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DOI:
10.1016/j.jmaa.2017.10.071

Citation for published version (APA):
Chazal, M., Loeffen, R., & Patie, P. (2018). Smoothness of continuous state branching with immigration semigroups. *Journal of Mathematical Analysis and Applications*, *459*(2), 619-660. https://doi.org/10.1016/j.jmaa.2017.10.071

Published in:
Journal of Mathematical Analysis and Applications

SMOOTHNESS OF CONTINUOUS STATE BRANCHING WITH IMMIGRATION SEMIGROUPS

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Abstract. In this work we develop an original and thorough analysis of the (non)-smoothness properties of the semigroups, and their heat kernels, associated to a large class of continuous state branching processes with immigration. Our approach is based on an in-depth analysis of the regularity of the absolutely continuous part of the invariant measure combined with a substantial refinement of Ogura's spectral expansion of the transition kernels. In particular, we find new representations for the eigenfunctions and eigenmeasures that allow us to derive delicate uniform bounds that are useful for establishing the uniform convergence of the spectral representation of the semigroup acting on linear spaces that we identify. We detail several examples which illustrate the variety of smoothness properties that CBI transition kernels may enjoy and also reveal that our results are sharp. Finally, our technique enables us to provide the (eventually) strong Feller property as well as the rate of convergence to equilibrium in the total variation norm.

Contents

1. Introduction and main results

The objective of this paper is to develop an original approach to obtain detailed information regarding the representation and the regularity properties of the solution to the parabolic

Key words and phrases. Continuous state branching processes with immigration, Bernstein functions, non-self-adjoint integro-differential operators, Laguerre polynomials, Markov semigroups, spectral theory
2010 Mathematical Subject Classification: 41A60, 47G20, 33C45, 47D07, 37A30.
The authors are grateful to an anonymous referee for the careful reading of the manuscript and the insightful comments that helped improving the quality of the paper. This work was partially supported by the Actions de Recherche Concertées IAPAS, a fund of the Communautée francaise de Belgique.

evolution equation

$$\frac{d}{dt}u_t - (\mathbf{D}+\mathbf{L})\,u_t = 0,\ u_0 = f, \tag{1.1}$$

where, for a smooth function f on $x>0$,

$$\mathbf{D}f(x) = \sigma^2 x f''(x) + (\mathbf{b}-\mathbf{m}x)\,f'(x) - (qx+a)f(x) \tag{1.2}$$

and

$$\mathbf{L}f(x) = \int_0^\infty \left(f(x+y)-f(x)-yf'(x)\mathbb{I}_{\{y<1\}}\right)\mathbf{K}(x,dy), \tag{1.3}$$

with the parameters $\sigma^2, q, a \geq 0$, $\mathbf{m},\mathbf{b}\in\mathbb{R}$ and the Lévy kernel $\mathbf{K}(x,dy) = x\mathbf{\Pi}(dy)+\mu(dy)$ defined in terms of the Lévy measures $\mathbf{\Pi}$ and μ, satisfying some mild conditions that are detailed in (1.26)-(1.28) below. It is already worth pointing out that our analysis includes the situations where $\overline{\overline{\mathbf{\Pi}}}(0^+) = \int_0^\infty \mathbf{\Pi}(dy) = \infty$ and where $\mathbf{A}=\mathbf{D}+\mathbf{L}$ is purely non-local, that is $\mathbf{D}\equiv 0$.

The linear operator $\mathbf{A}$ turns out to be the generator of the Feller semigroup of a continuous state branching process with immigration (for short CBI). There exists a rich and fascinating literature devoted to the study of fine analytical and stochastic properties of CBI-semigroups, see e.g. Handa [20], Stannat [46], Li and Ma [33], Foucart and Bravo [17], Duhalde et al. [14], Caballero et al. [6], Abraham and Delmas [1], Lambert [30] and the monograph [32]. However, little seems to be known regarding the regularity properties of (at least) their transition kernels, something which may be attributed to the fact that (non)-regularity properties of general non-local Markov semigroups and their heat kernels are fragmentally understood due to the lack of a comprehensive theory from both functional and stochastic analysis.
In the framework of diffusions, that is, semigroups associated to differential operators, many theories have been successively designed to study this type of question. We mention for instance the techniques based on Malliavin Calculus, see e.g. Hairer [19] for a recent and general account on these approaches, and, also analytical techniques based on Hölder estimates, see the monograph [28] for a thorough account.

Due to the generic role played by non-local operators in the class of operators satisfying the maximum principle, see Courrège [12] for more details, the recent years have witnessed a fast growing literature devoted to the study of smoothness properties of the heat kernels or of the solution to parabolic problems associated to non-local Markov generators. The approaches can be split into three main categories.
The first one is based on classical Fourier analysis which requires precise information regarding the asymptotic decays for large arguments of the Fourier transform of the semigroup in order to derive smoothness properties. This approach has been successfully developed by Hartman and Wintner [21] for providing a sufficient condition expressed in terms of their symbol for the continuity of the densities of Lévy processes, see also Knopova and Schilling [26] and the references therein for more detailed and interesting results in this direction. It was also used recently in Filipovic et al. [16] to study smoothness properties of the transition distributions of affine processes. In particular for the CBI transition kernels that we study, the authors obtain a smoothness property on the real line in the case where there is a diffusion component, i.e. $\sigma^2>0$ in (1.2) above. This latter result reveals two noteworthy limitations of the use of Fourier analysis in the context of transition kernels whose supports are not the entire real line, namely the positive real line for CBI kernels. On the one hand, this technique does not

provide the optimal regularity property on the support of kernels (or their derivatives) that do not vanish at one end of the support. On the other hand, it also requires precise information regarding the asymptotic behavior of the (modulus) of the Fourier transform which are often difficult to get without restrictive assumptions. For these reasons we are lead to develop an original approach that focuses on the smoothness property of the density (or its absolute continuous part) of the CBI kernels within their supports, that is, allowing possible explosions at 0 for the kernels or their successive derivatives.
A second approach is based on Malliavin calculus which has been extended under various conditions to study regularity properties of the solutions to stochastic differential equations driven by processes with jumps. However, these developments, in the context of Markov kernels, considered merely dynamics with either a Lévy kernel that is homogeneous in space or of finite intensity, and/or with a diffusion component, see e.g. Picard [39], Cass [8] or the original paper by Fournier [18] and the references therein.
Finally there are some substantial results of analytical nature based on Harnack inequalities which have been obtained recently for getting Hölder continuity properties of the solution to parabolic equations involving integro-differential operators, see e.g. Caffarelli and Silvestre [7]. However all these techniques are not general enough to be applied in our context. This is due to either a lack of symmetry and/or non-homogeneity of the Lévy kernel, or, unboundness of the drift coefficient, or, the possible absence of diffusion part, or, simply, the non-local feature of the generators.

We propose an alternative approach that stems on a combination of an original methodology that is developed to deal with the smoothness properties of distributions on $\mathbb{R}^+ = (0, \infty)$ and the spectral expansion of CBI-semigroups. For the former, we show that the invariant measure of a CBI-semigroup is solution to a convolution equation that we study in depth to derive the regularity properties. We already emphasize that these developments could be used in a larger context as, for instance, for the class of positive infinitely divisible laws. For the second part, we exploit and develop substantially a spectral expansion of the kernel of CBI-semigroups whose initial form was found by Ogura [35]. The novelty of this result lies on the lack of a spectral theorem for non-self-adjoint operator and Ogura overcomes this difficulty by first suggesting a candidate as an eigenvalues expansion for their transition kernel and by means of the Laplace transform techniques show that it indeed corresponds to the right one. These series expansions involve three spectral components, the eigenvalues which take in this case a simple form, the eigenfunctions and the eigenmeasures, that is, when these latter are absolutely continuous they correspond, in some sense, to the eigenfunctions of the adjoint semigroup. Unfortunately Ogura's representation of these spectral functions does not allow one to study their regularity properties. Thus, we start by providing new appropriate representations for the eigenfunctions as well as for the eigenmeasures that are useful for deriving delicate uniform bounds and for investigating their smoothness properties. This approach which offers another view of Ogura's work also enables us to describe linear functional spaces for which the spectral expansion of the semigroup remains valid. This is critical to determine regularity properties of the semigroup, that is formally the solution to the associated Cauchy problem (1.1).

Our new developments on the spectral decomposition of these semigoups also enable to apply some transforms that are known to carry over the spectral expansion. For instance, by means of a tensorization procedure, our results extend directly to any dimension $d \geq 2$. Furthermore this latter could be associated to the subordination in the sense of Bochner in order to obtain similar results for a larger class of d-dimensional assymetric Markov processes with two-sided jumps. We shall not present the details of these standard arguments therein but we refer the interested reader to the excellent monograph [3] for a description.

Now writing $\mathbb{R}^+_{\geq 0} = [0,\infty)$, denoting by $B_b(\mathbb{R}^+_{\geq 0})$ the set of bounded Borel measurable functions on $\mathbb{R}^+_{\geq 0}$ and by $C_0(\mathbb{R}^+_{\geq 0})$ the set of continuous functions on $\mathbb{R}^+_{\geq 0}$ vanishing at infinity and following Kawazu and Watanabe [24], we say that $P = (P_t)_{t\geq 0}$ is a CBI-semigroup (resp. CB-semigroup) if it is a non-negative strongly continuous contraction semigroup on $C_0(\mathbb{R}^+_{\geq 0})$, satisfying

$$P_t(\Lambda_0) \subset \widetilde{\Lambda_0} \quad (\text{resp. } P_t(\Lambda_0) \subset \Lambda_0), \text{ for all } t \geq 0, \tag{1.4}$$

where, writing $e_\lambda(\cdot) = e^{-\lambda \cdot}$, we have set

$$\Lambda_0 = (e_\lambda)_{\lambda\geq 0}, \quad \widetilde{\Lambda_0} = (ce_\lambda)_{c,\lambda\geq 0}.$$

From [24], it is well known that any CBI-semigroup P is characterized, via its Laplace transform, by a unique couple of functions (ψ, ϕ) that are respectively called the branching and immigration mechanisms and we denote it by $\mathrm{CBI}(\psi,\phi)$ and write simply $\mathrm{CB}(\psi) = \mathrm{CBI}(\psi, 0)$.

We point out that *the main results of this paper hold under the general conditions, stated in* (1.26)-(1.28) *below, on the mechanisms* $(\overline{\psi}, \overline{\phi})$*, defined in terms of the parameters* $\mathbf{b}, \mathbf{m}, q, a$ *and* $\mathbf{\Pi}$ *that appear in the expression of the generator in* (1.2). However, for sake of presentation, we prefer to state first these results in a slightly more restrictive setting that we now describe. Note that Proposition 1.9(1) explains that the semigroups under the two sets of conditions are related by a Doob h-transform which enable to readily transfer the properties from one semigroup to the h-transformed one.

We will denote by $\mathcal{N}$ the set of functions $\psi : \mathbb{R}^+_{\geq 0} \mapsto \mathbb{R}^+_{\geq 0}$ defined, for $u \geq 0$, by

$$\psi(u) = \sigma^2 u^2 + mu + \int_0^\infty \left(e^{-ur} - 1 + ur\right) \Pi(dr), \tag{1.5}$$

where $\sigma \geq 0$, $m > 0$ and Π is a non-negative Borel measure on $\mathbb{R}^+ = (0,\infty)$, and, which satisfy the following two conditions

$$\int^\infty \frac{du}{\psi(u)} < \infty \text{ and } \psi \in \mathcal{H}(R_\psi) \text{ with } R_\psi > 0, \tag{1.6}$$

where throughout $\mathcal{H}(R)$ (resp. $\mathcal{H}_{(a,b)}$) is the set of functions holomorphic on the open disk $D(0,R)$ (resp. on the strip $a < \Re(z) < b$), where we understand that R is the radius of convergence of their Taylor series at 0. Note that the second requirement in (1.6) implies that $\int_0^\infty (r \wedge r^2)\Pi(dr) < \infty$ and that the integral in (1.5) is well-defined. Of course, here and below, we mean that ψ as defined in (1.5) extends to a holomorphic function and we keep the same notation for its extension. Note that, by Sato [43, Theorem 25.17], $\psi \in \mathcal{H}(R_\psi)$ is equivalent to assume that ψ is holomorphic on the half-plane $\Re(z) > -R_\psi$. This standard equivalence also holds for Laplace transforms. Next, we denote by $\mathcal{B}$ the set of Bernstein functions on $\mathbb{R}^+_{\geq 0}$, that is, the functions $\phi : \mathbb{R}^+_{\geq 0} \mapsto \mathbb{R}^+_{\geq 0}$ such that

$$\phi(u) = bu + \int_0^\infty \left(1 - e^{-ur}\right) \mu(dr) = u\left(b + \int_0^\infty e^{-ur}\overline{\mu}(r)dr\right) \tag{1.7}$$

where $b \geq 0$ and μ is a non-negative Borel measure on $\mathbb{R}^+$ satisfying $\int_0^\infty (1 \wedge r)\mu(dr) < \infty$ and such that

$$\phi \in \mathcal{H}(R_\phi) \text{ with } R_\phi > 0. \tag{1.8}$$

Here in (1.7) we have set $\overline{\mu}(r) = \int_r^\infty \mu(dy)$, for all $r > 0$. Note that for any $\psi \in \mathcal{N}$, we have

$$\begin{aligned} \psi(u) &= u\phi_p(u) = u\left(\sigma^2 u + m + \int_0^\infty (1 - e^{-ur})\overline{\Pi}(r)dr\right) \\ &= u\left(\sigma^2 u + m + u\int_0^\infty e^{-ur}\overline{\overline{\Pi}}(r)dr\right), \end{aligned} \tag{1.9}$$

where $\overline{\Pi}(y) = \int_y^\infty \Pi(dr)$, $\overline{\overline{\Pi}}(y) = \int_y^\infty \overline{\Pi}(r)dr$ and $\phi_p \in \mathcal{B}$ is the descending ladder height exponent. Under these conditions, the CBI(ψ, ϕ) semigroup is conservative and subcritical with

$$\psi(0) = 0, \psi^{(1)}(0) = m > 0, \int_\lambda^\infty \frac{du}{\psi(u)} < \infty \text{ for all } \lambda > 0 \text{ and } \int_0^\infty \frac{du}{\psi(u)} = \infty. \tag{1.10}$$

where we have used the notation $f^{(\mathfrak{p})}(x) = \frac{d^{\mathfrak{p}}}{dx^{\mathfrak{p}}} f(x)$ for the $\mathfrak{p}$-th derivative of f, for some integer $\mathfrak{p}$. We refer to e.g. [32, Theorem 3.8] for a proof of (1.10). We also point out that, in fact, it is easy to check that the condition $\psi^{(1)}(0) = m > 0$ yields that $\int_0^\infty \frac{du}{\psi(u)} = \infty$. From these considerations, we deduce that the mapping

$$\lambda \mapsto A(\lambda) = \exp\left(-m\int_\lambda^\infty \frac{du}{\psi(u)}\right) \tag{1.11}$$

is an increasing bijection from $\mathbb{R}^+_{\geq 0}$ to $[0,1)$ with inverse function denoted by B, i.e. $B : [0,1) \longmapsto \mathbb{R}^+_{\geq 0}$ satisfies

$$\exp\left(-m\int_{B(z)}^\infty \frac{du}{\psi(u)}\right) = z.$$

In addition, under the assumptions (1.6) and (1.8), and writing

$$\Phi_\nu(\lambda) = \int_0^\lambda \frac{\phi(u)}{\psi(u)} du, \tag{1.12}$$

we shall show in Lemma 3.5 below that, there exists some $0 < R_0 \leq 1$ such that, for all $x \geq 0$, the function

$$z \longmapsto G_x(z) = e^{-xB(z) + \Phi_\nu(B(z))} \in \mathcal{H}(R_0). \tag{1.13}$$

Then, with this notation, we have the following representation of the Laplace transform of the Feller semigroup P which is due to Ogura [35, Proposition 1.2] and valid for any $t, x, \lambda \geq 0$,

$$P_t e_\lambda(x) = e^{-\Phi_\nu(\lambda)} G_x(A(\lambda)e^{-mt}). \tag{1.14}$$

Next, we set

$$T_0 = -\frac{\ln(R_0)}{m} \tag{1.15}$$

and we denote by W the so-called scale function associated to the spectrally negative Lévy process whose law is determined by its Laplace exponent ψ. More precisely, the function $W : \mathbb{R}^+_{\geq 0} \to \mathbb{R}^+_{\geq 0}$ is characterized by its Laplace transform as follows, for any $u > 0$,

$$\psi(u) \int_0^\infty e^{-uy} W(y)dy = 1. \tag{1.16}$$

We shall check, in Lemma 2.3 below, that the function W satisfies $W(0) = 0$ and is in $C^1(\mathbb{R}^+)$ with the derivative being positive on $\mathbb{R}^+$. With b the drift parameter and μ the Lévy measure associated to ϕ, see (1.7) and recalling $\overline{\mu}(r) = \int_r^\infty \mu(dy)$ we define, for all $y > 0$,

$$\kappa(y) = bW^{(1)}(y) + \int_0^y W^{(1)}(y-r)\overline{\mu}(r)dr, \tag{1.17}$$

and, we set $\underline{\kappa}(0^+) = \underline{\lim}_{y\to 0} \kappa(y)$, $\overline{\kappa}(0^+) = \overline{\lim}_{y\to 0} \kappa(y)$ and we define the integer $\underline{\kappa}$ by

$$\underline{\kappa} = \lceil \underline{\kappa}(0^+) \rceil - 1, \tag{1.18}$$

where $\lceil . \rceil$ is the ceiling function and we understand that $\lceil \infty \rceil = \infty$. Note that when $\phi \equiv 0$, i.e. P is a CB(ψ), obviously $\kappa \equiv 0$. Next, we denote for $t > T_0$,

$$\bar{\lambda}_t = \begin{cases} \min\left(-B\left(2 - e^{m(t-T_0)}\right), R_\phi\right) & \text{if } T_0 < t < T_0 + \frac{1}{m}\ln(2 - A(-R_A)), \\ R_A \wedge R_\phi & \text{if } t \geq T_0 + \frac{1}{m}\ln(2 - A(-R_A)), \end{cases} \tag{1.19}$$

where R_A is the radius of convergence of the Taylor series at 0 of A. Note that in Lemma 3.5 it is shown that $A \in \mathcal{H}(R_A)$ with $R_A > 0$ and thus B, the inverse of A, is well-defined for $z > A(-R_A)$. Writing $fe_\lambda(x) = f(x)e^{-\lambda x}$, $x \geq 0$, we set

$$\mathcal{D}_t = \{f : \mathbb{R}^+_{\geq 0} \to \mathbb{R} \ \text{ measurable; } fe_\lambda \in L^\infty(\mathbb{R}^+) \text{ for some } \lambda < \bar{\lambda}_t\}, \tag{1.20}$$

that is, $f \in \mathcal{D}_t$ if there exists $C > 0$ and $\lambda < \bar{\lambda}_t$ such that $|f(x)| \leq Ce^{\lambda x}$, for almost every (a.e.) $x \geq 0$. Plainly $\mathcal{D}_t$ is a linear space and since $B(0) = 0$, $\mathcal{D}_t$ contains the set of bounded measurable functions if $t > T_0 + \frac{\ln 2}{m}$. We write

$$L^1_{loc}(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R} \text{ measurable; for any } a > 0, \int_0^a |f(y)|dy < \infty\},$$
$$L^1(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R} \text{ measurable; } \int_0^\infty |f(y)|dy < \infty\}.$$

Further, for $E \subseteq \mathbb{R}$, $C(E)$, respectively $C^p(E)$ for $p = 1, 2, \cdots, \infty$ stand for the space of continuous, respectively p times continuously differentiable functions on E. Similarly, for any $E_i \subseteq \mathbb{R}$, $i = 1, 2, 3$, $C^{\infty^2,k}(E_1 \times E_2 \times E_3)$ denote the space of infinitely continuously differentiable functions with respect to the two first variables and k times with respect to third one on $E_1 \times E_2 \times E_3$. We also denote by $C_b(\mathbb{R}^+_{\geq 0})$, the set of bounded continuous functions on $\mathbb{R}^+_{\geq 0}$ and we set

$$\Lambda = \mathrm{Span}(e_\lambda)_{\lambda > 0},$$

where Span of a set stands for its linear hull. Note that, by the Stone-Weierstrass theorem, Λ is dense in $C_0(\mathbb{R}^+_{\geq 0})$. Finally, for all $t, x \geq 0$, we denote by $P_t(x, dy)$ the transition kernel of the CBI-semigroup P and by $\delta_a(dy)$ the dirac measure at a. We are now ready to state the first main result of this paper.

Theorem 1.1. *Let P be a CBI(ψ, ϕ) semigroup with $(\psi, \phi) \in \mathcal{N} \times \mathcal{B}$. Then, for all $f \in \mathcal{D}_t \cup \Lambda$, $z \mapsto P_z f \in \mathcal{H}_{(T_0,\infty)}$. Moreover, for any $t > T_0$, the following hold.*

(1) $P_t f \in C_b(\mathbb{R}^+_{\geq 0}) \cap C^\infty(\mathbb{R}^+_{\geq 0})$ *for all* $f \in \mathcal{D}_t \cup \Lambda$.

