi(a)\phi(x)$ for any $\psi \in H^\vee$.

Proposition 5.30. *Let (H, j, p, s) be a coquasi-Hopf algebra in $\mathscr{V}$, and suppose the comodule W is invertible. Then any isomorphism*

$$\tau : (H \xrightarrow{p^*} H \otimes H \xrightarrow{1 \otimes \mathsf{W}} H \xrightarrow{d\bar{d}^*} H) \cong (H \xrightarrow{p^*} H \otimes H \xrightarrow{\mathsf{W} \otimes 1} H \xrightarrow{\bar{d}d^*} H)$$

in $\mathbf{Comod}(\mathscr{V})$*, where $d = s_*$, gives rise to a convolution invertible arrow $\alpha : H \to I$ in $\mathscr{V}$ satisfying*



(5.11)

where $b : I \to H$ is as in Definition 5.1.

Proof. Denote by $\bar{s}$ and $\tilde{s}$ the inverse of s when regarded as a coalgebra morphism $H^\circ \to H$ and $H \to H^\circ$ respectively. Also denote by $\hat{s}$ the morphism s when regarded as a morphism $H \to H^\circ$. Then, $\bar{s} = \hat{s}^{-1}$ and $s = \tilde{s}^{-1}$ in $\mathbf{Comon}(\mathscr{V})$, and hence $\bar{d} \cong \bar{s}_* \cong \hat{s}^*$ and $d \cong \tilde{s}^*$. Thus

$$d\bar{d}^* \cong \tilde{s}^*\bar{s}^* \cong (\bar{s}\tilde{s})^* \cong (s^{-2})^* \qquad \bar{d}d^* \cong \hat{s}^*s^* \cong (s\hat{s})^* \cong (s^2)^*,$$

where $s^{-2}, s^2 : H \to H$ in $\mathbf{Comon}(\mathscr{V})$. We can rewrite the isomorphism τ as an isomorphism $(p(s^{-2} \otimes b))^* \cong (p(b \otimes s^2))^*$, and hence it is induced by an isomorphism $\alpha : p(b \otimes s^2) \cong p(s^{-2} \otimes b)$ in $\mathbf{Comon}(\mathscr{V})$, in the sense that $\tau = (\alpha \otimes H)\Delta$, and this is exactly our result.

$\square$

Observation 5.31. Recall the following standard notation in Hopf Algebra Theory. If C is a coalgebra, its dual $C^\vee$ acts on C on the left and on the right by

$$\gamma \otimes x \mapsto \gamma \rightharpoonup x = \sum x_1\gamma(x_2) \qquad x \otimes \gamma \mapsto x \leftharpoonup \gamma = \sum \gamma(x_1)x_2.$$

Then, the equality (5.11) is often written as

$$b \cdot s^2(x \leftharpoonup \alpha) = s^{-2}(\alpha \rightharpoonup x) \cdot b.$$

Lemma 5.32. *Let k be a commutative ring and $\mathscr{U} = k\text{-Mod}^{\mathrm{op}}$. Denote by $\mathbf{Comod}(\mathscr{U})(B, E)_f$ the category of B-E-bicomodules whose underlying k-module has a dual. Then any natural transformation*

$$\tau : \mathbf{Comod}(\mathscr{U})(M, I) \Rightarrow \mathbf{Comod}(\mathscr{U})(N, I)$$

between functors $\mathbf{Comod}(\mathscr{U})(D, I)_f \to \mathbf{Comod}(\mathscr{U})(C, I)$ is of the form

$$\tau = \mathbf{Comod}(\mathscr{U})(\gamma, I)$$

for a unique comodule morphism $\gamma : M \to N$.

Proof. Write $D\text{-Mod}_f$ for the category of D-modules whose underlying k-module has a dual. This category is the *Cauchy completion* (*i.e.*, the completion under absolute colimits) of its full subcategory determined by D. Observe that $\mathbf{Comod}(\mathscr{U})(D, I)_f$ is just $D\text{-Mod}_f^{\mathrm{op}}$, the category of D-modules with a dual within k-Mod. This is because a comonoid in $\mathscr{U}$ is a k-algebra and a comodule in $\mathscr{U}$ a k-module. Then $\mathbf{Comod}(\mathscr{U})(D, I)_f$ is the completion under absolute limits of its full subcategory determined by the regular comodule D. Hence τ is of the form $\gamma\square_D-$ for a unique morphism of left C-comodules $\gamma : M \to N$ which can be easily seen to be also a morphism of right D-comodules. $\square$

Lemma 5.33. *Let k be a commutative ring.*

1. *A k-module L is invertible if and only if $L_{\mathfrak{m}}$ is a $k_{\mathfrak{m}}$-vector space of dimension one for each maximal ideal $\mathfrak{m}$ of k.*

2. *If a k-module L has (categorical) dual $L^{\vee}$ and M is a k-module such that $M \otimes L \cong L^{\vee}$, then M is invertible.*

Proof. (1) Localisation is a (normal) monoidal functor k-Mod $\to k_{\mathfrak{m}}$-Mod; this is just the well known fact that $M_{\mathfrak{m}} \otimes_{k_{\mathfrak{m}}} N_{\mathfrak{m}}$ is canonically isomorphic to $(M \otimes N)_{\mathfrak{m}}$. Therefore, the evaluation $\mathrm{ev} : L \otimes L^{\vee} \to k$ gives evaluations $L_{\mathfrak{m}} \otimes_{k_{\mathfrak{m}}} L^{\vee}_{\mathfrak{m}} \to k_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$, each of which is an isomorphism since $L_{\mathfrak{m}}$ has dimension one over $k_{\mathfrak{m}}$. If follows that $\mathrm{ev}_{\mathfrak{m}}$ is an isomorphism for each $\mathfrak{m}$, and then ev is an isomorphism.

(2) For each maximal ideal $\mathfrak{m}$, we have $M_{\mathfrak{m}} \otimes_{k_{\mathfrak{m}}} L_{\mathfrak{m}} \cong (L_{\mathfrak{m}})^{\vee}$, where the dual is taken in the category of $k_{\mathfrak{m}}$-vector spaces. Then, $\dim_{k_{\mathfrak{m}}} M_{\mathfrak{m}} = 1$ for every maximal ideal $\mathfrak{m}$, and hence M is an invertible k-module. $\square$

A quasi-Hopf algebra H in k-Mod is a coquasi-Hopf algebra in $\mathscr{U} = k\text{-Mod}^{\mathrm{op}}$. If H has dual as a k-module, we can consider the 1-cell $\mathsf{W} : H \to I$ in $\mathbf{Comod}(\mathscr{U})$ as before. This 1-cell is just a left H-module, and when W is invertible the action of H is given by a convolution invertible multiplicative functional $\beta : H \to k$.

If Δ is the comultiplication of a quasibialgebra, we will write $\Delta(x) = \sum x_1 \otimes x_2$. However, the usual computations with Sweedler's notation do not apply since Δ is not associative.

Theorem 5.34. *Let k be a commutative ring and H a k-algebra with multiplication $m : H \otimes H \to H$ and unit $u \in H$. Assume H is projective and finitely generated as k-module, and equipped with a quasi-Hopf algebra structure with comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to k$ and antipode $s : H^{\circ} \to H$. Then there exists an invertible $h \in H$ such that*

$$\sum \beta(x_1) h \cdot s^2(x_2) = \sum \beta(x_2) s^{-2}(x_1) \cdot h, \tag{5.12}$$

where $\beta : H \to k$ is the invertible multiplicative functional given by the H-module structure of W.

Proof. Write $\mathscr{U}$ for k-Mod$^{\mathrm{op}}$. The 1-cell $\mathsf{W} : H \to I$ in $\mathbf{Comod}(\mathscr{U})$ satisfies $\mathsf{W} \otimes 1_H \cong 1_H^{\triangleleft}$; in particular, there is an isomorphism of k-modules $\mathsf{W} \otimes H \cong H^{\vee}$, and therefore, W is an invertible k-module by Lemma 5.33. By Lemma 5.26 we deduce that W is invertible in $\mathbf{Comod}(\mathscr{U})(H, I) \simeq H$-Mod$^{\mathrm{op}}$.

On the other hand, by means of Lemma 5.32 and Corollary 4.16 we obtain an isomorphism

$$(H \xrightarrow{p^*} H \otimes H \xrightarrow{1 \otimes \mathsf{W}} H \xrightarrow{d\bar{d}^*} H) \cong (H \xrightarrow{p^*} H \otimes H \xrightarrow{\mathsf{W} \otimes 1} H \xrightarrow{\bar{d}d^*} H)$$

while Proposition 5.30 tells us that (5.12) holds. □

Now we turn to the case of coquasi-Hopf algebras.

Lemma 5.35. *If we write $\mathscr{M} = \mathbf{Comod}(\mathbf{Vect})$, any natural transformation $\tau : \mathscr{M}(f^*, I) \Rightarrow \mathscr{M}(g^*, I) : \mathscr{M}(D, I)_f \to \mathscr{M}(C, I)_f$, where $f, g : D \to C$ are coalgebra morphisms and D and C have dual in* **Vect**, *is of the form $\mathscr{M}(\alpha^*, I)$ for a unique 2-cell $\alpha : g \Rightarrow f$ in* $\mathbf{Comon}(\mathbf{Vect})$. *Moreover, τ is invertible if and only if α is too.*

Proof. Set $\mathscr{V} = \mathbf{Vect}$. There are isomorphisms between $\mathbf{Comod}(\mathscr{V})(D, I)_f$ and $\mathbf{Mod}(\mathscr{V})(D^{\vee}, I)_f$ sending a comodule to the module with same underlying space but with the action of $D^{\vee}$. Under these isomorphisms, τ becomes a natural transformation

$$\mathbf{Mod}(\mathscr{V})(f^{\vee *}, I) \Rightarrow \mathbf{Mod}(\mathscr{V})(g^{\vee *}, I) : \mathbf{Mod}(\mathscr{V})_f(D^{\vee}, I) \to \mathbf{Mod}(\mathscr{V})_f(C^{\vee}, I)$$

where $f^{\vee *}$ is the bimodule with left and right actions

$$C^{\vee} \otimes D^{\vee} \xrightarrow{f^{\vee} \otimes 1} D^{\vee} \otimes D^{\vee} \to D^{\vee} \qquad \text{and} \qquad D^{\vee} \otimes D^{\vee} \to D^{\vee},$$

and analogously for g. This is just a natural transformation $\tau : ((f^{\vee})^* \otimes_{D^{\vee}} -) \Rightarrow ((g^{\vee})^* \otimes_{D^{\vee}} -)$ between functors from $D^{\vee}$-Mod$_f$ to $C^{\vee}$-Mod$_f$, and any such transformation is of the form $(\beta \otimes_{D^{\vee}} -)$ for a unique morphism of bimodules $\beta : (f^{\vee})^* \to (g^{\vee})^*$. To give β is the same as to give a morphism of bicomodules $\sigma : g_* \to f_*$; in fact, $\sigma^{\vee} = \beta$ as morphisms in **Vect**. Finally, $\tau = \mathscr{M}(\alpha^*, I)$ where $\alpha_* = \sigma$. □

Theorem 5.36. *If H is a finite dimensional coquasi-Hopf algebra (in* **Vect***),* *then there exists a convolution-invertible functional* $\alpha : H \to k$ *such that*

$$b \cdot s^2(x \leftharpoonup \alpha) = s^{-2}(\alpha \rightharpoonup x) \cdot b.$$

Proof. First we have to check that the comodule W is invertible. This is clear since $W \otimes H \cong H^\vee$, and hence $\dim \mathsf{W} = 1$, and using Lemma 5.26. Now, Lemma 5.35 together with Corollary 4.16 provide us with an isomorphism of comodules as in Proposition 5.30, and hence with our result. $\square$

Chapter 6

Pseudo-commutative enriched monads and monoidal structures

We would like to apply the theory of autonomous pseudomonoids developed in previous chapters to $\mathscr{V}$-categories with certain class of (co)limits, or more generally, to 2-categories of algebras and pseudomorphisms for a 2-monad. To this aim, we must endow a 2-category of the form T-Alg with a monoidal structure compatible in certain sense with T. Of course, it will not be possible to accomplish this for any 2-monad T. An answer to our needs is given by the *pseudo-commutative* 2-monads introduced in [37].

The main result of this chapter is Theorem 6.35, where we show that each lax-idempotent or KZ 2-monad and each colax-idempotent 2-monad has a canonical structure of a pseudo-commutative 2-monad. This considerably extends the set of examples given in [37].

As we already mentioned, our main examples, which will be treated in the next chapter, are $\mathscr{V}$-categories with certain (co)limits. This amounts to considering 2-monads on the 2-category $\mathscr{V}$-**Cat**. As a result the constructions in [37] cannot be applied without modification. The problem arises in that in the mentioned paper the authors construct monoidal structures on 2-categories of the form T-Alg for a pseudo-commutative 2-monad T on **Cat**. If one tries to replace **Cat** by $\mathscr{V}$-**Cat** one finds that several additional hypotheses are required on $\mathscr{V}$, which essentially amount to the monadicity of $\mathscr{V}$ over **Set**. To avoid these undesirable restrictions we are forced to develop the theory of pseudo-commutative $\mathscr{W}$-enriched monads, where $\mathscr{W}$ is a symmetric monoidal closed **Cat**-enriched category (2-category). When $\mathscr{W}$ is $\mathscr{V}$-**Cat** we obtain our main examples. The price we have to pay for this generality is to extend some of the basic theory of 2-monads to the $\mathscr{W}$-enriched case.

The philosophy of [37] is that in many examples of monoidal closed structures, the closed structure is easier to describe and behaves better that the monoidal structure. The simplest example is the category of vector spaces with the usual monoidal closed structure. The authors define a notion of *pseudo-closed* 2-category that is a semi-strict version of closedness, in the sense that it is as strict as the examples of interest allow. The weakness in this definition is introduced in the conditions involving the unit object. Under certain conditions, a pseudo-closed structure induces a (weak or pseudo) monoidal structure by means of biadjunctions.

The structure on a 2-monad T on **Cat** that corresponds to a pseudo-closed structure on T-Alg is called a *pseudo-commutativity*. Pseudo-commutative 2-monads are the 2-dimensional analogue of the commutative monads introduced in [48] and further studied in [49, 50].

For a complete and cocomplete symmetric monoidal closed **Cat**-category $\mathscr{W}$, we develop an enriched version of many of the results in [37]. In Section 6.1 we give the basic definitions and properties of $\mathscr{W}$-limits and their relation with 2-limits. In particular, we look at the 2-dimensional aspects of $\mathscr{W}$-limits.

In Section 6.2 we define for a $\mathscr{W}$-enriched monad T a $\mathscr{W}$-category T-Alg. We study the preservation of $\mathscr{W}$-limits and colimits of the "inclusion" $\mathscr{W}$-functor $J : T\text{-Alg}_s \to T\text{-Alg}$ with domain the usual Eilenberg-Moore $\mathscr{W}$-category of T. These constructions capture the usual elements of the 2-monad theory of [7] in the case $\mathscr{W} = \mathbf{Cat}$.

Section 6.3 introduces pseudo-closed $\mathscr{W}$-categories and pseudo-commutative $\mathscr{W}$-monads, while Section 6.4 gives an alternative description of a pseudo-commutativity essential in our proof of the pseudo-commutativity of lax-idempotent 2-monads.

In Section 6.5 we describe the pseudo-closed structure of the $\mathscr{W}$-category T-Alg for a pseudo-commutative $\mathscr{W}$-monad T, and in Section 6.6 we induce monoidal structures from pseudo-closed ones.

Finally, Section 6.7 shows that lax-idempotent and colax-idempotent $\mathscr{W}$-monads are equipped with canonical pseudo-commutativities. As a consequence, for a finitely presentable monoidal category $\mathscr{V}$, the 2-category of $\mathscr{V}$-categories with finite (co)limits and finitely (co)continuous $\mathscr{V}$-functors has canonical pseudo-closed and monoidal structures. To keep this chapter of a reasonable size, we study this consequence and others in the next chapter.

6.1 $\mathscr{W}$-limits when $\mathscr{W}$ is a 2-category

Fix a complete and cocomplete monoidal closed 2-category (*i.e.*, **Cat**-category) $\mathscr{W}_1$. We will write $\mathscr{W}_0$ for the underlying category of $\mathscr{W}_1$, which is a complete and cocomplete monoidal closed category. We can consider $\mathscr{W}$-categories, $\mathscr{W}$-functors and $\mathscr{W}$-natural transformations by enriching in $\mathscr{W}_0$. We write $\mathscr{W}$ for the $\mathscr{W}$-category $\mathscr{W}$, with enriched structure induced by the closed structure.

Denote by W the 2-functor $\mathscr{W}_1(I,-) : \mathscr{W}_1 \to \mathbf{Cat}$. The corresponding underlying functor $W_0 : \mathscr{W}_0 \to \mathbf{Cat}_0$ is lax monoidal, and then it induces a 2-functor $(-)_1 = (W_0)_* : \mathscr{W}\text{-}\mathbf{Cat} \to \mathbf{2\text{-}Cat}$. The image of a $\mathscr{W}$-category $\mathscr{K}$ under $(-)_1$ is denoted by $\mathscr{K}_1$ and called the *underlying 2-category* of $\mathscr{K}$. Similarly, if F is a $\mathscr{W}$-functor, we call F_1 the *underlying 2-functor* of F. The underlying 2-category of $\mathscr{W}$ is $\mathscr{W}_1$. Observe that the underlying category of the 2-category $\mathscr{K}_1$ is $\mathscr{K}_0$, the underlying category of the enriched category $\mathscr{K}$.

Since W_0 has left adjoint given by taking tensor product with the neutral object $I \in \mathscr{W}$, the 2-functor $(-)_1$ has a left adjoint $\mathcal{F} : \mathbf{2\text{-}Cat} \to \mathscr{W}\text{-}\mathbf{Cat}$.

Observation 6.1. If X is an object of a $\mathscr{W}$-category $\mathscr{K}$, the representable 2-functor $\mathscr{K}_1(X,-) : \mathscr{K}_1 \to \mathbf{Cat}$ equals the composition of $\mathscr{K}(X,-)_1 : \mathscr{K}_1 \to \mathscr{W}_1$ with $W : \mathscr{W}_1 \to \mathbf{Cat}$.

Given $\mathscr{W}$-functors $F, G : \mathscr{K} \to \mathscr{L}$, with $\mathscr{K}$ small, there is a canonical isomorphism between the categories $W([\mathscr{K},\mathscr{L}](F,G))$ and $\mathscr{W}\text{-}\mathbf{Cat}(\mathscr{K},\mathscr{L})(F,G)$.

Observation 6.2. For any 2-category $\mathscr{A}$ and $\mathscr{W}$-category $\mathscr{K}$ there is a $\mathscr{W}$-category of 2-functors $\mathscr{A} \to \mathscr{K}_1$, denoted by $[\mathscr{A},\mathscr{K}]$. The $\mathscr{W}$-enriched homs are given by the usual end formula: $[\mathscr{A},\mathscr{K}](f,g) = \int_{x\in\mathscr{A}} \mathscr{K}(fx,gx)$. In fact, $\mathscr{W}$-**Cat** is monoidal closed and hence canonically a ($\mathscr{W}$-**Cat**)-category, and hence a (**2-Cat**)-category via the 2-functor $(-)_1 : \mathscr{W}\text{-}\mathbf{Cat} \to \mathbf{2\text{-}Cat}$. The $\mathscr{W}$-category $[\mathscr{A},\mathscr{K}]$ is just the cotensor product of the 2-category $\mathscr{A}$ with the $\mathscr{W}$-category $\mathscr{K}$. In particular, there are canonical isomorphisms between $[\mathscr{A},\mathscr{K}]$ and the usual $\mathscr{W}$-category of $\mathscr{W}$-functors $[\mathcal{F}(\mathscr{A}),\mathscr{K}]$.

Let $\phi : \mathscr{P} \to \mathbf{Cat}$ be a 2-functor. Denote by $\bar{\phi} : \mathcal{F}(\mathscr{P}) \to \mathscr{W}$ the unique

$\mathscr{W}$-functor such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}(\mathscr{P})_1 \arrow[r, "\bar{\phi}_1"] & \mathscr{W}_1 \\ \mathscr{P} \arrow[u, "N"] \arrow[r, "\phi"'] & \mathbf{Cat} \arrow[u, "-*I"'] \end{tikzcd} \tag{6.1}$$

Here $N : \mathscr{P} \to \mathcal{F}(\mathscr{P})_1$ is the unit of the adjunction $\mathcal{F} \dashv (-)_1$. Let $G : \mathcal{F}(\mathscr{P}) \to \mathscr{K}$ be a $\mathscr{W}$-functor corresponding to a 2-functor $F : \mathscr{P} \to \mathscr{K}_1$. We define the $\mathscr{W}$-limit $\{\phi, F\}_{\mathscr{W}}$ of F weighted by ϕ as the usual weighted limit $\{\bar{\phi}, G\}$.

Lemma 6.3. *Let $\mathscr{K}$ be a $\mathscr{W}$-category and $F : \mathscr{P} \to \mathscr{K}_1$ be a 2-functor with corresponding $\mathscr{W}$-functor $G : \mathcal{F}(\mathscr{P}) \to \mathscr{K}$. If $\mu : \bar{\phi} \to \mathscr{K}(L, G-)$ is a $\mathscr{W}$-limiting cylinder, then*

$$\phi \xrightarrow{\text{counit}\phi} W(-*I)\phi = W\bar{\phi}_1 N \xrightarrow{W\mu N} W\mathscr{K}(L, G-)_1 N = \mathscr{K}_1(L, F-) \tag{6.2}$$

is a **Cat***-limiting cylinder.*

Proof. By hypothesis, μ induces $\mathscr{W}$-natural isomorphisms

$$\mathscr{K}(Y, L) \cong [\mathcal{F}(\mathscr{P}), \mathscr{W}](\bar{\phi}, \mathscr{K}(Y, G-)).$$

These isomorphisms constitute an arrow in the category $\mathscr{W}$-$\mathbf{Cat}(\mathscr{K}^{\mathrm{op}}, \mathscr{W})$. Applying the 2-functor $(-)_1 : \mathscr{W}\text{-}\mathbf{Cat} \to \mathbf{2\text{-}Cat}$ we get an arrow in $\mathbf{2\text{-}Cat}(\mathscr{K}_1^{\mathrm{op}}, \mathscr{W}_1)$, and composing with the 2-functor $W : \mathscr{W}_1 \to \mathbf{Cat}$ we get the first 2-natural isomorphism in the chain of isomorphisms below (see Observation 6.1).

$$\begin{aligned} \mathscr{K}_1(Y, L) &\cong [\mathcal{F}(\mathscr{P}), \mathscr{W}]_1(\bar{\phi}, \mathscr{K}(Y, G-)) \\ &\cong \mathscr{W}\text{-}\mathbf{Cat}(\mathcal{F}(\mathscr{P}), \mathscr{W})(\bar{\phi}, \mathscr{K}(Y, G-)) \\ &\cong \mathbf{2\text{-}Cat}(\mathscr{P}, \mathscr{W}_1)(\bar{\phi}_1 N, \mathscr{K}(Y, G-)_1 N) \\ &= \mathbf{2\text{-}Cat}(\mathscr{P}, \mathscr{W}_1)((-*I)\phi, \mathscr{K}(Y, G-)_1 N) \\ &\cong \mathbf{2\text{-}Cat}(\mathscr{P}, \mathbf{Cat})(\phi, W\mathscr{K}(W, G-)_1 N) \\ &\cong \mathbf{2\text{-}Cat}(\mathscr{P}, \mathbf{Cat})(\phi, \mathscr{K}_1(Y, F-))) \end{aligned}$$

The third isomorphism is induced by $\mathcal{F} \dashv (-)_1$, the equality by (6.1), the fifth isomorphism is induced by the adjunction $(-*I) \dashv W : \mathscr{W}_1 \to \mathbf{Cat}$, and the last isomorphism by Observation 6.1 and definition of G. Now it is easy to see that the 2-natural transformation $\phi \to \mathscr{K}_1(L, F-)$ corresponding to the identity 1-cell $1 \in \mathscr{K}_1(L, L)$ is (6.2). $\square$

Observation 6.4. Let $\phi : \mathscr{P} \to \mathbf{Cat}$ be a weight, $\bar{\phi} : \mathcal{F}(\mathscr{P}) \to \mathscr{W}$ the induced $\mathscr{W}$-functor and $\mathscr{K}$ a $\bar{\phi}$-complete $\mathscr{W}$-category. Recall from Observation 6.2 that there is a $\mathscr{W}$-category of 2-functors $[\mathscr{P}, \mathscr{K}]$. Then $\{\phi, -\}$ is the underlying 2-functor of a $\mathscr{W}$-functor $[\mathscr{P}, \mathscr{K}] \to \mathscr{K}$, namely, the composition of the isomorphism $[\mathscr{P}, \mathscr{K}] \cong [\mathcal{F}(\mathscr{P}), \mathscr{K}]$ and $\{\bar{\phi}, -\}$.

Given a set of **Cat**-weights Φ, *i.e.*, a set of 2-functors $\phi : \mathscr{P} \to \mathbf{Cat}$ with $\mathscr{P}$ small, we denote by $\bar{\Phi}$ the set of $\mathscr{W}$-weights $\bar{\phi}$ with $\phi \in \Phi$.

Corollary 6.5. *Let $\mathscr{K}$ be a $\mathscr{W}$-category. The 2-category $\mathscr{K}_1$ is Φ-complete whenever $\mathscr{K}$ is $\bar{\Phi}$-complete.*

6.2 Enriched categories of algebras

We shall call monads (T, η, μ) in the 2-category $\mathscr{W}$-**Cat** *$\mathscr{W}$-enriched monads.* When there is no chance of confusion we omit the unit η and the multiplication μ of the monad and write only T. As usual, we have the Eilenberg-Moore $\mathscr{W}$-category of algebras, which we will denote by $T\text{-Alg}_s$, and we recall briefly here.

Let T be a $\mathscr{W}$-enriched monad on the $\mathscr{W}$-category $\mathscr{K}$. The objects of $T\text{-Alg}_s$, called (strict) T-algebras, are T_0-algebras, where T_0 is the underlying (ordinary) monad of T on the category $\mathscr{K}_0$. We will write T-algebras as pairs (A, a) where $a : TA \to A$ is the algebra structure, or simply as A when there is no place to confusion. The enriched hom $T\text{-Alg}_s((A, a), (B, b))$ is given by the equalizer of the following pair of arrows in $\mathscr{W}_0$.

$$\begin{tikzcd} \mathscr{K}(A, B) \arrow[r, "T"] \arrow[rr, bend right, "\mathscr{K}(a,1)"'] & \mathscr{K}(TA, TB) \arrow[r, "\mathscr{K}(1,b)"] & \mathscr{K}(TA, B) \end{tikzcd} \tag{6.3}$$

The fact that $\mathscr{W}_1$ is a 2-category provides us with an extra dimension, allowing us to define the enriched analogous of many 2-categorical constructions.

