Twistor theory at fifty: from contour integrals to twistor strings

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Dedicated to Roger Penrose and Nick Woodhouse at 85 and 67 years.

Abstract

We review aspects of twistor theory, its aims and achievements spanning the last five decades. In the twistor approach, space–time is secondary with events being derived objects that correspond to compact holomorphic curves in a complex three–fold – the twistor space. After giving an elementary construction of this space we demonstrate how solutions to linear and nonlinear equations of mathematical physics: anti-self-duality (ASD) equations on Yang–Mills, or conformal curvature can be encoded into twistor cohomology. These twistor correspondences yield explicit examples of Yang–Mills, and gravitational instantons which we review. They also underlie the twistor approach to integrability: the solitonic systems arise as symmetry reductions of

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ASD Yang–Mills equations, and Einstein–Weyl dispersionless systems are reductions of ASD conformal equations.

We then review the holomorphic string theories in twistor and ambitwistor spaces, and explain how these theories give rise to remarkable new formulae for the computation of quantum scattering amplitudes. Finally we discuss the Newtonian limit of twistor theory, and its possible role in Penrose's proposal for a role of gravity in quantum collapse of a wave function.

Contents

1 Twistor Theory

Twistor theory was originally proposed as a new geometric framework for physics that aims to unify general relativity and quantum mechanics [173, 174, 184, 182, 183]. In the twistor approach, space–time is secondary with events being derived objects that correspond to compact holomorphic curves in a complex three–fold, the *twistor space*. The mathematics of twistor theory goes back to the 19th century Klein correspondence, but we shall begin our discussion with a formula for solutions to the wave equation in (3+1)–dimensional Minkowski space–time put forward by Bateman in 1904 [31]

$$\phi(x, y, z, t) = \oint_{\Gamma \subset \mathbb{CP}^1} f((z + ct) + (x + iy)\lambda, (x - iy) - (z - ct)\lambda, \lambda)d\lambda. \quad (1.1)$$

This is the most elementary of Penrose's series of twistor integral formulae for massless fields [175]. The closed contour $\Gamma \subset \mathbb{CP}^1$ encloses some poles of a meromorphic function f. Differentiating (1.1) under the integral sign yields

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (1.2)$$

The twistor contour integral formula (1.1) is a paradigm for how twistor theory should work and is a good starting point for discussing its development over the last five decades. In particular one may ask

- *What does this formula mean geometrically?*

  The integrand of (1.1) is a function of three complex arguments and we will see in §2 that these arise as local affine coordinates on projective twistor space $\mathbb{PT}$ which we take to be $\mathbb{CP}^3 - \mathbb{CP}^1$. In (1.1) the coordinates on $\mathbb{PT}$ are restricted to a line with affine coordinate λ. The Minkowski space arises as a real slice in the four-dimensional space of lines in $\mathbb{PT}$.

  The map (1.1) from functions f to solutions to the wave equation is not one to one: functions holomorphic inside Γ can be added to f without changing the solution ϕ. This freedom in f was understood in the 1970s in a fruitful interaction between the Geometry and Mathematical Physics research groups in Oxford [15]: twistor functions such as f in (1.1) should be regarded as elements of Čech sheaf cohomology groups. Rigorous theorems establishing twistor correspondences for the wave equation, and higher spin linear equations have now been established [81, 218, 108, 29]. The concrete realisations of these theorems lead to (contour) integral formulae.

- *Do 'similar' formulae exist for nonlinear equations of mathematical physics, such as Einstein or Yang–Mills equations?*

  The more general integral formulae of Penrose [175] give solutions to both linearised Einstein and Yang–Mills equations. In the case that

the linearised field is anti-self-dual (i.e., circularly polarised or right handed) these cohomology classes correspond to linearised deformations of the complex structure of twistor space for gravity [176, 18] or of a vector bundle in the Yang-Mills case [210]. We shall review these constructions in §3 and §5.

These constructions give an *'in principle'* general solution to the equations in the sense that locally every solution can be represented locally in terms of free data on the twistor space as in the original integral formula. Indeed this leads to large classes of explicit examples (e.g. Yang-Mills and gravitational instantons which we shall review in the gravitational case in §33.4) although it can be hard to implement for general solutions.

It turns out [213] that most known integrable systems arise as symmetry reductions of either the anti–self–dual Yang–Mills or the anti–self–dual (conformal) gravity equations. The twistor constructions then reduce to known (inverse scattering transform, dressing method, . . .) or new solution generation techniques for soliton and other integrable equations [158, 75]. We shall review some of this development in §6.

As far as the full Einstein and Yang-Mills equations are concerned, the situation is less satisfactory. The generic nonlinear fields can be encoded in terms of complex geometry in closely related ambitwistor spaces. In these situations the expressions of the field equations are less straightforward and they no longer seem to provide a general solution generation method. Nevertheless, they have still had major impact on the understanding of these theories in the context of perturbative quantum field theory as we will see in §7.

- *Does it all lead to interesting mathematics?*

  The impacts on mathematics have been an unexpected major spin–off from the original twistor programme. These range over geometry in the study of hyper–Kähler manifolds [18, 107, 138, 139, 191], conformal, CR and projective structures [101, 26, 100, 48, 144, 83, 144, 88, 59, 50, 27, 82, 32, 219], exotic holonomy [49, 165, 166, 167], in representation theory [29, 60] and differential equations particularly in the form of integrable systems [158, 75]. We will make more specific comments and references in the rest of this review.

- *Is it physics?*

  Thus far, the effort has been to reformulate conventional physics in twistor space rather than propose new theories. It has been hard to give a complete reformulation of conventional physics on twistor space in the form of nonlinear generalizations of (1.1). Nevertheless, in just the past 13 years, holomorphic string theories in twistor and ambitwistor spaces have provided twistorial formulations of a full range of theories that are

commonly considered in particle physics. They also provide remarkable new formulae for the computation of scattering amplitudes. Many technical issues that remain to be resolved to give a complete reformulation of conventional physics ideas even in this context of peturbative quantum field theory. Like conventional string theories, these theories do not, for example, have a satisfactory non-perturbative definition. Furthermore, despite recent advances at one and two loops, their applicability to all loop orders has yet to be demonstrated. See §7 for a full discussion.

The full (non-anti–self–dual) Einstein and Yang–Mills equations are not integrable and so one does not expect a holomorphic twistor description of their solutions that has the simplicity of their integrable self-dual sectors. It is hoped that the full, non–perturbative implementation of twistor theory in physics is still to be revealed. One set of ideas builds on Penrose's proposal for a role of gravity in quantum collapse of a wave function [178, 180]. This proposal only makes use of Newtonian gravity, but it is the case that in the Newtonian limit the self–dual/anti–self–dual constraint disappears from twistor theory and all physics can be incorporated in the $c \to \infty$ limit of $\mathbb{PT}$ [80], see §8.

- *Does it generalise to higher dimensions?*

  There are by now many generalisations of twistors in dimensions higher than four [191, 124, 125, 126, 168, 194, 37, 200, 69, 91, 209]. One definition takes twistor space to be the projective pure spinors of the conformal group. This definition respects full conformal invariance, and there are analogues of (1.1) for massless fields. However, the (holomorphic) dimension of such twistor spaces goes up quadratically in dimension and become higher than the dimension of the Cauchy data (i.e., one less than the dimension of space–time). Thus solutions to the wave equation and its non–linear generalisations do not map to unconstrained twistor data and this is also reflected in the higher degree of the cohomology classes in higher dimensions that encode massless fields. These do not seem to have straightforward nonlinear extensions.

  Another dimension agnostic generalisation of twistor theory is via *ambitwistors*. Indeed some of the ambitwistor string models described in §7 are only critical in 10 dimensions, relating closely to conventional string theory, although without the higher massive modes.

  Twistor theory has many higher dimensional analogues for space-times of restricted holonomy [191]. The hyper-Kähler case of manifolds of dimension $4k$ with holonomy in $SU(2) \times SP(2k)$ admit a particularly direct generalisation of Penrose's original nonlinear graviton construction and now has wide application across mathematics and physics.

This review celebrates the fifty years of twistor theory since the publica-

tion of the first paper on the subject[1] by Roger Penrose [173]. We apologize to the many researchers whose valuable contributions have been inadvertently overlooked.

2   Twistor space and incidence relation

Twistor theory is particularly effective in dimension four because of an interplay between three isomorphisms. Let M be a real oriented four–dimensional manifold with a metric g of arbitrary signature.

- The Hodge $*$ operator is an involution on two-forms, and induces a decomposition

$$\Lambda^2(T^*M) = \Lambda^2_+(T^*M) \oplus \Lambda^2_-(T^*M) \tag{2.3}$$

   of two-forms into self-dual (SD) and anti-self-dual (ASD) components, which only depends on the conformal class of g.

- Locally there exist complex rank-two vector bundles $\mathbb{S}, \mathbb{S}'$ (spin-bundles) over M equipped with parallel symplectic structures $\varepsilon, \varepsilon'$ such that

$$T_{\mathbb{C}}M \cong \mathbb{S} \otimes \mathbb{S}' \tag{2.4}$$

   is a canonical bundle isomorphism, and

$$g(p_1 \otimes q_1, p_2 \otimes q_2) = \varepsilon(p_1, p_2)\varepsilon'(q_1, q_2) \tag{2.5}$$

   for $p_1, p_2 \in \Gamma(\mathbb{S})$ and $q_1, q_2 \in \Gamma(\mathbb{S}')$. The isomorphism (2.4) is related to (2.3) by

$$\Lambda^2_+ \cong \mathbb{S}'^* \odot \mathbb{S}'^*, \quad \Lambda^2_- \cong \mathbb{S}^* \odot \mathbb{S}^*.$$

- The orthogonal group in dimension four is not simple:

$$SO(4, \mathbb{C}) \cong (SL(2, \mathbb{C}) \times \widetilde{SL}(2, \mathbb{C}))/\mathbb{Z}_2 \tag{2.6}$$

   where $\mathbb{S}$ and $\mathbb{S}'$ defined above are the representation spaces of $SL(2)$ and $\widetilde{SL}(2)$ respectively. There exist three real slices: In the Lorentzian signature $Spin(3, 1) \cong SL(2, \mathbb{C})$ and both copies of $SL(2, \mathbb{C})$ in (2.6) are related by complex conjugation. In the Riemannian signature $Spin(4, 0) = SU(2) \times \widetilde{SU}(2)$. In $(2, 2)$ (also called neutral, or ultra-hyperbolic signature) $Spin(2, 2) \cong SL(2, \mathbb{R}) \times \widetilde{SL}(2, \mathbb{R})$. Only in this signature there exists a notion of real spinors, and as we shall see in §3.2 real twistors.

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[1] See also the programme and slides from the meeting, New Horizons in Twistor Theory in Oxford January 2017 that celebrated this anniversary along with the 85th birthday or Roger Penrose and the 67th of Nick Woodhouse. http://www.maths.ox.ac.uk/groups/mathematical-physics/events/twistors50.