(2) $P_t f \in C_0(\mathbb{R}^+_{\geq 0}) \cap C^\infty(\mathbb{R}^+_{\geq 0})$ *for all* $f \in (\mathcal{D}_t \cap C_0(\mathbb{R}^+_{\geq 0})) \cup \Lambda$.

(3) $P_t f \in C_b(\mathbb{R}^+_{\geq 0})$ *for all* $f \in B_b(\mathbb{R}^+_{\geq 0})$*, that is,* P *is (eventually) strong Feller.*

(4) For all $x \geq 0$*, there exists a function* $y \mapsto p_t(x, y)$ *such that*

$$P_t(x, dy) = e^{-\overline{\Phi}_\nu} G_x(e^{-mt})\delta_0(dy) + p_t(x, y)dy, \quad y \in \mathbb{R}. \tag{1.21}$$

Note that one can take $p_t(x, y) = 0$ *for all* $y < 0$*. Thus, for all* $x \geq 0$*,* $P_t(x, dy)$ *is absolutely continuous if and only if*

$$\overline{\Phi}_\nu = \lim_{\lambda \to \infty} \Phi_\nu(\lambda) = \infty, \tag{1.22}$$

which holds if $\underline{\kappa} > 0$*. Moreover, in any case,*

a) if $\underline{\kappa} \geq 1$*, then* $(t, x, y) \longmapsto p_t(x, y)$ *is* $C^{\infty^2, \underline{\kappa}-1}((T_0, \infty) \times \mathbb{R}^+ \times \mathbb{R})$*,*

b) $(t, x, y) \longmapsto p_t(x, y) \in C^{\infty^2, \underline{\kappa}+\bar{\mathfrak{q}}}((T_0, \infty) \times \mathbb{R}^+ \times \mathbb{R}^+)$ *where*

b1) $\bar{\mathfrak{q}} = 0$ *if either* $\underline{\kappa} \geq 1$ *or* $\underline{\kappa} = 0$ *and* $\overline{\kappa}(0^+) < \infty$*,*

b2) $\bar{\mathfrak{q}} = \sup\{q \geq 1;\ \kappa, W \in C^{\mathfrak{q}}(\mathbb{R}^+)\}$ *if* $\kappa \in C^1(\mathbb{R}^+)$*,* $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$ *and* $\underline{\kappa}(0^+) = \overline{\kappa}(0^+)$*.*

Remark 1.2. Note that in Theorem 1.1(4(b)2) the condition $\kappa \in C^1(\mathbb{R}^+)$ ensures that in this case $\bar{\mathfrak{q}} \geq 1$ as we shall prove that, in our setting, $W \in C^1(\mathbb{R}^+)$, see Lemma 2.3 below.

Remark 1.3. We also point out that in a recent paper Li and Ma [33] have shown by means of an elegant coupling argument the strong Feller property of CBI semigroups satisfying the first condition in (1.6) and having a linear immigration, i.e. $\phi(u) = bu$.

Another interesting by-product of our analysis is the following precise estimate regarding the speed of convergence to stationarity in the total variation norm, which we recall to be defined for a signed measure μ on $\mathbb{R}^+_{\geq 0}$ by $||\mu||_{\mathrm{TV}} = \sup_{E \in \mathcal{B}(\mathbb{R}^+_{\geq 0})} |\mu(E)|$, with $\mathcal{B}(\mathbb{R}^+_{\geq 0})$ the set of Borelians of $\mathbb{R}^+_{\geq 0}$.

Proposition 1.4. *Let* P *be a* $\mathrm{CBI}(\psi, \phi)$ *semigroup with* $(\psi, \phi) \in \mathcal{N} \times \mathcal{B}$*, then* P *admits a unique invariant probability measure* $\mathcal{V}$ *on* $\mathbb{R}^+_{\geq 0}$*. Moreover* P *is exponentially ergodic, in the sense that there exist* $C > 0$ *and* $\overline{B} > 0$ *such that, for any* $x \geq 0$ *and* $t > \underline{T} = T_0 + \frac{1}{m}\ln(2 - A(-R_A))$*, we have*

$$||P_t^{\mathcal{V}}(x)||_{\mathrm{TV}} \leq C e^{\overline{B}x} \frac{e^{-m(t-\underline{T})}}{1 - e^{-m(t-\underline{T})}} \tag{1.23}$$

where we have set $P_t^{\mathcal{V}}(x)(.) = P_t(x, .) - \mathcal{V}(.)$*.*

Remark 1.5. We point out that the exponential ergocity of CBI semigroups have been studied recently under various restrictive conditions. For instance, Li and Ma [33] (resp. Jin and al. [23]) proved this fact by means of a coupling argument (resp. a Forster-Lyapunov function argument) when the immigration mechanism is linear, i.e. $\phi(u) = bu, b > 0$ (resp. the branching mechanism is quadratic, i.e. $\psi(u) = \sigma^2 u^2 + mu$).

We proceed by stating some sufficient conditions, expressed in terms of the characteristics of both mechanisms, for the mapping κ defined in (1.17) to satisfy the specific conditions appearing in the smoothness properties of the absolutely continuous part of the transition kernel.

Proposition 1.6. *We have the following.*

(1) *(i) If* $\sigma^2 + b > 0$ *then*

$$\underline{\kappa}(0^+) = \overline{\kappa}(0^+) = \frac{b}{\sigma^2} \in [0, \infty]. \tag{1.24}$$

(ii) If $\sigma^2 + b = 0$ *then*

$$\underline{\kappa}(0^+) \geq \varliminf_{y \to 0} \overline{\mu}(y) W(y). \tag{1.25}$$

In particular $\underline{\kappa}(0^+) = \infty$ *if* $\overline{\Pi}(y) \overset{0}{=} \mathrm{O}(y^{-\alpha})$ *and* $\frac{1}{\bar{\mu}(y)} \overset{0}{=} \mathrm{O}(y^{\beta})$ *with* $1 < \alpha < 1 + \beta < 2$*, where throughout* $f \overset{a}{=} \mathrm{O}(g)$ *for* $a \in [-\infty, \infty]$ *means that* $\varlimsup_{x \to a} \left| \frac{f(x)}{g(x)} \right| < \infty$*. Furthermore,* $\overline{\kappa}(0^+) = 0$ *if* $\bar{\mu}(0^+) < \infty$*.*

(2) Assume that $\underline{\kappa}(0^+) = \overline{\kappa}(0^+) < \infty$*. Then* $\kappa \in C^1(\mathbb{R}^+)$ *and* $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$ *if one of the following holds.*

(i) $\sigma = 0$, $W^{(1)}, \bar{\mu} \in C^1(\mathbb{R}^+)$ *and for some* $\delta > 0$*, we have that* $W^{(2)}$ *is non-positive on* $(0, \delta)$ *and*

$$-\int_0^\delta \frac{y \bar{\mu}^{(1)}(y) dy}{\int_0^y \overline{\overline{\Pi}}(r) dr} dy < \infty.$$

(ii) $\sigma = 0$, $\bar{\mu}(0^+) < \infty$, $\bar{\mu} \in C^1(\mathbb{R}^+)$ *and* $\bar{\mu}^{(1)} \in L^1_{loc}(\mathbb{R}^+)$*.*

(iii) $\sigma > 0$ *and* $\bar{\mu} \in C(\mathbb{R}^+)$*. Moreover, if in addition* $b = 0$, $\bar{\mu}(0^+) < \infty$, $\bar{\mu} \in C^1(\mathbb{R}^+)$ *and* $\bar{\mu}^{(1)} \in L^1_{loc}(\mathbb{R}^+)$ *then* $\kappa \in C^2(\mathbb{R}^+)$*.*

Remark 1.7. Note that if $\psi(u) = u^2 + u$, then $W^{(1)}(y) = e^{-y}$ and thus

$$\kappa(y) = e^{-y} \left(b + \int_0^y e^{-r} \overline{\mu}(r) dr \right),$$

which implies $\kappa(0^+) = b$ as in Proposition 1.6(1i). Assume further that $\mu(dr) = \delta_1(dr)$. Then $\overline{\mu}(r) = \mathbb{I}_{\{r \leq 1\}} \notin C(\mathbb{R}^+)$ and

$$y \mapsto \kappa(y) = e^{-y} \left(b + \left(1 - e^{-y \wedge 1} \right) \right) \notin C^1(\mathbb{R}^+),$$

which shows that Proposition 1.6(2iii) is sharp.

Remark 1.8. Proposition 1.6 gives some conditions under which κ is in $C^1(\mathbb{R}^+)$ and $\kappa^{(1)}$ is in $L^1_{loc}(\mathbb{R}^+)$. In part (2i), it is assumed that $W^{(2)}$ exists, is in $C(\mathbb{R}^+)$ and is further negative in a neighbourhood of zero. Unfortunately, not much is known about which Lévy measures Π imply these conditions on the scale function (in the situation where $\sigma = 0$ and $\int_0^1 r \Pi(dr) = \infty$). It is known that if $\overline{\overline{\Pi}}$ is log-convex, then $W^{(1)}$ is non-increasing (but not necessarily in $C^1(\mathbb{R}^+)$), whereas if $\overline{\Pi}$ is completely monotone, then $W^{(1)}$ is completely monotone, see [44, Chap. 11]. Recall that a non-negative function f is completely monotone if it is in $C^\infty(\mathbb{R}^+)$ and $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0$ and $n \in \mathbb{N}$. Higher order differentiability properties of κ can be straightforwardly deduced from the expressions for $\kappa^{(1)}$ and $\kappa^{(2)}$, see e.g. (2.12) below, given in Proposition 1.6 in combination with Lemma 2.5 below, upon imposing higher order continuous differentiability on $W^{(1)}$ and $\bar{\mu}$. If $\sigma > 0$, the problem of higher order (non-)differentiability of $W^{(1)}$ is studied in Chan et al. [9]. In particular, Theorem 2 in [9] says that if the Blumenthal-Getoor lower index $\inf\{\beta > 0; \int_0^1 r^\beta \Pi(dr) < \infty\} < 2$, then $W^{(2)} \in C^{n+1}(\mathbb{R}^+)$ if and only if $\overline{\Pi} \in C^n(\mathbb{R}^+)$. When $\sigma = 0$, again little is known about the existence of higher order derivatives except in the aforementioned case where $\overline{\Pi}$ is completely monotone.

We emphasize that in fact the main results of this paper extend to the larger class of CBI$(\overline{\psi}, \overline{\phi})$ semigroups whose mechanisms $(\overline{\psi}, \overline{\phi})$ are in $\overline{\mathcal{N}} \times \overline{\mathcal{B}}$, which corresponds to the set of functions of the form

$$\overline{\psi}(u) = \sigma^2 u^2 + \mathbf{m}u + \int_0^\infty \left(e^{-ur} - 1 + ur\mathbb{I}_{\{|r|<1\}}\right) \mathbf{\Pi}(dr) - q, \tag{1.26}$$

where $\sigma^2, q \geq 0, \mathbf{m} \in \mathbb{R}$ and $\mathbf{\Pi}$ is a Lévy measure satisfying $\int_0^\infty (1 \wedge r^2)\mathbf{\Pi}(dr) < \infty$ and

$$\overline{\phi}(u) = \phi(u) + a,$$

for some $a \geq 0$ and ϕ of the form (1.7), that satisfy the following conditions

$$\int^\infty \frac{du}{|\overline{\psi}(u)|} < \infty \text{ and either } \theta > 0 \text{ or } \overline{\psi}, \overline{\phi} \in \mathcal{H}(R) \text{ for some } R > 0 \text{ and } \overline{\psi}^{(1)}(0) > 0, \tag{1.27}$$

where θ is the largest root of the equation $\overline{\psi}(u) = 0$, i.e.

$$\theta = \sup\{u \geq 0; \overline{\psi}(u) = 0\} \in [0, \infty). \tag{1.28}$$

Note that since $\int^\infty \frac{du}{|\overline{\psi}(u)|} < \infty$, we must have $\lim_{u\to\infty} \overline{\psi}(u) = \infty$ (see Lemma 2.1 and its proof below) and thus there exists at least one root of $\overline{\psi}$ as $\overline{\psi}(0) \leq 0$. Next, denote, for any $\eta \geq 0$, $\mathcal{E}_\eta$ the η-Esscher transform, which is defined for a function $f : \mathbb{R}^+_{\geq 0} \mapsto \mathbb{R}$, by $\mathcal{E}_\eta f(u) = f(u+\eta) - f(\eta)$. It is well-known and easy to prove that, with θ as in (1.28), $\mathcal{E}_\theta \overline{\mathcal{N}} \subseteq \mathcal{N}$ and $\mathcal{E}_\theta\, \overline{\mathcal{B}} \subseteq \mathcal{B}$, see e.g. [43, Example 33.14]. Then, we define the following transform

$$\begin{aligned} \mathcal{E} \;:\; & \overline{\mathcal{N}} \times \overline{\mathcal{B}} \to \mathcal{N} \times \mathcal{B} \\ & (\overline{\psi}, \overline{\phi}) \mapsto (\psi, \phi) = \mathcal{E}(\overline{\psi}, \overline{\phi}) = (\mathcal{E}_\theta \overline{\psi}, \mathcal{E}_\theta \overline{\phi}). \end{aligned} \tag{1.29}$$

An interesting motivation underlying the introduction of $\mathcal{E}_\theta$ is the two time-space Doob's transforms that leave invariant the set of CBI-semigroups that are described in Proposition 1.9 below. The first transform seems to be original whereas the second one was proved by Roelly and Rouault in [41]. These transforms serve to simplify the notation and are useful to derive the smoothness properties of general CBI-semigroups in $\overline{\mathcal{N}} \times \overline{\mathcal{B}}$ from the one of CBI-semigroups in $\mathcal{N} \times \mathcal{B}$. They are proved in subsection 2.1.

Proposition 1.9. *(1) Let $\overline{P}$ be a* CBI$(\overline{\psi}, \overline{\phi})$ *semigroup where $(\overline{\psi}, \overline{\phi}) \in \overline{\mathcal{N}} \times \overline{\mathcal{B}}$ and let P be the* CBI(ψ, ϕ) *semigroup where $(\psi, \phi) = \mathcal{E}(\overline{\psi}, \overline{\phi})$. Writing $f_\theta(x) = e^{\theta x} f(x)$, we have, for all $f \in B_b(\mathbb{R}^+)$ and $t, x \geq 0$,*

$$\overline{P}_t f(x) = e^{-\theta x} e^{-\overline{\phi}(\theta) t} P_t f_\theta(x). \tag{1.30}$$

Consequently, by replacing f by f_θ in the statements (1), (2), (3) *and* (4)*, Theorem 1.1 also holds for $\overline{P}$.*

(2) *Let* $\overline{P}$ *be a* $\mathrm{CB}(\overline{\psi})$ *semigroup. Then, there exists a* $\mathrm{CBI}(\psi,\phi)$*semigroup* P *where* $(\psi,\phi) = (\mathcal{E}_\theta\overline{\psi}, (\mathcal{E}_\theta\overline{\psi})^{(1)} - m)$ *with* $m = \psi^{(1)}(0)$ *such that, for any* $t, x > 0$ *and* $f \in B_b(\mathbb{R}^+)$

$$\overline{P}_t f(x) = xe^{-\theta x} e^{-mt} P_t \bar{f}_\theta(x). \tag{1.31}$$

where $\bar{f}_\theta(x) = \frac{f_\theta(x)}{x}$.

The proof of Theorem 1.1 relies on a combination of an in-depth analysis of the smoothness properties of the invariant measure and a substantial refinement of the spectral decomposition of the transition kernels of CBI-semigroups which was originally studied by Ogura [35] and that we now state. To this end, we need to introduce further notation. First, let $(\mathcal{L}_n)_{n\geq 0}$ be the family of Sheffer polynomials whose generating function is $G_x(z)$ given by (1.13), i.e. for any $x \geq 0$,

$$G_x(z) = \sum_{n=0}^{\infty} \mathcal{L}_n(x) z^n, \quad |z| < R_0. \tag{1.32}$$

We let ν, respectively ω, be a non-negative integrable function on $\mathbb{R}^+$ whose Laplace transform takes the form

$$\int_0^\infty e^{-\lambda y} \nu(y) dy = e^{-\Phi_\nu(\lambda)} - e^{-\overline{\Phi}_\nu}, \quad \lambda \geq 0, \tag{1.33}$$

respectively,

$$\int_0^\infty e^{-\lambda y} \omega(y) dy = 1 - A(\lambda), \quad \lambda \geq 0, \tag{1.34}$$

where we recall that A, Φ_ν and $\overline{\Phi}_\nu$ are defined in (1.11), (1.12) and (1.22). It will be shown in Proposition 3.1 and Corollary 3.4 below that the functions ν and ω are well-defined. We further set for $n \geq 1$,

$$\mathcal{W}_n(y) = \sum_{j=1}^{n} \binom{n}{j} (-1)^j \omega^{*j}(y), \tag{1.35}$$

where $\omega^{*1} = \omega$, and, for any $n \geq 2$,

$$\omega^{*n}(y) = \omega^{*n-1} * \omega(y),$$

where $*$ stands for the standard convolution, i.e. $f * g(y) = \int_0^y f(y-x)g(x)dx$. Also, we set

$$\lambda_n = mn,$$

where we recall that $m = \psi^{(1)}(0)$.

Theorem 1.10. *Let* P *be a* $\mathrm{CBI}(\psi,\phi)$ *semigroup with* $(\psi,\phi) \in \mathcal{N} \times \mathcal{B}$. *Then, for any* $t > T_0$, $f \in \mathcal{D}_t \cup \Lambda$, *and, for all integers* $\mathfrak{m}, \mathfrak{p} \geq 0$, *we have*

$$\frac{d^{\mathfrak{m}}}{dt^{\mathfrak{m}}} \left(P_t f\right)^{(\mathfrak{p})}(x) = \sum_{n=\mathfrak{p}}^{\infty} (-\lambda_n)^{\mathfrak{m}} e^{-\lambda_n t} \mathcal{L}_n^{(\mathfrak{p})}(x) \mathcal{V}_n f, \quad x \geq 0, \tag{1.36}$$

where the series is locally uniformly convergent in x, t *and, for any* $n \geq 0$, $\mathcal{V}_n f = \int_0^\infty f(y) \mathcal{V}_n(dy)$ *with*

$$\mathcal{V}_n(dy) = e^{-\overline{\Phi}_\nu} \delta_0(dy) + \nu_n(y) dy, \tag{1.37}$$

where for $n \geq 1$,

$$\nu_n(y) = e^{-\overline{\Phi}_\nu}\mathcal{W}_n(y) + \mathcal{W}_n * \nu(y) + \nu(y), \quad y > 0, \tag{1.38}$$

and $\nu_0 = \nu$. In particular, for all $t > T_0$, $x, y \geq 0$, $P_t(x, dy) = e^{-\overline{\Phi}_\nu} G_x(e^{-mt})\delta_0(dy) + p_t(x, y)dy$ with

$$p_t(x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\mathcal{L}_n(x)\nu_n(y). \tag{1.39}$$

Finally, for all integers $\mathfrak{m}, \mathfrak{p} \geq 0$ and $0 \leq \mathfrak{q} \leq \underline{\kappa} + \bar{\mathfrak{q}}$, with $\bar{\mathfrak{q}}$ as in Theorem 1.1(4b)*, we have*

$$\frac{d^{\mathfrak{m}}}{dt^{\mathfrak{m}}} p_t^{(\mathfrak{p},\mathfrak{q})}(x, y) = \sum_{n=\mathfrak{p}}^{\infty} (-\lambda_n)^{\mathfrak{m}} e^{-\lambda_n t}\mathcal{L}_n^{(\mathfrak{p})}(x)\nu_n^{(\mathfrak{q})}(y) \tag{1.40}$$

where, when $0 \leq \mathfrak{q} \leq \underline{\kappa} - 1$, the $\mathfrak{q}$-th derivative of ν_n is given by $\nu_n^{(\mathfrak{q})}(y) = \mathcal{W}_n * \nu^{(\mathfrak{q})}(y)$.

Remark 1.11. Note that from the Doob's transform (1.30) in Proposition 1.9 we get the following identity between the corresponding heat kernels

$$\bar{P}_t(x, dy) = e^{-\theta(x-y)}e^{-\bar{\phi}(\theta)t}P_t(x, dy), \quad t, x, y \geq 0. \tag{1.41}$$

Remark 1.12. We point out that the phenomena that the linear functional space, here $\mathcal{D}_t$, for which the spectral representation is valid increases with respect to time, has been observed in recent works dealing with the spectral representation of non-self-adjoint (NSA) Markov semigroups, see e.g. [36], [38] and [37]. This may be explained by the fact that in opposition to the self-adjoint case where, by the spectral theorem, a resolution of the identity is available, the invariant subspaces of NSA operators do not form in general a basis of the Hilbert space yielding to convergent spectral expansion only on a subspace of the full Hilbert space.