6.2.1 Lax and pseudo morphisms

For each object $X \in \mathscr{K}$ and each T-algebra (B, b) define a 1-cell in $\mathscr{W}$

$$\sigma_{X,B} : \mathscr{K}(X, B) \xrightarrow{T} \mathscr{K}(TX, TB) \xrightarrow{\mathscr{K}(1,b)} \mathscr{K}(TX, B). \tag{6.4}$$

When A, B are T-algebras, the 1-cells $\sigma_{A,B}$ are the components of a $\mathscr{W}$-natural transformation $\sigma : \mathscr{K}(U_s-, U_s-) \Rightarrow \mathscr{K}(TU_s-, U_s-) : T\text{-Alg}_s^{\mathrm{op}} \otimes T\text{-Alg}_s \to \mathscr{W}$. Observe that σ satisfy the following equations:

$$\sigma_{TA,B}\sigma_{A,B} = \mathscr{K}(\mu_A, B)\sigma_{A,B} \qquad \mathscr{K}(\eta_A, B)\sigma_{A,B} = 1 \tag{6.5}$$

Definition 6.1. Given two T-algebras (A, a) and (B, b) in $\mathscr{K}_1$, define a 1-cell $p : L \to \mathscr{K}(A, B)$ and a 2-cell

$$\begin{tikzcd} & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] & \\ L \arrow[ur, "p"] \arrow[dr, "p"'] & \Downarrow\gamma & \mathscr{K}(TA,B) \\ & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] & \end{tikzcd} \tag{6.6}$$

The pair (p, γ) is defined as the universal such pair satisfying the equalities in Figure 6.1, in the following sense. If $(q : M \to \mathscr{K}(A, B), \delta)$ is another pair satisfying the same conditions, then there exists a unique $f : L' \to L$ such that $q = pf$ and $\delta = \gamma q$. Moreover, suppose that $(r : N \to \mathscr{K}(A, B), \epsilon)$ is yet another pair satisfying the conditions and denote by $g : N \to L$ the corresponding 1-cell. If $\varpi : q \Rightarrow r$ is a 2-cell compatible with δ and ϵ in the sense that

$$\begin{tikzcd} & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] & \\ N \arrow[ur, bend left=60, "q", ""{name=Q, below}] \arrow[ur, "r"', ""{name=R}] \arrow[dr, "r"'] & \Downarrow\delta & \mathscr{K}(TA,B) \\ & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] & \arrow[from=Q, to=R, phantom, "\Downarrow\varpi"] \end{tikzcd} = \begin{tikzcd} & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] & \\ L \arrow[ur, "q"] \arrow[dr, "q", ""{name=Q2}] \arrow[dr, bend right=60, "r"', ""{name=R2, above}] & \Downarrow\varepsilon & \mathscr{K}(TA,B) \\ & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] & \arrow[from=Q2, to=R2, phantom, "\Downarrow\varpi"] \end{tikzcd}$$

then there exists a unique 2-cell $\lambda : f \Rightarrow g$ such that $p \cdot \lambda = \varpi$. We will denote the object L by $T\text{-Alg}_\ell(A, B)$, and call it the *object of lax morphisms* from A to B.

Similarly, if we add the requirement that γ be *invertible*, we obtain an object in $\mathscr{W}_1$ which we will denote by $T\text{-Alg}(A, B)$, and call it the *object of pseudomorphisms* from A to B.

Observation 6.6. The universal pair $(p : T\text{-Alg}_\ell(A, B) \to \mathscr{K}(A, B), \gamma)$ can be constructed using an inserter and two equifers in the complete 2-category $\mathscr{W}_1$. Hence this universal pair always exists. Analogously, $T\text{-Alg}(A, B)$ can be constructed using an iso-inserter and two equifers.

Proposition 6.7. *There is a $\mathscr{W}$-category $T\text{-Alg}_\ell$ with objects the T-algebras and*

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[r, "\sigma_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma"] & \mathscr{K}(TA,B) \arrow[dr, "\sigma_{TA,B}"] & \\
L \arrow[ur, "p"] \arrow[r, "p"] \arrow[dr, "p"'] & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a{,}1)"'] \arrow[dr, "\sigma A{,}B"] \arrow[d, phantom, "\Downarrow\gamma"] & & \mathscr{K}(T^2A,B) \\
 & \mathscr{K}(A,B) \arrow[r, "\mathscr{K}(a{,}1)"'] & \mathscr{K}(TA,B) \arrow[ur, "\mathscr{K}(Ta{,}1)"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma"] & & \\
L \arrow[ur, "p"] \arrow[dr, "p"'] & {} & \mathscr{K}(TA,B) \arrow[r, "\mathscr{K}(\mu_A{,}1)"] & \mathscr{K}(T^2A,B) \\
 & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a{,}1)"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma"] & & \\
L \arrow[ur, "p"] \arrow[dr, "p"'] & {} & \mathscr{K}(TA,B) \arrow[r, "\mathscr{K}(\eta_A{,}1)"] & \mathscr{K}(A,B) \; = 1 \\
 & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a{,}1)"'] & &
\end{tikzcd}
$$

Figure 6.1: Axioms for T-Alg(A, B).

enriched homs given by the objects $T\text{-Alg}_\ell(A,B)$*. Similarly, there is a* $\mathscr{W}$*-category* T-Alg *with objects the* T*-algebras and enriched homs given by* $T\text{-Alg}(A,B)$*. There are obvious* forgetful $\mathscr{W}$*-functors* $U_\ell : T\text{-Alg}_\ell \to \mathscr{K}$*,* $U : T\text{-Alg} \to \mathscr{K}$ *and an identity on objects* inclusion $\mathscr{W}$*-functor* $T\text{-Alg} \to T\text{-Alg}_\ell$*.*

Proof. We give only a very short outline of the proof. Let $(A,a),(B,b),(C,c)$ be T-algebras, and $(p : L \to \mathscr{K}(A,B),\gamma)$, $(q : M \to \mathscr{K}(A,B),\delta)$ the respective universal objects as in Definition 6.1. Using γ and δ we can form a 2-cell

$$\begin{tikzcd}
 & \mathscr{K}(B,C)\otimes\mathscr{K}(A,B) \arrow[r, "\mathrm{comp}"] \arrow[d, phantom, "\Downarrow"] & \mathscr{K}(A,C) \arrow[dr, "\sigma_{A,C}"] & \\
L\otimes M \arrow[ur, "q\otimes p"] \arrow[dr, "q\otimes p"'] & & & \mathscr{K}(TA,C) \\
 & \mathscr{K}(B,C)\otimes\mathscr{K}(A,B) \arrow[r, "\mathrm{comp}"'] & \mathscr{K}(A,C) \arrow[ur, "\mathscr{K}(a,1)"'] &
\end{tikzcd}$$

obtaining a 1-cell $L \otimes M \to T\text{-Alg}_l(A,C)$. This is the composition of the $\mathscr{W}$ category $T\text{-Alg}_\ell$. Similarly, if we denote the identity of A in $\mathscr{K}$ by $\mathrm{id} : I \to \mathscr{K}(A,A)$, the identity 2-cell $\mathscr{K}(TA,a)T_{A,A}\mathrm{id} \Rightarrow \mathscr{K}(a,A)\mathrm{id}$ induces a 1-cell $I \to T\text{-Alg}_\ell(A,A)$. This is the identity of (A,a) in $T\text{-Alg}_\ell$. $\square$

Definition 6.2. Define a $\mathscr{W}$-functor $J : T\text{-Alg}_s \to T\text{-Alg}$ as follows. On objects J is the identity. On homs, $J_{A,B} : T\text{-Alg}_s(A,B) \to T\text{-Alg}(A,B)$ is induced by the universal property of $T\text{-Alg}(A,B)$. If $(p : T\text{-Alg}(A,B) \to \mathscr{K}(A,B),\gamma)$ is an universal pair then $J_{A,B}$ is defined by the requirement that $pJ_{A,B}$ be the equalizer of (6.3), and $\gamma J_{A,B}$ be an identity 2-cell. Similarly, there is a $\mathscr{W}$-functor $J_\ell : T\text{-Alg}_s \to T\text{-Alg}_\ell$. Observe that $UJ = U_s = U_\ell J_\ell : T\text{-Alg}_s \to \mathscr{W}$.

6.2.2 Preservation of limits

Now we study the preservation of limits and colimits of $J : T\text{-Alg}_s \to T\text{-Alg}$ and $J : T\text{-Alg}_s \to T\text{-Alg}_\ell$.

The objects $T\text{-Alg}_\ell(A,B)$ and $T\text{-Alg}(A,B)$ can be described as weighted limits in one step, as follows. Consider the graph G

$$\begin{tikzcd}
\mathsf{0} \arrow[r, shift left=2, "x"] \arrow[r, "v"'] & \mathsf{1} \arrow[l, shift left=2, "u"] \arrow[r, shift left=2, "y"] \arrow[r, "z"'] \arrow[r, shift right=2, "w"'] & \mathsf{2}
\end{tikzcd}$$

and let $\mathscr{F}$ be the free 2-category on G with relations $yv = zx$, $yx = wx$, $zv = wv$, $ux = uv$. For T-algebras A,B the following diagram in $\mathscr{W}_1$ defines a 2-functor

$F_{A,B} : \mathscr{F} \to \mathscr{W}_1$.

$$\mathscr{K}(A,B) \overset{\sigma_{A,B}}{\underset{\mathscr{K}(a,1)}{\rightrightarrows}} \mathscr{K}(TA,B) \overset{\sigma_{TA,B}}{\underset{\mathscr{K}(Ta,1)}{\rightrightarrows}} \mathscr{K}(T^2A,B) \qquad (6.7)$$
$$\mathscr{K}(A,B) \xleftarrow{\mathscr{K}(\eta_A,1)} \mathscr{K}(TA,B) \xrightarrow{\mathscr{K}(\mu_A,1)} \mathscr{K}(T^2A,B)$$

Now, the Yoneda embedding $Y : \mathscr{F} \to [\mathscr{F}^{\mathrm{op}}, \mathbf{Cat}]$ defines a diagram of shape G in $[\mathscr{F}^{\mathrm{op}}, \mathbf{Cat}]$, satisfying the relations in the definition of $\mathscr{F}$. Thanks to these relations we can form a (iso) inserter and two equifers analogous to the ones in Observation 6.6 to obtain a 2-functor (χ) $\chi_\ell : \mathscr{F}^{\mathrm{op}} \to \mathbf{Cat}$. Using the Yoneda lemma one can easily deduce that the limit $(\{\chi, F_{A,B}\})$ $\{\chi_\ell, F_{A,B}\}$ is just $(T\text{-Alg}(A,B))$ $T\text{-Alg}_\ell(A,B)$.

We claim that the correspondence $A, B \mapsto F_{A,B} : \mathscr{F} \to \mathscr{W}_1$ is the object part of a $\mathscr{W}$-functor

$$F : T\text{-Alg}_s^{\mathrm{op}} \otimes T\text{-Alg}_s \to [\mathscr{F}, \mathscr{W}] \qquad (6.8)$$

where the codomain is the $\mathscr{W}$-category of 2-functors from $\mathscr{F}$ to $\mathscr{W}$ of Observation 6.2. Indeed, this $\mathscr{W}$-functor corresponds to a 2-functor

$$\mathscr{F} \to [T\text{-Alg}_s^{\mathrm{op}} \otimes T\text{-Alg}_s, \mathscr{W}]_1 \qquad (6.9)$$

and hence to a graph morphism $\mathsf{G} \to [T\text{-Alg}_s^{\mathrm{op}} \otimes T\text{-Alg}_s, \mathscr{W}]_0$ satisfying the relations given above. This morphism is defined by the diagrams (6.7), that are clearly $\mathscr{W}$-natural in A, B.

Proposition 6.8. *The $\mathscr{W}$-functors J and J_ℓ preserve limits.*

Proof. We only treat the case of J; the proof for J_ℓ is completely analogous. We will show that the $\mathscr{W}$-functors $T\text{-Alg}(A, J-) : T\text{-Alg}_s \to \mathscr{W}$ preserve limits. By the discussion above, $T\text{-Alg}(A, J-)$ is just the composition

$$T\text{-Alg}_s \xrightarrow{F(A,-)} [\mathscr{F}, \mathscr{W}] \xrightarrow{\{\chi,-\}} \mathscr{W}.$$

(Recall that $\{\phi, -\}$ is a $\mathscr{W}$-functor by Observation 6.4). Since $\{\phi, -\}$ is continuous, it suffices to show that $F(A,-)$ is so. To show this, recall that $F(A,-)$ is defined by sending B to the diagram (6.7), or rather to the diagram of shape $\mathscr{F}$ in $\mathscr{W}$ constructed from (6.7). A direct inspection of the diagram (6.7) shows that the assignment $B \mapsto$ (6.7) preserves limits; since the diagram $F(A,B)$ is given by composition of the arrows in (6.7), it follows that $F(A,-)$ preserves limits because limits in $[\mathscr{F}, \mathscr{W}]$ are computed point-wise. $\square$

Proposition 6.9. *If $T : \mathscr{K} \to \mathscr{K}$ preserves ψ-colimits, then J, J_ℓ have the same property.*

Proof. The proof is similar to that of Proposition 6.8. We have to show that $T\text{-Alg}(J-, B) : T\text{-Alg}_s^{\mathrm{op}} \to \mathscr{W}$ preserves ψ-limits, or equivalently, that $F(-, B) : T\text{-Alg}_s^{\mathrm{op}} \to [\mathscr{F}, \mathscr{W}]$ does so. This last property holds since $F(-, B)$ is defined by the diagrams (6.7) and T preserves ψ-colimits. □

Corollary 6.10. *The $\mathscr{W}$-functor J has a left adjoint if and only if its underlying 2-functor J_1 has a left adjoint, if and only if its underlying functor J_0 has a left adjoint.*

Hence, J has left adjoint when the 2-category $T\text{-Alg}_{s,1}$ has codescent objects. In particular, this holds if $\mathscr{K}_1$ is cocomplete and T has a rank. See [53].

6.2.3 Flexible replacement

Now fix a monad T on $\mathscr{K}$ in $\mathscr{W}$-**Cat** and assume that J has a left adjoint. We can reproduce some of the results on flexibility in [7] in the context of $\mathscr{W}$-categories. It is not our intention to develop a theory of monads enriched in 2-categories, but only prove the results we will need later in this work. We will use the notations in [7].

Denote by $p_A : A \to A'$ and $q_A : A' \to A$ the unit and counit of the adjunction $(-)' \dashv J$. We know $q_A p_A = 1$ and q_A is a retract equivalence in $T\text{-Alg}_1$. We say that a T-algebra A is *flexible* when it is a flexible T_1-algebra in $\mathscr{K}_1$ in the sense of [7]. In other words, when q_A has a section in $T\text{-Alg}_{s,1}$. As observed in [7], T-algebras of the form A' are flexible.

The following lemma is a slight generalisation of [7, Theorem 4.7].

Lemma 6.11. *For a T-algebra A, $q : A' \to A$ is an equivalence in $T\text{-Alg}_{s,1}$ if and only if the 1-cell $J_{A,B} : T\text{-Alg}_s(A, B) \to T\text{-Alg}(A, B)$ is an equivalence in $\mathscr{W}$ for all T-algebras B. In particular, this is the case if A is a flexible algebra.*

Proof. Suppose q_A is an equivalence in $T\text{-Alg}_{s,1}$, with chosen pseudoinverse $k : A \to A'$. As p_A is a pseudoinverse for q_A in $T\text{-Alg}_1$, it follows that k and p_A are isomorphic. The pseudoinverse for $J_{A,B}$ is

$$T\text{-Alg}(A, B) \xrightarrow{\cong} T\text{-Alg}_s(A', B) \xrightarrow{T\text{-Alg}_s(k,1)} T\text{-Alg}_s(A, B).$$

In the proof one uses that $J_{A,B}$ is monic in $\mathscr{W}_0$, as $U_{A,B} J_{A,B} = (U_s)_{A,B} : T\text{-Alg}_s(A, B) \to \mathscr{K}(A, B)$ is a regular mono. □

Proposition 6.12. *Let $G : T\text{-Alg} \to \mathscr{L}$ be a $\mathscr{W}$-functor and suppose $H : \mathscr{L} \to T\text{-Alg}_s$ is a left adjoint for GJ, with unit $s : 1 \to GJH$. Then the components of the $\mathscr{W}$-natural transformation*

$$t_{L,A} : T\text{-Alg}(JH(L), A) \xrightarrow{G} \mathscr{L}(GJH(L), A) \xrightarrow{\mathscr{L}(s,1)} \mathscr{L}(L, GA)$$

are retract equivalences in $\mathscr{W}$.

Proof. The proof is by inspection of the composition

$$\mathscr{L}(L, GJ(A)) \cong T\text{-Alg}_s(H(L), A) \xrightarrow{J_{H(L),A}} T\text{-Alg}(JH(L), J(A)) \to$$
$$\xrightarrow{G_{JH(L),J(A)}} \mathscr{L}(GJH(L), GJ(A)) \xrightarrow{\mathscr{L}(s_L,1)} \mathscr{L}(L, GJ(A)).$$

This composition is an identity. On the other hand, the composition of the last two arrows, $G_{JH(L),J(A)}$ and $\mathscr{L}(s_L, 1)$, is just $t_{L,A}$. It follows that $t_{L,A}J_{H(L),A}$ is an isomorphism. But $H(L)$ is flexible by [7, Theorem 5.1], and hence $J_{H(L),A}$ is an equivalence by Lemma 6.11. Therefore, $t_{L,A}$ is an equivalence, and since it has a section, in fact it is part of a retract equivalence. □

Corollary 6.13. *The components of the $\mathscr{W}$-natural transformation*

$$T\text{-Alg}(FX, A) \xrightarrow{U_{FX,A}} \mathscr{K}(TX, UA) \xrightarrow{\mathscr{K}(\eta_X,1)} \mathscr{K}(X, A)$$

are retract equivalences in $\mathscr{W}$.

We do not go as far as to define $\mathscr{W}$-biadjunctions and saying something about the pseudo-$\mathscr{W}$-naturality of the equivalences in the corollary above.

6.2.4 Preservation of colimits

For a moment we go back to the case of a 2-monad S on a 2-category $\mathscr{K}$. Assume the left adjoint to $J : S\text{-Alg}_s \to S\text{-Alg}$ has a left adjoint $(-)'$. In [53] the existence of $(-)'$ is related to the existence of certain colimits in $S\text{-Alg}_s$ called *codescent objects*. We recall below only the definitions needed in this work. In particular we consider strict algebras and not lax algebras as in [53].

A *strict coherence data* in a 2-category $\mathscr{K}$ is a diagram

$$\begin{tikzcd} X_3 \arrow[r, shift left=2, "p"] \arrow[r, "q" description] \arrow[r, shift right=2, "r"'] & X_2 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_1 \arrow[l, "e" description] \end{tikzcd}$$

satisfying equations: $de = 1_{X_1} = ce$, $dp = dq$, $cr = cq$, $cp = dr$. A *lax codescent object* for the strict coherence data is a pair (x, ξ) where $x : X_1 \to X$ and $\xi : xd \Rightarrow xc$ universal (in a 2-categorical sense) with respect to the conditions $(\xi p) \cdot (\xi r) = \xi q$ and $\xi e = 1$. A *codescent object* is defined in the same way buy insisting that the 2-cell ξ must be an isomorphism. Codescent objects of strict coherence data were called "strict coherence objects" in [79].

By [53], the existence of $(-)'$ amounts to the existence of codescent objects for the *strict* coherence data below, for each S-algebra (A, a).

$$\begin{tikzcd} T^3A \arrow[r, shift left=2, "\mu_{TA}"] \arrow[r, "T\mu_A" description] \arrow[r, shift right=2, "T^2a"'] & T^2A \arrow[r, shift left=2, "\mu_A"] \arrow[r, leftarrow, "T\eta_A" description] \arrow[r, shift right=2, "Ta"'] & TA \end{tikzcd} \tag{6.10}$$

There is an obvious 2-category $\mathscr{D}$ such that strict codescent objects in a 2-category $\mathscr{L}$ are in bijection with 2-functors $\mathscr{D} \to \mathscr{L}$. Moreover, there is a weight $\gamma : \mathscr{D}^{\mathrm{op}} \to \mathbf{Cat}$ such that for any $F : \mathscr{D} \to \mathscr{L}$ the colimit $\gamma * F$ is a codescent object for the strict codescent object defined by F. The same construction in the case of lax codescent objects can be found in [53]; we omit the details.

Proposition 6.14. *Let $\phi : \mathscr{P}^{\mathrm{op}} \to \mathbf{Cat}$ be a 2-functor. If S preserves ϕ-colimits, then $(-)'J$ preserves ϕ-colimits.*

Proof. First observe that since S preserve ϕ-colimits, the forgetful $U_s : S\text{-Alg}_s \to \mathscr{K}$ creates such colimits. Now, the diagrams (6.10) define a 2-functor $C : S\text{-Alg}_s \to [\mathscr{D}, S\text{-Alg}_s]$, which is easily shown to preserve ϕ-colimits. Hence the composition of C with $(\gamma * -) : [\mathscr{D}, S\text{-Alg}_s] \to S\text{-Alg}_s$ preserves ϕ-colimits. This composition is just $(-)'J$. □

6.3 Pseudo-closed $\mathscr{W}$-categories and pseudo-commutative $\mathscr{W}$-monads

This section recalls the concepts of pseudo-closed $\mathscr{W}$-category and pseudo-commutative $\mathscr{W}$-category. We say recall because, although [37] considers 2-categories the relevant definitions are exactly the same for $\mathscr{W}$-categories.

6.3.1 Pseudo-closed enriched categories

In the same way that pseudo-closed 2-categories were defined in [37], we can define pseudo-closed $\mathscr{W}$-categories.

Definition 6.3. A *pseudo-closed $\mathscr{W}$-category* is a $\mathscr{W}$-category $\mathscr{K}$ equipped with the following data: $\mathscr{W}$-functors $V : \mathscr{K} \to \mathscr{W}$ and $[-,-] : \mathscr{K}^{\mathrm{op}} \otimes \mathscr{K} \to \mathscr{K}$, an object $I \in \mathscr{K}$, $\mathscr{W}$-natural transformations $j_A : I \to [A,A]$, $e_A : [I,A] \to A$, $i_A : A \to [I,A]$, $k_{A,B,C} : [B,C] \to [[A,B],[A,C]]$. This data must satisfy axioms completely analogous to [37, Definition 1]. Explicitly, this data is subject to the commutativity of the diagrams in Figure 6.2 and

- $V[-,-] = \mathscr{K}(-,-) : \mathscr{K}^{\mathrm{op}} \otimes \mathscr{K} \to \mathscr{W}$;
- the 1-cell $I \xrightarrow{j_A} \mathscr{K}(I,[A,A]) = V[I,[A,A]] \xrightarrow{Ve_{[A,A]}} V[A,A] = \mathscr{K}(A,A)$ is the identity of A;
- there are retract equivalences $i_A \dashv e_A$ in the 2-category $\mathscr{K}_1$;
- the 1-cell $\mathscr{W}(I, V(i_A e_A)) : \mathscr{K}_1(I,A) \to \mathscr{K}_1(I,A)$ in **Cat** takes each $f : I \to A$ in $\mathscr{K}_1$ to $e_A[p,A]j_A : I \to [A,A] \to [I,A] \to A$.

$$\begin{tikzcd}
I \arrow[r, "j_B"] \arrow[dr, "j_{[A,B]}"'] & {[B,B]} \arrow[d, "k_A"] \\
& {[[A,B],[A,B]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[A,C]} \arrow[r, "k_{A,A,C}"] \arrow[d, equal] & {[[A,A],[A,C]]} \arrow[d, "{[j_A,1]}"] \\
{[A,C]} & {[I,[A,C]]} \arrow[l, "e_{[A,C]}"]
\end{tikzcd}$$

$$\begin{tikzcd}
{[C,D]} \arrow[r, "k_{A,C,D}"] \arrow[d, "k_{B,C,D}"'] & {[[A,C],[A,D]]} \arrow[r, "k"] & {[[[A,B],[A,C]],[[A,B],[A,D]]]} \arrow[d, "{[k_{A,B,C},1]}"] \\
{[[B,C],[B,D]]} \arrow[rr, "{[1,k_{A,B,D}]}"'] & & {[[B,C],[[A,B],[A,D]]]}
\end{tikzcd}$$

$$\begin{tikzcd}
{[A,B]} \arrow[r, "k_{I,A,B}"] \arrow[dr, "{[e_A,1]}"'] & {[[I,A],[I,B]]} \arrow[d, "{[1,e_B]}"] \\
& {[[I,A],B]}
\end{tikzcd}$$

Figure 6.2: Some of the axioms of a pseudo-closed $\mathscr{W}$-category.

When $\mathscr{W} = \mathbf{Cat}$ we recover the pseudo-closed 2-categories of Hyland-Power [37].

Lemma 6.15. *If $\mathscr{K}$ is a pseudo-closed $\mathscr{W}$-category then $\mathscr{K}_1$ has an induced structure of a pseudo-closed 2-category.*

Proof. Apply $(-)_1 : \mathscr{W}\textbf{-Cat} \to \textbf{2-Cat}$ to the pseudo-closed structure of $\mathscr{K}$. $\square$

A *closed $\mathscr{W}$-functor* $G : \mathscr{K} \to \mathscr{L}$ between pseudo-closed $\mathscr{W}$-categories is a $\mathscr{W}$-functor equipped with 1-cells $\phi_{X,Y} : G[X,Y] \to [GX, GY]$, $\phi^0 : I \to GI$ satisfying the usual axioms of a closed functor that we recall from [26] or [37, Definition 3]. So, G is said to be closed when it is equipped with a $\mathscr{W}$-natural transformation $\phi_{X,Y} : G[X,Y] \to [GX, GY]$ and an arrow $\phi_0 : I \to GI$ satisfying the axioms depicted in Figure 6.3.

$$\begin{tikzcd}
I \arrow[r, "j"] \arrow[d, "\phi_0"'] & {[GA,GA]} \\
GI \arrow[r, "Gj"'] & G[A,A] \arrow[u, "\phi_{A,A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
G[I,A] \arrow[r, "\phi_{I,A}"] \arrow[d, "Ge_A"'] & {[GI,GA]} \arrow[d, "{[\phi_0,1]}"] \\
GA & {[I,GA]} \arrow[l, "e_{GA}"]
\end{tikzcd}$$

$$\begin{tikzcd}
G[B,C] \arrow[r, "Gk"] \arrow[d, "\phi_{B,C}"'] & G[[A,B],[A,C]] \arrow[r, "\phi_{[A,B],[A,C]}"] & {[G[A,B],G[A,C]]} \arrow[d, "{[1,\phi_{A,C}]}"] \\
{[GB,GC]} \arrow[r, "k"'] & {[[GA,GB],[GA,GC]]} \arrow[r, "{[\phi_{A,B},1]}"'] & {[G[A,B],[GA,GC]]}
\end{tikzcd}$$

Figure 6.3: Axioms of a closed functor.

We will need the concept of a pseudo-closed transformation mentioned in [37].

Definition 6.4. Let $G, H : \mathscr{K} \to \mathscr{L}$ be two closed $\mathscr{W}$-functors between pseudo-closed $\mathscr{W}$-categories. A *pseudo-closed $\mathscr{W}$-natural transformation* $\tau : G \to H$ is a $\mathscr{W}$-natural transformation equipped with invertible 2-cells

$$\begin{tikzcd}
G[X,Y] \arrow[r, "\phi"] \arrow[dd, "\tau_{[X,Y]}"'] \arrow[ddr, phantom, "\bar{\tau}_{X,Y}\Downarrow"] & {[GX,GY]} \arrow[d, "{[1,\tau_Y]}"] \\
 & {[GX,HY]} \\
H[X,Y] \arrow[r, "\phi"'] & {[HX,HY]} \arrow[u, "{[\tau_X,1]}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\phi_0"] \arrow[dr, "\phi_0"'] \arrow[dr, phantom, bend left, "\Swarrow\bar{\tau}_0"] & GI \arrow[d, "\tau_I"] \\
 & HI
\end{tikzcd}$$

satisfying three axioms depicted in Figure 6.4.