2.1 Incidence relation

The projective twistor space $\mathbb{PT}$ is defined to be $\mathbb{CP}^3 - \mathbb{CP}^1$. The homogeneous coordinates of a twistor are $(Z^0, Z^1, Z^2, Z^3) \sim (\rho Z^0, \rho Z^1, \rho Z^2, \rho Z^3)$, where $\rho \in \mathbb{C}^*$ and $(Z^2, Z^3) \neq (0,0)$. The projective twistor space (which we shall call twistor space from now on) and Minkowski space are linked by the incidence relation

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} ct+z & x+iy \\ x-iy & ct-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \tag{2.7}$$

where $x^\mu = (ct, x, y, z)$ are coordinates of a point in Minkowski space. If two points in Minkowski space are incident with the same twistor, then they are connected by a null line. Let

$$\Sigma(Z, \overline{Z}) = Z^0 \overline{Z^2} + Z^1 \overline{Z^3} + Z^2 \overline{Z^0} + Z^3 \overline{Z^1}$$

be a $(+ + - -)$ Hermitian inner product on the non–projective twistor space $\mathbb{T} = \mathbb{C}^4 - \mathbb{C}^2$. The orientation–preserving endomorphisms of the twistor space which preserve Σ form a group $SU(2,2)$ which is locally isomorphic to the conformal group $SO(4,2)$ of Minkowski space. The twistor space $\mathbb{T}$ is divided into three parts depending on whether Σ is positive, negative or zero. This partition descends to the projective twistor space. In particular the hypersurface

$$\mathcal{PN} = \{[Z] \in \mathbb{PT}, \Sigma(Z, \overline{Z}) = 0\} \subset \mathbb{PT}$$

is preserved by the conformal transformations of the Minkowski space which can be verified directly using (2.7). The five dimensional manifold $\mathcal{PN} \cong S^2 \times \mathbb{R}^3$ is the space of light rays in the Minkowski space. Fixing the coordinates x^μ of a space–time point in (2.7) gives a plane in the non–projective twistor space $\mathbb{C}^4 - \mathbb{C}^2$ or a projective line $\mathbb{CP}^1$ in $\mathbb{PT}$. If the coordinates x^μ are real this line lies in the hypersurface $\mathcal{PN}$. Conversely, fixing a twistor in $\mathcal{PN}$ gives a light–ray in the Minkowski space.

So far only the null twistors (points in $\mathcal{PN}$) have been relevant in this discussion. General points in $\mathbb{PT}$ can be interpreted in terms of the *complexified Minkowski space* $M_\mathbb{C} = \mathbb{C}^4$ where they correspond to α–planes, i. e. null two–dimensional planes with self–dual tangent bi-vector. This, again, is a direct consequence of (2.7) where now the coordinates x^μ are complex:
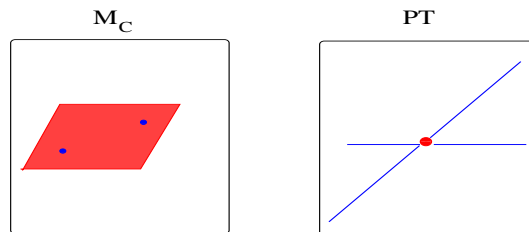

Figure 1. *Twistor incidence relation*

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$$
\begin{array}{rcl}
\text{Complexified space-time } M_{\mathbb{C}} & \longleftrightarrow & \text{Twistor space } \mathbb{PT} \\
\text{Point } p & \longleftrightarrow & \text{Complex line } L_p = \mathbb{CP}^1 \\
\text{Null self-dual } (=\alpha) \text{ two-plane} & \longleftrightarrow & \text{Point.} \\
p_1, p_2 \text{ null separated} & \longleftrightarrow & L_1, L_2 \text{ intersect at one point.}
\end{array}
$$

2.2 Robinson Congruence

The non–null twistors can also be interpreted in the real Minkowski space, but this is somewhat less obvious [173]: The inner product Σ defines a vector space $\mathbb{T}^*$ dual to the non–projective twistor space. Dual twistors are the elements of the projective space $\mathbb{PT}^*$. Consider a twistor $Z \in \mathbb{PT} \setminus \mathcal{PN}$. Its dual $\overline{Z} \in \mathbb{PT}^*$ corresponds to a two–dimensional complex projective plane $\mathbb{CP}^2$ in $\mathbb{PT}$. This holomorphic plane intersects the space of light rays $\mathcal{PN}$ in a real three–dimensional locus corresponding to a three–parameter family of light–rays in the real Minkowski space. The family of light rays representing a non–null twistor is called the Robinson congruence.
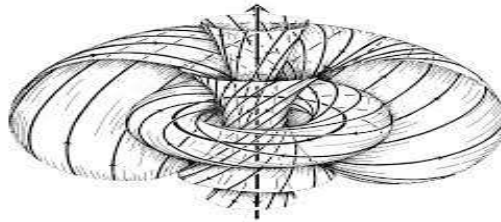
Figure 2. *Robinson congruence of twisting light rays.*

$$
\begin{array}{rcl}
\text{Null ray in } M & \longleftrightarrow & \text{Point in } \mathcal{PN} \\
\text{Robinson congruence } \{\overline{Z}\} \cap \mathcal{PN} & \longleftrightarrow & \text{Point in } \mathbb{PT} \setminus \mathcal{PN}.
\end{array}
$$

The Robinson congruence in Figure 2 is taken from the front cover of [184]. It consists of a system of twisted oriented circles in $\mathbb{R}^3$: a light–ray is represented by a point in $\mathbb{R}^3$ together with an arrow indicating the direction of the ray's motion. It is this twisting property of circles in Figure 2 which gave rise to a term 'twistor' for points of $\mathbb{PT}$. An account of congruences in general relativity which motivated initial progress in twistor theory is given in [134, 189, 190].

2.3 Cohomology

The twistor interpretation of Penrose's contour integral formula (1.1) is as follows. Cover the twistor space $\mathbb{T} = \mathbb{C}^4 - \mathbb{C}^2$ by two open sets: U_0 defined by $Z^2 \neq 0$ and U_1 defined by $Z^3 \neq 0$. Consider a function on the non–projective twistor space $f = f(Z^0, Z^1, Z^2, Z^3)$ which is holomorphic on $U_0 \cap U_1$ and homogeneous of degree -2 in Z^α. Restrict this function to a two–dimensional

plane in $\mathbb{T}$ defined by the incidence relation (2.7) with (x, y, z, t) fixed. This gives rise to an element of the cohomology group[2] $H^1(L_p, \mathcal{O}(-2))$ on the projective twistor space, where $L_p \cong \mathbb{CP}^1$ is the curve corresponding, via the incidence relation (2.7), to a point $p \in M_{\mathbb{C}}$. Integrate the cohomology class along a contour Γ in L_p. This gives (1.1) with $\lambda = Z^3/Z^2$. For example $f = (P_\alpha Z^\alpha)^{-1}(Q_\beta Z^\beta)^{-1}$, where $\alpha, \beta = 0, \ldots, 3$ and (P_α, Q_β) are constant dual twistors gives rise to a fundamental solution to the wave equation (1.2). The Theorem of [81] states that solutions to the wave equation (1.2) which holomorphically extend to a future tube domain in $M_{\mathbb{C}}$ are in one–to–one correspondence with elements of the cohomology group $H^1(\mathbb{PT}, \mathcal{O}(-2))$. This correspondence extents to solutions of zero–rest–mass equations with higher spin, and elements of $H^1(\mathbb{PT}, \mathcal{O}(k))$ where k is an integer. See [81, 108, 184, 218, 121] for further details.

3 Twistors for curved spaces

The twistor space of complexified Minkowski space $M_{\mathbb{C}} = \mathbb{C}^4$ was defined by the incidence relation (2.7) as the space of all α–planes in $\mathbb{C}^4$. Let $(M_{\mathbb{C}}, g)$ be a holomorphic four–manifold with a holomorphic Riemannian metric and a holomorphic volume form. Define an α–surface to be a two–dimensional surface in $M_{\mathbb{C}}$ such that its tangent plane at every point is an α–plane. If the metric g is curved, there will be integrability conditions coming from the Frobenius Theorem for an α–plane to be tangent to a two–dimensional surface.

3.1 The Nonlinear Graviton construction

Define $\mathbb{PT}$ to be the space of α–surfaces ζ in $(M_{\mathbb{C}}, g)$. The Frobenius theorem implies that for $X, Y \in T\zeta \to [X, Y] \in T\zeta$, and there are obstruction in terms of the curvature of g. This gives rise to the Nonlinear Graviton Theorem

Theorem 3.1 (Penrose [176]) *There exists a three–parameter family of α–surfaces in $M_{\mathbb{C}}$ iff the the Weyl tensor of g is anti–self–dual, i.e.*

$$C_{abcd} = -\frac{1}{2}\epsilon_{ab}{}^{pq}C_{cdpq}. \tag{3.8}$$

The anti–self–duality of the Weyl tensor is the property of the whole conformal class

$$[g] = \Omega^2 g, \quad \Omega : M_{\mathbb{C}} \to \mathbb{C}^*$$

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[2]The cohomology group $H^1(\mathbb{CP}^1, \mathcal{O}(k))$ is the space of functions f_{01} holomorphic on $U_0 \cap U_1$ and homogeneous of degree k in coordinates $[Z^2, Z^3]$ modulo addition of coboundaries (functions holomorphic on U_0 and U_1). In a trivialisation over U_0 we represent f_{01} by a holomorphic function f on $\mathbb{C}^*$. In the trivialisation over U_1, f_{01} is represented by $\lambda^{-k}f$, where $\lambda = Z^3/Z^2$. Here, and in the rest of this paper $\mathcal{O}(k)$ denotes a line bundle over $\mathbb{CP}^1$ with transition function λ^{-k} in a trivialisation over U_0. Alternatively it is defined as the $(-k)$th tensor power of the tautological line bundle.

rather than any particular metric. Points in an ASD conformal manifold $(M_\mathbb{C}, [g])$ correspond to rational curves in $\mathbb{PT}$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, and points in $\mathbb{PT}$ correspond to α–surfaces in $M_\mathbb{C}$. The ASD conformal structure on $M_\mathbb{C}$ can be defined in terms of algebraic geometry of curves in twistor space: $\mathbb{PT}$ is three dimensional, so two curves in $\mathbb{PT}$ generically do not intersect. Two points in $M_\mathbb{C}$ are null separated if and only if the corresponding curves in $\mathbb{PT}$ intersect at one point.

Theorem 3.2 (Penrose [176]) *Let $M_\mathbb{C}$ be a moduli space of all rational curves with the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ in some complex three–fold $\mathbb{PT}$. Then $M_\mathbb{C}$ is a complex four–fold with a holomorphic conformal metric with anti–self–dual curvature. Locally all ASD holomorphic conformal metrics arise from some $\mathbb{PT}$.*

More conditions need to be imposed on $\mathbb{PT}$ if the conformal structure contains a Ricci-flat metric. In this case there exists a holomorphic fibration $\mu : \mathbb{PT} \to \mathbb{CP}^1$ with $\mathcal{O}(2)$-valued symplectic form on the fibres. Other curvature conditions (ASD Einstein [211, 109, 142, 5, 120], Hyper–Hermitian [131, 73], scalar–flat Kähler [187], null Kähler [74]) can also be encoded in terms of additional holomorphic structures on $\mathbb{PT}$. Some early motivation for Theorem 3.2 came from complex general relativity, and theory of $\mathcal{H}$–spaces. See [170, 186].

3.2 Reality conditions

The real ASD conformal structures are obtained by introducing an involution on the twistor space. If the conformal structure has Lorentizian signature, then the anti–self–duality implies vanishing of the Weyl tensor, and thus g is conformally flat. This leaves two possibilities: Riemannian and neutral signatures. In both cases the involutions act on the twistor lines, thus giving rise to maps from $\mathbb{CP}^1$ to $\mathbb{CP}^1$: the antipodal map which in stereographic coordinates is given by $\lambda \to -1/\overline{\lambda}$, or a complex conjugation which swaps the lower and upper hemispheres preserving the real equator. The antipodal map has no fixed points and corresponds to the positive–definite conformal structures. The conjugation corresponds to the neutral case.

