Quantum Wiener chaos

Mateusz Jurczyński

Department of Mathematics and Statistics

Lancaster University

This dissertation is submitted for the degree of

Doctor of Philosophy

November 2017

I would like to dedicate this thesis to my parents and my siblings.

Niniejszą pracę dedykuję moim Rodzicom i mojemu Rodzeństwu.

Declaration

Hereby I declare that the present thesis was prepared by me and none of its contents was obtained by means that are against the law.

I also declare that the present thesis is a part of a PhD Programme at Lancaster University. The thesis has never before been a subject of any procedure of obtaining an academic degree.

The thesis contains research carried out jointly: Chapter 3-4 and the Appendix forms the basis of the paper [43] co-authored with J. M. Lindsay. Hereby I declare that I made a full contribution to all aspects of this research and the writing of these papers.

Mateusz Jurczyński

November 2017

Acknowledgements

As much as a PhD is a piece of independent research, it is an undertaking I would have found impossible to complete without an innumerable amount of people both in and out of Lancaster.

I would like to thank my parents, Krystyna and Jerzy, and my siblings, Aleksandra, Łukasz, Milena and Sandra, for being a constant and unwavering source of support and belief that I could always rely on, despite the distance which usually separated us.

I would like to thank the members of IX and LUCI, two mysterious acronyms (or, more precisely, initialisms!) who helped me extensively and provided far more welcome, fun and support than I thought possible.

There are many people who have made Lancaster an unforgettable experience for me who I want to give my thanks to. It would be impossible to list them all, but in no particular order: to Michał, for the brotherhood in PhD, to Jak, for introducing me to approximately three quarters of my current hobbies, to Jess, for welcoming me to IX, to Ricky, for all the conversations wrapped in teas, coffees and food, to fellow occupiers of Westham Street, Tom, Rhianna, Henry, for consistently making it truly a home I was happy to go back to, to Jayna, David, Jason, James for being awesome in general.

To all those who made the Mathematics and Statistics department a wonderful place to be - among others, Tomek, Daria, Richard, Alec, James, Lefteris, Chris, Hattie, Jason, Shane, Harrish, Biswarup, BK - for all the conversations, mathematical and otherwise.

To Ania, for so many things I cannot possibly express. To all the Polish friends, in Poland and out of it, for helping me remember that there is a place where it's the British who are the foreigners.

To my supervisor Martin, for guiding me well at my best and forgiving my sloppiness at my worst.

And finally to Judy, for her love and support and for making me look forward to every coming day.

Abstract

In this thesis we develop the theory of quantum Wiener integrals on the bosonic Fock space. We study multiple quantum Wiener integrals as an algebra of unbounded operators, investigating its properties, including closedness, common domains and multiplication formulas. We show the applications of the new formalism by providing new proofs to the established theory of quantum stochastic calculus and new conditions for generating quantum stochastic cocycles and quantum stochastic evolutions. The corresponding quasifree case is also studied and the constructions extended to fit in that formalism.

We construct the multiple quantum Wiener integral as one operator on a family of operators which we dub operator kernels. This in particular covers the case of quantum stochastic cocycles and evolutions. We show that the family of quantum Wiener integrals forms a WOT-dense algebra of unbounded operators on the bosonic Fock space. We provide more general conditions for an operator kernel to be multiple quantum Wiener integrable, which allows us to treat multiple quantum Wiener integrals as an algebra. We explore the influence of an initial space on the theory. Our setting gives natural conditions for a product of two cocycles (evolutions) to still be a cocycle (an evolution). We apply our theory by solving quantum stochastic differential equations (QSDEs) and by finding more elementary proofs of structure conditions on the generator of a quantum stochastic evolution and of the fundamental estimate in the proof of quantum stochastic Lie–Trotter formula. We also show how our theory unifies and generalises the theory of integral kernels and chaotic representation properties, proving in particular that every Hilbert–Schmidt operator is a quantum Wiener integral.

Table of contents

Introduction

The first appearance of polynomial chaos dates back to the 1938 paper of Norbert Wiener [82]. Its interpretation as multiple Wiener integrals and its applications outside of Brownian motion are due to Itô [40]. It is defined as follows: consider the standard Brownian motion B with its canonical probability space $(\Omega, \Sigma, \mathbb{P})$. Then, in fact, the space of square integrable random variables on Ω consists exactly of limits of polynomials of Brownian motion, and the polynomials of different orders are orthogonal. In Wiener's original work, the polynomials in question were the Hermite polynomials of Brownian motion. It is worth noting this approach was continued and made more approachable by Cameron and Martin in [15]. In the language of Itô calculus, the polynomials are the multiple Wiener integrals of functions defined on the positive real line. Thus,

$$L^2(\Omega) = \bigoplus_{n \geqslant 0} I_n(L^2_{\text{sym}}(\mathbb{R}^n_+)).$$

Today this property has been dubbed the chaotic representation property (CRP) and has been extensively studied - work has been done to find wider classes of martingales with CRP [4], for example, the compensated Poisson process [18] and some of the Azemá martingales [71]. It implies the predictable representation property and forms the foundation of the theory of Malliavin calculus, cf. [69].

Applications of the CRP include numerical approximations of stochastic processes, *e.g.* [67] for its application to propagators, [62] for the nonlinear filtering problem, [25] for its application in solving backward stochastic differential equations and [26] for solving elliptic equations with random coefficients.

CRP is just one of the many brilliant features of Itô stochastic calculus [39], which has since found a multitude of applications, including mathematical finance [14], cancer research [80], climate modelling [34] and molecular biology [74].

What is remarkable as well is the probabilistic connection this theory gives between martingales with the CRP and the bosonic Fock space. Indeed, since $L^2_{\mathrm{sym}}(\mathbb{R}^n_+)$ is just the n-th symmetric tensor power of $L^2(\mathbb{R}_+)$, the CRP can also be stated as

$$L^2(\Omega) = \mathcal{F},$$

where $\mathcal{F}$ denotes the symmetric Fock space over $L^2(\mathbb{R}_+)$. This cements the fundamental place Fock space plays in quantum probability theory.

The theory of quantum stochastic calculus was started by Hudson and Partasarathy in the 1970s [36], [37], [70]. It is a natural extension of the Itô stochastic calculus to the noncommutative setup. Since then, the usual construction of quantum stochastic integral goes by an appropriate generalisation of the Skorokhod integral [78], [24], [69], rather than through Riemann-Stieltjes sums [37]. In their foundational papers Hudson and Parthasarathy established the quantum analogue of a Wiener process as a combination of the annihilation and creation processes and the (weak version of) quantum Itô product formula. They also formalised and developed the theory of quantum stochastic differential equations (QSDEs). Since then, QSDEs have found applications in, for example, quantum control theory [41] and have been subject to extensive research [20], [21], [49]. It is worth mentioning the related theory of quantum Lévy processes [23] and the sister theory of noncommutative stochastic calculus on the free Fock space [46].

The idea of making polynomial Wiener chaos noncommutative has appeared in literature before, with the theme featuring quite prominently in the idea of integral kernels. Fundamental here is the Guichardet point of view on the Fock space [32]. This theory was initiated by Maassen [63] and then developed by Meyer [66] and further on by Lindsay [54]. Dermoune [17] studied the case of Fock space with multiplicities. This approach sits very strongly in the "coordinate" approach to quantum stochastic calculus, which has since been superceded by the coordinate-free framework [31]. Quantum Wiener ideas, although not necessarily named as such, have appeared in *e.g.* [58] and [60]. Polynomial chaos has recently met with renewed interest with an appearance in free probability, cf. [45], in which the prominent integral is, of course, the Wigner integral, demonstrating how the Wigner process corresponds to the Wiener process in classical probability. A more complete theory of free Wigner chaos is available in [13]. These ideas follow along the lines of the Itô-Clifford integral [6]. It is important to note that the noncommutative Wiener chaos in the sense of this thesis is truly different than free Wigner chaos, as the quantum Brownian motion and free Brownian motion are not special cases of each other, but genuinely different noncommutative stochastic processes. The difference will also be exemplified in our convolution formula, in which the fact that free stochastic calculus utilises noncrossing partitions while we do not will become very apparent.

In the appendix we also treat the quasifree case. The quasifree case arises from the different representations of the CCR algebra, in contrast to the usual representation on the symmetric Fock space. The mathematical formalism of quasifree quantum stochastic calculus was first established by Lindsay in his PhD thesis [51], and since then Lindsay and Margetts have written a complete theory of quasifree stochastic calculus in [55], [56] with a very nice presentation in the latter's PhD thesis [65]. The corresponding theory of quasifree random walks was developed by Belton, Gnacik and Lindsay in [9], [10], [27].

Description of the contents

The contents of the thesis are presented in four chapters and an appendix. In Chapter 1 we introduce quantum stochastic calculus. We define the bosonic Fock space via the Guichardet construction and discuss the basic properties of the space and of the representation. We present the fundamental annihilation, creation and preservation operators and briefly recall the properties of Weyl operators. Then we construct the quantum stochastic integrals in the coordinate free setup.

In Chapter 2 we build the multiple quantum Wiener integral theory, starting from the vector and operator kernels. We develop the language and notations that will be used throughout the thesis. We study the algebra and measure theory of these kernels to allow us to study natural conditions for quantum Wiener integrability.

In Chapter 3 we apply the theory from Chapter 2 to construct quantum Wiener integrals of operator kernels. We study their properties as operators, including conditions for their closedness and their natural cores. We also present fundamental identities which will be of use throughout the thesis, with mentions on how they correspond to the classical quantum stochastic calculus. The chapter ends with a note about representation of Hilbert-Schmidt operators through quantum Wiener integrals.

In Chapter 4 we apply the quantum Wiener integrals to quantum stochastic calculus. We start by exploring the special case of quantum Wiener integrals which are formed from product operator kernels, which correspond to quantum stochastic cocycles and evolutions. Using our setup we obtain an easy proof of the relation between contractivity of the process and nonnegativity of the series product of its generator with its adjoint. This result is known in the cocycle theory, but not in the case of evolutions. We also show how our setup allows one to easily prove the quantum cocycle Trotter product. Next we show how the quantum Wiener integrals descend to the classical probability theory, in particular how our general product formula gives the Wiener and the Poisson product. We also show how they generalise the Maassen–Lindsay chaos expansion kernels.

In the Appendix we explore the quasifree case. We develop the language and notation needed to construct our multiple quantum Wiener integrals in the quasifree case, in particular extending the partial transpose to a partial transpose of "column kernels". We show how the theory extends to this setup and prove some structure results.

Notation and conventions

Throughout the thesis the Hilbert spaces are assumed to be separable and all inner products are linear with respect to second variable. The importance of separability will become clear upon closer inspection of measurability.

For two sets X, Y the space of all functions between X and Y is denoted by $F(X, Y)$. For a Hilbert space k we write $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$ and for vectors $k \in \mathsf{k}$ we put $\widehat{\mathsf{k}} \ni \widehat{k} := \binom{1}{k}$. For a vector $k \in \mathsf{k}$ the maps $\langle k| \colon \mathsf{k} \to \mathbb{C}, |k\rangle \colon \mathbb{C} \to \mathsf{k}$,

$$\langle k|l = \langle k, l\rangle, |k\rangle\alpha = \alpha k,$$

called bra and ket, respectively, play a fundamental role. Upon introducing another Hilbert space h, these maps will be freely ampliated with identity on the left or right, giving

$$I_{\mathsf{h}} \otimes |k\rangle =: E_k, I_{\mathsf{h}} \otimes \langle k| =: E^k.$$

No separate notation for the tensoring from either side will be introduced, but it will always be clear from context.

We extensively use the Guichardet measure space construction. Thus let $I \subset \mathbb{R}_+$ be measurable and let us denote by $\Gamma_n(I)$ the family of all n-element subsets of I and let $\Gamma(I) = \bigcup_{n \geqslant 0} \Gamma_n(I)$. We identify each n-element set $\{s_1 < ... < s_n\}$ with the point $(s_1, ..., s_n) \in \mathbb{R}^n$. This introduces a natural measure on $\Gamma(I)$, by taking Lebesgue measure on each $\Gamma_n(I)$ and treating the $\{\emptyset\}$ as an atom of measure 1. If $I = \mathbb{R}_+$ we suppress it in the notation, thus obtaining the sets Γ_n, Γ. When convenient to do so we will consider function spaces over $\Gamma(I)$ as subspaces of similar function spaces over Γ, obtained via extending the appropriate functions by 0. Similarly, we will sometimes abuse notation by writing Γ_t for $\Gamma([0, t))$. We trust this will not lead to confustion and it will always be clear from context if we are slicing "on time" or "on chaos".

For some elementary properties of Guichardet space calculus, we refer the reader *e.g.* to [3]. The most fundamental property is the integral-sum identity, which reads as follows:

Theorem 1. *Let $f: \Gamma \times \Gamma \to \mathbb{C}$ be a measurable function. Then its integrable if and only if the function $\sigma \mapsto \sum_{\alpha \subset \sigma} f(\alpha, \sigma \setminus \alpha)$ is integrable and the following holds:*

$$\int_\Gamma \int_\Gamma f(\alpha, \beta) d\alpha d\beta = \int_\Gamma \sum_{\alpha \subset \sigma} f(\alpha, \sigma \setminus \alpha) d\sigma.$$

Due to the amount and volume of calculations involved, the tensor sign $\otimes$ between vectors will sometimes be omitted. It will always be clear from context what is meant. For tensor products of spaces, $\underline{\otimes}$ will always denote the algebraic and $\overline{\otimes}$ the ultraweak tensor product.

Chapter 1

Preliminaries

This chapter collects the basics of quantum stochastic calculus on the symmetric Fock space, along with the Guichardet viewpoint. At the end we also quote facts from the measure theory of Hilbert- and Banach-space valued functions which will accompany us throughout the thesis. We do not include the majority of the proofs, however we provide appropriate references. For more details about quantum stochastics we refer the Reader to *e.g.* [48] and for measurability questions to [5].

Firstly, let us recall a well-known inequality for integrals, the generalized Minkowski inequality, which will be useful in our approximations. We prove it here in the required generality for the Reader's convenience. For more information we refer the Reader to [83], Equation (9.12) and [33], Theorem 202.

Proposition 1.0.1. *Let $(X, \mu), (Y, \tau)$ be two measure spaces and let $F\colon X \times Y \to \mathbb{R}$ be measurable (in the product measure). Furthermore, let $1 \leqslant q \leqslant p < \infty$. Then*

$$\left(\int_X \left(\int_Y |F(x,y)|^q \tau(dy)\right)^{\frac{p}{q}} \mu(dx)\right)^{\frac{1}{p}} \leqslant \left(\int_Y \left(\int_X |F(x,y)|^p \mu(dx)\right)^{\frac{q}{p}} \tau(dy)\right)^{\frac{1}{q}}. \quad (1.1)$$

In particular, for $q = 1$, we get

$$\left(\int_X \left(\int_Y |F(x,y)|\tau(dy)\right)^p \mu(dx)\right)^{\frac{1}{p}} \leqslant \int_Y \left(\int_X |F(x,y)|^p \mu(dx)\right)^{\frac{1}{p}} \tau(dy). \qquad (1.2)$$

Proof. Note that if $p = q$ then Equation (1.1) is trivial. Assume $q < p$, in particular $p > 1$. We also see that once we prove the case $q = 1$, $q > 1$ follows by taking

$$F_0(x,y) = |F(x,y)|^q,$$

$$p_0 = \frac{p}{q}.$$

Thus assume that $1 = q < p$.

Let $J(x) = (\int_Y |F(x,y)|\tau(dy))$. Then we have

$$\begin{aligned}
\int_X \left(\int_Y |F(x,y)|^q \tau(dy)\right)^{\frac{p}{q}} \mu(dx)^p &= \int_X J(x)^p \mu(dx) = \int_X J(x)^{p-1} J(x)\mu(dx) \\
&= \int_X J(x)^{p-1} \int_Y |F(x,y)|\tau(dy)\mu(dx) \\
&= \int_Y \int_X J(x)^{p-1} |F(x,y)|\mu(dx)\tau(dy)
\end{aligned}$$

by the Fubini's theorem. Taking $r = \frac{p}{p-1}$, we apply Hölder's inequality to the inner integral

$$\begin{aligned}
LHS^p &\leqslant \int_Y \left(\int_X J(x)^{(p-1)r} \mu(dx)\right)^{\frac{1}{r}} \left(\int_X |F(x,y)|^p \mu(dx)\right)^{\frac{1}{p}} \tau(dy) \\
&= \int_Y \left(\int_X J(x)^p \mu(dx)\right)^{\frac{p-1}{p}} \left(\int_X |F(x,y)|^p \mu(dx)\right)^{\frac{1}{p}} \tau(dy).
\end{aligned}$$

Dividing through by $(\int_X J(x)^p \mu(dx))^{\frac{p-1}{p}}$, we get

$$\left(\int_X J(x)^p \mu(dx)\right)^{\frac{1}{p}} \leqslant \int_Y \left(\int_X |F(x,y)|^p \mu(dx)\right)^{\frac{1}{p}} \tau(dy),$$

as required. □

Throughout the thesis we will often use the notion of positive operators.

Definition 1.0.2. By a positive operator T on a Hilbert space H we mean a closed operator satisfying

$$\forall_{h\in\mathrm{Dom}(T)}\langle h, Th\rangle \geqslant 0$$

and in that case we write $T \geqslant 0$. Note that in the case of T being bounded with $\mathrm{Dom}(T) = \mathsf{H}$ its closedness is automatic.

We need their following property:

Theorem 1.0.3. *Let $T \in B(H_1 \oplus H_2)$. Then $T \geqslant 0$ if and only if it can be represented in the following form:*

$$T = \begin{bmatrix} A & \sqrt{A}V\sqrt{D} \\ \sqrt{D}V^*\sqrt{A} & D \end{bmatrix},$$

where $A, D \geqslant 0, \|V\| \leqslant 1$.

A proof of this can be found in *e.g.* [28] with historical notes in [22].

We also need some results about cores of unbounded operators. The first is a classical theorem, cf. [81], Theorem 4.11 a.

Lemma 1.0.4. *Let T be a closed and densely defined operator on a Hilbert space H. Then* $\mathrm{Dom}(T^*T)$ *is a core for T.*

We improve upon this lemma with the following folklore result.

Lemma 1.0.5. *Let T be a closed and densely defined operator on a Hilbert space H and let $\mathcal{D}$ be a dense subset of* $\mathrm{Dom}(T^*T)$ *(in the norm of H). Then $\mathcal{D}$ is a core for T.*

Proof. We need to prove that $\mathcal{D}$ is dense in $\mathrm{Dom}(T)$ in the graph norm. It suffices to check it is such in $\mathrm{Dom}(T^*T)$, as a dense subset of a dense subset is necessarily dense in the whole space. Thus let $\xi \in \mathrm{Dom}(T^*T) \setminus \mathcal{D}$ have the property that

$$\langle \xi, \eta \rangle + \langle T\xi, T\eta \rangle = 0$$

for all $\eta \in \mathcal{D}$. Then

$$\langle (I + T^*T)\xi, \eta \rangle = 0.$$

By density of $\mathcal{D}$ in H, this implies $(I + T^*T)\xi = 0$. However, injectivity of $I + T^*T$ gives $\xi = 0$. $\square$

1.1 Symmetric Fock space

In this section we introduce the Fock space and the Guichardet viewpoint. The material here is standard - for treatment of Fock spaces we refer the Reader to [48] and for Guichardet to [3]. Let H be a Hilbert space. The full Fock space over H is defined by

$$\Phi(H) = \bigoplus_{n \geqslant 0} H^{\otimes n}$$

and the symmetric Fock space by

$$\Gamma(H) = \bigoplus_{n \geqslant 0} H^{\vee n}.$$

where $H^{\vee n}$ is the symmetric n-fold tensor product of H, that is,

$$H^{\vee n} = \overline{\mathrm{Lin}}\{u \otimes \cdots \otimes u \colon u \in H\}.$$

A special class of vectors in $\Gamma(H)$ are the exponential vectors, which, for $u \in H$, are defined by

$$\varepsilon(u) := \left(\frac{1}{\sqrt{n!}} u^{\otimes n}\right)_{n \geqslant 0} = \left(1, u, \frac{u \otimes u}{\sqrt{2}}, \frac{u \otimes u \otimes u}{\sqrt{3!}}, \cdots\right) \in \Gamma(H).$$

We can normalise them by putting $\varpi(u) = \frac{1}{\|\varepsilon(u)\|}\varepsilon(u)$.

Exponential vectors form a linearly independent and total set in $\Gamma(H)$ and moreover

$$\langle \varepsilon(u), \varepsilon(v) \rangle = e^{\langle u, v \rangle} \text{ for all } u, v \in H.$$

It is also easily seen via

$$\|\varepsilon(u) - \varepsilon(v)\|^2 = e^{\|u\|^2} - 2\operatorname{Re} e^{\langle u, v \rangle} + e^{\|v\|^2}$$

that the map $u \mapsto \varepsilon(u)$ is continuous.

As a special case we have the vacuum vector $\varepsilon(0)$, which will from now on be denoted by Ω.

Thus, if for $S \subset H$ we denote

$$\mathcal{E}(S) := \operatorname{span}\{\varepsilon(u) \colon u \in S\},$$

then we can see that if S is dense in H, then $\mathcal{E}(S)$ is dense in $\Gamma(H)$.

Fock space also enjoys the exponential property, which means that for any Hilbert spaces H_1, H_2 we have

$$\Gamma(H_1 \oplus H_2) = \Gamma(H_1) \otimes \Gamma(H_2).$$

A subset $S \subset L^2(\mathbb{R}_+;\mathsf{k})$ is called *admissible* if $\mathcal{E}(S)$ is dense in $\Gamma(L^2(\mathbb{R}_+;\mathsf{k}))$ and for $f \in S, t \geqslant 0$ $f_{[0,t)} \in S$, where for $I \subset \mathbb{R}_+$ f_I denotes f multiplied by $\mathbf{1}_I$ the indicator function of I. We will also treat f_I as an element of $L^2(I;\mathsf{k})$ when appropriate.

For us it will be useful to employ the following identification of the symmetric Fock space in the case when $H = L^2(\mathbb{R}_+;\mathsf{k})$ for k a Hilbert space. We then have

$$H^{\vee n} \cong L^2_{\text{sym}}(\mathbb{R}^n_+;\mathsf{k}^{\otimes n}).$$

Let Γ denote the space of all finite subsets of $\mathbb{R}_+$. We create a measure μ on Γ by taking $\{\emptyset\}$ to have measure 1 and identifying each set $\{s_1 < ... < s_n\}$ with a point $(s_1, ..., s_n) \in \mathbb{R}^n_+$ and using Lebesgue measure. Thus, for example,

$$\mu(\{\sigma \in \Gamma \colon \sigma \subset [0,t]\}) = e^t, t \in \mathbb{R}_+.$$

In this setup we can identify

$$\mathcal{F}^{\mathsf{k}} := \Gamma(L^2(\mathbb{R}_+;\mathsf{k})) = \{f \in L^2(\Gamma;\Phi(\mathsf{k}))\colon f(\sigma) \in \mathsf{k}^{\otimes \#\sigma}\}$$

by

$$\varepsilon(f) \leftrightarrow \pi_f,$$

where

$$\pi_f(\{s_1 < ... < s_n\}) = f(s_1) \otimes ... \otimes f(s_n)$$

for $f \in L^2(\mathbb{R}_+;\mathsf{k})$.

It is also useful to introduce some notations for splitting the Fock space "in time" - thus, for $t \geqslant 0$, we will denote

$$\mathcal{F}^{\mathsf{k}}_{t)} := \Gamma(L^2([0,t);\mathsf{k})), \mathcal{F}^{\mathsf{k}}_{[t} = \Gamma(L^2([t,\infty);\mathsf{k})),$$

$$\mathcal{F}_t^{\mathsf{k}} := \Gamma(L^2([0,t];\mathsf{k})), \mathcal{F}_{(t}^{\mathsf{k}} = \Gamma(L^2((t,\infty);\mathsf{k})).$$

Let us notice that, due to the exponential property of Fock space, we have

$$\mathcal{F}^{\mathsf{k}} = \mathcal{F}_{t)}^{\mathsf{k}} \otimes \mathcal{F}_{[t}^{\mathsf{k}} = \mathcal{F}_t^{\mathsf{k}} \otimes \mathcal{F}_{(t}^{\mathsf{k}}.$$

1.1.1 Fock space operators

In this section we present some important Fock space linear operators.

The two classical operators of annihilation and creation are defined as closed, mutually self-adjoint operators with a core $\mathcal{E}(H)$, on which

$$a(u)\varepsilon(v) = \langle u, v\rangle\, \varepsilon(v), \tag{1.3}$$

$$a^\dagger(u)\varepsilon(v) = \frac{d}{dt}\varepsilon(v+tu)|_{t=0}, \qquad \text{for all } u, v \in H. \tag{1.4}$$

For an operator $T \in B(H)$ we can define $\Gamma_0(T)$ on $\mathcal{E}(H)$ by

$$\Gamma_0(T)\varepsilon(u) = \varepsilon(Tu).$$

By extending this to a closed operator $\Gamma(T)$ we get the second quantisation of T. This operator is contractive or (co)isometric if and only if T is. However, we can see that in general $\Gamma(T)$ need not be bounded.

In Guichardet language for $H = L^2(\mathbb{R}_+;\mathsf{k})$ we can write the second quantisation as

$$\Gamma(T)k(\sigma) = T^{\otimes \#\sigma}k(\sigma).$$

If we furthermore assume $\mathsf{k} = \mathbb{C}$, annihilation and creation operators become

$$a(u)k(\sigma) = \int_{\mathbb{R}_+} \overline{u(s)}k(\sigma \cup \{s\})ds,$$

$$a^\dagger(u)k(\sigma) = \sum_{s\in\sigma} u(s)k(\sigma \setminus s).$$

We will develop notation and framework to express these operators for nontrivial k in later chapters.

Another important family of operators is the family of Fock-Weyl operators. Fix $u \in H$ and let $W_0(u)$ be defined on $\mathcal{E}(H)$ via

$$W_0(u)\varepsilon(v) = e^{-\frac{1}{2}\|u\|^2 - \langle u,v\rangle}\varepsilon(u+v).$$

These operators turn out to be closable - in fact, they extend to a bounded operator on all of $\Gamma(H)$, which we denote by $W(u)$. In fact, $W(u)$ are unitary operators. They possess the important property that

$$\varpi(u) = W(u)\Omega.$$

The Weyl operators arise naturally in quantum physics through the unitary semigroups generated by the position and momentum operators. More precisely, given the momentum operator $p(u)$,

$$W(tu) = e^{-itp(u)},$$

where $t \geqslant 0$. Thus their central position in the mathematics is of no surprise. We will see the role they play in our research later. For more information on the Fock-Weyl operators in quantum stochascic calculus see e.g. [70].

1.2 Quantum stochastics

Let us fix Hilbert spaces h - called the *initial* space - and k - called the *noise dimension* space.

Definition 1.2.1. Let $\mathcal{D}$ be a dense subspace of h and $S \subset L^2(\mathbb{R}_+;\mathsf{k})$ be admissible. A family of linear operators $X = (X_t)_{t\geqslant 0}, X_t \colon \mathcal{D}\underline{\otimes}\mathcal{E}(S) \to \mathsf{h}\otimes\mathcal{F}^{\mathsf{k}}$ is called an *operator process* if:

- X_t is weakly measurable, *i.e.* $t \mapsto \langle x, X_t y\rangle$ is measurable for all $y \in \mathcal{D}\underline{\otimes}\mathcal{E}(S), x \in \mathsf{h}\otimes\mathcal{F}^{\mathsf{k}}$;
- The process is *adapted*, *i.e.* for each $t \geqslant 0$ there exists $X_{t)} \colon \mathcal{D}\underline{\otimes}\mathcal{E}(S|_{[0,t)}) \to \mathsf{h}\otimes\mathcal{F}^{\mathsf{k}}_{t)}$ such that

$$X_t = X_{t)} \otimes I_{\mathcal{E}(S|_{(t,\infty)})}.$$

Such a process is called *continuous* if the function $t \mapsto X_t x$ is continuous and *(weakly) measurable* if it is (WOT) measurable for every $x \in \bigcap_{t\geqslant 0} \mathrm{Dom}(X_t)$.

Remark 1.2.2. We will also talk about operator processes defined on $\widehat{\mathsf{k}}\otimes\mathcal{D}\otimes\mathcal{E}(S)$ for another Hilbert space $\widehat{\mathsf{k}}$ - in that case one can think of $\widehat{\mathsf{k}}\otimes\mathcal{D}$ as the dense subspace of $\widehat{\mathsf{k}}\otimes\mathsf{h}$ and take $\widehat{\mathsf{k}}\otimes\mathsf{h}$ as the initial space.

Example 1.2.3. We can construct important examples of operator processes using the fundamental operators introduced earlier.

1. (Creation and annihilation processes) Recall the annihilation and creation operators, given in Equations (1.3) and (1.4). Define, for $u \in L^2(\mathbb{R}_+;\mathsf{k})$,

$$A_t(u) = a(u\mathbf{1}_{[0,t[}),$$

$$A_t^\dagger(u) = a^\dagger(u\mathbf{1}_{[0,t[}).$$

It is easily seen that $(A_t(u))_{t\geqslant 0}, (A_t^\dagger(u))_{t\geqslant 0}$ form continuous operator processes, called the annihilation process and the creation process, respectively.

2. (Preservation process) For a selfadjoint operator $H \in B(L^2(\mathbb{R}_+;\mathsf{k}))$, $\left(e^{-itH}\right)_{t\in\mathbb{R}}$ forms a unitary group of operators. Consider a second quantisation of these, *i.e.* the unique bounded operator $\Gamma(e^{-itH}) \in B(\mathcal{F}^{\mathsf{k}})$ such that

$$\Gamma(e^{-itH})\varepsilon(u) = \varepsilon(e^{-itH}u), u \in L^2(\mathbb{R}_+;\mathsf{k}).$$

It is easily seen that $(\Gamma(e^{-itH}))_{t\geqslant 0}$ is a (strongly continuous) unitary group and thus admits a self adjoint generator, so that

$$\Gamma(e^{-itH}) = e^{-it\lambda(H)}$$

for a (not necessarily bounded) selfadjoint operator $\lambda(H)$ on $\mathcal{F}^{\mathsf{k}}$. If we denote the projection $L^2(\mathbb{R}_+;\mathsf{k}) \to L^2([0,t[;\mathsf{k})$ by P_t and assume that H commutes with each P_t, then we can form the preservation process:

$$\Lambda_t(H) = \lambda(HP_t),$$

which is again easily seen to be a continuous operator process. This process is also called conservation or gauge process in the literature.

3. (Time process) Given a family of operators $H(t) \in B(\mathcal{F}_t^{\mathsf{k}})$ with the property that $t \mapsto H(t)\xi$ is Bochner integrable on some domain $\mathcal{D}$, we can form an operator process $(\xi \mapsto \int_0^t H(s)\xi ds)_{t\geqslant 0}$, the integral being taken in Bochner sense and $\xi \in \mathcal{D}$. Assuming this domain is dense, this gives us again a continuous operator process.

4. (Weyl process) In the same manner as before, given $f \in \mathcal{F}^{\mathsf{k}}$, we define

$$W_t(f) = W(f_{[0,t[}),$$

giving a continuous operator process called the Weyl process.

An important tensor product of operator spaces with applications to quantum stochastic calculus was constructed by Lindsay and Wills in [60].

Definition 1.2.4. For an operator space V in $B(\mathsf{H})$ and a Hilbert space h, we define

$$V \otimes_M B(\mathsf{h}) = \{T \in B(\mathsf{H} \otimes \mathsf{h}) \colon E^x T E_y \in V \text{ for all } x, y \in \mathsf{h}\}.$$

This defines an operator space in $B(\mathsf{H} \otimes \mathsf{h})$ called the h-matrix space over V.

This tensor product will feature prominently in Section 4.4, as it turns out to be a natural domain for defining mapping quantum stochastic processes.

The noncommutative integrals

In this section we will use the abstract gradient and divergence operators to construct noncommutative stochastic integrals. We follow the treatment of Lindsay in [48], originating from his paper [52]. It relies on the noncommutative version of the Skorokhod integral, as opposed to the original Riemann sum treatment of Hudson and Parthasarathy [37]. For Hilbert spaces h, k we define the divergence and gradient operator as closures of the following densely defined operators:

$$\mathcal{S}(g \otimes u \otimes \varepsilon(f)) = u \otimes a^\dagger(g)(\varepsilon(f)),$$

$$\nabla(u \otimes \varepsilon(f)) = f \otimes u \otimes \varepsilon(f),$$

where $g, f \in L^2(\mathbb{R}_+, \mathsf{k})$ and $u \in \mathsf{h}$. (We stress the similarity of ∇ to an ampliation of the annihilation operator)

We define the local versions of these operators to be

$$\mathcal{S}_t(g \otimes u \otimes \varepsilon(f)) = \mathcal{S}(g\mathbf{1}_{[0,t[} \otimes u \otimes \varepsilon(f)),$$

$$\nabla_t(u \otimes \varepsilon(f)) = f\mathbf{1}_{[0,t[} \otimes u \otimes \varepsilon(f).$$

Definition 1.2.5 (Quantum stochastic integral)**.** For an operator process z with noise dimension space k and initial space $\widehat{\mathsf{k}} \otimes \mathsf{h}$ we decompose z as

$$z(t) = \begin{pmatrix} z_{00}(t) & z_{01}(t) \\ z_{10}(t) & z_{11}(t) \end{pmatrix},$$

using the decomposition $\widehat{\mathsf{k}} \otimes \mathsf{h} = \mathsf{h} \oplus (\mathsf{k} \otimes \mathsf{h})$, and, under the assumption that

$$t \mapsto z_{00}(t), t \mapsto z_{01}(t) \quad \text{are locally Bochner integrable,} \tag{1.5}$$

$$t \mapsto z_{10}(t), t \mapsto z_{11}(t) \quad \text{are locally square integrable,} \tag{1.6}$$

we define the quantum stochastic integral of z to be the operator process Z with initial space h given by

$$\begin{aligned} Z(t)(u \otimes \varepsilon(f)) =& \mathcal{S}_t(z_{10}(s)u \otimes \varepsilon(f) + z_{11}(s)(f(s) \otimes u \otimes \varepsilon(f))) \\ &+ \int_0^t (z_{00}(s)(u\varepsilon(f)) + z_{01}(s)(f(s) \otimes u \otimes \varepsilon(f)))ds. \end{aligned}$$

Such an integral of the z_{00} part is known as the *time* integral, z_{10} - *creation* integral, z_{01} - *annihilation* and z_{11} - *preservation* integral. When the integrability conditions (1.5), (1.6) are satisfied, the process z is called QS-integrable.

For a process z we will denote this quantum stochastic integral by $\Lambda_t(z)$.

Remark 1.2.6. It is sometimes useful to talk about each of the four integrals separately and introduce separate notations - in these cases, we would denote them as

$$\int_0^t z_{00}(s)ds, \int_0^t z_{01}(s)dA_s, \int_0^t z_{10}(s)dA_s^*, \int_0^t z_{11}(s)d\Lambda_s,$$

and call them the time, annihilation, creation and preservation integral, respectively.

Remark 1.2.7 (Coordinate setup)**.** It is important to note that this is the *coordinate-free* setup - the dimension of the noise dimension space k does not play a direct role, as we perform all our operations looking at it as simply a Hilbert space, and changing the dimension does not modify the formulas. To understand how it corresponds to an 'annihilation' or 'creation' integral in a more intuitive sense, the following observation is useful.

On a weak level, an annihilation integral $\int_0^t(\cdot)dA_s(h)$ - with the extra coordinate $h \in \mathsf{k}$ identified with the constant h-valued function on k - would be expected to satisfy the following identity:

$$\begin{aligned}\langle u \otimes \varepsilon(f), \int_0^t X_s dA_s(h)(v \otimes \varepsilon(g))\rangle &= \int_0^t \langle u \otimes \varepsilon(f), X_s A_s(h)(v \otimes \varepsilon(g))\rangle ds \\ &= \int_0^t \langle h(s), g(s)\rangle \langle u \otimes \varepsilon(f), X_s(v \otimes \varepsilon(g))\rangle ds \\ &= \int_0^t \langle u \otimes \varepsilon(f), (\langle h(s)| \otimes X_s)(g(s) \otimes v \otimes \varepsilon(s))\rangle.\end{aligned}$$

Thus one can see that this expected equality is merely the annihilation integral as we defined it for an operator process given by $\langle h(t)| \otimes X_t$ with initial space $\mathsf{k} \otimes \mathsf{h}$ and values in $\mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}$. We can see that now taking an arbitrary operator process with this initial space we get exactly our integral as defined, without the need of specifying a vector $h \in L^2(\mathbb{R}_+; \mathsf{k})$ (which means without having to specify the coordinate we are working on). By the exact same reasoning we can motivate the names of creation and preservation integrals.

The original coordinate setup is developed in [70]. The coordinate-free viewpoint first appeared in [29] and was further elaborated on in [31].

Definition 1.2.8. We define the quantum Itô projection $\Delta \in B(\widehat{\mathsf{k}})$ via

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathsf{k}} \end{pmatrix}.$$

This operator will play a fundamental role throughout.

Of particular interest to us are the two fundamental formulas of quantum stochastic calculus.

Proposition 1.2.9 (First fundamental formula)**.** *Let X be an operator process with domain $\widehat{\mathsf{k}} \otimes \mathcal{D} \otimes \mathcal{E}(S)$ which is QS-integrable on $\mathbb{R}_+$ and let $\Lambda(X)$ be the operator process on $\mathcal{D} \otimes \mathcal{E}(S)$ which is the quantum stochastic integral of X. Then for $u \in \mathsf{h}, v \in \mathcal{D}, g \in K, f \in S$ we have*

$$\begin{aligned}\langle u \otimes \varepsilon(g), &\Lambda(X)(v \otimes \varepsilon(f))\rangle \\ &= \int ds \langle \widehat{g}(s) \otimes u \otimes \varepsilon(g), X_s(\widehat{f}(s) \otimes v \otimes \varepsilon(f))\rangle\end{aligned}$$

and

$$\begin{aligned}&\|\Lambda(X)(u \otimes \varepsilon(f))\| \\ &\quad \leqslant \int ds \|\Delta^{\perp} X_s(\widehat{f}(s) \otimes u \otimes \varepsilon(f))\| + C_f\Big(\int ds \|\Delta X_s(\widehat{f}(s) \otimes u \otimes \varepsilon(f)))\|^2\Big)^{\frac{1}{2}},\end{aligned}$$

where C_f is a constant depending on f.

Proposition 1.2.10 (Second Fundamental formula)**.** *Let X and Y be QS-integrable processes with domains $\widehat{\mathsf{k}}\otimes\mathcal{D}\otimes\mathcal{E}(S)$,$\widehat{\mathsf{k}}\otimes\mathcal{D}'\otimes\mathcal{E}(S')$, respectively. Then, for $u\in\mathcal{D}, u'\in\mathcal{D}', f\in S, g\in S'$ we have*

$$
\begin{aligned}
\langle \Lambda(X)(u\otimes\varepsilon(f)), &\Lambda(Y)(v\otimes\varepsilon(g))\rangle \\
&= \int dt\,\Big(\Big\langle \widehat{f}(t)\otimes\Lambda_t(X)u\otimes\varepsilon(f)), Y_t(\widehat{g}(t)\otimes v\otimes\varepsilon(g))\Big\rangle \\
&+ \Big\langle X_t(\widehat{f}(t)\otimes u\otimes\varepsilon(f)), \widehat{g}(t)\otimes\Lambda_t(Y)_t(v\otimes\varepsilon(g))\Big\rangle \\
&+ \Big\langle X_t(\widehat{f}(t)\otimes u\otimes\varepsilon(f)), \Delta Y_t(\widehat{g}(t)\otimes v\otimes\varepsilon(g))\Big\rangle\Big)\,.
\end{aligned}
$$

Remark 1.2.11. We should keep in mind the classical probability picture here, in which the Itô formula is intuitively understood as

$$(\mathrm{d}W_t)^2 = \mathrm{d}t,$$

i.e. the "multiplication table" of time and Wiener integral is:

	$\mathrm{d}W_t$	$\mathrm{d}t$
$\mathrm{d}W_t$	$\mathrm{d}t$	0
$\mathrm{d}t$	0	0

In a similar manner, the quantum Itô formula can be summarised by the following table:

	$\mathrm{d}t$	$\mathrm{d}A_t$	$\mathrm{d}A_t^*$	$\mathrm{d}N_t$
$\mathrm{d}t$	0	0	0	0
$\mathrm{d}A_t$	0	0	$\mathrm{d}t$	$\mathrm{d}A_t$
$\mathrm{d}A^*t$	0	0	0	0
$\mathrm{d}N_t$	0	0	$\mathrm{d}A_t^*$	$\mathrm{d}N_t$

Here we need to remember that order of integration matters - we consider the columns to be on the left of our multiplication, i.e. $\mathrm{d}N_t\,\mathrm{d}A_t^* = \mathrm{d}A_t^*$.

In other words, if integration happens in the following order:

$$\mathrm{d}A^*,\ \mathrm{d}N,\ \mathrm{d}A,$$

then no extra correction terms appear. This is sometimes referred to as the Wick ordering.

1.3 Maassen–Meyer–Lindsay kernels

Let us recall the theory of integral kernels. This theory was initiated by Maassen [63], who introduced two argument kernels to represent the creation and annihilation integrals. Meyer (cf. [66]) observed that the addition of a third argument takes care of the preservation integral and Lindsay added the fourth to represent the time integral.

In this presentation we rely primarily on Lindsay's paper [53]. All the integrals are taken over the whole Guichardet space, but one could just as well restrict oneself to local integrals by restricting ourselves to Γ_t for some positive t. Our noise dimension space is, for now, taken to be $\mathsf{k} = \mathbb{C}$.

The main idea guiding us in this section is the following. By the chaos completeness property of Brownian motion, all L^2 functions on the Wiener space can be expressed as multiple Wiener integrals. In particular, products of those functions which remain in L^2 are again multiple Wiener integrals. The work of Lindsay and Maassen shows how to express the integrand of the product by appropriately convolving the integrands of the terms. To express this, Guichardet presentation is essential.

Any function in $L^2(\Gamma)$ can be represented as

$$f = \int_\Gamma f(\sigma)\,\mathrm{d}W_\sigma,$$

where W_σ denotes the multiple Wiener integral. Then, multiplication of functions corresponds to the following convolution, called the Wiener product:

$$(f \star g)(\gamma) = \sum_{\alpha \subset \gamma} \int d\omega f(\alpha \cup \omega) g(\omega \cup (\gamma \setminus \alpha)),$$

i.e. $f(\sigma)g(\sigma) = \int (f \star g)(\gamma) dW_\gamma$.

Let us go quantum and try to similarly define operators. First, we need to split the Wiener integration into an annihilation and creation integral. Second, we need to add a number integral. Thus we seek to define an operator on $L^2(\Gamma)$ given by

$$X = \iiint x(\alpha, \beta, \gamma) dA^*_\alpha dN_\beta dA_\gamma.$$

Here $x(\alpha, \beta, \gamma)$ is a complex number and α, β, γ are pairwise disjoint (!). This particular order of integration operations is a consequence of the Wick ordering - cf. Remark 1.2.11.

By the Itô relations, without worrying about the analytical assumptions for now, we can define this operator by its action on a vector:

$$(Xf)(\sigma) = \sum_{\alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 = \sigma} \int d\omega x(\alpha_1, \alpha_2, \omega) f(\omega \cup \alpha_2 \cup \alpha_3).$$

We also have a corresponding convolution for the product of operators X, Y with kernels x, y:

$$(x \star y)(\alpha, \beta, \gamma) = \sum \int d\omega x(\alpha_2, \beta_1 \cup \beta_2 \cup \alpha_3, \gamma_1 \cup \omega \cup \gamma_3) y(\alpha_1 \cup \alpha_3 \cup \omega, \gamma_1 \cup \beta_2 \cup \beta_3, \gamma_2),$$

where the sum is over all partitions $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3, \beta = \beta_1 \sqcup \beta_2 \sqcup \beta_3, \gamma = \gamma_1 \sqcup \gamma_2 \sqcup \gamma_3$.