Since a pseudo-closed 2-category is a semi-strict kind of closed 2-category, it is reasonable to think that if each 2-functor $[X, -]$ has a left biadjoint we obtain a monoidal structure. This was studied in [37].

$$
\begin{tikzcd}
G[B,C] \arrow[r, "Gk"'] \arrow[d, "\tau"'] & G[[A,B],[A,C]] \arrow[r, "\phi"'] \arrow[d, "\tau"'] & {[G[A,B],G[A,C]]} \arrow[r, "{[1,\phi]}"'] \arrow[dd, "{[1,\tau]}"] & {[G[A,B],[GA,GB]]} \arrow[ddd, "{[1,[1,\tau]]}"] \\
H[B,C] \arrow[r, "Hk"] \arrow[d, "\phi"'] & H[[A,B],[A,C]] \arrow[d, "\phi"'] \arrow[r, phantom, "\Downarrow\bar\tau"] & \arrow[r, phantom, "\Downarrow{[1,\bar\tau]}"] & \\
{[HB,HC]} \arrow[d, "k"'] & {[H[A,B],H[A,C]]} \arrow[r, "{[\tau,1]}"'] \arrow[d, "{[1,\phi]}"] & {[G[A,B],H[A,C]]} \arrow[d, "{[1,\phi]}"'] & \\
{[[HA,HB]]} \arrow[r, "{[\phi,1]}"] & {[H[A,B],[HA,HC]]} \arrow[r, "{[\tau,1]}"] & {[G[A,B],[HA,HC]]} \arrow[r, "{[1,[\tau,1]]}"] & {[G[A,B],[GA,HC]]}
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
G[B,C] \arrow[r, "\phi"'] \arrow[d, "\tau"'] & {[GB,GC]} \arrow[r, "k"'] \arrow[d, "{[1,\tau]}"] & {[[GA,GB],[GA,GC]]} \arrow[r, "{[\phi,1]}"'] \arrow[dd, "{[1,[1,\tau]]}"] & {[G[A,B],[GA,GC]]} \arrow[ddd, "{[1,[1,\tau]]}"'] \\
H[B,C] \arrow[d, "\phi"'] \arrow[r, phantom, "\Downarrow\bar\tau"] & {[GB,HC]} \arrow[dr, "k"] & & \\
{[HB,HC]} \arrow[ur, "{[\tau,1]}"'] \arrow[r, "k"'] \arrow[d, "k"'] & {[[GA,HB],[GA,HC]]} \arrow[r, "{[[1,\tau],1]}"'] \arrow[d, "{[[\tau,1],1]}"'] & {[[GA,GB],[GA,HC]]} \arrow[dr, "{[\phi,1]}"] \arrow[d, phantom, "\Downarrow{[\bar\tau,1]}"] & \\
{[[HA,HB],[HA,HC]]} \arrow[r, "{[1,[\tau,1]]}"] & {[[HA,HB],[GA,HC]]} \arrow[r, "{[\phi,1]}"] & {[H[A,B],[GA,HC]]} \arrow[r, "{[\tau,1]}"] & {[G[A,B],[GA,HC]]}
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \arrow[r, "\phi_0"'] \arrow[rrr, bend left, "\phi_0"] \arrow[dddrr, bend right, "j"'] & GI \arrow[rr, "\tau"'] \arrow[rr, phantom, bend left=20, "\bar\tau_0\Uparrow"] \arrow[dr, "Gj"'] & & HI \arrow[dr, "Hj"] & \\
& & G[A,A] \arrow[rr, "\tau"] \arrow[dd, "\phi"] & & H[A,A] \arrow[d, "\phi"] \\
& & \arrow[rr, phantom, "\overset{\bar\tau}{\Rightarrow}"] & & {[HA,HA]} \arrow[d, "{[\tau,1]}"] \\
& & {[GA,GA]} \arrow[rr, "{[1,\tau]}"'] & & {[GA,HA]}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
HI \arrow[r, "Hj"] & H[A,A] \arrow[d, "\phi"] \\
I \arrow[u, "\phi_0"] \arrow[r, "j"] \arrow[d, "j"'] & {[HA,HA]} \arrow[d, "{[\tau,1]}"] \\
{[GA,GA]} \arrow[r, "{[1,\tau]}"'] & {[GA,HA]}
\end{tikzcd}
$$

$$
\begin{tikzcd}
G[I,A] \arrow[rr, "\phi"] \arrow[d, "\tau"'] & \arrow[d, phantom, "\Downarrow\bar\tau"] & {[GI,GA]} \arrow[d, "{[1,\tau]}"] \\
H[I,A] \arrow[r, "\phi"'] \arrow[d, "He"'] & {[HI,HA]} \arrow[r, "{[\tau,1]}"] \arrow[d, "{[\phi_0,1]}"'] & {[GI,HA]} \arrow[dl, bend left, "{[\phi_0,1]}"] \arrow[dl, phantom, "\overset{\Leftarrow}{[\bar\tau_0,1]}"'] \\
A & {[I,A]} \arrow[l, "e"] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
G[I,A] \arrow[r, "\phi"] \arrow[d, "Ge"'] & {[GI,GA]} \arrow[d, "{[\phi_0,1]}"] \\
GA \arrow[d, "\tau"'] & {[I,GA]} \arrow[l, "e"] \arrow[d, "{[1,\tau]}"] \\
HA & {[I,HA]} \arrow[l, "e"]
\end{tikzcd}
$$

Figure 6.4: Axioms of a pseudo-closed transformation

Theorem 6.16 ([37]). *Suppose $\mathscr{K}$ is a pseudo-closed 2-category. If for each pair of objects X, Y there is an object $X \otimes Y$ and an equivalence $d_Z : [X \otimes Y, Z] \to [X, [Y, Z]]$ 2-natural in Z such that the following diagram commutes*

$$\begin{tikzcd}
{[Z,W]} \arrow[r, "k"] \arrow[d, "k"'] & {[[Y,Z],[Y,W]]} \arrow[r, "k"] & {[[X,[Y,Z]],[X,[Y,W]]]} \arrow[d, "{[d_Z,1]}"] \\
{[[X\otimes Y,Z],[X\otimes Y,W]]} \arrow[rr, "{[1,d_W]}"'] & & {[[X\otimes Y,Z],[X,[Y,W]]}
\end{tikzcd}$$

then the assignment $(X, Y) \mapsto X \otimes Y$ extends to a (weak or pseudo) monoidal structure on $\mathscr{K}$ with unit object I.

If $\mathscr{K}, \mathscr{L}$ are pseudo-closed 2-categories with compatible monoidal structures as in Theorem 6.16: each closed 2-functor $\mathscr{K} \to \mathscr{L}$ has a canonical structure of a (weak) monoidal 2-functor, and each pseudo-closed transformation induces a (weak) monoidal 2-natural transformation.

6.3.2 Pseudo-commutative enriched monads

If $T : \mathscr{W} \to \mathscr{W}$ is a $\mathscr{W}$-functor, a *strength* for T is a $\mathscr{W}$-natural transformation $t_{X,Y} : X \otimes T(Y) \to T(X \otimes Y)$. Using the symmetry of $\mathscr{W}$, we obtain a $\mathscr{W}$-natural transformation $t'_{X,Y} : T(X) \otimes Y \to T(X \otimes Y)$. To give a strength t is the same as giving a $\mathscr{W}$-natural transformation $\bar{t} : T[X, Y] \to [X, TY]$ and the same as giving a $\mathscr{W}$-natural transformation $\mathsf{T} : [X, Y] \to [TX, TY]$. The bijection between these structures is given by the diagrams below (where i denotes the unit of the closedness adjunction of $\mathscr{W}_1$).

$$\begin{tikzcd}
{[TX,TY]\otimes TX} \arrow[r, "\mathrm{ev}"] & TY \\
{[X,Y]\otimes X} \arrow[r, "t"'] \arrow[u, "\mathsf{T}\otimes 1"] & T([X,Y]\otimes X) \arrow[u, "T\mathrm{ev}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\mathrm{i}"] \arrow[d, "\mathrm{i}"'] & {[Y,X\otimes Y]} \arrow[d, "\mathsf{T}"] \\
{[TY,X\otimes TY]} \arrow[r, "{[1,t]}"'] & {[TY,T(X\otimes Y)]}
\end{tikzcd}$$

$$\begin{tikzcd}
TX \arrow[r, "T\mathrm{i}"] \arrow[d, "\mathrm{i}"'] & {T[Y,X\otimes Y]} \arrow[d, "\bar{t}"] \\
{[Y,TX\otimes Y]} \arrow[r, "{[1,t']}"'] & {[Y,T(X\otimes Y)]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{T[X,Y]\otimes X} \arrow[r, "t'"] \arrow[d, "\bar{t}\otimes 1"'] & T([X,Y]\otimes X) \arrow[d, "T\mathrm{ev}"] \\
{[X,TY]\otimes X} \arrow[r, "\mathrm{ev}"'] & TY
\end{tikzcd}$$

We will consider $\mathscr{W}$*-enriched monads* T equipped with the canonical strength corresponding to the enrichment $[X, Y] \to [TX, TY]$.

Definition 6.5. A pseudo-commutativity for a $\mathscr{W}$-enriched monad T is an invertible modification

$$\begin{tikzcd}
TX \otimes TY \arrow[r, "t'_{X,TY}"] \arrow[d, "t_{TX,Y}"'] & T(X \otimes TY) \arrow[r, "Tt_{X,Y}"] \arrow[d, phantom, "\Downarrow\gamma_{X,Y}"] & T^2(X \otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
T(TX \otimes Y) \arrow[r, "Tt'_{X,Y}"'] & T^2(X \otimes Y) \arrow[r, "\mu_{X\otimes Y}"'] & T(X \otimes Y)
\end{tikzcd} \tag{6.11}$$

satisfying the axioms resulting from replacing in [37, Definition 5] the cartesian product in **Cat** by the tensor product $\otimes$ of $\mathscr{W}$. We do not write the axioms here, as these will not be explicitly used. An equivalent description of a pseudo-commutativity is given in Proposition 6.18 below, which can be taken itself as an alternative definition.

We call an strong enriched monad equipped with a pseudo-commutativity a *pseudo-commutative* monad.

When $\mathscr{W} = \textbf{Cat}$ with the symmetric monoidal closed structure induced by the cartesian product, we recover the pseudo-commutative 2-monads of [37].

Example 6.17. One basic example of pseudo-commutative 2-monad discussed in detail in [37] is the one of the free symmetric strict (unbiased) monoidal category 2-monad. This is the monad T on **Cat** given by the following description. If X is a category, TX has objects finite sequences of objects of X, and arrows $(x_1, \cdots, x_n) \to (y_1, \cdots, y_n)$ pairs $((f_1, \cdots, f_n), \pi)$ where π is an element of the symmetric group S_n and $f_i : x_1 \to y_{\pi i}$ is an arrow in X. There are no arrows between sequences of different length. Composition is defined by multiplying the elements of the symmetric group and then composing the arrows in the lists in the unique possible way. Identities are of the form $((1_{x_1}, \cdots, 1_{x_n}), 1)$. The tensor product of two lists of objects of X is obtained by appending the second list to the first. This is easily extended to arrows. The multiplication $\mu_X : T^2X \to TX$ is given on objects by removing parenthesis:

$$\mu_X((x_1^1, \cdots, x_{k_1}^1), (x_1^2, \cdots, x_{k_2}^2), \cdots, (x_1^n, \cdots, x_{k_n}^n)) = (x_1^1, \cdots, x_{k_1}^1, x_1^2, \cdots, x_{k_n}^n)$$

The unit $\eta_X : X \to TX$ is given on objects by $\eta_X(x) = (x)$. The description of these functors on arrows are obvious and omitted for simplicity.

The domain of the 2-cell (6.11) is the 2-functor sending an object

$$((x_1, \cdots, x_n), (y_1, \cdots, y_m)) \in TX \times TY \tag{6.12}$$

to

$$((x_1,y_1),(x_1,y_2),\cdots,(x_1,y_m),(x_2,y_1),\cdots,(x_2,y_m),\cdots,(x_n,y_1),\cdots(x_n,y_m))$$

while the codomain is the functor sending (6.12) to

$$((x_1,y_1),(x_2,y_1),\cdots,(x_n,y_1),(x_1,y_2),\cdots,(x_n,y_2),\cdots,(x_1,y_m),\cdots,(x_n,y_m)).$$

Therefore, domain and codomain of (6.11) are the functors sending (6.12) to the two lexicographic orders. The components of the natural isomorphism $\gamma_{X,Y}$ are of the form $((1,\cdots,1),\pi)$ where π is the permutation that mediates between the two lexicographic orders. More details can be found in [37].

Later we will use the following alternative description of pseudo-commutativities in [37, Proposition 8]. The basic observation is that 2-cells $\gamma_{X,Y}$ in (6.11) are in bijection with 2-cells

$$\begin{tikzcd} T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}_{X,Y}"'] & T[TX,TY] \arrow[r, "\bar{t}"] \arrow[d, phantom, "\Downarrow\bar{\gamma}_{X,Y}"] & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\ {[X,TY]} \arrow[r, "\mathsf{T}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]} \end{tikzcd} \tag{6.13}$$

and the axioms of a pseudo-commutativity translate accordingly into conditions on $\bar{\gamma}$.

The following result was proved in [37, Proposition 8] in the case of 2-monads, but it carries over unchanged to the case of enriched monads.

Proposition 6.18. *To give a pseudo-commutativity for a $\mathscr{W}$-enriched monad T is equivalent to give a modification $\bar{\gamma}$ as in (6.13) subject to the following conditions.*

1. $[X,\bar{\gamma}_{Y,Z}]\bar{t}_{X,[Y,Z]}$ *is the exponential transpose of* $[t,TZ]\bar{\gamma}_{X\otimes Y,Z}$.

2. $\bar{\gamma}_{X,Y}\eta_{[X,Y]}$ *is an identity.*

3. $[\eta_X,TY]\bar{\gamma}_{X,Y}$ *is an identity.*

4. $\bar{\gamma}_{X,Y}\mu_{[X,Y]}$ *is equal to the pasting*

$$\begin{tikzcd}
T^2[X,Y] \arrow[r, "T^2(\mathsf{T})"] \arrow[d, "T\bar{t}"'] & T^2[TX,TY] \arrow[r, "T\bar{t}"] \arrow[d, phantom, "\Downarrow T\bar{\gamma}_{X,Y}"] & T[TX,T^2Y] \arrow[d, "T[1,\mu_B]"] \\
T[X,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"'] & T[TX,T^2Y] \arrow[r, "T[1,\mu_Y]"] \arrow[d, "\bar{t}"'] & T[TX,TY] \arrow[d, "\bar{t}"] \\
{[X,T^2Y]} \arrow[d, "\mathsf{T}"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}_{X,TY}"] & {[TX,T^3Y]} \arrow[r, "{[1,T\mu_Y]}"] \arrow[d, "{[1,\mu_{TY}]}"] & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
{[TX,T^3Y]} \arrow[r, "{[1,\mu_{TY}]}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]}
\end{tikzcd}$$

5. $[\mu_X, TY]\bar{\gamma}_{X,Y}$ *is equal to the pasting*

$$\begin{tikzcd}
T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"'] & T[TX,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"] & T[T^2X,T^2Y] \arrow[r, "\bar{t}"] \arrow[d, phantom, "\Downarrow\bar{\gamma}_{TX,TY}"] & {[T^2X,T^3Y]} \arrow[d, "{[1,\mu_{TY}]}"] \\
{[X,TY]} \arrow[r, phantom, "\bar{\gamma}_{X,Y}\Downarrow"] \arrow[d, "\mathsf{T}"'] & {[TX,T^2Y]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,\mu_Y]}"] & T[T^2X,T^3Y] \arrow[r, "{[1,\mu_{TY}]}"] \arrow[d, "{[1,T\mu_Y]}"] & {[T^X,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
{[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]} \arrow[r, "\mathsf{T}"'] & {[T^2X,T^Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[T^2X,TY]}
\end{tikzcd}$$

6.4 A characterisation of pseudo-commutativity

In [37, Section 4.1] the authors show that a pseudo-commutativity on a strong 2-monad T on **Cat** induces a canonical structure of pseudomorphism on the functor $\sigma_{X,B} : [TX,b]\mathsf{T} : [X,B] \to [TX,TB] \to [TX,B]$, for $X \in \mathbf{Cat}$, $(B,b) \in T$-Alg. These pseudomorphisms satisfy certain properties necessary to ensure that T-Alg has the structure of a pseudo-closed 2-category. In this section we improve these observations in two ways. First, we work with monads enriched in a monoidal closed 2-category, and secondly we show that pseudomorphism structures on the functors $[TX,b]\mathsf{T}$ satisfying certain conditions are in bijection with pseudo-commutativities on T.

Let $T : \mathscr{W} \to \mathscr{W}$ be a $\mathscr{W}$-enriched monad, equipped with its canonical strength. Consider the 2-functors $[-,-],[T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_1 \to T\text{-Alg}_1$.

Observe that the 1-cells

$$\sigma_{X,B} : [X,B] \xrightarrow{\mathsf{T}} [TX,TB] \xrightarrow{[1,b]} [TX,B] \tag{6.14}$$

are part of a pseudonatural transformation

$$U[-,-] \Rightarrow U[T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to \mathscr{W}_1.$$

Indeed, if $f : B \to C$ is a 1-cell in T-Alg_1, the structural 2-cell σ_f corresponding to f is the pasting below.

$$\begin{tikzcd}
{[X,B]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,f]}"'] & {[TX,TB]} \arrow[r, "{[1,b]}"] \arrow[d, "{[1,Tf]}"'] \arrow[dr, phantom, "\bar{f}\Uparrow"] & {[TX,B]} \arrow[d, "{[1,f]}"] \\
{[X,C]} \arrow[r, "T"] & {[TX,TC]} \arrow[r, "{[1,c]}"] & {[TX,C]}
\end{tikzcd} \tag{6.15}$$

Observe that the pseudonatural transformation obtained by precomposing σ with $1 \times J_1 : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_{s,1} \to \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1$ is in fact 2-natural. In other words, σ is 2-natural on *strict* morphisms.

The conditions in the proposition below appear in [37].

Proposition 6.19. *There is a bijection between pseudo-commutativities on T and liftings of σ to a pseudonatural transformation $[-,-] \Rightarrow [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to T\text{-}\mathrm{Alg}_1$ satisfying the following conditions.*

1. $[\eta_X, B]\sigma_{X,B} = 1_{[X,B]}$ *in* T-Alg_1.

2. $\sigma_{TX,B}\sigma_{X,B} = [\mu_X, B]\sigma_{X,B}$ *in* T-Alg_1.

3. $[X, \sigma_{Y,B}] : [X,[Y,B]] \to [X,[TY,B]]$ *is the exponential transpose of the 1-cell* $[t,B]\sigma_{X\otimes Y,B} : [X \otimes Y, B] \to [X \otimes TY, B]$ *in* T-Alg_1.

4. *The composition of* $1 \times J_1 : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_{s,1} \to \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1$ *with* σ *is a 2-natural transformation.*

We split the proof of the proposition in several lemmas.

Lemma 6.20. *Let $T : \mathscr{W} \to \mathscr{W}$ be a $\mathscr{W}$-enriched monad, equipped with its canonical strength. There is a bijection between modifications $\bar{\gamma}$ as in* (6.13) *satisfying conditions 2 and 4 of Proposition 6.18 and liftings of σ to a pseudonatural transformation $[-,-] \Rightarrow [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to T\text{-}\mathrm{Alg}_1$ which composed with $\mathscr{W}_1^{\mathrm{op}} \times J$ are 2-natural.*

Proof. Given a modification $\bar{\gamma}$ as in 6.13, we can define 2-cells $\bar{\sigma}_{X,B}$ for $X \in \mathscr{W}$, $(B, b) \in T\text{-Alg}$ as the following composition.

$$
\begin{tikzcd}
T[X,B] \arrow[rr, "T(\mathsf{T})"] \arrow[dd, "\bar{t}"'] & & T[TX,TB] \arrow[r, "{T[1,b]}"] \arrow[d, "\bar{t}"] & T[TX,B] \arrow[d, "\bar{t}"] \\
& \arrow[phantom, "\Downarrow\bar{\gamma}_{X,B}"] & {[TX,T^2B]} \arrow[r, "{[TX,Tb]}"] \arrow[d, "{[1,\mu_B]}"] & {[TX,TB]} \arrow[dd, "{[1,b]}"] \\
{[X,TB]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,b]}"'] & {[TX,T^2B]} \arrow[r, "{[1,\mu_B]}"] \arrow[d, "{[1,Tb]}"] & {[TX,TB]} \arrow[dr, "{[1,b]}"'] & \\
{[X,B]} \arrow[r, "\mathsf{T}"'] & {[TX,TB]} \arrow[rr, "{[1,b]}"'] & & {[TX,B]}
\end{tikzcd}
$$

Each 2-cell $\bar{\sigma}_{X,B}$ endows $[TX, b]\mathsf{T}$ with the structure of a pseudomorphism of T-algebras: the condition involving the unit η follows from condition 2 of Proposition 6.18 and the condition involving the multiplication μ follows from condition 4 of the same proposition. With this pseudomorphism structure, the 2-cell (6.15) is a 2-cell in $T\text{-Alg}_1$; in other words, $(\sigma, \bar{\sigma})$ is a lifting of σ to a pseudonatural transformation between the 2-functors $[-,-], [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_1 \to T\text{-Alg}_1$. (If such a lifting exists, it is unique). Moreover, the composition of $(\sigma, \bar{\sigma})$ with $1 \times J_1 : \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_{s,1} \to \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_1$ is a 2-natural transformation.

Conversely, we now show that any lifting $(\sigma, \bar{\sigma})$ of σ whose composition with $\mathscr{W}_1^{\mathrm{op}} \times J_1$ is 2-natural, induces a modification $\bar{\gamma}$ as in (6.13). Given $\bar{\sigma}_{X,B}$ define $\bar{\gamma}_{X,Y}$ by

$$
\begin{tikzcd}
T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[dr, "{T[1,\eta_Y]}"] \arrow[dd, "\bar{t}"'] & T[TX,TY] \arrow[dr, "{T[1,T\eta_Y]}"] \arrow[drr, bend left, "1"] & & \\
& T[X,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"] & T[TX,T^2Y] \arrow[r, "{T[1,\mu_Y]}"] & T[TX,TY] \arrow[d, "\bar{t}"] \\
{[X,TY]} \arrow[r, "{[1,T\eta_Y]}"] \arrow[dr, bend right, "1"'] & {[X,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] & \arrow[phantom, "\Downarrow\bar{\sigma}_{X,TY}"] & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
& {[X,TY]} \arrow[r, "\mathsf{T}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]}
\end{tikzcd}
$$

To show that $\bar{\gamma}_{X,Y}$ is a modification, we use that $\bar{\sigma}$ is 2-natural on strict morphisms: for $f : Z \to Y$, $h : W \to X$ in $\mathscr{W}_1$,

$$\bar{\gamma}_{X,Y} T[h,f] = \bar{\sigma}_{X,TY}(T[X,\eta_Y])(T[h,f]) = \bar{\sigma}_{X,TY}(T[h,Tf])(T[W,\eta_Z]) \\ = [Th,Tf]\bar{\sigma}_{W,TZ}(T[W,\eta_Z]) = [Th,Tf]\bar{\gamma}_{W,Z}.$$

Condition 2 of Proposition 6.18 follows easily from the unit axiom of a pseudomorphism: $\bar{\gamma}_{X,Y}\eta_{[X,Y]} = \bar{\sigma}_{X,TY} T[X,\eta_Y]\eta_{[X,Y]} = \bar{\sigma}_{X,TY}\eta_{[X,TY]}[X,\eta_Y] = 1$. Condition 4 of the same proposition is a bit harder to prove, but routine nonetheless. We leave the verification to the reader; we only mention that the equality $[TX,\mu_Y]\bar{\sigma}_{X,T^2Y} = \bar{\sigma}_{X,TY} T[X,\mu_Y]$ and the multiplication axiom of a pseudomorphism must be used in the verification.

These constructions are inverses of each other: there is a bijection between modifications $\bar{\gamma}$ and liftings of σ to a pseudonatural transformation $(\sigma,\bar{\sigma})$ which composed with $\mathscr{W}_1^{\mathrm{op}} \times J_1$ are 2-natural.

□

Lemma 6.21. *Assume the hypotheses of Lemma 6.20. Then*

1. *Condition 3 of Proposition 6.18 holds if and only if $[\eta,-]\sigma$ is the identity pseudonatural transformation of $[-,-]$.*
2. *Condition 5 of Proposition 6.18 holds for $\bar{\gamma}$ if and only if $\sigma_{TX,B}\sigma_{X,B} = [\mu_X,B]\sigma_{X,B}$ for all $X \in \mathscr{W}_1$ and $B \in T\text{-}\mathrm{Alg}_1$.*
3. *Condition 1 of Proposition 6.18 holds for $\bar{\gamma}$ if and only if the pseudomorphism $[X,\sigma_{Y,B}] : [X,[Y,B]] \to [X,[TY,B]]$ corresponds to the pseudomorphism $[t,B]\sigma_{X\otimes Y,B} : [X\otimes Y,B] \to [X\otimes TY,B]$ under the closedness structure of $\mathscr{W}_1$.*

Proof. The proof of part 1 is obvious.

Now we show 2. Suppose that $\sigma_{TX,B}\sigma_{X,B} = [\mu_X,B]\sigma_{X,B}$. If $\bar{\gamma}$ is defined as in the proof of Lemma 6.20, condition 5 of Proposition 6.18 is the equality

$$\big([T^2X,\mu_Y]\bar{\sigma}_{X,T^2X} T([TX,\eta_Y\mu_Y]\mathsf{T})T[X,\eta_Y]\big)\cdot\big([T^2X,\mu_Y]\mathsf{T}\bar{\sigma}_{X,TY}T[X,\eta_Y]\big) \\ = [\mu_X,TY]\bar{\sigma}_{X,TY}T[X,\eta_Y] \tag{6.16}$$

Using the 2-naturality of σ with respect to strict morphisms,

$$[T^2X,\mu_Y]\bar{\sigma}_{X,T^2Y} = \bar{\sigma}_{X,TY} T[TX,\mu_Y]$$

and using this we can transform the left hand side of (6.16) into the pasting

$$\begin{tikzcd}
T[X,Y] \arrow[r, "{T[1,\eta_Y]}"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\bar\sigma_{X,TY}"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\bar\sigma_{TX,TY}"] & \bullet \arrow[d] \\
& \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

that is by hypothesis equal to $[\mu_X, TY]\bar\sigma_{X,TY}T[X,\eta_Y]$. Conversely, assuming condition 5 of Proposition 6.18, and defining $\bar\sigma$ in terms of $\bar\gamma$ as in the proof of Lemma 6.20, we have to show

$$\big([T^2X,b]\bar\gamma_{TX,B}T[TX,b]T(\mathsf{T})\big)\cdot\big([T^2X,b]\mathsf{T}[TX,b]\bar\gamma_{X,B}\big) = [\mu_X,B][TX,b]\bar\gamma_{X,B}$$

for $X \in \mathscr{W}$ and a T-algebra (B,b). Using the fact that $\bar\gamma$ is a modification, one can see that the left hand side in the equality above is equal to the pasting

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\bar\gamma_{X,B}"] & T[TX,TB] \arrow[r] \arrow[d] \arrow[dr, phantom, "\bar\gamma_{TX,TB}"] & \bullet \arrow[d] \\
& {[TX,T^2B]} \arrow[r] \arrow[d, "{[1,\mu]}"] & {[T^2X,T^B]} \arrow[d, "{[T^2X,b][T^2X,Tb]}"] \\
\bullet \arrow[r] & {[TX,TB]} \arrow[r, "{[T^2X,b][T^2X,Tb]\mathsf{T}}"'] & {[T^2X,B]}
\end{tikzcd}$$

that by hypothesis is just $[T^2X,b][\mu_X,TB]\bar\gamma_{X,B}$. This completes the proof of part 2.

Finally, we prove 3. It is not hard to show that at the level of 1-cells in $\mathscr{W}$, $[X,\sigma_{Y,B}]$ always corresponds to $[t,B]\sigma_{X\otimes Y,B}$, so we must only check the 2-dimensional aspect. Suppose condition 1 of Proposition 6.18 holds. The pseudo-morphism structure of $[X,\sigma_{Y,B}]$ is given by the 2-cell $[X,\bar\sigma_{Y,B}]\bar t_{X,[Y,B]}$, and then we must show that its exponential transpose is the 2-cell $[t_{X,Y},B]\bar\sigma_{X\otimes Y,B}$. This follows trivially from our hypothesis as the former is equal to the composition $[X,[TY,b]][X,\bar\gamma_{X,B}]t_{X,[Y,B]}$ and the latter is equal to $[X\otimes TY,b][t_{X,Y},TB]\bar\gamma_{X\otimes Y,B}$. Conversely, if we assume that the exponential transpose of $[X,\bar\sigma_{Y,B}]\bar t_{X,[Y,B]}$ is $[t_{X,Y},B]\bar\sigma_{X\otimes Y,B}$, it is clear that

$$[X,\bar\gamma_{Y,Z}]\bar t_{X,[Y,Z]} = [X,\bar\sigma_{X,TZ}][X,T[Y,\eta_Z]]\bar t_{X,[Y,Z]}$$

corresponds to

$$[t_{X,Y}, TZ]\bar{\gamma}_{X\otimes Y,Z} = [t_{X,Y}, TZ]\bar{\sigma}_{X\otimes Y,TZ}T[X \otimes Y, \eta_Z].$$

This concludes the proof of the lemma. □

6.5 The pseudo-closed $\mathscr{W}$-category T-Alg

In this section we show that for a pseudo-commutative $\mathscr{W}$-monad on $\mathscr{W}$, the $\mathscr{W}$-category T-Alg is pseudo-closed. We briefly discuss the closed 2-multicategory structure of T-Alg_1. Although we do not emphasise here the $\mathscr{W}$-multicategory structure of T-Alg, our view is that this is the most natural structure to consider on T-Alg. To save space, and because the description of the closed $\mathscr{W}$-multicategory structure is analogous to the 2-categorical case, we will only discuss the latter.