Remark 1.13. In Proposition 4.3 below, we state that the set $(e^{-\lambda_n t})_{n\geq 0}$ is part of the point spectrum of the (unique continuous extension of the) CBI operator P_t in the weighted Hilbert space

$$L^2(\mathcal{V}) = \left\{ f : \mathbb{R}^+_{\geq 0} \to \mathbb{R} \text{ measurable; } \int_0^\infty f^2(y)\mathcal{V}(dy) < \infty \right\}, \tag{1.42}$$

where $\mathcal{V} = \mathcal{V}_0$ and the latter is defined in (1.37). The characterization of the different components of the spectrum of P_t, that is the point, continuous and residual spectrum, see e.g. [15] for definition, seems to be a delicate issue and goes beyond the scope of this work. We refer the interested readers to the recent paper by Patie and Savov [36] where an approach based on the theory of Hilbert sequences has been developed to describe these different parts of the spectrum, including the algebraic and geometric multiplicities of the eigenvalues.

Remark 1.14. It is interesting to observe that the condition $\psi, \phi \in \mathcal{H}(R)$ for some $R > 0$, when $\theta = 0$, ensures that the CBI(ψ, ϕ) semigroup contains a countable set of isolated eigenvalues, that is, its (discrete) point spectrum is countable. Indeed, under this condition, the expansion of the holomorphic mapping G_x enables us to define the eigenfunctions. We point out that Ogura [35] shows that when this condition is not satisfied and assuming some (restrictive) additional technical conditions on the mechanisms then the transition kernel of the corresponding CBI-semigroup admits an integral representation. It would be interesting to relax Ogura's conditions in this situation and to study if the eigenvalues are part of the (continuous) point or the continuous spectrum. Regarding the second assumption $\int^\infty \frac{du}{\psi(u)} < \infty$ in (1.6), it ensures both the existence of eigenmeasures and the absolute convergence of the eigenvalues expansions. Finally, we remark that the analycity property of the mechanisms is, according to Lemma 3.5 below, equivalent to the existence of positive exponential moments of the associated Lévy measures, that is about the behavior of the Lévy measure at ∞ whereas, from Lemma 2.1 below, the second condition $\int^{\infty} \frac{du}{\psi(u)} < \infty$ in (1.6), when $\sigma^2 = 0$, is a condition on their behaviors at 0.

Remark 1.15. The main improvement of our spectral representation in comparison to Ogura's one in [35] is our original characterization of both the eigenfunctions $\mathcal{L}_n$, see Section 4, and of the eigenmeasures $\mathcal{V}_n$, which allows us to study, in particular, regularity properties of the CBI semigroup and its transition kernel. Besides providing the Lebesgue decomposition of $\mathcal{V}_n$, (1.37) and (1.38) also lead to the following bound on the Laplace transform of $|\mathcal{V}_n|$, the total variation measure of the signed measure $\mathcal{V}_n$,

$$\int_0^{\infty} e^{-\lambda y} |\mathcal{V}_n|(dy) \leq (2 - A(\lambda))^n e^{-\Phi_\nu(\lambda)}, \quad \lambda > -(R_A \wedge R_\phi),$$

see Proposition 5.1 below. This bound improves (since $A(\lambda) \in [0, 1)$ for $\lambda \geq 0$) the one from Ogura, see (2.2) in [35], which reads

$$\int_0^{\infty} e^{-\lambda y} |\mathcal{V}_n|(dy) \leq A(\lambda)^{-n} e^{-\Phi_\nu(\lambda)}, \quad \lambda > 0.$$

This improved bound on $|\mathcal{V}_n|$ allows us in particular to show that (1.36) holds for a wider class of functions f than can be concluded from the results in [35]. In this vein, it is worth mentioning that when in (1.19) $\bar{\lambda}_t = R_A \wedge R_\phi$ then, according to Lemma 3.5, the functional spaces $\mathcal{D}_t$ and $L^2(\mathcal{V})$ are comparable at least at infinity.

Example 1.16. Though our main focus is on studying smoothness properties, we now look at a short example to illustrate the ingredients of the spectral representation in Theorem 1.10. Note that in [10] we study how the eigenmeasures and eigenfunctions can be numerically computed. Examples that deal with the regularity of CBI-semigroups will follow in Section 7. We look at the CBI(ψ, ϕ)-semigroup with mechanisms given, for any $u \geq 0$, by

$$\begin{aligned} \psi(u) &= (u+1)^{1+\alpha} - (u+1), \\ \phi(u) &= (u+1)^{\alpha} - 1, \end{aligned} \tag{1.43}$$

with $0 < \alpha \leq 1$. We point out that when $\alpha = 1$, the CBI(ψ, ϕ) boils down to a linear diffusion which is the CIR process. We see that $m = \psi^{(1)}(0) = \alpha$, $A(\lambda) = 1 - \left(\frac{1}{\lambda+1}\right)^{\alpha}$, which by (1.34), leads to $\omega(y) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}$, where Γ is the gamma function. Therefore recalling the well-known fact that the convolution of two gamma distributions with the same scale parameter is again a gamma distribution with the same scale parameter but with shape parameter equal to the sum of the individual shape parameters, we have $\mathcal{W}_n(y) = e^{-y} \sum_{j=1}^{n} \binom{n}{j} (-1)^j \frac{y^{\alpha j - 1}}{\Gamma(\alpha j)}$. Further, $\Phi_\nu(\lambda) = \ln(\lambda+1)$ and thus $\nu(y) = e^{-y}$. Then, from the expression (1.38), we get that for $n \geq 0$,

$$\nu_n(y) = e^{-y} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{y^{\alpha j}}{\Gamma(\alpha j + 1)}, \ y > 0.$$

Note that with $\alpha = 1$, $\nu_n(y) = e^{-y} \mathrm{L}_n(x)$ where the L_n's are the classical Laguerre polynomials, see e.g. [31, 4.17.2].

We have B, the inverse of A, equals $B(z) = (1-z)^{-1/\alpha} - 1$. Thus $B \in \mathcal{H}(1)$ and hence $T_0 = 0$ in (1.15), and, for all $n \geq 0$, $x \geq 0$, from (1.32), we have $n!\mathcal{L}_n(x) = G_x^{(n)}(0)$ with

$$G_x(z) = (B(z)+1)e^{-xB(z)}.$$

Note that when $\alpha = 1$, $G_x(z) = (1-z)^{-1}e^{\frac{z}{1-z}x}$ which is the generating function of the classical Laguerre polynomials, that is, in this case, for all $n \geq 0$, $\mathcal{L}_n(x) = \mathrm{L}_n(x)$. Otherwise, since for $j \geq 0$, $(B(z)+1)^{(j)}{}_{|z=0} = \frac{\Gamma(\frac{1}{\alpha}+j)}{\Gamma(\frac{1}{\alpha})}$, we have, by Leibniz's formula, for any $n \geq 0$,

$$\mathcal{L}_n(x) = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{\Gamma(\frac{1}{\alpha}+j)}{\Gamma(\frac{1}{\alpha})}\left(e^{-xB(z)}\right)^{(n-j)}_{|z=0},$$

with, by means of the Faa Di Bruno's formula,

$$\left(e^{-xB(z)}\right)^{(n-j)}_{|z=0} = \sum_{k=1}^{n-j}(-x)^k\mathcal{B}_{n-j,k}\left(\frac{\Gamma(\frac{1}{\alpha}+1)}{\Gamma(\frac{1}{\alpha})}, \frac{\Gamma(\frac{1}{\alpha}+2)}{\Gamma(\frac{1}{\alpha})}, \cdots, \frac{\Gamma(\frac{1}{\alpha}+n-j-k+1)}{\Gamma(\frac{1}{\alpha})}\right),$$

where the $\mathcal{B}_{n,k}$'s are the Bell polynomials.

Finally, by Theorem 1.10, the CBI(ψ,ϕ)-semigroup transition kernel is given by

$$P_t(x,dy) = \sum_{n=0}^{\infty} e^{-\alpha n t}\mathcal{L}_n(x)\nu_n(y)dy, \quad y > 0, t > 0, x \geq 0.$$

The remaining part of the paper is mainly devoted to the proofs of the main statements, namely Theorem 1.1, Proposition 1.6, Proposition 1.9 and Theorem 1.10. They are split into several parts where each of them may be of independent interest. Indeed, in the next Section, we review some useful preliminary results including different criteria for the main conditions and general results regarding smoothness property of solutions of convolution equations. We also provide therein the proof of Proposition 1.6 and Proposition 1.9. Section 3 contains a thorough study of smoothness properties of the absolutely continuous part of the invariant measure. We proceed by studying in detail the spectral components of the CBI-semigroups, and, in particular, we establish some specific representations of each of them which allow us to derive some analytical properties. Note, as CBI-semigroups are in general non-self-adjoint operators, that the spectral components include a set of eigenfunctions and of eigenmeasures which when the latter are absolutely continuous may correspond to the sequence of eigenfunctions for the adjoint semigroups. More specifically, in Section 4, we present an original study of the sequence of eigenfunctions by relating them to the Sheffer polynomials. In Section 5 we provide explicit representations for the eigenmeasures which enable us to obtain both smoothness properties and uniform upper bounds for their absolutely continuous parts as well as for the successive derivatives of these latter, whenever they exist. Section 6 includes the last arguments required to prove Theorem 1.1 and Theorem 1.10. Finally, the last Section contains several instances of CBI-semigroups which illustrate the variety of smoothness properties that this class may enjoy and also reveal that our results are sharp.

2. Proof of Propositions 1.6 and 1.9 and preliminary results

2.1. **Proof of Proposition 1.9.** An application of [35, Proposition 1.2] shows that, for all $\lambda > \theta$,

$$\overline{P}_t e_\lambda(x) = e^{-\overline{\phi}(\theta)t} \exp\left(\int_{\overline{B}\left(\overline{A}(\lambda)e^{-\overline{\psi}^{(1)}(\theta)t}\right)}^{\lambda} \frac{\overline{\phi}(u)-\overline{\phi}(\theta)}{\overline{\psi}(u)}du - x\overline{B}\left(\overline{A}(\lambda)e^{-\overline{\psi}^{(1)}(\theta)t}\right)\right),$$

where $\overline{B}$ is the inverse of $\overline{A}(\lambda) = \exp\left(-\overline{\psi}^{(1)}(\theta)\int_\lambda^\infty \frac{du}{\overline{\psi}(u)}\right)$. Now, recalling that, for all $u > \theta$, $\overline{\phi}(u) - \overline{\phi}(\theta) = \phi(u-\theta)$ and $\overline{\psi}(u) = \psi(u-\theta)$, it is plain that $\overline{\psi}^{(1)}(\theta) = \psi^{(1)}(0) = m$ and $\overline{A}(\lambda) = A(\lambda-\theta)$ and $\overline{B}(z) = B(z)+\theta$. It then easy to check, by a change of variables and (1.14), that

$$\begin{aligned}
\overline{P}_t e_\lambda(x) &= e^{-\overline{\phi}(\theta)t} \exp\left(\int_{B(A(\lambda-\theta)e^{-mt})+\theta}^{\lambda} \frac{\phi(u-\theta)}{\psi(u-\theta)}du - x\left(B\left(A(\lambda-\theta)e^{-mt}\right)+\theta\right)\right) \\
&= e^{-\theta x}e^{-\overline{\phi}(\theta)t}G_x(A(\lambda-\theta)e^{-mt}) \\
&= e^{-\theta x}e^{-\overline{\phi}(\theta)t}P_t e_{\lambda-\theta}.
\end{aligned}$$

This proves the first part of Proposition 1.9. The last claim follows by first applying the transformation (1.30) and then the transformation [32, (3.37)] recalling that $m < \infty$ as it is imposed in this latter reference.

2.2. **Criteria for the main conditions.** The conditions as well as the criteria used in the description of the main results are given in terms of the CBI mechanisms or of the function κ defined in (1.17). In this part, we aim at providing equivalent criteria expressed directly in terms of the characteristic triplets of these mechanisms. We recall the following notation on asymptotic behaviours that will remain in force throughout the paper.

$$\begin{aligned}
f &\asymp g \text{ means that } \exists\, c > 0 \text{ such that } c \le \frac{f}{g} \le c^{-1}, \\
f &\overset{a}{\sim} g \text{ means that } \lim_{x\to a}\frac{f(x)}{g(x)} = 1, \text{ for some } a \in \mathbb{R}\cup\{\pm\infty\}.
\end{aligned}$$

We start with the following result dealing with the first condition in (1.6).

Lemma 2.1. *Let $\psi(u) = \sigma^2u^2 + mu + \int_0^\infty (e^{-ur}-1+ur)\,\Pi(dr)$ with $\sigma \ge 0$, $m > 0$ and $\int_0^\infty (r\wedge r^2)\Pi(dr) < \infty$. Then*

$$\int^\infty \frac{du}{\psi(u)} < \infty \Longleftrightarrow \sigma > 0 \text{ or } \int_0 \frac{dv}{\int_0^v \overline{\overline{\overline{\Pi}}}(r)dr} < \infty. \tag{2.1}$$

If for some $\epsilon > 0$, $\underline{\lim}_{y\downarrow 0}\overline{\overline{\overline{\Pi}}}(y)y^\epsilon > 0$ then (2.1) holds. However, if $\overline{\overline{\overline{\Pi}}}(0^+) < \infty$ and $\sigma = 0$, then (2.1) fails. When $\sigma^2 = 0$, (2.1) is not equivalent to $\overline{\overline{\overline{\Pi}}}(0^+) = \infty$ as $\psi(u) = u(\ln(u+1)+1) \in \mathcal{N}$ with $\overline{\overline{\overline{\Pi}}}(0^+) = \infty$ and $\int^\infty \frac{du}{u\ln(u+1)} = \infty$.

Remark 2.2. Note that the conditions in (2.1) ensure that the class of CBI-processes have paths of unbounded variation, as either $\sigma^2 > 0$ or $\overline{\overline{\overline{\Pi}}}(0^+) = \infty$.

Proof. First, recall from (1.9) that

$$\psi(u) = u\phi_p(u) = u\left(\sigma^2 u + u\int_0^\infty e^{-ur}(\overline{\Pi}(r) + m)dr\right) \tag{2.2}$$

and so $\phi_p \in \mathcal{B}$. Thus, [4, Proposition III.1] yields

$$\psi(u) = u\phi_p(u) \asymp \sigma^2 u^2 + mu + u^2 \int_0^{\frac{1}{u}} \overline{\overline{\Pi}}(r)dr. \tag{2.3}$$

From this estimate, we easily get the first statement where for the integral test we have performed a change of variables. Next, the condition $\underline{\lim}_{r\downarrow 0} \overline{\overline{\Pi}}(r)r^{\epsilon} > 0$ for some $\epsilon > 0$ implies that there exists $C > 0$ such that for r small enough, $\overline{\overline{\Pi}}(r) \geq Cyr^{-\epsilon}$ and as $\overline{\overline{\Pi}}$ is non-increasing, we get that $\int_0^v \overline{\overline{\Pi}}(r)dr \geq v\overline{\overline{\Pi}}(v) \geq Cv^{1-\epsilon}$, that is, from the preceding discussion,

$$\int^\infty \frac{du}{\psi(u)} \quad \leq \quad C^{-1}\int_0 v^{\epsilon-1}dv < \infty. \tag{2.4}$$

Next, since from (2.2) when $\sigma^2 = 0$, we get that $\phi_p(\infty) = m + \overline{\overline{\Pi}}(0^+)$ and hence, when $\overline{\overline{\Pi}}(0^+) < \infty$, $\psi(u) \leq u\phi_p(\infty), u \geq 0$, as ϕ_p is non-decreasing, which provides the statement in this case. Finally, observing that $\ln(u+1) = \int_0^\infty (1-e^{-ux})\frac{e^{-x}}{x}dx$, we easily deduce, by integration by parts, that $u\ln(u+1) + u = u\int_0^\infty (1-e^{-ux})\frac{e^{-x}}{x}dx + u = \int_0^\infty (e^{-ux} - 1 + ux)e^{-x}\frac{(x+1)}{x^2}dx + u \in \mathcal{N}$ with $\overline{\overline{\Pi}}(0^+) = \int_0^\infty \frac{e^{-x}}{x}dx = \infty$ and $\int^\infty \frac{du}{u\ln(u+1)+u} = \infty$, which completes the proof of the Lemma. □

We proceed with these known and basic facts regarding the scale function W defined in (1.16).

Lemma 2.3. *For any $\psi \in \mathcal{N}$, we have $W \in C^1(\mathbb{R}^+)$ with $W(0) = 0$ and $W^{(1)}(y) > 0$, for all $y > 0$. Moreover, for all $u > 0$,*

$$\frac{1}{\phi_p(u)} = \frac{u}{\psi(u)} \quad = \quad \int_0^\infty e^{-uy}W^{(1)}(y)dy, \tag{2.5}$$

that is $W^{(1)}(y)dy$ is the potential measure of the subordinator whose Laplace exponent is ϕ_p. Consequently, $\lim_{y\to\infty} W(y) = 1/m$ and $W^{(1)} \in L^1(\mathbb{R}^+)$.

Proof. Since, under the assumption $\int^\infty \frac{du}{\psi(u)} < \infty$, Lemma 2.1 ensures that the underlying Lévy process has paths of unbounded variation, $W(0) = 0$ and $W \in C^1(\mathbb{R}^+)$ follows from p.254 and Proposition 5.1 in [29]. The identities in (2.5) follow from (2.2) and an integration by parts respectively. Hence $W^{(1)}(y)dy$ is the potential measure of the subordinator, denoted by $(S_t)_{t\geq 0}$, whose Laplace exponent is ϕ_p, i.e. $W^{(1)}(y)dy = \int_0^\infty \mathbb{P}(S_t \in dy)dt$. It follows that $W^{(1)}(y) \geq 0$ for all $y > 0$. To show that actually $W^{(1)}(y) > 0$ for all $y > 0$, suppose instead that $W^{(1)}(y_0) = 0$ for some $y_0 > 0$. Since $W^{(1)}(y)/W(y)$ is a non-increasing function for $y > 0$ (see (8.26) in [27]), it follows that $W^{(1)}(y) = 0$ for all $y \geq y_0$. But then $\mathbb{P}(S_t \geq y_0) = 0$ for a.e. $t > 0$ which is absurd. For the last two claims, by (2.5) and the monotone convergence theorem,

$$\frac{1}{m} = \lim_{u\downarrow 0}\frac{1}{\phi_p(u)} = \lim_{u\downarrow 0}\int_0^\infty e^{-uy}W^{(1)}(y)dy = \lim_{y\to\infty} W(y) - W(0) = \lim_{y\to\infty} W(y)$$

and so $W^{(1)} \in L^1(\mathbb{R}^+)$ since it is a positive function. $\square$

2.3. **Derivatives and smoothness of convolutions.** In this section we present two lemmas on differentiability of convolutions, which will be used later on.

Lemma 2.4. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be absolutely continuous on $\mathbb{R}^+$ and $g \in L^1_{loc}(\mathbb{R}^+)$. Assume that $f' \in L^1_{loc}(\mathbb{R}^+)$, where f' denotes a version of the density of f and further assume that $f(0^+) = \lim_{y\downarrow 0} f(y) \in \mathbb{R}$. Then the convolution*

$$h(y) = \int_0^y f(y-r)g(r)dr$$

has a density on $\mathbb{R}^+$ and a version of it is given, for any $y > 0$, by

$$h'(y) = \int_0^y f'(y-r)g(r)dr + f(0^+)g(y).$$

Moreover, $h \in C^1(\mathbb{R}^+)$ with derivative given by $h^{(1)} = h'$ if $g \in C(\mathbb{R}^+)$ and either

(i) $f \in C^1(\mathbb{R}^+)$ or

(ii) $g \overset{0}{=} \mathrm{O}(1)$.