One can also define operators via kernels with four arguments, adding in the time integral. Thus now our operator is (formally) defined as

$$X = \iiiint x(\alpha, \beta, \gamma, \delta) dA^*_\alpha dN_\beta dA_\gamma d\delta.$$

The corresponding convolution is then given by a purely combinatorial formula

$$(x \star y)(\alpha, \beta, \gamma, \delta) = \sum x(\alpha_2, \beta_1 \cup \beta_2 \cup \alpha_3, \gamma_1 \cup \delta_2 \cup \gamma_3, \delta_1) y(\alpha_1 \cup \delta_2 \cup \alpha_3, \gamma_1 \cup \beta_2 \cup \beta_3, \gamma_2, \delta_3)$$

with analogous notation to the one before.

This theory can then be lifted to a Fock space with finite multiplicity $d > 1$. This was done by Dermoune in [17] - cf. also [66]. In his work, he makes use of the isomorphism

$$\mathcal{F}^{\mathbb{C}^d} = (\mathcal{F})^{\otimes d} = L^2(\Gamma)^{\otimes d} = L^2(\Gamma^d).$$

The fact that our domain is now Γ^d rather than Γ necessitates our four argument kernels not to take sets as arguments anymore - now the correct expression for a kernel is

$$x((A^\alpha_0), (A^\alpha_\beta), (A^\alpha_0), (A^0_0)) \in \mathbb{C},$$

where $\alpha, \beta \in \{1, ..., d\}$. A way of understanding this is that now a kernel takes as an argument a $(d+1) \times (d+1)$ matrix of sets $A = [A^\alpha_\beta]_{0 \leqslant \alpha, \beta \leqslant d}$, with A^0_0 term corresponding to the time integral, the A_0^i terms to the creation integral, A_j^0 to annihilation and the $d \times d$ block A_j^i to the number and exchange integrals (for $i, j > 0$). Again, in this presentation, A_α^β are assumed to be pairwise disjoint. The action of the kernel on the vector is given as follows. Let $\alpha_1 \ldots, \alpha_d$ be pairwise disjoint finite subsets of $\mathbb{R}_+$. We take:

$$X\xi(\alpha_1, \cdots, \alpha_d) = \int d\alpha_0 \sum_{\alpha_0 = \underset{j=0,\ldots,d}{\sqcup} \alpha_{0j}} \; \sum_{\alpha_i = \underset{j=0,\ldots,d+1}{\sqcup} \alpha_{ij}, 1 \leqslant i \leqslant d} x((\alpha_{i0}), (\alpha_{ij}), (\alpha_{0j}), \alpha_{00})$$
$$\xi((\underset{j=1,\cdots,d}{\cup} \alpha_{ji} \cup \alpha_{(d+1)i})_{i=1}^d).$$

We stress that this sum is taken over disjoin partitions of each α_i. In conclusion, our matrix A from the previous page takes the form of $[\alpha_{ij}]_{0 \leqslant i,j \leqslant d}$, where $\cup_j \alpha_{ij} = \alpha_i$ for $0 \leqslant i \leqslant d$. The notation here aims to help us keep track of the dimension within $\mathbb{C}^d$ - sets α_{0i} correspond to the dimension we integrate out, while α_{ij} are at the j-th coordinate of $\mathbb{C}^d$. Thus we see that the i-th dimension of our vector ξ is dependent upon the sets $(\underset{j=1,\cdots,\mathrm{d}}{\cup} \alpha_{ji} \cup \alpha_{(d+1)i})$ - i.e. all the α_{ji} sets where we act on ξ with the exchange and creation integrals and $\alpha_{(d+1)i}$ which is the set where ξ is not acted upon. This point of view will be elaborated upon in Chapter 4.

In Dermoune's presentation, it is assumed that both the kernel and the vectors from the domain of the integral kernel operator have compact support and satisfy the geometric condition:

$$|x((\alpha_{ij})_{i,j=0}^d)| \leqslant C_1 M_1^{\sum \#\alpha_{ij}}, \qquad |\xi((\alpha_i)_{i=1}^d)| \leqslant C_2 M_2^{\sum \#\alpha_i}.$$

We will see that, in fact, we can slightly improve upon these assumptions.

For completeness sake, we cite Dermoune's kernel convolution formula:

$$\begin{aligned}
&(x \star y)((T_0^\alpha),(T_\beta^\alpha),(T_0^\alpha),(T_0^0))\\
&\quad= \sum_{T_\sigma^\rho = A_\sigma^\rho \sqcup \bigsqcup_{\gamma\in\{0,\dots,d\}} B_{\gamma,\sigma}^{\rho;\gamma} \sqcup C_\sigma^\rho} x\left((A_0^\alpha),(A_\alpha^\beta \sqcup \bigsqcup_{\gamma>0} B_{\alpha,\gamma}^{\beta,\alpha}),(A_\alpha^0 \sqcup \bigsqcup_{\gamma} B_{\alpha,\gamma}^{0,\alpha}), A_0^0\right)\\
&\quad\cdot y\left((\bigsqcup_{\gamma} B_{\alpha,0}^{\gamma;\alpha} \sqcup C_0^\alpha),(\bigsqcup_{\gamma>0} B_{\alpha,0}^{\gamma;\alpha} \sqcup C_\alpha^\beta),(C_\alpha^0), C_0^0\right).
\end{aligned} \tag{1.7}$$

In simplest terms, this expression tells us that the intuition of arranging the $(d+1)^2$ sets into a $(d+1)\times(d+1)$ matrix is the correct one - a closer inspection of each term will readily show that the way of modifying each argument is exactly that of matrix multiplication, with a twist which, as it turns out, is just the Itô projection. In our work we will prove that our kernel framework is a more general case of this formula, extending it to infinite d and recovering the above identity as a special case. As we will not be using the isomorphism

$$L^2(\Gamma)^{\otimes d} = L^2(\Gamma^d),$$

we will be able to express our formula without resorting to (possibly infinite) matrices of sets. Finally, our work will endeavor to package Equation (1.7) into a more pleasing and easier to apply form.

1.4 Measurability of vector and operator valued functions

1.4.1 Classical measurability

As we are interested in integrating families of operators and vectors on Hilbert space, it is important to revisit some results about measurability of Hilbert and Banach-space valued functions. We will apply the results presented here to the particular case of vector and operator kernels on the Guichardet space in Chapter 2. In this presentation we rely primarily on the papers of Johnson [42], Badrikian, Johnson and Yoo [5] and Schlüchtermann [76].

Before we start, let us recall some classical definitions from topology.

Definition 1.4.1. A linear, locally convex topological space E is called a Fréchet space if its topology is induced by a complete, translation invariant metric. Equivalently, E is Hausdorff and there is a countable family of seminorms on E inducing the topology with respect to which E is complete.

Definition 1.4.2. A Hausdorff topological space Z is called a Lusin space if and only if it is the image of a Polish space under a continuous bijection.

In the realm of non-scalar valued functions, there are many notions of measurability and different authors have different naming conventions. We will go through the different notions carefully.

Definition 1.4.3. Let $(S,\Sigma),(S',\Sigma')$ be measurable spaces, T,T' be topological spaces Then:

(a) $f\colon S\to S'$ is measurable if $\forall_{A\in\Sigma'}\ f^{-1}(A)\in\Sigma$;

(b) $g\colon S\to T'$ is measurable if $\forall_{A\in\mathrm{Borel}(T')}\ g^{-1}(A)\in\Sigma$;

(c) $h\colon T \to T'$ is Borel if $\forall_{A \in \mathrm{Borel}(T')}\ h^{-1}(A) \in \mathrm{Borel}(T)$.

Remark 1.4.4. (i) Some authors use "Borel" for (b) too.

(ii) For (b) the following suffices:

$$\forall_{A \subset T',\ \text{open}}\ g^{-1}(A) \in \Sigma.$$

(iii) Let T' be second countable. Then, for (b), by Lindelöf's theorem (cf. [44]) the following suffices:

$$\forall_{A \in \mathcal{S}'}\ g^{-1}(A) \in \Sigma,$$

where $\mathcal{S}'$ is any subbase for the topology of T'.

(iv) (cf. [5]) Let $\mathcal{F}$ be a family of Borel maps from T to $\mathbb{C}$ where T is Lusin. If $\mathcal{F}$ is countable and separates the points of T, then

$$\sigma(\mathcal{F}) = \mathrm{Borel}(T).$$

(v) (cf. [5]) Let $\mathcal{F}$ be a family of continuous functions from T to T', where T is Lusin and T' is Hausdorff. If $\mathcal{F}$ separates the points of T, then so does some countable subset $\mathcal{F}_0$ of $\mathcal{F}$, so that

$$\mathrm{Borel}(T) = \sigma(\mathcal{F}_0) = \sigma(\mathcal{F}).$$

Moving on to the case of Fréchet spaces, let us introduce two more definitions of measurability:

Definition 1.4.5. Let (S, Σ) be a measurable space and E a Fréchet space. Then:

(a) $f\colon S \to E$ is scalarly or weakly measurable if

$$\forall_{\varphi' \in E^*}\ \varphi \circ f\colon S \to \mathbb{C}$$

is measurable;

(b) $g\colon S \to E$ is Bochner measurable if $\exists_{f_n\colon S \to E}\ f_n$ simple, measurable and $f_n \to f$ pointwise.

These two notions turn out to coincide if E is separable:

Proposition 1.4.6. *Let $f\colon S \to E$ for (S, Σ) a measurable space and E a separable Fréchet space. Then the following are equivalent:*

(i) f is weakly measurable;

(ii) f is Bochner measurable.

Proof. $(ii) \Rightarrow (i)$ is trivial. Let us consider the opposite direction. As E is a separable Fréchet space, it is trivially Lusin. Thus part (v) of Remark 1.4.4 applies. As E^* obviously separates points of E, this means that

$$\mathrm{Borel}(E) = \sigma(E^*). \tag{1.8}$$

Weak measurability of f means that for every $\varphi \in E^*$ and $U \in \mathrm{Borel}(\mathbb{C})$ we have

$$f^{-1}(\varphi^{-1}(U)) \in \Sigma.$$

But by Equation (1.8), sets of the form $\varphi^{-1}(U)$ generate $\mathrm{Borel}(E)$, so in fact weak measurability can be equivalently stated as

$$f^{-1}(V) \in \Sigma$$

whenever $V \in \text{Borel}(E)$.

Now, let $\{x_k\}_{k\in\mathbb{N}}$ be a countable dense subset of E. For $n,k \in \mathbb{N}$, we denote:

$$A_k^n = \begin{cases} B_{\frac{1}{n}}(x_1) & \text{for } k = 1 \\ B_{\frac{1}{n}}(x_k) \setminus \cup_{l=1}^{k-1} B_{\frac{1}{n}}(x_l) & \text{for } k > 1. \end{cases},$$

where $B_{\frac{1}{n}}(x_k)$ is the open ball of radius $\frac{1}{n}$ around x_k in the Fréchet metric d of E. We see that for each fixed $n \in \mathbb{N}$ $(A_k^n)_{k\in\mathbb{N}}$ are pairwise disjoint, Borel and $E = \cup_{k\in\mathbb{N}} A_k^n$ by density of $\{x_k\}_{k\in\mathbb{N}}$ in E. Thus

$$\forall_{n,k\in\mathbb{N}} f^{-1}(A_k^n) \in \Sigma.$$

Therefore the functions

$$f_{n,k} = \sum_{l=1}^{k} 1_{f^{-1}(A_l^n)} x_l$$

are simple, measurable functions and their pointwise limits

$$f_n = \lim_{k\to\infty} f_{n,k} = \sum_{k=1}^{\infty} 1_{f^{-1}(A_k^n)} x_k$$

are thus well-defined (as all A_k^n are disjoint) and Bochner measurable.

It is easily seen that $f_n \to f$ pointwise. Indeed, for $s \in \Sigma, \varepsilon > 0$ we can find $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. By density of $\{x_k\}_{k\in\mathbb{N}}$,

$$d(x_{k_0}, f(s)) < \frac{1}{n}$$

for some $k_0 \in \mathbb{N}$, so $f(s) \in A_k^n$ for some $k \leqslant k_0$ and in particular

$$d(x_k, f(s)) < \frac{1}{n}.$$

But this means that $f_n(s) = x_k$ and thus

$$d(f_n(s), f(s)) < \frac{1}{n} < \varepsilon,$$

as required.

But now, applying a simple diagonalization procedure, it is easily seen that $g_n = f_{n,n}$ is a sequence of simple functions which tend pointwise to f. Thus f is Bochner measurable, which ends the proof. □

We introduce a helplful bit of notation for the rest of this section.

Definition 1.4.7. For Fréchet spaces E and F, set

$$CL(E; F) = \{T \colon E \to F |\, T \text{ is continuous and linear}\},$$

$$CL_{st}(E; F) = (CL(E; F), SOT).$$

The following fact is fundamental:

Theorem 1.4.8 (Theorem 7 in [77] along with Theorem 1 in [5])**.** *Let E, F be separable Fréchet spaces. Then $CL_{st}(E; F)$ is a Lusin space.*

The following application is due to Badrikian *et al* [5]:

Theorem 1.4.9. *Let E and F be separable Fréchet spaces. Then*

$$\text{Borel}(CL_{st}(E; F)) = \sigma\{\varphi \circ \varepsilon_x \colon x \in E, \varphi \in F^*\},$$

where $\varepsilon_x \colon CL(E; F) \to F$ is the evaluation on x mapping, given by

$$\varepsilon_x(T) = Tx.$$

Proof. One can see that this follows from Remark 1.4.4. Indeed, as, by Theorem 1.4.8 $CL_{st}(E;F)$ is a Lusin space and maps $\varphi \circ \varepsilon_x$ obviously separate points in $CL(E;F)$, we immediately get the conclusion by part (v) of Remark 1.4.4. □

We can apply the machinery we have developed so far to obtain the following measurability result:

Theorem 1.4.10 (cf. [5])**.** *Let $f_i \colon S \to CL_{st}(E_i, E_{i+1})$ for $i = 1, 2$ for a measurable space (S, Σ) and separable Fréchet spaces E_1, E_2 and E_3. If f_1 and f_2 are measurable (in the topological sense - that is, in line with (b) in Definition 1.4.3), then so is*

$$f_2(\cdot)f_1(\cdot).$$

Proof. Set $f = f_2(\cdot)f_1(\cdot)$. Let $x \in E_1$ and $\varphi \in E_3^*$. Then

$$(\varphi \circ \varepsilon_x \circ f)(s) = \varphi(f_2(s)f_1(s)x) = (f_2(s)^*\varphi)(f_1(s)x),$$

where $f_2(s)^* \in CL(E_3^*; E_2^*)$ denotes the Fréchet dual of $f_2(s)$. By Proposition 1.4.6 applied to the function $s \mapsto f_1(s)x$, there is a sequence of simple measurable functions

$$\psi_n \colon S \to E_2$$

such that $\psi_n \to f_1(\cdot)x$ pointwise. For each $n \in \mathbb{N}$,

$$\varphi(f_2(\cdot)\psi_n(\cdot)) \colon S \to \mathbb{C}$$

is a simple measurable function and

$$\varphi(f_2(\cdot)\psi_n(\cdot)) \to \varphi \circ \varepsilon_x \circ F$$

pointwise. Therefore $\varphi \circ \varepsilon_x \circ f$ is measurable. The result therefore follows from Theorem 1.4.9. □

1.4.2 μ measurability

The previous section merely used the σ-algebra of the space S, with no mention of the actual measure on it. With the measure in play, we can define new notions of measurability. These will be more useful when working with measure equivalence classes of functions, rather than functions themselves.

Definition 1.4.11. Let $f\colon S \to X$ for a complete σ-finite measure space (S, Σ, μ) and a separable Banach space X.

- f is μ-measurable if there is a sequence

  $$f_n\colon S \to X, \text{ countably valued and measurable}$$

  such that $f_n \to f$ μ-a.e.

- f is weakly μ-measurable if

  $$\forall_{\varphi \in X^*}\ \varphi \circ f\colon S \to \mathbb{C} \text{ is } \mu\text{-measurable.}$$

Remark 1.4.12. (i) (cf. Section 3.5 in [35]) If (S, Σ, μ) is finite then μ-measurability of f is equivalent to the existence of a sequence

$$f_n\colon S \to X, \text{ simple and measurable,}$$

such that $f_n \to f$ μ-a.e.

(ii) μ-measurability is often called strong μ-measurability or strong measurability in the literature.

Finiteness of μ, in fact, allows us to say even more:

Theorem 1.4.13 (cf.Theorem IV.22 in [75])**.** *If μ is finite, then the following are equivalent:*

- *f is μ-measurable,*
- *f is measurable,*
- *f is weakly μ-measurable.*

Let us now move to the case when our f takes values in a space of bounded operators on Banach spaces. Let $f\colon S \to B(X;Y)$ for a complete σ-finite measure space (S, Σ, μ) and separable Banach spaces X and Y. Again, we denote $B(X;Y)$ with the strong operator topology by $B_{st}(X;Y)$.

Definition 1.4.14.

- f is strong operator μ-measurable if

$$\forall_{x\in X}\ f(\cdot)x\colon S \to Y \text{ is } \mu\text{-measurable},$$

- f is weak operator μ-measurable if

$$\forall_{x\in X}\ f(\cdot)x\colon S \to Y \text{ is weakly } \mu\text{-measurable},$$

in other words,

$$\forall_{x\in X}\forall_{\varphi\in Y^*}\ \varphi(f(\cdot)x)\colon S \to \mathbb{C} \text{ is measurable.}$$

The connection between this notion and ours is the following (again, the finiteness of μ playing a crucial role):

Theorem 1.4.15. *If μ is finite, then f is strongly operator μ-measurable if and only if f is measurable as a function*

$$f\colon (S,\Sigma) \to B_{st}(X;Y).$$

Proof. The last assertion means that

$$\forall_{U\in \mathrm{Borel}(B_{st}(X;Y))}\ f^{-1}(U) \in \Sigma.$$

By Theorem 1.4.9, this is equivalent to saying that

$$\forall_{x\in X,\varphi\in Y^*,U\in \mathrm{Borel}(\mathbb{C})}\ f^{-1}\varepsilon_x^{-1}\varphi^{-1}(U) \in \Sigma,$$

or in other words that

$$\forall_{x\in X,\varphi\in Y^*,U\in \mathrm{Borel}(\mathbb{C})}\ \{s \in S\colon \varphi(f(s)x) \in U\} \in \Sigma.$$

But that is equivalent to saying that f is weak operator μ-measurable, *i.e.* each $f(\cdot)x$ is weakly μ-measurable. However, since μ is finite, Theorem 1.4.13 tells us that that is equivalent to strong μ-measurability, which in turn is equivalent to strong operator μ-measurability of f. □

Corollary 1.4.16. *If μ is finite and $f\colon S \to B(X;Y)$ and $g\colon S \to B(Y;Z)$ are strong operator μ-measurable, then so is*

$$g \circ f\colon g(\cdot)f(\cdot)\colon S \to B(X;Z).$$

Proof. By Theorem 1.4.15 we can treat f, g as measurable functions with values in $B_{st}(X;Y), B_{st}(Y;Z)$, respectively - however, we know by Theorem 1.4.10 that compositions of such functions remain measurable when the measure in question is finite. □

It is worth noting that these final theorems assume finiteness of μ. In other words, when one works with measure equivalences of operators, in order to ensure that composition of measurable operator-valued functions yields a measurable operator-valued function one has to make sure that the measure spaces one considers are all finite.

Chapter 2

Vector quantum Wiener integrals

In this chapter we develop the language of vector and operator kernels. In the first section we consider their algebra, to move on to their measurability in the second section. Finally, we consider only their measure equivalence classes and finish this chapter by constructing a vector version of a quantum Wiener integral. The product quantum Wiener integral will be constructed in the following chapter.

2.1 Algebra of kernels

2.1.1 Kernels and placement

This work is inspired by results of Maassen and Lindsay ([54]). At the end of the next Chapter we present the exact correspondence between the kernels constructed by us and the original integral kernels from the aforementioned authors and *e.g.* Dermoune ([17]). In this we focus on constructing the kernel framework which will serve us for the rest of this thesis.

We require both vector kernels and operator kernels. The former are needed for amalgamating multiple Wiener integrals and multiple time integrals; the latter for directly defining QS Wiener integrals. Thus let h, H be Hilbert spaces and let $K(\mathsf{h}, \mathsf{H})$ denote the linear space of families

$$\zeta = (\zeta(\sigma) \in \mathsf{h} \otimes \mathsf{H}^{\otimes \# \sigma})_{\sigma \in \Gamma} .$$

When it is convenient to do so, we may regard $K(\mathsf{h}, \mathsf{H})$ as a subspace of $F(\Gamma; \mathsf{h} \otimes \Phi(\mathsf{H}))$, where we recall that $\Phi(\mathsf{H})$ denotes the full Fock space over $\mathsf{H} : \Phi(\mathsf{H}) = \bigoplus_{n \geq 0} \mathsf{H}^{\otimes n}$.

The subspace of constant vector kernels $K_{\text{const}}(\mathsf{h}, \mathsf{H})$ consists of those vector kernels ζ satisfying

$$\zeta = (\zeta_{\# \sigma})_{\sigma \in \Gamma} \qquad \text{for some family } (\zeta_n \in \mathsf{h} \otimes \mathsf{H}^{\otimes n})_{n \in \mathbb{Z}_+} .$$

This class is more relevant on $\Gamma_{[0,T]} (T \in \mathbb{R}_+)$ than on Γ itself. In Section 4 we identify further relevant subspaces.

Product vector kernels $v \otimes \pi_\varphi$ $(v \in \mathsf{h}, \varphi \in F(\mathbb{R}_+; \mathsf{H}))$ form a very important class of kernels. These are defined by

$$(v \otimes \pi_\varphi)(\sigma) = \begin{cases} v & \text{if } \sigma = \emptyset; \\ v \otimes \varphi(s_1) \otimes ... \otimes \varphi(s_n) & \text{if } \sigma = \{s_1 < ... < s_n\} \in \Gamma \setminus \{\emptyset\} \end{cases} .$$

The elementary properties contained in the lemma below are useful.

Lemma 2.1.1. *Given Hilbert spaces* h, H*, the following hold:*

1. *For* $T \subset \mathsf{h}$ *and* F_0 *a subspace of* $F(\mathbb{R}_+; \mathsf{H})$ *the set*

$$\{(v \otimes \pi_\varphi)(\sigma) : v \in T, \varphi \in F_0\}$$

is total in $\mathsf{h} \otimes \mathsf{H}^{\otimes \# \sigma}$ *for all* $\sigma \in \Gamma$*, provided that* T *is total in* h *and* $\{\varphi(s) \colon \varphi \in F_0\}$ *is total in* H *for all* $s \in \mathbb{R}_+$*.*

2. In the notation $\hat{\varphi}(s) = \widehat{\varphi(s)}$ for $\varphi \in F(\mathbb{R}_+; \mathsf{H})$, the set

$$\{\hat{\varphi}(s) \colon \varphi \in F(\mathbb{R}_+; \mathsf{H})\}$$

is total in $\widehat{\mathsf{H}}$.

3. For $\varphi, \psi \in L^2(\mathbb{R}_+; \mathsf{H})$ and $u, v \in \mathsf{h}, u \otimes \pi_\varphi, v \otimes \pi_\psi \in \mathsf{h} \otimes \mathcal{F}^{\mathsf{H}}$ and

$$\langle u \otimes \pi_\varphi, v \otimes \pi_\psi \rangle = \langle u, v \rangle e^{\langle \varphi, \psi \rangle}.$$

Definition 2.1.2 (Placement - vectors)**.** For a unit vector $e_0 \in \mathsf{H}$, element $\sigma \in \Gamma$ and $\alpha \subset \sigma$, the prescription

$$\pi_\varphi(\alpha) \mapsto \pi_\psi(\sigma), \qquad \psi = 1_\alpha \varphi + e_0 1_{\mathbb{R}_+ \setminus \alpha}, \varphi \in F(\mathbb{R}_+, \mathsf{H})$$

(in which 1_S denotes the indicator function of S) uniquely determines a linear isometry $J^{e_0}_{\alpha;\sigma} \in B(\mathsf{H}^{\otimes \# \alpha}; \mathsf{H}^{\otimes \# \sigma})$. Thus, for example

$$J^{e_0}_{\emptyset;\emptyset} = I_{\mathbb{C}}, J^{e_0}_{\emptyset;\sigma} = |e_0\rangle^{\otimes \# \sigma} \text{ and } J^{e_0}_{\sigma;\sigma} = I_{\mathsf{H}^{\otimes \# \sigma}} \qquad (\sigma \in \Gamma),$$

and if $s = \max \sigma$ we get

$$J^{e_0}_{\{s\};\sigma} = I_{|e_0\rangle^{\otimes(\#\sigma - 1)}} \otimes I_h.$$

For $\zeta \in K(\mathsf{h}, \mathsf{H})$ and sets $\sigma \in \Gamma, \alpha \subset \sigma$ set

$$\zeta(\alpha; \sigma, e_0) = (I_{\mathsf{h}} \otimes J^{e_0}_{\alpha;\sigma}) \zeta(\alpha) \in \mathsf{h} \otimes \mathsf{H}^{\otimes \# \sigma}.$$

For an operator space $V \subset B(\mathsf{h}_1;\mathsf{h}_2)$ and an ultraweakly closed operator space $Z \subset B(\mathsf{H}_1;\mathsf{H}_2)$ let $OK(V,Z)$ denote the linear space of families

$$x = (x(\sigma) \in V \overline{\otimes} Z^{\overline{\otimes} \# \sigma})_{\sigma \in \Gamma}.$$

For the majority of the thesis, V will be taken to be $B(\mathsf{h})$ and $Z = B(\widehat{\mathsf{k}})$ for fixed Hilbert spaces h, k. For example

$$\pi_F(\sigma) = \begin{cases} I_{\mathbb{C}} & \text{if } \sigma = \emptyset \\ F(s_1) \otimes \cdots \otimes F(s_n) & \text{if } \sigma = \{s_1 < ... < s_n\} \in \Gamma \setminus \{\emptyset\} \end{cases}$$

defines an operator kernel $\pi_F \in OK(\mathbb{C}, B(\mathsf{H}_1;\mathsf{H}_2))$ whenever $F \in F(\mathbb{R}_+; B(\mathsf{H}_1;\mathsf{H}_2))$. In the case when F is a constant function (*i.e.* $F \in B(\mathsf{H}_1;\mathsf{H}_2)$) we will write $F^{\otimes} := \pi_F$.

The subclass of constant operator kernels $OK_{\text{const}}(V,Z)$ consists of these kernels x which satisfy

$$x = (x_{\#\sigma})_{\sigma \in \Gamma} \quad \text{for a family } (x_n \in V \overline{\otimes} Z^{\overline{\otimes} n})_{n \in \mathbb{Z}_+}.$$

This class has already found applications in QS analysis, cf. [60].

Note that corresponding to Lemma 2.1.1, the set

$$\{\pi_F(\sigma) \colon F \in F(\mathbb{R}_+, B(\mathsf{H}_1;\mathsf{H}_2))\}$$

is ultraweakly total in $B(\mathsf{H}_1;\mathsf{H}_2)^{\overline{\otimes} \# \sigma} = B(\mathsf{H}_1^{\otimes \# \sigma}; \mathsf{H}_2^{\otimes \# \sigma})$.

Remark 2.1.3. For present purposes we are restricting to bounded operator valued kernels; for some applications one needs unbounded operator valued kernels. These can be handled with modifications which are reasonably straightforward, but are somewhat cumbersome.

Definition 2.1.4 (Placement - operators)**.** Let $F\colon \mathbb{R}_+ \to B(\mathsf{H}_1;\mathsf{H}_2), \sigma \in \Gamma$ and $\alpha \subset \sigma$. Moreover, let $Q \in B(\mathsf{H}_1;\mathsf{H}_2)$ be such that $\|Q\| = 1$ and define

$$\pi_F(\alpha) \mapsto \pi_G(\sigma), \qquad G = F\mathbf{1}_\alpha + Q\mathbf{1}_{\mathbb{R}_+\setminus\alpha}, F \in F(\mathbb{R}_+; B(\mathsf{H}_1;\mathsf{H}_2)).$$

This determines an ultraweakly continuous complete isometry

$$\iota^Q_{\alpha;\sigma}\colon B(\mathsf{H}_1^{\otimes\#\alpha};\mathsf{H}_2^{\otimes\#\alpha}) \to B(\mathsf{H}_1^{\otimes\#\sigma};\mathsf{H}_2^{\otimes\#\sigma}).$$

For $x \in OK(B(\mathsf{h}_1;\mathsf{h}_2), B(\mathsf{H}_1;\mathsf{H}_2))$, set

$$x(\alpha;\sigma,Q) := (\mathrm{id}_{B(\mathsf{h}_1;\mathsf{h}_2)} \overline{\otimes} \iota^Q_{\alpha;\sigma})(x(\alpha)) \qquad (\sigma \in \Gamma, \alpha \subset \sigma).$$

When $\mathsf{H}_2 = \mathsf{H}_1$ and $Q = I$ we abbreviate

$$x(\alpha;\sigma) := x(\alpha;\sigma,I).$$

Remark 2.1.5. Each operator kernel $x \in OK(V,Z)$ has an adjoint kernel $x^* \in OK(V^*,Z^*)$ defined pointwise: $x^*(\sigma) = x(\sigma)^*(\sigma \in \Gamma)$. Each vector kernel $\zeta \in K(\mathsf{h},\mathsf{H})$ determines mutually adjoint operator kernels $|\zeta(\cdot)\rangle \in OK(|\mathsf{h}\rangle, |\mathsf{H}\rangle)$ and $\langle\zeta(\cdot)| \in OK(\langle\mathsf{h}|, \langle\mathsf{H}|)$. The map

$$K(\mathsf{h},\mathsf{H}) \to OK(|\mathsf{h}\rangle, |\mathsf{H}\rangle), \quad \zeta \mapsto |\zeta(\cdot)\rangle$$

is manifestly a linear isomorphism. Moreover, when $\mathsf{H} = \widehat{\mathsf{k}}$ the placing notations enjoy the consistency

$$|\zeta(\cdot)\rangle(\alpha,\sigma,e_0) = |\zeta(\alpha;\sigma,\Delta^\perp)\rangle.$$

In practice the ordered pairs $(\mathsf{k}_1,\mathsf{k}_2)$ take one of the forms $(\mathsf{k},\mathsf{k}), (\mathsf{k},\mathbb{C}), (\mathbb{C},\mathsf{k})$.

Earlier we defined π_F for $F \in F(\mathbb{R}_+; B(\mathsf{H}_1; \mathsf{H}_2)$.

Example 2.1.6 (Product operator kernels with initial space)**.** Let $F \in F(\mathbb{R}_+; B(\mathsf{h} \otimes \mathsf{H}))$. Its associated product operator kernel $\pi_F \in OK(B(\mathsf{h}), B(\mathsf{H}))$ is defined by

$$\pi_F(\sigma) = \begin{cases} I_{\mathsf{h}} & \text{if } \sigma = \emptyset; \\ F(s_1;\sigma)\cdots F(s_n;\sigma) & \text{if } \sigma = \{s_1 < ... < s_n\} \in \Gamma \setminus \{\emptyset\}, \end{cases}$$

where for $\sigma \in \Gamma, s \in \sigma, F(s;\sigma) := F(s;\sigma, I_{\mathsf{H}})$. Sometimes we will want to apply the operators F in the reverse order, *i.e.* consider the product

$$F(s_n;\sigma)\cdots F(s_1;\sigma).$$

This product operator kernel will be denoted by ${}_F\pi$. It is important to note it only changes the order in which our operators F operate on the initial space h and not how the placement is performed on the noise dimension space. Thus $\pi_F = {}_F\pi$ *e.g.* whenever $F(s) = A \otimes B_s$ for some $A \in B(\mathsf{h}), B\colon \mathbb{R}_+ \to B(\mathsf{H})$. It is not true that $\pi_F = {}_F\pi$ whenever F is a constant function, however.

It is also worth noting that $(\pi_F)^* \supset {}_{F^*}\pi$. This will be important in our analysis of dual processes later.

For the majority of the thesis we are interested in the case where $\mathsf{H} = \widehat{\mathsf{k}}$ and $e_0 = \binom{1}{0} \in \widehat{\mathsf{k}}$, for some Hilbert space k, for which we abbreviate $\zeta(\alpha;\sigma, e_0)$ to $\zeta(\alpha;\sigma)$. Thus, for example,

$$\zeta(\emptyset;\sigma) = \zeta(\emptyset) \otimes e_0^{\otimes \#\sigma} \text{ and } \zeta(\sigma;\sigma) = \zeta(\sigma) \qquad (\sigma \in \Gamma),$$

and for $v \in \mathsf{h}, \psi \in F(\mathbb{R}_+; \mathsf{H}), \sigma \in \Gamma, \alpha \subset \sigma$

$$(v \otimes \pi_\psi)(\alpha;\sigma) = (v \otimes \pi_\chi)(\sigma), \qquad \text{where } \chi(s) = \begin{cases} \psi(s) & \text{if } s \in \alpha, \\ e_0 & \text{if } s \in \sigma \setminus \alpha \end{cases}.$$

For this case we also introduce a modification of the J^{e_0} isometry, namely the isometry

$$J_{\alpha;\sigma} \colon \mathsf{k}^{\otimes \# \alpha} \to \widehat{\mathsf{k}}^{\otimes \# \sigma}.$$

Thus it acts like $J^{\binom{1}{0}}$ with the exception that its domain is actually the natural isomorphic copy of k inside $\widehat{\mathsf{k}}$.

In the case of operator kernels, we abbreviate as follows:

$$x(\alpha;\sigma,\Delta^\perp) := x(\alpha;\sigma,|e_0\rangle\langle e_0|).$$

Recall that the Δ here is just the quantum Itô projection, which we defined in Definition 1.2.8.

Note here that $|e_0\rangle\langle e_0|$ is versatile enough to be viewed as an operator in $B(\widehat{\mathsf{k}}_1;\widehat{\mathsf{k}}_2)$; we occasionally write $|e_0\rangle\langle e_0|_{\widehat{\mathsf{k}}_1;\widehat{\mathsf{k}}_2}$ when it might be helpful.

We introduce a special piece of notation for the case when domains and codomains differ and $\mathsf{H} = \widehat{\mathsf{k}}$. For $\alpha \subset \sigma$ or $\alpha \supset \sigma$ and β disjoint from the two, consider $T(\alpha) \colon \widehat{\mathsf{k}}^{\otimes \# \alpha} \to \widehat{\mathsf{k}}^{\otimes \# \sigma}$. We will write

$$[T(\alpha);\alpha \cup \beta] \colon \widehat{\mathsf{k}}^{\otimes \#(\alpha \cup \beta)} \to \widehat{\mathsf{k}}^{\otimes \#(\sigma \cup \beta)}$$

for the ampliation of T, with the placement to be understood as before. Thus the set after the semicolon always signifies the tensor power of the domain of the operator in question. The fact that $\alpha \subset \sigma$ or $\alpha \supset \sigma$ here guarantees that there is no ambiguity in the notation.

Example 2.1.7 (Itô projection)**.** An important constant operator kernel is the one obtained from the constant function $\Delta = I_{\mathsf{h}} \otimes \begin{bmatrix} 0 & \\ & I_{\mathsf{k}} \end{bmatrix} \in B(\mathsf{h} \otimes \widehat{\mathsf{k}})$. Due to its ubiquitousness in the paper, we abbreviate for clarity

$$\Delta(\alpha;\sigma) := \pi_{\Delta}(\alpha;\sigma) \qquad \sigma \in \Gamma, \alpha \subset \sigma.$$

Thus

$$\Delta(\emptyset;\sigma) = I_{\mathsf{h}\otimes\widehat{\mathsf{k}}^{\otimes\#\sigma}}, \Delta(\sigma;\sigma) = I_{\mathsf{h}} \otimes \begin{bmatrix} 0 & \\ & I_{\mathsf{k}} \end{bmatrix}^{\otimes\#\sigma}.$$

The space k that Δ operates on will always be clear from context and thus we do not introduce a dependence on k in the notation for Δ.

$\Delta^{\perp}$ will denote $I_{\widehat{\mathsf{k}}} - \Delta$ and will be ampliated without change of notation similarly to Δ.

We introduce an easy lemma to support our placement notation.

Lemma 2.1.8. *Let $\alpha, \beta, \delta \subset \sigma \in \Gamma$, $x(\beta) \in B(\widehat{\mathsf{k}}^{\otimes\#\beta})$. Then the following identities hold:*

1. $J^*_{\alpha;\sigma} x(\beta;\sigma) = [J^*_{\alpha\cap\beta;\beta} x(\beta);\sigma][J^*_{\alpha\setminus\beta;\sigma\setminus\beta};\sigma]$

2. $x(\beta;\sigma) J_{\delta;\sigma} = [J_{\delta\setminus\beta;\sigma};\delta][x(\beta) J_{\delta\cap\beta;\beta};\delta]$

3. *If α, β, δ are disjoint and $\alpha \cup \beta \cup \delta = \sigma$, then, for $\alpha = \alpha_0 \cup \alpha_1$ and $\beta = \beta_0 \cup \beta_1$ with $\alpha_0, \alpha_1, \beta_0, \beta_1$ disjoint,*

$$[J^*_{\alpha_0;\alpha};\sigma]\Delta(\delta;\sigma)[J_{\beta_0;\beta};\sigma \setminus \beta_1] = [J_{\beta_0\cup\delta;\beta\cup\delta};\sigma \setminus (\alpha_1 \cup \beta_1)][J^*_{\alpha_0\cup\delta;\alpha\cup\delta};\sigma \setminus \beta_1].$$

Thus in particular, for $\xi(\beta) \in \mathsf{k}^{\otimes \# \alpha}$,

$$\|J^*_{\delta;\sigma} x(\beta;\sigma)\xi(\alpha;\sigma)\| \leqslant \|J^*_{\delta\cap\beta;\beta} x(\beta) J_{\beta\cap\alpha;\beta}\| \|\xi(\alpha)\|.$$

Proof. We only need to check the identities for x being a simple tensor. Thus let $x(\beta) = \bigotimes_{s\in\beta} x_s$ and consider $\xi = \bigotimes_{s\in\sigma} \xi_s \in \widehat{\mathsf{k}}^{\otimes\#\sigma}$. It is now easy to verify that

$$J^*_{\alpha;\sigma} x(\beta;\sigma)\xi = \prod_{t\in\sigma\setminus\alpha} v_t \bigotimes_{s\in\alpha} u_s,$$

where

$$u_s = \begin{cases} J^*_{s;s} x_s \xi_s & s \in \alpha\cap\beta \\ J^*_{s;s}\xi_s & s \in \alpha\setminus\beta \end{cases},$$

$$v_t = \begin{cases} \langle e_0, x_t\xi_t\rangle & t \in \beta\setminus\alpha \\ \langle e_0, \xi_t\rangle & t \in \sigma\setminus(\alpha\cup\beta) \end{cases}.$$

A straightforward calculation reveals the right hand side to be equal to this expression.

The next two identities are checked analogously. For the last one, it suffices to notice that each element of σ belongs to precisely one of the sets: $\alpha_0, \alpha_1, \beta_0, \beta_1, \delta$, on which one operator, the same on both sides of the equation, operates, as the action of $J_{\alpha;\sigma}$ coincides with the action of Δ on the tensor components corresponding to α. □

Finally, the following operation will turn out to be very useful in our considerations.

Definition 2.1.9 (Series product)**.** For functions $F, G \in F(\mathbb{R}_+; B(\widehat{\mathsf{k}}))$ we define the series product $F \triangleleft G$ by

$$F \triangleleft G = F + G + F\Delta G,$$

where Δ is the Itô projection. This product occurs naturally in the second fundamental formula of quantum stochastic calculus and it will play a fundamental role in the algebra of quantum Wiener integrals.

2.1.2 Convolutions

We next define the convolutions; first operator-vector convolutions.

Definition 2.1.10. Consider kernels $x \in OK(B(\mathsf{h}_1;\mathsf{h}_2), B(\widehat{\mathsf{k}}_1;\widehat{\mathsf{k}}_2))$ and $\zeta \in K(\mathsf{h}_1,\widehat{\mathsf{k}}_1)$ and let $Q \in B(\widehat{\mathsf{k}}_1;\widehat{\mathsf{k}}_2)$ be a projection, ampliated to h as needed. Recall that $e_0 = \binom{1}{0}$ as an element of $\widehat{\mathsf{k}}_1$. We define $x \,{}_{Q}\!\star\, \zeta \in K(\mathsf{h}_2;\widehat{\mathsf{k}}_2)$ by

$$(x \,{}_{Q}\!\star\, \zeta)(\sigma) = \sum_{\alpha\cup\beta=\sigma} x(\alpha;\sigma,Q)\Delta(\alpha\cap\beta;\sigma)\zeta(\beta;\sigma,e_0),$$

the sum being over all $3^{\#\sigma}$ internal covers of σ by subsets α,β. For operator kernels $x \in OK(B(\mathsf{h}_2;\mathsf{h}_3), B(\widehat{\mathsf{k}}_2;\widehat{\mathsf{k}}_3))$ and $z \in OK(B(\mathsf{h}_1;\mathsf{h}_2), B(\widehat{\mathsf{k}}_1;\widehat{\mathsf{k}}_2))$, we fix operators $Q \in B(\widehat{\mathsf{k}}_1,\widehat{\mathsf{k}}_2)$ and $Q' \in B(\widehat{\mathsf{k}}_2,\widehat{\mathsf{k}}_3)$. We define the operator kernel $x \,{}_{Q'}\!\star_{Q}\, z \in OK(B(\mathsf{h}_1;\mathsf{h}_3), B(\widehat{\mathsf{k}}_1;\widehat{\mathsf{k}}_3))$ by

$$(x \,{}_{Q'}\!\star_{Q}\, z)(\sigma) = \sum_{\alpha\cup\beta=\sigma} x(\alpha;\sigma,Q')\Delta(\alpha\cap\beta;\sigma)z(\beta;\sigma,Q).$$

In practice, those operators Q, Q' will practically always be of norm one, but that assumption is not necessary for next several results. In fact, in most future applications and next few results the operator Q in question will usually be the identity on the space of the argument, in which case we will omit it in the notation, writing $\star$ rather than ${}_{I}\!\star$.

Convolutions may also usefully be expressed in terms of partitions, rather than internal covers, as follows:

$$\begin{aligned}(x \;_{Q'}\!\star_Q z)(\sigma) &= \sum_{\sigma_1 \sqcup \sigma_2 \sqcup \sigma_3 = \sigma} x(\sigma_1 \cup \sigma_2; \sigma, Q')\Delta(\sigma_2; \sigma) z(\sigma_2 \cup \sigma_3; \sigma, Q) \\ &= \sum_{\alpha \sqcup \beta = \sigma} \sum_{\alpha_1 \sqcup \alpha_2 = \alpha} x(\alpha_1 \cup \beta; \sigma, Q')\Delta(\beta; \sigma) z(\beta \cup \alpha_2; \sigma, Q).\end{aligned}$$

We notice the following simple property for product kernels:

Lemma 2.1.11. *For product operator kernels $x = \pi_F, y = \pi_G \in OK(B(\mathsf{h}), B(\widehat{\mathsf{k}}))$ we have*

$$x \star y = \pi_{F+G+F\Delta G}$$

if either $\mathsf{h} = \mathbb{C}$ or, more generally, if F and G commute on the initial space, in the sense that for all $s_1, s_2 \in \mathbb{R}_+, s_1 \neq s_2, \sigma = \{s_1, s_2\}$:

$$F(s_1; \sigma)G(s_2; \sigma) = G(s_2; \sigma)F(s_1; \sigma).$$

Proof. We use the fact that

$$\pi_{F+G}(\sigma) = \sum_{\alpha \subset \sigma} \pi_F(\alpha; \sigma)\pi_G(\sigma \setminus \alpha; \sigma).$$

Let us notice that here we use the commutativity on the initial space.

Then, we have that

$$\begin{aligned}\pi_F \star \pi_G(\sigma) &= \sum_{\beta_0 \sqcup \beta_2 \sqcup \beta_2 = \sigma} \pi_F(\beta_0 \cup \beta_1; \sigma)\Delta(\beta_1; \sigma)\pi_G(\beta_1 \cup \beta_2; \sigma) \\ &= \sum_{\beta_0 \sqcup \beta_2 \sqcup \beta_2 = \sigma} \pi_F(\beta_0; \sigma)\pi_{F\Delta G}(\beta_1; \sigma)\pi_G(\beta_2; \sigma) \\ &= \pi_{F \lhd G}(\sigma),\end{aligned}$$

as required. □

Remark 2.1.12. Let us notice that a possible case where F and G commute on the initial space is when

$$F \in \mathcal{A} \otimes_M B(\widehat{\mathsf{k}}),$$

$$G \in \mathcal{B} \otimes_M B(\widehat{\mathsf{k}}),$$

where $\mathcal{A}, \mathcal{B} \subset B(\mathsf{h})$ are operator spaces such that $AB = BA$ for every $A \in \mathcal{A}, B \in \mathcal{B}$.