In the discussion below we shall make use of the double fibration picture

$$M_\mathbb{C} \xleftarrow{\ r\ } \mathcal{F} \xrightarrow{\ q\ } \mathbb{PT}, \tag{3.9}$$

where the five–complex–dimensional correspondence is defined by

$$\mathcal{F} = \mathbb{PT} \times M_\mathbb{C}|_{\zeta \in L_p} = M_\mathbb{C} \times \mathbb{CP}^1$$

where L_p is the line in $\mathbb{PT}$ that corresponds to $p \in M_\mathbb{C}$ and $\zeta \in \mathbb{PT}$ lies on L_p. The space $\mathcal{F}$ can be identified with a projectivisation $\mathbb{PS}'$ of the spin bundle $\mathbb{S}' \to M_\mathbb{C}$. It is equipped with a rank-2 distribution, the *twistor distribution*, which at a given point (p, λ) of $\mathcal{F}$ is spanned by horizontal lifts of vectors

spanning α–surface at $p \in M_\mathbb{C}$. The normal bundle to L_p consists of vectors tangent to p horizontally lifted to $T_{(p,\lambda)}\mathcal{F}$ modulo the twistor distribution $\mathcal{D}$. We have a sequence of sheaves over $\mathbb{CP}^1$

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathbb{C}^4 \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow 0.$$

Using the abstract index notation [184] (so that, for example, $\pi^{A'}$ denotes a section of $\mathbb{S}'$, and no choice of a local frame or coordinates is assumed) the map $\mathbb{C}^4 \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1)$ is given by $V^{AA'} \longrightarrow V^{AA'}\pi_{A'}$. Its kernel consists of vectors of the form $\pi^{A'}\rho^A$ with $\rho \in \mathbb{S}$ varying. The twistor distribution is therefore $\mathcal{D} = \mathcal{O}(-1) \otimes \mathbb{S}$ and so there is a canonical $L_A \in \Gamma(D \otimes \mathcal{O}(1) \otimes \mathbb{S})$, given by $L_A = \pi^{A'}\nabla_{AA'}$, where $A = 0, 1$.

- **Euclidean case.** The conjugation $\sigma : \mathbb{S}' \to \mathbb{S}'$ given by $\sigma(\pi_{0'}, \pi_{1'}) = (\overline{\pi_{1'}}, -\overline{\pi_{0'}})$ descends from $\mathbb{S}'$ to an involution $\sigma : \mathbb{PT} \to \mathbb{PT}$ such that $\sigma^2 = -\mathrm{Id}$. The twistor curves which are preserved by σ form a four–real parameter family, thus giving rise to a real four–manifold $M_\mathbb{R}$. If $\zeta \in \mathbb{PT}$ then ζ and $\sigma(\zeta)$ are connected by a unique *real curve*. The real curves do not intersect as no two points are connected by a null geodesic in the positive definite case. Therefore there exists a fibration of the twistor space $\mathbb{PT}$ over a real four–manifold $M_\mathbb{R}$. A fibre over a point $p \in M_\mathbb{R}$ is a copy of a $\mathbb{CP}^1$. The fibration is not holomorphic, but smooth.

  In the Atiyah–Hitchin–Singer [18] version of the correspondence the twistor space of the positive definite metric is a real six–dimensional manifold identified with the projective spin bundle $P(\mathbb{S}') \to M_\mathbb{R}$.

  Given a conformal structure $[g]$ on $M_\mathbb{R}$ one defines an almost–complex–structure on $P(\mathbb{S}')$ by declaring

  $$\{\pi^{A'}\nabla_{AA'}, \partial/\partial\overline{\lambda}\}$$

  to be the anti–holomorphic vector fields in $T^{0,1}(P(\mathbb{S}'))$.

  Theorem 3.3 (Atiyah–Hitchin–Singer [18]) *The six–dimensional almost–complex manifold*

  $$P(\mathbb{S}') \to M_\mathbb{R}$$

  parametrises almost–complex–structures in $(M_\mathbb{R}, [g])$. Moreover $P(\mathbb{S}')$ is complex iff $[g]$ is ASD.

- **Neutral case.** The spinor conjugation $\sigma : \mathbb{S}' \to \mathbb{S}'$ given by $\sigma(\pi_{0'}, \pi_{1'}) = (\overline{\pi_{0'}}, \overline{\pi_{1'}})$ allows an invariant decomposition of a spinor into its real and imaginary part, and thus definition of *real α-surfaces* [224, 74].

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In general $\pi = \mathrm{Re}(\pi) + i\mathrm{Im}(\pi)$, and the correspondence space $\mathcal{F} = P(\mathbb{S}')$ decomposes into two open sets

$$
\begin{aligned}
\mathcal{F}_+ &= \{(p,[\pi]) \in \mathcal{F}; \mathrm{Re}(\pi_{A'})\mathrm{Im}(\pi^{A'}) > 0\} = M_{\mathbb{R}} \times D_+, \\
\mathcal{F}_- &= \{(p,[\pi]) \in \mathcal{F}; \mathrm{Re}(\pi_{A'})\mathrm{Im}(\pi^{A'}) < 0\} = M_{\mathbb{R}} \times D_-,
\end{aligned}
$$

where $D_\pm$ are two copies of a Poincare disc. These sets are separated by a real correspondence space $\mathcal{F}_0 = M_{\mathbb{R}} \times \mathbb{RP}^1$. The correspondence spaces $\mathcal{F}_\pm$ have the structure of a complex manifold in a way similar to the AHS Euclidean picture. There exists an $\mathbb{RP}^1$ worth of real α–surfaces through each point in $M_{\mathbb{R}}$, and real twistor distribution consisting of vectors tangent to real α–surfaces defines a foliation of $\mathcal{F}_0$ with quotient $\mathbb{PT}_0$ which leads to a double fibration:

$$
M_{\mathbb{R}} \xleftarrow{\ r\ } \mathcal{F}_0 \xrightarrow{\ q\ } \mathbb{PT}_0.
$$

The twistor space $\mathbb{PT}$ is a union of two open subsets $\mathbb{PT}_+ = (\mathcal{F}_+)$ and $\mathbb{PT}_- = (\mathcal{F}_-)$ separated by a three-dimensional real boundary $\mathbb{PT}_0$.

These reality conditions are relevant in the twistor approach to integrable systems (see §6), integral geometry, twistor inspired computations of scattering amplitudes (see §7), as well as recent applications [16] of the Index Theorem [23] which do not rely on positivity of the metric. The discussion in this subsection has assumed real analyticity of $M_{\mathbb{R}}$. The approach of LeBrun and Mason [148] based on holomorphic discs can weaken this assumption.

3.3   Kodaira Deformation Theory

One way of obtaining complex three–manifolds with four–parameter families of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ curves comes from the Kodaira deformation theory applied to $\mathbb{PT} = \mathbb{CP}^3 - \mathbb{CP}^1$
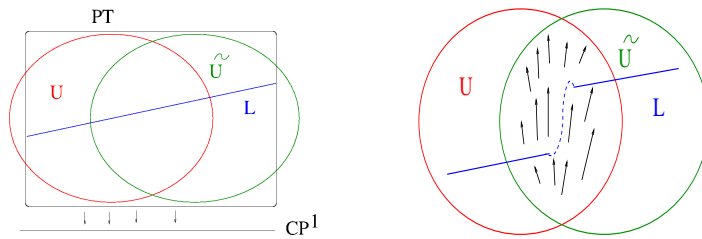
Figure 3. *Curvature on $(M_{\mathbb{C}}, g)$ corresponds to deformations of $\mathbb{PT}$*

The normal bundle $N(L_p) \equiv T(PT)|_{L_p}/TL_p \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ satisfies

$$
H^1(L_p, N(L_p)) = 0.
$$

The Kodaira theorems [136] imply that there exist infinitesimal deformations of the complex structure of $\mathbb{PT}$ which preserve the four parameter family $M_{\mathbb{C}}$

of $\mathbb{CP}^1$s, as well as the type of their normal bundle. Moreover this deformed family admits an isomorphism

$$H^0(L_p, N(L_p)) \cong T_p M_{\mathbb{C}}$$

identifying tangent vectors to $M_{\mathbb{C}}$ with pairs of linear homogeneous polynomials in two variables. This identification allows to construct a conformal structure on $M_{\mathbb{C}}$ arising from a quadratic condition that both polynomials in each pair have a common zero. There are some examples of ASD Ricci flat metrics arising from explicit deformations - see [121, 75]. A method of constructing such examples was pioneered by George Sparling.

3.3.1 Twistor solution to the holonomy problem.

The Kodaira approach to twistor theory has given rise to a complete classification of manifolds with exotic holonomy groups (holonomy groups of affine connections which are missing from Berger's list). The first landmark step was taken by Robert Bryant [49] who generalised the Kodaira theorems and the twistor correspondence to Legendrian curves. Complex contact three–folds with 4–parameter family of Legendrian rational curves with normal bundle $\mathcal{O}(2) \oplus \mathcal{O}(2)$ correspond to four manifolds $M_{\mathbb{C}}$ such that $T_p M_{\mathbb{C}} \cong \mathbb{C}^2 \odot \mathbb{C}^2 \odot \mathbb{C}^2$ and there exists a torsion–free connection with holonomy group $GL(2, \mathbb{C})$. The theory was extended by Merkulov to allow Legendrian deformations of more general submanifolds [165, 166]. This work lead to a complete classification by Merkulov and Schwachhofer [167].

3.4 Gravitational Instantons

Gravitational instantons are solutions to the Einstein equations in Riemannian signature which give complete metrics whose curvature is concentrated in a finite region of a space-time. The non–compact gravitational instantons asymptotically 'look like' flat space. While not all gravitational instantons are (anti)–self–dual (e.g. the Euclidean Schwarzchild solution is not) most of them are, and therefore they arise from Theorems 3.2 and 3.3.

- There exists a large class of gravitational instantons which depend on a harmonic function on $\mathbb{R}^3$:

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}(d\tau + A)^2, \quad \text{where} \quad *_3 dV = dA. \quad (3.10)$$

  where V and A are a function and a one–form respectively which do not depend on τ. This is known as the Gibbons-Hawking ansatz [96]. The resulting metrics are hyper–Kähler (or equivalently anti–self–dual and Ricci flat). The Killing vector field $K = \partial/\partial \tau$ is tri–holomorphic - it preserves the sphere of Kähler forms of g. It gives rise to a holomorphic vector field on the corresponding twistor space which preserves the $\mathcal{O}(2)$–valued symplectic structure on the fibres of $\mathbb{PT} \to \mathbb{CP}^1$. Therefore

there exists an associated $\mathcal{O}(2)$ valued Hamiltonian, and the Gibbons–Hawking twistor space admits a global fibration over the total space of $\mathcal{O}(2)$. Conversely, any twistor space which admits such fibration leads to the Gibbons–Hawking metric on the moduli space of twistor curves [207, 115].

- An example of a harmonic function in (3.10) which leads to the Eguchi–Hanson gravitational instanton is $V = |\mathbf{r} + \mathbf{a}|^{-1} + |\mathbf{r} - \mathbf{a}|^{-1}$. It is asymptotically locally Euclidean (ALE), as it approaches $\mathbb{R}^4/\mathbb{Z}^2$ for large $|\mathbf{r}|$. The corresponding twistor space has been constructed by Hitchin [107].