6.5.1 Parametrised pseudomaps

For a $\mathscr{W}$-enriched monad T on $\mathscr{W}$, we can define objects of parametrised pseudomaps. Given objects $X_1, \cdots, X_n$, in $\mathscr{W}$, we denote by

$$t_i : X_1 \otimes \cdots \otimes TX_i \otimes \cdots \otimes X_n \to T(X_1 \otimes \cdots \otimes X_n)$$

the unique arrow obtained by compositions of components of t and t'. We recall from [37] the notion of parametrised pseudomap. If (A_i, a_i) and (B, b) are T-algebras, a pseudomap parametrised by $X_1, \cdots, X_{i_1}$ and $X_{i+1}, \cdots, X_n$ is a 1-cell $f : X_1 \otimes \cdots \otimes X_{i-1} \otimes A_i \otimes X_{i+1} \otimes \cdots \otimes X_n \to B$ equipped with an invertible 2-cell $b(Tf)t_i \cong f(X_1 \otimes \cdots \otimes a \otimes \cdots \otimes X_n)$ satisfying pseudomap axioms. With the obvious notion of 2-cell between parametrised pseudomaps, we obtain categories T_1-$\mathrm{Alg}(X_1, \cdots, A_i, \cdots, X_n, B)$. Parametrised pseudomaps and cotensor products are related in the following way. If A, B are T-algebras and X is an object of $\mathscr{W}$, there are 2-natural isomorphisms

$$T_1\text{-}\mathrm{Alg}(X, A; B) \cong T_1\text{-}\mathrm{Alg}(A, [X, B]) \tag{6.17}$$

where $[X, B]$ is the usual cotensor product in T-Alg.

For an enriched monad S on a monoidal category $\mathscr{V}$ one can define parametrised maps of algebras. There is an obvious multicategory $\mathscr{V}^T$ with objects the T-

algebras and multimaps $A_1, \cdots, A_n \to B$ the arrows $A_1 \otimes \cdots \otimes A_n \to B$ which are parametrised maps in each variable. However, in the two-dimensional case we have two obvious choices of a 2-multicategory, as we explain below.

For a pseudo-commutative $\mathscr{W}$-monad T on $\mathscr{W}$, there are two 2-multicategories of T-algebras one can consider. The simplest is the 2-multicategory whose objects are the T-algebras and multihom-categories $T_1\text{-Alg}^b(A_1, \cdots, A_n, B)$ the categories of 1-cells $A_1 \otimes \cdots \otimes A_n \to B$ that are parametrised pseudomaps in each variable, and 2-cells compatible with this structure. The other 2-multicategory is the one considered in [37], denoted by T_1-Alg. It has as objects the T-algebras and multihoms $T_1\text{-Alg}(A_1, \cdots, A_n, B)$ the full subcategory of the corresponding multihom of $T_1\text{-Alg}^b$ determined by the 1-cells which are partial maps in each variable with the additional condition that each partial map structure must *commute* with the others. To state this property, a pseudo-commutativity on T is required. For more details see [37]. There is an obvious inclusion morphism of 2-multicategories $T_1\text{-Alg} \to T_1\text{-Alg}^b$.

From these two 2-multicategories the most interesting is T-Alg, as it carries a closed structure. This is explained below.

6.5.2 The T-algebra of pseudomorphisms

In this section we construct the internal hom that will be part of a pseudo-closed structure on the $\mathscr{W}$-category T-Alg. The underlying object of this T-algebra is the $\mathscr{W}$-object of pseudomorphisms constructed in Definition 6.1. First we look at an example.

Example 6.22. Let T be the 2-monad on **Cat** of Example 6.17, whose algebras are symmetric strict unbiased monoidal categories. For each pair of T-algebras A, B, the category $T\text{-Alg}(A, B)$ has an obvious symmetric strict monoidal structure. If $f_1, \cdots, f_n : A \to B$ are symmetric strong monoidal functors, their tensor product is the functor sending $a \in A$ to $f_1(a) \otimes \cdots \otimes f_n(a)$. This functor is symmetric stong monoidal thanks to the existence of the pseudo-commutativity of T.

We need to give, for each pair of T-algebras A, B, an T-algebra $[\![A, B]\!]$ with underlying object $T\text{-Alg}(A, B)$. The latter is defined by means of (PIE)*-limits in $\mathscr{W}_1$, (limits that can be constructed from products, inserters and equifers) and these are created by the forgetful $\mathscr{W}$-functor $U_1 : T\text{-Alg}_1 \to \mathscr{W}_1$, and in this way $T\text{-Alg}(A, B)$ is endowed with a canonical structure of a T-algebra. We explain this in more detail below.

Recall from Section 6.2.2 that the enriched hom $T\text{-Alg}(A,B)$ can be obtained as a limit $\{\chi, F_{A,B}\}$ in $\mathscr{W}_1$, where $F_{A,B} : \mathscr{F} \to \mathscr{W}_1$ is certain 2-functor defined by (6.7) and χ is a weight in the class (PIE)*. The diagram in (6.7) in this case ($\mathscr{K} = \mathscr{W}$) is

$$
\begin{tikzcd}
{[A,B]} \arrow[r, shift left=2, "\sigma_{A,B}"] \arrow[r, shift left=0.5, "{[a,1]}"'] & {[TA,B]} \arrow[l, shift left=1.5, "{[\eta_A,1]}"] \arrow[r, shift left=2, "\sigma_{TA,B}"] \arrow[r, shift left=0.5, "{[Ta,1]}"'] \arrow[r, shift right=1.5, "{[\mu_A,1]}"'] & {[T^2A,B]}
\end{tikzcd}
\tag{6.18}
$$

Proposition 6.19 tells us that the 1-cell $\sigma_{A,B}$ of (6.14) is a pseudomorphism, and hence the diagram above lies in $T\text{-Alg}_1 = T_1\text{-Alg}$. Hence $F_{A,B} : \mathscr{F} \to \mathscr{W}_1$ factors through a 2-functor $\hat{F}_{A,B} : \mathscr{F} \to T_1\text{-Alg}$ as $U\hat{F}_{A,B}$. Since U creates (PIE)*-limits, it follows that $T\text{-Alg}(A,B)$ is the underlying object of a canonical T-algebra, namely $\{\chi, \hat{F}_{A,B}\}$. Moreover, the universal $p : [\![A,B]\!] \to [A,B]$ is a strict morphism.

The assignment $A, B \mapsto [\![A,B]\!]$ extends to a $\mathscr{W}$-functor $[\![-,-]\!] : T\text{-Alg}^{\mathrm{op}} \otimes T\text{-Alg} \to T\text{-Alg}$. Indeed, we need to exhibit 1-cells

$$T\text{-Alg}(C,A) \otimes T\text{-Alg}(B,D) \to T\text{-Alg}([\![A,B]\!], [\![C,D]\!]) \tag{6.19}$$

in $\mathscr{W}$, or equivalently, a parametrised pseudomap of algebras

$$T\text{-Alg}(C,A) \otimes [\![A,B]\!] \otimes T\text{-Alg}(B,D) \to [\![C,D]\!]. \tag{6.20}$$

As a 1-cell, this parametrised pseudomap is just the obvious composition 1-cell. The other piece of data we have to provide is an invertible 2-cell between the 1-cells

$$
\begin{tikzcd}
T\text{-Alg}(C,A) \otimes T[\![A,B]\!] \otimes T\text{-Alg}(B,D) \arrow[r, "t_2"] \arrow[d, "1\otimes \mathrm{act} \otimes 1"] & T(T\text{-Alg}(C,A) \otimes [\![A,B]\!] \otimes T\text{-Alg}(B,D)) \arrow[r, "T\mathrm{comp}"] & T[\![C,D]\!] \arrow[d, "\mathrm{act}"] \\
T\text{-Alg}(C,A) \otimes [\![A,B]\!] \otimes T\text{-Alg}(B,D) \arrow[rr, "\mathrm{comp}"'] & & {[\![C,D]\!]}
\end{tikzcd}
$$

This 2-cell corresponds, upon composition with the universal $[\![C,D]\!] \to [C,D]$, to the 2-cell obtained from the 2-cell in Figure 6.5.

Example 6.23. Although it might seem a bit complicated, the definition of $[\![-,-]\!]$ on homs (6.19) has a very simple content. This can be exemplified by the case when $\mathscr{W}$ is **Cat** and T is the 2-monad whose algebras are symmetric strict (unbiased) monoidal categories. See Examples 6.17 and 6.22. In this case, (6.20) is the composition functor and with parametrised pseudomap structure given by

$$
\begin{tikzcd}
T\llbracket A,B\rrbracket\otimes\llbracket B,D\rrbracket \arrow[r, "Tp\otimes p"] \arrow[d, "\bar{t}(Tp)\otimes 1"'] & T[A,B]\otimes[B,D] \arrow[r, "t'"] & T([A,B]\otimes[B,D]) \arrow[r, "T\mathrm{comp}"] & T[A,D] \arrow[d, "\bar{t}"] \\
{[A,TB]\otimes\llbracket B,D\rrbracket} \arrow[r, "1\otimes p"] \arrow[d, "1\otimes p"'] \arrow[dr, phantom, "\cong"] & {[A,TB]\otimes[B,D]} \arrow[r, "1\otimes \mathsf{T}"] \arrow[d, "1\otimes\sigma_{B,D}"] & {[A,TB]\otimes[TB,TD]} \arrow[r, "\mathrm{comp}"] \arrow[dl, "1\otimes[1,d]"] & {[A,TD]} \arrow[d, "{[1,d]}"] \\
{[A,TB]\otimes[B,D]} \arrow[r, "1\otimes[b,1]"] \arrow[dr, "{[1,b]\otimes 1}"'] & {[A,TB]\otimes[TB,D]} \arrow[rr, "\mathrm{comp}"] & & {[A,D]} \\
& {[A,B]\otimes[B,D]} \arrow[urr, "\mathrm{comp}"'] & &
\end{tikzcd}
$$

Figure 6.5: 2-cell that induces the parametrised pseudomap structure of the composition.

the canonical isomorphism

$$h(g_1 \otimes \cdots \otimes g_n)f \cong (hg_1f) \otimes \cdots \otimes (hg_nf)$$

induced by the strong monoidal structure of h, for $f : C \to A$, $g_i : A \to B$, $h : B \to D$.

Observation 6.24. The $\mathscr{W}$-functor $\llbracket -,- \rrbracket$ restricts to a $\mathscr{W}$-functor $T\text{-}\mathrm{Alg}_s^{\mathrm{op}} \otimes T\text{-}\mathrm{Alg}_s \to T\text{-}\mathrm{Alg}_s$. Indeed, in the construction of the 1-cell (6.19) we only used the universal isomorphism $\sigma_{B,D}p \cong [b,D]p$ defining $\llbracket B,D \rrbracket$. This means that by restricting to $T\text{-}\mathrm{Alg}_s(B,D)$ in (6.19) we obtain a 1-cell

$$T\text{-}\mathrm{Alg}(C,A) \otimes T\text{-}\mathrm{Alg}_s(B,D) \to T\text{-}\mathrm{Alg}_s(\llbracket A,B \rrbracket, \llbracket C,D \rrbracket).$$

In other words, each $\mathscr{W}$-functor $\llbracket A,- \rrbracket$ sends strict morphisms to strict morphisms and each $\mathscr{W}$-functor $\llbracket -,B \rrbracket$ send all pseudomorphisms to strict morphisms.

The proof of the following proposition is the same as the proof of [37, Theorem 10].

Proposition 6.25. *If T is a pseudo-commutative $\mathscr{W}$-enriched monad on $\mathscr{W}$, then $T\text{-}\mathrm{Alg}_1$ has a canonical structure of a closed 2-multicategory .*

For later use we include the description of the isomorphisms of categories $T\text{-}\mathrm{Alg}_1(A,B;C) \cong T\text{-}\mathrm{Alg}_1(A,\llbracket B,C \rrbracket)$ given in [37]. A multimap $(f,\bar{f}_A,\bar{f}_B) : A,B \to C$ is given by the following data in $\mathscr{W}$: a 1-cell $f : A \otimes B \to C$, and invertible 2-cells $\bar{f}_A : (Tf)t' \Rightarrow f(a \otimes B)$ and $\bar{f}_B : (Tf)t \Rightarrow f(A \otimes b)$ satisfying axioms. If $g : A \to [B,C]$ is the exponential transpose of f and $\bar{g}$ the transpose of $\bar{f}_A$, then the condition on $\bar{f}_A$ expressing the fact that f is a partial map in the first variable translates into the statement that $(g, \bar{g})$ is a pseudomap $A \to [B,C]$. The transpose of $\bar{f}_B$ gives a 2-cell $\alpha : \sigma_{B,C} g \Rightarrow [b, C] g$ because f_A commutes with f_B. Finally , α lifts to a 2-cell between 1-cells into $[\![B,C]\!]$ as a translation of the pseudomap condition on the second variable.

6.5.3 T-Alg as a pseudo-closed $\mathscr{W}$-category

Proposition 6.26. *If T is a pseudo-commutative $\mathscr{W}$-enriched monad on $\mathscr{W}$, then T-Alg has a canonical structure of a pseudo-closed $\mathscr{W}$-category.*

Proof. We have to provide the data in Definition 6.3. The $\mathscr{W}$-functor $T\text{-Alg} \to \mathscr{W}$ will be the forgetful $\mathscr{W}$-functor U, the internal hom will be the one described in the preceding section, denoted by $[\![-,-]\!]$, and the unit object will be FI, the free T-algebra on the unit object of $\mathscr{W}$. The 1-cell $j_A : FI \to [\![A,A]\!]$ is the strict morphism corresponding to the identity 1-cell $I \to T\text{-Alg}(A,A)$. The 1-cell e_A is the strict morphism

$$[\![FI, A]\!] \xrightarrow{p} [TI, A] \xrightarrow{[\eta_I, 1]} [I, A] \xrightarrow{\cong} A.$$

The pseudoinverse of e_A is $i_A : A \xrightarrow{\cong} [I,A] \xrightarrow{\simeq} [\![FI, A]\!]$. The composition 1-cell $k_{A,B,C} : [\![B,C]\!] \to [\![[\![A,B]\!],[\![A,C]\!]]\!]$ corresponds to the composition multi pseudomap $[\![B,C]\!] \otimes [\![A,B]\!] \to [\![A,C]\!]$. Observe that j_A, e_A and k are strict maps of T-algebras. Checking the axioms of a pseudo-closed $\mathscr{W}$-category is now a matter of routine. This can be found for the case $\mathscr{W} = \mathbf{Cat}$ in [37]. □

Recall that the unit and counit η, ε of the $\mathscr{W}$-adjunction $F \dashv U_s : T\text{-Alg}_s \to \mathscr{W}$ induce $\mathscr{W}$-natural transformations $1 \to UF :$ and $FU \to 1$, which we still name η and ε.

Proposition 6.27. *The $\mathscr{W}$-functors $U : T\text{-Alg} \to \mathscr{W}$ and $F : \mathscr{W} \to T\text{-Alg}$ have canonical closed structures. The unit $\eta : 1 \to UF$ is a closed $\mathscr{W}$-natural transformation and the counit $\varepsilon : FU \to 1$ is a pseudo-closed $\mathscr{W}$-natural transformations.*

Proof. The forgetful $\mathscr{W}$-functor U is closed by definition of the pseudo-closed structure of T-Alg: the closed constraints are given by the universal $U[\![A,B]\!] = T\text{-Alg}(A,B) \to \mathscr{W}(A,B)$ and the unit $\eta_I : I \to UFI$.

The closed structure of F is given in the following way. The arrow $\phi : F[X,Y] \to [\![FX, FY]\!]$ is the unique strict morphisms of T-algebras corresponding to the 1-cell $[X,Y] \to T\text{-Alg}(FX,FY)$ induced by the identity 2-cell

$$\begin{tikzcd}
 & {[TX,TY]} \arrow[r, "T"] & {[T^2X,T^2Y]} \arrow[dr, "{[1,\mu_Y]}"] & \\
{[X,Y]} \arrow[ur, "T"] \arrow[dr, "T"'] & & & {[T^2X,TY]} \\
 & {[TX,TY]} \arrow[urr, "{[\mu_X,1]}"'] & &
\end{tikzcd}$$

The unit constraint is just the identity $FI \to FI$.

The closedness of η follows from the commutativity of the diagrams below.

$$\begin{tikzcd}
{[X,Y]} \arrow[rr, "{[1,\eta_Y]}"] \arrow[d, "\eta_{[X,Y]}"'] \arrow[drr, "T"] & & {[X,UFY]} \\
{UF[X,Y]} \arrow[r, "U\phi"'] & {U\llbracket FX,FY\rrbracket} \arrow[r, "Up"'] & {[UFX,UFY]} \arrow[u, "{[\eta_X,1]}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I \arrow[dl, "1"'] \arrow[dr, "\eta_I"] & \\
I \arrow[r, "\eta_I"'] & UFI \arrow[r, "1"'] & UFI
\end{tikzcd}$$

Now we show that the counit ε is a pseudo-closed $\mathscr{W}$-natural transformation in the sense of Definition 6.4. We exhibit an isomorphism depicted in the diagram on the left hand side below, and show that the triangle on the right hand side commutes.

$$\begin{tikzcd}
{FU\llbracket A,B\rrbracket} \arrow[r] \arrow[d, "\varepsilon_{\llbracket A,B\rrbracket}"'] & {F[UA,UB]} \arrow[r, "\phi"] \arrow[d, phantom, "\cong"] & {\llbracket FUA,FUB\rrbracket} \arrow[d, "{\llbracket 1,\varepsilon_B\rrbracket}"] \\
{\llbracket A,B\rrbracket} \arrow[rr, "{\llbracket \varepsilon_A,1\rrbracket}"'] & & {\llbracket FUA,B\rrbracket}
\end{tikzcd}
\qquad
\begin{tikzcd}
FI \arrow[r, "1"] \arrow[dr, "1"'] & FI \arrow[r, "F\eta_I"] & FUFI \arrow[dl, "\varepsilon_{FI}"] \\
 & FI &
\end{tikzcd}$$

The unit condition is obvious. To define the isomorphism in the diagram on the left hand side, we observe that all the arrows are strict morphisms of T-algebras, and hence it suffices to define an isomorphism between $\llbracket \varepsilon_A, B\rrbracket$ and

$$U\llbracket A,B\rrbracket \xrightarrow{p} [UA,UB] \xrightarrow{F} \llbracket FUA,FUB\rrbracket \xrightarrow{\llbracket 1,\varepsilon_B\rrbracket} \llbracket FUA,B\rrbracket. \tag{6.21}$$

By using the 2-dimensional aspect of the universal property of the limit $\llbracket FUA,B\rrbracket$, we reduce the problem to defining an isomorphism between the composition of (6.21) and $\llbracket \varepsilon_A, B\rrbracket$ with the universal $p : \llbracket FUA,B\rrbracket \to [TA,B]$, compatible with the universal isomorphism $[T^2A,b]Tp \cong [\mu_A,B]p$ (see Definition 6.1). The composition of (6.21) and $\llbracket \varepsilon A, B\rrbracket$ with p are respectively

$$\llbracket A,B\rrbracket \to [A,B] \xrightarrow{\mathsf{T}} [TA,TB] \xrightarrow{[1,b]} [TA,B] \quad \text{and} \quad \llbracket A,B\rrbracket \to [A,B] \xrightarrow{[a,1]} [TA,B]$$

and the required isomorphism between these is the universal isomorphism in the definition of $[\![A, B]\!]$. One can verify that this isomorphism satisfies the axioms of a pseudo-closed transformation.

$\square$

6.6 Monoidal structures

Proposition 6.28. *Given a monad T on $\mathscr{W}$ in $\mathscr{W}$-**Cat**, suppose we have the following commutative diagram in $\mathscr{W}$-**Cat**.*

$$\begin{tikzcd} T\text{-Alg}_s \arrow[r, "G"] \arrow[d, "J"'] & T\text{-Alg}_s \arrow[d, "U_s"] \\ T\text{-Alg} \arrow[r, "T\text{-Alg}(A,-)"'] & \mathscr{W} \end{tikzcd}$$

If J has a left adjoint, T-Alg_s admits tensor products and coequalizers, then G has a left adjoint too.

Proof. First of all, observe that in presence of cotensor products, the existence of coequalizers in T-Alg_s is equivalent to the existence of coequalizers in the ordinary category $(T\text{-}\mathrm{Alg}_s)_0$. See [42, Section 3.8].

Since U_s creates limits, G preserves limits if and only if $U_sG = T\text{-}\mathrm{Alg}(A, J-)$ preserves limits. But this is true as J preserves limits (Proposition 6.8). Then G has a left adjoint if and only if the functor G_0 does. We can use the Adjoint Triangle Theorem, or for example [2, Theorem 7.3.b], to prove that G_0 has a left adjoint. Indeed, $U_{s,0}$ is monadic and $T\text{-}\mathrm{Alg}(A, J-)_0 \cong T\text{-}\mathrm{Alg}_s(A', -)_0$ has a left adjoint given by taking $- * A'$ (tensor product with A') and $(T\text{-}\mathrm{Alg}_s)_0$ has coequalizers. $\square$

Corollary 6.29. *Let T be a $\mathscr{W}$-enriched pseudo-commutative monad on $\mathscr{W}$ and $[\![-,-]\!]$ the internal hom of the induced pseudo-closed structure on T-Alg. If J has left adjoint and T-Alg_s has cotensor products and coequalizers, then the $\mathscr{W}$-functor $[\![B,-]\!] : T\text{-}\mathrm{Alg}_s \to T\text{-}\mathrm{Alg}_s$ has a left adjoint $- \oslash B$, for all T-algebras B. In particular, the result holds if T is pseudo-commutative and has a rank.*

Corollary 6.30. *The $\mathscr{W}$-functors $- \oslash B$ extend to a $\mathscr{W}$-functor*

$$\oslash : T\text{-}\mathrm{Alg}_s^{\mathrm{op}} \otimes T\text{-}\mathrm{Alg}_s \to T\text{-}\mathrm{Alg}_s$$

cocontinuous in the first variable. Moreover, if for a weight ψ, T preserves ψ-colimits, then $\oslash$ preserves ψ-colimits in the second variable.

Proof. The first part of the statement is obvious from Corollary 6.29 above. The last part is equivalent to claiming that each $\mathscr{W}$-functor $[\![-, C]\!] : T\text{-Alg}_s^{\mathrm{op}} \to T\text{-Alg}_s$ preserves ψ-limits whenever T preserves ψ-colimits. Since $U_s : T\text{-Alg}_s \to \mathscr{W}$ creates limits, this is equivalent to saying that $U_s[\![-, C]\!] = T\text{-Alg}(J-, C)$ preserves ϕ-limits whenever T preserves ψ-colimits. This last statement holds true by Proposition 6.9. $\square$

Proposition 6.31. *Assume the hypotheses of Corollary 6.29. Then each 2-functor $[\![B, -]\!]_1 : T\text{-Alg}_1 \to T\text{-Alg}_1$ has a left biadjoint $- \boxtimes B$. Moreover, the pseudofunctor $- \boxtimes -$ is part of a (pseudo-)monoidal structure making $T\text{-Alg}_1$ a (pseudo-)monoidal pseudo-closed 2-category.*

Proof. We use Theorem 6.16 to deduce our result. Define $A \boxtimes B = J(A' \oslash B)$. Since $(- \oslash B)_1$ is a left adjoint to

$$(T\text{-Alg}_{s,1} \xrightarrow{J_1} T\text{-Alg}_1 \xrightarrow{[\![B,-]\!]_1} T\text{-Alg}_1) = (T\text{-Alg}_{s,1} \xrightarrow{[\![B,-]\!]_1} T\text{-Alg}_{s,1} \xrightarrow{J_1} T\text{-Alg}_1)$$

we have that $(- \boxtimes B)_1$ is a left biadjoint to the 2-functor $[\![B, -]\!]_1$. Moreover,

$$d_C : [\![J(A' \oslash B), C]\!] \xrightarrow{k_B} [\![[\![B, J(A' \oslash B)]\!], [\![B, C]\!]]\!] \xrightarrow{[\![\mathrm{unit}, 1]\!]} [\![A, [\![B, C]\!]]\!]$$

is a retract equivalence in $\mathscr{W}$ (see Proposition 6.12). The commutativity of the diagram in Theorem 6.16 is equivalent to the commutativity in $\mathscr{W}$ of the diagram below, which is easy to verify.

$$\begin{tikzcd}
{[\![A\boxtimes B,C]\!]\otimes[\![C,D]\!]} \arrow[r, "d_C\otimes 1"] \arrow[d, "\mathrm{comp}"'] & {[\![A,[\![B,C]\!]]\!]\otimes[\![C,D]\!]} \arrow[r, "1\otimes k"] & {[\![A,[\![B,C]\!]]\!]\otimes[\![[\![B,C]\!],[\![B,D]\!]]\!]} \arrow[d, "\mathrm{comp}"] \\
{[\![A\boxtimes B,D]\!]} \arrow[rr, "d_D"'] & & {[\![A,[\![B,D]\!]]\!]}
\end{tikzcd}$$

Now we apply Theorem 6.16. $\square$

Corollary 6.32. *The 2-functors F_1, U_1 and the unit and counit η, ε of the biadjunction $F_1 \dashv_{\mathrm{b}} U_1$, have canonical (pseudo) monoidal structures.*

Proof. We already mentioned at the end of Section 6.3.1 that when pseudo-closed structures induce monoidal structures, closed 2-functors and pseudo-closed 2-natural transformations acquire structures of (pseudo) monoidal functors and (pseudo) monoidal transformations, respectively. In the case of the biadjunction $F_1 \dashv_{\mathrm{b}} U_1$, this means that F_1, U_1 together with the unit and the counit are (pseudo) monoidal. $\square$

Observation 6.33. 1. The universal property of $A \boxtimes B$ can be expressed in the following way. There is a multimap $A \otimes B \to A \boxtimes B$ inducing an equivalence between the category of pseudomorphisms $A \boxtimes B \to C$ and the category of multimaps $A \otimes B \to C$, for all T-algebras C.

2. A standard argument shows that the monoidal constraint $F_1(X) \boxtimes F_1(Y) \to F_1(X \otimes Y)$ is an equivalence in T-Alg.

6.7 Lax-idempotent monads are pseudo-commutative

In this section we show that every lax-idempotent $\mathscr{W}$-enriched monad has a canonical pseudo-commutativity.

Recall that a 2-monad (T, η, μ) on a 2-category $\mathscr{K}$ is *lax-idempotent*, or *Kock-Zöberlein*, when any 1-cell $f : A \to B$ between T-algebras has a unique structure of a lax morphism of T-algebras. This is equivalent to the condition that a 1-cell $a : TA \to A$ is a T-algebra structure if and only if there exists an retract adjunction $a \dashv_{\mathrm{r}} \eta_A$ (*i.e.*, and adjunction with counit an identity). Another equivalent condition is the existence of a modification $\delta : T\eta \to \eta T$ satisfying

$$\delta\eta = 1 \qquad \text{and} \qquad \mu\delta = 1. \tag{6.22}$$

Many more equivalent conditions are given in [44, Theorem 6.2]. Also, the forgetful 2-functor $U_\ell : T\text{-Alg} \to \mathscr{K}$ is locally fully faithful.

If A, B are T-algebras, the unique lax morphism structure on a 1-cell $f : A \to B$ in $\mathscr{K}$ is given by the following 2-cell, where the arrows denote the counit and unit of the respective adjunctions.

$$\begin{tikzcd} TA \arrow[r, equal] \arrow[d, "a"'] & TA \arrow[rr, "Tf"] \arrow[d, phantom, "\Uparrow"] & & TB \arrow[d, "b"] \\ A \arrow[ur, "\eta_A"'] \arrow[rr, "f"'] & & B \arrow[ur, "\eta_B"] \arrow[r, equal] \arrow[u, phantom, "\Uparrow"] & B \end{tikzcd} \tag{6.23}$$

It follows that a 1-cell $f : A \to B$ has a (unique) structure of a pseudomorphism of T-algebras if and only if 6.23 is invertible. Also, the forgetful 2-functor U :

T-Alg $\to \mathscr{K}$ is locally injective on objects (*i.e.*, U is injective on 1-cells) and locally fully faithful.