Taking ζ to be the kernel $u \otimes \delta_\emptyset (u \in \mathsf{h})$,

$$(x \;_Q\!\star\; (u \otimes \delta_\emptyset))(\sigma) = x(\sigma) u \otimes e_0^{\otimes \# \sigma} \qquad (\sigma \in \Gamma)$$

for any bounded operator Q and kernel $x \in OK(B(\mathsf{h}; \mathsf{h}'); B(\widehat{\mathsf{k}}, \widehat{\mathsf{k}}'))$. Thus any vector kernel $\zeta \in K(\mathsf{h}, \widehat{\mathsf{k}})$ may be obtained from one of the form $u \otimes \delta_\emptyset, u \neq 0$ by convolving with a suitable operator kernel $x \in OK(B(\mathsf{h}), B(\widehat{\mathsf{k}}))$:

$$\zeta = x \;_Q\!\star\; (u \otimes \delta_\emptyset), \qquad \text{where } x := \frac{1}{\|u\|^2} (|\zeta(\sigma)\rangle\langle u \otimes e_0^{\otimes \# \sigma}|)_{\sigma \in \Gamma}.$$

In view of an earlier identity, the operator kernel $I_{\mathsf{h}} \otimes \delta_\emptyset \in OK(B(\mathsf{h}); B(\widehat{\mathsf{k}}, \widehat{\mathsf{k}}'))$ acts as follows under convolution:

$$x \;_{Q'}\!\star_Q\; (I_{\mathsf{h}} \otimes \delta_\emptyset) = x \pi_Q$$

for bounded operators Q, Q' and compatible kernel $x \in OK(B(\mathsf{h}; \mathsf{h}'); B(\widehat{\mathsf{k}}', \widehat{\mathsf{k}}''))$ (the product on the right being pointwise defined) and so also

$$I_{\mathsf{h}} \otimes \delta_\emptyset \;_{Q'}\!\star_Q\; z = \pi_{Q'} z,$$

$$I_{\mathsf{h}} \otimes \delta_\emptyset \;_{Q'}\!\star_0\; \zeta = \pi_{Q'} \zeta$$

for kernels $z \in OK(B(\mathsf{h}';\mathsf{h});B(\widehat{\mathsf{k}}',\widehat{\mathsf{k}}))$ and $\zeta \in K(\mathsf{h},\widehat{\mathsf{k}})$. In particular, $I_{\mathsf{h}} \otimes \delta_{\emptyset}$ is an identity for the convolution $\star$ on $OK(B(\mathsf{h}),B(\widehat{\mathsf{k}}))$ and for the convolution $\star$ on $K(\mathsf{h},\widehat{\mathsf{k}})$ (when the appropriate $Q = I$).

Note the consistency of the notations:

$$|(x \,{}_{Q}\!\star\, \zeta)(\cdot)\rangle = x \,{}_{Q}\!\star\, |\zeta(\cdot)\rangle,$$

the adjoint relations for compatible kernels:

$$(x \,{}_{Q'}\!\star_{Q}\, z)^* = z^* \,{}_{Q^*}\!\star_{Q'^*}\, x^*,$$

and the following identities for $\zeta, \eta \in K(\mathsf{h},\widehat{\mathsf{k}})$:

$$\begin{aligned}(\langle\zeta(\cdot)| \star |\eta(\cdot)\rangle)(\sigma) &= \sum_{\alpha\cup\beta=\sigma} \langle\zeta(\alpha;\sigma)|\Delta(\alpha\cap\beta;\sigma)|\eta(\beta;\sigma)\rangle \\ &= \sum_{\sigma_1\sqcup\sigma_2\sqcup\sigma_3=\sigma} \langle\zeta(\sigma_1\cup\sigma_2;\sigma)|\Delta(\sigma_2;\sigma)|\eta(\sigma_2\cup\sigma_3;\sigma)\rangle.\end{aligned}$$

Introduce the notations

$$\tilde{c} = \tilde{J}_{\mathsf{k}} c := \begin{pmatrix} 0 \\ c \end{pmatrix}, \tilde{g}(s) = \widetilde{g(s)} \text{ and } \tilde{k}(\sigma) = (I_{\mathsf{h}} \otimes (\tilde{J}_{\mathsf{k}})^{\otimes \#\sigma})k(\sigma)$$

for $c \in \mathsf{k}, g \in F(\mathbb{R}_+;\mathsf{k}), k \in K(\mathsf{h},\mathsf{k})$ and $\sigma \in \Gamma$. Then the above identities specialise as follows:

$$\begin{aligned}(\langle\zeta(\cdot)| \star |\tilde{k}(\cdot)\rangle)(\sigma) &= \sum_{\alpha\subset\sigma} \langle\zeta(\sigma), \tilde{k}(\alpha;\sigma)\rangle; \\ (\langle\tilde{k}_1(\cdot)| \star |\tilde{k}_2(\cdot)\rangle)(\sigma) &= \langle k_1(\sigma), k_2(\sigma)\rangle I_{B(\mathbb{C})}\end{aligned}$$

for $\zeta \in K(\mathsf{h},\widehat{\mathsf{k}}), k, k_1, k_2 \in K(\mathsf{h},\mathsf{k})$.

The composition of convolutions is then applied by multiplication of operators.

The generalised form of associativity enjoyed by these convolutions is given next.

Theorem 2.1.13 (General associativity)**.** *For projections Q, R, T with $RT = R = QR, R\Delta = \Delta R$ and operator kernels x, y, z we have*

$$x \;{}_Q\!\star_R\, (y \;{}_R\!\star_T\, z) = (x \;{}_Q\!\star_R\, y) \;{}_R\!\star_T\, z. \tag{2.1}$$

The common value of these kernels at $\sigma \in \Gamma$ is

$$\sum_{\alpha\cup\beta\cup\gamma=\sigma} x(\alpha;\sigma,Q)\Delta(\alpha\cap(\beta\cup\gamma);\sigma)y(\beta;\sigma,R)\Delta((\alpha\cup\beta)\cap\gamma;\sigma)z(\gamma;\sigma,T).$$

Proof. Set $w_1 := x \;{}_Q\!\star_R\, (y \;{}_R\!\star_T\, z)$ and $w_2 = (x \;{}_Q\!\star_R\, y) \;{}_R\!\star_T\, z$. Let $\sigma \in \Gamma$. Then, for $\delta \subset \sigma$,

$$\begin{aligned}(y \;{}_R\!\star_T\, z)(\delta;\sigma,R) &= \sum_{\beta\cup\gamma=\delta} (((y(\beta;\delta,R))\Delta(\beta\cap\gamma;\delta)(z(\gamma;\delta,T)));\sigma,R)\\ &= \sum_{\beta\cup\gamma=\delta} y(\beta;\sigma,R)\Delta(\beta\cap\gamma;\sigma)z(\gamma;\sigma,T),\end{aligned}$$

as $RT = R$. Therefore, since $\Delta((\alpha\cap\gamma)\setminus\beta;\sigma) \smile y(\beta;\sigma,R)_\eta$ and $((\alpha\cap\gamma)\setminus\beta)\cup(\beta\cap\gamma) = (\alpha\cup\beta)\cap\gamma$,

$$\begin{aligned}w_1(\sigma) &= \sum_{\alpha\cup\delta=\sigma} x(\alpha;\sigma,Q)\Delta(\alpha\cap\delta;\sigma)(y \;{}_R\!\star_T\, z)(\delta;\sigma,R)\\ &= \sum_{\alpha\cup\beta\cup\gamma=\sigma} x(\alpha;\sigma,Q)\Delta(\alpha\cap(\beta\cup\gamma);\sigma)y(\beta;\sigma,R)\Delta(\beta\cap\gamma;\sigma)z(\gamma;\sigma,T)\\ &=: w(\sigma).\end{aligned}$$

By the symmetry of the formula for w and the fact that $QR = R$, this also implies that

$$w^* = z^* \;{}_T\!\star_R\, (y^* \;{}_R\!\star_Q\, x^*)$$

and so, using the adjoint relation,

$$w = (y^* \,{}_R\star_Q\, x^*)^* \,{}_R\star_T\, z = (x \,{}_Q\star_R\, y) \,{}_R\star_T\, z = w_2. \qquad \square$$

Remark 2.1.14. As a sum over partitions of σ, rather than internal covers, the common value of these kernels at $\sigma \in \Gamma$ is:

$$\sum_{\sigma=\omega_1\sqcup\ldots\sqcup\omega_7} x(\omega_{1467};\sigma,Q)\Delta(\omega_{467};\sigma)y(\omega_{2457};\sigma,R)\Delta(\omega_{567};\sigma)z(\omega_{3567};\sigma,T),$$

where $\omega_{ijkl} = \omega_i \cup \omega_j \cup \omega_k \cup \omega_l$, or

$$\sum_{\sigma=\alpha\sqcup\beta\sqcup\gamma}\ \sum_{\alpha=\alpha_1\sqcup\ldots\sqcup\alpha_5} x(\alpha_1\cup\alpha_4\cup\beta\cup\gamma;\sigma,Q)\Delta(\alpha_4\cup\beta\cup\gamma;\sigma)$$
$$y(\alpha_2\cup\alpha_4\cup\alpha_5\cup\gamma;\sigma,R)\Delta(\alpha_5\cup\beta\cup\gamma;\sigma)z(\alpha_3\cup\alpha_5\cup\beta\cup\gamma;\sigma,T).$$

Corollary 2.1.15. *Let Q, R, T be projections.*

- *For operator kernels x, y and z bracketing is superfluous in the following cases:*

  $$x \star y \star z \text{ and } x \,{}_Q\star_R\, y \,{}_R\star_T\, z$$

  when $QR = R = RT$ and $R\Delta = \Delta R$, so e.g. *when $Q = T = I$;*

- *For kernels $\zeta_1 \in K(\mathsf{h}_1;\widehat{\mathsf{k}}_1), x \in OK(B(\mathsf{h}_1;\mathsf{h}_2);B(\widehat{\mathsf{k}}_1,\widehat{\mathsf{k}}_2))$ and $\zeta_2 \in K(\mathsf{h}_2;\widehat{\mathsf{k}}_2)$, taking $Q = \Delta^\perp$, we have*

  $$\langle\zeta_1(\cdot)| \,{}_Q\star\, |(x\star\zeta_2)(\cdot)\rangle = \langle(x^*\star\zeta_1)(\cdot)| \,{}_Q\star\, |\zeta_2(\cdot)\rangle.$$

  Moreover,

1. *their common value at* $\sigma \in \Gamma$ *is*

$$\sum_{\alpha\cup\beta\cup\gamma=\sigma} \langle \zeta_1(\alpha;\sigma)|\Delta(\alpha\cap(\beta\cup\gamma);\sigma)x(\beta;\sigma)$$
$$\Delta((\alpha\cup\beta)\cap\gamma;\sigma)|\zeta_2(\gamma;\sigma)\rangle$$
$$= \sum_{\sigma=\sigma_1\sqcup\ldots\sqcup\sigma_7} \langle \Delta(\sigma_{467};\sigma)\zeta_1(\sigma_{1467};\sigma)|$$
$$x(\sigma_{2457};\sigma)|\Delta(\sigma_{567};\sigma)\zeta_2(\sigma_{3567};\sigma)\rangle;$$

2. *If* $\zeta_i = \tilde{k}_i$ *for* $k_i \in K(\mathsf{h}_i;\mathsf{k}_i)(i=1,2)$ *then the common value at* $\sigma \in \Gamma$ *is*

$$\sum_{\alpha\cup\beta\cup\gamma=\sigma} \langle \tilde{k}_2(\alpha;\sigma)|x(\beta;\sigma)|\tilde{k}_1(\gamma;\sigma)\rangle,$$

in particular,

$$(\langle \tilde{k}_2(\cdot)| \star |(x \star \tilde{k}_2)(\cdot)\rangle)(\sigma)$$
$$= \sum_{\sigma=\sigma_1\sqcup\ldots\sqcup\sigma_4} \langle \tilde{k}_2(\sigma_1\cup\sigma_2;\sigma)|x(\sigma)|\tilde{k}_1(\sigma_2\cup\sigma_3;\sigma)\rangle.$$

We finish this section with an interesting bound on the norms of convolutions of kernels.

Proposition 2.1.16. *Let* x, y *be two operator kernels on a Hilbert space* $B(\widehat{\mathsf{k}})$ *which are product bounded,* i.e. *there exist functions* $\varphi, \psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|x(\sigma)\| \leqslant \pi_\varphi(\sigma), \|y(\sigma)\| \leqslant \pi_\psi(\sigma)$$

for all $\sigma \in \Gamma$. *Then* $(x \star y)$ *is also product bounded and the bounding function is given by the series product of* φ *and* ψ, *namely* $\varphi \lhd \psi = \varphi + \psi + \varphi\psi$.

Proof. Let $\sigma \in \Gamma$. We have that

$$\begin{aligned}
\|(x \star y)(\sigma)\| &\leqslant \sum_{\sigma_0 \sqcup \sigma_1 \sqcup \sigma_2 = \sigma} \|x(\sigma_0 \cup \sigma_1; \sigma)\Delta(\sigma_1; \sigma)y(\sigma_1 \cup \sigma_2; \sigma)\| \\
&\leqslant \sum_{\sigma_0 \sqcup \sigma_1 \sqcup \sigma_2 = \sigma} \pi_\varphi(\sigma_0 \cup \sigma_1)\pi_\psi(\sigma_1 \cup \sigma_2) \\
&= \sum_{\sigma_0 \sqcup \sigma_1 \sqcup \sigma_2 = \sigma} \pi_\varphi(\sigma_0) \cdot \pi_{\varphi\psi}(\sigma_1) \cdot \pi_\psi(\sigma_2) \\
&= \pi_{\varphi+\psi+\varphi\psi}(\sigma).
\end{aligned}$$

□

2.2 Measurability

Here we will consider the measurability of our kernels. We do it here so that in the forthcoming chapters we can consider our kernels to be measurable and be secure in the knowledge that all operations we do on them preserve that measurability. We will denote the n-th Cartesian product of Γ by Γ^n. We also introduce the following notation:

$$\Gamma^{(n)} = \{(\sigma_1, \cdots, \sigma_n) \in \Gamma^n \colon \sigma_i \cap \sigma_j = \emptyset \text{ for } i \neq j\},$$

We may also write Γ_t rather than Γ when we are interested in subsets of $[0, t)$ instead of $\mathbb{R}_+$. The set $\{1, \cdots, n\}$ will be denoted by $\overline{n}$ and we will write $\mathcal{P}_n(\overline{m})$ for the family of n-element subsets of an m element set ($m \geqslant n$). We will write $[j, k]$ for the subset $\{j, j+1, \cdots, k\}$ of $\overline{n}$. For a set $\sigma \in \Gamma$ we write $[\sigma]_i$ for its i-th element, when its elements are written in increasing order. In other words, if $\sigma = \{s_1 < \cdots < s_n\}$, then $[\sigma]_i = s_i$.

For $S \in \mathcal{P}_n(\overline{n+m})$ we write

$$\varphi(S) = \{(\alpha, \beta) \in \Gamma_n \times \Gamma_m \colon S = \{i \in \overline{n+m} \colon [\alpha \cup \beta]_i \in \alpha\} = \{i \in \overline{n+m} \colon [\alpha \cup \beta]_i \notin \beta\}\}.$$

Let us notice that requiring both of those equalities to hold in particular implies $\alpha \cap \beta = \emptyset$. Also, it is useful to note that

$$\varphi(S) = \{(\alpha, \beta) \in \Gamma_n \times \Gamma_m \colon v_1 < \cdots < v_{n+m}\} \cap \Gamma^{(2)}, \tag{2.2}$$

where v_i's are uniquely determined by requiring that $v_i \in \alpha$ for $i \in S$ and $v_i \in \beta$ otherwise.

Proposition 2.2.1. *For any $S \in \mathcal{P}_n(\overline{n+m})$ $\varphi(S)$ is measurable.*

Proof. Looking at Equation (2.2) as a subset of $\mathbb{R}_+^{n+m}$, it is clear that $\varphi(S)$ is open, so measurable. □

We also write χ_S for the permutation of $\overline{n+m}$ which acts as follows:

$$\chi_S(i) = \begin{cases} [s]_i & \text{if } i \leqslant n \\ [\overline{n+m} \setminus S]_j & \text{if } i = n+j \end{cases}.$$

Thus, for example, for $n = 2, m = 3$ and $S = \{2, 4\}$, χ_S is the following permutation:

$$\chi_S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

The n, m in question will be clear from context, so we believe there is no need of introducing them in the notation for χ.

It is worth noting that a subset $A \subset \Gamma$ is measurable in the Guichardet measure if and only if each $A \cap \Gamma_n$ is measurable in the n-dimensional complete Lebesgue σ-algebra - in other words, a set is measurable if and only if it is measurable on each chaos. Also, $\Gamma^n \setminus \Gamma^{(n)}$ has measure zero for each $n \in \mathbb{N}$.

To start off, we investigate the union operation:

Lemma 2.2.2. *Let $\mathcal{N} \subset \Gamma$ be a null set. Then the following sets are null too:*

- $\mathcal{N}' = \{\alpha \in \Gamma \colon \beta \subset \alpha$ *for some* $\beta \in \mathcal{N}\}$.
- $\mathcal{N}'' = \{(\alpha, \beta) \in \Gamma \times \Gamma \colon \alpha \cup \beta \in \mathcal{N}\}$

Proof. Denote the Guichardet measure by μ. We have:

$$\begin{aligned}\mu(\mathcal{N}') = \int_\Gamma \mathbf{1}_{\mathcal{N}'}(\alpha) d\alpha &= \int_\Gamma \min\{1, \sum_{\beta \subset \alpha} \mathbf{1}_{\mathcal{N}}(\beta)\} d\alpha \\ &\leqslant \int_\Gamma \sum_{\beta \subset \alpha} \mathbf{1}_{\mathcal{N}}(\beta) d\alpha \leqslant \int_\Gamma \mathbf{1}_{\mathcal{N}}(\beta) d\beta \int_\Gamma 1 d\alpha = 0.\end{aligned}$$

Reading from right to left, this implies that $\mathbf{1}_{\mathcal{N}'}$ is measurable of integral 0, so $\mathcal{N}'$ is null. Analogously,

$$\begin{aligned}\mu(\mathcal{N}'') = \int_\Gamma \int_\Gamma \mathbf{1}_{\mathcal{N}''}(\alpha, \beta) d\alpha &= \int_\Gamma \sum_{\alpha \subset \sigma} 1_{\mathcal{N}}(\sigma) d\sigma \\ &= \int_\Gamma 2^{\#\sigma} 1_{\mathcal{N}}(\sigma) d\sigma = 0.\end{aligned}$$

By an analogous reasoning, $\mathcal{N}''$ is of measure zero. □

Proposition 2.2.3. *Let* $\mathcal{U} \colon \Gamma^2 \to \Gamma$ *be the union operator,* i.e. $(\alpha, \beta) \mapsto \alpha \cup \beta$. *Then* $\mathcal{U}$ *is measurable.*

Proof. Let $U \subset \Gamma_n$ be measurable. Without loss of generality, $U = ((U_1 \times \cdots \times U_m) \cap \Gamma_n) \cup \mathcal{N}$ for measurable $U_j \subset \mathbb{R}_+$ and a null set $\mathcal{N}$. We are interested in $\mathcal{U}^{-1}(U)$. We have the following string of identities, with $\mathcal{M}$ denoting another null set and $\mathcal{N}''$ coming from Lemma 2.2.2:

$$\begin{aligned}\mathcal{U}^{-1}(U) &= \{(\alpha, \beta) \in \Gamma^{(2)} \colon \alpha \cup \beta \in U\} \cup \mathcal{M} \\ &\{(\alpha, \beta) \in \Gamma^{(2)} \colon \alpha \cup \beta \in U\} \cup \mathcal{M} \\ &= \cup_{k=0}^n \{(\alpha, \beta) \in \Gamma_k \times \Gamma_{n-k} \colon \alpha \cup \beta \in U\} \cup \mathcal{M}\end{aligned}$$

$$= \cup_{k=0}^{n} \cup_{S\in\mathcal{P}_k(\overline{m})} \{(\alpha,\beta) \in \Gamma_k \times \Gamma_{n-k} \colon \alpha \in \bigtimes_{i\in S} U_i, \beta \in \bigtimes_{i\notin S} U_i\} \cup \mathcal{M} \cup \mathcal{N}'',$$

which is easily seen to be Lebesgue measurable as a finite union of measurable sets and a null set. □

We notice a following property of the integral-sum identity, with its proof adapted from [57].

Proposition 2.2.4. *Consider functions $f\colon \Gamma^n \to H, g\colon \Gamma \to H$ such that*

$$g(\sigma) = \sum_{\sigma_1\sqcup\cdots\sqcup\sigma_n=\sigma} f(\sigma_1,\cdots,\sigma_n).$$

Then g is measurable if f is.

Proof. Let us notice we only need to prove it for $n = 2$. Thus we need to prove that if $\Gamma^{(2)} \ni (\alpha,\beta) \mapsto f(\alpha,\beta)$ is measurable, then so is $\sigma \mapsto \sum_{\alpha\subset\sigma} f(\alpha, \sigma\setminus\alpha)$. Consider particular $\sigma, \#\sigma = N$. Let $f_k \colon \mathbb{R}_+^k \times R_+^{N-k} \to H$ be the function which is symmetric with respect to its first k coordinates and its second $N-k$ coordinates and coincides with f on $\Gamma^k \times \Gamma^{N-k}$. We consider it as a function on $\mathbb{R}_+^N$. Then f_k is obviously Lebesgue measurable and we see that

$$\begin{aligned} g(\sigma) &= \sum_{\alpha\subset\sigma} f(\alpha,\sigma\setminus\alpha) \\ &= \sum_{k=0}^{N} \sum_{\substack{\alpha\subset\sigma \\ \#\alpha=k}} f(\alpha,\sigma\setminus\alpha) \\ &= \sum_{k=0}^{N} \sum_{S\in\mathcal{P}_k(\overline{n})} f_k(\chi_S(\sigma)), \end{aligned}$$

where we use the identification of a set σ with the point in $\mathbb{R}_+^N$ and the permutation χ_S defined earlier.

Now, as the map $\sigma \to \chi_S(\sigma)$ is Lebesgue measurable for any $S \in \mathcal{P}_k(\overline{n})$ and each f_k is measurable, we see that g is measurable as a finite combination of measurable functions. □

Remark 2.2.5. It is important to note that the reverse is not true. Indeed, let $f_0 \colon \{(s_1, s_2) \in \mathbb{R}_+^2 \colon s_1 < s_2\} \to \mathsf{H}$ be any non-measurable function. We can then define a function $f_1 \colon \mathbb{R}_+^2 \to \mathsf{H}$ by the antisymmetric extension and 0 on the diagonal - and it is easily seen this function is still non-measurable. We then define f via

$$f(\{s_1\}, \{s_2\}) = f_1(s_1, s_2).$$

We see as a function $f \colon \Gamma^2 \to \mathbb{R}_+$ f is supported on $\Gamma_1 \times \Gamma_1$. Then f is non-measurable (as the Guichardet space inherits the measure structure from $\mathbb{R}_+^2$), but it is easily seen that

$$\sum_{\sigma_1 \sqcup \sigma_2 = \sigma} f(\sigma_1, \sigma_2) = 0$$

for every σ. Indeed, the only case when it might not be zero is when $\sigma = \{s_1, s_2\}$, but then the sum is equal to

$$f(\{s_1\}, \{s_2\}) + f(\{s_2\}, \{s_1\}) = f_1(s_1, s_2) - f_1(s_1, s_2) = 0$$

by the antisymmetry of f_1.

We are interested in exploring the measurability of vector and operator kernels. For operator kernels we mean strong operator measurability, *i.e.* if, given $\xi \in \mathsf{h} \otimes \Phi(\widehat{\mathsf{k}})$, the mapping

$$\Gamma \ni \sigma \mapsto x(\sigma)\xi$$

is measurable. Let us notice that this is equivalent to saying: for every $n \in \mathbb{N}, \xi \in \mathsf{h} \otimes \Phi(\widehat{\mathsf{k}})$ supported on the n-th chaos, the mapping

$$\Gamma_n = \{\sigma \in \Gamma \colon \#\sigma = n\} \ni \sigma \mapsto x(\sigma)\xi$$

is measurable.

The following notation for tensor flips will be useful:

For $n \in \mathbb{N}$, Hilbert space H and a permutation $\tau \in S_n$, we write Π_τ for the unitary operator on $H^{\otimes n}$ which implements this permutation, *i.e.*

$$\Pi_\tau \xi_1 \otimes \cdots \otimes \xi_n = \xi_{\tau(1)} \otimes \cdots \xi_{\tau(n)}.$$

For $\alpha \subset \sigma \in \Gamma$, $\#\alpha = k, \#\sigma = n$ let $S_{\alpha;\sigma} := \{i \in \overline{n} \colon [\sigma]_i \in \alpha\}$. Then we define $\Pi_{\alpha;\sigma}$ to be the tensor flip implemented by permutation $\chi(S_{\alpha;\sigma})$.

We recall Corollary 1.4.16 from Chapter 1.

Now, we can build up the repertoire of results which ensure that if we start with a measure equivalence class of a measurable operator kernel, then its quantum Wiener integral, which we will define in the next chapter, will also be measurable.

We will very often use the isometry J and its adjoint, along with our general placement notation. The forthcoming three results ensure that these operations do not violate measurability.

Lemma 2.2.6. *Let $F \colon \mathbb{R}_+ \to B(\widehat{\mathsf{k}})$ be strongly measurable. Then the product kernel π_F is measurable.*

Proof. We see that for a product vector $\xi = \xi_1 \otimes \cdots \xi_n \in \widehat{\mathsf{k}}^{\otimes n}$ and a Borel set $U = U_1 \times \cdots U_n \subset \widehat{\mathsf{k}}^{\otimes n}$ we have

$$(\pi_F(\cdot)\xi)^{-1}(U) = \times_{k=1}^{n} (F(\cdot)\xi_k)^{-1}(U_k) \cap \Gamma^n,$$

which is measurable by strong measurability of F. □

Lemma 2.2.7. *The prescription*

$$\Pi\colon (\alpha,\beta) \mapsto \Pi_{\alpha;\alpha\cup\beta}\,\boldsymbol{1}_{\alpha\cap\beta=\emptyset}$$

is strongly measurable.

Proof. Let $\xi \in H^{\otimes n}$. We see that for any $U \in \mathrm{Borel}(H^{\otimes n})$,

$$(\Pi(\cdot,\cdot)\xi)^{-1}(U) = \cup_{k=0}^{n} \cup_{S\in\mathcal{P}_k(\overline{n})} \Phi(S)\mathbf{1}_{\Pi_{\chi(S)}\xi\in U},$$

where the indicator signifies that we take the $\Phi(S)$ summand if the proposition is true and we do not otherwise.

In any case, we see that $\Pi^{-1}(U)$ is a finite sum of sets of the form $\Phi(S)$, which are measurable by Proposition 2.2.1 □

Lemma 2.2.8. *Let x be a measurable operator kernel and $\xi \in \mathsf{h}\otimes\Phi(\widehat{\mathsf{k}})$ with both having compact support and let $Q \in B(\widehat{\mathsf{k}}), e \in B(\widehat{\mathsf{k}})$ with $\|Q\| = 1 = \|e\|$. Then $(\alpha,\beta) \mapsto x(\alpha;\alpha\cup\beta,Q)$ is strongly measurable and $\xi(\alpha;\alpha\cup\beta,e)$ is measurable.*

Proof. Since

$$x(\alpha;\alpha\cup\beta) = \Pi^*_{\alpha;\alpha\cup\beta}x(\alpha)\otimes Q^{\otimes\#(\beta\setminus\alpha)}\Pi_{\alpha;\alpha\cup\beta},$$

$$\xi(\alpha;\alpha\cup\beta,e) = \Pi^*_{\alpha;\alpha\cup\beta}\xi(\alpha)\otimes e^{\otimes\#(\beta\setminus\alpha)}\Pi_{\alpha;\alpha\cup\beta},$$

we see that both of them are measurable as compositions of measurable maps on a finite measure space. □

Lemma 2.2.9. *Let $t \geqslant 0$. Then*

$$\Gamma_t \ni (\alpha,\beta) \mapsto J^*_{\alpha;\alpha\cup\beta}$$

is measurable as an operator on $\cup_{n\geqslant 0}H^{\otimes n}$ *for a Hilbert space* H*.*

Proof.

$$J^*_{\alpha;\alpha\cup\beta} = \Pi^*_{\alpha;\alpha\cup\beta}\underline{\Delta}(\alpha;\alpha\cup\beta)\underline{\Delta}^{\perp}(\beta;\alpha\cup\beta)\Pi_{\alpha;\alpha\cup\beta},$$

where $\underline{\Delta}, \underline{\Delta}^{\perp}$ denotes $\Delta, \Delta^{\perp}$ composed with the the projections $\widehat{\mathsf{k}} \to \mathsf{k}$ and $\widehat{\mathsf{k}} \to \mathbb{C}$, respectively. Thus, as a composition of product kernels (of constant maps!) on a finite measure space, it is measurable. □

This allows us to strengthen the statement of Lemma 2.2.6:

Lemma 2.2.10. *Let* $F\colon \mathbb{R}_+ \to B(\mathsf{h}\otimes\widehat{\mathsf{k}})$ *be measurable and compactly supported. Then the product kernel* π_F *is measurable.*

Proof. It is easily seen that for $n \in \mathbb{N}, k \in \{1,\cdots,n\}$ and $\sigma \in \Gamma, \sigma = \{s_1,\cdots,s_n\}$ the function $\sigma \mapsto F(s_k;\sigma)$ is measurable as a composition of a tensor product of F with $I_{\widehat{\mathsf{k}}}$ with a fixed tensor flip. But then $\pi_F(\sigma)$ is measurable as a product of measurable operators. Thus for every $n \in \mathbb{N}$

$$\pi_F|_{\Gamma^{(n)}} \text{ is measurable,}$$

so by our discussion earlier π_F is measurable. □

Finally, we are ready to talk about measurability of operator-operator and operator-vector convolutions.

Proposition 2.2.11. *Let* x, y *be measurable, compactly supported operator kernels. Then* $x \star y$ *is measurable and for any compactly supported, measurable* $\xi \in \mathsf{h}\otimes\Phi(\widehat{\mathsf{k}})$ $x \star \xi$ *is measurable.*

Proof. We will only prove the measurability of $x \star \xi$. Measurability of $x \star y$ is proven analogously.

We have that:

$$x \star \xi(\sigma) = \sum_{\alpha\cup\beta=\sigma} x(\alpha;\sigma)\Delta(\beta;\sigma)\xi(\beta;\sigma).$$

By integral-sum identity, this is measurable if

$$(\alpha,\beta,\gamma) \mapsto x(\alpha\cup\beta;\sigma)\Delta(\beta;\sigma)\xi(\beta\cup\gamma;\sigma)$$

is measurable, where $\sigma = \alpha\cup\beta\cup\gamma$.

We know that the function $(\alpha,\beta) \mapsto (\alpha\cup\beta)$ is measurable, so obviously $(\alpha,\beta,\gamma) \mapsto (\alpha\cup\beta,\gamma)$ is. Thus, by the previous lemma,

$$(\alpha,\beta,\gamma) \mapsto x(\alpha\cup\beta;\sigma)$$

is measurable. Analogously, we see that

$$(\alpha,\beta,\gamma) \mapsto \Delta(\beta;\sigma),\ (\alpha,\beta,\gamma) \mapsto \xi(\beta\cup\gamma;\sigma)$$

are measurable. Thus the result is measurable as a composition of measurable maps on a finite measure space. □

Corollary 2.2.12. *Let x be a measurable operator kernel. Then, for $\xi \in \mathsf{h}\otimes\mathcal{F}^{\mathsf{k}}$ and $t > 0$ the function*

$$(x\star\xi)'(\alpha,\beta) := J^*_{\alpha;\alpha\cup\beta}(x\boldsymbol{1}_{\Gamma_t}\star\tilde{\xi})(\alpha\cup\beta)$$

is measurable. Thus in particular we can talk about $(x\star\xi)'$ integrability. If $(x\star\xi)'$ is integrable over β, then $\int(x\star\xi)'(\cdot,b)\mathrm{d}\beta$ is measurable with respect to α.

This operation will lie at the centre of our definition of quantum Wiener integral.

2.3 Multiple Wiener-time integrals

The aim of this section is to amalgamate multiple 'Wiener' integrals and multiple 'time' integrals, both in a common setting suitable for quantum stochastic generalisation. Our goal then is to realise the square-norm of the resulting hybrid Wiener-time integral as an integral of the convolution of bra and ket forms of the Wiener-time integrand.

To this end define isometries

$$J_{\alpha;\sigma} := J^{e_0}_{\alpha;\sigma} \tilde{J}^{\otimes \# \alpha} \in B(\mathsf{k}^{\otimes \# \alpha}; \widehat{\mathsf{k}}^{\otimes \# \sigma}),$$

for $\sigma \in \Gamma$ and $\alpha \subset \sigma$, where $\tilde{J} \colon \mathsf{k} \mapsto \widehat{\mathsf{k}}$ denotes the isometry $c \mapsto \tilde{c} := \binom{0}{c}$ and $J^{e_0}_{\alpha;\sigma}$ is the placing notation introduced in Subsection 2.1.1. Thus

$$J_{\alpha;\sigma} \pi_\varphi(\alpha) = \pi_{\tilde{\varphi}}(\alpha; \sigma) = \pi_\psi(\sigma) \text{ for } \varphi \in F(\mathbb{R}_+; \mathsf{k}), \tag{2.3}$$

where $\psi = \tilde{\varphi} \mathbf{1}_\alpha + e_0 \mathbf{1}_{\mathbb{R}_+ \setminus \alpha}$. In particular,

$$J_{\emptyset,\emptyset} = I_{\mathbb{C}}, J_{\emptyset,\sigma} = |e_0\rangle^{\otimes \# \sigma} \text{ and } J_{\sigma;\sigma} = \tilde{J}^{\otimes \# \sigma}; J_{\{s_1\},\{s_1 < s_2\}} k = \begin{pmatrix} 0 \\ k \end{pmatrix} \otimes e_0. \tag{2.4}$$

From now on, and *for the rest of the thesis*, we will consider our kernels (both vector and operator) to be measure equivalence classes of kernels.

Lemma 2.3.1. *The collection of isometries* $\{J_{\alpha;\sigma} \colon \sigma \in \Gamma, \alpha \subset \sigma\}$ *satisfy the orthogonality relations*

$$\widehat{\mathsf{k}}^{\otimes \# \sigma} = \bigoplus_{\alpha \subset \sigma} J_{\alpha;\sigma} \mathsf{k}^{\otimes \# \alpha} \qquad (\sigma \in \Gamma).$$

Proof. The mutual orthogonality of the ranges of the isometries $\{J_{\alpha;\sigma} \colon \alpha \subset \sigma\}$ follows from the fact that $e_0 \perp \operatorname{Ran} \tilde{J}$. The fact that their orthogonal sum equals $\widehat{\mathsf{k}}^{\otimes \# \sigma}$ follows from the binomial-type identity

$$\pi_{\hat{\chi}}(\sigma) = \sum_{\alpha \subset \sigma} J_{\alpha;\sigma} \pi_{\chi}(\alpha) \qquad (\chi \in F(\mathbb{R}_+; \mathsf{k}))$$

and the totality of $\{\pi_{\hat{\chi}}(\sigma) \colon \chi \in F(\mathbb{R}_+; \mathsf{k})\}$ in $\widehat{\mathsf{k}}^{\otimes \# \sigma}$. □

From now on we freely ampliate, so that

$$J_{\alpha;\sigma} \in B(\mathsf{h} \otimes \mathsf{k}^{\otimes \# \alpha}; \mathsf{h} \otimes \widehat{\mathsf{k}}^{\otimes \# \sigma}).$$

The lemma above remains valid with obvious adjustments.

We define the following operator which, as it will turn out, is naturally dual to our integration.

Proposition 2.3.2. *Let $t \geq s \geq 0$ and $k \in \mathcal{F}^{\mathsf{k}}$. The prescription*

$$(\hat{D}_t k)(\sigma) = \mathbf{1}_{\Gamma_{[0,t[}}(\sigma) \sum_{\alpha \subset \sigma} J_{\alpha;\sigma} k(\alpha)$$

when ampliated defines an operator $\hat{D}_t \in B(\mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}; \mathsf{h} \otimes \mathcal{F}^{\widehat{\mathsf{k}}})$ satisfying the following properties:

1. *$\hat{D}_t v \otimes \varepsilon(g) = v \otimes \varepsilon(\hat{g}_{[0,t[})$ for $g \in L^2(\mathbb{R}_+; \mathsf{k})$;*

2. *$e^{-\frac{t}{2}} \hat{D}_t$ is a partial isometry with initial space $\mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}_t$ and final space*

$$\overline{\mathrm{Lin}}\{v \otimes \varepsilon(1_{[0,t[}\hat{g}) \colon g \in L^2(\mathbb{R}_+; \mathsf{k}), v \in \mathsf{h}\};$$

3. *$(e^t - e^r)^{-\frac{1}{2}}(\hat{D}_t - \hat{D}_r)$ is a partial isometry with initial space $\mathsf{h} \otimes (\mathcal{F}^{\mathsf{k}}_t \ominus \mathcal{F}^{\mathsf{k}}_r)$ and final space*

$$\overline{\mathrm{Lin}}\{v \otimes (\varepsilon(1_{[0,t[}\hat{g}) - \varepsilon(1_{[0,r[}\hat{g})) \colon g \in L^2(\mathbb{R}_+; \mathsf{k}), v \in \mathsf{h}\}.$$

Here $\mathsf{H} \ominus \mathsf{K}$ *for a Hilbert space* H *and its subspace* K *denotes the orthogonal complement of* K *in* H*.*

Proof. Let $g \in L^2(\mathbb{R}_+; k)$. Then, in view of the identity $\hat{c} = e_0 + \tilde{c} = e_0 + \tilde{J}c (c \in k)$,

$$\pi_{\hat{g}}(\sigma) = \sum_{\alpha \subset \sigma} J^{e_0}_{\alpha;\sigma} \pi_{\tilde{g}}(\alpha) = \sum_{\alpha \subset \sigma} J^{e_0}_{\alpha;\sigma} \tilde{J}^{\otimes \# \alpha} \pi_g(\alpha) = \sum_{\alpha \subset \sigma} \pi_g(\alpha)$$

for $\sigma \in \Gamma$. (1) follows.

Let $k \in \mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}$. Then, by the orthogonality relations

$$\| \sum_{\alpha \subset \sigma} J_{\alpha;\sigma} k(\alpha) \|^2 = \sum_{\alpha \subset \sigma} \|k(\alpha)\|^2.$$

Set $S = \Gamma_{[0,t[} \setminus \Gamma_{[0,r[}$. By the integral-sum identity

$$\begin{aligned} \|(\hat{D}_t - \hat{D}_r)k\|^2 &= \int_S d\sigma \| \sum_{\alpha \subset \sigma} J_{\alpha;\sigma} k(\alpha)\|^2 = \int_S \sum_{\alpha \subset \sigma} \|k(\alpha)\|^2 \\ &= \int_S d\alpha \int_S d\beta \|k(\alpha)\|^2 = |S| \int d\alpha \|1_s k(\alpha)\|^2. \end{aligned}$$

Since $|S| = e^t - e^r$ and $\{1_S k \colon k \in \mathcal{F}^{\mathsf{k}}\} = \mathcal{F}^{\mathsf{k}}_t \ominus \mathcal{F}^{\mathsf{k}}_r$, (3) follows. (2) follows similarly. $\square$

Remark 2.3.3. Note that the operators $\hat{D}_t$ are ampliations of multiples of partial isometries in $B(\mathcal{F}^{\mathsf{k}}; \mathcal{F}^{\widehat{\mathsf{k}}})$.

For a vector kernel $\zeta \in K(\mathsf{h}, \widehat{\mathsf{k}})$ define an associated function

$$\zeta' \colon \Gamma \times \Gamma \to \mathsf{h} \otimes \Phi^{\mathsf{k}}, (\alpha, \beta) \mapsto J^*_{\alpha;\alpha\cup\beta} \zeta(\alpha \cup \beta).$$

Lemma 2.3.4. *Let* $\zeta \in K(h, \widehat{\mathsf{k}})$*.*

1. *For* $S \subset \Gamma$, $(\mathbf{1}_S \zeta)' = \mathbf{1}_{S'} \zeta'$*, where*

$$S' = \{(\alpha, \beta) \in \Gamma \times \Gamma | \alpha \cup \beta \in S\}.$$

2. *If ζ is measurable as a function $\Gamma \to h \otimes \Phi^{\widehat{\mathsf{k}}}$, then ζ' is measurable too.*

3. *If ζ is almost everywhere zero, then ζ' is too.*

Proof. (1) follows from the identity $\mathbf{1}_S(\alpha \cup \beta) = \mathbf{1}_{S'}(\alpha, \beta)$ for $\alpha, \beta \in \Gamma$. (2) follows from the fact that if ζ is measurable, then so is $(\alpha, \beta) \mapsto \zeta(\alpha \cup \beta)$. At the same time, $(\alpha, \beta) \mapsto J^*_{\alpha;\alpha\cup\beta}$ is obviously measurable by treating it as a product function of two variables (the product in one variable being the projection $\widehat{\mathsf{k}} \to \mathsf{k}$ and in the other $\widehat{\mathsf{k}} \to \mathsf{k}^\perp$. Thus the result is measurable as a product of two measurable functions.

(3) follows from the integral-sum identity: for $S \subset \Gamma$ measurable,

$$|S'| = \int d\alpha \int d\beta \mathbf{1}_{S'}(\alpha, \beta) = \int d\alpha \int d\beta \mathbf{1}_S(\alpha \cup \beta)$$
$$= \int d\sigma \sum_{\alpha \subset \sigma} \mathbf{1}_S(\sigma) = \int_S d\sigma 2^{\#\sigma},$$

so $|S'| = 0$ if $|S| = 0$. □

Note that

$$\mathsf{h} \otimes \mathcal{F}^{\mathsf{k}} = L^2(\Gamma, \mathsf{h} \otimes \Phi^{\mathsf{k}}) \cap K(\mathsf{h}, \mathsf{k}).$$

Definition 2.3.5. Let $p, q \in [1, \infty]$, X, Y be measure spaces and H be a Hilbert space. We define $L^{p,q}(X \times Y; H)$ to be:

$$L^{p,q}(X \times Y; H) := \{f \colon X \times Y \to H | f \text{ measurable},$$
$$\|f\|_{p,q} = (\int_X (\int_Y \|f(x,y)\|^q dy)^{\frac{p}{q}} dx)^{\frac{1}{p}} < \infty\}.$$

If $X = Y = \Gamma$ and $a, b > 0$, we introduce weighted spaces as follows:

$$L^{p,q}_{a.b}(\Gamma \times \Gamma; H) := \{f \colon \Gamma \times \Gamma \to H | \int_\Gamma (b^{\#\beta} \int_\Gamma \|a^{\#\alpha} f(\beta, \alpha)\|^q d\alpha)^{\frac{p}{q}} d\beta < \infty\}.$$

Let us notice that if $a > c, b > d$ then $L^{2,1}_{a,b}(\Gamma \times \Gamma; \Phi_k) \subset L^{2,1}_{c,d}(\Gamma \times \Gamma; \Phi_k)$.