- A general gravitational instanton is called ALE if it approaches $\mathbb{R}^4/\Gamma$ at infinity, where Γ is a discrete subgroup of $SU(2)$. Kronheimer [138, 139] has constructed ALE spaces for finite subgroups

$$A_k, D_k, E_6, E_7, E_8$$

of $SU(2)$. In each case the twistor space is a three–dimensional hyper–surface

$$F(X, Y, Z, \lambda) = 0$$

in the rank three bundle $\mathcal{O}(p) \oplus \mathcal{O}(q) \oplus \mathcal{O}(r) \to \mathbb{CP}^1$, for some integers (p, q, r), where F is a singularity resolution of one of the Klein polynomials corresponding to the Platonic solids

$$XY - Z^k = 0, X^2 + Y^2 Z + Z^k = 0, \ldots, X^2 + Y^3 + Z^5 = 0 \quad \text{(isocahedron.)}$$

The twistor spaces of these ALE instantons admit a holomorphic fibration over the total space of $\mathcal{O}(2n)$ for some $n \geq 1$. In case of A_k one has $n = 1$ and the metric belongs to the Gibbons–Hawking class. In the remaining cases $n > 1$, and the resulting metrics do not admit any tri–holomporphic Killing vector. They do however admit hidden symmetries (in the form of tri–holomorphic Killing spinors), and arise from a generalised Legendre transform [150, 77, 41, 45].

- There are other types of gravitational instantons which are not ALE, and are characterised by different volume growths of a ball of the given geodesic radius [66, 106]. They are ALF (asymptotically locally flat), and 'inductively' named ALG, ALH spaces. Some ALF spaces arise from the Gibbons–Hawking ansatz (3.10) where

$$V = 1 + \sum_{m=1}^{N} \frac{1}{|\mathbf{x} - \mathbf{x}_m|}$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are fixed points in $\mathbb{R}^3$ (the corresponding twistor spaces are known), but others do not. In [65] some progress has been made in constructing twistor spaces for D_k ALF instantons, but finding the twistor spaces, or explicit local forms for the remaining cases is a open problem.

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There also exist compact examples of Riemannian metrics with ASD conformal curvature. The round S^4 and $\mathbb{CP}^2$ with the Fubini–Study metric are explicit examples where the ASD metric is also Einstein with positive Ricci scalar. A Ricci–flat ASD metric is known to exist on the $K3$ surface, but the explicit formula for the metric is not known.

LeBrun has proven [147] that there are ASD metrics with positive scalar curvature on any connected sum $N\overline{\mathbb{CP}}^2$ of reversed oriented complex projective planes. This class, together with a round four–sphere exhaust all simply connected possibilities. The corresponding twistor spaces can be constructed in an algebraic way. The strongest result belongs to Taubes [202]. If M is any compact oriented smooth four–manifold, then there exists some $N_0 > 0$ such that

$$M_N = M \# N\overline{\mathbb{CP}}^2$$

admits an ASD metric for any $N \geq N_0$.

4 Local Twistors

There exists at least three definitions of a twistor which agree in a four–dimensional flat space. The first, twistors as α–planes, was used in the last section, where its curved generalisation lead to the Nonlinear Graviton construction and anti–self–duality. The second, twistors as spinors for the conformal group, relies heavily on maximal symmetry and so does not generalise to curved metrics. The last, twistors as solutions to the twistor equation, leads to interesting notions of a local twistor bundle and a local twistor transport [182, 70, 184, 196] which we now review.

We shall make use of the isomorphism (2.4). Let Z^α be homogeneous coordinates of a twistor as in §2. Set $Z^\alpha = (\omega^A, \pi_{A'})$. Differentiating the incidence relation (2.7) yields

$$\nabla_{A'}{}^{(A}\omega^{B)} = 0, \tag{4.11}$$

where $\nabla_{A'}{}^A = \epsilon^{AB}\nabla_{BA'}$, and $\epsilon^{AB} = -\epsilon^{BA}$ is a (chosen) symplectic forom on $\mathbb{S}^*$ used to raise and lower indices.

The space–time coordinates (x, y, z, t) are constants of integration resulting from solving this equation on $M_{\mathbb{C}}$. Let us consider (4.11) on a general curved four–manifold, where it is called *the twistor equation.* It is conformally invariant under the transormations of the metric $g \to \Omega^2 g$. The prolongation of the twistor equation leads to a connection on a rank–four vector bundle $\mathbb{S} \otimes \mathcal{E}[1] \oplus \mathbb{S}'$ called the *local twistor bundle*. Here $\mathcal{E}[k]$ denotes a line bundle of conformal densities of weight k. This connection also called the *local twistor transport*, and is given by [70]

$$\mathcal{D}_a \left(\begin{array}{c} \omega^B \\ \pi_{B'} \end{array} \right) = \left(\begin{array}{c} \nabla_a \omega^B - \epsilon_A{}^B \pi_{A'} \\ \nabla_a \pi_{B'} - \mathrm{P}_{ABA'B'} \omega^B \end{array} \right),$$

where P_{ab} is the Schouten tensor of conformal geometry given by

$$P_{ab} = \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}.$$

The holonomy of the local twistor transport obstructs existence of global twistors on curved four manifolds (all local normal forms of Lorentzian metrics admitting solutions to (4.11) have been found in [149]).

The tractor bundle is isomorphic to the exterior square of the local twistor bundle. It is a rank–six vector bundle $\mathbf{T} = \mathcal{E}[1] \oplus T^*M \oplus \mathcal{E}[-1]$, and its connection induced from the local twistor transform is

$$\mathcal{D}_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + P_{ab}\sigma + g_{ab}\rho \\ \nabla_a \rho - P_a{}^b \mu_b \end{pmatrix}. \tag{4.12}$$

This connection does not arise from a metric, but is related to a pull back of the Levi–Civita connection of the so-called ambient metric to a hypersurface. See [88, 59] as well as [100] for discussion of the ambient construction.

The point about the connection (4.12) is that it also arises as a prolongation connection for the conformal to Einstein equation

$$(\nabla_a \nabla_b + P_{ab})_0 \sigma = 0, \tag{4.13}$$

where $(\dots)_0$ denotes the trace–free part. If σ satisfies (4.13) where ∇_a and P_{ab} are computed from g, then $\sigma^{-2}g$ is Einstein [146]. Therefore the holonomy of (4.12) leads to obstructions for an existence of an Einstein metric in a given conformal class [30, 146, 99, 78]. The Bach tensor is one of the obstructions arising from a requirement that a parallel tractor needs to be annihilated by the curvature of (4.12) and its covariant derivatives.

Conformal geometry is a particular example of a parabolic geometry - a curved analog of a homogeneous space G/P which is the quotient of a semi-simple Lie group G by a parabolic subgroup P. Other examples include projective, and CR geometries. All parabolic geometries admit tractor connections. See [60] for details of these construction, and [101, 98, 84] where conformally invariant differential operators have been constructed. Examples of such operators are the twistor operator $\omega_A \to \nabla_{A'(A}\omega_{B)}$ underlying (4.11) and the operator acting on $\mathrm{Sym}^4(\mathbb{S}^*)$

$$C_{ABCD} \to (\nabla^C_{(A'}\nabla^D_{B')} + P^{CD}_{A'B'})C_{ABCD}.$$

This operator associates the conformally invariant Bach tensor to the anti–self–dual Weyl spinor.

5   Gauge Theory

The full second–order Yang Mills equations on $\mathbb{R}^4$ are not integrable, and there is no twistor construction encoding their solutions in an unconstrained

holomorphic data on $\mathbb{PT}$ - there do exist *ambitwistor constructions* [220, 127, 105] in terms of formal neighbourhods of spaces of complex null geodesics, but they do not lead to any solution generation techniques. As in the case of gravity, the anti–self–dual sub–sector can be described twistorialy, this time in terms of holomorphic vector bundles over $\mathbb{PT}$ rather than deformations of its complex structures.

5.1 ASDYM and the Ward Correspondence

Let $A \in \Lambda^1(\mathbb{R}^4) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is some Lie algebra, and let

$$F = dA + A \wedge A.$$

The anti–self–dual Yang–Mills equations are

$$*F = -F \tag{5.14}$$

where $* : \Lambda^2 \to \Lambda^2$ is the Hodge endomorphism depending on the flat metric and the orientation on $\mathbb{R}^4$. These equations together with the Bianchi identity $DF := dF + [A, F] = 0$ imply the full Yang–Mills equations $D * F = 0$.

Let us consider (5.14) on the complexified Minkowski space $M_{\mathbb{C}} = \mathbb{C}^4$ with a flat holomorphic metric and a holomorphic volume form. Equations (5.14) are then equivalent to the vanishing of F on each α–plane in $M_{\mathbb{C}}$. Therefore, given $\zeta \in \mathbb{PT}$, there exists a vector space of solutions to

$$l^a D_a \Phi = 0, \quad m^a D_a \Phi = 0, \quad \text{where} \quad \zeta = \mathrm{span}\{l, m\} \in TM_{\mathbb{C}}. \tag{5.15}$$

The converse of this construction is also true, and leads to a twistor correspondence for solutions to ASDYM equations

Theorem 5.1 (Ward [210]) *There is a one-to-one correspondence between:*

1. *Gauge equivalence classes of ASD connections on $M_{\mathbb{C}}$ with the gauge group $G = GL(n, \mathbb{C})$,*

2. *Holomorphic rank–n vector bundles E over twistor space $\mathbb{PT}$ which are trivial on each degree one section of $\mathbb{PT} \to \mathbb{CP}^1$.*

The splitting of the patching matrix F_E for the bundle E into a product of matrices holomorphic on U_0 and U_1 is the hardest part of this approach to integrable PDEs. When the Ward correspondence is reduced to lower dimensional PDEs as in §6, the splitting manifests itself as the Riemann–Hilbert problem in the dressing method.

To obtain real solutions on $\mathbb{R}^4$ with the gauge group $G = SU(n)$ the bundle must be compatible with the involution σ preserving the Euclidean slice (compare §33.2). This comes down to $\det F_E = 1$, and

$$F_E{}^*(\zeta) = F_E(\sigma(\zeta)),$$

where $*$ denotes the Hermitian conjugation, and $\sigma : \mathbb{PT} \to \mathbb{PT}$ is the anti–holomorphic involution on the twistor space which restricts to an antipodal map on each twistor line. See [218, 223].

5.2   Lax pair

Consider the complexified Minkowski space $M_{\mathbb{C}} = \mathbb{C}^4$ with coordinates $w, z, \tilde{w}, \tilde{z}$, and the metric and orientation

$$g = 2(dz d\tilde{z} - dw d\tilde{w}), \quad \text{vol} = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}.$$

The Riemannian reality conditions are recovered if $\tilde{z} = \overline{z}, \tilde{w} = -\overline{w}$, and the neutral signature arises if all four coordinates are taken to be real. The ASDYM equations (5.14) arise as the compatibility condition for an overdetermined linear system $L\Psi = 0, M\Psi = 0$, where

$$L = D_{\tilde{z}} - \lambda D_w, \qquad M = D_{\tilde{w}} - \lambda D_z, \tag{5.16}$$

where $D_\mu = \partial_\mu + [A_\mu, \cdot]$, and $\Psi = \Psi(w, z, \tilde{w}, \tilde{z}, \lambda)$ is the fundamental matrix solution. Computing the commutator of the Lax pair (L, M) yields

$$[L, M] = F_{\tilde{z}\tilde{w}} - \lambda(F_{w\tilde{w}} - F_{z\tilde{z}}) + \lambda^2 F_{wz} = 0,$$

and the vanishing of the coefficients of various powers of λ gives (5.14). The geometric interpretation of this is as follows: for each value of $\lambda \in \mathbb{CP}^1$ the vectors $l = \partial_{\tilde{z}} - \lambda\partial_w, m = \partial_{\tilde{w}} - \lambda\partial_z$ span a null plane in $M_{\mathbb{C}}$ which is self–dual in the sense that $\omega = \text{vol}(l, m, \ldots, \ldots)$ satisfies $*\omega = \omega$. The condition (5.14) takes the equivalent form $\omega \wedge F = 0$, thus F vanishes on all α planes. For a given YM potential A, the lax pair (5.16) can be expressed as $L = l + l \lrcorner A, M = m + m \lrcorner A$.