Proof. We have, for any $y > 0$,

$$\begin{aligned}\int_0^y \int_0^r |f'(r-v)g(v)|dvdr &= \int_0^y \int_r^y |f'(v-r)|dv|g(r)|dr = \int_0^y \int_0^{y-r} |f'(v)|dv|g(r)|dr \\ &\leq \int_0^y |g(r)|dr \int_0^y |f'(r)|dr < \infty,\end{aligned}$$

where we used the fact that f' and g are in $L^1_{loc}(\mathbb{R}^+)$. Hence Fubini Theorem yields

$$\begin{aligned}\int_0^y \int_0^r f'(r-v)g(v)dv + f(0^+)g(r)dr &= \int_0^y \int_v^y f'(r-v)drg(v)dv + f(0^+)\int_0^y g(r)dr \\ &= \int_0^y (f(y-r) - f(0^+))g(r)dr + f(0^+)\int_0^x g(r)dr \\ &= h(y).\end{aligned}$$

Hence the function h has a density on $\mathbb{R}^+$ which is given by h' as stated in the Lemma. Now assume that g is in $C(\mathbb{R}^+)$ and let $y > 0$. If $f \in C^1(\mathbb{R}^+)$, then $f^{(1)}$ and g are bounded on sets of the form $[a, b] \subset \mathbb{R}^+$ and therefore by the dominated convergence theorem,

$$\begin{aligned}\lim_{\delta\to 0} h'(y+\delta) &= \lim_{\delta\to 0} \int_0^{\frac{y+\delta}{2}} f^{(1)}(y+\delta-r)g(r)dr + \lim_{\delta\to 0}\int_0^{\frac{y+\delta}{2}} g(y+\delta-r)f^{(1)}(r)dr + f(0^+)g(y) \\ &= \int_0^{\frac{y}{2}} f^{(1)}(y-r)g(r)dr + \int_0^{\frac{y}{2}} g(y-r)f^{(1)}(r)dr + f(0^+)g(y) \\ &= h'(y).\end{aligned}$$

If instead g is bounded in a neighbourhood of zero, then by the dominated convergence theorem,

$$\begin{aligned}\lim_{\delta\to 0} h'(y+\delta) &= \lim_{\delta\to 0}\int_0^{y+\delta} g(y+\delta-y)f'(y)dy + f(0^+)g(y)\\ &= \int_0^{y} g(y-y)f'(y)dy + f(0^+)g(y)\\ &= h'(y).\end{aligned}$$

Hence, in both cases, h' is in $C(\mathbb{R}^+)$, which implies by the fundamental theorem of calculus that h is (continuously) differentiable on $\mathbb{R}^+$ with derivative given by $h^{(1)} = h'$. □

Lemma 2.5. *Let $f, g \in C^{p-1}(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$ for some $p \geq 1$ and assume that the p-th derivatives $f^{(p)}$ and $g^{(p)}$ exist on $\mathbb{R}^+$ and are bounded on compact (with respect to $\mathbb{R}$) subsets of $\mathbb{R}^+$. Then $h \in C^p(\mathbb{R}^+)$ where, for any $y > 0$,*

$$h(y) = \int_0^y f(y-r)g(r)dr$$

and

$$\begin{aligned}h^{(p)}(y) &= \int_0^{\frac{y}{2}} f^{(p)}(y-r)g(r)dr + \int_0^{\frac{y}{2}} g^{(p)}(y-r)f(r)dr\\ &+ \frac{1}{2}\sum_{j=0}^{p-1}\Big(f^{(j)}(\tfrac{y}{2})g(\tfrac{y}{2}) + f(\tfrac{y}{2})g^{(j)}(\tfrac{y}{2})\Big)^{(p-1-j)}.\end{aligned}$$

Proof. First we prove the claims for $p = 1$. Let $y > 0$. We can write for $\delta > 0$,

$$\begin{aligned}\frac{h(y+\delta)-h(y)}{\delta} &= \int_0^{\frac{y}{2}} \frac{f(y+\delta-r)-f(y-r)}{\delta}g(r)dr + \int_{\frac{y}{2}}^{\frac{y+\delta}{2}} \frac{f(y+\delta-r)}{\delta}g(r)dr\\ &+ \int_0^{\frac{y}{2}} \frac{g(y+\delta-r)-g(y-r)}{\delta}f(r)dr + \int_{\frac{y}{2}}^{\frac{y+\delta}{2}} \frac{g(y+\delta-r)}{\delta}f(r)dr.\end{aligned} \tag{2.6}$$

By the mean value theorem, we get, for all $r \in [0, \frac{y}{2}]$ and $\delta \in [0, \frac{y}{2}]$,

$$\left|\frac{f(y+\delta-r)-f(y-r)}{\delta}\right| \leq \sup_{r\in[\frac{y}{2},\frac{3y}{2}]} f^{(1)}(r).$$

Thus, by the assumption that $f^{(1)}$ is bounded on sets of the form $[a, b] \subset \mathbb{R}^+$ and the dominated convergence theorem, we obtain that

$$\lim_{\delta\downarrow 0}\int_0^{\frac{y}{2}} \frac{f(y+\delta-r)-f(y-r)}{\delta}g(r)dr = \int_0^{\frac{y}{2}} f^{(1)}(y-r)g(r)dr.$$

Due to the continuity of f and g, we have by the mean-value theorem again that for each $\delta > 0$ sufficiently small there exists $r_\delta \in \left[\frac{y}{2}, \frac{y+\delta}{2}\right]$ such that

$$\lim_{\delta\downarrow 0}\int_{\frac{y}{2}}^{\frac{y+\delta}{2}} \frac{f(y+\delta-r)}{\delta}g(r)dr = \lim_{\delta\downarrow 0} f(y+\delta-r_\delta)g(r_\delta)\frac{\frac{y+\delta}{2}-\frac{y}{2}}{\delta} = \tfrac{1}{2}f(\tfrac{y}{2})g(\tfrac{y}{2}).$$

Similarly, we can treat the other two terms on the right-hand side of (2.6), which leads to

$$\lim_{\delta\downarrow 0}\frac{h(y+\delta)-h(y)}{\delta}=\int_0^{\frac{y}{2}} f^{(1)}(y-r)g(r)dr+\int_0^{\frac{y}{2}} g^{(1)}(y-r)f(r)dr+f(\tfrac{y}{2})g(\tfrac{y}{2}).$$

Similarly, one shows that

$$\lim_{\delta\uparrow 0}\frac{h(y+\delta)-h(y)}{\delta}=\int_0^{\frac{y}{2}} f^{(1)}(y-r)g(r)dr+\int_0^{\frac{y}{2}} g^{(1)}(y-r)f(r)dr+f(\tfrac{y}{2})g(\tfrac{y}{2}).$$

This proves the claims for $p=1$. The results for any $p\geq 1$ then follows by a straightforward induction argument using the same steps as for the $p=1$ case. □

2.4. **Proof of Proposition 1.6.**

2.4.1. *Proof of Proposition 1.6*(1). First note that (2.5) and the estimate in [4, Proposition III.1] yield that, for any $y>0$,

$$W(y)\asymp\frac{1}{\phi_p\left(\frac{1}{y}\right)}\asymp\frac{y}{\sigma^2+my+\int_0^y\overline{\overline{\Pi}}(r)dr}. \tag{2.7}$$

Thus, $\underline{\lim}_{y\to 0}\frac{W(y)}{y}\geq c\,\underline{\lim}_{y\to 0}\frac{1}{y\phi_p\left(\frac{1}{y}\right)}$ for some $c>0$. As when $\sigma^2=0$, we have $\phi_p\left(u\right)\overset{\infty}{=}o(u)$, see also [4, Proposition III.1], we easily deduce that in this case $W^{(1)}(0^+)=\infty$. Otherwise, combining (2.5) with [4, Theorem III.2.5], we get that $W^{(1)}(0^+)=\sigma^{-2}$. Next, let us consider the case $\sigma^2+b>0$. When $\sigma^2=0$ and $b>0$ the statement follows from the preceding discussion as $\lim_{y\to 0}\kappa(y)\geq\lim_{y\to 0}bW^{(1)}(y)=\infty$. Otherwise if $\sigma^2>0$ we have, as W is increasing,

$$0\leq\lim_{y\to 0}\int_0^y W^{(1)}(y-r)\overline{\mu}(r)dr\leq\sup_{r\in[0,y]}W^{(1)}(r)\lim_{y\to 0}\int_0^y\overline{\mu}(r)dr=0$$

as, when $\sigma>0$, $W^{(1)}(0^+)=\sigma^{-2}$, and, by [9, Theorem 1], $W^{(1)}\in C^1(\mathbb{R}^+)$. Hence, for all $b\geq 0$,

$$\lim_{y\to 0}\kappa(y)=\lim_{y\to 0}bW^{(1)}(y)+\int_0^y W^{(1)}(y-r)\overline{\mu}(r)dr=\frac{b}{\sigma^2},$$

which completes the proof of the statement for the case $\sigma^2+b>0$. Next, since $\overline{\mu}$ is non-increasing, we have, for all $y>0$,

$$\kappa(y)=bW^{(1)}(y)+\int_0^y W^{(1)}(y-r)\overline{\mu}(r)dr\geq bW^{(1)}(y)+\overline{\mu}(y)\int_0^y W^{(1)}(r)dr=bW^{(1)}(y)+\overline{\mu}(y)W(y).$$

Thus, if $\sigma^2+b=0$,

$$\underline{\kappa}(0^+)=\underline{\lim}_{y\to 0}\kappa(y)\geq\underline{\lim}_{y\downarrow 0}\overline{\mu}(y)W(y).$$

Thus if $\overline{\Pi}(y)\overset{0}{=}\mathrm{O}(y^{-\alpha})$ and $\frac{1}{\bar{\mu}(y)}\overset{0}{=}\mathrm{O}(y^{\beta})$, we observe from (2.7) that for some $C_{\alpha,\beta}>0$,

$$\underline{\kappa}(0^+)\geq\underline{\lim}_{y\downarrow 0}\overline{\mu}(y)W(y)\geq\underline{\lim}_{y\downarrow 0}\frac{C_{\alpha,\beta}\bar{\mu}(y)}{m+\frac{1}{y}\int_0^y\overline{\overline{\Pi}}(r)dr}=\underline{\lim}_{y\downarrow 0}\frac{C_{\alpha,\beta}y^{-\beta}}{m+\frac{1}{\alpha-1}\frac{1}{2-\alpha}y^{1-\alpha}}, \tag{2.8}$$

which shows that $\underline{\kappa}(0^+) = \infty$ if $1 < \alpha < 1 + \beta < 2$. Finally if $\overline{\mu}(0^+) < 0$, since $\overline{\mu}$ is non-increasing, we have, for all $y > 0$,

$$\overline{\mu}(y)W(y) \leq \kappa(y) \leq \overline{\mu}(0^+)W(y),$$

which gives the last claim as $W(0) = 0$.

2.4.2. *Proof of Proposition 1.6*(2). Assume first that $\sigma = 0$, then necessarily $b = 0$ as otherwise $\underline{\kappa} = \infty$. Under the assumption that $W^{(1)}, \bar{\mu} \in C^1(\mathbb{R}^+)$, we can use Lemma 2.5 to deduce that $\kappa \in C^1(\mathbb{R}^+)$, and, for any $y > 0$,

$$\kappa^{(1)}(y) = \int_0^{\frac{y}{2}} W^{(2)}(y-r)\bar{\mu}(r)dr + \int_0^{\frac{y}{2}} \bar{\mu}^{(1)}(y-r)W^{(1)}(r)dr + W^{(1)}(\tfrac{y}{2})\bar{\mu}(\tfrac{y}{2}).$$

Now we show that $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$. First, by integration by parts and the fact that W is a non-decreasing function,

$$\begin{aligned} \frac{\int_0^x W^{(1)}(\frac{y}{2})\bar{\mu}(\frac{y}{2})dy}{2} &= \lim_{\epsilon \downarrow 0} \int_\epsilon^{\frac{x}{2}} W^{(1)}(r)\bar{\mu}(r)dr \\ &\leq W(\tfrac{x}{2})\bar{\mu}(\tfrac{x}{2}) - \int_0^{\frac{x}{2}} W(r)\bar{\mu}^{(1)}(r)dr < \infty, \end{aligned} \tag{2.9}$$

where the last line follows by the integral assumption and the fact that, in this case, $W(y) \asymp \frac{y}{m+\int_0^y \overline{\overline{\Pi}}(r)dr}$, see (2.7). Next, fix $x > 2\delta$. By the assumptions we have $|W^{(2)}(y)| \leq C$ with $0 < C < \infty$ for all $y \in [\delta, x]$ and $|W^{(2)}(y)| = -W^{(2)}(y)$ for all $0 < y < \delta$. Therefore by Fubini Theorem,

$$\begin{aligned} \int_0^x \left| \int_0^{\frac{y}{2}} W^{(2)}(y-r)\bar{\mu}(r)dr \right| dy &\leq \int_0^{\frac{x}{2}} \left(\int_{2r}^x \left| W^{(2)}(y-r) \right| dy \right) \bar{\mu}(r)dr \\ &= \int_0^{\frac{x}{2}} \left(\int_r^{x-r} \left| W^{(2)}(v) \right| dv \right) \bar{\mu}(r)dr \\ &\leq \int_0^{\frac{x}{2}} \left(\int_r^{x-r} \left(C - W^{(2)}(v)\mathbb{I}_{\{v<\delta\}} \right) dv \right) \bar{\mu}(r)dr \\ &= C \int_0^{\frac{x}{2}} (x-2r)\bar{\mu}(r)dr - \int_0^\delta (W^{(1)}(\delta) - W^{(1)}(r))\bar{\mu}(r)dr \\ &< \infty, \end{aligned}$$

where in the last line we used (2.9). Similarly, one shows that

$$\int_0^x \left| \int_0^{\frac{y}{2}} \bar{\mu}^{(1)}(y-r)W^{(1)}(r)dr \right| dy < \infty$$

and we conclude that $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$ which completes the proof of (2i). Under the assumptions of (2ii) we have by Lemma 2.4, for any $y > 0$,

$$\kappa^{(1)}(y) = \int_0^y \bar{\mu}^{(1)}(y-r)W^{(1)}(r)dr + \bar{\mu}(0)W^{(1)}(y),$$

and, thus $\kappa \in C^1(\mathbb{R}^+)$ and $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$, that is, (2ii) holds. Assume now that $\sigma > 0$ and $\bar{\mu} \in C(\mathbb{R}^+)$. Then, by [9, Theorem 1], $W^{(2)} \in C(\mathbb{R}^+)$. Moreover, recalling, from (2.5), that W

can be seen as the integrated potential measure of a subordinator whose Lévy measure has a density given by $\overline{\Pi}$, it follows by [9, Corollary 9] that, for any $y>0$,

$$W^{(2)}(y) = \frac{1}{\sigma^2}\sum_{n=1}^{\infty}\left(-\frac{1}{\sigma^2}\right)^n \overline{\overline{\Pi}}^{*n}(y). \tag{2.10}$$

Since $\int_0^y \overline{\overline{\Pi}}^{*n}(r)dr = \left(\int_0^y \overline{\overline{\Pi}}(r)dr\right)^n$ and $\lim_{y\downarrow 0}\int_0^y \overline{\overline{\Pi}}(r)dr = 0$, it follows that $\int_0^y |W^{(2)}(r)|dr < \infty$ for $y>0$ small enough and thus for all $y>0$ since $W^{(2)}$ is locally bounded. Hence $W^{(2)} \in L^1_{loc}(\mathbb{R}^+)$. As $W^{(1)}(0^+) = \sigma^{-2}$, we can use Lemma 2.4 to deduce that for any $y>0$,

$$\kappa^{(1)}(y) = bW^{(2)}(y) + \int_0^y W^{(2)}(y-r)\bar{\mu}(r)dr + \sigma^{-2}\bar{\mu}(y). \tag{2.11}$$

Therefore $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$ as $W^{(2)} \in L^1_{loc}(\mathbb{R}^+)$ and, by Lemma 2.4 again, $\kappa \in C^1(\mathbb{R}^+)$ as $W^{(2)} \in L^1_{loc}(\mathbb{R}^+) \cap C(\mathbb{R}^+)$ and by assumption $\bar{\mu} \in C(\mathbb{R}^+)$. This proves the first claim of (2iii). In order to prove the last claim, assume further that $b=0$, $\bar{\mu}(0)<\infty$, $\bar{\mu} \in C^1(\mathbb{R}^+)$ and $\bar{\mu}^{(1)} \in L^1_{loc}(\mathbb{R}^+)$. Then we can use Lemma 2.4 on (2.11) to deduce that $\kappa^{(1)} \in C^1(\mathbb{R}^+)$ with, for $y>0$,

$$\kappa^{(2)}(y) = \int_0^y W^{(2)}(y-r)\bar{\mu}^{(1)}(r)dr + \frac{1}{\sigma^2}\bar{\mu}^{(1)}(y) + \bar{\mu}(0)W^{(2)}(y). \tag{2.12}$$

3. Smoothness of the invariant measure

In this section we investigate fine distributional properties of the density of the absolutely continuous part of the invariant measure of the CBI-semigroups. We already point out that although we restraint our analysis to the framework of this paper, our results extend modulo mild modifications to the most general case. For sake of completeness, we start by stating and providing a short and original proof of some basic properties of this invariant measure whose study traces back to the work of Pinsky [40], see also [25] and [32] for more recent references.

Proposition 3.1. *Let $(\psi,\phi) \in \mathcal{N}\times\mathcal{B}$.*

(1) The CBI(ψ,ϕ) *semigroup P admits a unique invariant probability measure $\mathcal{V}$ on $\mathbb{R}^+_{\geq 0}$, in the sense that, for any $f \in C_0(\mathbb{R}^+_{\geq 0})$,*

$$\mathcal{V}P_t f = \mathcal{V}f. \tag{3.1}$$

The measure $\mathcal{V}$ is infinitely divisible and its Laplace exponent is the function Φ_ν, see (1.12)*, which is the following Bernstein function*

$$\Phi_\nu(\lambda) = \int_0^\infty (1-e^{-\lambda r})\frac{\kappa(r)}{r}dr. \tag{3.2}$$

where κ, defined in (1.17)*, satisfies $\int_0^\infty \kappa(r)dr<\infty$.*

(2) There exists $\nu \in L^1(\mathbb{R}^+)$ such that

$$\mathcal{V}(dy) = e^{-\overline{\Phi}_\nu}\delta_0(dy) + \nu(y)dy, \tag{3.3}$$

where we recall that $\overline{\Phi}_\nu = \lim_{\lambda\to\infty}\Phi_\nu(\lambda)$. Moreover, if $\phi\equiv 0$, then $\nu=0$ a.e. and $\nu>0$ a.e. otherwise.

Remark 3.2. If $(\overline{\psi}, \overline{\phi}) \in \overline{\mathcal{N}} \times \overline{\mathcal{B}}$ with $(\psi, \psi) = \mathcal{E}(\overline{\psi}, \overline{\phi})$ then the measure $\mathcal{V}_\theta(dx) = e^{\theta x}\mathcal{V}(dx)$ is a $\overline{\phi}(\theta)$-stationary measure for the CBI$(\overline{\psi}, \overline{\phi})$semigroup $\overline{P}$, in the sense that, for any $f \in C_0(\mathbb{R}^+_{\geq 0})$,

$$\mathcal{V}_\theta \overline{P}_t f = e^{-\overline{\phi}(\theta)t}\mathcal{V}_\theta f.$$

Proof. We shall prove that Φ_ν is a Bernstein function. By (1.7) we have, for all $u > 0$,

$$\frac{\phi(u)}{\psi(u)} = \frac{u}{\psi(u)} \left(\int_0^\infty e^{-ur} \left(b\delta_0(dr) + \overline{\mu}(r)dr\right) \right). \tag{3.4}$$

Thus, using the relation (2.5), by convolution and injectivity of the Laplace transform, we get

$$\frac{\phi(u)}{\psi(u)} = \int_0^\infty e^{-ur}\kappa(r)dr \tag{3.5}$$

where κ is defined in (1.17). It is clear that $\kappa(r) \geq 0$, for all $r > 0$ and since $\phi, \psi \in \mathcal{H}(R)$ for some $R > 0$, with $\phi(0) = \psi(0) = 0$, $m > 0$ one has $\lim_{u\downarrow 0} \frac{\phi(u)}{\psi(u)} = \frac{\phi^{(1)}(0)}{m} < \infty$ and hence $\int_0^\infty \kappa(r)dr < \infty$. Now, integrating (3.5) yields, for all $\lambda \geq 0$,

$$\Phi_\nu(\lambda) = \int_0^\lambda \frac{\phi(u)}{\psi(u)}du = \int_0^\infty (1 - e^{-\lambda r})\frac{\kappa(r)}{r}dr,$$

and, one can check that $\int_0^\infty (1 \wedge r)\frac{\kappa(r)}{r}dr \leq \int_0^\infty \kappa(r)dr < \infty$. Thus, there exists an infinitely divisible measure $\mathcal{V}$ on $\mathbb{R}^+_{\geq 0}$ such that, for all $\lambda \geq 0$,

$$\mathcal{V}e_\lambda = e^{-\Phi_\nu(\lambda)}. \tag{3.6}$$

Since $\Phi_\nu(0) = 0$, the measure $\mathcal{V}$ is a probability measure. Now, from (1.14), we have for all $t, \lambda \geq 0$,

$$\mathcal{V}P_t e_\lambda = e^{-\Phi_\nu(\lambda)}e^{\Phi_\nu(B(A(\lambda)e^{-mt}))}\mathcal{V}e_{B(A(\lambda)e^{-mt})} = \mathcal{V}e_\lambda.$$

Since, by the Stone-Weistrass Theorem, $\Lambda = \text{Span}(e_\lambda)_{\lambda>0}$ is dense in $C_0(\mathbb{R}^+_{\geq 0})$, this proves (3.1), i.e. that $\mathcal{V}$ is an invariant probability measure for P. One obtains that it is unique by showing, from (3.1), that its Laplace transform is the unique solution to some ordinary differential equation with initial condition and invoking injectivity property of the Laplace transform. We refer the reader to Ogura [35, Proposition 1.1] for more details. Finally we see, from Sato [43, Theorem 27.7], that a sufficient condition for $\mathcal{V}$ to be absolutely continuous is $\int_0^\infty \frac{\kappa(r)}{r}dr = \infty$, that is, $\overline{\Phi}_\nu = \Phi_\nu(\infty) = \infty$. Moreover, in this case, we have from [22, Theorem 1] that a.e. $\nu > 0$. Assume now that $\int_0^\infty \frac{\kappa(r)}{r}dr < \infty$. Then $\mathcal{V}$ is a compound Poisson distribution associated to the Lévy measure $\frac{\kappa(r)}{r}dr$. If $\phi \equiv 0$, then $\kappa \equiv 0$, which implies $\mathcal{V}(dy) = \delta_0(dy)$ and so $\nu = 0$ a.e. Otherwise, when $b > 0$ or $0 < \bar{\mu}(0^+) \leq \infty$, then from (1.17) combined with $W^{(1)} > 0$ (see Lemma 2.3), we note that for all $r > 0$, $\frac{\kappa(r)}{r} > 0$. Then, by means of [43, Remark 27.3] (with $t = 1$ and recalling that therein the notation $\nu^{*0} = \delta_0$ is used), and using the fact that in our case the Lévy measure has a density, we conclude that there exists some positive density ν on $\mathbb{R}^+$ such that $\mathcal{V}(dy) = e^{-\overline{\Phi}_\nu}\delta_0(dy) + \nu(y)dy$. □

We proceed by giving (a necessary and) sufficient conditions for the absolute continuity of the invariant measure, that is, conditions for $\overline{\Phi}_\nu = \infty$ in (3.3).