In [51] it is shown that left adjoint 1-cells between algebras for a doctrine are pseudomaps. In our case, the same is true. If A, B are T-algebras and $f \dashv f^* : B \to A$ is an adjunction in $\mathscr{K}$, then f^*, just as any 1-cell, is a lax morphism and hence f has a structure of a colax morphism of T-algebras. It follows from [44, Lemma 6.5] that the colax structure $fa \Rightarrow bTf$ is invertible and its inverse is a pseudomorphism structure on f.

Lemma 6.34. *Let $T : \mathscr{W} \to \mathscr{W}$ be a $\mathscr{W}$-enriched monad and assume that the underlying 2-monad $T_1 : \mathscr{W}_1 \to \mathscr{W}_1$ is lax-idempotent. Then the 1-cell in* (6.14)

$$\sigma_{X,B} : [X,B] \xrightarrow{\mathsf{T}} [TX,TB] \xrightarrow{[TX,b]} [TX,B]$$

is part of a coretract adjunction with right adjoint $[\eta_X, B] : [TX,B] \to [X,B]$. In particular, (6.14) *is a pseudomorphism.*

Proof. We have $[\eta_X,B][TX,b]\mathsf{T} = [X,b][\eta_X,TB]\mathsf{T} = [X,b][X,\eta_X] = 1$ by $\mathscr{W}$-naturality of η. So indeed we can define the unit of our adjunction as the identity. Now define the counit as the following 2-cell

$$\begin{tikzcd}
& {[X,B]} \arrow[drr, bend left, "\mathsf{T}"] & & & \\
{[TX,B]} \arrow[ur, "{[\eta_X,1]}"] \arrow[r, "\mathsf{T}"] \arrow[rrr, bend right, "{[1,\eta_B]}"'] & {[T^2X,TB]} \arrow[rr, bend left=15, "{[T\eta_X,1]}"] \arrow[rr, bend right=15, "{[\eta_{TX},1]}"'] \arrow[rr, phantom, "\Downarrow"] & & {[TX,TB]} \arrow[r, "{[1,b]}"] & {[TX,B]}
\end{tikzcd}$$

where the unlabelled 2-cell is $[\delta_X, 1]$. Now we check the axioms of an adjunction. First, $[\eta_X,B][TX,b][\delta_X,TB]\mathsf{T} = [X,b][\delta_X\eta_X,TB]\mathsf{T} = 1$ by (6.22). The other axioms is again follows from (6.22):

$$\begin{aligned}
[TX,b][\delta_X,TB]\mathsf{T}[TX,b]\mathsf{T} &= [\delta_X,B][T^2X,b][T^2X,Tb]\mathsf{T}\mathsf{T} \\
&= [\delta_X,B][T^2X,b][T^2X,\mu_B]\mathsf{T}\mathsf{T} \\
&= [\delta_X,B][T^2X,b][\mu_X,TB]\mathsf{T} \\
&= [\delta_X,B][\mu_X,B][TX,b]\mathsf{T} \\
&= 1.
\end{aligned}$$

□

Theorem 6.35. *Every $\mathscr{W}$-enriched monad $T : \mathscr{W} \to \mathscr{W}$ such that its underlying 2-monad T_1 is lax-idempotent is pseudo-commutative. Moreover, the pseudo-commutativity is unique.*

Proof. We have to check the conditions in Proposition 6.19. By Lemma 6.34 above, σ lifts to a pseudonatural transformation $[-,-] \Rightarrow [T-,-] : \mathscr{W}_1^{\mathrm{op}} \otimes T\text{-Alg}_1 \to T\text{-Alg}_1$. Moreover this lifting is unique because $U_1 : T\text{-Alg}_1 \to \mathscr{W}_1$ is injective on 1-cells and locally fully faithful. The conditions (1) to (4) in Proposition 6.19 hold trivially, since U_1 is injective in 1-cells; in other words, these conditions hold if and only if they hold in $\mathscr{W}$. The uniqueness of the pseudo-commutativity is equivalent to the uniqueness of the pseudomorphism structure on each $\sigma_{X,B}$, which holds by the properties of U_1 already mentioned. $\square$

Corollary 6.36. *If $T : \mathscr{W} \to \mathscr{W}$ is a $\mathscr{W}$-enriched monad with lax-idempotent underlying 2-monad, then T-*Alg *has a canonical structure of a pseudo-closed $\mathscr{W}$-category.*

Proof. It is a consequence of Theorem 6.35 together with Proposition 6.26. $\square$

Example 6.37. There are pseudo-commutative 2-monads which are *not* lax-idempotent. For example, the 2-monad T on **Cat** whose algebras are the symmetric strict monoidal categories. See [37] for a detailed description of the pseudo-commutativity for this 2-monad, or Example 6.17. One of the several possible ways of seeing that T is not lax-idempotent is to show that there can not be a 2-natural transformation $\delta_X : T\eta_X \Rightarrow \eta_{TX} : TX \to T^2X$. For, if $(x_1, \cdots, x_n) \in TX$, the corresponding component of δ_X should be an arrow $((x_1), \cdots, (x_n)) \to ((x_1, \cdots, x_n))$. But there are no such arrows in T^2X unless $n = 1$, as there are arrows only between strings of the same length.

Example 6.38. There are property-like 2-monads which are *not* pseudo-commutative. Property-like 2-monads were defined in [44] as those 2-monads for which every algebra structure is unique up to isomorphism and every pseudomorphism structure on a 1-cell is unique. For example, if T is the 2-monad on **Cat** that freely adds chosen finite products and finite coproducts, T is property-like but is not pseudo-commutative (as products do not always commute with coproducts).

Recall from Section 6.5 the 2-multicategories T_1-Alg and T_1-Alg^b.

Proposition 6.39. *If T is lax-idempotent, the canonical morphism of 2-multicategories $T\text{-Alg}_1 \to T\text{-Alg}_1^b$ is an isomorphism.*

Proof. For simplicity we explain why the inclusion

$$T\text{-Alg}_1(A, B; C) \longrightarrow T\text{-Alg}_1^b(A, B; C)$$

is an isomorphism. In other words, we must show that every 1-cell $f : A \otimes B \to C$ equipped with structures of a partial map in each variable automatically satisfies the commutation relation necessary to be a multimap. As we noted at the end of Section 6.5.2 of the partial map structures give a pseudomap structure to the exponential transpose $g : A \to [B, C]$ of f. The other partial map structure correspond to a 2-cell $\alpha : \sigma_{B,C} g \Rightarrow [b, C]g$ in $\mathscr{W}$. The two partial map structures commute with each other if and only if α is a 2-cell in T-Alg$_1$. Now, by Lemma 6.34, the domain and codomain of α are pseudomorphisms, and hence α is a 2-cell in T-Alg$_1$ since for the lax-idempotent 2-monad T_1 the forgetful 2-functor $T\text{-Alg}_1 = T_1\text{-Alg} \to \mathscr{W}$ is locally full.

□

Chapter 7

Categories with finite (co)limits

Now it is time to apply the theory of the previous chapters to the example of the 2-category of $\mathscr{V}$-categories with finite (co)limits equipped with the (weak or pseudo) monoidal structure constructed in Chapter 6.

After discussing pseudo-closed structures on 2-categories of $\mathscr{V}$-categories with a class of colimits, we look closely at the corresponding monoidal structures in the case of finite colimits. In Section 7.3 we recall Deligne's tensor product of abelian categories (introduced in [19]) and prove that our tensor product of categories with finite colimits coincides with Deligne's tensor product on a special class of abelian categories: those abelian categories for which Deligne's tensor product is proven to exist in [19].

In Section 7.5 we deduce the "Radford's formula for finite tensor categories" of [27] from the general theory of previous chapters. In particular our proof is independent of the Perron-Frobenius dimension argument used in [27]. In the rest of the chapter we consider the case of semisimple categories, and give a characterisation of semisimplicity of a autonomous category enriched in vector spaces in bicategorical terms (the existence of certain adjunction). Explicit descriptions of various constructions are provided.

7.1 The case $\mathscr{W} = \mathscr{V}$-Cat

In this section we apply the results developed so far to the case of the symmetric monoidal closed 2-category $\mathscr{W} = \mathscr{V}$-**Cat**. Here $\mathscr{V}$ is a complete an cocomplete symmetric monoidal closed category, and $\mathscr{V}$-**Cat** the 2-category of small $\mathscr{V}$-categories. We use the constructions and follow the notations of [45].

Let Φ be a small class of weights. Recall from [42, Section 5.5] that the free completion of a $\mathscr{V}$-category A under Φ-colimits can be obtained as the closure under Φ-colimits of the representables in $[A^{\mathrm{op}}, \mathscr{V}]$. This Φ-cocomplete $\mathscr{V}$-category is usually denoted by ΦA. The Yoneda embedding $\mathsf{y}_A : A \to \Phi A$ induces equivalences of $\mathscr{V}$-categories $\Phi\text{-}\mathbf{Cocts}[\Phi A, B] \simeq [A, B]$ for all Φ-cocomplete $\mathscr{V}$-category B, with pseudoinverse given by taking left Kan extension along y_A. Here $\Phi\text{-}\mathbf{Cocts}[C, D]$ denotes the $\mathscr{V}$-category of Φ-cocontinuous $\mathscr{V}$-functors $C \to D$; these are the enriched homs of a $\mathscr{W}$-category with objects the Φ-cocomplete small $\mathscr{V}$-categories.

Let us denote by $\Phi\text{-}\mathbf{Colim}_1$ be the 2-category of $\mathscr{V}$-categories with chosen Φ-colimits, $\mathscr{V}$-functors strictly preserving these and $\mathscr{V}$-natural transformations. This is the underlying 2-category of a $\mathscr{W}$-category Φ-**Colim** with enriched homs $\Phi\text{-}\mathbf{Colim}(A, B)$ the full sub-$\mathscr{V}$-category of $[A, B]$ determined by the $\mathscr{V}$-functors that strictly preserve Φ-colimits. There is an obvious forgetful $\mathscr{W}$-functor U_s : $\Phi\text{-}\mathbf{Colim} \to \mathscr{W}$. The main result of [45] is the monadicity of $(U_s)_1$, as a 2-functor. If T is the associated 2-monad on $\mathscr{W}$, with unit $\eta : 1 \to T$, there is an equivalence of $\mathscr{V}$-categories making the following diagram commutative.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathsf{y}_A"'] & TA \arrow[d, "\simeq"] \\ & \Phi A \end{tikzcd}$$

Below we explain the necessary modifications to prove that the $\mathscr{W}$-functor U_s is monadic.

Lemma 7.1. *The $\mathscr{W}$-functor U_s has a left adjoint.*

Proof. By [45, Theorem 5.1] we know the 2-functor $(U_s)_1$ has a left adjoint. Hence, it is enough to prove that Φ-**Colim** has cotensor products preserved by U_s. As we aim to prove U_s is monadic, U_s will in fact create cotensor products, and that is what we shall show. This amounts to saying that for any $\mathscr{V}$-category X and any $\mathscr{V}$-category with chosen Φ-colimits A, the $\mathscr{V}$-category $[X, A]$ has a canonical choice of colimits and the unit $X \to \Phi\text{-}\mathbf{Colim}([X, A], A)$ is a cotensor product.

Let A be a $\mathscr{V}$-category with chosen Φ-colimits and X be any $\mathscr{V}$-category. Given a weight $\phi : D \to \mathscr{V}$ in Φ and a functor $G : D \to [X, A]$, we want to choose a colimit $\phi * G$. What follows is the explanation of how to chose the colimits point-wise. Denote by $\hat{G} : X \to [D, A]$ and $G' : D \otimes X \to A$ the $\mathscr{V}$-functors induced by G, and define

$$\phi \diamond G = (X \xrightarrow{\hat{G}} [D, A] \xrightarrow{\phi * -} A).$$

Now we need a cylinder $\nu : \phi \to [X, A](G-, \phi \diamond G) = \int_{x \in X} A(G'(-, x), \phi \diamond G(x)) = \int_{x \in X} A(G'(-, x), \phi * G'(-, x))$. For each $x \in X$ we have a chosen colimiting cylinder $\mu_x : \phi \to A(\hat{G}(x)(-), \phi * (\hat{G}(x)))$. If follows easily that μ_x is dinatural in x and hence it induces a unique $\mathscr{V}$-natural transformation into the end, which we take as ν. We leave to the reader the rest of the verification of the fact that $X \to \Phi\text{-}\mathbf{Colim}([X, A], A)$ has the universal property of a cotensor product. □

Proposition 7.2. *The $\mathscr{W}$-functor U_s is monadic.*

Proof. At the beginning of [45, Section 6] it is shown that the functor $(U_s)_0$ creates coequalizers of $(U_s)_0$-contractible pairs of arrows. But we know that Φ-**Colim** has cotensor products, which is enough to ensure that coequalizers in $\Phi\text{-}\mathbf{Colim}_0$ are coequalizers in the $\mathscr{W}$-category Φ-**Colim**. From the enriched version of Beck's monadicity theorem [25], we deduce that U_s is monadic. □

Denote by T the $\mathscr{W}$-enriched monad on $\mathscr{W} = \mathscr{V}\text{-}\mathbf{Cat}$ whose Eilenberg-Moore construction is $U_s : \Phi\text{-}\mathbf{Colim} \to \mathscr{W}$. Theorem 6.3 of [45] asserts that the 2-monad T_1 on the 2-category $\mathscr{W}_1$ of small $\mathscr{V}$-categories, $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations is *lax-idempotent*. Henceforth, Theorem 6.35 gives the following corollary.

Corollary 7.3. *The $\mathscr{V}$-**Cat**-monad T on $\mathscr{V}$-**Cat** whose algebras are the $\mathscr{V}$-categories with chosen Φ-colimits is pseudo-commutative. The same holds if we replace colimits by limits.*

Proof. Only the case of limits needs a proof. If L is the $\mathscr{W}$-monad on $\mathscr{W} = \mathscr{V}\text{-}\mathbf{Cat}$ whose algebras are categories with chosen Φ-limits, L_1 is colax-idempotent. Now, if we write $\mathscr{U}_1$ for $\mathscr{W}_1^{\mathrm{co}}$, L_1^{co} has an obvious structure of a lax-idempotent 2-monad on $\mathscr{U}_1$. Therefore, L_1^{co} is a pseudo-commutative 2-monad on $\mathscr{U}_1$ by Theorem 6.35, and hence L is pseudo-commutative too. □

Corollary 7.4. *For T as in the corollary above, the $\mathscr{W}$-category T-*Alg*, and hence the 2-category Φ-$\mathbf{Cocts}_c$, are pseudo-closed.*

Example 7.5. As observed in [70], there are lax-idempotent 2-monads which are *not* equivalent to a 2-monad given by freely adding chosen colimits as described above. Any such 2-monad must have a fully faithful unit. The example in [70, Example 10] is the 2-monad T on **Cat** constant on the terminal category. The unit of this 2-monad is the unique possible functor $\eta_X : X \to 1$, which is not always fully faithful.

7.2 Finite limits and colimits

Let $\mathscr{V}$ be a complete and cocomplete locally finitely presented closed symmetric monoidal category, Φ be a class of finite weights (see [43]) and Φ-Colim the $\mathscr{W}$-category of $\mathscr{V}$-categories with chosen finite colimits, $\mathscr{V}$-functors strictly preserving them and $\mathscr{V}$-natural transformations.

We want to show that the forgetful 2-functor $U_s : \Phi\text{-Colim} \to \mathscr{V}\text{-}\mathbf{Cat}$ creates filtered colimits. The forgetful 2-functor $\mathscr{V}\text{-}\mathbf{Cat}_0 \to \mathscr{V}\text{-}\mathbf{Gph}_0$ into the category of $\mathscr{V}$-graphs is finitarily monadic, as shown in [46]. Colimits in $\mathscr{V}\text{-}\mathbf{Gph}_0$ have the following simple description. If $D : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}_0$ is a functor with $\mathscr{J}$ small, write $\mathscr{G}_j = D(j)$. Define $\mathrm{ob}\,\mathscr{G}$ as $\mathrm{colim}_j\,\mathrm{ob}\,\mathscr{G}_j$, with universal cone $q_j : \mathrm{ob}\,\mathscr{G}_j \to \mathrm{ob}\,\mathscr{G}$. Define $\mathscr{G}(X,Y)$ as the colimit in $\mathscr{V}$ of the functor $G : \mathscr{J} \to \mathscr{V}$ defined on objects by sending $j \in \mathscr{J}$ to $\sum_{q_j(U)=X, q_j(V)=Y} \mathscr{G}_j(U,V)$ and on arrows in the obvious way. We obtain morphisms of $\mathscr{V}$-graphs $q_j : \mathscr{G}_j \to \mathscr{G}$ forming a colimiting cone. Details, along with a more conceptual description using the bicategory $\mathscr{V}\text{-}\mathbf{Mat}$ of $\mathscr{V}$-matrices, can be found in [46].

Let $D : \mathscr{J} \to \Phi\text{-Colim}_0$ be an ordinary functor with $\mathscr{J}$ filtered. We shall also denote by D the functor $\mathscr{J} \to \mathscr{V}\text{-}\mathbf{Cat}_0$ resulting from composing with $(U_s)_0$. To abbreviate, we denote $D(j)$ by $\mathscr{C}_j$. We know that D has a colimit since the 2-category $\mathscr{V}\text{-}\mathbf{Cat}$ is cocomplete; that is, there exists a $\mathscr{V}$-category $\mathscr{C}$ and a natural transformation $q_j : \mathscr{C}_j \to \mathscr{C}$ inducing an isomorphism $\mathscr{V}\text{-}\mathbf{Cat}(\mathscr{C},\mathscr{B}) \cong \lim_j \mathscr{V}\text{-}\mathbf{Cat}(\mathscr{C}_j,\mathscr{B})$ *2-natural* in $\mathscr{B}$. Then $\mathscr{C}$ is *a fortiori* a colimit in the ordinary category $\mathscr{V}\text{-}\mathbf{Cat}_0$ and hence in $\mathscr{V}\text{-}\mathbf{Gph}_0$. As $\mathscr{J}$ is filtered, the $\mathscr{V}$-enriched homs $\mathscr{C}(X,Y)$ have a simpler description than in the general case. Pick $j_1 \in \mathscr{J}$ such that there exist $X_1, Y_1 \in \mathscr{C}_{j_1}$ with $q_{j_1}(X_1) = X$ and $q_{j_1}(Y_1) = Y$. Consider the functor $H : j_1{\downarrow}\mathscr{J} \to \mathscr{V}$ defined on objects by $H(\alpha : j_1 \to j) = \mathscr{C}_j(D\alpha(X_1), D\alpha(Y_1))$. On arrows, $H(\gamma)$ is given by the effect of $D\gamma$ on homs. If $P : j_1{\downarrow}\mathscr{J} \to \mathscr{J}$ is the projection functor, we have a natural transformation $\tau : H \Rightarrow GP$ with components

$$\tau_{(\alpha:j_1\to j)} : \mathscr{C}_j(D\alpha(X_1), D\alpha(Y_1)) \to \sum_{q_j(U)=X, q_j(V)=Y} \mathscr{C}_j(U, V)$$

the canonical arrows into the coproduct.

Lemma 7.6. *1. The natural transformation τ induces an isomorphism*

$$\operatorname{colim} H \cong \mathscr{C}(X, Y).$$

2. For any $\mathscr{V}$-functor $F : \mathscr{P} \to \mathscr{C}_{j_1}$ and $X \in \operatorname{ob}\mathscr{C}_{j_1}$ consider the functor $F_\# : j_1 \downarrow \mathscr{J} \to [\mathscr{P}^{\mathrm{op}}, \mathscr{V}]_0$ sending $\alpha : j_1 \to j$ to $\mathscr{C}_j((D\alpha)F, (D\alpha)(X))$. There exists a canonical isomorphism $\operatorname{colim} F_\# \cong \mathscr{C}(q_{j_1}F(-), q_{j_1}(X))$.

Proof. The functor $P : j_1 \downarrow \mathscr{J} \to \mathscr{J}$ is final. This gives a canonical isomorphism $\operatorname{colim} G \cong \operatorname{colim} GP$. We shall show that τ induces a bijection between the cones $\sigma_\alpha : H(\alpha : j_1 \to j) \to Z$ and cones $\rho_\alpha : GP(\alpha) = G(j) \to Z$ for any $Z \in \mathscr{V}$.

Given σ, define ρ in the following way. To give ρ_α is to give arrows $\rho_\alpha^{U,V} : \mathscr{C}_j(U, V) \to Z$ for every $U, V \in \operatorname{ob}\mathscr{C}_j$ such that $q_j(U) = X$ and $q_j(V) = Y$. Given such U, V, choose some arrow $\beta : j \to k$ in $\mathscr{J}$ such that $D\beta(U) = D(\beta\alpha)(X_1)$ and $D\beta(V) = D(\beta\alpha)(Y_1)$. Here $X_1, Y_1 \in \mathscr{C}_{j_1}$ are the objects used in the definition of H. Set

$$\rho_\alpha^{U,V} = \mathscr{C}_j(U, V) \xrightarrow{D\beta} \mathscr{C}_k(D\beta(U), D\beta(V)) = \mathscr{C}_k(D(\beta\alpha)(X_1)), D(\beta\alpha)(Y_1)) \xrightarrow{\sigma_{\beta\alpha}} Z.$$

Using the fact that $\mathscr{J}$ is filtered and the naturality of σ, it is easy to show that $\rho_\alpha^{U,V}$ does not depend on the choice of the arrow $\beta : j \to k$. Moreover, $\rho_\alpha \tau_\alpha = \sigma_\alpha$. Naturality of ρ can be easily established using the same techniques. Next we show ρ is unique. Suppose $\tilde{\rho}$ is another cone satisfying $\tilde{\rho}\tau = \sigma$. Then, for any $U, V \in \operatorname{ob}\mathscr{C}_j$, if we choose $\beta : j \to k$ as above, we have: $\tilde{\rho}_\alpha^{U,V} = \tilde{\rho}_{\beta\alpha}^{D\beta(U), D\beta(V)} D\beta = (\tilde{\rho}\tau)_{\beta\alpha} D\beta = (\rho\tau)_{\beta\alpha} D\beta = \rho_\alpha^{U,V}$. It follows that $\rho = \tilde{\rho}$. This finishes the proof of the first part of the lemma.

Consider the cone $q_\# : F_\# \to \mathscr{C}(q_{j_1}F(-), q_{j_1}(X))$ with components $(q_\#)_\alpha = q_j : \mathscr{C}_j((D\alpha)F, D(\alpha)(X)) \to \mathscr{C}(q_{j_1}F(-), q_{j_1}(X))$. Given any other cone $r : F_\# \to G$, the part (1) of the lemma gives arrows $\hat{r}_P : \mathscr{C}(q_{j_1}F(P), q_{j_1}(X)) \to G(P)$ unique with the property that $\hat{r}_P q_j = (r_\alpha)_P$ for each $P \in \operatorname{ob}\mathscr{P}$. The $\mathscr{V}$-naturality of $\hat{r}$

can be expressed as the commutativity of

$$\begin{tikzcd}
\mathscr{C}(q_{j_1}F(P), q_{j_1}(X)) \otimes \mathscr{C}(P', P) \arrow[r] \arrow[d, "\hat{r}_P \otimes 1"'] & \mathscr{C}(q_{j_1}F(P'), q_{j_1}(X)) \arrow[d, "\hat{r}_{P'}"'] \\
G(P) \otimes \mathscr{C}(P', P) \arrow[r] & G(P')
\end{tikzcd}$$

where the horizontal arrows correspond to the respective $\mathscr{V}$-functor structures. The fact that this diagram does commute follows from the analogous diagrams for each r_α and the fact that $\otimes$ preserves colimits in each variable. $\square$

For a class of finite weights Φ, denote by R the monad induced by the monadic $\mathscr{W}$-functor $U_s : \Phi\text{-}\mathbf{Colim} \to \mathscr{W}$.

Proposition 7.7. *The forgetful $\mathscr{W}$-functor $U_s : \Phi\text{-}\mathbf{Colim} \to \mathscr{V}\text{-}\mathbf{Cat}$ creates filtered colimits. Equivalently, the $\mathscr{W}$-monad R is finitary.*

Proof. Since U_s creates cotensor products, it is enough to prove that the ordinary functor $(U_s)_0 : \Phi\text{-}\mathbf{Colim}_0 \to \mathscr{W}_0$ creates filtered colimits.

Let $D : \mathscr{J} \to \Phi\text{-}\mathbf{Colim}_0$ be a functor with $\mathscr{J}$ filtered. As before, denote $D(j)$ and $\operatorname{colim} U_s D$ by $\mathscr{C}_j$ and $\mathscr{C}$ respectively, with colimiting cone $q_j : \mathscr{C}_j \to \mathscr{C}$. We have to show that the q_j form a colimiting cone in $\Phi\text{-}\mathbf{Colim}_0$.

First we show that $\mathscr{C}$ has finite colimits. Let $\phi : \mathscr{P} \to \mathscr{V}$ be a weight in Φ and $G : \mathscr{P} \to \mathscr{C}$ a $\mathscr{V}$-functor. Being a finite $\mathscr{V}$-category, $\mathscr{P}$ is finitely presented in $\mathscr{V}\text{-}\mathbf{Cat}_0$; hence, G factors through some q_j as $G = q_j G_j$ for some $G_j : \mathscr{P} \to \mathscr{C}_j$. Consider the unit of the colimit $\eta_j : \phi \to \mathscr{C}_j(G_j(-), \phi * G_j)$. We claim that

$$\eta : \phi \xrightarrow{\eta_j} \mathscr{C}_j(G_j(-), \phi * G_j) \xrightarrow{q_j} \mathscr{C}(G(-), q_j(\phi * G_j)) \tag{7.1}$$

is the unit of a colimit, or in other words, (7.1) induces a $\mathscr{V}$-natural isomorphism $\mathscr{C}(q_j(\phi * G_j), C) \cong [\mathscr{P}, \mathscr{V}](\phi, \mathscr{C}(G(-), C))$. First, observe that η does not depend on the factorisation of G. For, if $G_k : \mathscr{P} \to \mathscr{C}_k$ is another factorisation, there exist $\beta : j \to \ell, \gamma : k \to \ell$ in $\mathscr{J}$ such that $(D\beta)G_j = (D\gamma)G_k$, and therefore $q_j(\phi * G_j) = q_\ell(D\beta)(\phi * G_j) = q_\ell(\phi * (D\beta)G_j) = q_\ell(\phi * (D\gamma)G_k) = q_\ell(D\gamma)(\phi * G_k) = q_k(\phi * G_k)$ and $q_j\eta_j = q_\ell(D\beta)\eta_j = q_\ell\eta_\ell = q_\ell(D\gamma)\eta_k = q_k\eta_k$.