5.3   Instantons

Instantons, i. e. solutions to ASDYM such that

$$\int_{\mathbb{R}^4} \text{Tr}(F \wedge *F) < \infty$$

extend from $\mathbb{R}^4$ to S^4. The corresponding vector bundles extend from $\mathbb{PT}$ to $\mathbb{CP}^3$. The holomorphic vector bundles over $\mathbb{CP}^3$ have been extensively studied by algebraic geometers. All such bundles (and thus the instantons) can be generated by the monad construction [19].

One way to construct holomorphic vector bundles is to produce extensions of line bundles, which comes down to using upper-triangular matrices as patching functions. Let E be a rank–two holomorphic vector bundle over $\mathbb{PT}$ which arises as an extension of a line bundle $L \otimes \mathcal{O}(-k)$ by another line bundle $L^* \otimes \mathcal{O}(k)$

$$0 \longrightarrow L \otimes \mathcal{O}(-k) \longrightarrow E \longrightarrow L^* \otimes \mathcal{O}(k) \longrightarrow 0. \tag{5.17}$$

If $k > 1$ then the YM potential A is given in terms of a solution to the linear zero–rest–mass field equations with higher helicity.

Theorem 5.2 (Atiyah–Ward [22]) *Every $SU(2)$ ASDYM instanton over $\mathbb{R}^4$ arises from a holomorphic vector bundle of the form (5.17)*

Advances made in anti–self–dual gauge theory using the twistor methods lead to the studies of moduli spaces of connections. Such moduli spaces continue to play an important role in mathematical physics [153], and gave rise to major advances in the understanding of topology of four–manifolds [71].

5.4   Minitwistors and magnetic monopoles

Another gauge theoretic problem which was solved using twistor methods [110, 111] is the construction of non–abelian magnetic monopoles.

Let (A, ϕ) be a $\mathfrak{su}(n)$–valued one–form and a function respectively on $\mathbb{R}^3$, and let $F = dA + A \wedge A$. The non–abelian monopole equation is a system of non–linear PDEs

$$d\phi + [A, \phi] = *_3 F \tag{5.18}$$

These are three equations for three unknowns as (A, ϕ) are defined up to gauge transformations

$$A \longrightarrow gAg^{-1} - dg\ g^{-1}, \qquad \phi \longrightarrow g\phi g^{-1}, \qquad g = g(x, y, z) \in SU(n) \tag{5.19}$$

and one component of A can always be set to zero.

Following Hitchin [110] define the mini twistor space $\mathcal{Z}$ to be the space of oriented lines in $\mathbb{R}^3$. Any oriented line is of the form $\mathbf{v} + s\mathbf{u}, \quad s \in \mathbb{R}$ where $\mathbf{u}$ is a unit vector giving the direction of the line, and $\mathbf{v}$ is orthogonal to $\mathbf{u}$ and joints the line with some chosen point (say the origin) in $\mathbb{R}^3$. Thus

$$\mathcal{Z} = \{(\mathbf{u}, \mathbf{v}) \in S^2 \times \mathbb{R}^3, \ \mathbf{u}.\mathbf{v} = 0\}.$$

For each fixed $\mathbf{u} \in S^2$ this space restricts to a tangent plane to S^2. The twistor space is the union of all tangent planes – the tangent bundle TS^2 which is also a complex manifold $T\mathbb{CP}^1$.
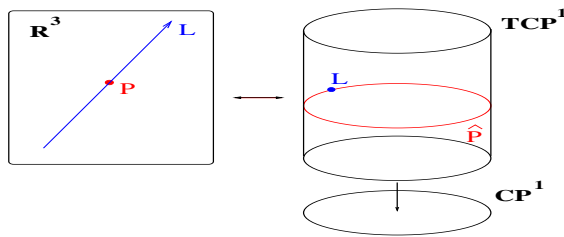
Figure 4. *Minitwistor Correspondence.*

Given (A, ϕ) solve a matrix ODE along each oriented line $\mathbf{x}(s) = \mathbf{v} + s\mathbf{u}$

$$\frac{dV}{ds} + (u^j A_j + i\phi)V = 0.$$

This ODE assigns a complex vector space $\mathbb{C}^n$ to each point of $\mathcal{Z}$, thus giving rise to a complex vector bundle over the mini–twistor space. Hitchin shows

[109] that monopole equation (5.18) on $\mathbb{R}^3$ holds if and only if this vector bundle is holomorphic.

The mini–twistor space of Hitchin can also be obtained as a reduction of the twistor space $\mathbb{PT} = \mathbb{CP}^3 - \mathbb{CP}^1$ by a holomorphic vector field corresponding to a translation in $\mathbb{R}^4$. An analogous reduction of ASDYM on $\mathbb{R}^4$ by a rotation gives nonabelian hyperbolic monopoles [14].

In the next section we shall discuss how more general reductions of $\mathbb{PT}$ give rise to solution generation techniques for lower dimensional integrable systems.

6   Integrable Systems

Most lower dimensional integrable systems arise as symmetry reductions of anti–self–duality equations on $(M, [g])$ in $(2, 2)$ or $(4, 0)$ signature.

The solitonic integrable systems are reductions of ASDYM as their linear systems (Lax pairs) involve matrices. The program of reducing the ASDYM equations to various integrable equations has been proposed and initiated by Ward [213] and fully implemented in the monograph [158]. The dispersionless integrable systems are reductions of anti–self–duality equations on a conformal structure [76, 75]. A unified approach combining curved backgrounds with gauge theory has been developed by Calderbank [57].

In both cases the reductions are implemented by assuming that the Yang–Mills potential or the conformal metric are invariant with respect to a subgroup of the full group of conformal symmetries. Conformal Killing vectors on $M_{\mathbb{C}}$ correspond to holomorphic vector fields on $\mathbb{PT}$. The resulting reduced system will admit a (reduced) Lax pair with a spectral parameter coming from the twistor α–plane distribution. It will be integrable by a reduced twistor correspondence of Theorem 3.2 or Theorem 5.1.

6.1   Solitonic equations

The general scheme and classification of reductions of ASDYM on the complexified Minkowski space involves a choice of subgroup of the complex conformal group $PGL(4, \mathbb{C})$, a real section (hyperbolic equations arise from ASDYM in neutral signature), a gauge group and finally canonical forms of Higgs fields.

We have already seen one such symmetry reduction: ASDYM on $\mathbb{R}^4$ invariant under a one–dimensional group of translations generated by $K = \partial/\partial x^4$ reduce to the non–abelian monopole equation (5.18). The Higgs field on $\mathbb{R}^3$ is related to the gauge potential A on $\mathbb{R}^4$ by $\phi = K \lrcorner A$. The analogous reduction from $\mathbb{R}^{2,2}$ leads to Ward's integrable chiral model on $\mathbb{R}^{2,1}$ [214]. It is solved by a minitwistor construction, where the minitwistor space $\mathcal{Z}$ from the description of monopoles is instead equipped with an anti–holomorphic involution fixing a real equator on each twistor line [215]. The solitonic

solutions are singled out by bundles which extend to compactified mini–twistor spaces [217, 185]. Below we give some examples of reductions to two and one dimensions.

- Consider the $SU(2)$ ASDYM in neutral signature and choose a gauge $A_{\tilde{z}} = 0$. Let $T_\alpha, \alpha = 1, 2, 3$ be two by two constant matrices such that $[T_\alpha, T_\beta] = -\epsilon_{\alpha\beta\gamma}T_\gamma$. Then ASDYM equations are solved by the ansätze

$$A_w = 2\cos\phi\, T_1 + 2\sin\phi\, T_2, \quad A_{\tilde{w}} = 2T_1, \quad A_z = \partial_z\phi\, T_3$$

  provided that $\phi = \phi(z, \tilde{z})$ satisfies

$$\phi_{z\tilde{z}} + 4\sin\phi = 0$$

  which is the Sine–Gordon equation. Analogous reductions of ASDYM with gauge group $SL(3, \mathbb{R})$ or $SU(2, 1)$ lead to the Tzitzeica equations and other integrable systems arising in affine differential geometry [75]. A general reduction by two translations on $\mathbb{R}^4$ lead to Hitchin's self–duality equations which exhibit conformal invariance and thus extend to any Riemann surface [112].

- Mason and Sparling [156] have shown that any reduction to the ASDYM equations on $\mathbb{R}^{2,2}$ with the gauge group $SL(2, \mathbb{R})$ by two translations exactly one of which is null is gauge equivalent to either the KdV or the Nonlinear Schrodinger equation depending on whether the Higgs field corresponding to the null translation is nilpotent or diagonalisable. In [157] and [158] this reduction has been extended to integrable hierarchies.

- By imposing three translational symmetries one can reduce ASDYM to an ODE. Choose the Euclidean reality condition, and assume that the YM potential is independent on $x^j = (x^1, x^2, x^3)$.

  Select a gauge $A_4 = 0$, and set $A_j = \Phi_j$, where the Higgs fields Φ_j are real $\mathfrak{g}$–valued functions of $x^4 = t$. The ASDYM equations reduce to the Nahm equations

$$\dot{\Phi}_1 = [\Phi_2, \Phi_3], \qquad \dot{\Phi}_2 = [\Phi_3, \Phi_1], \qquad \dot{\Phi}_3 = [\Phi_1, \Phi_2].$$

  These equations admit a Lax representation which comes from taking a linear combination of L and M in (5.16). Let

$$B(\lambda) = (\Phi_1 + i\Phi_2) + 2\Phi_3\lambda - (\Phi_1 - i\Phi_2)\lambda^2.$$

  Then

$$\begin{aligned} \dot{B} &= [\Phi_2 - i\Phi_1, \Phi_3] + 2[\Phi_1, \Phi_2]\lambda - [\Phi_2 + i\Phi_1, \Phi_3]\lambda^2 \\ &= [B, -i\Phi_3 + i(\Phi_1 - i\Phi_2)\lambda] \\ &= [B, C], \qquad \text{where} \quad C = -i\Phi_3 + i(\Phi_1 - i\Phi_2)\lambda. \end{aligned}$$

The Nahm equations with the group of volume–preserving diffeomorphism of some three-manifold as the gauge group are equivalent to ASD vacuum equations [13].

- Reductions of ASDYM by three–dimensional abelian subgroups of the complexifed conformal group $PGL(4, \mathbb{C})$ lead to all six Painlevé equations [158]. The coordinate–independent statement of the Painleve property for ASDYM was first put forward by Ward [212]: If a solution of ASDYM on $M_{\mathbb{C}} = \mathbb{C}^4$ has a non- characteristic singularity, then that singularity is at worst a pole. Another twistor approach to the Painlevé equation is based on $SU(2)$–invariant anti–self–dual conformal structures [205, 113, 164].

6.2 Dispersionless systems and Einstein–Weyl equations

There is a class of integrable systems in 2+1 and three dimensions which do not fit into the framework described in the last section. They do not arise from ASDYM and there is no finite–dimensional Riemann–Hilbert problem which leads to their solutions. These dispersionless integrable systems admit Lax representations which do not involve matrices, like (5.16), but instead consist of vector fields.