Lemma 3.3. *If $\phi \equiv 0$, i.e. P is a* $\mathrm{CB}(\psi)$ *semigroup, then $\overline{\Phi}_\nu = 0$. Otherwise, we have $\overline{\Phi}_\nu < \infty$ (resp. $\overline{\Phi}_\nu = \infty$) if and only if $\int_0^1 \frac{\kappa(r)}{r}dr < \infty$ (resp. $\int_0^1 \frac{\kappa(r)}{r}dr = \infty$). A sufficient condition for $\overline{\Phi}_\nu = \infty$ is $\underline{\kappa}(0^+) = \underline{\lim}_{r\downarrow 0}\kappa(r) > 0$.*

Proof. The first claim is obvious and the necessary and sufficient condition can be easily deduced from the proof of Proposition 3.1. Next, assume that $\underline{\kappa}(0^+) > 0$, then there exists $C, \epsilon > 0$ such that for small $0 < r < \epsilon$, $\kappa(r) > C$ which implies that $\int_0^1 \frac{\kappa(r)}{r}dr \geq C\int_0^\epsilon \frac{dr}{r} = \infty$ and completes the proof of the Lemma. □

Corollary 3.4. *Let $\psi \in \mathcal{N}$. Then there exists a proper probability density function ω on $\mathbb{R}^+$ such that*

$$\int_0^\infty e^{-\lambda y}\omega(y)dy = 1 - A(\lambda), \quad \lambda \geq 0,$$

where we recall that $A(\lambda) = \exp\left(-m\int_\lambda^\infty \frac{du}{\psi(u)}\right)$.

Proof. The function A satisfies the differential equation $\psi(\lambda)A^{(1)}(\lambda) = mA(\lambda)$. Differentiating on both sides and rearranging gives $A^{(2)}(\lambda) = -\frac{\psi^{(1)}(\lambda)-m}{\psi(\lambda)}A^{(1)}(\lambda)$, which leads to

$$A^{(1)}(\lambda) = A^{(1)}(0)\exp\left(-\int_0^\lambda \frac{\psi^{(1)}(u)-m}{\psi(u)}du\right).$$

Since $(\psi, \psi^{(1)} - m) \in \mathcal{N}\times\mathcal{B}$, Proposition 3.1 yields that there exists a probability measure $\underline{\mathcal{V}}$ on $\mathbb{R}^+_{\geq 0}$ such that

$$\int_0^\infty e^{-\lambda y}\underline{\mathcal{V}}(dy) = \frac{A^{(1)}(\lambda)}{A^{(1)}(0)}. \tag{3.7}$$

Moreover, by assumption, $\lim_{u\to\infty}\psi(u) = \infty$ and $\int_1^\infty \frac{du}{\psi(u)} < \infty$, which yields

$$\int_0^\infty \frac{\psi^{(1)}(u)-m}{\psi(u)}du = \int_0^1 \frac{\psi^{(1)}(u)-m}{\psi(u)}du + [\log\psi(u)]_{u=1}^\infty - m\int_1^\infty \frac{du}{\psi(u)} = \infty.$$

This implies by Proposition 3.1 that $\underline{\mathcal{V}}$ is absolutely continuous on $\mathbb{R}^+_{\geq 0}$ and we denote its probability density function by $\underline{\nu}$. Since, from (1.10), $\lim_{\lambda\to 0}\int_\lambda^\infty \frac{du}{\psi(u)} = \infty$, we get $A(0) = 0$ and thus it follows by Tonelli Theorem,

$$\begin{aligned}1 - A(\lambda) = 1 - \int_0^\lambda A^{(1)}(u)du =& 1 - A^{(1)}(0)\int_0^\lambda \left(\int_0^\infty e^{-uy}\underline{\nu}(y)dy\right)du \\ =& 1 - A^{(1)}(0)\int_0^\infty \left(1 - e^{-\lambda y}\right)\frac{\underline{\nu}(y)}{y}dy.\end{aligned}$$

Because $\lim_{\lambda\to\infty}A(\lambda) = 1$, we therefore must have

$$A^{(1)}(0)\int_0^\infty \frac{\underline{\nu}(y)}{y}dy = 1 \tag{3.8}$$

and hence the corollary has been proved with

$$\omega(y) = \frac{\underline{\nu}(y)}{y\int_0^\infty \frac{\underline{\nu}(r)}{r}dr}.$$

□

Lemma 3.5. *First, we have $\psi \in \mathcal{H}(R)$ (resp. $\phi \in \mathcal{H}(R)$) if and only if $\int^{\infty} e^{ur}\Pi(dr) < \infty$ (resp. $\int^{\infty} e^{ur}\mu(dr) < \infty$) for all $u < R$. Next, let $(\psi, \phi) \in \mathcal{N} \times \mathcal{B}$. Then $R_\psi, R_\phi > 0$ and $\Phi_\nu \in \mathcal{H}(R_{\Phi_\nu})$ with $\Phi_\nu(0) = 0$ and $R_{\Phi_\nu} = R_\psi \wedge R_\phi \wedge \underline{\theta}$ where $\underline{\theta} = \inf\{u > 0; \psi(-u) = 0\} \in (0, \infty]$. Similarly $A \in \mathcal{H}(R_A)$ with $R_A = R_\psi \wedge \underline{\theta}$, $A(0) = 0$ and $A^{(1)}(\lambda) > 0$ for any $\lambda > -R_A$. As a by-product, $B \in \mathcal{H}(R_B)$ with $0 < R_B \leq 1$ and $B(0) = 0$. Finally, there exists some $0 < R_0 \leq 1$, such that, for all $x \geq 0$, $G_x \in \mathcal{H}(R_0)$.*

Proof. The first claim follows readily from [43, Theorem 25.17]. Next, under the assumptions (1.6), we have $\psi \in \mathcal{H}(R_\psi)$ with $\psi(0) = \phi(0) = 0$ and $\psi^{(1)}(0) = m > 0$. Moreover, as from Lemma 2.3, $u \mapsto \frac{u}{\psi(u)}$ is the Laplace transform of a positive measure, by a standard result on the Laplace transform, its first singularity, as a function of the complex variable, occurs on the (negative) real line. Since $\psi(0) = 0$, $\psi^{(1)}(0) = m > 0$ and $\psi \in \mathcal{H}(R_\psi)$, we deduce that the first singularity of $z \mapsto \frac{z}{\psi(z)}$ in the disc $\{z \in \mathbb{C}; |z| < R_\psi\}$ can be only, if it exists, a zero $-\underline{\theta} < 0$ of ψ, which is isolated from 0. Thus, $\frac{z}{\psi(z)} \in \mathcal{H}(R_\psi \wedge \underline{\theta})$, and, since $\phi \in \mathcal{H}(R_\phi)$, we conclude that $\Phi_\nu \in \mathcal{H}(R_{\Phi_\nu})$. From the proof of Corollary 3.4, we easily get, by combining (3.8) with (3.7), that $A^{(1)} \in \mathcal{H}(R_A)$ with $R_A = R_\psi \wedge \underline{\theta} > 0$ and $A^{(1)}(\lambda) > 0$ for any $\lambda > -R_A$. Therefore $A \in \mathcal{H}(R_\psi \wedge \underline{\theta})$ as well. Then by the Lagrange inversion theorem, the inverse function B of A belongs to $\mathcal{H}(R_B)$ with $R_B > 0$ and satisfies $B(0) = 0$ and $B^{(1)}(0) \neq 0$. Finally, $R_B \leq 1$ as $\lim_{\lambda\to\infty} A(\lambda) = \lim_{\lambda\to\infty} e^{-m\int_\lambda^\infty \frac{du}{\psi(u)}} = 1$. The last statement follows readily from the previous ones. □

Remark 3.6. Our results provide the smoothness of the transition kernel for $t > T_0$. Looking at (1.15) and (1.13), we see that in order to determine T_0, we need to know R_B, the radius of convergence of the Taylor series at 0 of B. Though we know by Lemma 3.5 that $0 < R_B \leq 1$, it would be interesting to find the precise value of R_B when A can not be inverted explicitly. In order to get an idea of the value of $T_0 = -\ln(R_0)/m$ in specific examples, one can instead proceed by numerically computing R_0, the radius of convergence of the power series (1.32). Note that in [10] we have given an algorithm for computing the eigenfunctions $\mathcal{L}_n(x)$.

With the aim of studying regularity properties of the heat kernel of CBI-semigroups, we need to deepen significantly the analysis concerning fine distributional properties of the invariant measure and in particular derive smoothness properties of its absolutely continuous part. We provide both a smoothness result on $\mathbb{R}$ as well as on $\mathbb{R}^+$. The question of how smooth infinitely divisible distributions are on $\mathbb{R}$ has been well-studied, see Section 28 of [43] for an overview. However for distributions with support on $\mathbb{R}^+$, the approaches developed in the literature are limited to the case when the density (or its derivatives) vanishes at 0 and to the best of our knowledge, we are not aware of any available techniques in the literature to deal with the smoothness on $\mathbb{R}^+$. To this purpose, we shall derive a convolution equation that the absolutely continuous part of the invariant measure satisfies and then apply the two lemmas in Section 2.3 to establish its degree of smoothness. The same technique will be used again in Lemma 5.3 to derive smoothness properties of the function $\mathcal{W}_n$. The next results can be seen as a significant complement of the previous works on the study of invariant measures of CBI-semigroups.

Proposition 3.7. *Recall from* (3.3) *the notation* $\mathcal{V}(dy) = e^{-\overline{\Phi}_\nu}\delta_0(dy) + \nu(y)dy, y > 0$. *We extend* $\nu : \mathbb{R}^+ \to \mathbb{R}$ *to* $\mathbb{R}$ *by setting* $\nu(y) = 0$ *for* $y \leq 0$. *Then, the density* ν *can be chosen such that it satisfies the following smoothness properties.*

(a) If $\underline{\kappa} \in [1, \infty]$, *then* $\nu \in C^{\underline{\kappa}-1}(\mathbb{R})$ *where we recall that* $\underline{\kappa}$ *is defined in* (1.18).

(b) Assume $\underline{\kappa} < \infty$.

(b1) If $\underline{\kappa} \geq 1$, *then* $\nu \in C^{\underline{\kappa}}(\mathbb{R}^+)$ *with* $\nu^{(\underline{\kappa})} \in L^1(\mathbb{R}^+)$ *and if* $\underline{\kappa} = 0$ *and* $\overline{\kappa}(0^+) < \infty$, *then* $\nu \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$.

(b2) If $\underline{\kappa}(0^+) = \overline{\kappa}(0^+)$, $\kappa \in C^q(\mathbb{R}^+)$ *for some* $q \geq 1$ *and* $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$, *then* $\nu \in C^{\underline{\kappa}+q}(\mathbb{R}^+)$.

Proof. The first claim follows from [47, Theorem 6], see also [43, Theorem 28.4]. Next assume that $\underline{\kappa} < \infty$. We first show that $\kappa \in C(\mathbb{R}^+)$. Since $\bar{\mu}$ is non-increasing on $\mathbb{R}^+$, it is continuous almost everywhere. Hence, by the dominated convergence theorem, one obtains, from (1.17), that for every $y > 0$,

$$\begin{aligned}
\lim_{\delta\to 0} \kappa(y+\delta) &= \lim_{\delta\to 0}\left(bW^{(1)}(y+\delta) + \int_0^{\frac{y+\delta}{2}} W^{(1)}(y+\delta-r)\bar{\mu}(r) + \bar{\mu}(y+\delta-r)W^{(1)}(r)dr\right) \\
&= bW^{(1)}(y) + \int_0^{\frac{y}{2}} W^{(1)}(y-r)\bar{\mu}(r)dr + \int_0^{\frac{y}{2}} \bar{\mu}(y-r)W^{(1)}(r)dr \\
&= \kappa(y)
\end{aligned}$$

where we used, for the second identity, the fact that $W^{(1)}$ is continuous, see Lemma 2.3. Next, differentiating (3.6), we get, for $\lambda > 0$,

$$\int_0^\infty e^{-\lambda y} y\nu(y)dy = \int_0^\infty e^{-\lambda y}\kappa(y)dy\, e^{-\Phi_\nu(\lambda)},$$

that is the density ν is (can be chosen as) the solution to the convolution equation, for any $y > 0$,

$$y\nu(y) = \int_0^y \nu(y-r)\kappa(r)dr + e^{-\overline{\Phi}_\nu}\kappa(y) = \int_0^y \kappa(y-r)\nu(r)dr + e^{-\overline{\Phi}_\nu}\kappa(y). \tag{3.9}$$

If $\underline{\kappa} \geq 1$, then, from Lemma 3.3, $\overline{\Phi}_\nu = \infty$ and, from item (a), $\nu \in C^{\underline{\kappa}-1}(\mathbb{R})$. Thus, we can use Lemma 2.4 (repeatedly if necessary) to deduce,

$$(y\nu(y))^{(\underline{\kappa}-1)} = \int_0^y \kappa(y-r)\nu^{(\underline{\kappa}-1)}(r)dr.$$

By [47, Theorem 6], $\nu^{(\underline{\kappa}-1)}$ has a derivative $\nu^{(\underline{\kappa})}$ that lies in $L^1(\mathbb{R}^+)$. Then since $\kappa \in C(\mathbb{R}^+)$ and $\nu^{(\kappa-1)}(0) = 0$, Lemma 2.4 yields that $\nu \in C^{\underline{\kappa}}(\mathbb{R}^+)$ and

$$(y\nu(y))^{(\underline{\kappa})} = \int_0^y \nu^{(\underline{\kappa})}(y-r)\kappa(r)dr = \int_0^y \kappa(y-r)\nu^{(\underline{\kappa})}(r)dr. \tag{3.10}$$

If $\underline{\kappa} = 0$ and $\overline{\kappa}(0^+) < \infty$, then since $\kappa \in C(\mathbb{R}^+)$, we get by an application of the dominated convergence theorem and (3.9) that $\nu \in C(\mathbb{R}^+)$. This proves (b1). For the last claim, assume that $\underline{\kappa}(0^+) = \overline{\kappa}(0^+)$, $\kappa \in C^1(\mathbb{R}^+)$ and $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$. Then by applying Lemma 2.4 to (3.9) in the case where $\underline{\kappa} = 0$ and to (3.10) in the case where $\underline{\kappa} \geq 1$, we get

$$(y\nu(y))^{(\underline{\kappa}+1)} = \int_0^y \kappa^{(1)}(y-r)\nu^{(\underline{\kappa})}(r)dr + \underline{\kappa}(0^+)\nu^{(\underline{\kappa})}(y) + e^{-\overline{\Phi}_\nu}\kappa^{(1)}(y).$$

Hence $\nu \in C^{\underline{\kappa}+1}(\mathbb{R}^+)$. If further $\kappa \in C^q(\mathbb{R}^+)$ for some $q \geq 2$, then one can use Lemma 2.5 and an induction argument to deduce that $\nu \in C^{\underline{\kappa}+q}(\mathbb{R}^+)$. □

4. Eigenfunctions: existence, properties and uniform bounds

In this part we investigate analytical properties of the sequence of eigenfunctions for CBI-semigroups.

Proposition 4.1. *For any $n = 0, 1, \ldots,$ and $x, t \geq 0$, we have*

$$P_t\mathcal{L}_n(x) = e^{-\lambda_n t}\mathcal{L}_n(x) \tag{4.1}$$

where we recall that $(\mathcal{L}_n)_{n\geq 0}$ is the family of Sheffer polynomials whose generating function is G_x, and, for all $\mathfrak{p} \geq 0$,

$$\frac{d^{\mathfrak{p}}}{dx^{\mathfrak{p}}}G_x(z) = \sum_{n=\mathfrak{p}}^{\infty} \mathcal{L}_n^{(\mathfrak{p})}(x)z^n, \quad |z| < R_0, \tag{4.2}$$

where the series is locally uniformly convergent in x. Moreover, for any $R \in (0, R_0)$, there exist $C = C(R) > 0$ and $\overline{B} = \max_{|z|=R}|B(z)| \in \mathbb{R}_+$ such that, for any $x \geq 0$, $n \geq 0$,

$$|\mathcal{L}_n(x)| \quad \leq \quad C\frac{e^{\overline{B}x}}{R^{n+1}}. \tag{4.3}$$

Remark 4.2. If $(\overline{\psi}, \overline{\phi}) \in \overline{\mathcal{N}} \times \overline{\mathcal{B}}$ with $(\psi, \psi) = \mathcal{E}(\overline{\psi}, \overline{\phi})$, then, writing $\mathcal{L}_n^\theta = e_\theta\mathcal{L}_n$, we have, for all $t, x \geq 0$, $n \geq 0$,

$$\overline{P}_t\mathcal{L}_n^\theta(x) = e^{-\lambda_n t}\mathcal{L}_n^\theta(x).$$

Proof. Since, from Lemma 3.5, we have for all $x \geq 0$, $G_x(z) = e^{\Phi_\nu(B(z))-xB(z)} \in \mathcal{H}(R_0)$, $R_0 > 0$, an application of the Cauchy's formula yields that

$$\mathcal{L}_n(x) = \frac{1}{2\pi i}\oint \frac{G_x(z)}{z^{n+1}}dz \tag{4.4}$$

where the contour is a circle centered at 0 and of radius $R < R_0$. Since the functions B and $\Phi_\nu \circ B \in \mathcal{H}(R_0)$, they are bounded on this contour and we get that

$$|\mathcal{L}_n(x)| \quad \leq \quad \frac{1}{2\pi}\oint \left|\frac{G_x(z)}{z^{n+1}}\right| dz \leq \frac{e^{\overline{B}x}}{2\pi}\oint \left|\frac{e^{\Phi_\nu(B(z))}}{z^{n+1}}\right| dz \leq CR^{-n-1}e^{\overline{B}x} \tag{4.5}$$

where $C > 0$. This proves (4.3). Next, since $B \in \mathcal{H}(R_0)$ with $B(0) = 0$, we can choose R such that $\overline{B} = \max_{|z|=R}|B(z)| < R_0$. From (1.14), we have, for any $\Re(z) > 0$,

$$P_t e_z(x) = e^{-\Phi_\nu(z)}G_x(A(z)e^{-mt}).$$

Since $z \mapsto e^{-\Phi_\nu(z)}G_x(A(z)e^{-mt}) \in \mathcal{H}(R_0)$, by the principle of analytical continuation, for all $x, t \geq 0$, $z \longmapsto P_t e_z(x) \in \mathcal{H}(R_0)$ and thus from (4.3) we get that $P_t|\mathcal{L}_n|(x) \leq CR^{-n-1}P_t e_{-\overline{B}}(x) < \infty$ as $\overline{B} < R_0$. Thus, by Fubini Theorem, one gets, for any $x \geq 0$,

$$\begin{aligned} P_t\mathcal{L}_n(x) &= \frac{1}{2\pi i}\oint \frac{P_t(G_.(z))(x)}{z^{n+1}}dz = \frac{1}{2\pi i}\oint \frac{G_x(ze^{-mt})}{z^{n+1}}dz \\ &= e^{-mnt}\frac{1}{2\pi i}\oint_t \frac{G_x(\bar{z})}{\bar{z}^{n+1}}d\bar{z} \\ &= e^{-\lambda_n t}\mathcal{L}_n(x) \end{aligned}$$

where we performed an obvious change of variable and we used the following identities, with the obvious notation,

$$P_t(G_.(z))(x) = e^{\Phi_\nu(B(z))}P_t e_{B(z)}(x) = e^{\Phi_\nu(B(z))}e^{-\Phi_\nu(B(z))}G_x(ze^{-mt}) = G_x(ze^{-mt}). \tag{4.6}$$

This completes the proof of the first statement with $\mathfrak{p} = 0$. The case $\mathfrak{p} \geq 1$ follows again from Cauchy's formula after observing that the mapping $z \mapsto \frac{d^{\mathfrak{p}}}{dx^{\mathfrak{p}}}G_x(z) = (-1)^{\mathfrak{p}}B^{\mathfrak{p}}(z)G_x(z) \in \mathcal{H}(R_0)$. □

We proceed by providing some additional properties regarding the CBI-semigroups and the Scheffer polynomials considered in this paper.