Given $C \in \operatorname{ob}\mathscr{C}$, we can choose j_1 such that $C = q_{j_1}(X)$ for some $X \in \operatorname{ob}\mathscr{C}_{j_1}$ and $G = q_{j_1}G_{j_1}$. Using the fact that finite weights $\phi : \mathscr{P}^{\mathrm{op}} \to \mathscr{V}$ are finitely presented objects in $[\mathscr{P}^{\mathrm{op}}, \mathscr{V}]$ and Lemma 7.6, one sees that the arrow $\mathscr{C}(q_{j_1}(\phi * G_{j_1}), C) \to [\mathscr{P}, \mathscr{V}](\phi, \mathscr{C}(G(-), C))$ is the composition below, and hence an isomorphism.

$$\begin{aligned}
\operatorname*{colim}_{\alpha:j_1\to j} \mathscr{C}_j(D\alpha(\phi * G_{j_1}), D\alpha(X)) &\cong \operatorname*{colim}_{\alpha:j_1\to j} \mathscr{C}_j(\phi * ((D\alpha)G_{j_1})), D\alpha(X)) \\
&\cong \operatorname*{colim}_{\alpha:j_1\to j} [\mathscr{P}, V](\phi, \mathscr{C}_j((D\alpha)G_{j_1}, D\alpha(X))) \\
&\cong [\mathscr{P}, \mathscr{V}](\phi, \mathscr{C}(q_{j_1}G_{j_1}, C))
\end{aligned}$$

Now we equip $\mathscr{C}$ with *chosen* finite colimits. So far we have showed that $\mathscr{C}$ has finite colimits and each $q_j : \mathscr{C}_j \to \mathscr{C}$ preserves finite colimits. For each finite weight $\phi : \mathscr{P}^{\mathrm{op}} \to \mathscr{V}$ in Φ and $G : \mathscr{P} \to \mathscr{C}$ set $\phi * G = q_j(\phi * G_j)$ where $G_j : \mathscr{P} \to \mathscr{C}_j$ is a factorisation of G through q_j. To make sense, $\phi * G$ has to be independent of the choice of j. Suppose G_k is a factorisation of G through $\mathscr{C}_k$. Since $\mathscr{J}$ is filtered and $\mathscr{P}$ is finite, there are arrows $\beta : j \to \ell$ and $\gamma : k \to \ell$ such that $(D\beta)G_j = (D\gamma)G_k$. Hence $q_j(\phi * G_j) = q_\ell(D\beta)(\phi * G_j) = q_\ell(\phi * ((D\beta)G_j)) = q_\ell(\phi * ((D\gamma)G_k)) = q_\ell((D\gamma)(\phi * G_k)) = q_k(\phi * G_k)$. With this choice of finite colimits each q_j strictly preserves colimits, so that them form a cone in Φ-$\mathbf{Colim}_0$. Now it is easy to show that q is a colimiting cone. Suppose $p_j : \mathscr{C}_j \to \mathscr{B}$ is a cone in Φ-$\mathbf{Colim}_0$ and let $F : \mathscr{C} \to \mathscr{B}$ the corresponding $\mathscr{V}$-functor. We only have to show that F preserves chosen colimits. For any finite weight $\phi : \mathscr{P}^{\mathrm{op}} \to \mathscr{V}$ in Φ and $G : \mathscr{P} \to \mathscr{C}$, we have $F(\phi * G) = Fq_j(\phi * G_j) = p_j(\phi * G_j) = \phi * (p_j G_j) = \phi * (Fq_j G_j) = \phi * (FG)$. Hence, $\mathscr{C}$ is a colimit in Φ-$\mathbf{Colim}_0$. □

Corollary 7.8. *Let Φ be a class of finite weights. The 2-categories Φ-$\mathbf{Cocts}_c$ and Φ-$\mathbf{Cts}_c$ have canonical structures of pseudo-monoidal pseudo-closed 2-categories. Furthermore, the right biadjoint forgetful 2-functors into $\mathscr{V}$-$\mathbf{Cat}$ are part of monoidal pseudo-closed biadjunctions.*

Proof. Recall that from [45] that there are isomorphisms of 2-categories making the following diagrams commute.

$$\begin{tikzcd}
R\text{-Alg} \arrow[rr, "\cong"] \arrow[dr, "U"'] & & \Phi\text{-}\mathbf{Cocts}_c \arrow[dl] \\
& \mathscr{V}\text{-}\mathbf{Cat} &
\end{tikzcd}
\qquad
\begin{tikzcd}
L\text{-Alg} \arrow[rr, "\cong"] \arrow[dr, "U"'] & & \Phi\text{-}\mathbf{Cts}_c \arrow[dl] \\
& \mathscr{V}\text{-}\mathbf{Cat} &
\end{tikzcd}$$

Here R, L are the 2-monads (and in fact $(\mathscr{V}$-$\mathbf{Cat})$-monads) on $\mathscr{V}$-$\mathbf{Cat}$ with Eilenberg-Moore constructions Φ-$\mathbf{Colim}$ and Φ-$\mathbf{Lim}$ respectively. Both R and L are finitary by Proposition 7.7, and then R-Alg_s and L-Alg_s are cocomplete by the results in [7]. Now we apply the results obtained in Section 6.6. □

Example 7.9. Let k be a commutative ring, A a k-algebra and ΣA the one-object k-Mod-category defined by A. Then, $[(\Sigma A)^{\mathrm{op}}, k\text{-Mod}]$ is the category of left A-modules. When A is finitely generated as a k-module, $\Phi\Sigma A$ is the category of finitely generated A-modules $A\text{-Mod}_f$.

Example 7.10. Suppose $\mathscr{V} = k$-Mod, the category of modules over a commutative ring k. If A and B are k-algebras of finite rank, $A\text{-Mod}_f \boxtimes B\text{-Mod}_f$ can be taken to be $A \otimes B\text{-Mod}_f$. To see this, define a $\mathscr{V}$-functor right exact in each variable $A\text{-Mod}_f \otimes B\text{-Mod}_f \to A \otimes B\text{-Mod}_f$ sending (M, N) to the $A \otimes B$-module $M \otimes N$. If $\mathscr{C}$ is a $\mathscr{V}$-category with finite colimits, a functor $F : A\text{-Mod}_f \otimes B\text{-Mod}_f \to \mathscr{C}$ right exact in each variable is the same, up to an equivalence, as an object C of $\mathscr{C}$ with an action of A and an action of B, each one commuting with the other; in other words, it is the same as a $\mathscr{V}$-functor $\Sigma(A \otimes B) \to \mathscr{C}$, or a right exact $\mathscr{V}$-functor $A \otimes B\text{-Mod}_f \to \mathscr{C}$. Therefore there is an equivalence and an isomorphism as depicted in the following diagram.

$$\begin{tikzcd}
A\text{-Mod}_f \otimes B\text{-Mod}_f \arrow[r] \arrow[dr, "\otimes"'] & A\text{-Mod}_f \boxtimes B\text{-Mod}_f \arrow[d, "\simeq"] \arrow[dl, phantom, "\cong"] \\
 & A \otimes B\text{-Mod}_f
\end{tikzcd}$$

7.3 Deligne's tensor product

Let k be a commutative ring and $\mathscr{U} = k$-Mod category of k-modules. Of course, $\mathscr{U}$ is a complete and cocomplete symmetric monoidal closed category. Moreover, $\mathscr{U}$ is locally finitely presentable; we write $\mathscr{U}_f$ for the full (monoidal) subcategory of finitely presentable objects, as usual.

If $\mathscr{A}$,$\mathscr{B}$ are abelian $\mathscr{U}$-categories, their Deligne's tensor product, introduced in [19], is a k-bilinear functor $\mathscr{A} \times \mathscr{B} \to \mathscr{A} \bullet \mathscr{B}$, where $\mathscr{A} \bullet \mathscr{B}$ is another abelian category, that induces equivalences between the category of right exact functors $\mathscr{A} \bullet \mathscr{B} \to \mathscr{C}$ and the category of functors $\mathscr{A} \times \mathscr{B} \to \mathscr{C}$ right exact in each variable, for all abelian $\mathscr{U}$-categories $\mathscr{C}$.

The property of $\mathscr{A} \bullet \mathscr{B}$ can be rewritten in the following manner. There are equivalences

$$\Phi\text{-}\mathbf{Cocts}(\mathscr{B}, [\![\mathscr{A}, \mathscr{C}]\!]) \simeq \Phi\text{-}\mathbf{Cocts}(\mathscr{A} \bullet \mathscr{B}, \mathscr{C})$$

pseudonatural in $\mathscr{C}$, where Φ is the class of weights of finite colimits, or a class whose closure is the class of finite colimits. However, as a result of the requirement that $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{A} \bullet \mathscr{B}$ be abelian, $\mathscr{A} \bullet \mathscr{B}$ need not be equivalent to the finitely cocomplete $\mathscr{A} \boxtimes \mathscr{B}$ defined by Proposition 6.31, at least *a priori*.

Although in [19] the tensor product is defined for arbitrary abelian categories, it is only shown to exist for certain special abelian categories, namely those satisfying the following conditions.

Condition 1. The ground commutative ring k is a field, all the objects have finite length and the homs are finite-dimensional.

We shall show that for this special type of abelian categories, $\mathscr{A} \boxtimes \mathscr{B}$ has the defining property of $\mathscr{A} \bullet \mathscr{B}$. In other words, for the kind of abelian categories that $\mathscr{A} \bullet \mathscr{B}$ is shown to exist in [19], $\mathscr{A} \boxtimes \mathscr{B}$ and $\mathscr{A} \bullet \mathscr{B}$ coincide. This gives evidence that the right product to consider would be $\boxtimes$.

Recall that a *sub-quotient* of an object in an abelian category is a quotient of a subobject. An *abelian* subcategory is closed under sub-quotients exactly when it is closed under subobjects, or dually, when it is closed under quotients.

Observation 7.11. If $\mathscr{D}$ is an abelian category satisfying Condition 1 above, the inclusion of a full abelian subcategory closed under sub-quotients $i : \mathscr{C} \to \mathscr{D}$ has a left adjoint i^ℓ; the left adjoint sends an object X of $\mathscr{D}$ to the greatest quotient of X lying in $\mathscr{C}$. When $\mathscr{D}$ has a projective generator P, $\mathscr{D}$ is canonically equivalent to $\mathscr{D}(P,P)$-Mod_f via $X \mapsto \mathscr{D}(P,X)$. Moreover, $i^\ell(P)$ is a projective generator in $\mathscr{C}$, so that $\mathscr{C} \simeq \mathscr{C}(i^\ell(P), i^\ell(P))$-$\mathrm{Mod}_f$. There is an isomorphism

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathscr{C}(i^\ell P, i^\ell P)\text{-}\mathrm{Mod}_f \arrow[d] \\
\mathscr{D} \arrow[r, "\simeq"'] & \mathscr{D}(P,P)\text{-}\mathrm{Mod}_f
\end{tikzcd}$$

with components $\mathscr{C}(i^\ell P, X) \cong \mathscr{D}(P, iX)$; here the functor on the right hand side is the one induced by the morphism of algebras $i^\ell_{P,P} : \mathscr{D}(P,P) \to \mathscr{C}(i^\ell P, i^\ell P)$.

We consider the base category $\mathscr{U} = k$-Mod equipped with chosen finite coproducts and coequalizers given by the usual constructions. This gives chosen finite colimits by the usual construction of colimits out of coproducts and coequalizers. For any k-algebra A, the category A-Mod_f of finitely presentable A-modules inherits a choice of finite colimits.

Observation 7.12. Given algebra k-morphisms $f : A \to A'$ and $g : B \to B'$ call f^* and g^* the functors given by restriction of scalars. If the four algebras involved are finitely presentable as k-modules and Noetherian as algebras, we can prove that $f^* \boxtimes g^*$ is an exact functor.

Consider the diagram

$$\begin{tikzcd}
A'\text{-Mod}_f \otimes B'\text{-Mod}_f \arrow[r] \arrow[d, "f^*\otimes g^*"'] & A'\text{-Mod}_f \boxtimes B'\text{-Mod}_f \arrow[r, "\simeq"] \arrow[d, "f^*\boxtimes g^*"'] \arrow[l, phantom, "\cong"] & A' \otimes B'\text{-Mod}_f \arrow[d, "(f\otimes g)^*"] \\
A\text{-Mod}_f \otimes B\text{-Mod}_f \arrow[r] & A\text{-Mod}_f \boxtimes B\text{-Mod}_f \arrow[r, "\simeq"] & A \otimes B\text{-Mod}_f
\end{tikzcd}$$

where the equivalences are the ones of Example 7.10. If A is a Noetherian algebra, the category of finitely presented A-modules A-Mod_f is not only finitely cocomplete but it is closed under kernels in A-Mod. Hence it makes sense to say that $f^* \boxtimes g^*$ is exact (as a tensor product of Noetherian algebras is again Noetherian). Since the outside rectangle commutes up to an isomorphism, we deduce that there exist an isomorphism filling in the square on the right hand side. Therefore, the exactness of $f^* \boxtimes g^*$ follows from the exactness of $(f \otimes g)^*$.

Now suppose that $\mathscr{A}$ is an abelian category satisfying Condition 1 above. Using [19, 5.12] $\mathscr{A}$ can be shown to be a filtered colimit of full abelian subcategories $\mathscr{A}_i$ closed under sub-quotients, such that each $\mathscr{A}_i$ is equivalent to category of modules of finite rank over a k-algebra of finite rank (depending on i). Following the notation of [19], denote by $\langle X \rangle$ the full subcategory closed non-empty finite direct sums and under subquotients of $\mathscr{A}$ generated by the object X. Define a filtered category $\mathscr{J}$ with $\mathrm{ob}\mathscr{J} = \mathrm{ob}\mathscr{A}$ and an arrow $X \to Y$ if an only if X is a direct summand of Y in $\mathscr{A}$, and a functor $\mathscr{J} \to \mathscr{U}$-**Cat** by sending $X \to Y$ to the inclusion $\langle X \rangle \hookrightarrow \langle Y \rangle$. Clearly, $\mathscr{A}$ is a (filtered) colimit of this functor. By [19, 2.14, 2.17], each category $\langle X \rangle$ has a projective generator P_X and there is an equivalence $\langle X \rangle \simeq \mathscr{A}(P_X, P_X)$-$\text{Mod}_f$ (see Observation 7.11).

Let R be the 2-monad on $\mathscr{U}$-**Cat** whose algebras are the $\mathscr{U}$-categories with chosen finite colimits. Suppose that the abelian category $\mathscr{A}$ in the paragraph above is equipped with chosen finite colimits. Then each subcategory $\mathscr{A}_i$ is a subobject of $\mathscr{A}$ in the category $(R\text{-Alg}_s)_0$ of categories with chosen finite colimits and functors strictly preserving them. Since R is finitary, $R\text{-Alg}_s \to \mathscr{U}$-**Cat** creates filtered colimits, and $\mathscr{A}$ is a filtered colimit of the subcategories $\mathscr{A}_i$ in $R\text{-Alg}_s$.

Theorem 7.13. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are abelian categories with chosen finite colimits and satisfying Condition 1. Then $\mathscr{A} \boxtimes \mathscr{B}$ not only has chosen finite colimits but is also abelian. Therefore, the monoidal structure $\boxtimes$ coincides on such abelian categories with the tensor product defined in [19].*

Proof. Suppose $\mathscr{A}$,$\mathscr{B}$ are $\mathscr{U}$-categories satisfying Condition 1 and with chosen finite colimits. We only need to prove that $\mathscr{A} \boxtimes \mathscr{B}$ is an abelian category.

As observed above, $\mathscr{A}$ and $\mathscr{B}$ are filtered colimit of filtered diagrams of subcategories $\mathscr{A}_i$ and $\mathscr{B}_j$ in R-Alg_s, respectively. By Proposition 6.14, $\mathscr{A}'$ is filtered colimit of the diagram $\mathscr{A}_i'$. The 2-functor $\oslash : R\text{-}\mathrm{Alg}_s \times R\text{-}\mathrm{Alg}_s \to R\text{-}\mathrm{Alg}_s$ of Corollary 6.30 preserves filtered colimits in each variable (since R does so) and hence $\mathscr{A} \boxtimes \mathscr{B} = \mathscr{A}' \oslash \mathscr{B}$ will be the filtered colimit of the $\mathscr{A}_i \boxtimes \mathscr{B} = \mathscr{A}_i' \oslash \mathscr{B}_j$ in R-Alg_s, and in $\mathscr{U}$-**Cat**.

We have seen in Observation 7.11 that each inclusion $\mathscr{A}_i \hookrightarrow \mathscr{A}_j$ is, after composing with certain equivalences and up to isomorphism, a restriction of scalars functor between categories of finite-dimensional modules, and likewise for the $\mathscr{B}_j$'s. Hence, each functor $\mathscr{A}_i \boxtimes \mathscr{B}_u \to \mathscr{A}_j \boxtimes \mathscr{B}_v$ is exact by Observation 7.12. It follows that $\mathscr{A} \boxtimes \mathscr{B}$ is abelian, since the colimit in $\mathscr{U}$-**Cat** of a filtered diagram of abelian categories and exact functors is an abelian category.

□

7.4 Tensor products from sketches

In this section we briefly explain the relationship between our definition of the tensor product of two $\mathscr{V}$-categories with chosen colimits and the work of Kelly on essentially algebraic theories.

For the purposes of this section, $\mathscr{V}$ will be a locally finitely presentable monoidal category (see [43]); however, in some parts less is needed (for example, locally bounded).

Denote by R the 2-monad on $\mathscr{V}$-**Cat** whose algebras are the $\mathscr{V}$-categories with chosen finite colimits. Then R-Alg is (isomorphic to) the 2-category of $\mathscr{V}$-categories with chosen finite colimits, finitely cocontinuous $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations. In this section we explain the relationship between the monoidal structure on the 2-category R-Alg defined in the previous sections and work by Kelly [42, 43].

For a $\mathscr{V}$-category $\mathscr{X}$, denote by $\Phi(\mathscr{X})$ the closure under finite colimits in $[\mathscr{X}^{\mathrm{op}}, \mathscr{V}]$ of the representables. One could consider any class of colimits here, but the finite ones will suffice for our purposes. Denote by $\mathsf{y} : \mathscr{X} \to \Phi(\mathscr{X})$ the Yoneda embedding. It has a universal property: composition with y induces equivalences from the $\mathscr{V}$-category of finitely cocontinuous $\mathscr{V}$-functors $\mathbf{FinCocts}[\Phi(\mathscr{X}), \mathscr{A}]$ to $[\mathscr{X}, \mathscr{A}]$, for any finitely cocomplete $\mathscr{V}$-category $\mathscr{A}$. By construction of R [45], there exits an equivalence making the following diagram commutative.

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\eta"] \arrow[dr, "\mathsf{y}"'] & R(\mathscr{X}) \arrow[d, "\simeq"] \\ & \Phi(\mathscr{X}) \end{tikzcd}$$

In particular, both y and η are dense.

The $\mathscr{V}$-category $[\mathscr{X}^{\mathrm{op}}, \mathscr{V}]$ is locally finitely presentable, and $\Phi(\mathscr{X})$ is the full subcategory of finitely presented objects.

A $\mathscr{V}$-*sketch* is a pair $(\mathscr{A}, \Psi)$ where $\mathscr{A}$ is a $\mathscr{V}$-category and $\Psi = \{\Psi_\gamma : \psi_\gamma \Rightarrow \mathscr{A}(A_\gamma, S_\gamma(-))\}_{\gamma\in\Gamma}$ is a set of $\mathscr{V}$-natural transformations, where $\psi_\gamma : \mathscr{D}_\gamma \to \mathscr{V}$ and $S : \mathscr{D}_\gamma \to \mathscr{A}$. A *model* for the sketch $(\mathscr{A}, \Psi)$, or a Ψ-model, is a $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ such that the transformations $\psi_\gamma \Rightarrow \mathscr{A}(A_\gamma, S_\gamma(-)) \Rightarrow \mathscr{B}(F(A_\gamma), FS_\gamma(-))$ are the units of a weighted limit $\{\psi_\gamma, FS_\gamma(-)\}$. We denote by Ψ-Mod$[\mathscr{A}, \mathscr{B}]$ the full subcategory of $[\mathscr{A}, \mathscr{B}]$ determined by the Ψ-models. Then, a Ψ-model is a functor that sends each Ψ_γ to a limit. We denote Ψ-Mod$[\mathscr{A}, \mathscr{V}]$ by Ψ-Alg, and call its objects Ψ-algebras.

Given a sketch $(\mathscr{A}^{\mathrm{op}}, \Psi)$, define the $\mathscr{V}$-category

$$\Psi\text{-Com}[\mathscr{A}, \mathscr{B}] = \Psi\text{-Mod}[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}].$$

The objects of this $\mathscr{V}$-category are called Ψ-*comodels*. A sketch $(\mathscr{A}^{\mathrm{op}}, \Psi)$ can be identified with a set of transformations $\{\phi_\gamma \Rightarrow \mathscr{A}(G_\gamma(-), A_\gamma)\}_{\gamma\in\Gamma}$ where $\phi_\gamma : \mathscr{D}_\gamma^{\mathrm{op}} \to \mathscr{V}$ and $G : \mathscr{D}_\gamma \to \mathscr{A}$. A Ψ-comodel is a $\mathscr{V}$-functor that sends each one of these transformations to a colimit.

It is shown in [42, 6.3] that for a sketch $(\mathscr{A}^{\mathrm{op}}, \Psi)$, Ψ-Alg is a reflective subcategory of $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$. In particular, it is cocomplete. Denote by $K : \mathscr{A} \to \Psi$-Alg the composition of the Yoneda embedding with the reflection. It is easy to see that K is dense. Moreover, K is a Ψ-comodel. To see this, note that a functor $G : \mathscr{A} \to \mathscr{B}$ is a Ψ-comodel if and only if G is a K-comodel in the sense of [42, 5.12] by [42, Theorem 6.11] and the last paragraph of [42, 6.2].

Now suppose Φ is a set of weights and that all the elements ψ_γ of the sketch Ψ belong to Φ. Denote by $\mathscr{K}$ the closure under Φ-colimits of the image of K in Ψ-Alg, and $Z : \mathscr{A} \to \mathscr{K}$ corresponding factorisation of K.

Theorem 7.14 ([42, Theorem 6.23])**.** *Composition with $Z : \mathscr{A} \to \mathscr{K}$ induces equivalences Φ-Cocts$[\mathscr{K}, \mathscr{B}] \simeq \Psi$-Com$[\mathscr{A}, \mathscr{B}]$. The inverse equivalence is given* *by taking left Kan extension along Z.*

Given two $\mathscr{V}$-sketches $(\mathscr{A}^{\mathrm{op}}, \Psi_{\mathscr{A}})$ and $(\mathscr{B}^{\mathrm{op}}, \Psi_{\mathscr{B}})$, one can define a new $\mathscr{V}$-sketch $(\mathscr{A} \otimes \mathscr{B}, \Psi_{\mathscr{A}} \otimes_{\Phi} \Psi_{\mathscr{B}})$ consisting of the $\mathscr{V}$-natural transformations

$$\phi \xrightarrow{\tau} \mathscr{A}(G(-), A) \xrightarrow{1 \otimes 1_B} \mathscr{A}(G(-), A) \otimes \mathscr{B}(B, B) = \mathscr{A} \otimes \mathscr{B}((G(-), B), (A, B))$$

$$\phi \xrightarrow{\sigma} \mathscr{B}(H(-), B) \xrightarrow{1_A \otimes 1} \mathscr{A}(A, A) \otimes \mathscr{B}(H(-), B) = \mathscr{A} \otimes \mathscr{B}((A, H(-)), (A, B))$$

where $\tau \in \Psi_{\mathscr{A}}$, $\sigma \in \Psi_{\mathscr{B}}$, A is an object of $\mathscr{A}$ and B an object of $\mathscr{B}$.

Consider the example when Φ is the class of all finite weights, or a class whose closure is the class of finite weights (for example, Φ can consists of the weights for coequalizers, binary coproducts, initial object and tensor products with objects in a small strong generator of $\mathscr{V}$ included in $\mathscr{V}_f$). Take two finitely cocomplete $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$ and $\mathscr{V}$-sketches $\Psi_{\mathscr{A}}$ and $\Psi_{\mathscr{B}}$ consisting, respectively, of the units of the colimits in $\mathscr{A}$ and $\mathscr{B}$ with weight in Φ. Then, $\Psi_{\mathscr{A}} \otimes_{\Phi} \Psi_{\mathscr{B}}$-Com$[\mathscr{A} \otimes \mathscr{B}, \mathscr{C}]$ is the $\mathscr{V}$-category of $\mathscr{V}$-functors $\mathscr{A} \otimes \mathscr{B} \to \mathscr{V}$, finitely cocontinuous in each variable. The $\mathscr{V}$-category $\Psi_{\mathscr{A}} \otimes_{\Phi} \Psi_{\mathscr{B}}$-Alg is Lex$[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}; \mathscr{V}]$, the $\mathscr{V}$-category of $\mathscr{V}$-functors $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}^{\mathrm{op}} \to \mathscr{V}$ finitely continuous in each variable. The dense $\mathscr{V}$-functor $Z : \mathscr{A} \otimes \mathscr{B} \to \mathscr{K}$ in this case has the universal property of the tensor product corresponding to the pseudo-closed structure on R-Alg.

Suppose that the base monoidal category $\mathscr{V}$ is equipped with chosen finite colimits. We choose finite colimits in $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}^{\mathrm{op}}, \mathscr{V}]$ point-wise, and use the reflection $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}^{\mathrm{op}}, \mathscr{V}] \to$ Lex$[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}; \mathscr{V}]$ to equip the codomain with chosen finite colimits. We equip the $\mathscr{V}$-category $\mathscr{K}$ of Theorem 7.14 with the chosen colimits corresponding to the ones of Lex$[\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}; \mathscr{V}]$. Hence, if $\mathscr{A}$ and $\mathscr{B}$ have chosen finite colimits, we have $\mathscr{K} \simeq \mathscr{A} \boxtimes \mathscr{B}$.

7.5 Radford's formula for finitely complete autonomous categories

Throughout this section we will denote the category of vector spaces by $\mathscr{V}$, and the category of finite-dimensional vector spaces by $\mathscr{V}_f$. Let L be the 2-monad on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen finite limits, and denote by $\boxtimes$ the tensor product corresponding to its pseudo-closed structure as in Section 6.6 (see Theorem 6.35). The neutral object for this pseudo-monoidal structure is the free $\mathscr{V}$-category with chosen finite limits $L(I)$ over the unit $\mathscr{V}$-category I. We shall identify $F(I)$ with $\mathscr{V}_f^{\mathrm{op}}$ via the canonical equivalence $L(I) \to \mathscr{V}_f^{\mathrm{op}}$ that makes the following triangle commute (see the beginning of Section 7.1).

$$\begin{tikzcd} I \arrow[r, "\eta_I"] \arrow[dr, "\mathsf{y}"'] & L(I) \arrow[d, "\simeq"] \\ & \mathscr{V}_f^{\mathrm{op}} \end{tikzcd}$$

The unit constraint $\mathscr{V}_f^{\mathrm{op}} \boxtimes \mathscr{C} \to \mathscr{C}$ corresponds to the cotensor product $\mathscr{V}$-functor $\{-,-\} : \mathscr{V}_f^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{C}$.

If X is an object of $\mathscr{C}$ and Y an object of $\mathscr{D}$, we will denote the image of the object (X, Y) under the universal multimap $\mathscr{C} \otimes \mathscr{D} \to \mathscr{C} \boxtimes \mathscr{D}$, the canonical $\mathscr{V}$-functor left exact in each variable, by $X \boxtimes Y$.

We denote the dual of a finite-dimensional vector space W by $W^\vee$.

A pseudomonoid in L-Alg is the same as a monoidal $\mathscr{V}$-category $\mathscr{C}$ which is finitely complete and whose tensor product is left exact in each variable. We will denote the multiplication and unit by $P : \mathscr{C} \boxtimes \mathscr{C} \to \mathscr{C}$ and $J : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{C}$, respectively, and by $\mathbf{1}$ the object of $\mathscr{C}$ defined by J.

From now on, we will consider only $\mathscr{V}$-categories with homs lying in the full subcategory of finite-dimensional vector spaces $\mathscr{V}_f$.

Lemma 7.15. *Any 1-cell $F : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{C}$ in L-Alg has a right adjoint given by*

$$\mathscr{C} \xrightarrow{\mathscr{C}(Fk,-)} \mathscr{V}_f \xrightarrow{(-)^\vee} \mathscr{V}_f^{\mathrm{op}}. \tag{7.2}$$

In particular, J has right adjoint.

Proof. The isomorphism $\mathscr{C}(Fk, X) \cong \mathscr{V}_f(\mathscr{C}(Fk, X)^\vee, k) \cong \mathscr{V}_f^{\mathrm{op}}(k, \mathscr{C}(Fk, X)^\vee)$ exhibits the left exact (7.2) as right adjoint to F. □

Example 7.16. Let k be a perfect field (*e.g.* a field of characteristic zero), $\mathscr{A}$ be a category equivalent to A-Mod$_f$, the category of finite-dimensional modules over a finite-dimensional k-algebra A, and suppose $\mathscr{A}$ has the structure of an autonomous monoidal category. We shall show that any such category is an autonomous map pseudomonoid in L-Alg, so that the theory developed in previous chapters apply. In particular, our results apply to the *finite tensor categories* considered in [27].

We only have to prove that the multiplication $P : \mathscr{A} \boxtimes \mathscr{A} \to \mathscr{A}$ has right adjoint. Via the left dual functor, $\mathscr{A}$ is equivalent to $\mathscr{A}^{\mathrm{op}}$. Hence, by a dualised instance of Example 7.10, the tensor product functor $A\text{-Mod}_f^{\mathrm{op}} \otimes A\text{-Mod}_f^{\mathrm{op}} \to$

$A \otimes A\text{-Mod}_f^{\mathrm{op}}$ provides us with a choice of $\mathscr{A} \boxtimes \mathscr{A}$. Now, the monoidal product of $\mathscr{A}$ corresponds to a functor $A\text{-Mod}_f^{\mathrm{op}} \otimes A\text{-Mod}_f^{\mathrm{op}} \to A\text{-Mod}_f^{\mathrm{op}}$, exact in each variable; by [19, 5.7], the induced functor $A \otimes A\text{-Mod}_f^{\mathrm{op}} \to A\text{-Mod}_f^{\mathrm{op}}$ is exact. It follows that the multiplication $P : \mathscr{A} \boxtimes \mathscr{A} \to \mathscr{A}$ is not only left exact but also right exact. Therefore P has a right adjoint, because both domain and codomain of P are equivalent to categories of finite-dimensional modules over a finite-dimensional algebra, as we explain below.