Given a four–dimensional conformal structure $(M, [g])$ with a non-null conformal Killing vector K, the three–dimensional space $\mathcal{W}$ of trajectories of K inherits a conformal structure $[h]$ represented by a metric

$$h = g - \frac{K \otimes K}{|K^2|}.$$

The ASD condition on $[g]$ results in an additional geometrical structure on $(\mathcal{W}, [h])$; it becomes an Einstein-Weyl space [130]. There exists a torsion–free connection D which preserves $[h]$ in the sense that

$$Dh = \omega \otimes h \tag{6.20}$$

for some one–form ω, and the symmetrised Ricci tensor of D is proportional to $h \in [h]$. These are the Einstein–Weyl equations [62]. They are conformally invariant: If $h \longrightarrow \Omega^2 h$ then $\omega \longrightarrow \omega + 2d(\log \Omega)$.

Most known dispersionless integrable systems in $2 + 1$ and 3 dimensions arise from the EW equations. Consult [75, 57, 86] for the complete list. Here we shall review the twistor picture, and examples of integrable reductions.

Theorem 6.1 (Hitchin [109]) *There is a one–to–one correspondence between solutions to Einstein–Weyl equations in three dimensions, and two–dimensional complex manifolds $\mathcal{Z}$ admitting a three parameter family of rational curves with normal bundle $\mathcal{O}(2)$.*
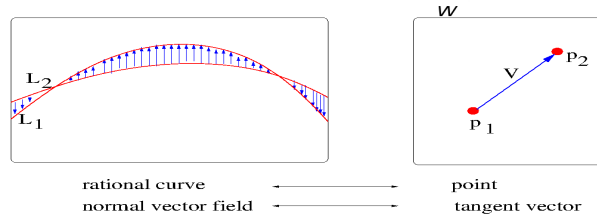
Figure 5. *Einstein–Weyl twistor correspondence.*

In this twistor correspondence the points of $\mathcal{W}$ correspond to rational $\mathcal{O}(2)$ curves in the complex surface $\mathcal{Z}$, and points in $\mathcal{Z}$ correspond to null surfaces in $\mathcal{W}$ which are totally geodesic with respect to the connection D.

To construct the conformal structure $[h]$ define the null vectors at p in $\mathcal{W}$ to be the sections of the normal bundle $N(L_p)$ vanishing at some point to second order. Any section of $\mathcal{O}(2)$ is a quadratic polynomial, and the repeated root condition is given by the vanishing of its discriminant. This gives a quadratic condition on $T\mathcal{W}$.

To define the connection D, let a direction at $p \in \mathcal{W}$ be a one–dimensional space of sections of $\mathcal{O}(2)$ which vanishing at two points ζ_1 and ζ_2 on a line L_p. The one–dimensional family of twistor $\mathcal{O}(2)$ curves in $\mathcal{Z}$ passing through ζ_1 and ζ_2 gives a geodesic in $\mathcal{W}$ in a given direction. The limiting case $\zeta_1 = \zeta_2$ corresponds to geodesics which are null with respect to $[h]$ in agreement with (6.20). The special surfaces in $\mathcal{W}$ corresponding to points in $\mathcal{Z}$ are totally geodesic with respect to the connection D. The integrability conditions for the existence of totally geodesic surfaces is equivalent to the Einstein–Weyl equations [62].

The dispersionless integrable systems can be encoded in the twistor correspondence of Theorem 6.1 if the twistor space admits some additional structures.

- If $\mathcal{Z}$ admits a preferred section of $\kappa^{-1/2}$, where κ is the canonical bundle of $\mathcal{Z}$, then there exist coordinates (x, y, t) and a function u on $\mathcal{W}$ such that
$$h = e^u(dx^2 + dy^2) + dt^2, \quad \omega = 2u_t dt$$
and the EW equations reduce to the $SU(\infty)$ Toda equation [216, 147]
$$u_{xx} + u_{yy} + (e^u)_{tt} = 0.$$

  This class of EW spaces admits both Riemannian and Lorentzian sections (for the later replace t by it or x by ix), which corresponds to two possible real structures on $\mathcal{Z}$. It can be characterised on $\mathcal{W}$ by the existence of twist–free shear–free geodesic congruence [206, 58].

- If $\mathcal{Z}$ admits a preferred section of $\kappa^{-1/4}$, then there exist coordinates (x, y, t) and a function u on $\mathcal{W}$ such that
$$h = dy^2 - 4dxdt - 4udt^2, \quad \omega = -4u_x dt \qquad (6.21)$$

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and the EW equations reduce to the dispersion-less Kadomtsev-Petviashvili (dKP) equation [76]

$$(u_t - uu_x)_x = u_{yy}.$$

This class of EW spaces can be real only in the Lorentzian signature. The corresponding real structure on $\mathcal{Z}$ is an involution which fixes an equator on each $\mathbb{CP}^1$ and interchanges the upper and lower hemisphere. The vector field $\partial/\partial x$ is null and covariantly constant with respect to the Weyl connection, and with weight $1/2$. This vector field gives rise to a parallel real weighted spinor, and finally to a preferred section of $\kappa^{-1/4}$. Conversely, any Einstein–Weyl structure which admits a covariantly constant weighted vector field is locally of the form (6.21) for some solution u of the dKP equation.

The most general Lorentzian Einstein–Weyl structure corresponds [79] to the Manakov–Santini system [152]. Manakov and Santini have used a version of the non–linear Riemann–Hilbert problem and their version of the inverse scattering transform to give an analytical description of wave breaking in $2 + 1$ dimensions. It would be interesting to put their result in the twistor framework. The inverse scattering transform of Manakov and Santini is intimately linked to the Nonlinear Graviton construction. The coordinate form of the general conformal anti–self–duality equation [79] gives the master dispersionless integrable system in $(2 + 2)$ dimensions, which is solvable by methods developed in [33, 225].

7   Twistors and scattering amplitudes

Although there has been a longstanding programme to understand scattering amplitudes in twistor space via 'twistor diagrams' [182, 116], the modern developments started with Witten's twistor-string [221] introduced in 2003. The fallout has now spread in many directions. It encompasses recursion relations that impact across quantum field theory but also back on the original twistor-diagram programme, Grassmannian integral formulae, polyhedral representations of amplitudes, twistor actions and ambitwistor–strings.

7.1   Twistor-strings

The twistor string story starts in the 1980's with a remarkable ampitude formula due to Parke and Taylor [172], and its twistorial interpretation by Nair [169]. Consider n massles gluons, each carrying a null momentum $p_i{}^\mu, i = 1, \ldots, n$. The isomorphism (2.4) and the fomula (2.5) imply that null vectors are two-by-two matrices with zero determinant, and thus rank one. Any such matrix is a tensor product of two spinors

$$p_i{}^\mu = \pi_i{}^{A'} \tilde{\pi}_i^A.$$

In spinor variables, the tree level amplitude for two negative helicity gluons and $n - 2$ positive leads to [172]

$$\mathcal{A}_n = \frac{< \pi_i \pi_j >^4 \, \delta^4(\sum_{k=1}^n p_k)}{< \pi_1 \pi_2 >< \pi_2 \pi_3 > \cdots < \pi_{n-1} \pi_n >< \pi_n \pi_1 >}, \tag{7.22}$$

where $< \pi_k \pi_l > := \epsilon_{A'B'} \pi_k{}^{A'} \pi_l{}^{B'}$, and ith and jth particles are assumed to have negative helicity, and the remaining particles have positive helicity. Nair [169] extended this formula to incorporate $\mathcal{N} = 4$ supersymmetry and expressed it as an integral over the space of degree one curves (lines) in twistor space using a current algebra on each curve.

Witten [221] extended this idea to provide a formulation of $\mathcal{N} = 4$ super Yang–Mills as a string theory whose target is the super–twistor space $\mathbb{CP}^{3|4}$ (see, e.g. [87]). This space has homogeneous coordinates $\mathcal{Z}^I = (Z^\alpha, \chi^a)$ with Z^α the usual four bosonic homogeneous coordinates and χ^b four anti–commuting Grassmann coordinates $b = 1, \ldots, 4$. The model is most simply described [35] as a theory of holomorphic maps[3] $\mathcal{Z} : C \to \mathbb{T}$ from a closed Riemann surface C to nonprojective twistor space. It is based on the world-sheet action

$$S = \int_C \mathcal{W}_I \bar\partial Z^I + a \mathcal{W}_\mathcal{I} \mathcal{Z}^I \, .$$

Here a is a $(0, 1)$-form on the worldsheet C that is a gauge field for the scalings $(\mathcal{W}, \mathcal{Z}, a) \to (\mathbf{e}^{-\alpha} \mathcal{W}, \mathbf{e}^\alpha \mathcal{Z}, a + \bar\partial\alpha)$. A prototype of this action was introduced in [192]

To couple this to Yang–Mills we introduce a d-bar operator $\bar\partial + A$ on a region $\mathbb{PT}$ in super twistor space with A a $(0, 1)$-form taking values in some complex Lie algebra, and the field J which is a $(1, 0)$ form with values in the same Lie algebra on the worldsheet, and generates the current algebra[4]. When expanded out in the fermionic variables χ, such A with $\bar\partial A = 0$ give a Dolbeault cohomology classes in $H^1(\mathbb{PT}, \mathcal{O}(p))$ for $p = 0, \ldots, -4$ corresponding via the Penrose transform to the full multiplet for $\mathcal{N} = 4$ super Yang-Mills from the positive helicty gluon to the fermions and scalars down to the negative helicity gluon.

The standard string prescription for amplitudes leads to a proposal for tree amplitudes for n particles as correlators of n 'vertex operators' when $\Sigma = \mathbb{CP}^1$. These take the form $V_i = \int_C \text{tr}(A_i J)$ where A_i is the 'wave function', i.e., twistor cohomology class $H^1(\mathbb{PT}, \mathcal{O})$, of the ith particle in the scattering, usually taken to be a (super-)momentum eigenstate and tr is the Killing form on the Lie algebra. The correlators break up into contributions corresponding to the different degrees d of the map from $C = \mathbb{CP}^1$ into twistor space. The degree d contribution gives the so-called N^{k}MHV amplitudes with

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[3]These are D1 instantons in Witten's original B-model formulation.

[4]To turn this into a string theory one includes a chiral analogue of worldsheet gravity which when gauge is fixed just lead to ghosts and a BRST operator.

$d = k + 1$ where the MHV degree[5] k corresponds to $k + 2$ of the external particles having negative helicity with the rest positive.

Let us give a flavour of the formula. If C is a degree-d curve in twistor space, and $\mathcal{M}^d$ is the moduli space of curves containing C, then, for any $(0,1)$ form A on $\mathbb{CP}^{3|4}$ with values in $\mathfrak{gl}(N, \mathbb{C})$, we can restrict A to C and consider $\bar{\partial}_A|_C$. The perturbative YM scattering amplitude is then obtained from the generating function

$$\int_{\mathcal{M}_\mathbb{R}^d} \det(\bar{\partial}_A|_C) d\mu,$$

where $d\mu$ is a holomorphic volume form on $\mathcal{M}^d$ and $\mathcal{M}_\mathbb{R}^d$ a real slice of $\mathcal{M}^d$. The role of $\det(\bar{\partial}_A|_C)$ is as the generating function of current algebra correlation functions on the curve C. The nth variation of this functional with respect to A as a $(0,1)$-form on C gives the current algebra correlator that forms the denominator of the Parke–Taylor factor when C is a line (together with some multitrace terms). In the full formula, the nth variation with respect to A with values in the cohomology classes of linearized gauge fields for momentum eigenstates, is then the scattering amplitude as a function of these linear fields. In principle, the genus of C determines the number of loops in the perturbative series, but the formulae have not been verified beyond genus 0.