Proposition 4.3. *Assume that $\phi \neq 0$. Then P extends to a strongly continuous contraction semigroup on the Hilbert space $L^2(\mathcal{V})$, defined in (1.42), which is still denoted by P. Moreover, for any $t > 0$,*

$$(e^{-\lambda_n t})_{n\geq 0} \subseteq \mathrm{S}_p(P_t) = \{z \in \mathbb{C};\ P_t - z\mathbf{I} \text{ is not one-to-one in } L^2(\mathcal{V})\},$$

that is the point spectrum of P_t, and $(\mathcal{L}_n)_{n\geq 0}$ is a complete sequence of eigenfunctions of P_t in $L^2(\mathcal{V})$. However, the sequence $(\mathcal{L}_n)_{n\geq 0}$ is formed of orthogonal polynomials in some weighted L^2 space if and only if $(\mathcal{L}_n)_{n\geq 0}$ is the sequence of Laguerre polynomials, i.e. $\psi(u) = \sigma^2 u + mu, \sigma^2 > 0, m \geq 0$, and $\phi(u) = bu$, $b > 0$. Hence, beyond this case, P_t is a non-self-adjoint contraction semigroup in $L^2(\mathcal{V})$. Finally, the algebra of polynomials $\mathcal{A}$ is a core of its generator $\mathbf{L}$.

Remark 4.4. Although our proof for the non-self-adjointness property of CBI-semigroups is rather straightforward, we mention that, in a recent and interesting paper, Handa [20] has shown that the invariant measure of the so-called CIR semigroup, that is the Gamma distribution, is the only reversible probability measure within the entire class of CBI invariant probability measures, which means that beyond the diffusion case they are non-self-adjoint.

Proof. Since from Proposition 3.1, $\mathcal{V}$ is an invariant measure and P is a Feller semigroup, we deduce from Theorem 5.8 in [13] the first claim, that is, P admits a unique contraction extension in $L^2(\mathcal{V})$. Next, since, for any $\lambda > 0$, $\mathcal{V}e_\lambda = e^{-\Phi_\nu(\lambda)}$ with, from Lemma 3.5, $\Phi_\nu \in \mathcal{H}(R_{\Phi_\nu})$, we get that for any $0 < \epsilon < R_{\Phi_\nu}$, $\int_0^\infty e^{\epsilon y}\mathcal{V}(dy) < \infty$. Hence, the probability measure $\mathcal{V}$ is moment determinate and according to [2, Corollary 2.3] the sequence $(\mathcal{L}_n)_{n\geq 0}$ is complete in $L^2(\mathcal{V})$. Since for all $n \geq 0$, $\mathcal{L}_n \in L^2(\mathcal{V})$, the property of the point spectrum is a direct consequence of the relation (4.1). The statement regarding the orthogonality property of the polynomials follows from a result of Chahira [11], see also [34], stating that the only sequence of orthogonal polynomials on a (weighted) Hilbert space L^2 with support on $\mathbb{R}_+$ and whose generating function is of the form $B(z)e^{xA(z)}$, are the Laguerre polynomials, i.e. when

$\psi(u) = \sigma^2 u + mu, \sigma^2 > 0$ and $\phi(u) = bu$. This implies that, beyond these cases, the CBI-semigroups are non-self-adjoint in $L^2(\mathcal{V})$. Using (4.1), we get, in the $L^2(\mathcal{V})$ topology, that, for any $n \geq 0$,

$$\mathbf{L}\mathcal{L}_n(x) = \lim_{t\downarrow 0} \frac{P_t\mathcal{L}_n(x) - \mathcal{L}_n(x)}{t} = \lim_{t\downarrow 0} \frac{e^{-\lambda_n t} - 1}{t}\mathcal{L}_n(x) = -\lambda_n \mathcal{L}_n(x),$$

which completes the proof. □

5. Eigenmeasures: characterization, properties and uniform bounds

The next result provides the existence as well as explicit representations of the eigenmeasures of the CBI-semigroup, see (5.2) below for definition. As a by-product, we derive sufficient conditions for these eigenmeasures to be absolutely continuous with a smooth density. We also establish a uniform upper bound which will be needed later for obtaining smoothness properties of the transition densities.

Proposition 5.1. *Let $(\psi, \phi) \in \mathcal{N} \times \mathcal{B}$ and $n \geq 0$. The following bound*

$$|\mathcal{V}_n| e_\lambda \leq e^{-\Phi_\nu(\lambda)}(2 - A(\lambda))^n \tag{5.1}$$

holds for any $\lambda > -(R_A \wedge R_\phi)$ where $|\mathcal{V}_n|$ stands for the total variation of the (signed) measure $\mathcal{V}_n$, which we recall is defined in (1.37).

Moreover, $\mathcal{V}_n$ is an eigenmeasure of the CBI(ψ, ϕ) semigroup P, in the sense that, for any $f \in \mathcal{D}_{R_A \wedge R_\phi} = \{f : \mathbb{R}^+_{\geq 0} \to \mathbb{R}$ measurable; $f e_\lambda \in L^\infty(\mathbb{R}_+)$ for some $\lambda < R_A \wedge R_\phi\}$, and $t \geq 0$,

$$\mathcal{V}_n P_t f = e^{-\lambda_n t}\mathcal{V}_n f. \tag{5.2}$$

Next, recall from (1.37) that $\mathcal{V}_n(dy) = e^{-\overline{\Phi}_\nu}\delta_0(dy) + \nu_n(y)dy$. We have the following smoothness properties of ν_n.

a) If $\underline{\kappa} \in [1, \infty]$, then $\nu_n \in C^{\underline{\kappa}-1}(\mathbb{R})$ with, for any integer $0 \leq \mathfrak{q} \leq \underline{\kappa} - 1$ and $y \geq 0$,

$$\nu_n^{(\mathfrak{q})}(y) = \mathcal{W}_n * \nu^{(\mathfrak{q})}(y) + \nu^{(\mathfrak{q})}(y), \quad n \geq 1.$$

Further, for all $0 \leq \mathfrak{q} \leq \underline{\kappa} - 1$, $K > 0$, $\lambda > 0$ there exists $C = C_{K,\lambda}(\mathfrak{q}) > 0$ such that, for all $n \geq 1$,

$$\sup_{y\in[0,K]} \left|\nu_n^{(\mathfrak{q})}(y)\right| \leq C(2 - A(\lambda))^n.$$

b) Assume $\underline{\kappa} < \infty$.

b1) If $\underline{\kappa} \geq 1$, then $\nu_n \in C^{\underline{\kappa}}(\mathbb{R}_+)$ and if $\underline{\kappa} = 0$ and $\overline{\kappa}(0^+) < \infty$, then $\nu_n \in C(\mathbb{R}_+)$. In both cases we have that for all $K > 0$, $\lambda > 0$ there exists $C = C_{K,\lambda}(\underline{\kappa}) > 0$ such that, for all $n \geq 1$,

$$\sup_{y\in[K^{-1},K]} \left|\nu_n^{(\underline{\kappa})}(y)\right| \leq Cn(2 - A(\lambda))^n. \tag{5.3}$$

b2) If $\kappa \in C^1(\mathbb{R}_+)$, $\kappa' \in L^1_{loc}(\mathbb{R}_+)$ and $\underline{\kappa}(0^+) = \overline{\kappa}(0^+)$, then $\nu_n \in C^{\underline{\kappa}+\bar{\mathfrak{q}}}(\mathbb{R}_+)$ with $\bar{\mathfrak{q}}$ as in Theorem 1.1(4(b)2). *Moreover, for all $1 \leq l \leq \bar{\mathfrak{q}}$, $K > 0$, $\lambda > 0$ there exists $C = C_{K,\lambda}(\underline{\kappa} + l) > 0$ such that, for all $n \geq 1$,*

$$\sup_{y\in[K^{-1},K]} \left|\nu_n^{(\underline{\kappa}+l)}(y)\right| \leq Cn^{l+1}(2 - A(\lambda))^n. \tag{5.4}$$

Remark 5.2. Note that if $(\overline{\psi}, \overline{\phi}) \in \overline{\mathcal{N}} \times \overline{\mathcal{B}}$ with $(\psi, \phi) = \mathcal{E}(\overline{\psi}, \overline{\phi})$ then the measure $\overline{\mathcal{V}}_n(dy) = e^{\theta y}\mathcal{V}_n(dy)$ is an eigenmeasure for the CBI$(\overline{\psi}, \overline{\phi})$ semigroup $\overline{P}$, in the sense that $\overline{\mathcal{V}}_n\overline{P}_t f = e^{-(\overline{\phi}(\theta)+\lambda_n)t}\overline{\mathcal{V}}_n f$.

5.1. **Proof of Proposition 5.1.** We split the proof into several steps. Note that for $n = 0$ we have $\lambda_n = 0$ and $\mathcal{V}(dy) = \mathcal{V}_0(dy) = e^{-\overline{\Phi}_\nu}\delta_0(dy) + \nu_0(y)dy$ with $\nu_0(y) = \nu(y)$. Hence this case corresponds to the study of the invariant measure $\mathcal{V}$ which was addressed in Proposition 3.1.

5.1.1. *Proof of* (5.2). By the definitions (1.35) and (1.34), a classical property of the Laplace transform of a convolution and the binomial theorem,

$$\begin{aligned}\int_0^\infty e^{-\lambda y}\mathcal{W}_n(y)dy &= \sum_{j=1}^n \binom{n}{j}(-1)^j\left(\int_0^\infty e^{-\lambda y}\omega(y)dy\right)^j \\ &= -1 + \sum_{j=0}^n \binom{n}{j}(-1)^j(1 - A(\lambda))^j \\ &= A(\lambda)^n - 1,\end{aligned} \tag{5.5}$$

where $\lambda > -R_A$. By the triangle inequality $|\mathcal{W}_n(y)| \leq \sum_{j=1}^n \binom{n}{j}\omega^{*j}(y)$ and so by the same arguments that led to (5.5),

$$\int_0^\infty e^{-\lambda y}|\mathcal{W}_n(y)|dy \leq (2 - A(\lambda))^n - 1. \tag{5.6}$$

Thus, combining (1.33), (1.37), (1.38), Lemma 3.5, (5.5) and (5.6) we get, for any $\lambda > -(R_A \wedge R_\phi)$, that

$$\mathcal{V}_n e_\lambda = \left(e^{-\overline{\Phi}_\nu} + \int_0^\infty e^{-\lambda y}\nu(y)dy\right)\left(1 + \int_0^\infty e^{-\lambda y}\mathcal{W}_n(y)dy\right) = e^{-\Phi_\nu(\lambda)}A(\lambda)^n. \tag{5.7}$$

and the inequality (5.1). On the other hand, from the expression (1.14) of the Laplace transform of the semigroup, we obtain that, for any $t, \lambda > 0$ and $n \geq 0$,

$$\begin{aligned}\mathcal{V}_n P_t e_\lambda &= e^{-\Phi_\nu(\lambda)}\int_0^\infty G_x(A(\lambda)e^{-mt})\mathcal{V}_n(dx) \\ &= e^{-\Phi_\nu(\lambda)}e^{\Phi_\nu(B(A(\lambda)e^{-mt}))}\mathcal{V}_n e_{B(A(\lambda)e^{-mt})} \\ &= e^{-\Phi_\nu(\lambda)}e^{\Phi_\nu(B(A(\lambda)e^{-mt}))}e^{-\Phi_\nu(B(A(\lambda)e^{-mt}))}A(\lambda)^n e^{-\lambda_n t} \\ &= e^{-\Phi_\nu(\lambda)}A(\lambda)^n e^{-\lambda_n t} = e^{-\lambda_n t}\mathcal{V}_n e_\lambda\end{aligned} \tag{5.8}$$

where the third equality is obtained by means of the identity (5.7) and recalling that B is the inverse of A. We complete the proof of (5.2) by combining the identities (5.7) and (5.8) and invoking the injectivity property of the Laplace transform.

5.1.2. *A key lemma.* Before we continue with the proof of Proposition 5.1, we need the following lemma on the function $\mathcal{W}_n$ defined in (1.35).

Lemma 5.3. *For any $n \geq 1$, $\mathcal{W}_n \in C^1(\mathbb{R}_+)$. Further, if $W \in C^p(\mathbb{R}_+)$ for some $p \geq 2$, then for any $n \geq 1$, $\mathcal{W}_n \in C^p(\mathbb{R}_+)$ and for all $0 \leq l \leq p$, $K > 0$, $\lambda > 0$ there exists $0 < C = C_{K,\lambda}(l) < \infty$ such that, for all $n \geq 1$,*

$$\sup_{y \in [K^{-1},K]} \left|\mathcal{W}_n^{(l)}(y)\right| \leq Cn^{l+1}(2 - A(\lambda))^n. \tag{5.9}$$

Proof. Recalling that $A(\lambda) = e^{-m\int_\lambda^\infty \frac{du}{\psi(u)}}$ and $\lambda_n = mn$, we have, for any $n \geq 0$ and $\lambda > 0$, the identity

$$\frac{d}{d\lambda}A^n(\lambda) = \frac{\lambda_n}{\psi(\lambda)}A^n(\lambda).$$

Then the Laplace transform inversion combined with (1.16) and (5.5) yield, for any $y > 0$,

$$-y\mathcal{W}_n(y) = \lambda_n\left(\int_0^y W(y-r)\mathcal{W}_n(r)dr + W(y)\right). \tag{5.10}$$

Since $W \in C(\mathbb{R}^+)$, we get by an application of the dominated convergence theorem to (5.10) that $\mathcal{W}_n \in C(\mathbb{R}^+)$. Moreover, since $W^{(1)} \in L^1(\mathbb{R}_+)$ by Lemma 2.3 and $\mathcal{W}_n \in L^1(\mathbb{R}_+)$ by (5.6), Lemma 2.4 in combination with (5.10) yields that $\mathcal{W}_n \in C^1(\mathbb{R}_+)$ with

$$(-y\mathcal{W}_n(y))^{(1)} = \lambda_n\left(\int_0^y W^{(1)}(y-r)\mathcal{W}_n(r)dr + W^{(1)}(y)\right). \tag{5.11}$$

Assume now that $W \in C^p(\mathbb{R}_+)$ for some $p \geq 2$. Then by induction and Lemma 2.5 combined with Leibniz's formula, we deduce that $\mathcal{W}_{\lambda_n} \in C^p(\mathbb{R}_+)$, and, for any $n > 0$,

(5.12)

$$\begin{aligned}
-\frac{y\mathcal{W}_n^{(p)}(y) + p\mathcal{W}_n^{(p-1)}(y)}{\lambda_n} &= -\frac{(y\mathcal{W}_n(y))^{(p)}}{\lambda_n} \\
&= \int_0^{\frac{y}{2}} W^{(p)}(y-r)\mathcal{W}_n(r)dr + \int_0^{\frac{y}{2}} \mathcal{W}_n^{(p-1)}(y-r)W'(r)dr + W^{(p)}(y) \\
&\quad + \tfrac{1}{2}\sum_{j=0}^{p-2}\left(W^{(j+1)}(\tfrac{y}{2})\mathcal{W}_n(\tfrac{y}{2}) + W'(\tfrac{y}{2})\mathcal{W}_n^{(j)}(\tfrac{y}{2})\right)^{(p-2-j)} \\
&= \int_0^{\frac{y}{2}} W^{(p)}(y-r)\mathcal{W}_n(r)dr + \int_0^{\frac{y}{2}} \mathcal{W}_n^{(p-1)}(y-r)W'(r)dr + W^{(p)}(y) \\
&\quad + \tfrac{1}{2}\sum_{j=0}^{p_0}(\tfrac{1}{2})^{p_j}\sum_{k=0}^{p_j}\binom{p_j}{k}\left(W^{(p-1-k)}(\tfrac{y}{2})\mathcal{W}_n^{(k)}(\tfrac{y}{2}) + W^{(1+k)}(\tfrac{y}{2})\mathcal{W}_n^{(p_k)}(\tfrac{y}{2})\right),
\end{aligned}$$

where for $k = 0, 1, \ldots$, we have set $p_k = p - 2 - k$. Next, we prove the uniform bound (5.9) by induction in l. First assume that $l = 0$. Then the identity (5.10) combined with the fact that

W is non-negative and increasing entail, writing for $j = 1, 2$, $I_j = [\frac{1}{jK}, K]$ here and below, that

$$
\begin{aligned}
\sup_{y\in I_1} |y\mathcal{W}_n(y)| &\leq \lambda_n W(K)\left(\int_0^K |\mathcal{W}_n(r)|dr + 1\right) \\
&\leq \lambda_n W(K)\left(e^{\lambda K}\int_0^\infty e^{-\lambda r}|\mathcal{W}_n(r)|dr + 1\right) \\
&\leq mW(K)ne^{\lambda K}(2 - A(\lambda))^n,
\end{aligned} \tag{5.13}
$$

where we used the inequality (5.6) in the last line. Hence (5.9) holds for $l = 0$. Now assume that $l = 1$. Then starting from the relation (5.11) and using similar arguments as for the previous case plus (5.13), we get

$$
\begin{aligned}
\sup_{y\in I_1}\left|y\mathcal{W}_n^{(1)}(y)\right| &\leq \sup_{y\in I_1}|\mathcal{W}_n(y)| \\
&+ \lambda_n \sup_{y\in I_1}\left|\int_0^{\frac{y}{2}} W^{(1)}(y-r)\mathcal{W}_n(r) + \mathcal{W}_n(y-r)W^{(1)}(r)dr + W^{(1)}(y)\right| \\
&\leq Cn(2-A(\lambda))^n + \lambda_n\left(e^{\lambda K}(2-A(\lambda))^n \sup_{y\in I_2} W^{(1)}(y) + W(K)\sup_{y\in I_2}\mathcal{W}_n(y)\right) \\
&\leq n(2-A(\lambda))^n\left(C + me^{\lambda K}\sup_{y\in I_2} W^{(1)}(y) + \lambda_n CW(K)\right),
\end{aligned}
$$

where $C > 0$ is a generic constant. Hence (5.9) holds for $l = 1$. Now assume that (5.9) is true for $l = 0, 1, \ldots, k-1$ with $2 \leq k \leq p$. Then by (5.12) and the induction hypothesis, we get, writing $k_j = k - 2 - j$, for $j = 0, 1 \ldots$,

$$
\begin{aligned}
\sup_{y\in I_1}\left|y\mathcal{W}_n^{(k)}(y)\right| &\leq \sup_{y\in I_1}\left|k\mathcal{W}_n^{(k-1)}(y)\right| + \lambda_n\left(\int_0^{\frac{K}{2}}|\mathcal{W}_n(r)|\,dr\sup_{y\in I_2}\left|W^{(k)}(y)\right|\right. \\
&+ \sup_{y\in I_2}\left|\mathcal{W}_n^{(k-1)}(y)\right|\int_0^{\frac{K}{2}} W^{(1)}(r)dr + \sup_{y\in I_1}\left|W^{(k)}(y)\right| \\
&+ \left.\frac{1}{2}\sum_{j=0}^{k_0}(\tfrac{1}{2})^{k_j}\sum_{i=0}^{k_j}\binom{k_j}{i}\sup_{y\in I_2}\left|W^{(k_i)}(y)\mathcal{W}_n^{(i)}(y)\right| + \sup_{y\in I_2}\left|W^{(1+i)}(y)\mathcal{W}_n^{(k_i)}(y)\right|\right) \\
&\leq n(2-A(\lambda))^n\left(n^{k-1}kC(k) + m\sup_{y\in I_2}\left|W^{(k)}(y)\right|e^{\lambda K} + n^k mC(k)W(K)\right. \\
&+ \left. mk_0!\sum_{j=0}^{k_0}\sum_{i=0}^{k_j} n^{i+1}C(i)\sup_{y\in I_2}\left|W^{(k_i)}(y)\right| + n^{k_i+1}C(k_i)\sup_{y\in I_2}\left|W^{(1+i)}(y)\right|\right),
\end{aligned}
$$

where the $C(k)$'s are generic positive constants and we have used that $(2 - A(\lambda))^n$ is increasing in $n \geq 1$ because $A(\lambda) \in (0, 1)$ for $0 < \lambda < \infty$. It follows that (5.9) holds for $l = k$, which completes the proof. □

5.1.3. *Proof of Proposition 5.1*(a). Assume that $\underline{\kappa} \in [1, \infty]$. Then $\overline{\Phi}_\nu = \infty$, see Lemma 3.3 and so

$$\nu_n(y) = \mathcal{W}_n * \nu(y) + \nu(y).$$

By Proposition 3.7(a), $\nu \in C^{\underline{\kappa}-1}(\mathbb{R})$. Hence, for all $0 \leq \mathfrak{q} \leq \underline{\kappa} - 1$, $\nu^{(\mathfrak{q})}(0) = 0$ and since $\mathcal{W}_n$ is in $C(\mathbb{R}_+)$ by Lemma 5.3, we have by using Lemma 2.4 repeatedly,

$$\nu_n^{(\mathfrak{q})}(y) = \mathcal{W}_n * \nu^{(\mathfrak{q})}(y) + \nu^{(\mathfrak{q})}(y), \tag{5.14}$$

and, in particular, $\nu_n \in C^{\underline{\kappa}-1}(\mathbb{R})$. To prove the uniform bound, note that by (5.14) and (5.6),

$$\begin{aligned} |\nu_n^{(\mathfrak{q})}(y)| \leq & \left(e^{\lambda y} \int_0^\infty e^{-\lambda r} |\mathcal{W}_n(r)| dr + 1 \right) \sup_{0 \leq r \leq y} |\nu^{(\mathfrak{q})}(r)| \\ \leq & e^{\lambda y} (2 - A(\lambda))^n \sup_{0 \leq r \leq y} |\nu^{(\mathfrak{q})}(r)|. \end{aligned}$$

Hence the result follows as $\nu^{(\mathfrak{q})} \in C(\mathbb{R})$.