Definition 2.3.6. A kernel $\zeta \in K(\mathsf{h}, \widehat{\mathsf{k}})$ is Wiener-time integrable if $\zeta' \in L^{2,1}(\Gamma \times \Gamma; h \otimes \Phi^{\mathsf{k}})$, in which case its multiple Wiener-time integral $\hat{\mathcal{W}}\zeta$ is the element of $\mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}$ defined almost everywhere by

$$(\widehat{\mathcal{W}}\zeta)(\sigma) = \int_{\Gamma} d\alpha \zeta'(\alpha, \sigma).$$

The kernel ζ is locally Wiener time integrable if $\mathbf{1}_{\Gamma_{[0,t[}}\zeta$ is Wiener time integrable for all $t \in \mathbb{R}_+$, in which case we set $\widehat{\mathcal{W}}_t\zeta = \widehat{\mathcal{W}}(1_{\Gamma_{[0,t[}}\zeta)$. We denote these two classes of kernels by $\widehat{\mathbb{I}}^W(\mathsf{h}, \mathsf{k})$ and $\widehat{\mathbb{I}}^W_{\text{loc}}(\mathsf{h}, \mathsf{k})$, respectively.
Thus if $\zeta \in \widehat{\mathbb{I}}^W(\mathsf{h}, \mathsf{k})$ then $\widehat{\mathcal{W}}\zeta \in \mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}$ and

$$\|\widehat{\mathcal{W}}\zeta\| \leq \|\zeta'\|_{1,2}.$$

It is easily observed that

Proposition 2.3.7. *If $\xi' \in L^{2,1}_{a,1}(\Gamma \times \Gamma; \Phi_{\mathsf{k}})$ for $a > 1$, then $\widehat{\mathcal{W}}(\xi) \in Dom(\sqrt{a}^N)$.*

Proof. We evaluate:

$$\sqrt{\int_{\Gamma} d\alpha a^{\frac{\#\alpha}{2}} \|\widehat{\mathcal{W}}(\xi)(\alpha)\|^2} \leqslant \int_{\Gamma} d\beta \sqrt{\int d\alpha \|a^{\#\alpha}\xi'(\alpha, \beta)\|^2} = \|\xi'\|^{a,1}_{2,1}.$$

□

Remark 2.3.8. The Wiener-time integral is a hybrid of its two extreme cases, the Wiener and time integrals:

- If $\zeta = \tilde{k}$ for some $k \in \mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}$, then

$$\zeta'(\alpha, \beta) = \mathbf{1}_{\emptyset}(\alpha)k(\beta), \text{ so } \zeta \in \widehat{\mathbb{I}}^W(\mathsf{h}, \mathsf{k}) \text{ and } \widehat{\mathcal{W}}\zeta = k,$$

  in particular, $\|\widehat{\mathcal{W}}\zeta\| = \|k\| = \|\zeta'\|_{1,2}$.

- If $\zeta(\sigma) = a(\sigma) \otimes e_0^{\otimes \# \sigma}$ for some $a \in L^1(\Gamma; \mathsf{h})$ then

$$\zeta'(\alpha, \beta) = \mathbf{1}_\emptyset(\beta) a(\alpha),$$

so $\zeta \in \widehat{\mathbb{I}}^W(h, k)$ and

$$\widehat{\mathcal{W}}\zeta = \left(\int_\Gamma d\alpha a(\alpha)\right) \delta_\emptyset,$$

in particular,

$$\|\widehat{\mathcal{W}}\zeta\| = \| \int_\Gamma d\alpha a(\alpha)\| \leq \|a\|_1 = \|\zeta'\|_{1,2}.$$

Proposition 2.3.9. *Let $\xi, \eta \in \widehat{\mathbb{I}}^W(\mathsf{h}, \mathsf{k})$. Then the function*

$$\Gamma \times \Gamma \times \Gamma \to \mathbb{C},\ (\sigma_1, \sigma_2, \sigma_3) \mapsto \langle \zeta'(\sigma_1, \sigma_3), \eta'(\sigma_2, \sigma_3)\rangle$$

is integrable and

$$\langle \widehat{\mathcal{W}}\zeta, \widehat{\mathcal{W}}\eta\rangle = \int d\sigma_1 \int d\sigma_2 \int d\sigma_3 \langle \zeta'(\sigma_1, \sigma_3), \eta'(\sigma_2, \sigma_3)\rangle.$$

Proof. Since $\zeta'(\sigma_1, \sigma_3), \eta'(\sigma_2, \sigma_3) \in \mathsf{h} \otimes \mathsf{k}^{\otimes \# \sigma_3}$ for all $\sigma_1, \sigma_2, \sigma_3 \in \Gamma$, the function is well-defined; its integrability follows from the fact that $\zeta', \eta' \in L^{2,1}(\Gamma \times \Gamma; \mathsf{h} \otimes \Phi^k)$. On the other hand,

$$\langle (\widehat{\mathcal{W}}\zeta)(\sigma_3), (\widehat{\mathcal{W}}\eta)(\sigma_3)\rangle = \int d\sigma_1 \int d\sigma_2 \langle \zeta'(\sigma_1, \sigma_3), \eta'(\sigma_2, \sigma_3)\rangle$$

for a.a. $\sigma_3 \in \Gamma$ and so the identity follows by integration, courtesy of Fubini's theorem. □

Remark 2.3.10. Again this is consistent with Wiener–Itô isometry, since if $\zeta = \eta = \tilde{k}$ for some $k \in \mathsf{h} \otimes \mathcal{F}^k$ then

$$\langle \zeta'(\sigma_1, \sigma_3), \zeta'(\sigma_2, \sigma_3)\rangle = \delta_\emptyset(\sigma_1)\delta_\emptyset(\sigma_2)\|k(\sigma_3)\|^2,$$

so

$$\int d\sigma_1 \int d\sigma_2 \int d\sigma_3 \langle \zeta'(\sigma_1, \sigma_3), \zeta'(\sigma_2, \sigma_3)\rangle = \int d\sigma_3 \|k(\sigma_3)\|^2 = \|k\|^2.$$

Lemma 2.3.11. *Let* $\zeta \in K(\mathsf{h}, \widehat{\mathsf{k}})$ *and* $t \geq r \geq 0$*. Then*

$$\|(\boldsymbol{1}_{\Gamma_{[0,t[}\setminus\Gamma_{[0,r[}}\zeta)\|_{1,2} \leq \sqrt{e^t - e^r}\left(\int_{\Gamma_{[0,t[}\setminus\Gamma_{[0,r[}} d\sigma 2^{\#\sigma}\|\zeta(\sigma)\|^2\right)^{\frac{1}{2}}.$$

Proof. Set $S = \Gamma_{[0,t[} \setminus \Gamma_{[0,r[}$ and note that

$$\mathbf{1}_S(\alpha \cup \beta) = \mathbf{1}_S(\alpha)\mathbf{1}_S(\beta) \text{ for all } \alpha, \beta \in \Gamma.$$

Therefore, by the Cauchy-Schwarz inequality and integral-sum identity,

$$\begin{aligned}
\int d\beta \left(\int d\alpha \|(\mathbf{1}_S\zeta)'(\alpha, \beta)\|\right)^2 &= \int d\beta \left(\int d\alpha \mathbf{1}_S(\alpha \cup \beta)\|\zeta(\alpha \cup \beta)\|\right)^2 \\
&= \int_S d\beta \left(\int_S d\alpha \|\zeta(\alpha \cup \beta)\|\right)^2 \\
&\leq \int_S d\beta |S| \int_S d\alpha \|\zeta(\alpha \cup \beta)\|^2 \\
&= |S| \int_S d\sigma \sum_{\alpha \subset \sigma} \|\zeta(\sigma)\|^2 \\
&= |S| \int_S d\sigma 2^{\#\sigma}\|\zeta(\sigma)\|^2.
\end{aligned}$$

□

Proposition 2.3.12. *Let* $\zeta \in \mathrm{Dom}(I_{\mathsf{h}} \otimes \sqrt{2}^N)$*. Then*

1. $\zeta \in \widehat{\mathbb{I}}^W_{\mathrm{loc}}(\mathsf{h}, \mathsf{k})$ *and*

2. $\langle k, \widehat{\mathcal{W}}_t \zeta \rangle = \langle \hat{D}_t k, \zeta \rangle$ *for all* $t \in \mathbb{R}_+$ *and* $k \in \mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}$.

Proof. 1. follows from Lemma 2.3.11.

2. Let $k \in \mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}$ and $t \in \mathbb{R}_+$. In view of the previous part, the left hand side is well-defined. By the integral-sum identity it equals

$$\begin{aligned} &\int_{\Gamma_{[0,t[}} d\beta \langle k(\beta), \int_{\Gamma_{[0,t[}} d\alpha J^*_{\beta,\alpha\cup\beta} \zeta(\alpha \cup \beta) \rangle \\ =& \int_{\Gamma_{[0,t[}} \int_{\Gamma_{[0,t[}} d\alpha \langle J_{\beta;\alpha\cup\beta} k(\beta), \zeta(\alpha \cup \beta) \rangle \\ =& \int_{\Gamma_{[0,t[}} d\sigma \sum_{\beta \subset \sigma} \langle J_{\beta;\sigma} k(\beta), \zeta(\sigma) \rangle, \end{aligned}$$

which coincides with the right hand side as required.

□

Thus, as Hilbert space operators,

$$\widehat{\mathcal{W}}_t|_{Dom(I_{\mathsf{h}} \otimes \sqrt{2}^N)} = \hat{D}^*_t|_{Dom(I_{\mathsf{h}} \otimes \sqrt{2}^N)}.$$

In particular, this gives the estimate

$$\|\widehat{\mathcal{W}}_t \zeta\| \leq e^{\frac{t}{2}} \|\zeta\| \qquad (\zeta \in Dom(I_{\mathsf{h}} \otimes \sqrt{2}^N)),$$

improving on that of Remark 2.3.8.

Lemma 2.3.13. *Let* $\Phi \in F(\mathbb{R}_+; \mathsf{k})$. *Then, for* $\sigma \in \Gamma$ *and* $\alpha \subset \sigma$,

$$J_{\alpha,\sigma} \pi_\Phi(\alpha) = \pi_{\tilde{\Phi}}(\alpha; \sigma)_0 \text{ and } J^*_{\alpha;\sigma} \pi_{\hat{\Phi}}(\sigma) = \pi_\Phi(\alpha).$$

Proof. The first identity was noted earlier, in (2.3). Let $\psi \in F(\mathbb{R}_+; k)$ and note that, applying the first identity with ψ in place of Φ,

$$\begin{aligned}\langle \pi_\psi(\alpha), J^*_{\alpha;\sigma}\pi_{\hat\Phi}(\sigma)\rangle &= \langle \pi_{\tilde\psi}(\alpha;\sigma)_0, \pi_{\hat\Phi}(\sigma)\rangle \\ &= \langle \pi_{\tilde\psi}(\alpha), \pi_{\tilde\Phi}(\sigma)\rangle = \langle \pi_\psi(\alpha), \pi_\Phi(\alpha)\rangle.\end{aligned}$$

Since $\{\pi_\psi(\alpha) \colon \psi \in F(\mathbb{R}_+; k)\}$ is total in $\mathsf{k}^{\otimes \#\alpha}$, the second identity follows. □

Theorem 2.3.14. *Let* $\zeta, \eta \in \widehat{\mathbb{I}}^W(\mathsf{h}, \mathsf{k})$. *Then*

1. $\langle \zeta(\cdot)|\ {}_0\!\star_0\ |\eta(\cdot)\rangle \in L^1(\Gamma; B(\mathbb{C}))$, *and*
2. $\langle \widehat{\mathcal{W}}\zeta, \widehat{\mathcal{W}}\eta\rangle = \mathcal{T}(\langle \zeta(\cdot)|\ {}_{\Delta^\perp}\!\star_{\Delta^\perp}\ |\eta(\cdot)\rangle)$, *where*

$$\mathcal{T} \colon L^1(\Gamma; B(\mathbb{C})) \to \mathbb{C}$$

denotes the integral functional

$$a \mapsto \int_\Gamma a(\alpha)\mathrm{d}\alpha.$$

Proof. Let $\sigma \in \Gamma$. Then, by Lemma 2.1.8, for any partition $\sigma = \alpha \sqcup \beta \sqcup \gamma$,

$$\begin{aligned}&\langle \zeta(\alpha \cup \beta; \sigma), \Delta(\beta; \sigma)\eta(\beta \cup \gamma; \sigma)\rangle \\ =&\langle \zeta(\alpha \cup \beta), (J^{e_0}_{\alpha\cup\beta;\sigma})^*\Delta(\beta;\sigma)J^{e_0}_{\beta\cup\gamma;\sigma}\eta(\beta\cup\gamma)\rangle \\ =&\langle J^*_{\beta;\alpha\cup\beta}\zeta(\alpha\cup\beta), J^*_{\beta;\beta\cup\gamma}\eta(\beta\cup\gamma)\rangle = \langle \zeta'(\alpha,\beta), \eta'(\gamma,\beta)\rangle.\end{aligned}$$

Thus, by (2.1.8),

$$(\langle \zeta(\cdot)|\star|\eta(\cdot)\rangle)(\sigma) = \sum_{\sigma=\sigma_1\sqcup\sigma_2\sqcup\sigma_3} \langle \zeta'(\sigma_2,\sigma_3), \eta'(\sigma_2,\sigma_3)\rangle,$$

and so the result follows from Corollary 2.1.15 and the integral-sum identity. □

Chapter 3

Quantum Wiener Integrals

3.1 Kernel conditions

We are now ready to define a multiple Wiener-time integral of an operator and thus to make our theory fully noncommutative. We will need corresponding conditions to $L^{2,1}$ from the previous section. Also quantum integrals may be unbounded operators and thus we will need to be careful with our domain considerations.

Definition 3.1.1. For a Hilbert space k we define the following subspaces of the symmetric Fock space $\mathcal{F}^{\mathsf{k}}$:

$$\mathrm{Exp} = \mathrm{Lin}\{\pi_f \colon f \in L^2(\mathbb{R}_+; \mathsf{k})\};$$

$$\mathrm{Geom} = \{\xi \in \mathcal{F}^{\mathsf{k}} \colon \exists_{a,C>0} \forall_{\sigma \in \Gamma} \|\xi(\sigma)\| \leqslant a C^{\#\sigma}\};$$

$$\mathrm{Num} = \bigcap_{a>0} \mathrm{Dom}(a^N);$$

$$\mathrm{Fin} = \{\xi \in \mathcal{F}^{\mathsf{k}} \colon \exists_{N \in \mathbb{N}} \forall_{\sigma \in \Gamma, \#\sigma \geqslant N} \xi(\sigma) = 0\}.$$

Definition 3.1.2. Let $p, q, r, s \in [1, \infty]$. We will say that an operator kernel x satisfies the (local) $L^{p,q,r,s}$ condition if there exist nonnegative functions $k \in L^p_{(\mathrm{loc})}(\mathbb{R}_+)$, $l \in$

$L^q_{(\mathrm{loc})}(\mathbb{R}_+), m \in L^r_{(\mathrm{loc})}(\mathbb{R}_+), n \in L^s_{(\mathrm{loc})}(\mathbb{R}_+)$ such that for every $\alpha, \beta \in \Gamma, \alpha \cap \beta = \emptyset, \delta \subset \alpha \cup \beta$ we have

$$\|J^*_{\alpha;\alpha\cup\beta} x(\alpha \cup \beta) J_{\delta;\alpha\cup\beta}\| \leqslant \pi_k(\beta \setminus \delta)\pi_l(\alpha \setminus \delta)\pi_m(\beta \cap \delta)\pi_n(\alpha \cap \delta).$$

We also introduce an auxiliary function c such that $n = c - 1$.

In applications the most important case is $p = 1, q = r = 2, s = \infty$.

Proposition 3.1.3. *Let $x \in OK((B(\mathsf{h}_2; \mathsf{h}_3); B(\widehat{\mathsf{k}})), y \in OK(B(\mathsf{h}_1; \mathsf{h}_2); B(\widehat{\mathsf{k}}))$ satisfy the (local) $L^{1,2,2,\infty}$ condition. Then so does $x \star y$.*

Proof. Let $\alpha, \beta \subset \sigma \in \Gamma$. Let the appropriate functions from the (local) $L^{1,2,2,\infty}$ property of x, y be denoted by $k^x, l^x, m^x, n^x, k^y, l^y, m^y, n^y$, respectively, with $n^x = c^x - 1, n^y = c^y - 1$. We calculate, using Lemma 2.1.8:

$$\begin{aligned}
\|J^*_{\alpha;\sigma}(x \star y)(\sigma) J_{\beta;\sigma}\| &\leqslant \sum_{\delta_1 \sqcup \delta_2 \sqcup \delta_3 = \sigma} \|J^*_{\alpha;\sigma} x(\delta_1 \cup \delta_2; \sigma)\Delta(\delta_2; \sigma) y(\delta_2 \cup \delta_3; \sigma) J_{\beta;\sigma}\| \\
&= \sum_{\delta_1 \sqcup \delta_2 \sqcup \delta_3 = \sigma} \|[J^*_{\alpha\cap(\delta_1\cup\delta_2);\delta_1\cup\delta_2} x(\delta_1 \cup \delta_2); (\alpha \cap \delta_3) \cup \delta_1 \cup \delta_2][J^*_{\alpha\cap\delta_3;\delta_3}; \sigma]\Delta(\delta_2; \sigma) \\
&[J_{\beta\cap\delta_1;\delta_1}; (\beta \cap \delta_1) \cup \delta_2 \cup \delta_3][y(\delta_2 \cup \delta_3) J_{\beta\cap(\delta_2\cup\delta_3);\delta_2\cup\delta_3}; \beta]\| \\
&= \sum_{\delta_1 \sqcup \delta_2 \sqcup \delta_3 = \sigma} \|J^*_{\alpha\cap(\delta_1\cup\delta_2);\delta_1\cup\delta_2} x(\delta_1 \cup \delta_2); (\alpha \cap \delta_3) \cup \delta_1 \cup \delta_2] \\
&[J_{\delta_2\cup(\beta\cap\delta_1);\delta_1\cup\delta_2}; (\beta \cap \delta_1) \cup \delta_2 \cup (\alpha \cap \delta_3)][J^*_{\delta_2\cup(\alpha\cap\delta_3);\delta_2\cup\delta_3}; (\beta \cap \delta_1) \cup \delta_2 \cup \delta_3] \\
&[y(\delta_2 \cup \delta_3) J_{\beta\cap(\delta_2\cup\delta_3);\delta_2\cup\delta_3}; \beta]\| \\
&\leqslant \sum_{\delta_1 \sqcup \delta_2 \sqcup \delta_3 = \sigma} \|J^*_{\alpha\cap(\delta_1\cup\delta_2);\delta_1\cup\delta_2} x(\delta_1 \cup \delta_2) J_{(\delta_1\cap\beta)\cup\delta_2;\delta_1\cup\delta_2}\| \\
&\|J^*_{(\delta_3\cap\alpha)\cup\delta_2;\delta_2\cup\delta_3} y(\delta_2 \cup \delta_3) J_{\beta\cap(\delta_2\cup\delta_3);\delta_2\cup\delta_3}\|.
\end{aligned}$$

Applying the $L^{1,2,2,\infty}$ properties of x, y we now obtain estimates as follows:

$$\|J^*_{\alpha\cap(\delta_1\cup\delta_2);\delta_1\cup\delta_2} x(\delta_1\cup\delta_2) J_{(\delta_1\cap\beta)\cup\delta_2;\delta_1\cup\delta_2}\| \leqslant |\pi_{k^x}(\delta_1\setminus(\alpha\cup\beta))|$$
$$|\pi_{l^x}((\alpha\setminus\beta)\cap\delta_1)||\pi_{m^x}((\delta_1\cap(\beta\setminus\alpha))\cup(\delta_2\setminus\alpha))||\pi_{n^x}((\alpha\cap\delta_2)\cup(\alpha\cap\beta\cap\delta_1))|;$$

$$\|J^*_{(\delta_3\cap\alpha)\cup\delta_2} y(\delta_2\cup\delta_3) J_{\beta\cap(\delta_2\cup\delta_3)}\| \leqslant |\pi_{k^y}(\delta_3\setminus(\alpha\cup\beta))|$$
$$|\pi_{l^y}(((\alpha\setminus\beta)\cap\delta_3)\cup(\delta_2\setminus\beta))||\pi_{m^y}(\delta_3\cap(\beta\setminus\alpha))||\pi_{n^y}((\alpha\cap\beta\cap\delta_3)\cup(\delta_2\cap\beta))|.$$

Thus, seeing that:

$$\begin{aligned}
\sigma\setminus(\alpha\cup\beta) &= (\delta_1\cup\delta_2\cup\delta_3)\setminus(\alpha\cup\beta)\\
&= (\delta_1\setminus(\alpha\cup\beta))\cup(\delta_3\setminus(\alpha\cup\beta))\cup((\delta_2\setminus\alpha)\cap(\delta_2\setminus\beta))\\
\alpha\setminus\beta &= ((\alpha\setminus\beta)\cap\delta_1)\cup((\alpha\setminus\beta)\cap\delta_3)\cup((\alpha\cap\delta_2)\cap(\delta_2\setminus\beta))\\
\beta\setminus\alpha &= ((\beta\setminus\alpha)\cap\delta_1)\cup((\beta\setminus\alpha)\cap\delta_3)\cup((\delta_2\setminus\alpha)\cap(\delta_2\cap\beta))\\
\alpha\cap\beta &= (\delta_1\cap\alpha\cap\beta)\cup(\delta_3\cap\alpha\cap\beta)\cup(\delta_2\cap\alpha\cap\beta),
\end{aligned}$$

referring to Lemma 2.1.11 and using nonnegativity of k, l, m, n, it is now easily seen that the appropriate functions to approximate $z = x \star y$ are:

$$\begin{aligned}
k^z &= k^x + k^y + m^x l^y \in L^1_{(\mathrm{loc})}(\mathbb{R}_+;\mathbb{R}_+);\\
l^z &= l^x + l^y + n^x l^y \in L^2_{(\mathrm{loc})}(\mathbb{R}_+;\mathbb{R}_+);\\
m^z &= m^x + m^y + m^x n^y \in L^2_{(\mathrm{loc})}(\mathbb{R}_+;\mathbb{R}_+);\\
n^z &= n^x + n^y + n^x n^y \in L^\infty_{(\mathrm{loc})}(\mathbb{R}_+;\mathbb{R}_+),
\end{aligned}$$

or, using c:

$$k^z = k^x + k^y + m^x l^y \in L^1_{(\mathrm{loc})}(\mathbb{R}_+;\mathbb{R}_+);$$
$$l^z = l^x + c^x l^y \in L^2_{(\mathrm{loc})}(\mathbb{R}_+;\mathbb{R}_+);$$
$$m^z = m^y + m^x c^y \in L^2_{(\mathrm{loc})}(\mathbb{R}_+;\mathbb{R}_+);$$
$$c^z = c^x c^y \in L^\infty_{(\mathrm{loc})}(\mathbb{R}_+;\mathbb{R}_+).\square$$

Remark 3.1.4. As we see later, the local $L^{1,2,2,\infty}$ condition is a natural condition to ensure that the domains of our quantum Wiener integrals are big enough. We could instead require the following to hold:

$$\|J^*_{\alpha;\sigma}x(\sigma)J_{\beta;\sigma}\| \leqslant f(\sigma\setminus(\alpha\cup\beta))g(\beta\setminus\alpha)h(\alpha\setminus\beta)k(\alpha\cap\beta)$$

for some functions $f \in L^1_{\mathrm{loc}}(\Gamma), g, h \in L^2_{\mathrm{loc}}(\Gamma), k \in L^\infty_{\mathrm{loc}}(\Gamma)$ (thus forgoing the assumption of them being product functions). However, while this does guarantee our domain to contain, for example, exponential vectors, this condition is not closed under the convolution action. Thus in our context it is more natural to assume the condition with product functions.

The non-product version of this condition will make an appearance in Theorem 3.4.3. Let us recall the $\sim$ notation from Section 2.3.

Definition 3.1.5. Let $x \in OK(B(\mathsf{h}),\widehat{\mathsf{k}})$ be measurable. We define the quantum time-Wiener integral of x to be the operator on $\mathsf{h}\otimes\Phi^{L^2(\mathbb{R}_+;\mathsf{k})}$ with domain

$$\mathrm{Dom}(\mathcal{Q}_t(x)) = \{\xi \in \mathsf{h}\otimes\Phi^{L^2(\mathbb{R}_+;\mathsf{k})} \colon (x\mathbf{1}_{\Gamma_t}\star\tilde{\xi}) \in \mathbb{I}^W\},$$

given by the formula

$$\mathcal{Q}_t(x)(\xi) = \widehat{\mathcal{W}}(x\mathbf{1}_{\Gamma_t}\star\xi).$$

We also define the global quantum time-Wiener integral by putting

$$\mathrm{Dom}(\mathcal{Q}(x)) = \{\xi \in \mathsf{h} \otimes \Phi^{L^2(\mathbb{R}_+;\mathsf{k})} \colon (x \star \xi) \in \mathbb{I}^W\},$$

$$\mathcal{Q}(x)(\xi) = \widehat{\mathcal{W}}(x \star \tilde{\xi})$$

and a two-parameter family, which will come into play in the next chapter, via

$$\mathrm{Dom}(\mathcal{Q}_{s,t}(x)) = \{\xi \in \mathsf{h} \otimes \Phi^{L^2(\mathbb{R}_+;\mathsf{k})} \colon (x\mathbf{1}_{\Gamma_{s,t}} \star \xi) \in \mathbb{I}^W\},\ s < t,$$

$$\mathcal{Q}_{s,t}(x)(\xi) = \widehat{\mathcal{W}}(x\mathbf{1}_{\Gamma_{s,t}} \star \tilde{\xi})\ s < t$$

Remark 3.1.6. We note that by Corollary 2.2.12 and the locality of definition of $\mathcal{Q}_t(x)$ and $\mathcal{Q}_{s,t}(x)$, this indeed gives a well-defined operator.

Example 3.1.7. Let $f \in L^2(\mathbb{R}_+;\mathsf{k})$. Define the product operator kernel $x_f = \pi_F$ by

$$F(s) = \begin{pmatrix} -\frac{1}{2}\|f(s)\|^2 & -\langle f(s)| \\ |f(s)\rangle & 0 \end{pmatrix}.$$

Then it is easily seen that $\mathcal{Q}_t(x_f)$ is just the Weyl process $W(f_{[0,t[})$.

Proof. For simplicity of notation, let us consider functions $f, g \in L^2(\mathbb{R}_+;\mathsf{k})$ supported on the interval $[0,t[$ and let $F(s)$ be as in the example. To show that $\mathcal{Q}_t(x_f) = W(f)$, it suffices to show that:

$$\mathcal{Q}_t(x_f)\varpi(g) = e^{-\mathrm{Im}\langle f,g\rangle}\varpi(f+g).$$

To make the following calculations easier, let us write out the result of applying function $F(s)$ to $g(s)$ and $\widehat{0}$:

$$F(s)g(s) = \begin{pmatrix} -\langle f(s), g(s)\rangle \\ 0 \end{pmatrix} =: \xi_s,$$

$$F(s)\widehat{0} = \begin{pmatrix} -\frac{1}{2}||f(s)||^2 \\ f(s) \end{pmatrix} =: \zeta_s.$$

Now, for $\alpha \in \Gamma_t$, we calculate:

$$\begin{aligned}
\mathcal{Q}_t(x_f)\varpi(g)(\alpha) &= e^{-\frac{\|g\|^2}{2}} \int_{\Gamma_t} J^*_{\alpha;\alpha\cup\beta}(x_f \star \varepsilon(g))(\alpha \cup \beta) \\
&= e^{-\frac{\|g\|^2}{2}} \int_{\Gamma_t} J^*_{\alpha;\alpha\cup\beta} \sum_{\gamma_0\cup\gamma_1=\alpha\cup\beta} x_f(\gamma_0; \alpha\cup\beta)\varepsilon(g)(\gamma_1;\alpha\cup\beta) \\
&= e^{-\frac{\|g\|^2}{2}} \int_{\Gamma_t} J^*_{\alpha;\alpha\cup\beta} \sum_{\gamma_0\cup\gamma_1=\alpha\cup\beta} \bigotimes_{s\in\alpha\cup\beta} \eta(s),
\end{aligned}$$

where

$$\eta(s) = \begin{cases} \xi_s & s \in \gamma_1 \cap \gamma_0, \\ \zeta_s & s \in \gamma_0 \setminus \gamma_1, \\ g(s) & s \in \gamma_1 \setminus \gamma_0. \end{cases}$$

This lets us split the summation into sums over disjoint unions:

$$\begin{aligned}
J^*_{\alpha;\alpha\cup\beta} \sum_{\gamma_0\cup\gamma_1=\alpha\cup\beta} \otimes_{s\in\alpha\cup\beta}\eta(s) = &\sum_{\alpha_0\sqcup\alpha_1=\alpha} \sum_{\beta_0\sqcup\beta_1=\beta} \varepsilon(f)(\alpha_0;\alpha,\varepsilon(g)) \\
&\cdot \prod_{s\in\beta_0} (-\langle f(s), g(s)\rangle) \prod_{s\in\beta_1} \left(-\frac{1}{2}\|f(s)\|^2\right), \\
&= \varepsilon(f+g)(\alpha) \sum_{\beta_0\sqcup\beta_1=\beta} \prod_{s\in\beta_0} (-\langle f(s), g(s)\rangle) \prod_{s\in\beta_1} \left(-\frac{1}{2}\|f(s)\|^2\right)
\end{aligned}$$

Upon integrating, this becomes:

$$\begin{aligned}
\mathcal{Q}_t(x_f)\varpi(g) &= e^{-\frac{\|g\|^2}{2} - \langle f,g\rangle - \frac{\|f\|^2}{2}} \varepsilon(f+g) \\
&= e^{-\frac{\|g\|^2}{2} - \langle f,g\rangle - \frac{\|f\|^2}{2} + \frac{\|f+g\|^2}{2}} \varpi(f+g) \\
&= e^{-\mathrm{Im}\langle f,g\rangle} \varpi(f+g),
\end{aligned}$$

which ends the proof. □

Remark 3.1.8. It might seem like a more natural way of defining the quantum Wiener integral would be by putting

$$\mathcal{Q}_t(x)(\xi) = \widehat{\mathcal{W}}_t(x \star \xi)$$

(with the appropriate domain adjustment). However, such an operator is vacuum-adapted, while we work in the identity-adapted setup. This requires the move of the indicator function. We do not treat the vacuum-adapted case here. We refer the Reader to [7] and [8].

We will be mainly studying the properties of the local-time quantum Wiener integral. Most of the results hold in the global case by dropping the subscripts t and the locality conditions.

Remark 3.1.9. Let us write out the definition of $\mathcal{Q}_t(x)\xi$ to notice a helpful algebraic identity. By the definition of our convolution, we have:

$$\mathcal{Q}_t(x)\xi(\alpha) = \int_{\Gamma_t} \mathrm{d}\beta J^*_{\alpha;\alpha\cup\beta} \sum_{\beta_0\cup\beta_1=\alpha\cup\beta} x(\beta_0;\alpha\cup\beta)\mathbf{1}_{\Gamma_t}(\beta_0)\Delta(\beta_0\cap\beta_1;\alpha\cup\beta)\xi(\beta_1;\alpha\cup\beta).$$

Assume that $\beta_1 \setminus \beta_0 \cap \beta \neq \emptyset$. Then that means that a part of ξ which is not being acted on by x is on the tensor component corresponding to β. But by the action of $J^*_{\alpha;\alpha\cup\beta}$ we see that such terms would be made equal to 0. Thus in further calculations we can, in fact, always assume that the partition of $\alpha\cup\beta$ enjoys the property

$$\beta_1 \setminus \beta_0 \cap \beta = \emptyset.$$

The first important and surprising property of our $\mathcal{Q}_t(x)$ is its closedness.

Theorem 3.1.10. *Let x satisfy the $L^{1,2,2,\infty}$ condition locally and $t \geqslant 0$. Then $\mathcal{Q}_t(x)$, treated as an operator with the domain*

$$\{\xi \in \mathcal{F}^{\mathsf{k}} \colon (x\boldsymbol{1}_{\Gamma_t} \star \xi) \textit{ is time-Wiener integrable}\}$$

is a closed operator.

Proof. Assume $\xi_n \to \xi$, $\mathcal{Q}_t(x)\xi_n \to y, \xi_n \in \mathrm{Dom}(\mathcal{Q}_t(x))$. Firstly, let us notice that upon passing to a subsequence we can assume these convergences hold almost surely, which in particular implies the measurability of the function

$$\alpha \mapsto J^*_{\alpha;\alpha\cup\beta}(x\mathbf{1}_{\Gamma_t} \star \xi)(\alpha \cup \beta).$$

Let us notice that if we prove that

$$\int_{\Gamma_t} d\beta \| J^*_{\alpha;\alpha\cup\beta}(x \star (\xi - \xi_n))(\alpha \cup \beta)\| \to 0 \tag{3.1}$$

outside a null set, then we will be done. Indeed, that would mean that for almost all α $\beta \mapsto J^*_{\alpha;\alpha\cup\beta}(x \star \xi)(\alpha \cup \beta)$ is an integrable function and its integral by the convergence assumption must be equal to $y(\alpha)$ for almost all α. Thus indeed $\mathcal{Q}_t(x) = y$. Therefore, the only thing to show is Equation 3.1 for sets $\alpha \subset \Gamma_t$, as our operator is identity adapted.

Let $\xi - \xi_n = \eta_n$. We perform the following calculation:

$$\begin{aligned}
\int_{\Gamma_t} d\beta \| J^*_{\alpha;\alpha\cup\beta}(x \star (\eta_n))(\alpha \cup \beta)\| &\leqslant \int_{\Gamma_t} \sum_{\beta_0 \cup \beta_1 = \alpha\cup\beta} \| J^*_{\alpha;\alpha\cup\beta} x(\beta_0; \alpha \cup \beta) J_{\beta_1;\alpha\cup\beta} \eta_n(\beta_1)\| d\beta \\
&\leqslant \int_{\Gamma_t} \sum_{\beta_0 \cup \beta_1 = \alpha\cup\beta} \| J^*_{\alpha\cap\beta_0;\beta_0} x(\beta_0) J_{\beta_1\cap\beta_0;\beta_0}\| \|\eta_N(\beta_1)\| d\beta \\
&\leqslant \sum_{\alpha_0 \sqcup \alpha_1 \sqcup \alpha_2 = \alpha} \int_{\Gamma_t} \sum_{\beta_0 \sqcup \beta_1 = \beta} |\pi_k(\beta_0)| |\pi_l(\beta_1)| |\pi_m(\alpha_0)|
\end{aligned}$$

$$
\begin{aligned}
&\cdot |\pi_w(\alpha_1)| \|\eta_n(\alpha_1 \cup \alpha_2 \cup \beta_1)\| d\beta \\
&\leqslant \sum_{\alpha_0 \sqcup \alpha_1 \sqcup \alpha_2 = \alpha} C(k) |\pi_m(\alpha_0)| |\pi_w(\alpha_1)|) \int_{\Gamma_t} |\pi_l(\beta_1)| \|\eta_n(\alpha_1 \cup \alpha_2 \cup \beta_1)\| d\beta.
\end{aligned}
$$

Here $C(k) = \exp \int_0^t |k(s)| ds$. Our expression is a finite sum over subsets of α involving constants independent of β, so we merely need to prove that

$$
\int_{\Gamma_t} |\pi_l(\beta)| \|\eta_n(\alpha \cup \beta)\| d\beta \to 0
$$

for almost all $\alpha \in \Gamma_t$. We have:

$$
\begin{aligned}
\int_{\Gamma_t} \int_{\Gamma_t} |\pi_l(\beta)| \|\eta_n(\alpha \cup \beta)\| d\beta d\alpha &= \int_{\Gamma_t} \sum_{\beta \subset \alpha} |\pi_l(\beta)| \|\eta_n(\alpha)\| d\alpha \\
&= \int_{\Gamma_t} |\pi_{1+l}(\alpha)| \|\eta_n(\alpha)\| d\alpha \leqslant \exp \frac{\|(1+l)_t\|_2}{2} \|\eta_n \mathbf{1}_{[0,t[}\| \to 0.
\end{aligned}
$$

Since the double integral is tending to 0, the inner integral - as a function of α - is tending to 0 almost everywhere, say outside of a set $\mathcal{N}$. We then see that 3.1 holds for α outside of the set

$$
\{\alpha \in \Gamma_t \colon \exists_{\beta \in \mathcal{N}} \beta \subset \alpha\},
$$

which is a null set by Lemma 2.2.2. □

The following is a norm estimate for our quantum Wiener integrals which we will use many times throughout this paper.

Lemma 3.1.11. *If x satisfies the $L^{1,2,2,\infty}$ condition locally, with the positive coefficient functions $k, l, m, n = c - 1$, respectively, and $\xi \in \mathrm{Dom}(\mathcal{Q}_t(x)) \cap \mathrm{Num}$, then we have*

$$
\|\mathcal{Q}_t(x)\xi\| \leqslant e^{\sqrt{2}\|k_t\|_1 + \|l_t\|_2^2 + \frac{3}{2}\|m_t\|_2^2} \| \left(\sqrt{1+3c_t}\right)^N \xi\|,
$$

where k_t, l_t, m_t, c_t *denote the respective functions cut at time* t. *Moreover, in fact* $\mathrm{Num} \subset \mathrm{Dom}(\mathcal{Q}_t(x))$.

Proof. Let us notice that proving that $\mathcal{Q}_t(x)\xi \in \mathsf{h} \otimes \mathcal{F}^{\mathsf{k}}$ is sufficient to prove that $\mathrm{Num} \subset \mathrm{Dom}(\mathcal{Q}_t(x))$.

We have that

$$\|\mathcal{Q}_t(x)\xi\| = \sqrt{\int_\Gamma d\alpha \|\int_{\Gamma_t} d\beta J^*_{\alpha;\alpha\cup\beta}(x\mathbf{1}_{\Gamma_t} \star \xi)(\alpha \cup \beta)\|^2}.$$

By the generalized Minkowski's inequality (1.1), this is less than or equal to

$$\int_{\Gamma_t} d\beta \sqrt{\int_\Gamma d\alpha \|J^*_{\alpha;\alpha\cup\beta}(x\mathbf{1}_{\Gamma_t} \star \xi)(\alpha \cup \beta)\|^2}.$$

By writing out the convolution, applying Lemma 2.1.8 and taking the sum out of the square norm, we get an upper estimate

$$\int_{\Gamma_t} d\beta \sqrt{2}^{\#\beta} \sum_{\beta_0 \sqcup \beta_1 = \beta} \sqrt{\int_\Gamma d\alpha 3^{\#\alpha} \sum_{\substack{\alpha_0 \sqcup \alpha_1 \sqcup \alpha_2 = \alpha \\ \alpha_0, \alpha_1 \subset \Gamma_t}} (|\pi_k(\beta_0)||\pi_l(\beta_1)||\pi_m(\alpha_0)||\pi_n(\alpha_1)|\|\xi(\alpha_1 \cup \alpha_2 \cup \beta)\|)^2}.$$

Using the integral-sum identity and the fact that the integrand function is positive (being a norm), we continue:

$$\begin{aligned}
\|\mathcal{Q}_t(x)\xi\| &\leqslant \int_{\Gamma_t} d\beta_0 \sqrt{2}^{\#\beta_0} |\pi_k(\beta_0)| \int_{\Gamma_t} d\beta_1 \sqrt{2}^{\#\beta_1} |\pi_l(\beta_1)| \sqrt{\int_\Gamma d\alpha_0 3^{\#\alpha_0} |\pi_m(\alpha_0)|^2} \\
&\quad \cdot \sqrt{\int_{\Gamma_t} d\alpha_1 3^{\#\alpha_1} |\pi_n(\alpha_1)|^2 \int_\Gamma d\alpha_2 3^{\#\alpha_2} \|\xi(\alpha_1 \cup \alpha_2 \cup \beta_1)\|^2} \\
&= e^{\int_0^t \sqrt{2}k(s)ds} e^{\frac{3}{2}\int_0^t |m(s)|^2 ds} \int_{\Gamma_t} d\beta |\pi_{\sqrt{2}l}(\beta)| \sqrt{\int_\Gamma d\alpha 3^{\#\alpha} \sum_{\substack{\alpha_1 \subset \alpha \\ \alpha_1 \subset [0,t[}} |\pi_{n^2}(\alpha_1)| \|\xi(\alpha \cup \beta)\|^2} \\
&\leqslant e^{\int_0^t \sqrt{2}k(s)ds} e^{\frac{3}{2}\int_0^t |m(s)|^2 ds} e^{\int_0^t |l(s)|^2 ds} \sqrt{\int_\Gamma d\beta \int_\Gamma d\alpha 3^{\#\alpha} \pi_{1+n_t^2}(\alpha) \|\xi(\alpha \cup \beta)\|^2} \\
&\leqslant e^{\sqrt{2}\|k_t\|_1 + \|l_t\|_2^2 + \frac{3}{2}\|m_t\|_2^2} \sqrt{\int_\Gamma d\alpha \pi_{4+3*n_t^2}(\alpha) \|\xi(\alpha)\|^2}
\end{aligned}$$

$$= e^{\sqrt{2}\|k_t\|_1+\|l_t\|_2^2+\frac{3}{2}\|m_t\|_2^2}\| \left(\sqrt{1+3c_t}\right)^N \xi\|,$$

as required. □

3.2 Closedness and core of Wiener integral

Using the findings from the previous Section, we can now prove the following theorem:

Theorem 3.2.1. *$\mathcal{Q}_t(x)$ has the following properties:*

1. *$\mathcal{Q}_t(x)$ is a closed operator.*
2. *If x satisfies the $L^{1,2,2,\infty}$ condition locally, then*

$$\mathrm{Exp} \cup \mathrm{Num} \subset \mathrm{Dom}(\mathcal{Q}_t(x))$$

   for all $t \geqslant 0$.

3. *If x satisfies the $L^{1,2,2,2}$ condition locally, then* $\mathrm{Geom} \subset \mathrm{Dom}(\mathcal{Q}_t(x))$.
4. *If x satisfies the $L^{1,2,2,\infty}$ condition locally and $\xi \in \mathrm{Num}$, then $\mathcal{Q}_t(x)\xi \in \mathrm{Num}$.*
5. *If $\xi \in \widehat{\mathbb{I}}^W_{\mathrm{loc}}(\mathsf{h},\mathsf{k})$ and $\widehat{\mathcal{W}}_t(\xi) \in Dom(\mathcal{Q}_t(x))$, then $x \star_0 \xi \in \widehat{\mathbb{I}}^W_{\mathrm{loc}}(\mathsf{h},\mathsf{k})$ and $\mathcal{Q}_t(x)\widehat{\mathcal{W}}_t(\xi) = \widehat{\mathcal{W}}_t(x \star_0 \xi)$.*

Proof. For the second part we need to check that if $\xi \in \mathrm{Exp}$, then $x \star_0 \xi \in \widehat{\mathbb{I}}^W_{\mathrm{loc}}(\mathsf{h},\mathsf{k})$, i.e. that $(x \star_0 \xi)' \in L^{2,1}_{\mathrm{loc}}$. Let $\xi = u\varepsilon(f), u \in \mathsf{h}, f \in L^2(\mathbb{R}_+;\mathsf{k})$. For $x \in L^{1,2,2,\infty}_{\mathrm{loc}}$ let k, l, m, n be the corresponding locally L^1, L^2, L^2, L^∞ functions. By Lemma 3.1.11:

$$\left(\int_{\Gamma_t} d\alpha \left(\int_{\Gamma_t} d\beta \|J^*_{\alpha;\alpha\cup\beta}(x \star_0 \xi)(\alpha \cup \beta)\|\right)^2\right)^{\frac{1}{2}}$$
$$\leqslant C(k,l,m,t)\| \left(\sqrt{4+3n_t}\right)^N u\varepsilon(f)\| = C(k,l,m,t)\|u\varepsilon((\sqrt{4+3n_t})f)\| < \infty,$$

as n_t is an essentially bounded function, so that $(\sqrt{4+3n_t})f \in L^2(\mathbb{R}_+;\mathsf{k})$.

The fact that Num $\subset$ Dom$(\mathcal{Q}_t(x))$ is an immediate consequence of Lemma 3.1.11.

The calculation for $\xi \in$ Geom is very similar, but rather than using Lemma 3.1.11 we twist it to flip the roles of ξ and w.

To prove that $\mathcal{Q}_t(x)\xi \in$ Num, we notice that the estimates in $a^N\mathcal{Q}_t(x)\xi$ will be exactly the estimates from 3.1.11, with m, n, ξ substituted with $am, an, a^N\xi$. Since $\xi \in$ Num, the result follows.