Any right exact functor $G : A\text{-Mod}_f \to B\text{-Mod}_f$ is of the form $(M \otimes_A -)$ for a (unique up to isomorphism) left B right A finite-dimensional bimodule M. Hence, the functor $\mathrm{Hom}_B(M, -)$ is a right adjoint to G.

Lemma 7.17. *If $\mathscr{C}$ and $\mathscr{D}$ are pseudomonoids in L-Alg, then $\mathscr{C} \boxtimes \mathscr{D}$ so is.*

Proof. The pseudomonoidal structure is induced by the monoidal structure of $\mathscr{C} \otimes \mathscr{D}$. $\square$

Note that in the proof above we are not allowed to say that that the pseudofunctor $(- \boxtimes -)$ is monoidal and hence preserves pseudomonoids. This is because, although there are equivalences $\mathscr{C} \boxtimes \mathscr{D} \simeq \mathscr{D} \boxtimes \mathscr{C}$ induced by the isomorphisms $\mathscr{C} \otimes \mathscr{D} \cong \mathscr{D} \otimes \mathscr{C}$, we can not say that the former provide a braiding, making L-Alg a braided monoidal 2-category, as we lack an established definition of braiding for general monoidal 2-categories.

Lemma 7.18. *If $X \in \mathscr{C}$ and $Y \in \mathscr{D}$ have left duals ${}^*X$ and ${}^*Y$, then $X \boxtimes Y \in \mathscr{C} \boxtimes \mathscr{D}$ has left dual ${}^*X \boxtimes {}^*Y$.*

Proof. We have to show that $({}^*X \boxtimes {}^*Y) \otimes -$ is left adjoint to $(X \boxtimes Y) \otimes -$. The former functor is the image under $\boxtimes : L\text{-Alg} \times L\text{-Alg} \to L\text{-Alg}$ of the 1-cell $(({}^*X \otimes -), ({}^*Y \otimes -))$ while the latter is the image of $((X \otimes -), (Y \otimes -))$. Since pseudo-functors preserve adjunctions, we get our result. $\square$

Now we fix a map pseudomonoid $\mathscr{C}$ in L-Alg; this means that $\mathscr{C}$ is a finitely complete monoidal category whose (left exact) multiplication functor $P : \mathscr{C} \boxtimes \mathscr{C} \to \mathscr{C}$ has right adjoint P^* (the unit J has right adjoint by Lemma 7.15). We shall suppose that $\mathscr{C}$ is autonomous, and denote by $\bar{D} : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ the functor given by taking right dual; its pseudo-inverse $\bar{D}^* : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ is given by taking left dual.

For any object Y of $\mathscr{C}$ we shall write $\mathsf{L}_Y : \mathscr{C} \to \mathscr{C}$ for the left exact functor given by $\mathsf{L}_Y(X) = Y \otimes X$. This functor has left adjoint L_Y^{ℓ} isomorphic to $\mathsf{L}_{{}^*Y}$. Observe that there exist canonical isomorphisms : $\varpi : P(\mathsf{L}_Y \boxtimes \mathscr{C}) \cong \mathsf{L}_Y P$ whose components $\varpi_{X\boxtimes Z} : (Y \otimes X) \otimes Z \to Y \otimes (X \otimes Z)$ are just the associativity constraint of the tensor product. Analogously, there is a left exact functor $\mathsf{R}_Y : \mathscr{C} \to \mathscr{C}$ given by tensoring with Y on the right, and $\mathsf{R}_{Y^*} \dashv \mathsf{R}_Y$.

Lemma 7.19. *The 2-cells below*

$$\begin{tikzcd}
& \mathscr{C}^3 \arrow[rr, "P\boxtimes 1"] \arrow[dr, "1\boxtimes P"] \arrow[d, phantom, "\Downarrow"] & & \mathscr{C}^2 \arrow[rr, equal] \arrow[dr, "P"] \arrow[r, phantom, "\Downarrow"] & & \mathscr{C}^2 \\
\mathscr{C}^2 \arrow[ur, "1\boxtimes P^*"] \arrow[rr, equal] & & \mathscr{C}^2 \arrow[rr, "P"] \arrow[u, phantom, "\cong"] & & \mathscr{C} \arrow[ur, "P^*"'] &
\end{tikzcd} \tag{7.3}$$

and

$$\begin{tikzcd}
& \mathscr{C}^3 \arrow[rr, "1\boxtimes P"] \arrow[dr, "P\boxtimes 1"] \arrow[d, phantom, "\Downarrow"] & & \mathscr{C}^2 \arrow[rr, equal] \arrow[dr, "P"] \arrow[r, phantom, "\Downarrow"] & & \mathscr{C}^2 \\
\mathscr{C}^2 \arrow[ur, "P^*\boxtimes 1"] \arrow[rr, equal] & & \mathscr{C}^2 \arrow[rr, "P"] \arrow[u, phantom, "\cong"] & & \mathscr{C} \arrow[ur, "P^*"'] &
\end{tikzcd} \tag{7.4}$$

are given, on an object of the form $Y \boxtimes Z$, respectively by

$$\mathscr{C}^2(X, (P\boxtimes\mathscr{C})(\mathscr{C}\boxtimes P^*(Z))Y) = \mathscr{C}^2(X, (\mathsf{L}_Y\boxtimes\mathscr{C})(P^*Z)) \cong \mathscr{C}^2((\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X, P^*Z)$$
$$\cong \mathscr{C}(P(\mathsf{L}^\ell_Y \boxtimes \mathscr{C})X, Z) \cong \mathscr{C}(\mathsf{L}^\ell_Y PX, Z) \cong \mathscr{C}(PX, Y \otimes Z) \cong \mathscr{C}^2(X, P^*(Y \otimes Z)$$

and

$$\mathscr{C}^2(X, (\mathscr{C} \boxtimes P)(P^*(Y)\boxtimes\mathscr{C})Z) = \mathscr{C}^2(X, (\mathscr{C} \boxtimes \mathsf{R}_Z)P^*Y) \cong \mathscr{C}^2((\mathscr{C} \boxtimes \mathsf{R}^\ell_Z)X, P^*Y)$$
$$\cong \mathscr{C}(P(\mathscr{C} \boxtimes \mathsf{R}^\ell_Z)X, Y) \cong \mathscr{C}(\mathsf{R}^\ell_Z PX, Y) \cong \mathscr{C}(PX, Y \otimes Z) \cong \mathscr{C}^2(X, P^*(Y \otimes Z).$$

In particular, (7.3) *and* (7.4) *are invertible.*

Proof. First observe that the result for (7.4) follows from the one for (7.3) by considering the reverse monoidal category. Therefore it suffices to prove that the outer rectangle in the diagram (7.5) in Figure 7.1 commutes; and this happens when the diagram marked as (A) commutes. In fact, we can show that (A) commutes when we substitute P^*Z by any object of $\mathscr{C} \boxtimes \mathscr{C}$. In order to do this consider the diagram (7.6), where we changed P^*Z by $U \boxtimes V$; this suffices since, for fixed X, all the functors $\mathscr{C} \boxtimes \mathscr{C} \to \mathscr{V}_f$ involved are left exact. Hence, it is enough to prove the commutativity of (B). Now, the rectangle in (7.7) commutes by naturality of ϖ, and then it follows that (B) commutes too.

□

As a direct consequence of Lemma 2.43 Observation 2.45, we get the following.

Corollary 7.20. *For any autonomous monoidal $\mathscr{V}$-category $\mathscr{C}$ with homs in $\mathscr{V}_f$ and multiplication $P : \mathscr{C} \boxtimes \mathscr{C} \to \mathscr{C}$ with right adjoint,*

1. *The left exact functors $J^*P : \mathscr{C} \boxtimes \mathscr{C} \to \mathscr{V}_f^{\mathrm{op}}$ and $P^*J : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{C} \boxtimes \mathscr{C}$ form a bidual pair. In particular, $\mathscr{C}$ is a Frobenius pseudomonoid in L-*Alg*.*

2. *The left dual functor $D : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ is a right dualization with respect to the bidual pair*

$$E = (\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}} \xrightarrow{1 \boxtimes D} \mathscr{C} \boxtimes \mathscr{C} \xrightarrow{P} \mathscr{C} \xrightarrow{J^*} \mathscr{V}_f^{\mathrm{op}})$$
$$N : (\mathscr{V}_f^{\mathrm{op}} \xrightarrow{J} \mathscr{C} \xrightarrow{P^*} \mathscr{C} \boxtimes \mathscr{C} \xrightarrow{D^* \boxtimes 1} \mathscr{C}^{\mathrm{op}} \boxtimes \mathscr{C}).$$

3. *The right dual functor $\bar{D} : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ is a right dualization with respect to the bidual pair*

$$\bar{E} = (\mathscr{C}^{\mathrm{op}} \boxtimes \mathscr{C} \xrightarrow{\bar{D} \boxtimes 1} \mathscr{C} \boxtimes \mathscr{C} \xrightarrow{P} \mathscr{C} \xrightarrow{J^*} \mathscr{V}_f^{\mathrm{op}})$$
$$\bar{N} : (\mathscr{V}_f^{\mathrm{op}} \xrightarrow{J^*} \mathscr{C} \xrightarrow{P^*} \mathscr{C} \boxtimes \mathscr{C} \xrightarrow{1 \boxtimes \bar{D}^*} \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}).$$

We are now ready to deduce, for any autonomous monoidal k-linear category with finite-dimensional homs and left adjoint multiplication $P : \mathscr{C} \boxtimes \mathscr{C} \to C$, the main results in [27].

Let $N : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ be the coevaluation given in the corollary above, and denote by $H \in \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ the corresponding object. By Lemma 7.15, N has a right adjoint, and therefore by Section 4.2 there exists a unique up to isomorphism left exact $\mathscr{V}$-functor $\mathsf{W} : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{C}$ such that $(P \boxtimes \mathscr{C}^{\mathrm{op}})(\mathsf{W} \boxtimes H) \cong (W \boxtimes \mathbf{1}) \otimes H$ is isomorphic as an H-module to the left dual of H. By Lemma 7.15, W has right adjoint.

Proposition 7.21. *The left exact functor $\mathsf{W} : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{C}$ is invertible in the monoidal category L-*Alg*$(\mathscr{V}_f^{\mathrm{op}}, \mathscr{C})$; equivalently, the object W of $\mathscr{C}$ defined by W is invertible.*

Proof. The functor W has a right adjoint by 7.15. Then, a result dual to Proposition 4.12 shows W is invertible in L-Alg$(\mathscr{V}_f^{\mathrm{op}}, \mathscr{C})$. In other words, the object $\mathsf{W} \in \mathscr{C}$ is an invertible object. □

Observation 7.22. If $\mathscr{C}$ is a finite tensor category in the sense of [27] (and hence a map pseudomonoid in L-Alg by Example 7.16), the object W is called a *distinguished invertible object* of $\mathscr{C}$ (see [27, Definition 3.1]). The proposition above shows that the invertibility of W follows from abstract considerations and it is independent of the Frobenius-Perron dimension argument used by the authors of that paper.

Theorem 7.23. *There exists a monoidal isomorphism between endo-functors of* $V(\mathscr{C})$

$$^{*}W \otimes X^{**} \otimes W \cong {}^{**}X.$$

Proof. The coevaluation $N : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}} \boxtimes \mathscr{C}$ has a right adjoint by Lemma 7.15, and hence $1_{\mathscr{C}}$ has a left and a right dual in L-$\mathrm{Alg}(\mathscr{C}, \mathscr{C})$, and we already saw that W has a right adjoint. Therefore we can apply Theorem 4.15 to the autonomous pseudomonoid $(\mathscr{C}, J^*, P^*)$ in L-$\mathrm{Alg}^{\mathrm{op}}$, and, since left (respectively right) duals in L-$\mathrm{Alg}^{\mathrm{op}}(\mathscr{C}, \mathscr{V}_f^{\mathrm{op}})$ are the same as left (respectively right) duals in L-$\mathrm{Alg}(\mathscr{V}_f^{\mathrm{op}}, \mathscr{C})$, we obtain the result. $\square$

We include below some results that may shed some light on how the general theory applies to L-Alg.

Lemma 7.24. *Let $\mathscr{C}$ and $\mathscr{D}$ be autonomous $\mathscr{V}$-categories which are map pseudomonoids in* L-Alg *equipped with the right and left bidual in part 2 and 3 of Corollary 7.20. Then any left exact $F : \mathscr{C} \to \mathscr{D}$ has a left adjoint F^{ℓ} and the right and left bidual $F^{\circ}, F^{\vee} : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$ are given by $F^{\circ} = F^{\vee} \cong F^{\ell\mathrm{op}} \cong F^{\mathrm{op}*}$. In particular, F does not only preserve finite limits but all the limits that may exist in $\mathscr{C}$.*

Proof. The bidual of F is determined by the existence of an isomorphism $\bar{E}(\mathscr{D}^{\mathrm{op}} \boxtimes F) \cong \bar{E}(F^{\vee} \boxtimes \mathscr{C})$, or, evaluating on $X \boxtimes Y \in \mathscr{D}^{\mathrm{op}} \boxtimes \mathscr{C}$, $\mathscr{D}(X, FY)^{\vee} \cong \mathscr{C}(F^{\vee}X, Y)^{\vee}$. If follows that $F^{\vee\mathrm{op}}$ is a left adjoint for F. The proof with the right bidual is analogous. $\square$

Observation 7.25. 1. We shall describe the pseudomonoid structure of the bidual $\mathscr{C}^{\mathrm{op}}$ of $\mathscr{C}$. By Lemma 7.24, the unit object of $\mathscr{C}^{\mathrm{op}}$ is the same as the one of $\mathscr{C}$. The multiplication is $(P^*)^{\vee}$; because of the definition of the evaluation and coevaluation of the bidual pair in Corollary 7.20.3, and by using repetitively Lemma 7.19, this functor is isomorphic to $\bar{D}^* P(\bar{D} \boxtimes \bar{D})$. Hence, the multiplication of $\mathscr{C}^{\mathrm{op}}$ is isomorphic to

$$\mathscr{C}^{\mathrm{op}} \boxtimes \mathscr{C}^{\mathrm{op}} \xrightarrow{\mathrm{sw}} \mathscr{C}^{\mathrm{op}} \boxtimes \mathscr{C}^{\mathrm{op}} \xrightarrow{Q} \mathscr{C}^{\mathrm{op}}$$

where Q denotes the left exact functor induced by the opposite of the tensor product of $\mathscr{C}$, $\otimes^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$, and sw is the canonical equivalence $\mathrm{sw} : \mathscr{A} \boxtimes \mathscr{B} \simeq \mathscr{B} \boxtimes \mathscr{A}$.

2. Now we turn our attention to the (convolution) pseudomonoid structure on $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$. This structure was described in Lemma 4.1, and has unit object $\mathbf{1} \boxtimes \mathbf{1}$ and multiplication $(P \boxtimes ((P^*)^\vee \mathrm{sw}))(\mathscr{C} \boxtimes \mathrm{sw} \boxtimes \mathscr{C}^{\mathrm{op}})$. Then, the multiplication is just the composition $(P \boxtimes Q)(\mathscr{C} \boxtimes \mathrm{sw} \boxtimes \mathscr{C}^{\mathrm{op}})$; it sends objects of the form $X \boxtimes Y \boxtimes Z \boxtimes U$ in $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}} \boxtimes \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ to $(X \otimes Z) \boxtimes (Y \otimes U)$.

Observation 7.26. The convolution hom-category $L\text{-Alg}(\mathscr{V}_f^{\mathrm{op}}, \mathscr{D})$ is monoidally equivalent to $\mathscr{V}\text{-}\mathbf{Cat}(I, \mathscr{D})$, which is the underlying ordinary category of $\mathscr{D}$. Therefore, if $\mathscr{D}$ is autonomous, $L\text{-Alg}(\mathscr{V}_f^{\mathrm{op}}, \mathscr{D})$ is autonomous. Explicitly, if $F : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{D}$ is left exact, its left and right duals in $L\text{-Alg}(\mathscr{V}_f^{\mathrm{op}}, \mathscr{D})$ are the left exact $\mathscr{V}$-functors determined by the objects $\bar{D}^*(Fk)$ and $\bar{D}(Fk)$ respectively.

A number of the constructions we have done for a general pseudomonoid in Chapter 2 translate under the equivalence

$$L\text{-Alg}(\mathscr{C}, \mathscr{C}) \simeq L\text{-Alg}(\mathscr{V}_f^{\mathrm{op}}, \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}) \simeq \mathscr{V}\text{-}\mathbf{Cat}(I, \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}) = V(\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}) \tag{7.8}$$

into constructions on the ordinary category $V(\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}})$. This equivalence is monoidal with respect to two different monoidal structures. On $L\text{-Alg}(\mathscr{C}, \mathscr{C})$ we have the convolution monoidal structure and the composition monoidal structure. The former corresponds to the pseudomonoid structure on $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ described in Observation 7.25; the latter corresponds to the pseudomonoidal structure on $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ given by multiplication and unit

$$\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}} \boxtimes \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}} \xrightarrow{1 \boxtimes \bar{E} \boxtimes 1} \mathscr{C} \boxtimes \mathscr{V}_f^{\mathrm{op}} \boxtimes \mathscr{C}^{\mathrm{op}} \simeq \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}} \qquad H := \bar{N}(k) \in \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}. \tag{7.9}$$

Explicitly, the multiplication sends an object $X \boxtimes Y \boxtimes Z \boxtimes U$ to $\{\mathscr{C}(Y, Z)^\vee, X \boxtimes U\} \cong (X \boxtimes U)^{\dim \mathscr{C}(Y,Z)}$. The fact that the identity 1-cell is a monoid in the convolution category $L\text{-Alg}(\mathscr{C}, \mathscr{C})$ translates into the fact that the coevaluation $\bar{N}$ is a monoid in $L\text{-Alg}(\mathscr{V}_f^{op}, \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}})$, where $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ has the product described in Observation 7.25, or equivalently, that the object H determined by $\bar{N}$ is a monoid in the underlying ordinary category $V(\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}})$, and hence in $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$.

Definition 7.1. We denote the Hopf module construction for the pseudomonoid $\mathscr{C}$ in $L\text{-Alg}^{\mathrm{rev}}$ by $\mathscr{H} \to \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$. The finitely complete category $\mathscr{H}$ is called the *category of Hopf modules* of $\mathscr{C}$.

Observation 7.27. When we restrict ourselves to the finite tensor categories of [27], the category $\mathscr{H}$ defined above is what is called the *category of Hopf bimodules*. (Note that in that paper the authors use the symbol $\mathscr{C}^{\mathrm{op}}$ for what we would denote by $\mathscr{C}^{\mathrm{rev}}$: $\mathscr{C}$ equipped with the reverse tensor product.) Hence, our notion of category of Hopf modules generalises the one of [27].

The following result is a generalisation of [27, Prop. 2.3 (a)].

Proposition 7.28. *The functor $\mathscr{C} \to \mathscr{H}$ given by $X \mapsto (X \boxtimes \mathbf{1}) \otimes H$ is a monoidal equivalence, where the latter category has the monoidal structure induced by* (7.9)*.*

Proof. The functor in question is equivalent to the functor sending $X : I \to \mathscr{C}$ to $(P \boxtimes \mathscr{C}^{\mathrm{op}})(X \boxtimes H)$, which is induced by composition with the monoidal equivalence $(P \boxtimes \mathscr{C}^{\mathrm{op}})(\mathscr{C} \boxtimes N)$ (see Theorem 2.35). Hence, $X \mapsto (X \boxtimes \mathbf{1}) \otimes H$ is a monoidal equivalence as claimed, with monoidal structure induced by transport of structure. □

7.6 Semisimple categories and completion under biproducts

In this section we express the semisimplicity of a category enriched in vector spaces in terms of two 2-monads: the 2-monad D whose algebras are the categories with chosen biproducts and L whose algebras are the categories with chosen finite limits. Then we relate the semisimplicity of an autonomous monoidal category with finite dimensional homs with a purely bicategorical property; namely, the existence of a right adjoint to the right adjoint to the unit. See Theorem 7.39.

In this section $\mathscr{V}$ will still denote the category of vector spaces over a field k.

Let $\mathsf{Fin}\text{-}\mathbf{BP}$ be the 2-category of $\mathscr{V}$-categories with chosen biproducts, $\mathscr{V}$-functors preserving biproducts up to isomorphism, and $\mathscr{V}$-transformations. We will also call biproducts *direct sums*. Of course, any $\mathscr{V}$-functor preserves biproducts; in other words, the forgetful 2-functor $\mathsf{Fin}\text{-}\mathbf{BP} \to \mathscr{V}\text{-}\mathbf{Cat}$ is locally an isomorphism of categories. The 2-category $\mathsf{Fin}\text{-}\mathbf{BP}$ is isomorphic to D-Alg for certain 2-monad D on $\mathscr{V}\text{-}\mathbf{Cat}$ that we describe below.

If $\mathscr{X}$ is a $\mathscr{V}$-category, $D(\mathscr{X})$ has as objects finite sequences $(x_1, x_2, \cdots, x_n)$ of objects of $\mathscr{X}$. The elements of the $\mathscr{V}$-enriched hom

$$D(\mathscr{X})((x_1, \cdots, x_m), (y_1, \cdots, y_n))$$

are matrices $(f_{p,q})$, $1 \leq p \leq n$, $1 \leq q \leq m$, where $f_{p,q} \in \mathscr{X}(x_q, y_p)$. Composition is given by product of matrices, while identities are given by the corresponding identities matrices. Multiplication $D^2(\mathscr{X}) \to D(\mathscr{X})$ is given by deleting brackets, and the unit $\mathscr{X} \to D(\mathscr{X})$ by adding brackets.

We now examine the relationship between the 2-monad D and semisimple categories. First we fix some terminology. A *subobject* of an object X in $\mathscr{X}$ is a monic $S \rightarrowtail X$. Subobjects of a fixed object form a category $\mathbf{Sub}(X)$ with the obvious arrows. We call any subobject isomorphic to $1_X : X \to X$ or $0 : 1 \to X$ *trivial;* note that the zero subobject is defined only when $\mathscr{X}$ has a final object. We say that X has *finite length* if the length of the chains of subobjects of X is bounded; this is, if there exists an $n \geq 0$ such that, if there is an m-tuple of composable arrows in $\mathbf{Sub}(X)$, none of which is an isomorphism, then $m \leq n$. An object X is *simple* if it is not initial and its non-zero subobjects are isomorphisms. Clearly, simple objects have finite length. A $\mathscr{V}$-category $\mathscr{X}$ is *semisimple* if every object is isomorphic to a direct sum of simple objects.

Observation 7.29. Any object $(x_1, \cdots, x_n)$ in $D(\mathscr{X})$ is isomorphic to an object in a *normal form.* By this we mean that there exists a permutation σ of $\{1, 2, \cdots, n\}$ and positive integers $k_1, \cdots, k_m$ such that $\sum_{i=1}^{m} k_i = n$ and $x_{\sigma(j)} = x_{\sigma(j+1)}$ if and only if $\sum_{i=1}^{r} k_i \leq j \leq \sum_{i=1}^{r+1} k_i - 1$, $1 \leq r \leq m-1$. In other words, we can rearrange the x_i in a way such that all of them that are equal are grouped together. The permutation matrix associated to σ provides an isomorphism between $(x_1, \cdots, x_n)$ and $(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. In particular, any object of $D(\mathscr{X})$ is isomorphic to an object of the form $(y_1)^{k_1} \oplus (y_2)^{k_2} \oplus \cdots \oplus (y_m)^{k_m}$, with all the y_i distinct objects of $\mathscr{X}$. Here $(y)^k$ denotes the direct sum of k copies of the object (y) of $D(\mathscr{X})$; this direct sum is isomorphic to the list $(y, y, \cdots, y)$, of length k.

Given two objects of $D(\mathscr{X})$, it is clear from the discussion above that we can find normal forms for each one in a compatible way. In other words, one of the objects will be isomorphic to $(x_1)^{\ell_1} \oplus \cdots \oplus (x_r)^{\ell_r} \oplus (y_1)^{k_1} \oplus \cdots \oplus (y_s)^{k_s}$ and the other to $(x_1)^{h_1} \oplus \cdots \oplus (x_r)^{h_r} \oplus (z_1)^{m_1} \oplus \cdots \oplus (z_t)^{m_s}$, with all the x_i, y_i and z_i distinct pairwise.

Lemma 7.30. *If $\mathscr{B}$ has kernels, then for any pair of simple objects S, S', $\mathscr{B}(S, S')$ is a division algebra if and only if $S \cong S'$ and zero otherwise.*

Proof. Suppose $f : S \to S'$ is a non-zero arrow. The arrow $\ker f$ is a subobject of S, and it cannot be an isomorphism as $f \neq 0$. Therefore, $\ker f$ is zero, and then f is a non-zero subobject of S'. We deduce that f is an isomorphism. $\square$

Lemma 7.31. *Let $\mathscr{B}$ be a semisimple $\mathscr{V}$-category with kernels, and $\{S_\alpha\}_{\alpha\in I}$ a representative set of simple objects. If the objects of $\mathscr{B}$ have finite length, then $\mathscr{B}$ is equivalent to $D(\mathscr{X})$ where $\mathscr{X} = \coprod_{\alpha\in I} \mathscr{X}_\alpha$, $\mathrm{ob}(\mathscr{X}_\alpha) = \{x_\alpha\}$ and $\mathscr{X}_\alpha(x_\alpha, x_\alpha) = \mathscr{B}(S_\alpha, S_\alpha)$.*

Proof. The $\mathscr{V}$-category $\mathscr{B}$ is equivalent to its full sub $\mathscr{V}$-category $\mathscr{B}'$ with objects the direct sums of the S_α. So, it is enough to prove the result for $\mathscr{B}'$. Let $\mathscr{X}$ be the $\mathscr{V}$-category described in the statement. Define $\mathscr{X} \to \mathscr{B}'$ sending x_α to S_α and as the identity on homs. We shall prove this functor induces equivalences $\mathsf{Fin}\text{-}\mathbf{BP}(\mathscr{B}', \mathscr{C}) \simeq [\mathscr{X}, \mathscr{C}]$ for any $\mathscr{V}$-category $\mathscr{C}$ with finite direct sums.

Any arrow $f : S_{\alpha_1}^{r_1} \oplus \cdots \oplus S_{\alpha_m}^{r_m} \to S_{\alpha_1}^{s_1} \oplus \cdots \oplus S_{\alpha_m}^{s_m}$ is coproduct of arrows $f^i : S_{\alpha_i}^{r_i} \to S_{\alpha_i}^{s_i}$, by Lemma 7.30. Moreover, each f^i determines and is determined by a matrix $(f^i_{p,q})$ with entries in $\mathrm{End}(S_{\alpha_i})$.

Now suppose $F : \mathscr{X} \to \mathscr{C}$ is a $\mathscr{V}$-functor. Define $G : \mathscr{B}' \to \mathscr{C}$ on objects such that it preserves finite direct sums. On arrows, $G(f) = G(f^1) \oplus \cdots \oplus G(f^m)$, where $G(f^i) : G(S_{\alpha_i})^{r_i} \to G(S_{\alpha_i})^{s_i}$ is given by the matrix $(F(f^i_{p,q}))$. This defines a finite direct sum-preserving $\mathscr{V}$-functor, corresponding to F under composition with $\mathscr{X} \to \mathscr{B}'$. This is the object part of the required inverse equivalence. $\square$

Lemma 7.32. *Let $\mathscr{X}_\alpha$ be $\mathscr{V}$-categories with $\mathrm{ob}(\mathscr{X}_\alpha) = \{x_\alpha\}$ and $A_\alpha = \mathscr{X}(x_\alpha, x_\alpha)$ division algebras. Then, the $\mathscr{V}$-category $\mathscr{D} = D(\coprod_\alpha \mathscr{X}_\alpha)$ is semisimple and abelian, and has objects of finite length.*

Proof. By definition, every object of $\mathscr{D}$ is direct sum of the objects x_α. So, to show this category is semisimple, it suffices to prove that each x_α is simple. Let $f : (x_{\alpha_1}, \cdots, x_{\alpha_n}) \to x_\alpha$ be a non zero monomorphism. By definition, f is a matrix $(f_1, \cdots, f_n)$ with $f_i : x_{\alpha_i} \to x_\alpha$. All the f_i must be non-zero, otherwise f would not be monic. This forces $\alpha_i = \alpha$ for all i. Now, $f \in \mathrm{Mat}_{1\times n}(A_\alpha)$, and the condition of being a monomorphism implies that for any $g \in \mathrm{Mat}_{m\times n}(A_\alpha)$, $fg = 0$ implies $g = 0$. But A_α is a division ring, and then $n = 1$. We just showed that f comes from a non-zero arrow $x_\alpha \to x_\alpha$, and hence it is invertible. Therefore, the only non-zero subobjects of x_α are isomorphisms, and x_α is simple.