5.1.4. *Proof of Proposition 5.1*(b1). Recall that, for any $n \geq 1$ and $y > 0$,

$$\nu_n(y) = \int_0^y \mathcal{W}_n(y - r) \nu(r) dr + \nu(y) + e^{-\overline{\Phi}_\nu} \mathcal{W}_n(y). \tag{5.15}$$

Assume first that $1 \leq \underline{\kappa} < \infty$. Then, from Lemma 3.3, $\overline{\Phi}_\nu = \infty$ and thus by Proposition 3.7(a) and (b1) and Lemma 2.4, $\nu_n \in C^{\underline{\kappa}}(\mathbb{R}_+)$ with, for any $y > 0$,

$$\nu_n^{(\underline{\kappa})}(y) = \int_0^y \mathcal{W}_n(y - r) \nu^{(\underline{\kappa})}(r) dr + \nu^{(\underline{\kappa})}(y). \tag{5.16}$$

Then, we get that, for any $y > \epsilon > 0$, $\lambda > 0$ and $n \geq 1$,

$$\begin{aligned} |\nu_n^{(\underline{\kappa})}(y)| \quad \leq & \quad \sup_{r \in [y-\epsilon, y]} |\mathcal{W}_n(r)| \int_0^\epsilon |\nu^{(\underline{\kappa})}(r)| dr + \sup_{r \in [\epsilon, y]} |\nu^{(\underline{\kappa})}(r)| \int_\epsilon^y |\mathcal{W}_n(r)| dr + |\nu^{(\underline{\kappa})}(y)| \\ \leq & \quad C_\epsilon n (2 - A(\lambda))^n + C e^{\lambda y} (2 - A(\lambda))^n \end{aligned} \tag{5.17}$$

where for the second inequality we used (5.9) with $l = 0$, (5.6) and the fact that $\nu^{(\underline{\kappa})} \in L^1(\mathbb{R}_+)$, which is given in Proposition 3.7. Hence (5.3) follows for $\underline{\kappa} \geq 1$.

Second, if $\underline{\kappa} = 0$ and $\overline{\kappa}(0^+) < \infty$, then by Proposition 3.7(b1), the fact that $\mathcal{W}_n$ is in $C(\mathbb{R}_+)$ and an application of the dominated convergence Theorem to (5.15), $\nu_n \in C(\mathbb{R}_+)$. The bound (5.3) is derived by means of similar arguments as in the previous case but using the identity (5.15). This completes the proof of (b1).

5.1.5. *Proof of Proposition 5.1*(b2). Assume $\kappa \in C^1(\mathbb{R}_+)$, $\kappa' \in L^1_{loc}(\mathbb{R}_+)$ and $\underline{\kappa}(0^+) = \overline{\kappa}(0^+)$ and let $\bar{\mathfrak{q}} \geq 1$ be as in Theorem 1.1(4b). Then invoking Proposition 3.7(b2) and Lemma 5.3 and by applying Lemma 2.5 to the identity (5.15) in the case where $\underline{\kappa} = 0$ and to (5.16) in the case where $\underline{\kappa} \geq 1$, we get that $\nu_n \in C^{\underline{\kappa}+\bar{\mathfrak{q}}}(\mathbb{R}_+)$ with, for any $y > 0$,

$$\begin{aligned}\nu_n^{(\underline{\kappa}+\bar{\mathfrak{q}})}(y) &= \int_0^{\frac{y}{2}} \mathcal{W}_n^{(\bar{\mathfrak{q}})}(y-r)\nu^{(\underline{\kappa})}(r)dr + \int_0^{\frac{y}{2}} \nu^{(\underline{\kappa}+\bar{\mathfrak{q}})}(y-r)\mathcal{W}_n(r)dr + \nu^{(\underline{\kappa}+\bar{\mathfrak{q}})}(y) \\ &+ e^{-\overline{\Phi}_\nu}\mathcal{W}_n^{(q)}(y) + \tfrac{1}{2}\sum_{j=0}^{\bar{\mathfrak{q}}-1}\left(\nu^{(\underline{\kappa}+j)}(\tfrac{y}{2})\mathcal{W}_n(\tfrac{y}{2}) + \nu^{(\underline{\kappa})}(\tfrac{y}{2})\mathcal{W}_n^{(j)}(\tfrac{y}{2})\right)^{(\bar{\mathfrak{q}}-1-j)}.\end{aligned}$$

The bound (5.4) is obtained from this identity in the same way as the previous bound, the tedious details being left to the reader. This completes the proof of Proposition 5.1.

We end this part with the following results which provide additional but more specific properties regarding the set of eigenmeasures and complement the results on the set of eigenfunctions stated in Proposition 4.3. Notions introduced below are classical and their definitions can be found for instance in the textbooks [15] and [48].

Proposition 5.4. *Let $(\psi, \phi) \in \mathcal{N} \times \mathcal{B}$. Then, for all $n, m \in \mathbb{N}$, we have*

$$\mathcal{V}_m\mathcal{L}_n = \delta_{n,m} \tag{5.18}$$

where $\delta_{n,m}$ is the Kronecker symbol. Assume now that $\phi \neq 0$. Moreover, for all $n \geq 0$, the measure $\mathcal{V}_n$ is absolutely continuous with respect to $\mathcal{V}$ and we write $\overline{\mathcal{V}}_n$ for the corresponding Radon-Nykodim derivative. If for some $n \in \mathbb{N}$, $\overline{\mathcal{V}}_n \in L^2(\mathcal{V})$ then $e^{-\lambda_n t} \in \mathrm{S}_p(P_t^*)$, where P_t^* is the adjoint of P_t in $L^2(\mathcal{V})$. Moreover if $(\overline{\mathcal{V}}_n)_{n\geq 0} \in L^2(\mathcal{V})$ then the geometric and algebraic multiplicity of $e^{-\lambda_n t}$ is 1 for all $n \geq 0$. Finally, in this case, $(\mathcal{L}_n, \overline{\mathcal{V}}_n)_{n\geq 0}$ is a biorthogonal sequence in $L^2(\mathcal{V})$.

Remark 5.5. Note that showing that $\overline{\mathcal{V}}_n \in L^2(\mathcal{V})$ for some $n \in \mathbb{N}$ seems to be a difficult problem as one has to get precise asymptotic estimate for small and large values of the argument of the functions ν_n and ν. However, in the example 1.16, one easily gets that $\overline{\mathcal{V}}_n(y) = \sum_{j=0}^n \binom{n}{j}(-1)^j \frac{y^{\alpha j}}{\Gamma(\alpha j+1)} \in L^2(\mathcal{V})$ where we recall that in this case $\mathcal{V}(dy) = e^{-y}dy$.

Proof. First, observe, from (4.4), that, for all $n, m \in \mathbb{N}$,

$$\mathcal{V}_m\mathcal{L}_n = \int_0^\infty \frac{1}{2\pi i}\oint \frac{G_x(z)}{z^{n+1}}dz\mathcal{V}_m(dx).$$

Next, using (4.5), we have, choosing R such that $0 < \overline{B} < R_A$ with $\overline{B} = \max_{|z|=R}|B(z)|$,

$$\int_0^\infty \left|\oint \frac{G_x(z)}{z^{n+1}}dz\right| |\mathcal{V}_m|(dx) \leq C_R\int_0^\infty e^{\overline{B}x}|\mathcal{V}_m|(dx) < \infty$$

where $C_R > 0$ and the last inequality follows from (5.1). Thus, an application of Fubini Theorem yields

$$\begin{aligned}\mathcal{V}_m\mathcal{L}_n &= \frac{1}{2\pi i}\oint e^{\Phi_\nu(B(z))}\mathcal{V}_m e_{B(z)}\frac{dz}{z^{n+1}} \\ &= \frac{1}{2\pi i}\oint \frac{1}{z^{n-m+1}}dz = \delta_{n,m}.\end{aligned}$$

Next, let $f \in L^2(\mathcal{V})$ then, for any $n \in \mathbb{N}$, using (5.2),

$$\langle f, P_t^*\overline{\mathcal{V}}_n\rangle_{\mathcal{V}} = \langle P_t f, \overline{\mathcal{V}}_n\rangle_{\mathcal{V}} = e^{-nt}\langle f, \overline{\mathcal{V}}_n\rangle_{\mathcal{V}},$$

which shows that $e^{-nt} \in \mathrm{S}_p(P_t^*)$. The multiplicity of the eigenvalues is proved in [36, Proposition 2.27] whereas the last statement follows from (5.18).

□

6. Proof of Theorem 1.10, Proposition 1.4 and Theorem 1.1

6.1. **Proof of Theorem 1.10 and Theorem 1.1**(1)**,** (2) **and** (4)**.** The proof is split into several steps. We first establish that the CBI-semigroup P coincide on appropriate linear spaces with the following two (linear spectral) operators defined on $B_b(\mathbb{R}^+_{\geq 0})$, for any $x \geq 0$, by

$$S_t f(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \mathcal{V}_n f \, \mathcal{L}_n(x) \quad \text{and} \quad \overline{S_t} f(x) = \int_0^{\infty} f(y) S_t \delta_y(x), \tag{6.1}$$

where we use the notation $S_t\delta_y(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \mathcal{V}_n(dy)\, \mathcal{L}_n(x)$. First we show that $S_t f = P_t f$ on Λ and then that $\overline{S_t} f = S_t f$ on $\mathcal{D}_t$.

Lemma 6.1. *For all $t > T_0$, $x \geq 0$ and $f \in \Lambda$, we have $S_t f(x) = P_t f(x)$. Consequently, $(S_t)_{t>T_0}$ is a densely defined continuous semigroup on $C_0(\mathbb{R}^+_{\geq 0})$, endowed with the uniform topology $\|.\|_\infty$, i.e. for any $f \in \Lambda$, $\|S_t f\|_\infty \leq \|f\|_\infty$, with P_t its unique contraction semigroup extension on the closure of Λ, that is on $\overline{\Lambda} = C_0(\mathbb{R}^+_{\geq 0})$.*

Proof. Using successively the identities (5.7), (1.32) and (1.14), we have, for any $\lambda > 0$,

$$\begin{aligned} S_t e_\lambda(x) &= \sum_{n=0}^{\infty} e^{-\lambda_n t} \mathcal{V}_n e_\lambda \mathcal{L}_n(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} e^{-\Phi_\nu(\lambda)} A(\lambda)^n \mathcal{L}_n(x) \\ &= e^{-\Phi_\nu(\lambda)} G_x(A(\lambda) e^{-mt}) \\ &= P_t e_\lambda(x) \end{aligned}$$

where we note that (1.32) is valid since $A(\lambda) \in (0,1)$ on $\mathbb{R}_+$ and hence $0 < A(\lambda)e^{-mt} < e^{-mT_0} = R_0$. This completes the proof as plainly S_t is linear on Λ and P is a Feller semigroup on $\mathbb{R}^+_{\geq 0}$. □

Lemma 6.2. *For all $t > T_0$, $x \geq 0$ and $f \in \mathcal{D}_t$, we have $\overline{S_t} f(x) = S_t f(x)$. Note that $\Lambda_t = \mathrm{Span}(e_\lambda, \lambda > -\bar{\lambda}_t) \subset \mathcal{D}_t$, where $\bar{\lambda}_t$ is defined in* (1.19).

Proof. Let $f \in \mathcal{D}_t$, which by definition in (1.20), implies that there exists $C > 0$, chosen here for sake of clarity greater than $f(0)$, and $\lambda < \bar{\lambda}_t$ such that $|f(y)| \leq Ce^{\lambda y}$ for a.e. $y \geq 0$. Thus, we get that $|\mathcal{V}_n||f| \leq C|\mathcal{V}_n|e_{-\lambda}$, where we used the fact that $\mathcal{V}_n$ is absolutely continuous on $\mathbb{R}^+$. From (5.1) and the definition of $\bar{\lambda}_t$ we get that $|\mathcal{V}_n|e_{-\lambda} \leq e^{-\Phi_\nu(-\lambda)}(2 - A(-\lambda))^n$. Hence $e^{-\lambda_n t}|\mathcal{V}_n||f| \leq Ce^{-\Phi_\nu(-\lambda)}\left(e^{-mt}(2 - A(-\lambda))\right)^n$. From the definition again of $\bar{\lambda}_t$ and since from Lemma 3.5 A is increasing on $(-R_A, \infty)$, we have in both cases $e^{-mt}(2 - A(-\lambda)) < e^{-mt}(2 - A(-\bar{\lambda}_t)) \leq e^{-mT_0} = R_0$. Thus, we obtain that there exists $R \in (0, R_0)$ such that, for any $n \geq 0$,

$$e^{-\lambda_n t}|\mathcal{V}_n||f| \leq CR^n. \tag{6.2}$$

Hence, the representation (4.2) for $\mathfrak{p}=0$ in Proposition 4.1 yields

$$\int_0^\infty |f(y)| \sum_{n=0}^{\infty} e^{-\lambda_n t} |\mathcal{V}_n|(dy)\, |\mathcal{L}_n(x)| = \sum_{n=0}^{\infty} e^{-\lambda_n t} |\mathcal{V}_n||f|\, |\mathcal{L}_n(x)| < \infty,$$

where we recall that the series are locally uniformly convergent. By Fubini Theorem, this shows that the linear operator $\overline{S_t}$ is well defined and satisfies $\overline{S_t}f = S_t f$ on $\mathcal{D}_t$. The statement $\Lambda_t \subset \mathcal{D}_t$ is obvious. □

From Lemma 6.1 and Lemma 6.2, we have gained that, for any $t > T_0$, P_t and $\overline{S_t}$ share the same Laplace transform on $(-\bar{\lambda}_t, \infty)$. This is sufficient to claim, by injectivity of the Laplace transform, that for all $t > T_0$, $x \geq 0$,

$$P_t(x,dy) = S_t\delta_y(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\, \mathcal{L}_n(x)\, \mathcal{V}_n(dy) \tag{6.3}$$

and to get that

$$P_t f(x) = S_t f(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \mathcal{V}_n f\, \mathcal{L}_n(x) \text{ on } \mathcal{D}_t \cup \Lambda. \tag{6.4}$$

Next, for any $f \in \mathcal{D}_t \cup \Lambda$, $x \geq 0$ and $t > T_0$, and integers $\mathfrak{m}, \mathfrak{p}$, the equality (5.7) if $f \in \Lambda$ or the bound (6.2) if $f \in \mathcal{D}_t$ yield that in both cases there exists $R \in (0, R_0)$ such that for n large enough $|(-\lambda_n)^{\mathfrak{m}} e^{-\lambda_n t} \mathcal{V}_n f| \leq R^n$. Thus, there exists $C > 0$ such that

$$\sum_{n=\mathfrak{p}}^{\infty} \left| (-\lambda_n)^{\mathfrak{m}} e^{-\lambda_n t} \mathcal{V}_n f\, \mathcal{L}_n^{(\mathfrak{p})}(x) \right| \quad \leq \quad C \sum_{n=\mathfrak{p}}^{\infty} R^n \; \left| \mathcal{L}_n^{(\mathfrak{p})}(x) \right|.$$

Thus from (4.2) in Proposition 4.1, the series on the right-hand side is locally uniformly convergent in (t,x). Since we already showed that (6.4) holds, this proves by a classical result in analysis, see e.g. Theorem 7.17 in [42], the claim (1.36) in Theorem 1.10 and shows that $(t,x) \mapsto P_t f(x) \in C^{\infty^2}((T_0,\infty) \times \mathbb{R}^+_{\geq 0})$ for such f. Theorem 1.1(1) (resp. Theorem 1.1(2)) is an immediate consequence of this property combined with the fact P is a Markov operator (resp. that $\mathcal{D}_t \cap C_0(\mathbb{R}^+_{\geq 0}) \subset C_0(\mathbb{R}^+_{\geq 0})$ and P is a Feller semigroup). Recalling that $\mathcal{V}_n(dy) = e^{-\overline{\Phi}_\nu} \delta_0(dy) + \nu_n(y)dy$, the claim (1.39) is in fact the identity (6.3). Consequently, for any $x \geq 0$ and $t > T_0 \geq 0$, $P_t(x,dy)$ is absolutely continuous if and only if $e^{-\overline{\Phi}_\nu} e^{\Phi_\nu(B(e^{-mt}))} e^{-xB(e^{-mt})} = 0$ which is equivalent to $\overline{\Phi}_\nu = \infty$, as $B(e^{-mt}) = \infty$ if and only if $t = 0$, which is impossible. The claims (1.21) and (1.22) follow.

Next, for any $t > T_0$, $x \in [0, \mathrm{K}], \mathrm{K} > 0$, $\lambda < \min\{\bar{\lambda}_t, 0\}$ and $y \in [K^{-1}, K], K > 0$ and for any integers $\mathfrak{m}, \mathfrak{p}$, and, any integer $\mathfrak{q}$ such that $\mathfrak{q} \leq \underline{\kappa} - 1$ if $\underline{\kappa} \geq 1$, or $\mathfrak{q} \leq \underline{\kappa} + \bar{\mathfrak{q}}$ with $\bar{\mathfrak{q}}$ as in Theorem 1.1(4b), we get

$$\sum_{n=\mathfrak{p}}^{\infty} \left| (-\lambda_n)^{\mathfrak{m}} e^{-\lambda_n t} \nu_n^{(\mathfrak{q})}(y)\, \mathcal{L}_n^{(\mathfrak{p})}(x) \right| \quad \leq \quad C \sum_{n=\mathfrak{p}}^{\infty} \lambda_n^{\mathfrak{m}} e^{-\lambda_n t} n^a (2 - A(-\lambda))^n \; \left| \mathcal{L}_n^{(\mathfrak{p})}(x) \right|$$

where $C > 0$ and to estimate $|\nu_n^{(\mathfrak{q})}(y)|$ we used the bounds of Proposition 5.1 with $a = \max(\mathfrak{q} + 1 - \underline{\kappa}, 0)$. We conclude that the series is locally uniformly convergent in (t,x,y) by invoking again (4.2) after noting that $\lambda_n^{\mathfrak{m}} e^{-\lambda_n t} n^a (2 - A(-\lambda))^n < R^n$ for some $R < R_0$ and all n large enough. Combined with (6.3), this provides (via Theorem 7.17 in [42]) the expression (1.40) and the proof of Theorem 1.1(4)(4a)-(4b). This completes the proof of Theorem 1.10 and Theorem 1.1 (1), (2) and (4).

6.2. **Proof of Proposition 1.4.** Note that the first claim regarding the existence of an invariant probability measure was proved in Proposition 3.1. Next, let $t > \underline{T} = T_0 + \frac{1}{m}\ln(2-A(-R_A))$ and $E \in \mathcal{B}(\mathbb{R}^+_{\geq 0})$. Then, from the definition (1.19), we have $\overline{\lambda}_t = R_A \wedge R_\phi$ and $\mathbb{I}_E \in \mathcal{D}_t$ as clearly $\mathbb{I}_E \leq e_{-\lambda}$ for any $\lambda \in (0, R_A \wedge R_\phi)$. Therefore, noting that $\lambda_0 = 0$, $\mathcal{L}_0(x) = 1$ and $\mathcal{V}_0 = \mathcal{V}$, we get, from (1.36) in Theorem 1.10, that, for any $x \geq 0$,

$$|P_t(x,E) - \mathcal{V}(E)| = \left|\sum_{n=1}^{\infty} e^{-nmt}\mathcal{L}_n(x)\mathcal{V}_n(E)\right| \leq \sum_{n=1}^{\infty} e^{-nmt}\left|\mathcal{L}_n(x)\right|\left|\mathcal{V}_n(E)\right|.$$

Now since $\mathbb{I}_E \leq e_{-\lambda}$, we have $|\mathcal{V}_n(E)| \leq |\mathcal{V}_n|(E) \leq |\mathcal{V}_n| e_{-\lambda} \leq e^{-\Phi_\nu(-\lambda)}(2 - A(-\lambda))^n$ where the last inequality follows from (5.1) in Proposition 5.1 which is valid since $-\lambda > -R_A \wedge R_\phi$. Thus

$$|P_t(x,E) - \mathcal{V}(E)| \leq e^{-\Phi_\nu(-\lambda)} \sum_{n=1}^{\infty} |\mathcal{L}_n(x)| \left(e^{-mt}(2 - A(-\lambda))\right)^n.$$

Now, from the fact that A is increasing on $(-R_A, \infty)$ one sees that the choices of $t > \underline{T} = T_0 + \frac{1}{m}\ln(2 - A(-R_A))$ and of $\lambda < R_A \wedge R_\phi$ ensure that $0 < R = e^{-m\underline{T}}(2 - A(-\lambda)) = e^{-mT_0}\frac{2-A(-\lambda)}{2-A(-R_A)} < e^{-mT_0} = R_0$. Thus by (4.3) in Proposition 4.1, there exist $C = C(R) > 0$ and $\overline{B} = \max_{|z|=R}|B(z)| \in \mathbb{R}_+$ such that

$$\begin{aligned} |P_t(x,E) - \mathcal{V}(E)| &\leq e^{-\Phi_\nu(-\lambda)} C e^{\overline{B}x} \sum_{n=1}^{\infty} \left(\frac{e^{-mt}(2 - A(-\lambda))}{R}\right)^n \\ &= e^{-\Phi_\nu(-\lambda)} C e^{\overline{B}x} \sum_{n=1}^{\infty} e^{-nm(t-\underline{T})}, \end{aligned}$$

which plainly proves the Corollary.