For the last part of the lemma, let $\alpha \subset [0,t[$ and calculate:

$$\begin{aligned}
\mathcal{Q}_t(x)\widehat{\mathcal{W}}_t(\xi)(\alpha) &= \widehat{\mathcal{W}}_t(x\mathbf{1}_{[0,t[} \star \widehat{\mathcal{W}}_t(\xi))(\alpha) \\
&= \int_{\Gamma_t} d\beta J^*_{\alpha;\alpha\cup\beta} \sum_{\delta_0\cup\delta_1=\alpha\cup\beta} x(\delta_0;\alpha\cup\beta)\left(\int_{\Gamma_t} d\gamma J^*_{\delta_1;\delta_1\cup\gamma}\xi(\delta_1\cup\gamma);\alpha\cup\beta\cup\gamma\right) \\
&= \int_{\Gamma_t} d\beta \int_{\Gamma_t} d\gamma \sum_{\delta_0\cup\delta_1=\alpha\cup\beta} J^*_{\alpha;\alpha\cup\beta\cup\gamma} x(\delta_0;\alpha\cup\beta\cup\gamma)\Delta(\delta_1;\alpha\cup\beta\cup\gamma)\xi(\delta_1\cup\gamma;\alpha\cup\beta\cup\gamma) \\
&= \int_{\Gamma_t} d\beta \int_{\Gamma_t} d\gamma \sum_{\substack{\delta_0\cup\delta_1=\alpha\cup\beta \\ \delta_1\setminus\delta_0\subset\alpha}} J^*_{\alpha;\alpha\cup\beta\cup\gamma} x(\delta_0;\alpha\cup\beta\cup\gamma)\Delta(\delta_1;\alpha\cup\beta\cup\gamma)\xi(\delta_1\cup\gamma;\alpha\cup\beta\cup\gamma).
\end{aligned}$$

On the other hand, we have:

$$\widehat{\mathcal{W}}(x \star_0 \xi) = \int_{\Gamma_t} d\beta \sum_{\delta_0\cup\delta_1=\alpha\cup\beta} J^*_{\alpha;\alpha\cup\beta} x(\delta_0;\alpha\cup\beta)\Delta(\delta_0\cap\delta_1;\alpha\cup\beta)\xi(\delta_1;\alpha\cup\beta).$$

We see that the two sums coincide by the integral-sum identity and substituting $\delta_1\cup\gamma$ for δ_1. This in particular implies that one sum is convergent if and only if the other is, thus proving the last part of the lemma.

□

Remark 3.2.2. A natural condition for 5. from the preceding theorem to be satisfied is for ξ to be in appropriately modified Num, i.e. for the following to hold:

$$\forall_{a>0} \int_{\Gamma} d\alpha a^{\#\alpha}\|\xi(\alpha)\|^2 < \infty.$$

Indeed, on one hand this condition easily implies time-Wiener integrability and on the other, it follows from the generalized Minkowski inequality (Equation 1.2 in the Introduction) that then, for an arbitrary $a > 0$,

$$\begin{aligned}\|\sqrt{a}^N \widehat{\mathcal{W}}_t(\xi)\| &= \sqrt{\int_\Gamma d\alpha a^{\#\alpha} \|\int_{\Gamma_t} d\beta J^*_{\alpha;\alpha\cup\beta}\xi(\alpha\cup\beta)\|^2} \\ &\leqslant \int_{\Gamma_t} d\beta \sqrt{\int_\Gamma d\alpha a^{\#\alpha}\|J^*_{\alpha;\alpha\cup\beta}\xi(\alpha\cup\beta)\|^2} \\ &\leqslant \int_{\Gamma_t} d\beta \sqrt{\int_\Gamma d\alpha a^{\#\alpha}\|\xi(\alpha\cup\beta)\|^2}.\end{aligned}$$

Now, let us denote $\xi_{a,\beta}(\alpha) := a^{\#\alpha}\|\xi(\alpha\cup\beta)\|^2$. We see that then

$$\|\sqrt{a}^N \widehat{\mathcal{W}}_t(\xi)\| \leqslant \int_{\Gamma_t} d\beta \sqrt{\xi_{a,\beta}(\alpha)}.$$

However,

$$\begin{aligned}\int_\Gamma d\beta \xi_{a,\beta}(\alpha) &= \int_\Gamma d\beta \int_\Gamma d\alpha a^{\#\alpha}\|\xi(\alpha\cup\beta)\|^2 \\ &= \int_\Gamma d\sigma (1+a)^{\#\sigma}\|\xi(\sigma)\|^2 = \|\sqrt{1+a}^N \xi\|^2 < \infty,\end{aligned}$$

which we can then use, along with Cauchy-Schwarz, to conclude that

$$\begin{aligned}\int_{\Gamma_t} d\beta \sqrt{\xi_{a,\beta}(\alpha)} &\leqslant e^{\frac{t}{2}} \int_\Gamma d\beta \xi_{a,\beta}(\alpha) \\ &\leqslant e^{\frac{t}{2}} \|\sqrt{1+a}^N \xi\|^2 < \infty.\end{aligned}$$

This shows $\widehat{\mathcal{W}}_t(\xi) \in \mathrm{Num}$, which implies $\widehat{\mathcal{W}}_t(\xi) \in \mathrm{Dom}(\mathcal{Q}_t(x))$.

There is another natural domain to consider, the domain of finite particle vectors:

$$\mathrm{Fin} = \{\xi \in \mathcal{F}^{\mathsf{k}} \colon \mathrm{supp}\xi \subset \Gamma^{\leqslant n} \text{ for some } n \in \mathbb{N}\}.$$

Let us introduce a notation for the vectors of at most n particles:

$$\mathrm{Fin}_n = \{\xi \in \mathrm{Fin} \colon \mathrm{supp}\xi \subset \Gamma^{\leqslant n}\}.$$

We prove the following:

Lemma 3.2.3. *Let x satisfy the $L^{1,2,2,\infty}$ condition locally and let $t \geqslant 0$. Then we have:*

(i) $\mathrm{Fin}_n \subset \mathrm{Dom}(\mathcal{Q}_t(x))$ *and* $\mathcal{Q}_t(x)|_{\mathrm{Fin}_n}$ *is a bounded operator for every* $n \in \mathbb{N}$*,*

(ii) $\mathrm{Fin} \subset \mathrm{Dom}(\mathcal{Q}_t(x))$*,*

Proof. (i) Let k, l, m, w be the L^1, L^2, L^2, L^∞ functions witnessing the $L^{1,2,2,\infty}$ property of x and let $\xi \in \mathrm{Fin}_n$ for a fixed $n \in \mathbb{N}$. Using 3.1.11 and the fact that $\xi \in \mathrm{Fin}_n$, we calculate:

$$\begin{aligned}\|\mathcal{Q}_t(x)\xi\| &\leqslant e^{\sqrt{2}\|k_t\|_1 + \|l_t\|_2^2 + \frac{3}{2}\|m_t\|_2^2} \|\sqrt{4 + 3w_t}^N \xi\| \\ &\leqslant e^{\sqrt{2}\|k_t\|_1 + \|l_t\|_2^2 + \frac{3}{2}\|m_t\|_2^2} \sqrt{4 + 3\|w_t\|}^n \|\xi\| = C(k, l, m, w, n, t)\|\xi\|,\end{aligned}$$

where $C(k, l, m, w, n, t)$ is a finite constant depending on these arguments. Thus in particular the required integral is finite, thus $\xi \in \mathrm{Dom}(\mathcal{Q}_t(x))$, and $\|\mathcal{Q}_t(x)\|$ is bounded.

(Alternatively, we can observe that this is a restriction of a closed operator to a complete space, thus necessarily bounded.)

(ii) Follows immediately from (i).

□

We would like to state some results about the cores and adjoints of our quantum Wiener integrals. For this we invoke the Lemmas 1.0.4, 1.0.5. They lead us to a family of results about cores:

Theorem 3.2.4. *Let x satisfy the $L^{1,2,2,\infty}$ condition locally and let $t \geqslant 0$. Then any subset $\mathcal{D}$ of* Num *which is dense in Fock space is a core for $\mathcal{Q}_t(x)$. In particular,* $\mathrm{Exp}, \mathrm{Fin}, \mathrm{Geom}, \mathrm{Num}$ *are cores. We can form a core sitting inside all of these, by fixing an admissible subset D of* k *and taking*

$$\mathcal{E}(\mathrm{Fin})_\infty = \{\xi \in \mathrm{Fin}\colon \exists_{f \in \mathbb{S}(D)} \forall_{\sigma \in \mathrm{supp}(\xi)} \xi(\sigma) = \varepsilon(f)\sigma\},$$

where $\mathbb{S}(D)$ denotes the family of D-valued step functions.

Proof. It is well known all of these sets are dense subsets of the Fock space, included in Num. By Lemma 3.2.1 we know that $\mathcal{Q}_t(x)\,\mathrm{Num} \subset \mathrm{Num} \subset \mathrm{Dom}(\mathcal{Q}_t(x)^*)$, thus in fact $\mathrm{Num} \subset \mathrm{Dom}(\mathcal{Q}_t(x)^*\mathcal{Q}_t(x))$. □

This easily leads to a theorem about the structure of adjoints of our multiple Wiener integrals:

Theorem 3.2.5. *Let x satisfy the $L^{1,2,2,\infty}$ condition locally and let $t \geqslant 0$. Then $\mathcal{Q}_t(x)^* = \mathcal{Q}_t(x^*)$.*

Proof. It is easily seen that $\mathcal{Q}_t(x^*) \subset \mathcal{Q}_t(x)^*$. Indeed, for $\xi \in \mathrm{Dom}\,\mathcal{Q}_t(x), \eta \in \mathrm{Dom}\,\mathcal{Q}_t(x^*)$ we have the following, where for simplicity $\alpha_{ijk\ldots} = \alpha_i \cup \alpha_j \cup \alpha_k \cup \ldots$ et caetera and $\alpha = \cup_i \alpha_i$:

$$\begin{aligned}
\langle \mathcal{Q}_t(x)\xi, \eta\rangle &= \int_\Gamma d\alpha \int_{\Gamma_t} d\beta \langle (x\mathbf{1}_{\Gamma_t} \star \xi)(\alpha \cup \beta), J_{\alpha;\alpha\cup\beta}\eta(\alpha)\rangle \\
&= \int_{\Gamma_t^3} d\alpha_0 d\alpha_1 d\alpha_2 \int_{\Gamma_{>t}} d\alpha_3 \int_{\Gamma_t^2} d\beta_0 d\beta_1 \\
&\qquad \langle x(\alpha_{01} \cup \beta_{01}; \alpha \cup \beta)\xi(\alpha_{123} \cup \beta_1; \alpha \cup \beta), J_{\alpha;\alpha\cup\beta}\eta(\alpha)\rangle \\
&= \int_{\Gamma_t^3} d\alpha_0 d\alpha_1 d\alpha_2 \int_{\Gamma_{>t}} d\alpha_3 \int_{\Gamma_t^2} d\beta_0 d\beta_1 \\
&\qquad \langle \xi(\alpha_{123} \cup \beta_1), J^*_{\alpha_{123}\cup\beta_1;\alpha\cup\beta} x^*(\alpha_{01}\beta_{01}; \alpha \cup \beta)\eta(\alpha; \alpha \cup \beta)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Gamma_t^3} d\gamma_0 d\gamma_1 d\gamma_2 \int_{\Gamma>t} d\gamma_3 \int_{\Gamma_t^2} d\delta_0 d\delta_1 \\
&\qquad \langle \xi(\gamma), J^*_{\gamma;\gamma\cup\delta} x^*(\gamma_{01} \cup \delta_{01}; \gamma \cup \delta)\eta(\gamma_{123} \cup \delta_1; \gamma \cup \delta)\rangle \\
&= \int_{\Gamma} d\gamma \int_{\Gamma_t} d\delta \langle \xi(\gamma), J^*_{\gamma;\gamma\cup\delta}(x^* \star \eta)(\gamma \cup \delta)\rangle \\
&= \langle \xi, \mathcal{Q}_t(x^*)\eta\rangle.
\end{aligned}
$$

To see that in fact we have the reverse inclusion as well, let $\eta \in \mathrm{Dom}(\mathcal{Q}_t(x)^*)$ and let η_n be its projection onto the n particle space. Then $\eta_n \to \eta$ and $\eta_n \in \mathrm{Dom}(\mathcal{Q}_t(x^*))$; what's more, it is easily seen that $\mathcal{Q}_t(x^*)\eta_n \to \mathcal{Q}_t(x)^*\eta$ weakly. Indeed, for $\xi \in \mathrm{Dom}(\mathcal{Q}_t(x))$,

$$
\begin{aligned}
\langle \xi, \mathcal{Q}_t(x)^*\eta\rangle &= \langle \mathcal{Q}_t(x)\xi, \eta\rangle \\
&= \lim_{n\to\infty} \langle \mathcal{Q}_t(x)\xi, \eta_n\rangle \\
&= \lim_{n\to\infty} \langle \xi, \mathcal{Q}_t(x^*)\eta_n\rangle.
\end{aligned}
$$

As $\mathrm{Dom}(\mathcal{Q}_t(x))$ is dense in Fock space, this implies the weak convergence. By Banach-Saks theorem, we can extract a subsequence $(n_k)_{k\in\mathbb{N}}$ such that in fact

$$
\mathcal{Q}_t(x^*)\frac{1}{n_k}\sum_{n=1}^{n_k} \zeta_n \to \mathcal{Q}_t(x)^*\eta
$$

as $k \to \infty$ strongly. But, since $\sum_{n=1}^{n_k} \frac{1}{n_k}\zeta_n \to \zeta$, closedness of $\mathcal{Q}_t(x^*)$ yields that in fact $\eta \in \mathrm{Dom}(\mathcal{Q}_t(x^*)$ and $\mathcal{Q}_t(x)^*\eta = \mathcal{Q}_t(x^*)\eta$. This ends the proof. □

3.3 Fundamental formulas and examples

The First Fundamental Formula of quantum stochastic calculus is well-known. We adapt it to our setting as follows:

Proposition 3.3.1. *If* $x \in OK(B(\mathsf{h}), B(\widehat{\mathsf{k}})), u, v \in \mathsf{h}, f, g \in L^2(\mathbb{R}_+, \mathsf{k}), t \geqslant 0,$ *then*

$$\langle u\varepsilon(f), \mathcal{Q}_t(x)(v\varepsilon(g))\rangle = e^{\langle f,g\rangle}\langle u\varepsilon(\widehat{f}\boldsymbol{1}_{[0,t[}), x(v\varepsilon(\widehat{g}\boldsymbol{1}_{[0,t[}))\rangle.$$

Proof.

$$\begin{aligned}\langle u\varepsilon(f), \mathcal{Q}_t(x)(v\varepsilon(g))\rangle &= e^{\langle f_{[t,\infty[}, g_{[t,\infty[}\rangle} \int_{\Gamma_t} d\alpha \int_{\Gamma_t} \mathrm{d}\beta \sum_{\beta_0 \sqcup \beta_1 \sqcup \beta_2 = \alpha \cup \beta} \\ &\quad \langle u\varepsilon(f)(\alpha), J^*_{\alpha;\alpha\cup\beta} x(\beta_0 \cup \beta_1; \alpha \cup \beta) v\varepsilon(g)(\beta_1 \cup \beta_2; \alpha \cup \beta)\rangle \\ &= e^{\langle f_{[t,\infty[}, g_{[t,\infty[}\rangle} \int_{\Gamma_t} \mathrm{d}\alpha \int_{\Gamma_t} \mathrm{d}\beta \sum_{\beta_0 \sqcup \beta_1 \sqcup \beta_2 = \alpha \cup \beta} \\ &\quad \langle u\varepsilon(f)(\alpha; \alpha \cup \beta), x(\beta_0 \cup \beta_1; \alpha \cup \beta) v\varepsilon(g)(\beta_1 \cup \beta_2; \alpha \cup \beta)\rangle \\ &= e^{\langle f,g\rangle} \int_{\Gamma_t} \mathrm{d}\alpha \sum_{\beta_0, \beta_1 \subset \alpha} \langle u\varepsilon(f)(\beta_1; \alpha), x(\alpha) v\varepsilon(g)(\beta_0; \alpha)\rangle \\ &= e^{\langle f,g\rangle} \int_{\Gamma_t} \mathrm{d}\alpha \langle u\varepsilon(\widehat{f})(\alpha), x(\alpha) v\varepsilon(\widehat{g})(\alpha)\rangle \\ &= e^{\langle f,g\rangle} \langle u\varepsilon(\widehat{f}\mathbf{1}_{[0,t[}), x(v\varepsilon(\widehat{g}\mathbf{1}_{[0,t[}))\rangle.\end{aligned}$$

□

We can take quantum integrals - in the sense of Hudson and Parthasarathy - of operator kernels ampliated to the Fock space (or rather, of their values on the first chaos). By iteration we obtain multiple quantum stochastic integrals of operator kernels. It is natural to ask whether the operator obtained in this way coincides with our quantum Wiener integral. The next corollary answers this question.

Corollary 3.3.2. $x(\cdot) \otimes I_{\mathcal{F}^{\mathsf{k}}}$ *is a locally integrable quantum stochastic process. In particular, for* $\xi \in \mathrm{Exp}$ *we have* $\mathcal{Q}_t(x)\xi(\sigma) = \sum_{n \geqslant 0} \Lambda_t^n (x(\cdot) \otimes I_{\mathcal{F}_k})\xi(\sigma)$. *Thus also the following equations hold:*

$$\mathcal{Q}_t(x\boldsymbol{1}_{\Gamma_t^{(n)}}) = \Lambda_t^n (x(\sigma) \otimes I_{\mathcal{F}_k}),$$

$$\mathcal{Q}_t(x\boldsymbol{1}_{\Gamma_t^{(n)}}) = \Lambda_t(Q.(\widetilde{x\boldsymbol{1}_{[\Gamma_t^{(n-1)}}})),$$

where $\tilde{y}(s)(\sigma) = y(\sigma \cup \{s\})\boldsymbol{1}_{\sigma\subset[0,s]}$.

Proof. Immediate by comparing the formula from above with the first fundamental formula of quantum stochastic calculus (cf. [48]). □

Theorem 3.3.3 (Second fundamental formula)**.** *Let* $x, y \in L^{1,2,2,\infty}$ *and* $u, v \in \mathsf{h}, f, g \in L^2(\mathbb{R}_+; \mathsf{k}), t \geqslant 0$. *Then we have:*

$$\langle \mathcal{Q}_t(x)u \otimes \varepsilon(f), \mathcal{Q}_t(y)v \otimes \varepsilon(g)\rangle = \langle u \otimes \varepsilon(f), \mathcal{Q}_t((x^*) \star y)v \otimes \varepsilon(g)\rangle.$$

Proof. Let us first notice that if $x \in L^{1,2,2,\infty}$, then so does x^*. Moreover, if $y \in L^{1,2,2,\infty}$ and $\xi \in \mathrm{Exp}$, then $(\mathcal{Q}_t(y)\xi)' \in L^{2,1}$. Thus in particular $\mathcal{Q}_t(y)\xi \in \mathrm{Dom}(\mathcal{Q}_t(x^*))$. Therefore we can interpret the above theorem as:

$$\mathcal{Q}_t(x)\mathcal{Q}_t(y) = \mathcal{Q}_t(x \star y) \qquad \text{on Exp}.$$

It is, however, a direct corollary from royal associativity (Theorem 2.1.13) and Lemma 3.2.1. □

Remark 3.3.4. There is a simple algebraic way to convince oneself of the truth of the formula $\mathcal{Q}_t(x)\mathcal{Q}_t(y)\xi = \mathcal{Q}_t(x \star y)\xi$ (assuming everything is in the relevant domains). Namely, let η be an arbitrary vector and let us denote

$$z_\gamma(\alpha) = J^*_{\alpha;\alpha\cup\gamma}(y \star \xi)(\alpha \cup \gamma)$$

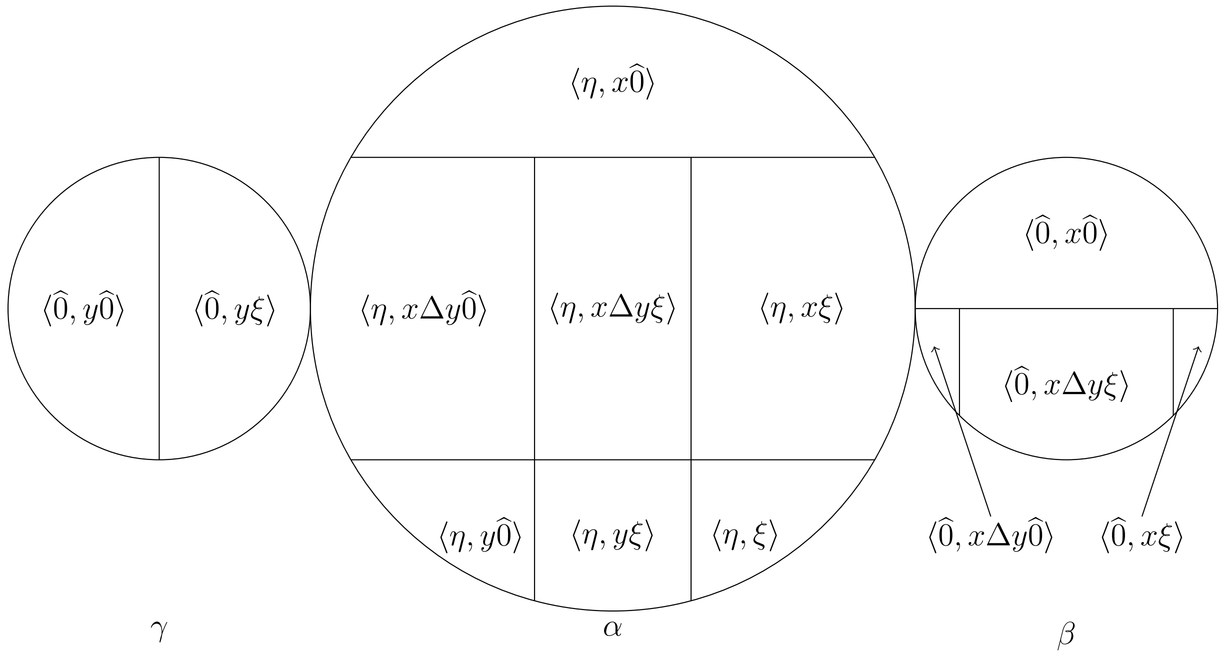
for $\alpha, \gamma \in \Gamma$. Consider the following formula:

$$\langle \eta, \mathcal{Q}_t(x)\mathcal{Q}_t(y)\xi\rangle = \int_{\Gamma_t} d\alpha \int_{\Gamma_t} d\beta \int_{\Gamma_t} d\gamma \langle \eta(\alpha), J^*_{\alpha;\alpha\cup\beta}(x \star z_\gamma)(\alpha \cup \beta)\rangle$$

$$= \int_{\Gamma_t} d\alpha \int_{\Gamma_t} d\beta \int_{\Gamma_t} d\gamma \sum_{\beta_0 \cup \beta_1 = \alpha \cup \beta} \sum_{\gamma_0 \cup \gamma_1 = \beta_1 \cup \gamma}$$

$$\langle \eta(\alpha), J^*_{\alpha;\alpha\cup\beta} x(\beta_0; \alpha \cup \beta)\Delta(\beta_0 \cap \beta_1; \alpha \cup \beta) J^*_{\beta_1;\beta_1\cup\gamma} y(\gamma_0; \beta_1 \cup \gamma)\xi(\gamma_1; \beta_1 \cup \gamma)\rangle.$$

Via integral sum identity, we can see that this formula is a sum over various partitions of $\alpha \cup \beta$. For each fixed partition, we can see what is happening on each tensor coordinate via the following diagram:

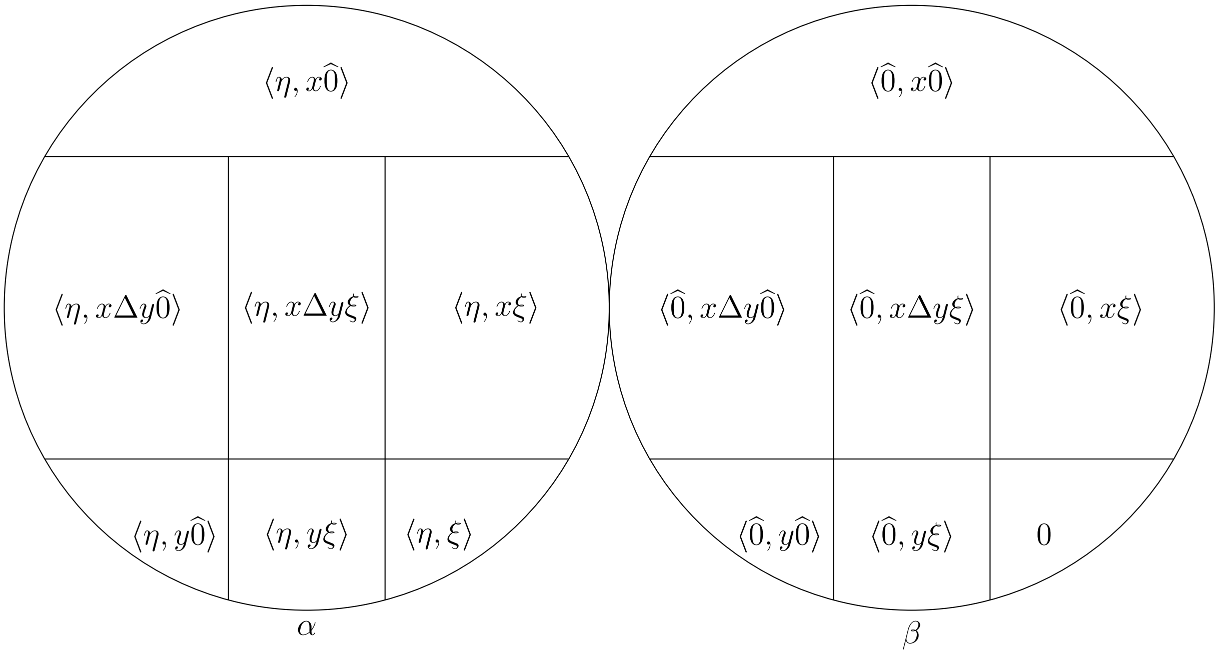


In an analogous manner, we can write out the right hand side of our equation:

$$\langle \eta, \mathcal{Q}_t(x \star y)\xi\rangle = \int_{\Gamma_t} d\alpha \int_{\Gamma_t} d\beta \sum_{\beta_0 \cup \beta_1 = \alpha \cup \beta} \sum_{\gamma_0 \cup \gamma_1 = \beta_1}$$

$$\langle \eta(\alpha), J^*_{\alpha;\alpha\cup\beta} x(\gamma_0; \alpha \cup \beta)\Delta(\gamma_0 \cap \gamma_1; \alpha \cup \beta) y(\gamma_1; \alpha \cup \beta)\xi(\beta_1; \alpha \cup \beta)\rangle.$$

The corresponding diagram looks as follows:



It is now clear from inspection that in fact the sums over all partitions yield the same result - in one case splitting the overarching set into three sets we integrate over (α, β, γ) and in the other into two (α, β).

Below we list some estimates for our multiple quantum stochastic integrals:

Proposition 3.3.5. *1. If* $x(\sigma) = |\xi(\sigma)\rangle\langle\widehat{0}|$ *for* $\xi \in \mathrm{Dom}(\sqrt{2}^N)$*, then* $\mathrm{Dom}(\sqrt{2}^N) \subset \mathrm{Dom}(\mathcal{Q}(x))$ *and for* $\eta \in \mathrm{Dom}(\sqrt{2}^N)$

$$\|\mathcal{Q}(x)\eta\| \leqslant \|\sqrt{2}^N\xi\|\|\sqrt{2}^N\eta\|.$$

2. If $x(\sigma) = |\widehat{0}\rangle\langle\xi(\sigma)|$ *for* $\xi \in \mathcal{F}^{\mathsf{k}}$*, then* $\mathrm{Dom}(\sqrt{2}^N) \subset \mathrm{Dom}(\mathcal{Q}(x))$ *and for* $\eta \in \mathrm{Dom}(\sqrt{2}^N)$

$$\|\mathcal{Q}(x)\eta\| \leqslant \|\xi\|\|\sqrt{2}^N\eta\|.$$

3. *By combining the previous two cases, when* $x(\sigma) = |\xi(\sigma)\rangle\langle\widehat{0}| + |\widehat{0}\rangle\langle\xi(\sigma)|$ *for* $\xi \in \mathrm{Dom}((1+\sqrt{2}^N))$*, then* $\mathrm{Dom}(\sqrt{2}^N) \subset \mathrm{Dom}(\mathcal{Q}(x))$ *and for* $\eta \in \mathrm{Dom}(\sqrt{2}^N)$

$$\|\mathcal{Q}(x)\eta\| \leqslant \|(1+\sqrt{2}^N)\xi\|\|\sqrt{2}^N\eta\|.$$

4. *If* $x(\sigma) = \Delta W(\sigma)\Delta$ *with* $W \in L^\infty_{\mathrm{loc}}(\Gamma; B(\mathsf{h}\otimes\mathsf{k}))$ $f \in L^1_{\mathrm{loc}}(\Gamma; B(\mathsf{h}))$*, then* $\mathcal{Q}_{s,t}(x)$ *is a bounded operator and*

$$\|\mathcal{Q}_{s,t}(x)(k)\| = \|W\|_\infty\|k\|.$$

5. *If* $x(\sigma) = \begin{pmatrix} f(\sigma) & 0 \\ 0 & 0 \end{pmatrix}$ *with* $f \in L^1_{\mathrm{loc}}(\Gamma; B(\mathsf{h}))$*, then* $\mathcal{Q}_{s,t}(x)$ *is a bounded operator and*

$$\|\mathcal{Q}_{s,t}(x)(k)\| = \|f\|_1\|k\|.$$

6. *If* $x(\sigma) = 0$ *for* $\#\sigma \geqslant N$*, then for any finite particle* $k \in Dom(\mathcal{Q}_{s,t}(x))$

$$\mathcal{Q}_{s,t}(x)k \in \mathrm{Fin}\,.$$

In other words, if x *is a finitely supported kernel, then the domain of finite particles is stable under quantum Wiener integration.*

7. *In fact, more can be said. If* $x(\sigma)|\widehat{0}(\sigma)\rangle = 0$ *for* $\sigma \geqslant N$*, then the same conclusion holds:*

$$\mathcal{Q}_{s,t}(x)\,\mathrm{Fin} \subset \mathrm{Fin}\,.$$

In other words, x *merely needs to be finitely supported "in its time and creation part".*

Remark 3.3.6. If our kernel is a sum of the kernels from the first four cases of the above lemma, then it in particular satisfies the local $L^{1,2,2,\infty}$ condition.

Example 3.3.7. It is useful to give an example of an operator which is NOT a quantum Wiener integral. For that, consider the shift operator.

Proposition 3.3.8. *Let $t \geqslant 0$ and T be the operator $T = T_t \otimes I_{[t}$ on Guichardet space, where $I_{[t}$ is the identity on $\Gamma_{[t}$ and T_t is given by*

$$T_t\xi(\sigma) = \xi(\sigma + 1),$$

where $\sigma + 1 = \{s + 1 \colon s \in \sigma\}$, for $\sigma \in \Gamma_t$. Then T is not of the form $Q_t(x)$ for any operator kernel x.

Proof. For ease of calculations, let us assume $t \geqslant 3$.

Suppose x is an operator kernel such that $T_t = Q_t(x)$ (we ignore the identity part, as by assumption all the operators are identity adapted). Consider the following functions $f_i \in L^2(\mathbb{R}_+)$:

$$f_0 = \mathbf{1}_{[0,1[},$$

$$f_1 = \mathbf{1}_{[1,2[},$$

$$f_2 = \mathbf{1}_{[2,3[}.$$

We have the following:

$$\langle \varepsilon(f_0), T_t\varepsilon(f_1)\rangle = e,$$

$$\langle \varepsilon(f_0), T_t\varepsilon(f_2)\rangle = 0.$$

On the other hand, if $T_t = Q_t(x)$, by the first fundamental formula we get:

$$\langle \varepsilon(f_0), T_t\varepsilon(f_1)\rangle = e\int_{\Gamma_t} d\sigma \langle \varepsilon(\widehat{f_0})(\sigma), x(\sigma)\varepsilon(\widehat{f_1})(\sigma)\rangle = e\int_{\Gamma_{[0,1[}} d\sigma \langle \varepsilon(\widehat{1})(\sigma), x(\sigma)\varepsilon(\widehat{0})(\sigma)\rangle,$$

so $\int_{\Gamma_{[0,1[}} d\sigma \langle \varepsilon(\widehat{1})(\sigma), x(\sigma)\varepsilon(\widehat{0})(\sigma)\rangle = 1$; on the other hand,

$$\langle \varepsilon(f_0), T_t \varepsilon(f_2)\rangle = e \int_{\Gamma_t} d\sigma \langle \varepsilon(\widehat{f_0})(\sigma), x(\sigma)\varepsilon(\widehat{f_2})(\sigma)\rangle = e \int_{\Gamma_{[0,1[}} d\sigma \langle \varepsilon(\widehat{1})(\sigma), x(\sigma)\varepsilon(\widehat{0})(\sigma)\rangle,$$

implying $\int_{\Gamma_{[0,1[}} d\sigma \langle \varepsilon(\widehat{1})(\sigma), x(\sigma)\varepsilon(\widehat{0})(\sigma)\rangle = 0$. This contradiction proves that T_t is not of the form $Q_t(x)$ for any x. □

3.4 Hilbert–Schmidt operator representation

We close this chapter with a theorem saying that every Hilbert–Schmidt operator is, in fact, a multiple quantum Wiener integral. For this we need a classical lemma about the representation of Hilbert–Schmidt operators.

Lemma 3.4.1. *Let T be a Hilbert–Schmidt operator on the space $L^2(\mathit{\Gamma}_t; H)$ for some separable Hilbert space H and $t > 0$. Then there exists a function $k \in L^2(\Gamma_t \times \Gamma_t; HS(H))$ such that*

$$(Tf)(\sigma) = \int_{\Gamma_t} d\beta k(\sigma, \beta) f(\beta).$$

Proof. As H is separable and the measure on $\mathit{\Gamma}$ is σ-finite and countably generated, $L^2(\mathit{\Gamma}_t; H)$ is separable. Let us fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\mathit{\Gamma}_t; H)$. By Parseval's identity, we have:

$$(Tf)(\sigma) = \sum_{n\in\mathbb{N}} \langle e_n, f\rangle (Te_n)(\sigma) = \sum_{n\in\mathbb{N}} \int_\Gamma d\beta \langle e_n(\beta)| f(\beta)(Te_n)(\sigma).$$

Now, by the fact that T is Hilbert–Schmidt, we know that in fact

$$\sum_{n\in\mathbb{N}} \|Te_n\|^2 < \infty,$$

so that the series $\sum_{n\in\mathbb{N}} Te_n$ converges in $L^2(\Gamma_t; H)$. Thus in particular, for almost all $\sigma, \beta \in \Gamma_t$ the series

$$k(\sigma,\beta) := \sum_{n\in\mathbb{N}} |Te_n(\sigma)\rangle\langle e_n(\beta)$$

is convergent; what's more, thus defined function k is measurable and square-integrable by Fubini's theorem. We see it is Hilbert–Schmidt valued via the isomorphism

$$HS(H) = H \otimes \overline{H}$$

and the fact that the series in question is square summable by the fact that T is Hilbert–Schmidt. This ends the proof. □

Remark 3.4.2.

1. This lemma is usually stated for $L^2(X,\mu)$ spaces. The proof however holds with no significant changes upon introducing a separable Hilbert space as the space of values.

2. In symbols, the isometric isomorphism invoked by the lemma is well known:

$$\begin{aligned} HS(L^2(\Gamma;H)) &= L^2(\Gamma;H) \otimes \overline{L^2(\Gamma;H)} = L^2(\Gamma)\otimes H \otimes \overline{L^2(\Gamma)} \otimes \overline{H} \\ &= L^2(\Gamma\times\Gamma)\otimes H\otimes\overline{H} = L^2(\Gamma\times\Gamma)\otimes HS(H) \\ &= L^2(\Gamma\times\Gamma; HS(H)), \end{aligned}$$

   but in the sequel we will need the explicit form of the function k.

3. Let T be an operator on $\mathsf{h}\otimes\mathcal{F}^{\mathsf{k}} = \{f \in L^2(\Gamma_t; \mathsf{h}\otimes\Phi(\mathsf{k})\colon f(\sigma) \in \mathsf{h}\otimes\mathsf{k}^{\otimes\#\sigma}\}$. Then we see that in fact

$$k(\alpha;\beta)\colon \mathsf{h}\otimes\mathsf{k}^{\otimes\#\beta} \to \mathsf{h}\otimes\mathsf{k}^{\otimes\#\alpha}.$$

Theorem 3.4.3. *Let T be an operator on $\mathsf{h}\otimes\mathcal{F}^{\mathsf{k}}$ such that $T = T_t \otimes I_{\mathcal{F}^{\mathsf{k}}_{[t,\infty[}}$ for some Hilbert–Schmidt operator T_t on $\mathsf{h}\otimes\mathcal{F}^{\mathsf{k}}_t$. Then $T = \mathcal{Q}_t(x)$ for an operator kernel x.*

Proof. We know that $\mathcal{Q}_t(x)$ is identity adapted, so the only thing to prove is that the equality holds for vectors f supported on Γ_t.

Let k be the L^2 function representing T_t as an integral operator. We introduce the placement notation $\mathsf{k}^{\alpha;\sigma} = J_{\alpha;\sigma}\mathsf{k}^{\#\alpha} \subset \widehat{\mathsf{k}}^{\otimes\#\sigma}$. We define x as:

$$x(\sigma) = \sum_{\alpha\cup\beta=\sigma} ([k(\alpha\setminus\beta, \beta\setminus\alpha); \beta]\pi_{-I}(\alpha\cap\beta;\beta)\mathbf{1}_{\mathsf{k}^{\beta;\sigma}}; \sigma).$$

This should be understood as an operator with domain $\mathsf{k}^{\otimes\#\sigma}$, which acts as function k on the coordinates corresponding to $\beta\setminus\alpha$ and as $-I$ on $\alpha\cap\beta$, with the final result extended by $\binom{1}{0}$ to land in $\mathsf{k}^{\otimes\#\sigma}$. Let us notice that the requirement that $\alpha\cup\beta=\sigma$ guarantees no ambiguity in the coordinate placement. Let us notice that

$$J^*_{\alpha;\sigma}x(\sigma)J_{\beta;\sigma} = [k(\alpha\setminus\beta,\beta\setminus\alpha);\beta]\pi_I(\alpha\cap\beta;\beta)\mathbf{1}_{\alpha\cup\beta=\sigma},$$

so that by square-integrability of k x satisfies the $L^{1,2,2,\infty}$ condition (the non-product $L^{1,2,2,\infty}$ condition, however!), so $\mathcal{Q}_t(x)$ is a densely defined operator. Also, we use the fact that

$$\sum_{\alpha\subset\sigma}\pi_{-I}(\alpha;\sigma) = \sum_{\alpha\subset\sigma}(-1)^{\#\alpha}I = \mathbf{1}_{\sigma=\emptyset}I. \tag{3.2}$$

We calculate, for $f\in\mathsf{h}\otimes\mathcal{F}^{\mathsf{k}}_t$:

$$\begin{aligned}\mathcal{Q}_t(x)f(\alpha) &= \int_{\Gamma_t} J^*_{\alpha;\alpha\cup\beta}(x\star f)(\alpha\cup\beta)d\beta = \int_{\Gamma_t} d\beta \sum_{\beta_0\cup\beta_1=\alpha\cup\beta} J^*_{\alpha;\alpha\cup\beta}x(\beta_0;\alpha\cup\beta)J_{\beta_1;\alpha\cup\beta}f(\beta_1)\\ &= \int_{\Gamma_t} d\beta \sum_{\beta_0\cup\beta_1=\alpha\cup\beta} (k(\beta_0\cap\alpha\setminus\beta_1, \beta_0\cap\beta_1\setminus\alpha);\beta_1)\\ &\cdot \pi_{-I}(\alpha\cap\beta_1\cap\beta_0;\beta_1)f(\beta_1)\end{aligned}$$

We can perform some simplifications now. We see from Remark 3.1.9 that in fact $\beta_1\setminus\beta_0\cap\beta=\emptyset$. It is also easy to see from the definition of our x that also $(\beta_0\setminus\beta_1)\cap\beta=\emptyset$.

Thus in fact β is not partitioned at all. On the other hand, we can write:

$$\sum_{\alpha_0 \cup \alpha_1 = \alpha} (k(\alpha_0 \setminus \alpha_1, \beta); \alpha_1 \cup \beta)\pi_{-I}(\alpha_0 \cap \alpha_1; \alpha_1 \cup \beta) f(\alpha_1 \cup \beta)$$
$$= \sum_{\alpha_0 \subset \alpha} \sum_{\alpha_1 \subset \alpha \setminus \alpha_0} (k(\alpha_0, \beta); \alpha \setminus \alpha_0 \cup \beta)\pi_{-I}(\alpha_1; \alpha \setminus \alpha_0 \cup \beta) f(\alpha \setminus \alpha_0 \cup \beta)$$
$$= \sum_{\alpha_0 \subset \alpha} (k(\alpha_0, \beta); \alpha \setminus \alpha_0 \cup \beta) \mathbf{1}_{\alpha \setminus \alpha_0 = \emptyset} f(\alpha \setminus \alpha_0 \cup \beta) = k(\alpha, \beta) f(\beta)$$

by Equation 3.2. Thus we get:

$$\mathcal{Q}_t(x) f(\alpha) = \int_{\Gamma_t} d\beta k(\alpha, \beta) f(\beta) = T_t f(\alpha),$$

as required. $\square$

Chapter 4

Applications of quantum Wiener integrals

4.1 Applications to quantum stochastic calculus

4.1.1 Dual processes

An important role in the sequel will be played by the time reversal process.

Definition 4.1.1. Let $t \geqslant 0$. We define the time reversal process R_t as the second quantisation of the operator $r_t\colon L^2(\mathbb{R}_+;\mathsf{k}) \to L^2(\mathbb{R}_+;\mathsf{k})$ given by

$$r_t f(s) = \begin{cases} f(t-s) & s \leqslant t \\ f(s) & s > t \end{cases}.$$

Thus

$$R_t \varepsilon(f) = \varepsilon(r_t f).$$

The operators will be applied in the same manner to $\widehat{f}$, via $r_t \widehat{f} = \widehat{r_t f}$.

For simplicity, for $\sigma \in \Gamma_t$ and $t \geqslant 0$ let

$$t - \sigma = \{t - s \colon s \in \sigma\}.$$

Lemma 4.1.2. *Let $F \colon \mathbb{R}_+ \to \mathcal{B}(\mathsf{h} \otimes \widehat{\mathsf{k}})$ satisfy the $L^{1,2,2,\infty}$ condition locally, $f, g \in L^2(\mathbb{R}_+; \mathsf{k}), u, v \in \mathsf{h}$. Then*

$$\langle u\varepsilon(f), R_t^* \mathcal{Q}_t(\pi_F) R_t v\varepsilon(g)\rangle = \langle u\varepsilon(f), \mathcal{Q}_t(\pi_G) v\varepsilon(g)\rangle,$$

where $G(s) = G(t - s)$. In particular, if F is constant, we have

$$R_t^* \mathcal{Q}_t(\pi_F) R_t = \mathcal{Q}_t({}_F\pi).$$

Proof.

$$\begin{aligned}
\langle u\varepsilon(f), R_t^* \mathcal{Q}_t(\pi_F) R_t v\varepsilon(g)\rangle &= e^{\langle f,g\rangle} \int_{\Gamma_t} d\sigma \langle u \otimes_{s\in\sigma} \widehat{f(t-s)}, \overrightarrow{\pi_F}(\sigma) v \otimes_{s\in\sigma} \widehat{g(t-s)}\rangle \\
&= e^{\langle f,g\rangle} \int_{\Gamma_t} d\sigma \langle u \otimes_{s\in\sigma} \widehat{f(s)}, \overrightarrow{\pi_G}(\sigma) v \otimes_{s\in\sigma} \widehat{g(s)}\rangle \\
&= \langle u\varepsilon(f), \mathcal{Q}_t(\pi_G) v\varepsilon(g)\rangle,
\end{aligned}$$

where we have performed the substitution $s \to t - s$ in the second equality and reversed the order of integration. It is also seen that since the placement of G is the reverse of placement of F that for constant F

$$\pi_G = {}_F\pi.$$

□

Definition 4.1.3. For a quantum stochastic process X_t, $X_t^\sharp := R_t^* X_t^* R_t$ will be called the dual process of X_t.