This correlation function was evaluated for momentum eigenstates by Roiban, Spradlin and Volovich (RSV) [188] as a remarkably compact integral formula for the full S-matrix of $\mathcal{N} = 4$ super Yang-Mills. The $\mathrm{N}^k\mathrm{MHV}$ component is expressed as an integral over the moduli of rational curves of degree $k - 1$ with n marked points, a space of bosonic|fermionic dimension $(4d + n)|4(d + 1)$. Furthermore, many of the integrals are linear[6] and can be performed explicitly reducing to a remarkable compact formula with $2d + 2 + n$ bosonic integrations against $2d + 6 + n|4(n - d - 1)$ delta-functions (the excess bosonic delta functions expressing momentum conservation). The resulting formula for the amplitude is, in effect, a sum of residues, remarkably simpler than anything that had been found before and difficult to imagine from the perspective of Feynman diagrams.

7.2   Twistor-strings for gravity

The Witten and Berkovits twistor string models also compute amplitudes of a certain conformal gravity theory with $\mathcal{N} = 4$ supersymmetry [36]. So it demonstrated the principle that gravity might be encoded in this way.

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[5]MHV stands for maximal helicity violating because amplitudes vanish when $k < 0$ or $k > n - 2$.

[6]It is simplfied by the fact that the space of rational curves of degree d in $\mathbb{CP}^{3|4}$ can be represented as homogeneous polynomials of degree d in homogeneous coordinates, $\mathbb{C}^2$ for $\mathbb{CP}^1$ and so it is a vector space of dimension $4(d + 1)|4(d + 1)$ modulo $GL(2)$ which also acts on the marked points.

However, it is a problem for the construction of loop amplitudes for super Yang-Mills as these conformal gravity modes will of necessity run in the loops, although it is in any case still not clear whether the model can be used to calculate loop amplitudes even with conformal supergravity. Furthermore, conformal gravity is widely regarded as a problematic theory, certainly quantum mechanically because it necessarily contains negative norm states.

Although there was an early version of the Parke Taylor MHV formulae for gravity found in the 1980s [34], the one given by Hodges [119] was the first to manifest permutation invariance and enough structure to suggest a version of the RSV Yang-Mills formula for maximally supersymmetric $\mathcal{N} = 8$ gravity tree amplitudes. This was discovered by Cachazo and Skinner [53], see [56] for a proof and further developments. An underlying twistor string theory for this formula has been constructed by Skinner [193], who showed that $\mathcal{N} = 8$ supergravity is equivalent to string theory with target $\mathbb{CP}^{3|8}$ but now with a supersymmetric worldsheet and some gauged symmetries.

7.3   The CHY formulae and ambitwistor strings

The twistor string theories of Witten, Berkovits, and Skinner gave a remarkable new paradigm for how twistor theory might encode genuine physics. However, their construction very much relies on maximal supersymmetry and it is unclear how more general theories might be encoded. The framework is also tied to four dimensions (for some this is a positive feature). Although the string paradigm suggested that multiloop processes should correspond to amplitudes built from higher genus Riemann surfaces, the details seem to be at best unclear and quite likely obstructed by anomalies.

In a parallel development, already before the Cachazo–Skinner formula for gravity amplitudes, Cachazo and coworkers had been exploring the relationship between the twistor-string amplitude formulae and a family of ideas, originating in conventional string theory, whereby gravity amplitudes can be expressed as the square of Yang-Mills amplitudes via the KLT relations [133] and their extensions to colour-kinematic duality [40]. This led to an independent twistor inspired formula for four dimensional gravity [52]. Cachazo, He and Yuan (CHY) subsequently refined and developed these ideas into a remarkably simple and elegant scheme for formulae analogous to those arising from the twistor-string for gauge theories and gravity (and a certain bi-adjoint scalar theory) in all dimensions [54]. The framework has by now been extended to a variety of theories [55]. In all of these formulae, the KLT idea of expressing gravity amplitudes as the square of Yang-Mills is essentially optimally realized (indeed much more elegantly than in KLT).

The essential observation on which all these formulae rely is that the residues on which all the twistor-string formulae of RSV and so on are supported are configurations of points on the Riemann sphere obtained from solutions to the *scattering equations*. These are equations for n points on the Riemann sphere given the data of n null momenta subject to momentum

conservation that can be in any dimension. They were first used in [85] to construct classical string solutions associated to n particle scattering, but also arose from strings at strong coupling in calculations of Gross and Mende [102]. On the support of the scattering equations, the complicated momentum kernel that forms the quadratic form in the gravity equals Yang-Mills squared in the KLT relations is diagonalized.

The question remained as to what the underlying string theory for these formulae might be. This was answered in [163] based in its simplest form on a chiral, infinite tension limit of the ordinary bosonic string

$$S = \int_C P \cdot \overline{\partial} X - eP^2/2 \qquad (7.23)$$

where (X, P) are coordinates on T^*M, and P is understood to take values in 1-forms on the Riemann surface C. The Lagrange multiplier e restricts the target space to a hypersurface $P^2 = 0$ and is a gauge field for the action of the geodesic spray $D_0 = P \cdot \nabla$. This is just a Lagrangian expression of the usual (holomorphic) symplectic reduction of the cotangent bundle to the space of (scaled) null geodesics $\mathbb{A}$. In four dimensions this has become known as ambitwistor space as it is both the cotangent bundle of projective twistor space and of projective dual twistor and so chooses neither chirality. However, it clearly exists in all dimensions as $\{T^*M|_{P^2=0}\}/D_0$, the space of complexified null geodesics. It can be defined for geodesically convex regions in a complexification of any analytic Lorentzian or Riemannian manifold of any dimension. If $\dim(M) = d$ then $\dim(\mathbb{PA}) = 2d - 3$.

The study of ambitwistor space for $d = 4$ started in 1978 with constructions by Witten and Isenberg, Yasskin and Green for Yang-Mills. In four dimensions, ambitwistor space of complexified Minkowski space $M_{\mathbb{C}}$ can be expressed as a quadric $Z^\alpha W_\alpha = 0$ in $\mathbb{PT} \times \mathbb{PT}^*$. The ambitwistor space is a complexification of the real hypersurface $\mathcal{PN} \subset \mathbb{PT}$ introduced in §2. In [220, 127, 105] (see also [135]) it was shown that generic analytic connections on bundles on Minkowski space correspond to topologically trivial bundles on ambitwistor space. The full Yang-Mills equations can be characterised as the condition that the corresponding holomorphic vector bundles on $\mathbb{PA}$ extend to a third formal neighbouroood in $\mathbb{PT} \times \mathbb{PT}^*$. The Witten version [220, 105] reformulated this third order extension condition to the simple requirement of the existence of the bundle on a supersymmetric extension of ambitwistor space built from $\mathcal{N} = 3$ supersymmetric twistor spaces. This generalises the Ward correspondence (Theorem 5.1) but unlike this Theorem, it has not yet led to any effective solution generating techniques.

Gravitational analogues of the Witten, Isenberg, Yasskin and Green were developed by LeBrun, Baston and Mason [30, 146].

Theorem 7.1 (LeBrun [143]) *The complex structure on $\mathbb{PA}$ determines the conformal structure $(M_{\mathbb{C}}, [g])$. The correspondence is stable under deformations of the complex structure on $\mathbb{PA}$ which preserve the contact structure $P \cdot dX$.*

The existence of 5th order formal neighbourhoods corresponds to vanishing of the Bach tensor of the space-time and, when the space-time is algebraically general, a 6th order extension corresponds to the space-time being conformally Einstein.

The string theories based on the action (7.23) have the property that they need to use ambitwistor cohomology classes arising from the ambitwistor Penrose transform in amplitude calculations; it is the explicit form of these cohomology classes that imposes the scattering equations. In order to obtain Yang-Mills and gravitational amplitudes from the theory, worldsheet supersymmetry needs to be introduced by analogy with the standard RNS superstring string and/or other worldsheet matter theories for other theories [63] (current algebras are required for the original biadjoint scalar and Yang-Mills formulae).

The ambitwistor string paradigm extends to many different geometrical realizations of ambitwistor space. Ambitwistor strings have also been constructed that are analogues of the Green-Schwarz string and the pure spinor string [28, 38]. In particular, the original twistor strings can be understood simply as arising from the four-dimensional realization of $\mathbb{A}$ as the cotangent bundle of projective twistor space. However, the approach of [92] uses the four dimensional realization of ambitwistor space as a subset of the product of twistor space with its dual to provide an ambidextrous twistorial ambitwistor string theory leading to new amplitude formulae for Yang-Mills and gravity that no longer depend on maximal supersymmetry.

In another direction, one can realize ambitwistor space geometrically as the cotangent bundle $T^*\mathfrak{I}$ of null infinity $\mathfrak{I}$, [2, 93]. This gives new insight into the relationship between asymptotic 'BMS' symmetries and the soft behaviour of scattering amplitudes, i.e., as the momentum of particles tends to zero.

An important feature of ambitwistor strings is that they lead to extensions of the CHY formulae to ones for loop amplitudes. The original models of [156] are critical in 26 dimensions for the bosonic string and in 10 for the type II RNS version with two worldsheet supersymmetries. This latter model was developed further in [3], where it was shown that this theory does indeed correspond to full type II supergravity in 10 dimensions and that anomalies all vanish for the computation of loop amplitudes. An explicit conjecture for a formula at one loop for type II supergravity amplitudes was formulated using scattering equations on a torus in a scheme that in principle extends to all loop orders. This formula was subsequently shown [94] to be equivalent to one on a Riemann sphere with double points that does indeed compute amplitudes at one loop correctly. Furthermore, on the Riemann sphere it is possible to see how to adapt the formulae to ones for many different theories in different dimensions with varying amounts of supersymmetry. There remains the challenge to extend this to a scheme that transparently extends to all loop orders, but although the framework does work at two loops, it already there needs new ideas [95].

Much remains to be understood about how ambitwistor strings reformulate conventional massless theories, particularly at higher loops or in the nonlinear regime. One key advance in the latter direction was the construction of ambitwistor strings on a curved background [4] providing a kind of Lax pair for the full type II supergravity equations in 10 dimensions in the sense that the quantum consistency of the constraints is equivalent to the full nonlinear supergravity equations.

7.4   Twistor actions

At degree $d = 0$ Witten argued that the effective field theory of the twistor-string is given by the twistor space action holomorphic Chern-Simons action

$$I = \int_{\mathbb{CP}^{3|4}} \Omega \wedge \mathrm{Tr}(A\overline{\partial}A + \frac{2}{3}A \wedge A \wedge A),$$

where $\Omega = d^{3|4}Z$ is the natural holomorphic super-volume-form on $\mathbb{CP}^{3|4}$ (which turns out to be super-Calabi-Yau), and A is a $(0,1)$-form. The field equations from this action are simply the integrability of the $\overline{\partial}$-operator, $\overline{\partial}_A^2 = 0$. Interpreting this via the Ward transform (Theorem 5.1) on *super*-twistor space now leads to the spectrum a of full $\mathcal{N} = 4$ super Yang–Mills, but with only anti-self-dual interactions.