Next we show that $\mathscr{D}$ has kernels and cokernels. In fact, only one of the two properties is needed, since by duality we get a proof for the other, as $D(\mathscr{Y})^{\mathrm{op}} \cong D(\mathscr{Y}^{\mathrm{op}})$ for any $\mathscr{V}$-category $\mathscr{Y}$. Let f be an arrow in $\mathscr{D}$. To the purpose of showing it has a kernel, by Observation 7.29, we may assume f is of the form $g_1 \oplus \cdots \oplus g_r \oplus 0$, where $g_i : (x_i)^{\ell_i} \to (x_i)^{h_i}$ and 0 is the zero morphism between certain objects. Hence, it suffices to prove that each g_i has a kernel. An arrow

$g : (x_{\alpha_0})^k \to (x_{\alpha_0})^h$ in $\mathscr{D}$ is a matrix $g_{p,q} \in \mathrm{Mat}_{h\times k}(A_{\alpha_0})$. Since A_{α_0} is a division algebra, and then every A_{α_0}-module is free, the matrix $(g_{p,q})$ has a kernel. This kernel is the matrix of the inclusion of the (submodule) kernel of $(g_{p,q})$, with respect to any basis in this submodule. Now it is easy to see that this kernel matrix is the kernel of g.

To prove that $\mathscr{D}$ is abelian it only remains to show that any arrow factors as a cokernel followed by a kernel. The argument is very similar to the one used in the previous paragraph. It is enough to prove that the arrows $(x_{\alpha_0})^k \to (x_{\alpha_0})^m$ have coker-ker factorisations, and this is true because matrices with entries in the division algebra A_{α_0} have a coker-ker factorisation. This finishes the proof. □

Recall that we denote by L the 2-monad on $\mathscr{V}$-**Cat** whose algebras are the $\mathscr{V}$-categories with chosen finite limits.

Proposition 7.33. *For a $\mathscr{V}$-category $\mathscr{B}$, the following are equivalent.*

1. *$\mathscr{B}$ is semisimple abelian with objects of finite length.*
2. *$\mathscr{B}$ is semisimple and has kernels and objects of finite length.*
3. *$\mathscr{B}$ is equivalent to $D(\coprod_\alpha \mathscr{X}_\alpha)$ where $\mathscr{X}_\alpha$ has one object x_α and its unique hom $\mathscr{X}_\alpha(x_\alpha, x_\alpha)$ is a division algebra.*
4. *$\mathscr{B}$ is equivalent to $L(\coprod_\alpha \mathscr{X}_\alpha)$ where $\mathscr{X}_\alpha$ has one object x_α and its unique hom $\mathscr{X}_\alpha(x_\alpha, x_\alpha)$ is a division algebra.*

Proof. (1) implies (2) trivially, (2) implies (3) by Lemma 7.31 together with Lemma 7.30, and (3) implies (1) by Lemma 7.32. To prove the equivalence of (3) and (4) it is enough to show that any $\mathscr{V}$-functor with domain $D(\coprod \mathscr{X}_\alpha)$ preserves kernels. But this is clear, since this category is semisimple abelian and every object in it is injective, and hence any kernel is, up to composing with isomorphisms, the the coprojection of a direct sum. □

Corollary 7.34. *Let $\mathscr{B}$ be a $\mathscr{V}$-category satisfying the properties of Proposition 7.33. Then, $\mathscr{B}$ is equivalent to $D(\mathscr{X})$ for a discrete $\mathscr{V}$-category $\mathscr{X}$ if and only if $\mathscr{B}(S,S) \cong k$ for every simple object S of $\mathscr{B}$. In this case, $\mathscr{B}$ is also equivalent to $L(\mathscr{X})$.*

The corollary above can be reinterpreted in the following way. Let $\mathscr{B}$ be as in Corollary 7.34, and let $\{S_\alpha\}_{\alpha\in\Lambda}$ be a set of representatives of the isomorphism classes of simple objects. Then, to give a $\mathscr{V}$-functor $\mathscr{B} \to \mathscr{A}$, where $\mathscr{A}$ has finite direct sums, is, up to isomorphism, to give an object of $\mathscr{A}$ for each S_α.

Example 7.35. Suppose the base field k is algebraically closed. Then any semisimple abelian $\mathscr{V}$-category $\mathscr{B}$ is as in Corollary 7.34. Indeed, for any simple object S, $\mathscr{B}(S,S)$ is a finite dimensional division algebra, and then isomorphic to k.

We now show that the results on semisimplicity above hold for $\mathscr{V}$-categories with finite homs and whose objects are injective. An object E is *injective* if for any monic $m : X \rightarrowtail Y$ the $\mathscr{V}$-natural transformation $\mathscr{X}(-,m) : \mathscr{X}(-,Y) \Rightarrow \mathscr{X}(-,X)$ is epi. In particular, if $s : S \rightarrowtail X$ is a subobject with S injective, then there is an arrow $r : X \to S$ such that $rs = 1_S$.

Lemma 7.36. *If $s : S \rightarrowtail X$ is a subobject with S injective in a $\mathscr{V}$-category $\mathscr{B}$ with kernels, then s is part of a biproduct diagram.*

Proof. We have to give a diagram

$$S \underset{r}{\overset{s}{\rightleftarrows}} X \underset{k}{\overset{p}{\rightleftarrows}} K.$$

Choose r such that $rs = 1_S$, and define $k = \ker r$ and p the unique arrow such that $kp = 1_X - sr$; this makes sense as $r(1_X - sr) = r - rsr = r - r = 0$. The only condition that remains to be checked is $pk = 1_K$. But this is also easy: $kpk = (1_X - sr)k = k - srk = k$, and k is mono. □

In the hypothesis of the Lemma above, we have

$$\dim \mathscr{B}(X,X) \geq \dim \mathscr{B}(S,S) + \dim(K,K) \geq \dim \mathscr{B}(S,S) + 1.$$

Lemma 7.37. *Suppose $\mathscr{B}$ has finite-dimensional homs and every object is injective. Then every object is Artinian and Noetherian, and has a simple subobject.*

Proof. Suppose X is not Artinian; that is, there is an infinite chain of subobjects $X \supsetneq S_1 \supsetneq S_2 \supsetneq \cdots$. Then, $\dim \mathscr{B}(X,X) \geq \dim \mathscr{B}(S_n,S_n) + n \geq n+1$ for all $n \geq 1$, which contradicts the finite-dimension of $\mathscr{B}(X,X)$. The proof for the Noetherian condition is analogous. Finally, if X had not a simple subobject, we could construct an infinite descending chain of subobjects, contradicting the Artinian property. □

Lemma 7.38. *Suppose $\mathscr{B}$ has finite-dimensional homs and every object is injective. Then every object is isomorphic to a finite direct sum of simple objects. If, in addition, $\mathscr{B}$ has kernels, then $\mathscr{B}$ is abelian.*

Proof. Every object is finite direct sum of simple objects by Lemmas 7.36 and 7.37. The rest follows from Proposition 7.33. $\square$

We now apply these results to the case of autonomous $\mathscr{V}$-categories. Recall that if $\mathscr{C}$ is a monoidal $\mathscr{V}$-category we denote by $J : \mathscr{V}_f^{\mathrm{op}} \to \mathscr{C}$ the left exact $\mathscr{V}$-functor corresponding to the unit of the monoidal structure.

Theorem 7.39. *Let $\mathscr{C}$ be an autonomous monoidal $\mathscr{V}$-category with finite-dimensional homs and finite limits. Then*

1. *If $J^* : \mathscr{C} \to \mathscr{V}_f^{\mathrm{op}}$ has a right adjoint, then $\mathscr{C}$ is semisimple abelian with objects of finite length.*

2. *J^* has right adjoint if and only if $J^* \dashv J$.*

3. *If $\mathscr{C}$ is semisimple and $\mathscr{C}(\mathbf{1}, \mathbf{1}) \cong k$, then $J^* \dashv J$.*

Proof. (1) If

$$J^* = (\mathscr{C} \xrightarrow{\mathscr{C}(-,\mathbf{1})} \mathscr{V}_f^{\mathrm{op}} \xrightarrow{(-)^\vee} \mathscr{V}_f)$$

(see Lemma 7.15) has right adjoint, then $\mathscr{C}(-, \mathbf{1}) : \mathscr{C} \to \mathscr{V}_f^{\mathrm{op}}$ is cocontinuous. In other words, $\mathscr{C}(-, \mathbf{1}) : \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$ is continuous, and hence $\mathbf{1}$ is injective. Therefore, every object in $\mathscr{C}$ is injective, as $\mathscr{C}(-, X)$ is isomorphic to $\mathscr{C}(- \otimes X^*, \mathbf{1})$. By Lemma 7.38, $\mathscr{C}$ is semisimple abelian with objects of finite length.

(2) Denote by G a right adjoint to J^*. We know that $\mathscr{C}$ is semisimple. Write $\mathbf{1} \cong \oplus_\alpha S_\alpha^{n_\alpha}$ and $G(k) \cong \oplus_\alpha S_\alpha^{m_\alpha}$, decompositions of $\mathbf{1}$ and $G(k)$ as direct sum of simple objects. There is an isomorphism $\mathscr{C}(X, G(k)) \cong \mathscr{V}_f^{\mathrm{op}}(J^*(X), k) \cong \mathscr{C}(\mathbf{1}, X)^\vee$, natural in X. Setting $X = S_{\alpha_0}$ a simple object, we get

$$\mathscr{C}(S_{\alpha_0}, S_{\alpha_0})^{\vee m_{\alpha_0}} \cong \mathscr{C}(S_{\alpha_0}, S_{\alpha_0})^{n_{\alpha_0}}$$

and hence $m_\alpha = n_\alpha$ for all α. Then we see that $G(k) \cong \mathbf{1}$ and therefore $G \cong J$.

(3) We have to prove the existence of a natural isomorphism $\mathscr{C}(\mathbf{1}, X)^\vee \cong \mathscr{C}(X, \mathbf{1})$. As $\mathscr{C}$ is semisimple, by Proposition 7.33, it is enough to show there exists an isomorphism between the restriction of the two functors to the full subcategory of $\mathscr{C}$ with objects a set of representatives of isomorphism classes of simple objects. Write $\mathbf{1}$ as the direct sum of simple objects as in the paragraph above. It is enough to show that there is an exact dinatural pairing $\mathscr{C}(S, \mathbf{1}) \otimes \mathscr{C}(\mathbf{1}, S) \to k$, where S is a simple object appearing in the decomposition of $\mathbf{1}$. If $\mathscr{C}(\mathbf{1}, \mathbf{1}) \cong k$, by the finiteness of the hom spaces, the composition is such a pairing. $\square$

Corollary 7.40. *For an autonomous monoidal $\mathscr{V}$-category $\mathscr{C}$ with finite limits and finite-dimensional homs and with $\mathscr{C}(\mathbf{1},\mathbf{1}) \cong k$, the following are equivalent.*

1. *$J^* : \mathscr{C} \to \mathscr{V}_f^{\mathrm{op}}$ is has a right adjoint.*
2. *$J^* \dashv J$.*
3. *$\mathscr{C}$ is semisimple abelian with objects of finite length.*

7.7 Semisimple pseudomonoids and fusion categories

In this section we show that any semisimple autonomous monoidal $\mathscr{V}$-category is unimodular. We work in the monoidal 2-category L-Alg of Section 7.5. As in the previous section, $\mathscr{V}$ will denote the category of vector spaces over a field k.

We have seen in Proposition 7.33 that a finitely complete semisimple $\mathscr{V}$-category is equivalent to $L(\coprod_\alpha \mathscr{X}_\alpha)$, where the $\mathscr{V}$-categories $\mathscr{X}_\alpha$ have one object and the unique $\mathscr{V}$-enriched hom is a division algebra under composition.

Proposition 7.41. *Assume k is algebraically closed. If the objects $\mathscr{A}$ and $\mathscr{B}$ in L-Alg are semisimple as $\mathscr{V}$-categories and have objects of finite length, then $\mathscr{A} \boxtimes \mathscr{B}$ is semisimple too. Moreover, if $\mathscr{A}$ and $\mathscr{B}$ have representative sets of simple objects $\{A_\alpha\}$ and $\{B_\beta\}$ respectively, then $\{A_\alpha \boxtimes B_\beta\}$ is a representative set of simple objects of $\mathscr{A} \boxtimes \mathscr{B}$.*

Proof. By Proposition 7.33, we may assume $\mathscr{A}$ and $\mathscr{B}$ are of the form $D(\coprod_\alpha \mathscr{X}_\alpha)$ and $D(\coprod_\beta \mathscr{Y}_\beta)$ respectively, where $\mathrm{ob}\,\mathscr{X}_\alpha = \{x_\alpha\}$, $\mathrm{ob}\,\mathscr{Y}_\alpha = \{y_\alpha\}$ and every $\mathscr{V}$-enriched hom $\mathscr{X}_\alpha(x_\alpha, x_\alpha)$ and $\mathscr{Y}_\beta(y_\beta, y_\beta)$ is isomorphic to k (since k is algebraically closed). The sets $\{(x_\alpha)\}$ and $\{(y_\beta)\}$ are the sets of simple objects in the respective categories. By Corollary 6.32, $L(\coprod_\alpha \mathscr{X}_\alpha) \boxtimes L(\coprod_\beta \mathscr{Y}_\beta)$ is equivalent to $L((\coprod_\alpha \mathscr{X}_\alpha) \otimes (\coprod_\beta \mathscr{Y}_\beta))$, and hence to $L(\coprod_{\alpha,\beta} \mathscr{X}_\alpha \otimes \mathscr{Y}_\beta)$. Under this equivalence $x_\alpha \boxtimes y_\beta$ corresponds to (x_α, y_β). Then, $\mathscr{A} \boxtimes \mathscr{B}$ is equivalent to $D(\coprod_{\alpha,\beta} \mathscr{X}_\alpha \otimes \mathscr{Y}_\beta)$, again by Proposition 7.33. The $\mathscr{V}$-categories $\mathscr{X}_\alpha \otimes \mathscr{Y}_\beta$ have one object (x_α, y_β), and its $\mathscr{V}$-enriched hom is clearly isomorphic to k. It follows from Proposition 7.33 that $\mathscr{A} \boxtimes \mathscr{B}$ is semisimple abelian. Finally, $\{(x_\alpha, y_\beta)\}$ is the set of simple objects of $D(\coprod_{\alpha,\beta} \mathscr{X}_\alpha \otimes \mathscr{Y}_\beta)$, and then $\{x_\alpha \boxtimes y_\beta\}$ is a set of representatives of the simple objects of $\mathscr{A} \boxtimes \mathscr{B}$. $\square$

Let $\mathscr{C}$ be semisimple and monoidal with objects of finite length and finitely many simple objects. Then $\mathscr{C}$ has associated *structural constants.* If $\{S_\alpha\}_{\alpha \in \Lambda}$ is a set of representatives of the isomorphism classes of simple objects, we can write

$$S_\alpha \otimes S_\beta \cong \oplus_\gamma S_\gamma^{n_\gamma^{\alpha,\beta}}.$$

The non negative integers $n_\gamma^{\alpha,\beta}$ are called the structural constants of $\mathscr{C}$.

Theorem 7.42. *Assume k is algebraically closed and let $\mathscr{C}$ be semisimple abelian autonomous monoidal $\mathscr{V}$-category with finite dimensional homs. Then, the multiplication $P : \mathscr{C} \boxtimes \mathscr{C} \to \mathscr{C}$ has a right adjoint if and only if $\mathscr{C}$ has finitely many isomorphism classes of simple objects. Moreover, $P^*(X) \cong \oplus_{\alpha\in\Lambda} S_\alpha \boxtimes (S_\alpha^* \otimes X) \cong \oplus_{\alpha\in\Lambda}(X \otimes S_\alpha) \boxtimes S_\alpha^*$, where $\{S_\alpha\}$ is a set of representatives of the isomorphism classes of simple objects.*

Proof. Let $\{S_\alpha\}_{\alpha\in\Lambda}$ be a set of representatives of classes of isomorphism of simple objects. Write α^* for the element of Λ such that $S_{\alpha^*} \cong S_\alpha^*$. Suppose P has a right adjoint P^*, and write $P^*(\mathbf{1}) \cong \oplus_{\alpha,\beta\in\Lambda}(S_\alpha \boxtimes S_\beta)^{n_{\alpha,\beta}}$, where all but finitely many of the non negative integers $n_{\alpha,\beta}$ are non zero. By Corollary 7.20, for any object X in $\mathscr{C}$ we have

$$X \cong \bigoplus_{\alpha,\beta\in\Lambda} \{\mathscr{C}(\mathbf{1}, S_\beta \otimes X)^\vee, S_\alpha^{n_{\alpha,\beta}}\} \cong \bigoplus_{\alpha,\beta\in\Lambda} \{\mathscr{C}({}^*S_\beta, X)^\vee, S_\alpha^{n_{\alpha,\beta}}\}$$

where $\{-,-\}$ denotes the cotensor product. Let S_γ be an arbitrary simple object and set $X = S_\gamma$ to obtain

$$S_\gamma \cong \bigoplus_{\alpha\in\Lambda} \{\mathscr{C}(S_\gamma, S_\gamma)^\vee, S_\alpha^{n_{\alpha,\gamma^*}}\} \cong \bigoplus_{\alpha\in\Lambda} S_\alpha^{n_{\alpha,\gamma^*}}$$

where we used $\mathscr{C}(S,S) \cong k$ for all simple objects S. We deduce that $n_{\alpha,\gamma^*} = 0$ if $\alpha \neq \gamma$ and $n_{\gamma,\gamma^*} = 1$. Then γ belongs to the finite set $\Lambda' = \{\alpha \in \Lambda \mid n_{\alpha,\alpha^*} \neq 0\}$. This proves that $\Lambda = \Lambda'$ is finite. Moreover,

$$P^*(\mathbf{1}) \cong \bigoplus_{\alpha\in\Lambda} S_\alpha \boxtimes S_\alpha^*.$$

Henceforth, by Corollary 7.20 (2) and (3), together with (2.20) and its dual, we deduce $P^*(X) \cong \oplus_\alpha (X \otimes S_\alpha) \boxtimes S_\alpha^* \cong \oplus_\alpha S_\alpha \boxtimes (S_\alpha^* \otimes X)$.

For the converse, suppose $\mathscr{C}$ has finitely many isomorphism classes of simple objects, and let $\{S_\alpha\}_{\alpha\in\Lambda}$ a set of representatives, and that $\mathscr{C}(S_\alpha, S_\alpha) \cong k$ for all $\alpha \in \Lambda$. We have to show that there exists a $\mathscr{V}$-functor $Q : \mathscr{C} \to \mathscr{C} \boxtimes \mathscr{C}$ and a natural isomorphism $\mathscr{C}(X \otimes Y, Z) \cong \mathscr{C} \boxtimes \mathscr{C}(X \boxtimes Y, Q(Z))$. By Corollary 7.34, to give Q is to give objects $Q(S_\alpha)$. Then we have to find objects $Q(S_\alpha)$ with a natural isomorphism $\mathscr{C}(X \otimes Y, S_\alpha) \cong \mathscr{C} \boxtimes \mathscr{C}(X \boxtimes Y, Q(S_\alpha))$. Applying again Corollary 7.34, we have to find isomorphisms

$$\mathscr{C}(S_\beta \otimes S_\gamma, S_\alpha) \cong \mathscr{C} \boxtimes \mathscr{C}(S_\beta \boxtimes S_\gamma, Q(S_\alpha)) \tag{7.10}$$

for all $\alpha, \beta, \gamma \in \Lambda$. Write $S_\beta \otimes S_\gamma \cong \oplus_\lambda S_\lambda^{n_\lambda^{\beta,\gamma}}$ and $Q(S_\alpha) \cong \oplus_{\lambda,\mu}(S_\lambda \boxtimes S_\mu)^{m^\alpha_{\lambda,\mu}}$, where $m^\alpha_{\lambda,\mu}$ are integers to be determined. Substituting in (7.10), we have to exhibit isomorphisms

$$\mathscr{C}(S_\alpha^{n_\alpha^{\beta,\gamma}}, S_\alpha) \cong \mathscr{C} \boxtimes \mathscr{C}(S_\beta \boxtimes S_\gamma, (S_\beta \boxtimes S_\gamma)^{m^\alpha_{\beta,\gamma}}).$$

Using that $\mathscr{C} \boxtimes \mathscr{C}(S_\beta \boxtimes S_\gamma, S_\beta \boxtimes S_\gamma)$ has dimension one, we deduce that setting $m^\alpha_{\beta,\gamma} = n_\alpha^{\beta,\gamma}$, we have an isomorphism as required. $\square$

Following [28], when k is algebraically closed, we call a semisimple abelian autonomous monoidal $\mathscr{V}$-category with finite dimensional homs, finitely many simple objects and simple unit object a *fusion category*.

Observation 7.43. The Theorem above can be reinterpreted in the following way: fusion categories are exactly the semisimple abelian autonomous monoidal k-linear categories $\mathscr{C}$ with simple unit object and such that the multiplication $P : \mathscr{C} \boxtimes \mathscr{C} \to \mathscr{C}$ has right adjoint.

The following result originally appears in [27, Corollary 6.4]. However, our proof is very different, as it follows from a general result on autonomous map pseudomonoids.

Theorem 7.44. *Let $\mathscr{C}$ be a fusion category regarded as an autonomous map pseudomonoid in L-*Alg*. Then, $\mathscr{C}$ is an unimodular autonomous pseudomonoid in L-*Alg*. In particular there exists a natural isomorphism*

$$^{**}X \cong X^{**}.$$

Proof. We can consider $\mathscr{C}$ as an autonomous map pseudomonoid in L-Alg by Theorem 7.42 and Corollary 7.20. By Corollary 7.40, there exists an adjunction $J^* \dashv J$, and this forces to $\mathscr{C}$ to be unimodular, as showed in Proposition 4.20. This means that the object W in Theorem 7.23 is isomorphic to the unit $\mathbf{1}$, yielding an isomorphism as claimed. $\square$

$$
\begin{tikzcd}
\mathscr{C}^2(X,(\mathsf{L}_Y\boxtimes\mathscr{C})P^*Z) \arrow[rrr, "\mathscr{C}^2(X{,}\eta)"] \arrow[d, "\cong"'] \arrow[rr, "P"'] & & & \mathscr{C}^2(X,P^*P(\mathsf{L}_Y\boxtimes\mathscr{C})P^*Z) \arrow[dl, "\cong"'] \arrow[dd, "\mathscr{C}^2(X{,}P^*\varpi_{P^*Z})"] \\
\mathscr{C}^2((\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X,P^*Z) \arrow[d, "P"'] \arrow[r, phantom, "\text{(A)}"] & {} & \mathscr{C}(PX,P(\mathsf{L}_Y\boxtimes C)P^*Z) \arrow[d, "\mathscr{C}(1{,}\cong)"] & \\
\mathscr{C}(P(\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X,PP^*Z) \arrow[r, "\cong"] \arrow[d, "\mathscr{C}(1{,}\varepsilon_Z)"] & \mathscr{C}(\mathsf{L}^\ell_Y PX,PP^*Z) \arrow[d, "\mathscr{C}(1{,}\varepsilon_Z)"] & \mathscr{C}(PX,\mathsf{L}_Y PP^*Z) \arrow[l, "\cong"'] \arrow[d, "\mathscr{C}(1{,}\mathsf{L}_Y\varepsilon_Z)"] & \mathscr{C}^2(X,P^*\mathsf{L}_Y PP^*Z) \arrow[l, "\cong"'] \arrow[d, "\mathscr{C}^2(1{,}P^*\mathsf{L}_Y\varepsilon_Z)"] \\
\mathscr{C}(P(\mathsf{L}_Y\boxtimes\mathscr{C})X,Z) \arrow[r, "\cong"'] & \mathscr{C}(\mathsf{L}^\ell_Y PX,Z) \arrow[r, "\cong"'] & \mathscr{C}(PX,\mathsf{L}_Y Z) \arrow[r, "\cong"'] & \mathscr{C}^2(X,P^*\mathsf{L}_Y Z)
\end{tikzcd}
\tag{7.5}
$$

$$
\begin{tikzcd}
\mathscr{C}^2(X,\mathsf{L}_Y(U)\boxtimes V) \arrow[rr, "P"] \arrow[dr, "\mathsf{L}^\ell_Y\boxtimes 1"] \arrow[dd, "\cong"'] & & \mathscr{C}(PX,P(\mathsf{L}_Y(U)\boxtimes V)) \arrow[d, "\mathscr{C}(1{,}\varpi)"] \\
& \mathscr{C}^2((\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X,\mathsf{L}^\ell_Y\mathsf{L}_Y(U)\boxtimes V) \arrow[dl, "\mathscr{C}^2(1{,}\varepsilon_U\boxtimes 1)"'] \arrow[d, "P"] \arrow[r, phantom, "\text{(B)}"] & \mathscr{C}(PX,\mathsf{L}_Y(U\otimes V)) \arrow[d, "\mathsf{L}^\ell_Y"] \\
\mathscr{C}^2((\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X,U\boxtimes V) \arrow[d] & \mathscr{C}(P(\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X,\mathsf{L}^\ell_Y\mathsf{L}_Y(U)\otimes V) \arrow[r, "\cong"] \arrow[dl, "\mathscr{C}(1{,}\varepsilon_U\boxtimes 1)"'] & \mathscr{C}(\mathsf{L}_Y PX,\mathsf{L}^\ell_Y\mathsf{L}_Y(U\otimes V)) \arrow[d, "\mathscr{C}(1{,}\varepsilon_{U\otimes V})"] \\
\mathscr{C}(P(\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X,U\otimes V) \arrow[rr, "\cong"] & & \mathscr{C}(\mathsf{L}_Y PX,U\otimes V)
\end{tikzcd}
\tag{7.6}
$$

$$
\begin{tikzcd}
\mathscr{C}^2(X,\mathsf{L}_Y(U)\boxtimes V) \arrow[r, "P"] \arrow[d, "\mathsf{L}^\ell_Y\boxtimes 1"] & \mathscr{C}(PX,\mathsf{L}_Y(U)\otimes V) \arrow[r, "\mathsf{L}^\ell_Y"] \arrow[d, "\mathscr{C}(1{,}\varpi_{U\boxtimes V})"] & \mathscr{C}(\mathsf{L}^\ell_Y PX,\mathsf{L}^\ell_Y(\mathsf{L}_Y(U)\otimes V)) \arrow[dd, "\mathscr{C}(1{,}\varpi^{-1}_{\mathsf{L}_Y(U)\boxtimes V})"] \\
\mathscr{C}^2((\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X,\mathsf{L}^\ell_Y\mathsf{L}_Y(U)\boxtimes V) \arrow[d, "P"'] \arrow[dr, phantom, "\text{(B)}"] & \mathscr{C}(PX,\mathsf{L}_Y(U\otimes V)) \arrow[dr, "\mathsf{L}^\ell_Y"] & \\
\mathscr{C}(P(\mathsf{L}^\ell_Y\boxtimes\mathscr{C})X,\mathsf{L}^\ell_Y\mathsf{L}_Y(U)\otimes V) \arrow[rr, "\mathscr{C}(\varpi_X^{-1}{,}1)"'] & & \mathscr{C}(\mathsf{L}^\ell_Y PX,\mathsf{L}^\ell_Y\mathsf{L}_Y(U)\otimes V)
\end{tikzcd}
\tag{7.7}
$$

Figure 7.1:

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