Our main reasons for studying dual processes are contained in the following lemma:

Lemma 4.1.4. *X_t is a (co)isometry if and only if the dual process $X_t^\dagger$ is a coisometry (isometry). If $X_t = \mathcal{Q}_t(\pi_F)$, then $X_t^\sharp = \mathcal{Q}_t(\pi_{F^*})$.*

Proof. Assume X_t is an isometry. Then X_t^* is a coisometry. As R_t is a unitary, this means that $X_t^\sharp$ is a coisometry. This ends the proof. The proof for X_t being a coisometry is analogous.

For the second part, we know that $X_t^* = \mathcal{Q}_t({}_{F^*}\pi)$. But we know that conjugating by the time reversal process reverses the order in the product operator kernel. That proves the claim. □

4.1.2 Quantum stochastic evolutions

In this section we shall focus on multiple quantum Wiener integrals of the form $\mathcal{Q}_t(\pi_F)$ for $F \in L^{1,2,2,\infty}_{\text{loc}}$. We are particularly interested in them as solutions to natural QSDEs.

Lemma 4.1.5. *Let $F\colon \mathbb{R}_+ \to B(\mathsf{h} \otimes \widehat{\mathsf{k}}), \pi_F \in L^{1,2,2,\infty}_{\text{loc}}$. Then $X_t = \mathcal{Q}_t(\pi_F)$ satisfies the QSDE:*

$$dX_t = X_s F_s d\Lambda_s. \tag{4.1}$$

Proof. We only need to check that

$$\langle u\varepsilon(f), (X_t - I)v\varepsilon(g)\rangle = \int_0^t ds \langle \hat{f}(s)u\varepsilon(f), (I_{\widehat{\mathsf{k}}} \otimes X_s)_\pi (F_s \otimes I_{\mathcal{F}})(\hat{g}(s)v\varepsilon(g))\rangle,$$

where $(I_{\widehat{\mathsf{k}}} \otimes X_s)_\pi$ denotes $(I_{\widehat{\mathsf{k}}} \otimes X_s)$ after the relevant tensor flip $\widehat{\mathsf{k}} \otimes \mathsf{h} \otimes \mathcal{F} \to \mathsf{h} \otimes \widehat{\mathsf{k}} \otimes \mathcal{F}$. We have:

$$\int_0^t ds \langle u\hat{f}(s)\varepsilon(f), (I_{\widehat{\mathsf{k}}} \otimes X_s)_\pi (F_s \otimes I_{\mathcal{F}})(v\hat{g}(s)\varepsilon(g))\rangle = e^{\langle f,g\rangle} \int_0^t ds \int_{\Gamma_s} d\sigma \langle u\pi_{\hat{f}}(\sigma)\hat{f}(s),$$
$$(I_{\widehat{\mathsf{k}}} \otimes \pi_F(\sigma))(F_s \otimes I_{\widehat{\mathsf{k}}\otimes\#\sigma})(v\pi_{\hat{g}}(\sigma)\hat{g}(s))\rangle$$
$$= e^{\langle f,g\rangle} \int_{\Gamma_t^{\geqslant 1}} \langle u\pi_{\hat{f}}(\sigma), \pi_F(\sigma)v\pi_{\hat{g}}(\sigma) = \langle u\varepsilon(f), (X_t - I)v\varepsilon(g)\rangle.$$

□

Definition 4.1.6. A quantum stochastic process $V = (V_{s,t})_{0\leqslant s\leqslant t}$ on $\mathsf{h} \otimes \mathcal{F}$ is called a quantum stochastic evolution if it satisfies the following:

1. $V_{r,s}V_{s,t} = V_{r,t}$ for $0 \leqslant r \leqslant s \leqslant t$;

2. V is bi-adapted in the sense that there exist operators $V_{(s,t)} \in B(\mathsf{h} \otimes \mathcal{F}_{[\int,\sqcup[})$ such that

$$V_{s,t} = \Pi^{-1} \circ (\mathrm{id}_{\mathcal{F}_{[0,s[}} \otimes V_{(s,t)} \otimes \mathrm{id}_{\mathcal{F}_{[t,\infty[}}) \circ \Pi,$$

   where $\Pi\colon \mathsf{h} \otimes \mathcal{F} \to \mathcal{F}_{[0,s[} \otimes \mathsf{h} \otimes \mathcal{F}_{[s,t[} \otimes \mathcal{F}_{[t,\infty[}$ is the tensor flip operation.

Remark 4.1.7. We notice that thus all processes satisfying QSDE 4.1 are in fact QS evolutions.

We now address the question of when is a contractive quantum stochastic evolution $(V_{s,t})_{0\leqslant s\leqslant t}$ a solution of a quantum stochastic differential equation. It is well known that it is the case when the functions $(s,t) \mapsto V_{s,t}$ are continuous in the hybrid norm-ultraweak topology (cf. *e.g.* [16]). Here we will present an alternative condition, which is weaker than Markov regularity, and prove that it is sufficient for a solution of the appropriate QSDE to exist. This condition is taken from [16], while some ideas for the proof of its sufficiency are inspired by [61].

Definition 4.1.8. A quantum stochastic evolution $(V_{s,t})_{0\leqslant s\leqslant t}$ on initial space h and noise dimension space k is called *elementary* if for each $\xi, \eta \in \widehat{\mathsf{k}}$ there exists a function $\varphi)\xi, \eta \in L^1(\mathbb{R}_+; B(\mathsf{h}))$ with the following property:

For every $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ there is $\varphi_{f,g} \in L^1(\mathbb{R}_+; B(\mathsf{h}))$ such that

$$E^{\varepsilon(f_{[s,t[})} V_{(s,t)} E_{\varepsilon(g_{s,t})} = \int_{\Gamma_{[s,t[}} \pi_{phi_{f(\cdot),g(\cdot)}}(\sigma) d\sigma.$$

If the function $\widehat{\mathsf{k}} \ni \xi \mapsto \varphi_{\xi,\eta}(s)(v)$ is strongly continuous for all $\eta \in \widehat{\mathsf{k}}, s \geqslant 0, v \in \mathsf{h}$, then we call the evolution strongly elementary.

Remark 4.1.9. It is easily noticed that the function $\widehat{\mathsf{k}} \ni \xi \mapsto \varphi_{\xi,\eta}(s)(v)$ is always weakly continuous.

Proposition 4.1.10. *Let $V = (V_{s,t})_{0\leqslant s\leqslant t}$ be a strongly measurable contractive quantum stochastic process on $\mathsf{h} \otimes \mathcal{F}$. Then the following are equivalent:*

(i) V is a strongly elementary quantum stochastic evolution;

(ii) $V_{s,t} = \mathcal{Q}_{s,t}(\pi_F)$ for $F\colon \mathbb{R}_+ \to B(\mathsf{h} \otimes \widehat{\mathsf{k}}), \pi_F$ quantum Wiener integrable;

(iii) V strongly satisfies the QSDE:

$$dV_t = V_t F_t d\Lambda_t$$

for some $F\colon \mathbb{R}_+ \to B(\mathsf{h} \otimes \widehat{\mathsf{k}})$ such that π_F is quantum Wiener integrable.

Proof. The implications $(ii) \Rightarrow (iii), (iii) \Rightarrow (i)$ are obvious, with $\varphi_{\xi,\eta}(s) = E^{\xi} F_s E_{\eta}$. Thus we only need to prove $(i) \Rightarrow (ii)$.

Now let us fix an orthonormal basis $T = \{f_\alpha \colon \alpha \in I\}, 0 \notin I$ of k. We will denote its embedding in $\widehat{\mathsf{k}}$ by $\tilde{T}$. For each of its finite subsets T_0 we define

$$F^{T_0} = \sum_{\alpha,\beta} E_{e_\alpha} \varphi_{e_\alpha, e_\beta} E^{e_\beta} \colon \mathbb{R}_+ \to B(\mathsf{h}),$$

where $e_\alpha = \binom{0}{f_a} \in \tilde{T_0}$ and $e_0 = \binom{1}{0}$. This is a well-defined bounded operator on $\mathsf{h} \otimes \mathrm{Lin}(\tilde{T_0} \cup \{e_0\})$ (as all the sums in question are finite).

We see that now for each $\xi, \eta \in \mathrm{Lin}(\tilde{T_0} \cup \{e_0\}), v \in \mathsf{h}$ and $s > 0$ we have $E^\xi F^{T_0}(s) v \eta = \varphi(\xi, \eta)(s) v$. If we take

$$J_{T_0} \colon \mathsf{h} \otimes \Gamma(L^2(\mathbb{R}_+; \mathrm{Lin}\, T_0)) \to \mathsf{h} \otimes \mathcal{F}$$

to be the second quantisation of the inclusion mapping $\mathrm{Lin}\, T_0 \hookrightarrow \mathsf{k}$ and define

$$V^{T_0} = ((J_{T_0})^* V J_{T_0})$$

then V^{T_0} is a contraction process which satisfies:

$$\langle u\varepsilon(f), (V^{T_0}_{s,t} - I) v\varepsilon(g) \rangle = \int_s^t \langle \widehat{f(r)} u\varepsilon(f), (I_{\widehat{\mathsf{k}}} \otimes V^{T_0}_{s,r} (F^{T_0}_r \otimes I_{\mathcal{F}_{T_0}}) (\widehat{g(r)} v\varepsilon(g) \rangle,$$

where $\mathcal{F}_{T_0}$ denotes the Fock space over $L^2(\mathbb{R}_+; \mathrm{Lin}\, T_0)$.

As V^{T_0} is contractive and strongly measurable, this in fact means that for each $u \in \mathsf{h}, f \in L^2(\mathbb{R}_+; \mathrm{Lin}\, T_0)$ we have

$$V^{T_0}_{s,t} u\varepsilon(f) - u\varepsilon(f) = \int_s^t (I_{\widehat{\mathsf{k}}} \otimes V^{T_0}_{s,r} (F^{T_0}_r \otimes I_{\mathcal{F}_{T_0}}) (\widehat{f(r)} u\varepsilon(f)).$$

This in turn implies that for almost all $s \in \mathbb{R}_+$

$$\|\Delta F_s^{T_0} v\eta\|^2 \leqslant -2\Re\langle v\eta, F_s^{T_0} v\eta\rangle = -2\Re\langle v, \varphi_{\eta,\eta}(s)v\rangle.$$

Let us now consider an orthonormal basis $\{e_0\} \cup \tilde{T}$. For each finite subset T_0 of it we have

$$\sum_{e_\alpha \in T_0} \|\varphi_{e_\alpha,\eta}(s)v\|^2 = \|\sum_{e_\alpha \in T_0} E_{e_\alpha} E^{e_\alpha} F_s^{T_0} v\eta\|^2 \leqslant \|\Delta F_s^{T_0} v\eta\|^2 \leqslant -2\Re\langle v, \varphi_{\eta,\eta}(s)v\rangle.$$

This means that the sum

$$\sum_{e_\alpha \in \{e_0\} \cup \tilde{T}} E_{e_\alpha} \varphi_{e_\alpha,\eta}(s)v$$

converges for almost all $s > 0$ and $v \in \mathsf{h}$. Thus the operator F_s, given by

$$F_s(v\eta) = \sum_{e_\alpha \in \{e_0\} \cup \tilde{T}} E_{e_\alpha} \varphi_{e_\alpha,\eta}(s)v$$

is well-defined, bounded and

$$E^{\hat{\xi}} F_s(v\hat{\eta}) = \varphi_{\xi,\eta}(s)v,$$

so that indeed

$$E^{\hat{\xi}} F_s E_{\hat{\eta}} = \varphi_{\xi,\eta}(s).$$

Thus the following QSDE is satisfied weakly (and so strongly because of strong measurability of $V_{s,t}$):

$$dV_t = V_t F_t d\Lambda_t.$$

This implies that in fact $F.E_{\widehat{f(\cdot)}} \in L^1_{\mathrm{loc}}(\mathbb{R}_+; B(\mathsf{h}; \mathsf{h} \otimes \widehat{\mathsf{k}}))$ for all $f \in L^2(\mathbb{R}_+; \mathsf{k})$. Therefore π_F is quantum Wiener integrable and it is now easily seen that $Q_{s,t}(\pi_F) = V_{s,t}$ on exponential domain of locally bounded step functions. But as $V_{s,t}$ is a contractive process (so, in particular, a bounded operator) and a solution to QSDE is unique, this implies that in fact $Q_{s,t}(\pi_F) = V_{s,t}$. This ends the proof. $\square$

It is well known that every Markov-regular quantum stochastic cocycles has a generator F. However, more in fact is true, which is summed up by the following theorem:

Corollary 4.1.11. *Let $(X_t)_{t\geqslant 0}$ be a Markov-regular quantum stochastic cocycle on the space $\mathsf{h} \otimes \mathcal{F}$. Then in fact*

$$X_t = \mathcal{Q}_t(\pi_F).$$

Proof. As X_t is a Markov-regular quantum stochastic cocycle, it has a bounded generator F. As F is a bounded operator, $\pi_F \in L^{1,2,2,\infty}_{\text{loc}}$. Under the above assumptions, X satisfies QSDE

$$dX_t = X_s F d\Lambda_t.$$

Thus the result follows by the above lemma and the uniqueness of a solution of a QSDE with a bounded constant coefficient (cf. *e.g.* [48], Theorem 4.2). $\square$

As contractivity of the evolution featured prominently, we will now look for properties of F which allow us to determine the contractivity of $\mathcal{Q}_t(\pi_F)$. For this, we recall the series product:

$$F \lhd G = F + G + F\Delta G.$$

Proposition 4.1.12. *Let $X_t = Q_t(\pi_F)$ for $F\colon \mathbb{R}_+ \to B(\mathsf{h} \otimes \widehat{\mathsf{k}}), \pi_F \in L^{1,2,2,\infty}_{\text{loc}}$. Then X_t is a (co)isometric process if and only if*

$$F_s^* \lhd F_s = 0$$

$$(F_s \lhd F_s^* = 0)$$

for almost all $s \in \mathbb{R}_+$.

Proof. Assume first X_t is coisometric, so that $X_t X_t^* = I$. This implies that $X_{s,t} = Q_{s,t}(\pi_F)$ is coisometric as well for all s, t by bi-adaptedness of the process X. For ease of notation, for $\sigma \subset \Gamma_{[s,r[}$ let σ_r denote $\sigma \cup \{r\}$. Then we have

$$\begin{aligned}
0 &= \frac{1}{t-s}\langle u\varepsilon(f), (X_{s,t}X_{s,t}^* - I)v\varepsilon(g)\rangle \\
&= e^{\langle f,g\rangle}\int_{\Gamma_{[s,t[}^{\geqslant 1}} \langle u\pi_{\widehat{f}}(\sigma), (\pi_F \star (\pi_F)^*)(\sigma)v\pi_{\widehat{g}}(\sigma)\rangle d\sigma \\
&= \frac{1}{t-s}\int_{\Gamma_{[s,t[}^{\geqslant 1}} \sum_{\alpha\cup\beta=\sigma} \langle u\pi_{\widehat{f}}(\sigma), \pi_F(\alpha;\sigma)\Delta(\alpha\cap\beta;\sigma)(\pi_F)^*(\beta;\sigma)v\pi_{\widehat{g}}(\sigma)\rangle d\sigma \\
&= \frac{1}{t-s}\int_0^t \int_{\Gamma_{[s,r[}} \sum_{\alpha\cup\beta=\sigma} \langle u\pi_{\widehat{f}}(\sigma_r), \pi_F(\alpha_r;\sigma_r)\Delta(\alpha\cap\beta;\sigma_r)(\pi_F)^*(\beta;\sigma_r)v\pi_{\widehat{g}}(\sigma_r)\rangle d\sigma dr \\
&+ \frac{1}{t-s}\int_0^t \int_{\Gamma_{[s,r[}} \sum_{\alpha\cup\beta=\sigma} \langle u\pi_{\widehat{f}}(\sigma_r), \pi_F(\alpha;\sigma_r)\Delta(\alpha\cap\beta;\sigma_r)(\pi_F)^*(\beta_r;\sigma_r)v\pi_{\widehat{g}}(\sigma_r)\rangle d\sigma dr \\
&+ \frac{1}{t-s}\int_0^t \int_{\Gamma_{[s,r[}} \sum_{\alpha\cup\beta=\sigma} \langle u\pi_{\widehat{f}}(\sigma_r), \pi_F(\alpha_r;\sigma_r)\Delta(\alpha\cap\beta_r;\sigma_r)(\pi_F)^*(\beta_r;\sigma_r)v\pi_{\widehat{g}}(\sigma_r)\rangle d\sigma dr \\
&= \frac{1}{t-s}\int_0^t \int_{\Gamma[s,r[} \sum_{\alpha\cup\beta=\sigma} \langle u\pi_{\widehat{f}}(\sigma_r), \\
&\quad \pi_F(\alpha;\sigma_r)(F_r + F_r^* + F_r\Delta F_r^*;\sigma_r)\Delta(\alpha\cap\beta;\sigma_r)(\pi_F)^*(\beta;\sigma_r)v\pi_{\widehat{g}}(\sigma\cup r)\rangle d\sigma dr.
\end{aligned}$$

Now by the properties of the quantum stochastic integrand, if we tend with t to s we get

$$0 = \langle u\widehat{f(s)}, (F_s + F_s^* + F_s\Delta F_s^*)v\widehat{g(s)}\rangle.$$

The first implication then follows by totality.

The other implication is now easily seen by our calculation, as if $F_r \triangleleft F_r^* = 0$, then the inner product from the first fundamental formula is zero as well and thus our process is coisometric.

If X_t is isometric, then its dual process is coisometric by Lemma 4.1.4. But by that result, $X_t^\sharp = \mathcal{Q}_t(\pi_{F^*})$. By the previous part, this means

$$F^* + F + F^* \Delta F = 0.$$

□

Before we proceed, we need the following lemma regarding measurability of operator-valued functions.

Lemma 4.1.13. *Consider measurable functions $F\colon \mathbb{R}_+ \to B(\mathsf{H}_1, \mathsf{H}_2), G\colon \mathbb{R}_+ \to B(\mathsf{H}_1)$ with the property that for each s $\mathrm{Ker}(G) \subset \mathrm{Ker}(F)$, so that the following function:*

$$V_s(Gu) = Fu$$

is well-defined (extending by 0 on $\mathrm{Ran}(G)^\perp$). Then V is measurable.

Proof. Let $\xi \in \mathsf{H}_1$. We are interested in seeing whether $s \mapsto V_s\xi$ is measurable. We can write ξ as $\xi = Gu + v$ for some $v \in \mathrm{Ran}(G)^\perp$. This implies that $V_s\xi = F_s u$. But we know that that function is measurable. Thus so is V. □

The following construction is well known - cf. [28]. We repeat the proof of the result here for clarity and to show that it can be proven using purely quantum Wiener methods.

Lemma 4.1.14 (Quantum stochastic dilation)**.** *If $F\colon \mathbb{R}_+ \to B(\widehat{\mathsf{k}}_0 \otimes \mathsf{h})$ is a measurable function with the property that*

$$F_s \lhd F_s^* \leqslant 0,$$

$$(F_s^* \lhd F_s \leqslant 0)$$

then there exists a measurable function $U\colon \mathbb{R}_+ \to B(\widehat{\mathsf{k}} \otimes \mathsf{h})$ *with* $\mathsf{k}_0 \subset \mathsf{k}$ *such that* U *is a dilation of* F*, in the sense that, denoting* P_{k_0} *to be the projection from* k *to* k_0,

$$P_{\mathsf{k}_0} U P_{\mathsf{k}_0} = F.$$

Moreover,

$$U_s \lhd U_s^* = U_s^* \lhd U_s = 0.$$

Thus in particular, for $\xi, \zeta \in \mathcal{F}^{\mathsf{k}_0}$,

$$\int_{\Gamma_t} d\alpha \langle \widehat{\xi(\alpha)}, \pi_U(\alpha)(\widehat{\zeta(\alpha)}\rangle = \int_{\Gamma_t} d\alpha \langle \widehat{\xi(\alpha)}, \pi_F(\alpha)(\widehat{\zeta(\alpha)}\rangle.$$

Moreover, if $F \in L^{1,2,2,\infty}_{\text{loc}}$*, then so is* U*.*

Proof. Without loss of generality, let us consider the case $F_s^* \lhd F_s \leqslant 0$. By Theorem 1.0.3, this implies that

$$F_s^* \lhd F_s = \begin{bmatrix} -A_s^2 & A_s V_s D_s \\ D_s V_s^* A_s & -D_s^2 \end{bmatrix}$$

for some contractive V_s and $A_s, D_s \geqslant 0$. Let us notice that A_s, D_s are automatically measurable (as being equal to PFP for fixed projections P). V is measurable by Lemma 4.1.13. It is easy to see via simple matrix calculations that this implies F to have the form:

$$F_s = \begin{bmatrix} iH_s - \frac{1}{2}(L_s^* L_s + A_s^2) & A_s V_s S_s - L_s^* N_s \\ L_s & N_s - I \end{bmatrix},$$

where $S = \sqrt{I - D^2}$ (and, in particular, $\|D\| \leqslant 1$). Let us put K, M to be operators such that $K_s = A_s V_s D_s + N_s^* S_s, M^* M = A_s(I - V_s V_s^*) A_s$. If we now consider the matrix

$$U_s = \begin{bmatrix} iH_s - \frac{1}{2}(L_s^* L_s + A_s^2) & A_s V_s S_s - L_s^* N_s & K_s & M_s^* \\ L_s & N_s - I & S_s & 0 \\ -V_s^* A_s & S_s & -N_s^* - I & 0 \\ M_s & 0 & 0 & 0 \end{bmatrix},$$

then it is easily seen that each U_s is a dilation of F_s and $U_s^* \lhd U_s = U_s \lhd U_s^* = 0$. But by our previous result, this implies that $\mathcal{Q}_t(\pi_U)$ is unitary - we will defer the proof that $U \in L^{1,2,2,\infty}_{\mathrm{loc}}$ for a moment. Considering $f_i, g_i \in L^2(\mathbb{R}_+; \mathsf{k}), u_i, v_i \in \mathsf{h}$ and $\lambda_i, \mu_i \in \mathbb{C}, i \leqslant n$ for some $n \in \mathbb{N}$, we see that by the First Fundamental Formula, the fact that U dilates F and the fact that $\mathcal{Q}_t(\pi_U)$ is unitary that

$$\begin{aligned} |\langle \sum_i \lambda_i u_i \varepsilon(f_i), \mathcal{Q}_t(\pi_F) \sum_i \mu_i v_i \varepsilon(g_i)\rangle| &= |\langle \sum_i \lambda_i u_i \varepsilon(f_i), \mathcal{Q}_t(\pi_U) \sum_i \mu_i v_i \varepsilon(g_i)\rangle| \\ &\leqslant |\langle \sum_i \lambda_i u_i \varepsilon(f_i), \sum_i \mu_i v_i \varepsilon(g_i)\rangle|, \end{aligned}$$

so that $\mathcal{Q}_t(\pi_F)$ gives a bounded sesquilinear form of norm 1. But this means that $\mathcal{Q}_t(\pi_F)$ itself is bounded of norm at most 1, which ends the proof.

For the $L^{1,2,2,\infty}_{\mathrm{loc}}$ part, one merely needs to observe that from $F \in L^{1,2,2,\infty}_{\mathrm{loc}}$ we can infer the following:

- $L \in L^2_{\mathrm{loc}}, N \in L^\infty_{\mathrm{loc}}$ (by definition);
- $D \in L^\infty_{\mathrm{loc}}, A \in L^2_{\mathrm{loc}}$ (by the previous, upon inspecting the upper and lower right corners of F);
- $H \in L^1_{\mathrm{loc}}$ (by the previous parts).

Now inspecting each entry in the matrix yields what we require, in that the terms in the first column and first row (except for the top left one) are in L^2_{loc} and the 3×3 bottom-right matrix is in L^∞_{loc}. Thus $U \in L^{1,2,2,\infty}_{\mathrm{loc}}$. □

Remark 4.1.15. The proposition above is well known in the case when F is an operator (rather than a function), but the presented proof is much more elementary, avoiding the usual mapping processes treatment. Also this proof extends to F being a function, so to the case of quantum stochastic evolutions, rather than merely quantum stochastic cocycles. Indeed, we can now say:

Corollary 4.1.16. *A Markov-regular quantum stochastic evolution is contractive if and only if its generator $F\colon \mathbb{R}_+ \to \mathcal{B}(\mathsf{h} \otimes \widehat{\mathsf{k}})$ satisfies*

$$F_s \lhd F_s^* \leqslant 0$$

for almost all $s \geqslant 0$.

4.1.3 Trotter product

As another application of the quantum Wiener formalism, we show an easy proof of the Trotter product formula from quantum stochastic calculus:

Corollary 4.1.17. *Let $x = F^{\otimes}, y = G^{\otimes}$ be the generators of two Markov regular quantum stochastic cocycles X, Y, respectively, and $F, G \in B(\mathsf{h} \otimes \widehat{\mathsf{k}})$. Then the Trotter product:*

$$Z_t^n = X_{\frac{t}{n}} Y_{\frac{t}{n}} \sigma_{\frac{t}{n}}(X_{\frac{t}{n}} Y_{\frac{t}{n}}) \ldots \sigma_{\frac{(n-1)t}{n}}(X_{\frac{t}{n}} Y_{\frac{t}{n}})$$

converges in the weak operator topology to $Z_t = \mathcal{Q}_t(z)$, where $z = F^{\otimes} \star G^{\otimes}$, for each $k \in \{u\varepsilon(f) : u \in \mathsf{h}, f \in L^2(\mathbb{R}_+; \mathsf{k})\}$.

Proof. The key element of the proof is proving that

$$\langle u\varepsilon(0), (\mathcal{Q}_t(F^{\otimes} \star G^{\otimes}) - \mathcal{Q}_t((F \lhd G)^{\otimes}))v\varepsilon(0)\rangle \leqslant t^2 C \|u\| \|v\|, \qquad u, v \in \mathsf{h}$$

(with C depending only on F, G). The rest then follows by standard treatment and properties of the Weyl process - for details see [50].

Fix $u, v \in \mathsf{h}$ and let w be a locally quantum Wiener integrable operator kernel with $\operatorname{supp} w \subset \Gamma^{(\geqslant 2)}$. We have

$$\begin{aligned}\langle u\varepsilon(0), (\mathcal{Q}_t(w)v\varepsilon(0)\rangle &= \langle u\varepsilon(\widehat{0}), w(v\varepsilon(\widehat{0}))\rangle \\ &= \int_{\Gamma_t} \mathrm{d}\alpha \langle u\varepsilon(\widehat{0})(\alpha), w(\alpha)(v\varepsilon(\widehat{0}))\rangle \\ &= \int_{\Gamma_t^{\geqslant 2}} \mathrm{d}\alpha \langle u\varepsilon(\widehat{0})(\alpha), w(\alpha)(v\varepsilon(\widehat{0}))\rangle \\ &\leqslant (e^t - t - 1)\|w\|\|u\|\|v\| \leqslant t^2 K\|w\|\|u\|\|v\|,\end{aligned}$$

where K is the appropriate constant from the Taylor expansion of the exponential function. Consider $w = (F \lhd G)^{\otimes} - F^{\otimes} \star G^{\otimes}$. We know that $\operatorname{supp} w \subset \Gamma^{(\geqslant 2)}$. Thus the key estimate and the rest of the proof follow. □

4.2 Applications to duality transforms

It is well-known that Fock space is isomorphic to the L^2-spaces of stochastic processes satisfying the so-called chaotic representation property. This chaotic representation is given by Wiener integrals and thus it is natural to ask how does this transform behave in the setup developed so far. As we have uncovered a lot of algebraic structure of mixed time Wiener integrals, it is natural to expect this algebraic structure to have a natural manifestation on the level of our stochastic process. This is explored below in the particular cases of Wiener and Poisson processes.

4.2.1 Wiener product

To develop the duality transform in multi dimensions, we refer the reader to [69]. Let H be a Hilbert space and denote by C the Gaussian space over H. For $f \in H$ let $\pi_f \in \Gamma(k)$ be the product function $\sigma \mapsto \vec{\bigotimes}_{s\in\sigma} f(s)$ and let $\varepsilon_f \in L^2(C)$ be given by $exp(\varphi_f - \frac{1}{2}\|f\|^2)$, where $\{\varphi_f : f \in H\}$ is the Gaussian process indexed by H. The complex linear spans Ψ, ε of $\{\pi_f : f \in H\}$ and $\{\varepsilon_f : f \in H\}$ are dense respectively in $\Gamma(k)$ and $L^2(C)$ and it is easily seen that

$$\langle \pi_f, \pi_g \rangle = exp\langle f, g\rangle = \langle \varepsilon_f, \varepsilon_g \rangle.$$

Thus the isomorphism $\Phi : \Gamma(k) \to L^2(C)$ is the continuous linear extension of the map $\pi_f \mapsto \varepsilon_f$.

Let $\widehat{\mathsf{k}} = \mathbb{C} \oplus \mathsf{k}$ with k being a complexification of a real Hilbert space $\mathsf{k}_{\mathbb{R}}$. We would like to define a time-Wiener product $\xi \star_W \eta$ of two time-Wiener integrands $\xi, \eta \in \Phi(\widehat{\mathsf{k}})$, which is again a time-Wiener integrand and corresponds to the multiplication of random variables under the duality transform. Consider the case $\xi, \eta \in \Phi(\mathsf{k})$ first, with ξ supported on Γ_t. We define

$$x_\xi(\sigma) = |\xi(\sigma)\rangle\langle \begin{pmatrix}1\\0\end{pmatrix}^{\otimes\#\sigma} | + | \begin{pmatrix}1\\0\end{pmatrix}^{\otimes\#\sigma} \rangle\langle \xi(\sigma)|,$$

$$\xi \star_W \eta = \mathcal{Q}_t(x_\xi)\eta,$$

where $\overline{\xi(\sigma)}$ is the conjugation coming from $\mathsf{k} = \mathsf{k}_{\mathbb{R}} \oplus i\mathsf{k}_{\mathbb{R}}$.

Lemma 4.2.1. *For a function $f : \Gamma \to \bigoplus k^{\otimes n}$ and $\alpha \in \Gamma$ let f^α denote the function given by $f^\alpha(\beta) = f(\alpha \cup \beta)$. If $f \in Dom(\sqrt{2}^N)$, then f^α is square-integrable for almost all $\alpha \in \Gamma$.*

Proof. Applying the integral-sum lemma, we have

$$\int_\Gamma d\beta \int_\Gamma d\alpha \|f(\alpha\cup\beta)\|^2 = \int_\Gamma d\sigma \sum_{\alpha\subset\sigma} \|f(\sigma)\|^2 = \int_\Gamma d\sigma 2^{\#\sigma}\|f(\sigma)\|^2 < \infty$$

and the claim follows. □

A natural question to ask is which classes of vectors can we Wiener-multiply to obtain a sensible result. The answer to this question is provided in the following theorem.

Theorem 4.2.2. *If* $\xi \in \mathrm{Dom}(\sqrt{3}^N)$, *then* $\mathrm{Dom}(\sqrt{3}^N) \subset \mathrm{Dom}(\mathcal{Q}_t(x_\xi))$.

Proof. Let us notice that $\|x_\xi(\sigma)\| = \|\xi(\sigma)\|$. Also, we see that

$$J^*_{\alpha;\alpha\cup\beta} x_\xi(\gamma;\alpha\cup\beta)\eta(\delta;\alpha\cup\beta) \neq 0 \;\Leftrightarrow\; \gamma = \omega\cup\beta, \delta = (\alpha\setminus\omega)\cup\beta \text{ for some } \omega\subset\alpha.$$

We calculate:

$$\begin{aligned}
\|\xi \star_W \eta\|^2 &= \int_\Gamma d\sigma \|Q(x_\xi)\eta(\sigma)\|^2 = \int_\Gamma d\sigma \| \int_\Gamma d\beta J^*_{\sigma;\sigma\cup\beta} x_\xi \star \eta(\alpha\cup\beta)\|^2 \\
&\leqslant \int_\Gamma d\sigma 2^{\#\sigma} \sum_{\alpha\subset\sigma} \| \int_\Gamma d\beta J^*_{\sigma;\sigma\cup\beta} x_\xi(\alpha;\sigma\cup\beta)\eta((\sigma\setminus\alpha)\cup\beta;\sigma\cup\beta)\|^2 \\
&\leqslant \int_\Gamma d\sigma 2^{\#\sigma} \sum_{\alpha\subset\sigma} (\int_\Gamma d\beta \|J^*_{\sigma;\sigma\cup\beta} x_\xi(\alpha;\sigma\cup\beta)\eta((\sigma\setminus\alpha)\cup\beta;\sigma\cup\beta)\|)^2 \\
&\leqslant \int_\Gamma d\sigma 2^{\#\sigma} \sum_{\alpha\subset\sigma} \int_\Gamma d\beta_1 \|x_\xi(\alpha\cup\beta_1)\|^2 \int_\Gamma d\beta_2 \|\eta((\sigma\setminus\alpha)\cup\beta_2)\|^2 \\
&= \int_\Gamma d\alpha \int_\Gamma d\beta_1 2^{\#\alpha} \|\xi(\alpha\cup\beta_1)\|^2 \int_\Gamma d\sigma \int_\Gamma d\beta_2 2^{\#\sigma} \|\eta(\alpha\cup\beta_2)\|^2 \\
&= \|\sqrt{3}^N \xi\|^2 \|\sqrt{3}^N \eta\|^2 < \infty.
\end{aligned}$$

□

Remark 4.2.3. 1. Let $\mathsf{k} = \mathbb{C}$. Then vectors in $\Phi(\mathsf{k})$ are just functions $f \in L^2(\Gamma)$. It is easily seen that then, for real-valued functions $f, g \in L^2(\Gamma)$,

$$\mathcal{Q}_t(x_f)g(\sigma) = \sum_{\alpha\subset\sigma} \int_{\Gamma_t} d\omega f(\alpha \cup \omega) g(\omega \cup (\sigma \setminus \alpha),$$

which is just the Wiener product as defined in *e.g.* [53].

2. It is easily seen that $x_\xi^* = x_{\overline{\xi}}$, where $\overline{\xi}(\sigma) = \overline{\xi(\sigma)}$. As $\mathcal{Q}_t(x^*) \subset \mathcal{Q}_t(x)^*$, we can see that in fact

$$\overline{\xi \star_W \eta} = \overline{\eta} \star_W \overline{\xi}, \langle \xi, \eta \star_W \zeta\rangle = \langle \overline{\eta} \star_W \xi, \zeta\rangle$$

whenever $\xi, \eta, \zeta \in \mathrm{Dom}(\sqrt{3}^N)$.

Corollary 4.2.4. *If $\xi, \eta \in Dom(\sqrt{3}^N)$, then*

$$\Phi(\xi \star_W \eta) = \Phi(\xi)\Phi(\eta).$$

Proof. The proof follows along similar lines as the proof in [53]. First of all let us notice that it holds for ξ, η being linear combinations of exponential vectors. Indeed, for $\xi = \varepsilon(f), \eta = \varepsilon(g)$, f, g alued in $\mathsf{k}_\mathbb{R}$ we have

$$\begin{aligned}
\varepsilon(f) \star_W \varepsilon(g)(\sigma) &= \int_\Gamma d\beta J^*_{\sigma;\sigma\cup\beta} \sum_{\alpha\subset\sigma} x_{\varepsilon(f)}(\alpha \cup \beta; \sigma \cup \beta)\varepsilon(g)((\sigma \setminus \alpha) \cup \beta; \sigma \cup \beta) \\
&= \sum_{\alpha\subset\sigma} \varepsilon(f)(\alpha;\sigma)\varepsilon(g)(\sigma \setminus \alpha;\sigma) \int_\Gamma d\beta \langle \varepsilon(f)(\beta), \varepsilon(g)(\beta)\rangle \\
&= e^{\langle f,g\rangle}\varepsilon(f+g),
\end{aligned}$$

so

$$\Phi(\varepsilon(f))\Phi(\varepsilon(g)) = \varepsilon_f\varepsilon_g = e^{\langle f,g\rangle}\varepsilon_{f+g} = e^{\langle f,g\rangle}\Phi(\varepsilon(f+g)) = \Phi(\varepsilon(f) \star_W \varepsilon(g)).$$

Now, for $\eta \in Dom(\sqrt{3}^N), \xi, \zeta \in \Psi$ we have:

$$\langle \Phi(\xi \star_W \eta), \Phi(\zeta)\rangle = \langle \eta, \overline{\xi} \star_W \zeta\rangle = \langle \Phi(\eta), \Phi(\overline{\xi})\Phi(\zeta)\rangle = \langle \Phi(\xi)\Phi(\eta), \Phi(\zeta)\rangle.$$

By the density of Ψ and the invariance of $\star_W$ under conjugation, this implies that $\Phi(\xi \star_W \eta) = \Phi(\xi)\Phi(\eta)$ whenever at least one of ξ, η is in Ψ. But that means that the above reasoning remains true or $\xi \in Dom(\sqrt{3}^N)$, so in fact the claim holds true for all $\xi, \eta \in Dom(\sqrt{3}^N)$, which ends the proof. □

Finally, we would like to discuss the case when ξ, η are $\widehat{\mathsf{k}}$ valued kernels. In this case we would like to define a 'hat'-Wiener convolution such that the following holds:

$$\Phi(\widehat{\mathcal{W}}(\xi \star_{\widehat{\mathcal{W}}} \eta)) = \Phi(\widehat{\mathcal{W}}(\xi))\Phi(\widehat{\mathcal{W}}(\eta)). \tag{4.2}$$

Corollary 4.2.5. *If ξ, η are time-Wiener integrands such that $\xi', \eta' \in L^{2,1}_{3,1}$, then (4.2) holds.*

Proof. An immediate corollary from Proposition 2.3.7. □

4.2.2 Poisson product

Now let us move to the duality between Fock space and Poisson space. For this let $\mathsf{k} = \mathbb{C}$ and let P be the Poisson space. It is known that the Poisson exponential of a real-valued, locally bounded function $f \in L^2(\mathbb{R}_+)$ with compact support is given by

$$\mathcal{E}_{\mathsf{P}}(f) = \exp\{-\int_0^\infty f(s)ds\} \prod_{s \geqslant 0}(1 + f(s)\Delta X_s),$$

where X_s denotes the cádlág version of the compensated Poisson process and $\Delta X_s = X_s - X_{s-}$.

Thus we see that for such functions

$$\mathcal{E}_{\mathsf{P}}(f)\mathcal{E}_{\mathsf{P}}(g) = e^{\langle f,g\rangle}\mathcal{E}_{\mathsf{P}}(f+g+fg).$$

(cf. [72]).

This gives us a natural candidate for our Poisson multiplication kernel. For $f \in L^2(\mathbb{R}_+)$ real-valued, locally bounded take

$$x_f^P(\sigma) = \otimes_{s\in\sigma}\begin{pmatrix} 0 & \langle f(s)| \\ |f(s)\rangle & |f(s)\rangle\langle 1| \end{pmatrix}.$$

Then it is easily seen that for $g \in L^2(\mathbb{R}_+)$ locally bounded we have:

$$\begin{aligned}(x_f^P \star_0 \varepsilon(g))(\sigma) &= \sum_{\alpha\cup\beta=\sigma} x_f^P(\alpha;\sigma)\varepsilon(g)(\beta;\sigma) \\ &= \sum_{\alpha\cup\beta=\sigma}\bigotimes_{s\in\alpha\setminus\beta}\begin{pmatrix} 0 \\ f(s)\end{pmatrix}\bigotimes_{s\in\alpha\cap\beta}\begin{pmatrix} f(s)g(s) \\ f(s)g(s)\end{pmatrix}\bigotimes_{s\in\beta\setminus\alpha}\begin{pmatrix} 0 \\ g(s)\end{pmatrix} \\ &= \varepsilon\begin{pmatrix} fg \\ f+g+fg\end{pmatrix}(\sigma),\end{aligned}$$

thus

$$\mathcal{Q}_t(x_f^P)\varepsilon(g) = \widehat{\mathcal{W}}(x_f^P\mathbf{1}_{\Gamma_{[0,t[}} \star_0 \varepsilon(g)) = e^{\langle f\mathbf{1}_{[0,t[},g\rangle}\varepsilon(f+g+fg)$$

as required.

Corollary 4.2.6. *It is easily seen that in fact*

$$\mathcal{Q}_t(x_f^P)\varepsilon(g)(\sigma) = \sum_{\alpha\cup\beta=\sigma}\int_{\Gamma_t} d\omega f(\alpha\cup\omega)g(\omega\cup\beta),$$

which is just the Poisson product as defined in e.g. *[53].*

4.3 Maassen–Meyer–Lindsay kernels

We will now show how our theory unifies and extends the theory of Maassen–Meyer–Lindsay kernels and Dermoune kernels. Let us start by elaborating on the isomorphism between the Guichardet space and Fock space of finite multiplicity.

Indeed, for $\mathsf{k} = \mathbb{C}^d$ we can look at the Fock space in the following ways:

$$\Gamma(L^2(\mathbb{R}_+;\mathbb{C}^d)) = \Gamma(\oplus_{n=1}^d L^2(\mathbb{R}_+)) = \Gamma(L^2(\mathbb{R}_+))^{\otimes d} = L^2(\mathit{\Gamma})^{\otimes d} = L^2(\mathit{\Gamma}^d).$$

Thus, rather than considering an element of our Fock space as a function of one set variable with values in an appropriate power of k, we look at it as a function of d set variables, but with complex values.

The isomorphism between the two spaces is easy to see. Namely, let e_i denote the usual orthonormal basis of $\mathbb{C}^d$. Then an orthonormal basis of $(\mathbb{C}^d)^{\otimes n}$ consists of vectors of the form $e_{i_1} \otimes ... \otimes e_{i_n}$. Given d parwise disjoint sets $\alpha_1, ..., \alpha_d \in \mathit{\Gamma}$ with $\cup_{i=1}^d \alpha_i = \{s_1 < \cdots < s_n\}$, we define a vector of this form via

$$e((\alpha_i)_{i=1}^d) = e(\alpha_1, \cdots, \alpha_d) = \otimes_{j=1}^n e(j), \quad e(j) = e_i \Leftrightarrow s_j \in \alpha_i.$$

We map a function $f \in \Gamma(L^2(\mathbb{R}_+;\mathbb{C}^d))$ to $\varphi(f) \in L^2(\mathit{\Gamma}^d)$ via

$$\varphi(f)(\alpha_1, \cdots, \alpha_d) = \langle e(\alpha_1, \cdots, \alpha_d), f(\cup_{i=1}^d \alpha_i)\rangle.$$

In words, this correspondence puts the i-th basis vector on the tensor components corresponding to the placement of the elements of α_i in $\cup_{j=1}^d \alpha_j$.

With that in mind, it is now easy to see that Dermoune's formalism is merely the lifting of this isomorphism into the realm of operator kernels. Indeed, given our operator kernel $x \in L^{1,2,2,\infty}$ let us define (recalling that our kernel in fact operates on multiplicity $(d+1)$, with the convention that the extra dimension has index 0):

$$x^{Der}((A_0^\alpha),(A_\beta^\alpha),(A_\alpha^0),(A_0^0)) = \langle e((\cup_{j=0}^d A_i^j))_{i=0}^d, x(\cup_{i,j}A_i^j)e((\cup_{j=0}^d A_j^i))_{i=0}^d\rangle.$$

Let us notice that, in the other direction, given a disjoint partition of a set σ into $(\alpha_0,\cdots,\alpha_d)$ and $(\beta_0\cdots,\beta_d)$ we have:

$$\langle e(\alpha_0,\cdots,\alpha_d), x(\sigma)e(\beta_0,\cdots,\beta_d) = x^{Der}((\alpha_i\cap\beta_0),(\alpha_i\cap\beta_j)_{i,j\geqslant 1},(\alpha_0\cap\beta_i),\alpha_0\cap\beta_0).$$

To see that truly this gives a Dermoune kernel, we need to verify its action on a vector $\xi\in\Gamma(L^2(\mathbb{R}_+;\mathbb{C}^d))$. Namely, we need to verify that:

$$\varphi(\mathcal{Q}_t(x)\xi) = X_t\varphi(\xi),$$

where by X_t we mean the kernel operator up to time t with its kernel given by x^{Der}.