The question arises as to whether one can complete this action to provide the full interactions of super Yang-Mills. This can be done by borrowing the leading part of the twistor-string computation and some experimentation shows that the interaction term

$$S_{int} = \int_{\mathcal{M}_{\mathbb{R}}^1} \log \, \det(\overline{\partial}_A|_C)d\mu,$$

gives the correct interactions for full maximally supersymmetric Yang-Mills theory [159, 43]. Here the integration is nonlocal in twistor space being firstly a nonlocal one over degree one rational curves (i.e., lines) and secondly over $\mathcal{M}_{\mathbb{R}}^1$ which is a real form of the complexified Minkowski space - for example the Euclidean space. This can be shown to be equivalent to the standard space-time action in Euclidean signature in a gauge that is harmonic on twistor lines as introduced in [223]. Despite its nonlocality, the resulting Feynman diagram formalism is remarkably tractable in an axial gauge in which a component of A is set to zero in the direction of some fixed reference twistor Z_*. This then leads [42] to the so-called MHV diagram formalism introduced informally in [51] by considering twistor-strings on degree-d curves that arise as disjoint unions of d lines. In this formalism, amplitudes are computed via $1/p^2$ propagators and interaction vertices built from (7.22) via a simple off-shell extension.

The formulation of these rules via a twistor action allowed them to be expressed with maximal residual symmetry beyond the choice of Z_*, and extended from amplitudes to more general correlation functions.

Perhaps the most important application of these ideas was to a proof of the conjectured amplitude/Wilson loop duality. This conjecture, originally due to Alday and Maldacena, arose from AdS/CFT considerations and stated that the planar amplitude (i.e., in the large N limit for gauge group $SU(N)$) for maximally supersymmetric Yang Mills is equivalent to a Wilson loop around a null polygon whose sides are made from the momenta of the particles in the scattering amplitude; the planarity condition means that there is a trace order for the gluons in the amplitude as in (7.22) that determines the ordering of the momenta around the polygon. This null polygon is particularly easy to represent in twistor space, as it corresponds naturally to a generic polygon there and one can express the Wilson-loop on space-time in terms of a holomorphic Wilson-loop in twistor space that can be computed via the twistor action. In the Feynman diagram framework that arises from the twistor action in the axial gauge, one finds that the Feynman diagrams for the holomorphic Wilson loop are dual to those for the amplitude in the sense of planar duality, giving a proof of the amplitude/Wilson loop correspondence at the level of the loop integrand [162].

Much of this and related material is reviewed in [1] although this does not cover more recent work on stress-energy correlators [67] and form factors [137, 68].

7.5   Recursion relations, Grassmannians and Polytopes

The BCFW recursion relations [47] were a separate major development that sprang from the twistor string. These use an elegant Cauchy theorem argument to construct tree amplitudes with n external particles from amplitudes with fewer external particles. The idea is to introduce a complex parameter z by shifting the spinor representation of the momenta of two of the particles

$$(\pi_1^{A'}\tilde\pi_1^A, \pi_n^{A'}\tilde\pi_n^A) \to (\pi_1^{A'}(\tilde\pi_1^A + z\tilde\pi_n^A), (\pi_n^{A'} - z\pi_1^{A'})\tilde\pi_n^A).$$

As a function of z, an amplitude only has simple poles where the momenta flowing through an internal propagator becomes null (i.e., where some partial sum of momentum becomes null as z varies). The residues at these poles are products of the tree amplitudes on each side of the propagator evaluated at the shifted momentum. Thus the amplitude can be expressed as the residue of $1/z$ times the amplitude at $z = 0$, but this can be expressed as the sum of the residues where $z \neq 0$ which are expressed in terms of lower order amplitudes (it turns out that there are many good situations where there is no contribution from $z = \infty$). These recursion formulae have subsequently been extended quite widely to include gravity and Yang-Mills in various dimensions and with varying amount of supersymmetry [6], and to loop integrands of planar gauge theories [9]. Andrew Hodges [117] expressed the recursion relations in terms of twistor diagrams providing a generating principle that had hitherto been missing.

Further work on expressing BCFW recursion in twistor space [7, 160] led to a Grassmannian contour formula for amplitudes and leading singularities (invariants of multiloop amplitudes) [8]. A related Grassmannian formula was obtained soon after, which with hindsight, gave the analagous Grassmannian contour integral formulae for the Wilson-loop, but, by the amplitude/Wilson-loop duality, gives an alternative but quite different contour integral formula for amplitudes [161] that are rather simpler than the original Grassmannian formulae.

This led to a programme to understand the residues that arise in the Grassmannian. It emerged that a key idea that is required is to think of the Grassmannians as real manifolds and study their positive geometry [10] leading to simple combinatorial characterisations of the residues.

The twistors for the Wilson-loop version of the amplitude were first introduced by Hodges [118] and called momentum twistors as they can be obtained locally from the momenta for the amplitude. He used them in a completely novel way to show that the redundancy and choices built into a BCFW decomposition of an amplitude could be understood geometrically in momentum twistor space as arising from different dissections of a polyhedron that describes the whole amplitude as one global object. This was originally only realized for NMHV amplitudes, but the programme has continued and matured into the *amplituhedron* [11] (although this is essentially the dual of Hodges' original picture which has still not been fully realized).

8 Gravitization of Quantum Mechanics and Newtonian Twistors

In [178, 180] Roger Penrose has argued that a collapse of the wave function is a real process taking place in time, and is not described by the Schrödinger equation. Gravity should have a role to play in explaining the nature of the quantum collapse, and the conventional views on quantum mechanics may need to be revisited in the process. The key idea is that the superposition of massive states must correspond to superposition of space–times. This makes the notion of the stationary states ambiguous - its definition depends on a choice of a time–like Killing vector. In [178] an essentially Newtonian calculation led to the conclusion that the timescale of instability of one stationary state is $\tau \approx \hbar/E_G$ where E_G is the gravitational energy need to separate two mass distributions. To attempt a twistor understanding of this relation one first has to take a Newtonian limit of the *relativistic* twistor correspondence. Analysing the incidence relation (2.7) in the flat case shows that such a limit corresponds to the jumping line phenomenon [80]. If $\mathbb{PT}_c$ is a family of twistor spaces corresponding to a flat Minkowski space parametrised by a finite speed of light c, then exploiting the holomorphic fibration $\mathbb{PT}_c \to \mathcal{O}(2)$ one finds
$$\lim_{c \to \infty} (\mathbb{PT}_c = \mathcal{O}(1) \oplus \mathcal{O}(1)) = \mathbb{PT}_\infty = \mathcal{O} \oplus \mathcal{O}(2).$$

In the Nonlinear Graviton and Ward correspondences (Theorems 3.2 and 5.1) the presence of jumping lines corresponds to singularities of the metric or gauge fields on a hypersurface [204, 114, 195]. In the Newtonian twistor theory all curves jump.

The curved twistor space in the Newtonian limit can be understood by considering a one–parameter family of Gibbons–Hawking metrics (3.10)

$$g(c) = (1 + c^{-2}V)(dx^2 + dy^2 + dz^2) + c^2(1 + c^{-2}V)^{-1}(d\tau + c^{-3}A)^2.$$

and taking a limit $c \to \infty$ in the Gibbons–Hawking twistor space. This leads to a Newton–Cartan theory [61, 208] on the moduli space M of $\mathcal{O}(2) \oplus \mathcal{O}$ curves. The limit of $\{c^{-2}g(c), g^{-1}(c), \nabla(c)\}$, where $\nabla(c)$ is the Levi–Civita connection of $g(c)$ is a triple consisting of one–form $\theta = d\tau$ giving a fibration $M \to \mathbb{R}$ (absolute time), a degenerate inverse metric $h^{ij} = \delta^{ij}$ on the fibres, and a torsion–free connection preserving h and θ with the only nonvanishing Christoffel symbols given by $\Gamma^i_{\ \tau\tau} = \frac{1}{2}\delta^{ij}\partial_j V$.
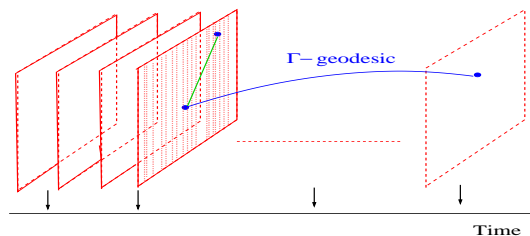

Figure 6. *Newton Cartan space-time.*

9   Other developments

Our presentation has omitted many interesting applications of twistor methods. A good reference are articles which appeared in Twistor Newsletter published by the Twistor group in Mathematical Institute between 1976 and 2000, and reprinted in [123, 155]. In particular they contain a discussion of the notoriously difficult and still unsolved *Googly Problem* of encoding self–dual (rather than anti–self–dual) and more general Einstein spaces in a geometry of $\mathbb{PT}$. See also [24, 141, 122, 179] (and [21, 17] for recent ideas of how four–manifolds can be used in constructing some geometric models of matter). Below we list some other developments.

In *integral geometry* the role of space–time and twistor space is turned round. The subject goes back to F. John [129] who considered the problem of characterising functions ϕ on the space or oriented lines in $\mathbb{R}^3$ such that $\phi(L) := \int_L f$ if $f : \mathbb{R}^3 \to \mathbb{R}$. The image of this integral transform is characterised by the kernel of the ultra-hyperbolic Laplacian. The resulting integral formula (see [129, 224]) is an analytic continuation of Penrose's contour integral formula (1.1). A general relationship between twistor theory and integral geometry has been explored in [97, 171, 25] and in the language of systems of 2nd order ODEs in [103, 64].

Quasi local mass. In the study of asymptotic properties of space–time one seeks satisfactory definitions of energy and momentum which make sense at extended, but finite regions of space–time, i. e. at the quasi local level. One definition associates these quantities to an arbitrary two–surface in space time, and makes use of twistor theory of such surfaces [177, 72, 199]. Another twistor approach to this problem [154] is based on the Ashtekar variables [12].

In *loop quantum gravity* twistors provide a description of spin network states. This approach has been developed by Simone Speziale and his collaborators. See [90, 140]. Their description of symplectic structures and canonical quantisation builds on a work of Tod [203].

Space-time points are derived objects in twistor theory. They become 'fuzzy' after quantisation which initially seemed to be an attractive framework for quantum gravity. One possible realisation of this may be a *non–commutative twistor theory* [201, 89, 132, 104, 46, 151] as well as the most recent twistorial contribution from Roger Penrose [181].

10 Conclusions

Twistor theory is a set of non-local constructions with roots in 19th century projective geometry. By now twistor ideas have been extended and generalized in many different directions and applied to many quite different problems in mathematics and physics. A unifying feature is the correspondence between points in space-time and holomorphic curves or some family of higher dimensional compact complex submanifolds in a twistor space, together with the encoding of space-time data into some deformed complex structure.

Complex numbers play an essential role. The local non–linearities of anti–self–dual Einstein and Yang–Mills equations in space–time are replaced by algebro–geometric problems in twistor–space or by the Cauchy–Riemann equations in the Atiyah–Hitchin–Singer picture adapted to Riemannian reality conditions. Twistor and ambitwistor string theories couple directly to the complex geometry of twistor and ambitwistor spaces; the cohomology classes on twistor and ambitwistor spaces that represent space-time fields restrict directly to give the vertex operators of the corresponding theories. These give a coherent twistorial formulation of many of the physical theories that lead to striking simplifications over their space-time formulations, particularly in amplitude calculations. There is now clear evidence that the conventional string is tied up with the geometry of twistors in ten dimensions [39].

We wish twistor theory and its founder all the best on the occasion of their respective anniversaries. We expect the future of twistor theory to be at least as productive as its past, and that there will much to celebrate in this 21st century.

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