We will utilise the isomorphism between $\Gamma(L^2(\mathbb{R}_+;\mathbb{C}^{d+1}))$ and $L^2(\mathit{\Gamma}^{d+1})$, putting the extra dimension as the first argument. We will also write $\underline{\alpha}=(\alpha_1,\cdots,\alpha_d)$, $\gamma\cap\underline{\alpha}=(\gamma\cap\alpha_1,\cdots,\gamma\cap\alpha_d)$ and abbreviate $(\alpha_0,\cdots,\alpha_d)=(\alpha_0,\underline{\alpha})$.

For $\alpha\in\Gamma_t$ we have:

$$\begin{aligned}\varphi(\mathcal{Q}_t(x)\xi)(\alpha) &= \langle e(\underline{\alpha}),\int_{\Gamma_t} d\beta J^*_{\alpha;\alpha\cup\beta}(x\star\xi)(\alpha\cup\beta)\rangle\\ &= \int_{\Gamma_t} d\beta\langle e(\beta,\underline{\alpha}),(x\star\xi)(\alpha\cup\beta)\rangle\\ &= \int_{\Gamma_t} d\beta \sum_{\gamma_0\cup\gamma_1=\alpha\cup\beta}\langle e(\beta,\underline{\alpha}), x(\gamma_0;\alpha\cup\beta);\xi(\gamma_1;\alpha\cup\beta)\rangle.\end{aligned}$$

We now use the expression:

$$\xi(\alpha) = \sum_{\alpha=\sqcup\underline{\alpha}} \varphi(\xi)(\underline{\alpha})e(\underline{\alpha}),$$

which is just a special case of Parseval's identity.

$$\begin{aligned}\varphi(\mathcal{Q}_t(x)\xi)(\alpha) = \int_{\Gamma_t} d\beta \sum_{\gamma_0\cup\gamma_1=\alpha\cup\beta} = \sum_{\gamma_1=\sqcup\underline{\gamma}_1} \langle e((\alpha\cup\beta)\setminus\gamma_1,\underline{\gamma}),\xi((\alpha\cup\beta)\setminus\gamma_1,\underline{\gamma})\rangle \\ \cdot\langle e(\beta,\underline{\alpha}), x(\gamma_0;\alpha\cup\beta)e((\alpha\cup\beta)\setminus\gamma_1,\underline{\gamma})\rangle \\ = \int_{\Gamma_t} d\beta \sum_{\gamma_0\cup\gamma_1=\alpha\cup\beta} = \sum_{\gamma_1=\sqcup\underline{\gamma}_1} \langle e((\alpha\cup\beta)\setminus\gamma_1,\underline{\gamma}),\xi(\gamma_1;\alpha\cup\beta)\rangle \\ \langle e(\beta,\underline{\alpha}\cap\gamma_0), x(\gamma_0)e(\gamma_0\setminus\gamma_1,\underline{\gamma}\cap\gamma_0)\rangle \\ = \int_{\Gamma_t} d\beta \sum_{\gamma_0\cup\gamma_1=\alpha\cup\beta} = \sum_{\gamma_1=\sqcup\underline{\gamma}_1} \varphi(\xi)(\underline{\gamma_1})x^D er((\alpha_{ij})_{i=0}^d),\end{aligned}$$

with α_{ij} defined as follows:

$$\begin{aligned}\alpha_{00} &= \beta\cap\gamma_0\setminus\gamma_1, \\ \alpha_{i0} &= \alpha\cap\gamma_0\setminus\gamma_1 \quad \text{for } i\geqslant 1, \\ \alpha_{0i} &= \beta\cap\gamma_1\cap\gamma_0 \quad \text{for } i\geqslant 1, \\ \alpha_{ij} &= \alpha\cap\gamma_1\cap\gamma_0 \quad \text{for } i,j\geqslant 1.\end{aligned}$$

Now it is easily seen that in fact,

$$\varphi(\mathcal{Q}_t(x)\xi)(\underline{\alpha}) = X_t\varphi(\xi)(\underline{\alpha}).$$

In particular, taking $d=1$, we recover Maassen–Meyer–Lindsay kernels:

$$x^M(\gamma,\beta,\alpha,\delta) = \langle e(\gamma\cup\delta,\alpha\cup\beta), x(\alpha\cup\beta\cup\gamma\cup\delta)e(\delta\cup\alpha,\gamma\cup\beta)\rangle.$$

We can summarise our findings in the following theorem:

Theorem 4.3.1. *Let $d < \infty$ and let $x^{Der}\colon \Gamma^{(d+1)\times(d+1)} \to \mathbb{C}$ be a measurable function satisfying the condition*

$$|x((\alpha_{ij})_{i,j=0}^{d})| \leqslant C_M^{\sum \#\alpha_{ij}}$$

for some $C, M > 0$ and let X denote the corresponding integral kernel operator on $\mathcal{F}^{\mathbb{C}^d}$. Then there exists an operator kernel x such that

$$\varphi(\mathcal{Q}_t(x)\xi) = X\varphi(\xi)$$

for all $\xi \in \mathrm{Geom}$.

Conversely, for a measurable operator kernel x on $\mathcal{F}^{\mathbb{C}^d}$ satisfying the local $L^{1,2,2,\infty}$ condition, defining

$$X_t\varphi(\xi) = \varphi(\mathcal{Q}_t(x)\xi)$$

gives an extension of the integral kernel operator in the sense of Dermoune, with the kernel x^{Der} defined as before. Indeed, the operator is given exactly in the integral kernel form and it is easily seen that

$$\mathrm{Dom}(X_t) \supset \varphi(\mathrm{Geom}).$$

Proof. The only non-trivial part is the fact that the resulting x satisfies the local $L^{1,2,2,\infty}$ condition. However, it is easily seen that the geometric condition is, in fact, stronger and, by taking $k = l = m = n = \max 1, C, M$ we get that x satisfies a local $L^{\infty,\infty,\infty,\infty}$ condition with these bounds. The fact that x is reconstructible from x^{Der} follows from the finiteness of d. □

4.4 Mapping case

Quantum mapping processes correspond to operator processes via a switch from the Schrödinger picture to Heisenberg picture in quantum mechanics. They were first studied by Evans and Hudson in [19]. Since then, quantum stochastic flows have met with considerable interest due to their connection to quantum dynamical semigroups [47], product systems [12], dilation theory [30] and classical probability [68]. In quantum stochastic calculus relevant works are [58], [59]. They also form the foundation of the theory of quantum Lévy processes and are elements of necessary machinery to perform quantum stochastic calculus on quantum groups [23]. [1] gives a nice overview of this perspective.

An operator kernel x can be considered as a map

$$x\colon \Gamma \to \oplus_{n\geqslant 0} B(\mathsf{h} \otimes \widehat{\mathsf{k}}^{\otimes n}) \supset \oplus_{n\geqslant 0} B(\mathsf{h}) \underline{\otimes} B(\widehat{\mathsf{k}}^{\otimes n}).$$

For the mapping case, let us instead fix an operator space/algebra/system $V \subset B(\mathsf{h})$ and, recalling Definition 1.2.4, let us define the notion of a mapping kernel as a map

$$j\colon V \times \Gamma \to \oplus_{n\geqslant 0} V \otimes_M B(\widehat{\mathsf{k}}^{\otimes n})$$

such that for each $a \in V$ $j(a)$, looked at as a map from Γ to $\oplus_{n\geqslant 0} V \underline{\otimes} B(\widehat{\mathsf{k}}^{\otimes n})$, is an operator kernel.

We define j to satisfy the $L^{1,2,2,\infty}$ condition if each $j(a)$ does.

Example 4.4.1. (i) Let $\varphi\colon V \to V \otimes B(\widehat{\mathsf{k}})$ be a completely bounded map and define its n-fold convolution via:

$$\varphi^{\circ n}\colon V \to V \otimes B(\widehat{\mathsf{k}}^{\otimes n}),$$

$$\varphi^{\circ n}(a) = (\varphi \otimes id_{B(\widehat{\mathsf{k}}^{\otimes(n-1)})}) \circ \cdots \circ (\varphi \otimes id_{B(\widehat{\mathsf{k}})}) \circ \varphi(a).$$

As φ is completely bounded, it is easily seen that this defines a mapping kernel π_φ satisfying the $L^{1,2,2,\infty}$ condition. This generates the mapping cocycle (Evans-Hudson flow).

(ii) This example can be generalised to families $\varphi_t \colon V \to V \otimes B(\widehat{\mathsf{k}}), \varphi_t \colon V \to B(\widehat{\mathsf{k}})$ with appropriate assumptions on the corners of φ, φ.

Remark 4.4.2. Cocycles of the form (i) are considered by Lindsay and Wills in [60] and they prove that the cocycle is weakly multiplicative if and only if, in our language,

$$\pi_\varphi(ab) = \pi_\varphi(a) \star \pi_\varphi(b).$$

It is worth noting that these algebraic identities for mapping cocycles have also been used in [11], where they have been applied to unbounded maps as well.
In our case we have the power to state the same results for φ depending non-trivially on time.

Remark 4.4.3. The machinery developed in this thesis is very powerful and fits into the framework of quantum stochastic cocycles and evolutions remarkably well. Further directions of research include applying it to the fermionic picture via [38] and allowing for unbounded kernels to allow for a wider family of quantum stochastic processes.

The Hilbert–Schmidt representation theorem (Theorem 3.4.3) gives hope that these quantum Wiener integrals may also reconstruct more general martingales.

Appendix A

Quasifree Wiener integrals

A.1 Quasifree stochastic calculus

The theory of quasifree quantum stochastic calculus was first started by Lindsay in his PhD thesis and then continued in his collaboration with Margetts ([55], [56]) and Gnacik ([27]). We will make heavy use of the machinery they have developed, which we present here.

A.1.1 The CCR algebra

We begin with a definition of the symplectic space.

Definition A.1.1. A vector space V is called a *symplectic space* if it is equipped with an antisymmetric, real-bilinear form $\sigma\colon V \times V \to \mathbb{R}$. We call σ a *symplectic form.* We write (V, σ) for a symplectic space with the associated symplectic form.

σ is nondegenerate if

$$\forall_{u \in V} \sigma(u, v) = 0 \Rightarrow v = 0$$

for any $v \in V$. In that case we call V a *nondegenerate* symplectic space.

Example A.1.2. Take V to be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and define $\sigma = \mathrm{Im}\langle\cdot,\cdot\rangle$. This is a fundamental example that will feature further on in our research.

Maps between two symplectic spaces which preserve the symplectic form are called *symplectic* maps.

Given a symplectic space (V,σ) we write $CCR(V,\sigma)$ for the C^*-algebra generated by unitaries $\{w_u : u \in V\}$ satisfying the Weyl form of canonical commutation relations:

$$w_u w_v = e^{-i\sigma(u,v)} w_{u+v}.$$

It turns out (cf. [79], [64]) that these relations define a unique C^*-algebra, which is simple, central and non-separable.

Proposition A.1.3. *For every symplectic map $T\colon V \to W$ between symplectic spaces (V,σ) and (W,τ) there is a unique C^*-monomorphism $\varphi_T\colon CCR(V,\sigma) \to CCR(W,\tau)$ satisfying*

$$\varphi_T(w_u) = w_{Tu}, \quad u \in V.$$

In the case when T is a symplectic automorphism, φ_T is known as the Bogoliubov transformation.

A.1.2 Partial transpose

Before we continue, we need the construction of a partial transpose. We will modify it for our purposes in the subsequent section. For details and motivations of the results quoted here we refer the Reader to [55].

We need the following notations and definitions to facilitate our treatment of unbounded operators.

Definition A.1.4. Let $\mathsf{H}_1, \mathsf{H}_2$ be two Hilbert spaces and $\mathcal{D}_1$ be a dense subset of H_1.

1. We denote the family of (unbounded) operators with domain $\mathcal{D}_1$ and values in H_2 by $\mathcal{O}(\mathcal{D}_1, \mathsf{H}_2)$.

2. For a dense subset $\mathcal{D}_2$ of H_2, we denote by $\mathcal{O}^\dagger(\mathcal{D}_1, \mathcal{D}_2)$ the family of adjointable operators T with domain $\mathcal{D}_1$, values in H_2 and such that $\mathcal{D}_2$ is in the domain of T^*. The restriction of T^* to $\mathcal{D}_2$ in this case will be denoted by $T^\dagger$.

Definition A.1.5. Let $\mathsf{h}_1, \mathsf{h}_2, \mathsf{H}$ be Hilbert spaces and M be a von Neumann algebra in $B(\mathsf{H})$. We say that a (possibly unbounded) operator T from $\mathsf{h}_1 \otimes \mathsf{H}$ to $\mathsf{h}_2 \otimes \mathsf{H}$ is affiliated to M, written $T \eta B(\mathsf{h}_1, \mathsf{h}_2) \overline{\otimes} \mathsf{M}$, if for all unitaries u in M' we have

$$(I_{\mathsf{h}_2} \otimes u^*) T (I_{\mathsf{h}_1} \otimes u) = T.$$

In accordance with our previous notation, we define the following families:

$$\mathcal{O}_{\mathsf{M}}(\mathsf{h}_1 \underline{\otimes} \mathcal{D}; \mathsf{h}_2 \otimes \mathsf{H}) := \{T \in \mathcal{O}(\mathsf{h}_1 \underline{\otimes} \mathcal{D}; \mathsf{h}_2 \otimes \mathsf{H})\colon T \eta B(\mathsf{h}_1, \mathsf{h}_2) \overline{\otimes} \mathsf{M}\};$$

$$\mathcal{O}^\dagger_{\mathsf{M}}(\mathsf{h}_1 \underline{\otimes} \mathcal{D}_1; \mathsf{h}_2 \otimes \mathcal{D}_2) := \{T \in \mathcal{O}^\dagger(\mathsf{h}_1 \underline{\otimes} \mathcal{D}_1; \mathsf{h}_2 \otimes \mathcal{D}_2)\colon T \eta B(\mathsf{h}_1, \mathsf{h}_2) \overline{\otimes} \mathsf{M}\}$$

for $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2$ dense subsets of H.

These notations allow us to succinctly state the vector operator correspondence of Tomita-Takesaki theory. Namely, let now M be a von Neumann algebra with a cyclic and separating vector ξ and let E_ξ be the ampliation of the ket map. In the usual form, the vector operator correspondence simply states that each operator affiliated with a von Neumann algebra is uniquely given by its value on the cyclic and separating vector. Here we need an "ampliated" version of this statement: that an operator $T \in \mathcal{O}_{\mathsf{M}}(\mathsf{h}_1 \underline{\otimes} \mathcal{D}; \mathsf{h}_2 \otimes \mathsf{H})$ is uniquely defined, up to its $\mathsf{h}_1 \mapsto \mathsf{h}_2$ part, by its value on vectors of the form $u \otimes \xi$ for $u \in \mathsf{h}_1$.

Let $\Xi = \mathsf{M}'\xi$. We have the following proposition:

Proposition A.1.6. *The map* $T \mapsto TE_\xi$ *is a linear isomorphism between families* $O_{\mathsf{M}}(\mathsf{h}_1\underline{\otimes}\Xi; \mathsf{h}_2 \otimes \mathsf{H})$ *and* $\mathcal{O}(\mathsf{h}_1; \mathsf{h}_2 \otimes \mathsf{H})$, *which restricts to an isomorphism between* $\mathcal{O}^\dagger_{\mathsf{M}}(\mathsf{h}_1\underline{\otimes}\mathcal{D}_1; \mathsf{h}_2 \otimes \mathcal{D}_2)$ *and*

$$\{B \in B(\mathsf{h}_1; \mathsf{h}_2 \otimes \mathsf{H})\colon \exists_{B_\dagger \in B(\mathsf{h}_2;\mathsf{h}_1\otimes \mathsf{H})} \forall_{x' \in \mathsf{M}'} B^* E_{x'\xi} = E^{x'^*\xi} B_\dagger\}.$$

We will write T^ξ *for its inverse.*

Let us note the surprising appearance of bounded operators. Indeed, for an unbounded closed operator T with domain $\mathsf{h}_1 \otimes \mathcal{D}_1$, we have that TE_ξ is everywhere defined and closed and thus bounded. Thus upon restricting to adjointable (and thus closed) operators we can easily see that their slices become automatically bounded.

Definition A.1.7. For a Hilbert space k $\overline{\mathsf{k}}$ denotes its Hilbert space conjugate. We write j for the map $u \mapsto \overline{u}, j\colon \mathsf{k} \to \overline{\mathsf{k}}$. For an operator $T \in \mathcal{O}(\mathcal{D}, \mathsf{k}_2), \mathcal{D} \subset \mathsf{k}_1$ we write $\overline{T} \in \mathcal{O}(\overline{\mathcal{D}}, \overline{\mathsf{k}_2})$ for

$$\overline{T} = j_2 T j_1^{-1}, \overline{c} \mapsto \overline{Tc},$$

where j_1, j_2 are conjugations on $\mathsf{k}_1, \mathsf{k}_2$ respectively.

The transpose map T^T is defined as

$$T^T = \overline{T^*}.$$

The transpose map, as a map which is not completely bounded, cannot easily be "tensored with identity". To circumvent this trouble, we note that transposition is a unitary operator between the Hilbert-Schmidt classes of operators, i.e.

$$U := (\cdot)^T\colon HS(\mathsf{k}_1; \mathsf{k}_2) \to HS(\overline{\mathsf{k}_2}; \overline{\mathsf{k}_1})$$

is a unitary map. Thus we can tensor with identity:

$$I \otimes U \colon HS(\mathsf{h}_1; \mathsf{h}_2) \otimes HS(\mathsf{k}_1; \mathsf{k}_2) = HS(\mathsf{h}_1 \otimes \mathsf{k}_1; \mathsf{h}_2 \otimes \mathsf{k}_2) \to HS(\mathsf{h}_1 \otimes \overline{\mathsf{k}_2}; \mathsf{h}_2 \otimes \overline{\mathsf{k}_1}).$$

Currently we are able to partially transpose Hilbert-Schmidt operator. We would like to be able to do that on a wider class of operators. The right classes are described by the following:

Definition A.1.8. Let M be a von Neumann algebra on $B(\mathsf{H})$ with a cyclic and separating vector ξ. For Hilbert spaces $\mathsf{k}_1, \mathsf{k}_2$ we define the $(\mathsf{k}_1, \mathsf{k}_2)$-matrix space via:

$$\mathsf{M}_{\mathsf{k}_1,\mathsf{k}_2}(\mathsf{M}, \xi) := \{T \in \mathcal{O}_{\mathsf{M}}(\mathsf{k}_1 \underline{\otimes} \Xi; \mathsf{k}_2 \otimes \mathsf{H}) \colon TE_\xi \in HS(\mathsf{k}_1; \mathsf{k}_2 \otimes \mathsf{H})\}.$$

More generally, for a subset $\mathcal{D} \subset \mathsf{H}$ we define:

$$\mathsf{M}_{\mathsf{k}_1,\mathsf{k}_2}(\mathsf{M}, \mathcal{D}) = \cap_{h \in \mathcal{D}} \mathsf{M}_{\mathsf{k}_1,\mathsf{k}_2}(\mathsf{M}, h).$$

In that case, we will denote the mapping $\mathcal{D} \ni \eta \to TE_\eta$ by $E_{\mathcal{D}}$ and its inverse by $(\cdot)^{\mathcal{D}}$. For $\mathsf{k}_1 = \mathbb{C}, \mathsf{k}_2 = \mathsf{k}$, we will write $\mathsf{M}_{\mathsf{k}_1,\mathsf{k}_2}(\mathsf{M}, \cdot)$ as $\mathcal{C}(\mathsf{M}, \cdot)$ (for column). Similarly, for $\mathsf{k}_1 = \mathsf{k}, \mathsf{k}_2 = \mathbb{C}$, we will write $\mathcal{R}(\mathsf{M}, \cdot)$ (for row).

In this class, we can define the partial transpose:

Definition A.1.9. Let $T \in \mathsf{M}_{\mathsf{k}_1,\mathsf{k}_2}(\mathsf{M}, \xi)$. We define:

$$T^T := ((I \otimes U)(TE_\xi))^\xi.$$

For $T \in \mathsf{M}_{\mathsf{k}_1,\mathsf{k}_2}(\mathsf{M}, \mathcal{D})$, we define

$$T^t := ((I \otimes U)(E_{\mathcal{D}}(T)))^{\mathcal{D}}.$$

We note the consistency of notations.

The properties of this map are elaborated in detail in e.g. [65], Chapter 4 and 5. We will only quote ones we need.

We give a final theorem before diving into the quasifree stochastic calculus proper:

Theorem A.1.10 (cf. [2])**.** *Let* H_0 *be a closed, real subspace of* H *such that* $\mathsf{H}_0 \oplus i\mathsf{H}_0$ *is dense in* H *and* $\mathsf{H}_0 \cap i\mathsf{H}_0 = \{0\}$*. Then*

$$\mathsf{M}'_{\mathsf{H}_0} = \mathsf{M}_{i\mathsf{H}_0^\perp},$$

where $\mathsf{M}_K = CCR(K)''$ *for a Hilbert space* K*.*

Moreover, denoting $\mathsf{M} = CCR(\mathsf{H})$*, if* ξ *is its cyclic and separating vector, then, under the natural identifications,* ξ *is also a cyclic and separating vector for* $\mathsf{M}_{\mathsf{H}_0}$ *and* $\mathsf{M}_{i\mathsf{H}_0^\perp}$*.*

This theorem is a special case of the Araki's Duality Theorem [2]. This form is taken from [73].

A.1.3 Quasifree stochastic calculus

Fix Hilbert spaces h, k and let $\overline{\mathsf{k}}$ denote the conjugate space to k. Let the conjugation $\mathsf{k} \to \overline{\mathsf{k}}$ be denoted by K and let $\iota \colon \mathsf{k} \to \mathsf{k} \oplus \overline{\mathsf{k}}$ be the operator

$$\iota = \begin{pmatrix} I \\ -K \end{pmatrix}$$

and $\widehat{\iota} \colon \mathsf{k} \to \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}}$ to be given by $\widehat{\iota}(f) = \widehat{\iota(f)}$. Moreover, fix a real subspace $\mathfrak{X} \subset \mathsf{k}$ and an operator Σ^0 on the subspace $\iota(\mathfrak{X}) \subset \mathsf{k} \oplus \overline{\mathsf{k}}$ with the following properties:

1. $\mathfrak{X}$ is dense in k;

2. Σ^0 is closable;

3. $\Sigma^0 \circ \iota$ is symplectic;

4. $\operatorname{Ran} \Sigma^0$ is dense in $\mathsf{k} \oplus \overline{\mathsf{k}}$;

5. $\overline{\Sigma^0 \iota \mathfrak{X}} =: H_1, H_2 := i(H_1)^{\Re\perp}$ have the following properties:

    - H_1 is a real, closed subspace of $\mathsf{k} \oplus \overline{\mathsf{k}}$;
    - $\mathsf{H}_1 \oplus \mathsf{H}_2$ is dense in $\mathsf{k} \oplus \overline{\mathsf{k}}$;
    - $\mathsf{H}_1 \cap i\mathsf{H}_2 = \{0\}$.

Let Σ be the closure of Σ^0.

Finally, let $\mathcal{A}$ be a von Neumann algebra acting on h with a cyclic and separating vector υ and N_Σ be the von Neumann algebra acting on $\mathcal{F}^{\mathsf{k}\oplus\overline{\mathsf{k}}}$ generated by the modified Weyl operators:

$$W(f) = W_0(\Sigma\iota(f)), \quad f \in L^2(\mathbb{R}_+; \mathsf{k}).$$

This noise algebra has, by the properties of Σ, Ω as its cyclic and separating vector, so the von Neumann algebra $A \overline{\otimes} N_\Sigma$ has a cyclic and separating vector $\upsilon \otimes \Omega =: \eta$. The commutant of the noise algebra can be represented using a conjugate operator Σ', obtained from Σ via modular conjugation. Thus

$$N'_\Sigma = N_{\Sigma'}.$$

We denote

$$M_W := \{a \otimes W(f) \colon a \in A, f \in L^2(\mathbb{R}_+; \mathsf{k})\}, \quad M'_W := \{a \otimes W'(f) \colon a \in A', f \in L^2(\mathbb{R}_+; \mathsf{k})\}.$$

A.2 Multiple quasifree Wiener integrals

A.2.1 Quasifree vector kernels

We recall the quasifree setup and notation from Chapter 3. To apply it in our case, we need some further notation.

Let $V^{(1)}$ denote the inclusion map $\mathsf{k} \oplus \overline{\mathsf{k}} \to \mathcal{F}^{\mathsf{k}\oplus\overline{\mathsf{k}}}$ and $s_\Omega = (V^{(1)})^* S_\Omega V^{(1)}$, where S_Ω denotes the Tomita-Takesaki sharp operator on N_Σ. On $\mathsf{k} \oplus \overline{\mathsf{k}}$ let π denote the sum flip $\pi\colon \mathsf{k} \oplus \overline{\mathsf{k}} \to \overline{\mathsf{k}} \oplus \mathsf{k}$. Recall the following proposition ([55], Thm. 4.2):

Proposition A.2.1.

$$s_\Omega \Sigma = \Sigma K^\pi.$$

Corollary A.2.2.

$$S_\Omega \pi_\Sigma = \pi_{\Sigma K^\pi}.$$

Proof. Follows from the fact that $S_\Omega = \Gamma(s_\Omega)$ (cf. [55]). □

For ease of use, let S_{h} denote the Tomita-Takesaki sharp operator on A and let S be the sharp operator on $A \overline{\otimes} N_\Sigma$, so that $S = S_{\mathsf{h}} \otimes S_\Omega$.

Let $K(\mathsf{h}, \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}})$ denote the family of measure equivalence classes of measurable $\widehat{\mathsf{k} \oplus \overline{\mathsf{k}}}$-valued vector kernels. We define

$$L^{1,2}(K(\mathsf{h}, \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}})) := \left\{ \xi \in K(\mathsf{h}, \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}})\colon \|\xi\|_{1,2} := \int_\Gamma d\beta \sqrt{\int_\Gamma d\alpha \| J^*_{\alpha;\alpha\cup\beta} \xi(\alpha \cup \beta)\|^2} < \infty \right\},$$

$$\mathrm{Dom}(\widehat{\mathcal{W}}^\Sigma) := \left\{ \xi \in K(\mathsf{h}, \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}})\colon \pi_{\widehat{\Sigma}} \xi \in L^{1,2}(K(\mathsf{h}, \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}})), \forall \beta \in \Gamma J^*_{\cdot;\cdot\cup\beta} \xi(\cdot \cup \beta) \in \mathrm{Dom}(S) \right\}.$$

Definition A.2.3. For $\xi \in \mathrm{Dom}(\widehat{\mathcal{W}}^\Sigma)$ we define

$$\widehat{\mathcal{W}}^\Sigma(\xi)(\alpha) = \int_\Gamma d\beta J^*_{\alpha;\alpha\cup\beta} (\pi_{\widehat{\Sigma}} \xi)(\alpha \cup \beta).$$

Remark A.2.4. 1. We can see from Corollary A.2.2 that $\widehat{\mathcal{W}}^{\Sigma}(\xi)$ is $\mathrm{Dom}(S)$-valued.

2. It is also easily seen that, since Σ is bounded,

$$\widehat{\mathcal{W}}^{\Sigma}(\xi) = \pi_{\widehat{\Sigma}} \widehat{\mathcal{W}}(\xi).$$

3. If we assume Σ is a time-independent bounded operator, then we only need to require $\xi \in L^{1,2}(K(\mathsf{h}, \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}}))$.

4. If ξ is $\mathrm{Dom}(S)$-valued, then

$$\widehat{\mathcal{W}}^{\Sigma}(\xi) = \pi_{\Sigma} \xi.$$

5. It is easily seen that the following version of the First Fundamental Formula holds:

$$\left\langle W'(f)u\Omega, \widehat{\mathcal{W}}^{\Sigma}(\xi) \right\rangle = \int_{\Gamma} d\sigma \left\langle \varpi(\widehat{\Sigma} \circ \widehat{\imath}(f))(\sigma), \pi_{\widehat{\Sigma}}(\sigma)\xi(\sigma) \right\rangle,$$

which for ξ supported only on compact intervals simplifies to

$$\left\langle W'(f)u\Omega, \widehat{\mathcal{W}}^{\Sigma}(\xi) \right\rangle = \left\langle \widehat{W'(f)u}\Omega, \pi_{\widehat{\Sigma}} \xi \right\rangle.$$

6. Since $\widehat{\mathcal{W}}(\xi)$ is $\mathrm{Dom}(S)$-valued, there exists an adjointable operator X_ξ affiliated to M with the property that

$$X_\xi \eta = \widehat{\mathcal{W}}^{\Sigma}(\xi).$$

This operator is closable, adjointable and $M'_W \eta$ is a core for both X_ξ and X^*_ξ.

Remark A.2.5. Equivalently, our operator X_ξ could be defined on the domain $(A\overline{N}_\Sigma)'$.

A.2.2 Operator kernels

Before we define the quantum Wiener operator integral, we need a partial transpose and convolution for column operators.

Let $\sigma \in \Gamma$. Then

$$x(\sigma) \colon \mathsf{h} \supset D \to \mathsf{h} \otimes \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}}^{\otimes \# \sigma},$$

so

$$(x(\sigma)) \in O_M(D \otimes \mathcal{F}^{\mathsf{k}\oplus\overline{\mathsf{k}}}; B(\mathbb{C}; \mathsf{h} \otimes \left(\widehat{\mathsf{k} \oplus \overline{\mathsf{k}}}\right)^{\otimes \# \sigma} \otimes \mathcal{F}^{\mathsf{k}\oplus\overline{\mathsf{k}}})),$$

so we can define for $\alpha \subset \sigma$

$$x^{T(\alpha)}(\sigma) := [\Delta[\alpha; \sigma] x(\sigma)]_{T(\alpha)} \in O_M(D; HS((\overline{\mathsf{k}} \oplus \mathsf{k})^{\otimes \# \alpha}; \mathsf{h} \otimes \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}}^{\otimes \#(\sigma \setminus \alpha)}))$$

via the column partial transpose of Margetts and Lindsay, where we treat $\mathbb{C}$ as k_1, $\mathsf{k} \oplus \overline{\mathsf{k}}^{\otimes \# \alpha}$ as k_2 and the rest as H. Thus for example, if $\sigma = \{s_1 < s_2\}, \alpha = \{s_1\}, \xi, \eta \in \mathsf{k} \oplus \overline{\mathsf{k}}, u, v \in \mathsf{h}$, then

$$\left\langle v\eta, x^{T(\alpha)}(\sigma) u\overline{\xi} \right\rangle = \langle v\xi\eta, x(\sigma)u \rangle .$$

We will need to supercede the convolution notation introduced before. We modify it as follows.

Definition A.2.6. For two operator kernels x, y and matrices Σ_1, Σ_2 we define their convolution by:

$$x \;{}_{\Sigma_1}\!\star_{\Sigma_2} y(\sigma)v := \sum_{\alpha \cup \beta = \sigma} \left(x^{T(\alpha\cap\beta)}(\alpha; \sigma) \Pi_{\alpha\cap\beta} \Big((\Sigma_1^* \Delta \Sigma_2)[\alpha \cap \beta; \sigma] y(\beta; \sigma) v \Big); \sigma \right),$$

where $v \in D$ and $\Pi_{\alpha\cap\beta}$ is the sum flip $\mathsf{k} \oplus \overline{\mathsf{k}} \to (\overline{\mathsf{k}} \oplus \mathsf{k})$ on the tensor components belonging to α. Similarly we define the convolution for an operator kernel x and vector kernel ξ:

$$x\ {}_{\Sigma}\!\star\ \xi(\sigma) := \sum_{\alpha\cup\beta=\sigma}\Big(\widehat{\Sigma}(\alpha\setminus\beta;\sigma)x^{T(\alpha\cap\beta)}(\alpha;\sigma)\Pi_{\alpha\cap\beta}\Big((\Sigma^*\Delta)[\alpha\cap\beta;\sigma]\xi(\beta)\Big);\sigma\Big).$$

Definition A.2.7. For a column kernel x coming from a vector kernel $\xi \in \mathrm{Dom}(\widehat{\mathcal{W}}^{\Sigma})$ we define the multiple quantum Wiener integral of x at time $t \geqslant 0$ as an unbounded operator by:

$$\mathrm{Dom}(\mathcal{Q}_t^{\Sigma}(x)) = \{\zeta \in \mathsf{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}} : (x\mathbf{1}_{\Gamma_t}\ {}_{\Sigma}\!\star\ \zeta) \in L^{1,2}(K(\mathsf{h},\widehat{\mathsf{k}\oplus\bar{\mathsf{k}}}))\},$$

$$\mathcal{Q}_t^{\Sigma}(x)\xi(\sigma) = \widehat{\mathcal{W}}(x\mathbf{1}_{\Gamma_t}\ {}_{\Sigma}\!\star\ \xi)(\sigma).$$

We note the similarity to Definition 3.1.5.

Definition A.2.8. Let $f \in L^2(\mathbb{R}_+;\mathsf{k})$ and $C_f = \begin{pmatrix} -\frac{1}{2}\|\Sigma\iota f(s)\|^2 \\ |f(s)\rangle \\ -|\overline{f(s)}\rangle > \end{pmatrix}$. Let $x_f = I_{\mathsf{h}} \otimes \pi_{C_f}$.

Proposition A.2.9. *With above notations, for* $\mathsf{h} = \mathbb{C}$*, an arbitrary operator kernel* x *coming from a vector* ξ *and function* $f \in L^2(\mathbb{R}_+;\mathsf{k})$ *we can extend* $\mathcal{Q}^{\Sigma}(x_f)$ *to a bounded operator, which is exactly* $W(f)$ *(thus also* $Q^{\Sigma'}(x_f)$ *can be extended to* $W'(f)$*), and*

$$x\ {}_{\Sigma}\!\star_{\Sigma'}\ x_f = x_f\ {}_{\Sigma'}\!\star_{\Sigma}\ x.$$

Proof. It suffices to show that $\varpi(\Sigma\iota(g)) \in \mathrm{Dom}(\mathcal{Q}^{\Sigma}(x_f))$ for all $g \in L^2(\mathbb{R}_+;\mathsf{k})$ and

$$\mathcal{Q}^{\Sigma}(x_f)\varpi(\Sigma\iota(g)) = e^{-i\,\mathrm{Im}\langle\Sigma\iota f,\Sigma\iota g\rangle}\varpi(\Sigma\iota(f+g)).$$

We will now omit the placement notation - all the following operators or vectors are placed within $\alpha\cup\beta$ as necessary.

We have:

$$\begin{aligned}
\mathcal{Q}^{\Sigma}(x_f)\varpi(\Sigma\iota(g)) &= \int_{\Gamma} d\beta\, J^*_{\alpha;\alpha\cup\beta}(x_f \,{}_{\Sigma}\!\star\, \varpi(\Sigma\iota(g)))(\alpha\cup\beta) \\
&= \int_{\Gamma} d\beta\, J^*_{\alpha;\alpha\cup\beta}\left(\sum_{\beta_0\sqcup\beta_1\sqcup\beta_2=\alpha\cup\beta} \Sigma(\beta_0)x_f^{T(\beta_1)}(\beta_0\cup\beta_1)\Pi_{\beta_1}((\Sigma^*\Delta)(\beta_1)\varpi(\Sigma\iota(g))(\beta_1\cup\beta_2)\right) \\
&= \int_{\Gamma} d\beta\, J^*_{\alpha;\alpha\cup\beta}\left(\sum_{\beta_0\sqcup\beta_1\sqcup\beta_2=\alpha\cup\beta} \bigotimes_{s\in\beta_0}\begin{pmatrix}-\frac{1}{2}\|\Sigma\iota f(s)\|^2\\ \Sigma\iota(f)\end{pmatrix} \bigotimes_{s\in\beta_1}\begin{pmatrix}-\langle\Sigma\iota(g(s),\Sigma\iota(f(s))\rangle\\ 0\end{pmatrix}\right. \\
&\qquad\qquad \left.\bigotimes_{s\in\beta_2}\Sigma\iota(g(s))\right),
\end{aligned}$$

where the last equality follows from the fact that

$$A^T E_c = E^{\overline{c}}A.$$

(cf. [27]) Thus for $s\in\beta_1$ we obtain

$$\begin{aligned}
x_f^T(s)\Pi(\Sigma^*\Delta\Sigma\iota(g(s))) &= E^{K\Pi(\Sigma^*\Delta\Sigma\iota(g(s)))}x_f(s) = E^{\Sigma^*\Delta\Sigma\iota(g(s))}K\Pi(x_f(s)) \\
&= E^{\Delta\Sigma\iota(g(s))}(-\Sigma x_f(s)) = \begin{pmatrix}-\langle\Sigma\iota(g(s)),\Sigma\iota(f(s))\rangle\\ 0\end{pmatrix}.
\end{aligned}$$

Now, it is easily seen that such a sum over all partitions is just a product vector of the sum of the components. Applying our J^* operator, we get

$$\begin{aligned}
\mathcal{Q}^{\Sigma}(x_f)\varpi(\Sigma\iota(g)) &= \int_{\Gamma} d\beta\varepsilon\left(-\frac{1}{2}\|\Sigma\iota f(\cdot)\|^2 - \langle\Sigma\iota(g(\cdot)),\Sigma\iota(f(\cdot))\rangle - \frac{1}{2}\|\Sigma\iota g(\cdot)\|^2\right)(\beta) \\
&\cdot\varepsilon(\Sigma\iota(f+g))(\alpha) = e^{-\frac{1}{2}(\|\Sigma\iota f\|^2+\|\Sigma\iota g\|^2+2\langle\Sigma\iota g,\Sigma\iota f\rangle)}\varepsilon(\Sigma\iota(f+g))(\alpha) \\
&= e^{-i\,\mathrm{Im}\langle f,g\rangle}\varepsilon(\Sigma\iota(f+g))(\alpha),
\end{aligned}$$

by the fact that $\Sigma\iota$ is symplectic.

Thus $\mathcal{Q}^{\Sigma}(x_f) = W(f)$ on exponential vectors and thus $W(f)$ is a bounded extension of $\mathcal{Q}^{\Sigma}(x_f)$.

To show the second part, we will make use of the fact that

$$K\Pi(\Sigma^*\Sigma'\iota'(g(s))) = -\Sigma^*\Sigma'\iota'(g(s)) \tag{A.1}$$

(cf. Lemma 1.2 of [56]). It implies that

$$K\Pi(\Sigma^*\Delta\Sigma' x_g(s)) = \Sigma^*\Sigma' x_{-g}(s).$$

Also we see that

$$\Pi(x_f^T(s)) = x_{-f}^*(s).$$

We have (again, omitting the placement notation and placing all vectors within σ):

$$\begin{aligned}
x \;{}_{\Sigma}\!\star_{\Sigma'}\; x_f(\sigma) &= \sum_{\alpha\cup\beta=\sigma} x^{T(\alpha\cap\beta)}(\alpha)\Pi_{\alpha\cap\beta}((\Sigma^*\Delta\Sigma')(\alpha\cap\beta)x_f(\beta)) \\
&= \sum_{\alpha\cup\beta=\sigma} E^{K\Pi((\Sigma^*\Delta\Sigma')(\alpha\cap\beta)x_f(\alpha\cap\beta))} x(\alpha)x_f(\beta\backslash\alpha) \\
&= \sum_{\alpha\cup\beta=\sigma} E^{-\Sigma^*\Delta\Sigma'(\alpha\cap\beta)x_f(\alpha\cap\beta)} x_f(\beta\backslash\alpha)x(\alpha) \\
&= \sum_{\alpha\cup\beta=\sigma} x_f(\beta\backslash\alpha)(-x_f^*\Sigma'^*\Delta\Sigma)(\alpha\cap\beta)x(\alpha) \\
&= \sum_{\alpha\cup\beta=\sigma} x_f(\beta\backslash\alpha)(x_{-f}^*\Sigma'^*\Delta\Sigma)(\alpha\cap\beta)x(\alpha) \\
&= \sum_{\alpha\cup\beta=\sigma} x_f(\beta\backslash\alpha)(\Pi(x_f^T)\Sigma'^*\Delta\Sigma)(\alpha\cap\beta)x(\alpha) \\
&= \sum_{\alpha\cup\beta=\sigma} x_f^{T(\alpha\cap\beta)}(\beta)\Pi(\alpha\cap\beta)((\Sigma'^*\Delta\Sigma)(\alpha\cap\beta)x(\alpha)) \\
&= x_f \;{}_{\Sigma'}\!\star_{\Sigma}\; x(\sigma).
\end{aligned}$$

$\square$

Corollary A.2.10. *For x as above, $\mathcal{Q}^\Sigma(x)$ is affiliated to $M = A \otimes N_\Sigma$ and thus in particular*

$$(A \otimes N_\Sigma)'\eta \subset \mathrm{Dom}(\mathcal{Q}^\Sigma(x)).$$

Proof. Let $\zeta = (a \otimes W'(f))\upsilon \otimes \Omega$ for some $a \in A'$, $f \in L^2(\mathbb{R}_+; \mathsf{k})$. Then we can see that $\zeta = Q^{\Sigma'}(a \otimes x_f)\upsilon \otimes \Omega$, so that

$$\begin{aligned}\mathcal{Q}^\Sigma(x)\zeta &= \widehat{\mathcal{W}}(x \;{}_\Sigma\!\star\; (\pi_{\Sigma'}\widehat{\mathcal{W}}(a\upsilon \otimes x_f\Omega))) = \widehat{\mathcal{W}}(x \;{}_\Sigma\!\star_{\Sigma'}\; (a\upsilon \otimes x_f\Omega)) \\ &= \widehat{\mathcal{W}}((a \otimes x_f) \;{}_{\Sigma'}\!\star_\Sigma\; x(\upsilon)) = a \otimes W'(f)(x(\cdot)(\upsilon)).\end{aligned}$$

Thus the convergence of right hand side implies convergence of the left hand side and thus $\zeta \in \mathrm{Dom}(\mathcal{Q}^\Sigma(x))$. Since operators of the form $a \otimes W'(f) \in A' \otimes N'_\Sigma$ generate $(A \otimes N_\Sigma)'$, we obtain that $\mathcal{Q}^\Sigma(x)$ is affiliated to M. □

Corollary A.2.11. *For x as above and $\zeta \in (A \otimes N_\Sigma)'\eta$ we have*

$$\mathcal{Q}^\Sigma(x)\zeta = X_\xi\zeta.$$

Proof. That is an immediate corollary from the previous observations and the fact that

$$\mathcal{Q}^\Sigma(x)\eta = X_\xi\eta.$$

□

To combine column kernels in greater generality we need the following conjugation operation:

Definition A.2.12. For a column kernel x coming from a vector $\xi \in \mathrm{Dom}(\widehat{\mathcal{W}}^\Sigma)$ we define

$$x^\dagger(\sigma) := (\Pi(x^{T(\sigma)}(\sigma)))^*.$$

Theorem A.2.13. *For such column kernels x, we have*

$$\mathcal{Q}^{\Sigma}(x^{\dagger})\upsilon\Omega = S\widehat{\mathcal{W}}^{\Sigma}(\xi),$$

$\mathrm{Dom}(\mathcal{Q}^{\Sigma}(x^{\dagger}) \subset \mathrm{Dom}((\mathcal{Q}^{\Sigma}(x)^{*}))$ *and*

$$\mathcal{Q}^{\Sigma}(x^{\dagger})\zeta = (\mathcal{Q}^{\Sigma}(x))^{*}\zeta$$

for $\zeta \in \mathrm{Dom}(\mathcal{Q}^{\Sigma}(x^{\dagger})$.

Proof. We omit the placement notation.

$$\begin{aligned}
&\left\langle \mathcal{Q}^{\Sigma}(x^{\dagger})\ (A \otimes W'(f))\eta, \eta\right\rangle = \\
&\quad = \int_{\Gamma} d\alpha \int_{\Gamma} d\beta \left\langle J^{*}_{\alpha;\alpha\cup\beta}\Sigma(\alpha\cup\beta)(\Pi(x^{T(\alpha\cup\beta)}(\alpha\cup\beta))^{*}\upsilon, (A^{*}\otimes W'(-f))\eta\right\rangle \\
&\quad = \int_{\Gamma} d\alpha \int_{\Gamma} d\beta \left\langle \upsilon, \Pi(x^{T(\alpha\cup\beta)}(\alpha\cup\beta))\Sigma^{*}(\alpha\cup\beta)J_{\alpha;\alpha\cup\beta}(A^{*}\otimes W'(-f))(\eta)(\alpha)\right\rangle \\
&\quad = \int_{\Gamma} d\alpha \int_{\Gamma} d\beta \left\langle \upsilon, x^{T(\alpha\cup\beta)}(\alpha\cup\beta)\Pi(\Sigma^{*}(\alpha\cup\beta)J_{\alpha;\alpha\cup\beta}(A^{*}\otimes W'(-f))(\eta)(\alpha))\right\rangle \\
&\quad = \int_{\Gamma} d\alpha \int_{\Gamma} d\beta \left\langle A\upsilon \otimes (K\Pi(\Sigma^{*}(\alpha\cup\beta)J_{\alpha;\alpha\cup\beta}W'(-f)\Omega(\alpha))), x(\alpha\cup\beta)\upsilon\right\rangle \\
&\quad = \int_{\Gamma} d\alpha \int_{\Gamma} d\beta \left\langle A\upsilon \otimes (\Sigma^{*}(\alpha\cup\beta)J_{\alpha;\alpha\cup\beta}W'(f)\Omega(\alpha)), x(\alpha\cup\beta)\upsilon\right\rangle \\
&\quad = \int_{\Gamma} d\alpha \int_{\Gamma} d\beta \left\langle A\upsilon \otimes W'(f)\Omega(\alpha)), J^{*}_{\alpha;\alpha\cup\beta}\Sigma(\alpha\cup\beta)x(\alpha\cup\beta)\upsilon\right\rangle \\
&\quad = \int_{\Gamma} d\alpha \left\langle (A\otimes W'(f))(\eta)(\alpha), \mathcal{Q}^{\Sigma}(x)\eta\right\rangle.
\end{aligned}$$

Thus in particular, by definition of the sharp operator and the fact that $Q^{\Sigma}(x^{\dagger}), (\mathcal{Q}^{\Sigma}(x))^{*}$ coincide on η, we obtain that

$$\mathcal{Q}^{\Sigma}(x^{\dagger})\upsilon\Omega = S\widehat{\mathcal{W}}^{\Sigma}(\xi).$$

□

Proposition A.2.14. *Let* $v\varepsilon(\Sigma'\iota(g)) \in \mathrm{Dom}(\mathcal{Q}_t^\Sigma(x)), u\varepsilon(\Sigma'(\iota(f)) \in \mathsf{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\overline{\mathsf{k}}}$*. Then we have:*

$$\langle u\varepsilon(\Sigma'(\iota(f)), \mathcal{Q}_t^\Sigma(x)v\varepsilon(\Sigma'\iota(g))\rangle = e^{\langle \Sigma'\iota(f), \Sigma'\iota(g)\rangle} \int_{\Gamma_t} d\sigma \langle I_{\mathsf{h}} \otimes \widehat{\Sigma'} u\widehat{\iota}(f-g)(\sigma), I_{\mathsf{h}} \otimes \Sigma x(\sigma)v\rangle,$$

which upon normalisation becomes, by the fact that Σ' *is symplectic:*

$$\langle uW'(f), \mathcal{Q}_t^\Sigma(x)v\widehat{\mathcal{W}}'(g)\rangle = e^{i\,\mathrm{Im}\langle f,g\rangle} \int_{\Gamma_t} d\sigma \langle I_{\mathsf{h}} \otimes \widehat{\Sigma'} u\widehat{\iota}(f-g)(\sigma), I_{\mathsf{h}} \otimes x(\sigma)v\rangle,$$

Proof. We will again make use of the identity in Equation A.1. We write, omitting placement notation as usual:

$$\langle u\varepsilon(\Sigma'(\iota(f)), \mathcal{Q}_t^\Sigma(x)vv\varepsilon(\Sigma'\iota(g))\rangle = \int_\Gamma d\alpha \int_{\Gamma_t} d\beta \sum_{\beta_0 \sqcup \beta_1 \sqcup \beta_2 = \alpha\cup\beta}$$
$$\langle J_{\alpha;\alpha\cup\beta} u\varepsilon(\Sigma'(\iota(f)), \widehat{\Sigma}(\beta_0;\sigma) x\mathbf{1}_{\Gamma_t}^{T(\beta_1)}(\beta_0\cup\beta_1;\sigma) \prod_{\beta_1} ((\Sigma^*(\beta_1)v\varepsilon(\Sigma'\iota(g))(\beta_1\cup\beta_2)) .$$

We see that the β_2 part is untouched by either the transposition or sum flip operation. As we are dealing with product vectors, we can take that out of the equation and it is easily seen that is where the $e^{\langle \Sigma'\iota(f), \Sigma'\iota(g)\rangle}$ part of our identity will come from (cf. Proposition 3.3.1 for an example of a similar reasoning). Thus now our sum takes the form

$$\sum_{\beta_0\sqcup\beta_1=\alpha\cup\beta} = \sum_{\delta\subset\alpha\cup\beta}$$

with δ taking the role of our β_1. But it is easily seen that if $\delta\cap\alpha \neq 0$, then on the right hand side we'll have an expression in $\mathsf{k}\oplus\overline{\mathsf{k}}^\perp$ (due to the transpose of x), while on the left we have something in $\mathsf{k}\oplus\overline{\mathsf{k}}$ (as we are in α). Thus in fact, we merely need to take $\delta\subset\beta$. In this new form, using Equation A.1 and the fact that $x\mathbf{1}_{\Gamma_t}$ is supported on Γ_t, we obtain

$$LHS = \int_{\Gamma_t} d\alpha \int_{\Gamma_t} d\beta \sum_{\delta\subset\beta} \langle u\varepsilon(h)(\alpha\cup\beta), x(\alpha\cup\beta)v\rangle,$$

where

$$h(s) = \begin{cases} \Sigma^*\Sigma'\iota(f)(s) & s\in\alpha \\ \Sigma^*\binom{1}{0} & s\in\beta\setminus\delta \\ \Sigma^*\Sigma'\iota(-g)(s) & s\in\delta. \end{cases}$$

But by integral-sum identity, this is equal to

$$\int_{\Gamma_t} \langle I_{\mathsf{h}}\otimes\widehat{\Sigma'}u\widehat{\iota}(f-g)(\sigma), I_{\mathsf{h}}\otimes\Sigma(\sigma)x(\sigma)v\rangle,$$

as required.

The normalisation part is easily seen by the fact that Σ' is symplectic. □

Theorem A.2.15. *Let x, y be column kernels coming from Σ-time-Wiener integrable vectors ξ_x, ξ_y, respectively. If $\widehat{\mathcal{W}}^\Sigma(\xi_y)\in\mathrm{Dom}(\mathcal{Q}^\Sigma(x^\dagger))^*$, then the vector*

$$(x\ {}_\Sigma{\star}_\Sigma\ y)\eta$$

is Σ-time-Wiener integrable and

$$(\mathcal{Q}^\Sigma(x^\dagger))^*\mathcal{Q}^\Sigma(y)\eta = \mathcal{Q}^\Sigma(x\ {}_\Sigma{\star}_\Sigma\ y)\eta.$$

Proof. Let us first expand the left hand side and the right hand side from the respective definitions.

$$LHS = \left\langle \mathcal{Q}^\Sigma(x^\dagger)(A\otimes W'(f))\eta, \mathcal{Q}^\Sigma(y)\eta\right\rangle$$

$$= \int_\Gamma d\alpha \int_\Gamma d\beta \int_\Gamma d\gamma \Big\langle J^*_{\alpha;\alpha\cup\beta} \Big(\sum_{\beta_0\cup\beta_1=\alpha\cup\beta} \widehat{\Sigma}(\beta_0 \backslash \beta_1; \alpha\cup\beta)(\Pi_{\beta_0}$$

$$\Big(x^{T(\beta_0)}(\beta_0))^*; \alpha\cup\beta\Big)^{T(\beta_0\cap\beta_1;\alpha\cup\beta)} \Pi_{\beta_0\cap\beta_1;\alpha\cup\beta}(\Sigma^*(\beta_0\cap\beta_1;\alpha\cup\beta)[(A\otimes W'(f))\eta(\beta_1;\alpha\cup\beta)]\Big),$$

$$J^*_{\alpha;\alpha\cup\gamma}\widehat{\Sigma}(\alpha\cup\gamma)y(\alpha\cup\gamma)\upsilon\Big\rangle$$

$$RHS = \Big\langle (A\otimes W'(f))\eta, \mathcal{Q}^\Sigma(x_\Sigma \star_\Sigma)\eta\Big\rangle$$

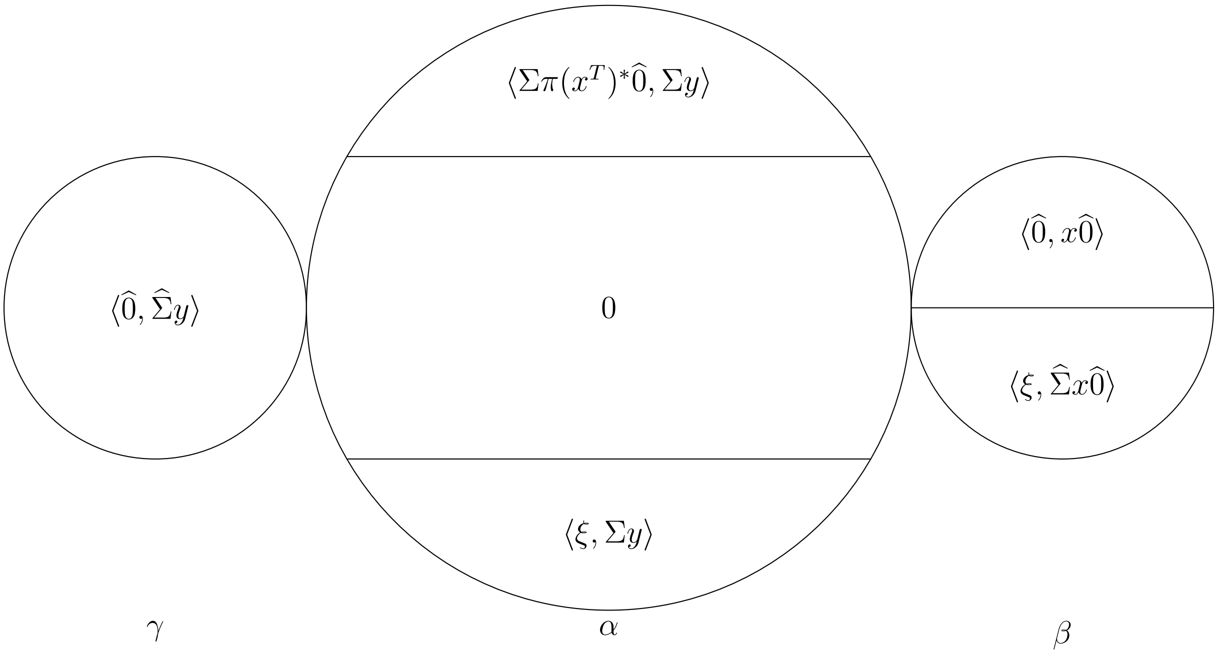
$$= \int_\Gamma d\alpha \int_\Gamma d\beta \sum_{\beta_0\cup\beta_1=\alpha\cup\beta} \Big\langle A\upsilon\otimes W'(f)(\Omega)(\alpha), J^*_{\alpha;\alpha\cup\beta}\Sigma(\beta_0\setminus\beta_1;\alpha\cup\beta)x^{T(\beta_0\cap\beta_1)}(\beta_0\cap\beta_1;\alpha\cup\beta)$$

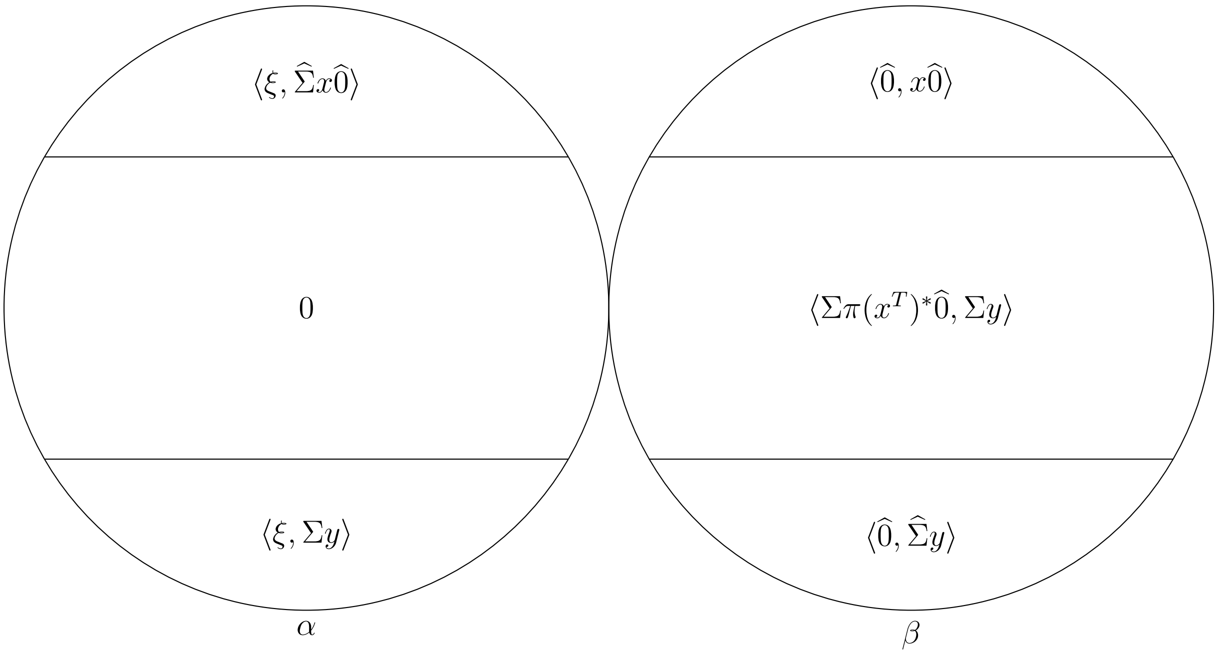
$$\Pi_{\beta_0\cap\beta_1}\left((\Sigma^*\Delta\Sigma)(\beta_0\cap\beta_1;\alpha\cup\beta)\Sigma(\beta_1\setminus\beta_0;\alpha\cup\beta)y(\beta_1;\alpha\cup\beta)\right)\rangle$$

The main idea of the calculation to follow can be explained via the following pair of diagrams. The first diagram symbolises the left hand side and how it decomposes into a sum over different partitions of the set $\alpha\cup\beta\cup\gamma$. In each part we write which operators are acting on the relevant tensor components.

The second diagram analogously portrays the right hand side of the equation.

It is easily seen from the diagrams that the equality holds via a relabeling of variables. This is, of course, merely a heuristic - in particular, we still need to justify the form of some of these operations on the relevant tensor components. Thus let us begin the detailed calculation. Firstly, we use the fact that $(A\otimes W'(f))\eta$ commutes with our quantum stochastic integral, and thus we can take it outside and then move it to the right hand side of the equation. Due to the interaction of $\Sigma^*\Sigma'$ with the partial transpose, we can write it as follows.





$$\Big\langle \mathcal{Q}^\Sigma(x^\dagger)\ (A\otimes W'(f))\eta, \mathcal{Q}^\Sigma(y)\eta\Big\rangle$$
$$= \int_\Gamma d\alpha \int_\Gamma d\beta \int_\Gamma d\gamma \Big\langle J^*_{\alpha;\alpha\cup\beta}\Bigg(\sum_{\beta_0\cup\beta_1=\alpha\cup\beta} \widehat{\Sigma}(\beta_0\setminus\beta_1;\alpha\cup\beta)(\Pi_{\beta_0}$$

$$\Big(x^{T(\beta_0)}(\beta_0))^*;\alpha\cup\beta\Big)^{T(\beta_0\cap\beta_1;\alpha\cup\beta)}\Pi_{\beta_0\cap\beta_1;\alpha\cup\beta}(\Sigma^*(\beta_0\cap\beta_1;\alpha\cup\beta)$$
$$[(A\otimes W'(f))\eta(\beta_1;\alpha\cup\beta)]),J^*_{\alpha;\alpha\cup\gamma}\widehat{\Sigma}(\alpha\cup\gamma)y(\alpha\cup\gamma)v\Big\rangle$$
$$=\int_\Gamma d\alpha\int_\Gamma d\beta\int_\Gamma d\gamma\sum_{\beta_0\cup\beta_1=\alpha\cup\beta}\Big\langle A\otimes W'(f)\eta(\beta_1\backslash\beta_0),\Pi(x^{T(\beta_0)}(\beta_0;\alpha\cup\beta))$$
$$\widehat{\Sigma}^*(\beta_0\backslash\beta_1;\alpha\cup\beta)J_{\alpha;\alpha\cup\beta}J^*_{\alpha;\alpha\cup\gamma}\widehat{\Sigma}(\alpha\cup\gamma)y(\alpha\cup\gamma)v\otimes\varpi(\Sigma^*\Sigma'\iota(-f))(\beta_0\cap\beta_1)\Big\rangle.$$

Continuing, we can now move the $x^\dagger$ to the right hand side, using the definition of the adjoint and partial transpose.

$$\Big\langle\mathcal{Q}^\Sigma(x^\dagger)\ (A\otimes W'(f))\eta,\mathcal{Q}^\Sigma(y)\eta\Big\rangle$$
$$=\int_\Gamma d\alpha\int_\Gamma d\beta\int_\Gamma d\gamma\sum_{\beta_0\cup\beta_1=\alpha\cup\beta}\langle Av\otimes(\Sigma^*(\beta_0\cap\beta_1;\beta_1)W'(f)(\beta_1)),$$
$$x^{T(\beta_0\backslash\beta_1)}(\beta_0;\alpha\cup\beta)\Pi_{\beta_0\backslash\beta_1}\Big(\widehat{\Sigma}^*(\beta_0\backslash\beta_1;\alpha\cup\beta)J_{\alpha;\alpha\cup\beta}J^*_{\alpha;\alpha\cup\gamma}\widehat{\Sigma}(\alpha\cup\gamma)y(\alpha\cup\gamma)v\Big)\Big\rangle$$
$$=\int_\Gamma d\alpha\int_\Gamma d\beta\int_\Gamma d\gamma\sum_{\beta_0\cup\beta_1=\alpha\cup\beta}\langle Av\otimes W'(f)(\Omega)(\beta_1),$$
$$\Sigma(\beta_0\cap\beta_1;\alpha\cup\beta)x^{T(\beta_0\backslash\beta_1)}(\beta_0;\alpha\cup\beta)\Pi_{\beta_0\backslash\beta_1}$$
$$\Big(\widehat{\Sigma}^*(\beta_0\backslash\beta_1;\alpha\cup\beta)J_{\alpha;\alpha\cup\beta}J^*_{\alpha;\alpha\cup\gamma}\widehat{\Sigma}(\alpha\cup\gamma)y(\alpha\cup\gamma)v\Big)\Big\rangle$$

Relabeling the variables and using the properties of J, we can now finish the calculation using only elementary steps.

$$\Big\langle\mathcal{Q}^\Sigma(x^\dagger)\ (A\otimes W'(f))\eta,\mathcal{Q}^\Sigma(y)\eta\Big\rangle$$
$$=\int_\Gamma d\sigma\sum_{\alpha\sqcup\beta\sqcup\gamma=\sigma}\ \sum_{\beta_0\cup\beta_1=\alpha\cup\beta}\langle Av\otimes W'(f)(\Omega)(\beta_1),$$
$$J^*_{\beta_1;\sigma}\Sigma(\beta_0\cap\beta_1;\sigma)x^{T(\beta_0\backslash\beta_1)}(\beta_0;\sigma)\Pi_{\beta_0\backslash\beta_1}\Big(\widehat{\Sigma}^*(\beta_0\backslash\beta_1;\sigma)\Delta[\alpha;\sigma]\widehat{\Sigma}(\alpha\cup\gamma)y(\alpha\cup\gamma)v\Big)\Big\rangle$$
$$=\int_\Gamma d\sigma\sum_{\delta_0\sqcup\ldots\sqcup\delta_4=\sigma}\langle Av\otimes W'(f)(\Omega)(\delta_0\cup\delta_2),$$
$$J^*_{\delta_0\cup\delta_2;\sigma}\Sigma(\delta_0;\sigma)x^{T(\delta_1\cup\delta_3)}(\delta_0\cup\delta_1\cup\delta_3;\sigma)\Pi_{\delta_3}\left((\Sigma^*\Delta\Sigma)(\delta_3;\sigma)y(\delta_2\cup\delta_3\cup\delta_4;\sigma)\right\rangle$$

$$= \int_\Gamma d\alpha \int_\Gamma d\beta \sum_{\beta_0 \cup \beta_1 = \alpha \cup \beta} \langle Av \otimes W'(f)(\Omega)(\alpha),$$

$$J^*_{\alpha;\alpha\cup\beta} \Sigma(\beta_0 \setminus \beta_1; \alpha \cup \beta) x^{T(\beta_0 \cap \beta_1)}(\beta_0 \cap \beta_1; \alpha \cup \beta)$$

$$\Pi_{\beta_0 \cap \beta_1} \left((\Sigma^* \Delta \Sigma)(\beta_0 \cap \beta_1; \alpha \cup \beta) \Sigma(\beta_1 \setminus \beta_0; \alpha \cup \beta) y(\beta_1; \alpha \cup \beta) \right) \rangle$$

$$= \left\langle (A \otimes W'(f))\eta, \mathcal{Q}^\Sigma(x_\Sigma \star_\Sigma)\eta \right\rangle$$

This ends the proof. □

A.2.3 Product kernels

In this section we will show how this theory specialises in the case of 'product' operator kernels and how that case relates to the classical setup of quantum stochastic calculus, referring to previous sections of this paper. We fix Hilbert spaces h, k.

Let $K\colon \mathbb{R}_+ \to B(\mathsf{h}), L\colon \mathbb{R}_+ \to \mathcal{C}(\mathsf{M}, \mathsf{h}), M\colon \mathbb{R}_+ \to \mathcal{C}_{\overline{\mathsf{k}}}(\mathsf{M}, \mathsf{h})$ be measurable with the property that the partial transposes of L and M are measurable as well. Let

$$F^{Qf} = \begin{pmatrix} K \\ L \\ M \end{pmatrix} : \mathsf{h} \to \mathsf{h} \otimes \widehat{\mathsf{k} \oplus \overline{\mathsf{k}}}.$$

We define the column product kernel x^{Qf} via our usual placement notation:

$$x^{Qf}(\sigma) = F(s_1; \sigma) \cdots F(s_n; \sigma)$$

for $\sigma = \{s_1, \cdots, s_n\}$. To it, we associate a product kernel $x = \pi_F \in OK(B(\mathsf{h}); B(\widehat{\mathsf{k} \oplus \overline{\mathsf{k}}}))$ via

$$F = \begin{pmatrix} K & (M\ L^T)(I_\mathsf{h} \otimes \Sigma^*) \\ (I_\mathsf{h} \otimes \Sigma)\binom{L}{M} & 0 \end{pmatrix}.$$

Theorem A.2.16. *For* $\xi \in \mathrm{Dom}(\mathcal{Q}_t^\Sigma), \xi = v\varepsilon(\Sigma'\iota(g))$ *for* $g \in L^2(\mathbb{R}_+;\bar{\mathsf{k}})$ *we have*

$$\mathcal{Q}_t(x)\xi = \mathcal{Q}_t^\Sigma(x^{Qf})\xi.$$

Proof. It suffices to notice that, taking $X = (I_{\mathsf{h}} \otimes \Sigma)\binom{L}{M}$, we have

$$F = \begin{pmatrix} K & \Pi X^* \\ X & 0 \end{pmatrix}.$$

We now use formulas from Proposition 3.3.1 and Proposition A.2.14. Let $\eta = u\varepsilon(\Sigma'\iota(f))$. From Proposition A.2.14 we obtain

$$\langle \eta, \mathcal{Q}_t^\Sigma(x^{Qf})\xi\rangle = e^{\langle \Sigma'\iota(f), \Sigma'\iota(g)\rangle} \int_{\Gamma_t} d\sigma \langle I_{\mathsf{h}} \otimes \widehat{\Sigma'}\pi_{\widehat{\iota}(f-g)}(\sigma), I_{\mathsf{h}} \otimes \Sigma x^{Qf}(\sigma)v\rangle.$$

On the other hand, it is easily seen that in fact

$$F = I_{\mathsf{h}} \otimes \Sigma F^{Qf} \oplus \Delta(F^{Qf})^T\Sigma^*$$

(where by $\oplus$ we mean a sum upon lifting to $\mathsf{h} \otimes \widehat{\mathsf{k} \oplus \bar{\mathsf{k}}}$), which implies

$$\langle u\varepsilon(\widehat{\iota}(f))(\sigma), x(\sigma)v\varepsilon(\widehat{\iota}(g))(\sigma)\rangle = \sum_{\alpha\subset\sigma} \langle \eta(\sigma), \Pi_{s\in\sigma} y(s)\xi(\sigma)\rangle,$$

where $y(s) = I_{\mathsf{h}} \otimes \Sigma F^{Qf}(s)$ for $s \in \alpha$, $y(s) = \Delta(F^{Qf})^T\Sigma^*$ for $s \in \sigma \setminus \alpha$. This gives But, using

$$AE_c = E^{\bar{c}}A^T$$

and Equation A.1, we get

$$\langle u\varepsilon(\widehat{\iota}(f))(\sigma), x(\sigma)v\varepsilon(\widehat{\iota}(g))(\sigma)\rangle = \sum_{\alpha\subset\sigma} \langle \Sigma^*\Sigma' u\varepsilon(\widehat{\iota}'(f-g))(\sigma), x^{Qf(\sigma)}v\rangle,$$

so we can apply Proposition 3.3.1 to get:

$$\begin{aligned}
\langle \eta, \mathcal{Q}_t(x)\xi\rangle &= e^{\langle \Sigma'\iota(f), \Sigma'\iota(g)\rangle} \int_{\Gamma_t} d\sigma \langle u\pi_{\widehat{\iota'}(f)}(\sigma), x(\sigma) v \pi_{\widehat{\iota'}(g)}(\sigma)\rangle \\
&= e^{\langle \Sigma'\iota(f), \Sigma'\iota(g)\rangle} \int_{\Gamma_t} d\sigma \langle I_{\mathsf{h}} \otimes \widehat{\Sigma'}\pi_{\widehat{\iota}(f-g)}(\sigma), I_{\mathsf{h}} \otimes \Sigma x^{Qf}(\sigma) v\rangle.
\end{aligned}$$

The conclusion now follows by density of vectors of the form $u\varepsilon(\Sigma'\iota(f))$ in $\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}$. □

List of Symbols

We add this list of symbols, both those standard in quantum stochastic calculus and those unique to the thesis, for the convenience of the Reader.

General notation

$\widehat{\mathsf{k}}$ For a Hilbert space k, this denotes Hilbert space $\mathbb{C} \oplus \mathsf{k}$

$\overline{\otimes}$ Ultraweak tensor product

$\underline{\otimes}$ Algebraic tensor product

$\widehat{k}$ For a vector k from a Hilbert space k, this is the vector $\binom{1}{k} \in \widehat{\mathsf{k}}$

$C(X, Y)$ Space of continuous functions from X to Y

$F(X, Y)$ Space of functions from X to Y

Quantum stochastic calculus

Δ Itô projection operator, $\Delta \in B(\widehat{\mathsf{k}})$

$\Gamma(I)$ Guichardet space on interval I

$\Gamma^n(I)$ n-th Cartesian product of the Guichardet space on interval I

$\Gamma^{(n)}(I)$ The subset of $\Gamma^n(I)$ consisting of n-tuples of pairwise disjoint sets.

$\Gamma_n(I)$ n-th Guichardet space on interval I

$\Lambda_t(X)$ Quantum stochastic integral of X up to time t

$\mathcal{F}^{\mathsf{k}}$ Symmetric Fock space over Hilbert space $L^2(\mathbb{R}_+; \mathsf{k})$

$\mathcal{F}^{\mathsf{k}}_I$ Symmetric Fock space over Hilbert space $L^2(I; \mathsf{k})$ for $I \subset \mathbb{R}_+$

$\mathcal{P}_n(\overline{m})$ Family of n-element subsets of $\overline{m} = \{1, \cdots m\}$

$\Phi(H)$ Full Fock space on Hilbert space H

$a(u), a^\dagger(u)$ Annihilation and creation operators on symmetric Fock space (for a vector u)

$A_t(u), A^*_t(u)$ Annihilation and creation processes

$F \lhd G$ Series product,

$$F \lhd G = F + G + F\Delta G$$

$J_{\alpha;\sigma}$ Linear isometry embedding $\mathsf{h} \otimes \mathsf{k}^{\otimes \# \alpha}$ in $\mathsf{h} \otimes \widehat{\mathsf{k}}^{\otimes \# \sigma}$ for given Hilbert spaces h, k and finite subsets $\alpha \subset \sigma$ of $\mathbb{R}_+$

N The number operator on Fock space

$V \otimes_M B(\mathsf{h})$ h-matrix space over V, defined as

$$\{T \in B(H \otimes \mathsf{h})\colon E^x T E_y \in V \text{ for all } x, y \in \mathsf{h}\}$$

$W(u)$ Fock-Weyl operator on symmetric Fock space (for a vector u)

Quantum Wiener chaos

$(x \;{}_Q\!\star_{Q'}\; y)$ Operator - operator convolution, given by

$$(x \;{}_Q\!\star_{Q'}\; y)(\sigma) = \sum_{\alpha \cup \beta = \sigma} x(\alpha; \sigma, Q)\Delta(\alpha \cap \beta; \sigma) y(\beta; \sigma, Q')$$

$(x\ {}_Q\!\star\ \zeta)$ Operator - vector convolution, given by

$$(x\ {}_Q\!\star\ \zeta)(\sigma) = \sum_{\alpha\cup\beta=\sigma} x(\alpha;\sigma,Q)\Delta(\alpha\cap\beta;\sigma)\zeta(\beta;\sigma,e_0)$$

$\mathcal{Q}_t(x)$ Quantum wiener integral of (an appropriate) vector kernel x. Its action is given by

$$\mathcal{Q}_t(x)(\xi) = \widehat{\mathcal{W}}(x\mathbf{1}_{\Gamma_t} \star \xi)$$

$\widehat{\mathcal{W}}$ Multiple time-Wiener integral, operating on (an appropriate) kernel $\zeta \in K(\mathsf{h}, \widehat{\mathsf{k}})$ by

$$(\widehat{\mathcal{W}}\zeta)(\sigma) = \int_\Gamma J^*_{\sigma;\sigma\cup\beta}\zeta(\sigma\cup\beta)d\beta$$

$\zeta(\alpha;\sigma,e_0)$ Placement notation for $\zeta(\alpha) \in \mathsf{h}\otimes H^{\otimes\#\alpha}$ inside $\mathsf{h}\otimes H^{\otimes\#\sigma}$. The e_0 is suppressed in the case $H = \widehat{\mathsf{k}}, e_0 = \binom{1}{0}$

$J^{e_0}_{\alpha;\sigma}$ Linear isometry in $B(H^{\otimes\#\alpha}; H^{\otimes\#\sigma})$, given by linear extension of

$$\pi_\varphi(\alpha) \mapsto \pi_\psi(\sigma),\ \psi = 1_\alpha\varphi + e_0 1_{\mathbb{R}_+\setminus\alpha}, \varphi \in F(\mathbb{R}_+, H)$$

$K(\mathsf{h}, H)$ Linear space of families

$$\{\zeta = (\zeta(\sigma) \in \mathsf{h}\otimes H^{\otimes\#\sigma})_{\sigma\in\Gamma}\}$$

$K_{\text{const}}(\mathsf{h}, H)$ Linear space of families $\{\zeta \in K(\mathsf{h}, H) \colon \zeta = (\zeta_{\#\sigma})_{\sigma\in\Gamma}\}$ for some $\zeta_n \in \mathsf{h}\otimes H^{\otimes n}, n\in\mathbb{N}$

$OK(V, Z)$ Linear space of families

$$\{x = (x(\sigma) \in V\overline{\otimes}Z^{\overline{\otimes}\#\sigma})_{\sigma\in\Gamma}\}$$

$OK_{\text{const}}(V, Z)$ Linear space of families $\{x \in OK(V, Z) \colon x = (x_{\#\sigma})_{\sigma\in\Gamma}\}$ for some $x_n \in V \underline{\otimes} Z^{\overline{\otimes} n}, n \in \mathbb{N}$

$v \otimes \pi_\varphi$ Product vector kernel

$x(\alpha; \sigma, Q)$ Placement notation for $x(\alpha) \in B(\mathsf{h}_1; \mathsf{h}_2) \otimes B(H_1; H_2)^{\otimes \#\alpha}$ inside $B(\mathsf{h}_1; \mathsf{h}_2) \otimes B(H_1; H_2)^{\otimes \#\sigma}$. Q suppressed in the case $H_1 = H_2, Q = I$

References

[1] D. Applebaum. Lévy processes: From probability to finance and quantum groups. *Notices of the American Mathematical Society*, 51(11):1336–1347, 2004.

[2] H. Araki. A lattice of von Neumann algebras associated with the quantum theory of a free Bose field. *Journal of Mathematical Physics*, 4(11):1343–1362, 1963.

[3] S. Attal. *Quantum Open Systems, Vol II: The Markovian approach*, chapter Quantum Noises, pages 79–148. Number 1881 in Lecture notes in mathematics. Springer Verlag, 2006.

[4] S. Attal and A. Belton. The chaotic representation property for a class of normal martingales. *Probability Theory and Related Fields*, pages 543–562, 2007.

[5] A. Badrikian, G.W. Johnson, and I. Yoo. The composition of operator-valued measurable functions is measurable. *Proceedings of the American Mathematical Society*, 123(6):1815–1820, 1995.

[6] C. Barnett, R. F. Streater, and I. F. Wilde. The Itô-Clifford integral. *Journal of Functional Analysis.*, 48(2):172–212, 1982.

[7] A. C.R. Belton, J. M. Lindsay, and A.G. Skalski. A vacuum-adapted approach to quantum Feynman-Kac formulae. *Communications on Stochastic Analysis*, 6(1):95–109, 2012.

[8] A.C.R. Belton. *A Matrix Formulation of Quantum Stochastic Calculus.* PhD thesis, University of Oxford, 1998.

[9] A.C.R. Belton, M. Gnacik, and J.M. Lindsay. Quasifree random walk approximation. In preparation.

[10] A.C.R. Belton, M. Gnacik, and J.M. Lindsay. Strong convergence of quantum random walks via semigroup decomposition. In preparation.

[11] A.C.R. Belton and S.J. Wills. An algebraic construction of quantum flows with unbounded generators. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques.*, 51(1):349–375, 2015.

[12] B.V.R. Bhat and K.R. Parthasarathy. Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 31(4):601–651, 1995.

[13] P. Biane and R. Speicher. Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. *Probability Theory and Telated Fields*, 112(3):373–409, 1998.

[14] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[15] R. H. Cameron and W. T. Martin. The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals. *Annals of Mathematics*, 48(2):385–392, 1947.

[16] B.K. Das and J.M. Lindsay. Elementary evolutions in Banach algebra. *Journal of the Ramanujan Mathematical Society*, 28:483–497, 2013.

[17] A. Dermoune. Formule de composition por une classe d'operateurs. *Seminaire de Probabilites XXIV*, 1988.

[18] M. Émery. Chaotic representation property of certain Azéma martingales. *Illinois Journal of Mathematics*, 50(1-4):395–411, 2006.

[19] M. Evans and R.L. Hudson. Multidimensional quantum diffusions. In *Quantum Probability and Applications III*, pages 69–88. Springer, 1988.

[20] F. Fagnola. Quantum stochastic differential equations. *Quantum Probability Communications: QP-PQ (Volumes XI & XII)*, pages 123–170, 2003.

[21] F. Fagnola and S.J. Wills. Solving quantum stochastic differential equations with unbounded coefficients. *Journal of Functional Analysis*, 198(2):279–310, 2003.

[22] C. Foias and A.E. Frazho. The commutant lifting approach to interpolation problems. *Operator Theory: Advances and Applications*, 44, 1990.

[23] U. Franz. Lévy processes on quantum groups and dual groups. In *Quantum Independent Increment Processes II*, pages 161–257. Springer, 2005.

[24] U. Franz, R. Léandre, and R. Schott. Malliavin calculus for quantum stochastic processes. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 328(11):1061–1066, 1999.

[25] C. Geiss and C. Labart. Simulation of BSDEs with jumps by Wiener chaos expansion. *Stochastic Processes and their Applications*, 126(7):2123–2162, 2016.

[26] R.G. Ghanem and P.D. Spanos. *Stochastic Finite Elements: A Spectral Approach*. Springer-Verlag, 1991.

[27] M. Gnacik. *Quantum Random Walks*. PhD thesis, Lancaster University, 2014.

[28] D. Goswami, J.M. Lindsay, K.B. Sinha, and S. Wills. Dilation of Markovian cocycles on a von Neumann algebra. *Pacific Journal of Mathematics*, 211(2):221–247, 2003.

[29] D. Goswami and K. B. Sinha. Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra. *Communications in Mathematical Physics*, 205(2):377–403, Aug 1999.

[30] D. Goswami and K. B. Sinha. Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra. *Communications in Mathematical Physics*, 205(2):377–403, 1999.

[31] D. Goswami and K. B. Sinha. *Quantum Stochastic Processes and Noncommutative Geometry*. Cambridge University Press, 2007.

[32] A. Guichardet. *Symmetric Hilbert Spaces and Related Topics*, volume 261 of *Lecture Notes in Mathematics*. Springer, 1972.

[33] G. H. Hardy, J.E. Littlewood, and G. Polya. *Inequalities*. Cambridge University Press, 1934.

[34] K. Hasselmann. Stochastic climate models part I. Theory. *Tellus*, 28(6):473–485, 1976.

[35] E. Hille and R. S. Phillips. *Functional Analysis and Semi-groups*, volume 31. American Mathematical Society, 1996.

[36] R.L. Hudson. On the early years of quantum stochastic calculus. *Communications on Stochastic Analysis*, 6(1):111–123, 2012.

[37] R.L. Hudson and K.R. Parthasarathy. Quantum Itô's formula and stochastic evolutions. *Communications in Mathematical Physics*, 93:301–323, 1984.

[38] R.L. Hudson and K.R. Parthasarathy. Unification of fermion and boson stochastic calculus. *Communications in Mathematical Physics*, 104(3):457–470, 1986.

[39] K. Itô. Stochastic integral. *Proceedings of the Imperial Academy*, 20:519–524, 1944.

[40] K. Itô. Multiple Wiener integral. *Journal of Mathematical Society of Japan*, 3(1):157–169, 1951.

[41] K. Jacobs, Y. Liu, F. Nori, R. Wu, and J. Zhang. Non-Markovian quantum input-output networks. *Physical Review A - Atomic, Molecular and Optical Physics*, 87:032117, Mar 2013.

[42] G.W. Johnson. The product of strong operator measurable functions is strong operator measurable. *Proceedings of the American Mathematical Society*, 117(4):1097–1104, 1993.

[43] M. Jurczyński and J.M. Lindsay. Quantum Wiener chaos. In preparation.

[44] J. L Kelley. *General topology*. Springer Science & Business Media, 1975.

[45] T. Kemp, T. Nourdin, G. Peccati, and R. Speicher. Wigner chaos and the fourth moment. *The Annals of Probability*, 40:1577–1635, 2012.

[46] B. Kümmerer and R. Speicher. Stochastic integration on the Cuntz algebra O_∞. *Journal of Functional Analysis*, 103(2):372–408, 1992.

[47] G. Lindblad. On the generators of quantum dynamical semigroups. *Communications in Mathematical Physics*, 48(2):119–130, 1976.

[48] J. M. Lindsay. *Quantum Independent Increment Processes I*, chapter Quantum Stochastic Analysis - an Introduction, pages 181–271. Number 1865 in Lecture Notes in Mathematics. Springer, 2005.

[49] J. M. Lindsay and A.G. Skalski. On quantum stochastic differential equations. *Journal of Mathematical Analysis and Applications*, 330(2):1093–1114, 2007.

[50] J.M. Lindsay. Quantum stochastic Lie-Trotter product formula II. In preparation.

[51] J.M. Lindsay. *A Quantum Stochastic Calculus*. PhD thesis, University of Nottingham, 1985.

[52] J.M. Lindsay. Quantum and non-causal stochastic calculus. *Probability Theory and Related Fields*, 97(1–2):65–80, 1993.

[53] J.M. Lindsay. Integral-sum kernel operators. *Quantum Probability Communications*, XII:1–22, 2003.

[54] J.M. Lindsay and H. Maassen. An integral kernel approach to noise. *Quantum Probability and Applications*, III:192–208, 1987.

[55] J.M. Lindsay and O. Margetts. Preprint on arXiv 1203.6693.

[56] J.M. Lindsay and O. Margetts. Quasifree stochastic analysis. To be published.

[57] J.M. Lindsay and K.R. Parthasarathy. Cohomology of power sets with applications in quantum probability. *Communications in Mathematical Physics*, 124:337–364, 1989.

[58] J.M. Lindsay and A.G. Skalski. Quantum stochastic convolution cocycles I. *Annales de l'Institut Henri Poincaré*, 41 (3):581–604, 2005.

[59] J.M. Lindsay and A.G. Skalski. Quantum stochastic convolution cocycles II. *Communications in Mathematical Physics*, 280(3):575–610, 2008.

[60] J.M. Lindsay and S. J. Wills. Homomorphic Feller cocycles on a C^*-algebra. *Journal of the London Mathematical Society*, 68(1):255–272, 2003.

[61] J.M. Lindsay and S.J. Wills. Quantum stochastic operator cocycles via associated semigroups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 142:535, 2007.

[62] S.V. Lototsky. Nonlinear filtering of diffusion processes in correlated noise: analysis by separation of variables. *Applied Mathematics & Optimization*, 47(2):167–194, 2003.

[63] H. Maassen. Quantum Markov processes on Fock space described by integral kernels. *Quantum Probability and Applications II*, 1984.

[64] J. Manuceau. C*-algébre des relations de commutation. *Annales de l'Institut Henri Poincaré*, 8:139–161, 1966.

[65] O. Margetts. *Quasifree Stochastic Analysis and E_0 Semigroups on Factors*. PhD thesis, Lancaster University, 2013.

[66] P.A. Meyer. *Quantum Probability for Probabilists*. Springer-Verlag, 1995.

[67] R. Mikulevicius and B.L. Rozovskii. Stochastic Navier-Stokes equations: Propagation of chaos and statistical moments. *Optimal Control and Partial Differential Equations: In Honour of Professor Alain Bensoussan's 60th Birthday*, pages 258–267, 2001.

[68] A. Mohari and K.R. Parthasarathy. On a class of generalised Evans-Hudson flows related to classical Markov processes. *Quantum Probability and Related Topics*, 7:221–249, 1992.

[69] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer-Verlag, 2006.

[70] K. R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus*. Birkhauser, 1992.

[71] K.R. Parthasarathy. Azéma martingales and quantum stochastic calculus. *Probability, Statistics and Design of Experiments*, pages 551–569, 1990.

[72] S. Pathmanathan. *The Poisson Process in Quantum Stochastic Calculus*. PhD thesis, University of Oxford, 2002.

[73] D. Petz. *An Invitation to the Algebra of Canonical Commutation Relations*. Leuven Notes in Mathematical and Theoretical Physics. Series A: Mathematical Physics 2. Leuven University Press, 1990.

[74] C. Priami, A. Regev, E. Shapiro, and W. Silverman. Application of a stochastic name-passing calculus to representation and simulation of molecular processes. *Information Processing Letters*, 80(1):25–31, 2001.

[75] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. vol. 1. Functional Analysis*. Academic, 2nd edition, 1980.

[76] G. Schlüchtermann. A note on multiplication of strong operator measurable functions. *Proceedings of the American Mathematical Society*, 123(9):2815–2816, 1995.

[77] L. Schwartz. *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, volume 6. Oxford University Press, 1973.

[78] A.V. Skorokhod. On a generalization of a stochastic integral. *Theory of Probability and its Applications*, 20:219–233, 1975.

[79] J. Slawny. On factor representations and the C*-algebra of canonical commutation relations. *Communications in Mathematical Physics*, 24:151–170, 1972.

[80] W.Y. Tan. *Stochastic Models of Carcinogenesis*. Marcel Dekker, 1991.

[81] J. Weidmann. *Linear Operators in Hilbert Spaces*, volume 68 of *Graduate Texts in Mathematics*. Springer, 1980.

[82] N. Wiener. The homogeneous chaos. *American Journal of Mathematics*, 60:897–936, 1938.

[83] A. Zygmund. *Trigonometric Series, Volume I*. Cambridge University Press, 1959.