6.3. **Proof of Theorem 1.1**(3)**.** Let us now prove that P is eventually strong Feller. We mention that Schilling and Wang in [45] provide interesting sufficient conditions on the transition kernel of Feller semigroups which imply the strong Feller property. However, we have not been able to use their criteria and instead we prove this property directly using Theorem 1.1(1). To this end, let $t > T_0$ and $f \in B_b(\mathbb{R}^+_{\geq 0})$. There exists a non-decreasing sequence $(g_n)_{n\geq 0}$ of non-negative functions in $C_c(\mathbb{R}^+_{\geq 0})$ converging pointwise to $g \equiv 1$ and by the monotone convergence Theorem we have $\lim_{n\to\infty} P_t g_n = P_t g$. As $g_n \in C_c(\mathbb{R}^+_{\geq 0})$, the space of continuous functions with compact support, by Theorem 1.1(1), we have $P_t g_n \in C(\mathbb{R}^+_{\geq 0})$. Moreover, by (1.4), $P_t g \in C(\mathbb{R}^+_{\geq 0})$. Then Dini Theorem shows that the convergence is uniform on any compact set in $\mathbb{R}^+_{\geq 0}$. Now let $x \in \mathbb{R}^+_{\geq 0}$ and $\epsilon > 0$. Consider a compact set K of the form $[0, \eta]$ if $x = 0$, $[x - \eta, x + \eta]$ otherwise, then there exists $n_0 = n_0(\epsilon) \geq 0$ such that

$$\sup_{x\in K} |P_t(g - g_{n_0})(x)| < \frac{\epsilon}{3\|f\|_\infty}.$$

Next, since $fg_{n_0} \in B_b(\mathbb{R}^+_{\geq 0})$ with compact support, Theorem 1.1(1) entails that $P_t f g_{n_0} \in C^\infty(\mathbb{R}^+_{\geq 0})$. Therefore, there exists $\eta_0 = \eta_0(n_0, \epsilon) \in (0, \eta)$ such that, for all $y \in U_0 = (0, \eta_0)$ if $x = 0$ and $y \in U_x = (x - \eta_0, x + \eta_0)$ otherwise,

$$|P_t f g_{n_0}(x) - P_t f g_{n_0}(y)| < \frac{\epsilon}{3}.$$

Since $U_x \subset K$, we get, using the previous estimates, that for all $y \in U_x$,

$$\begin{aligned} |P_t f(x) - P_t f(y)| &\leq |P_t f(g - g_{n_0})(x)| + |P_t f(g - g_{n_0})(y)| + |P_t f g_{n_0}(x) - P_t f g_{n_0}(y)| \\ &\leq |||f||_\infty \left(|P_t(g - g_{n_0})(x)| + |P_t(g - g_{n_0})(y)|\right) + |P_t f g_{n_0}(x) - P_t f g_{n_0}(y)| \\ &< \epsilon. \end{aligned}$$

Hence, from the contraction property of P_t, we conclude that for any $t > T_0$ and $f \in B_b(\mathbb{R}^+_{\geq 0})$, $P_t f \in C_b(\mathbb{R}^+_{\geq 0})$ which is the (eventually) strong Feller property.

7. Examples

We end this paper by detailing some examples of CBI-semigroups which illustrate the variety of smoothness properties that the absolutely continuous part of their transition kernel enjoy. The last example also reveals that our results are sharp in the sense that some instances of CBI-semigroups do not have a better regularity property on $\mathbb{R}_+$ than the one stated in Theorem 1.1.

7.1. **Handa CBI-semigroups.** In [20], Handa showed that that every generalized gamma convolution distribution is an invariant measure for a CBI-semigroup and then he studied the sector property of this class of CBI-semigroups and we refer to the aforementioned paper for definition. More specifically, let us consider the mechanisms

$$\psi(u) = \sigma^2 u^2 + mu + \int_0^\infty \left(e^{-ur} - 1 + ur\right) \Pi(dr) \quad \text{and} \quad \phi(u) = u \tag{7.1}$$

where $\sigma^2 \geq 0$, and

$$\Pi(dr) = \int_0^\infty v^2 e^{-vr} M(dv) dr, \tag{7.2}$$

for some measure M on $\mathbb{R}_+$ such that $\int_0^\infty \frac{M(dv)}{1+v} < \infty$ and which is associated to a Thorin measure τ by the following relation

$$\int_0^\infty \frac{\tau(dv)}{u+v} = \frac{1}{\frac{u}{\overline{\tau}_0} + \frac{1}{\overline{\tau}_1} + \int_0^\infty \frac{u}{u+v} M(dv)}$$

where for $k = 0, 1$, $\overline{\tau}_k = \int_0^\infty v^{-k} \tau(dv)$. We recall that a Thorin measure is a measure τ on $\mathbb{R}_+$ satisfying

$$\int_0^{\frac{1}{2}} |\log v| \tau(dv) + \int_{\frac{1}{2}}^\infty \frac{\tau(dv)}{v} < \infty. \tag{7.3}$$

Then according to [20, Theorem 4.3], we have that the Lévy measure of Φ_ν, the Laplace exponent of the invariant measure, is $\frac{\kappa(r)}{r}dr, r>0$, where

$$\kappa(r)=\int_0^\infty e^{-vr}\tau(dv)=W^{(1)}(r) \tag{7.4}$$

is completely monotone and hence the invariant measure is a generalized gamma convolution distribution. Note that (7.4) combined with (2.5) yields that

$$\phi_p(u)=\frac{u}{\overline{\tau}_0}+\frac{1}{\overline{\tau}_{-1}}+\int_0^\infty \frac{u}{u+v}M(dv).$$

Thus, if $M\equiv 0$ on $[0,R)$ for some $R>0$ then it is easy to check that $\phi_p\in\mathcal{H}(R)$ and also $\psi\in\mathcal{H}(R)$. Moreover, the condition (1.6) is according to Lemma 2.1 satisfied if $\sigma^2>0$, i.e. $\overline{\tau}_0<\infty$, or, for instance if there exists g positive and non-increasing such that $\frac{M(dv)}{dv}=g(v)\overset{\infty}{\sim}Cv^{-\alpha}$, $\alpha\in(1,2)$, as by classical arguments, $\overline{\overline{\overline{\Pi}}}(r)\overset{0}{\sim}C_\alpha r^{1-\alpha}$. From (7.4) we see that, as the Laplace transform of a measure on $\mathbb{R}_+$, κ is completely monotone. Hence $\kappa\in C^\infty(\mathbb{R}_+)$ and

$$\underline{\kappa}(0^+)=\overline{\kappa}(0^+)=\tau(\mathbb{R}_+)=W^{(1)}(0^+)=\frac{1}{\sigma^2}\in(0,\infty], \tag{7.5}$$

where for the last equality we have used Proposition 1.6(1i). Thus, we have that $\underline{\kappa}(0^+)>0$, which by Lemma 3.3 implies that $P_t(x,dy)=p_t(x,y)dy$. First, if $\tau(\mathbb{R}_+)=\infty$ then, by (7.5), $\underline{\kappa}=\infty$ and from Theorem 1.1 (4a), we have that $(t,x,y)\mapsto p_t(x,y)\in C^{\infty^3}((T_0,\infty)\times\mathbb{R}^+_{\geq 0}\times\mathbb{R})$. Assume now that $\tau(\mathbb{R}_+)<\infty$. Then, by (7.4), for any $a>0$,

$$\int_0^a|\kappa^{(1)}(r)|dr=\int_0^a\int_0^\infty ve^{-vr}\tau(dv)dr=\int_0^\infty(1-e^{-av})\tau(dv)<\tau(\mathbb{R}_+)<\infty,$$

so that $\kappa^{(1)}\in L^1_{loc}(\mathbb{R}^+_{\geq 0})$. Moreover, since by (7.4) $\kappa,W\in C^\infty(\mathbb{R}_+)$, we have that $\bar{\mathfrak{q}}=\infty$ in Theorem 1.1 (4b) and hence $(t,x,y)\mapsto p_t(x,y)\in C^{\infty^3}((T_0,\infty)\times\mathbb{R}^+_{\geq 0}\times\mathbb{R}_+)$.

7.2. **Tempered stable.** Let for any $\alpha\in(1,2]$ and $\beta\in(-1,1]$,

$$\psi(u)=a\left((u+\eta_\psi)^\alpha+\gamma u-\eta_\psi^\alpha\right)\ \text{ and }\ \phi(u)=ca\left|(u+\eta_\phi)^\beta-\eta_\phi^\beta\right|,$$

where $a,c,\eta_\psi,\eta_\phi>0$ and $\gamma>-\alpha\eta_\psi^{\alpha-1}$. Then ψ and ϕ are of the form (1.5) and (1.7) and satisfy (1.6) and (1.8) with $m=a(\alpha\eta_\psi^{\alpha-1}+\gamma)$ and

$$\sigma^2=\begin{cases}0 & \text{if } \alpha<2,\\ a & \text{if } \alpha=2,\end{cases}\qquad \Pi(dr)=\begin{cases}\frac{a\alpha(\alpha-1)}{\Gamma(2-\alpha)}e^{-\eta_\psi r}r^{-\alpha-1}dr & \text{if } \alpha<2,\\ 0 & \text{if } \alpha=2,\end{cases}$$

$$b=\begin{cases}0 & \text{if } \beta<1,\\ ca & \text{if } \beta=1,\end{cases}\qquad \mu(dr)=\begin{cases}\frac{ca}{\Gamma(-\beta)}e^{-\eta_\phi r}r^{-\beta-1}dr & \text{if } \beta<0,\\ \frac{ca\beta}{\Gamma(1-\beta)}e^{-\eta_\phi r}r^{-\beta-1}dr & \text{if } 0<\beta<1,\\ 0 & \text{if } \beta\in\{0,1\},\end{cases}$$

where for $a>0$, $\Gamma(a)=\int_0^\infty e^{-x}x^{a-1}dx$ is the gamma function. Note that $\bar{\mu}(0^+)=\infty$ if $0<\beta<1$. We fix $t>T_0$ and $x\geq 0$ and investigate the regularity of $y\mapsto p_t(x,y)$.

If $\alpha<\beta+1$, then by combining Proposition 1.6(1ii) and Theorem 1.1, we have $y\mapsto p_t(x,y)\in C^\infty(\mathbb{R})$. Suppose now $\alpha>\beta+1$. Then as $\overline{\overline{\Pi}}$ is completely monotone and $\overline{\mu}\in C^\infty(\mathbb{R}^+)$ it follows from Remark 1.8 and Proposition 1.6(2i) (if $\alpha<2$ and $\beta>0$), Proposition 1.6(2ii) (if

$\alpha < 2$ and $\beta \le 0$) or Proposition 1.6(2iii) (if $\alpha = 2$) that $W, \kappa \in C^\infty(\mathbb{R}^+)$ and $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$ and thus by Theorem 1.1, $y \mapsto p_t(x,y) \in C^\infty(\mathbb{R}^+)$, provided $\overline{\kappa}(0^+) = \underline{\kappa}(0^+) < \infty$. To show the latter, note that by Proposition 1.6(1i), $\overline{\kappa}(0^+) = 0$ if $\alpha = 2$ and by Proposition 1.6(1ii), $\overline{\kappa}(0^+) = 0$ if $\alpha < 2$ and $\beta \le 0$. If $\alpha < 2$ and $\beta > 0$, let $\ell(u) = \frac{u^\alpha}{\psi(u)}$. Then $\lim_{u\to\infty} \ell(u) = 1/a$ and thus in particular it is a slowly varying function at infinity. Since

$$\int_0^\infty e^{-uy} W^{(1)}(y)dy = \frac{u}{\psi(u)} = u^{1-\alpha}\ell(u),$$

and $W^{(1)}$ is non-increasing (since it is completely monotone), we have by [5, Theorem 1.7.1' and Theorem 1.7.2b] that

$$\lim_{y\downarrow 0} \frac{W^{(1)}(y)}{y^{\alpha-2}} = \frac{1}{a\Gamma(\alpha-1)}.$$

Let $\epsilon > 0$ be arbitrary and choose $\delta > 0$ be such that $\frac{a\Gamma(\alpha-1)W^{(1)}(y)}{y^{\alpha-2}} \in [1-\epsilon, 1+\epsilon]$ and $e^{-\eta_\phi y} \in [1-\epsilon, 1]$ for all $0 < y < \delta$. Then, writing $C_\epsilon = \frac{c\beta(1+\epsilon)}{\Gamma(\alpha-1)\Gamma(1-\beta)}$, we have, for $0 < y < \delta$,

$$\begin{aligned}
(7.6)\quad \kappa(y) &= \int_0^y W^{(1)}(y-r)\bar{\mu}(r)dr \le \int_0^y (y-r)^{\alpha-2} \int_r^\infty e^{-\eta_\phi u} u^{-\beta-1} du dr \\
&= C_\epsilon \int_0^y (y-r)^{\alpha-2} \left(\int_r^y e^{-\eta_\phi u} u^{-\beta-1} du + \int_y^\infty e^{-\eta_\phi u} u^{-\beta-1} du \right) dr \\
&\le C_\epsilon \left(\int_0^y (y-r)^{\alpha-2} \int_r^y u^{-\beta-1} du dr + \int_0^y (y-r)^{\alpha-2} \int_y^\infty e^{-\eta_\phi u} u^{-\beta-1} du dr \right) \\
&= C_\epsilon \left(\frac{1}{\beta} \int_0^y (y-r)^{\alpha-2} \left(r^{-\beta} - y^{-\beta} \right) dr + \frac{y^{\alpha-1}}{\alpha-1} \int_y^\infty e^{-\eta_\phi u} u^{-\beta-1} du \right) \\
&= C_\epsilon \left(\left(\frac{\Gamma(\alpha-1)\Gamma(1-\beta)}{\beta\Gamma(\alpha-\beta)} - \frac{1}{(\alpha-1)\beta} \right) y^{\alpha-\beta-1} + \frac{y^{\alpha-1}}{\alpha-1} \int_y^\infty e^{-\eta_\phi u} u^{-\beta-1} du \right),
\end{aligned}$$

where for the last equality we used, for $\eta_1 < 1$ and $\eta_2 < 1$, the identity

$$\int_0^y (y-r)^{-\eta_1} r^{-\eta_2} dr = \frac{\Gamma(1-\eta_1)\Gamma(1-\eta_2)}{\Gamma(2-\eta_1-\eta_2)} y^{1-\eta_1-\eta_2},$$

which can be easily proved via Laplace transforms. Similarly, writing $\overline{C}_{-\epsilon} = \frac{\Gamma(\alpha-1)\Gamma(1-\beta)}{\beta\Gamma(\alpha-\beta)} - \frac{1}{(\alpha-1)\beta}(1-\epsilon)$, we have the lower bound

$$(7.7)\qquad \kappa(y) \ \ge \ C_{-\epsilon} \left(\overline{C}_{-\epsilon} y^{\alpha-\beta-1} + \frac{y^{\alpha-1}}{\alpha-1} \int_y^\infty e^{-\eta_\phi u} u^{-\beta-1} du \right).$$

Next, by means of l'Hôpital's rule, we observe that

$$(7.8)\qquad \frac{y^{\alpha-1}}{\alpha-1} \int_y^\infty e^{-\eta_\phi u} u^{-\beta-1} du \overset{0}{\sim} \frac{e^{-\eta_\phi y} y^{-\beta-1}}{(\alpha-1)^2 y^{-\alpha}} \overset{0}{\sim} \frac{y^{\alpha-\beta-1}}{(\alpha-1)^2}.$$

Recalling that we are considering the case $\alpha > \beta + 1$, it follows from (7.6) and (7.8) that $\overline{\kappa}(0^+) = 0$ also when $\alpha < 2$ and $\beta > 0$. Thus, we conclude that $y \mapsto p_t(x,y) \in C^\infty(\mathbb{R}+)$ if

$\alpha > \beta + 1$. Now we consider the remaining case where $\alpha = \beta + 1$. If $\alpha = 2$, then by Lemma 1.6(1i), $\overline{\kappa}(0^+) = \underline{\kappa}(0^+) = c$. If $\alpha < 2$, then by (7.6)-(7.8), we have for any $\epsilon > 0$,

$$\overline{\kappa}(0^+) \leq c(1+\epsilon), \quad \underline{\kappa}(0^+) \geq c(1-\epsilon)^2 + \epsilon \frac{c\beta(1-\epsilon)}{\Gamma(\alpha-1)\Gamma(1-\beta)(\alpha-1)^2}.$$

Hence $\overline{\kappa}(0^+) = \underline{\kappa}(0^+) = c$. Then, by Theorem 1.1, $y \mapsto p_t(x,y) \in C^{k-1}(\mathbb{R}) \cap C^k(\mathbb{R}^+)$ for all integers $k \in [1, c)$. To deal with additional regularity on $\mathbb{R}^+$, note that if $\alpha = 2$, Theorem 1.1(4(b)2) applies via Lemma 1.6(2iii) and Remark 1.8 and thus $y \mapsto p_t(x,y) \in C^\infty(\mathbb{R}^+)$. However, if $\alpha < 2$ it is not obvious to us how to prove that $\kappa^{(1)} \in L^1_{loc}(\mathbb{R}^+)$ since the integral condition in Lemma 1.6(2i) is not satisfied. Therefore we restrict ourselves to the special case where $\eta_\psi = \eta_\phi$ and $\gamma = -\eta_\psi^{\alpha-1}$, which has also been considered in [10] and [35]. In that case, $\frac{\phi(u)}{\psi(u)} = \frac{c}{u+\eta_\psi}$ and so by Laplace inversion, $\kappa(y) = ce^{-\eta_\psi y}$. Then, recalling that W is in $C^\infty(\mathbb{R}^+)$, Theorem 1.1 (4b) yields $y \mapsto p_t(x,y) \in C^\infty(\mathbb{R}^+)$.

7.3. **Example of non-smoothness.** This example provides an instance when the absolutely continuous part of the transition density is not in $C^\infty(\mathbb{R}_+)$ in general. Let us consider the case where $\sigma = 1$, $m = 1$ and $\Pi(dr) = \delta_1(dr)$ and $b = 0$, $\mu \equiv 0$, that is, P is a CB semigroup. Note that from Proposition 1.9(ii), one may construct a CBI-semigroup whose transition kernel has the same smoothness properties as the kernel of the CB semigroup. We fix $t > T_0$ and $x \geq 0$. Since $W \in C^2(\mathbb{R}_+)$ when $\sigma > 0$, see in [9, Theorem 1], we know by Theorems 1.1 and 1.10 and Lemma 3.3 that in this case $P_t(x,dy) = \delta_0(dy) + p_t(x,y)dy$, where

$$y \mapsto p_t(x,y) = \sum_{n=0}^{\infty} \mathcal{L}_n(x) e^{-\lambda_n t} \mathcal{W}_n(y) \in C^2(\mathbb{R}_+),$$

with, with the obvious notation,

$$\begin{aligned} p_t^{(2)}(x,y) &= \sum_{n=0}^{\infty} \mathcal{L}_n(x) e^{-\lambda_n t} \mathcal{W}_n^{(2)}(y) \\ &= \sum_{n=0}^{\infty} \mathcal{L}_n(x) e^{-\lambda_n t} \left(\mathcal{W}_n^{(2)}(y) + \lambda_n \frac{W^{(2)}(y)}{y} \right) - \sum_{n=0}^{\infty} \mathcal{L}_n(x) e^{-\lambda_n t} \lambda_n \frac{W^{(2)}(y)}{y}. \end{aligned} \tag{7.9}$$

We are going to show that $y \mapsto p_t(x,y) \notin C^3(\mathbb{R}_+)$. By (2.10), we have, for $y > 0$,

$$W^{(2)}(y) = \overline{\overline{\Pi}}(y) + \sum_{n=2}^{\infty} (-1)^n \overline{\overline{\Pi}}^{*n}(y). \tag{7.10}$$

As $\overline{\overline{\Pi}}(y) = (1-y)\mathbb{I}_{\{0<y\leq 1\}}$ is an absolutely continuous function with bounded density, we get by Lemma 2.4, that $\overline{\overline{\Pi}}^{*n} \in C^1(\mathbb{R}_+)$ for $n \geq 2$. It is then an easy exercise to show that the infinite sum on the right hand side of (7.10) is in $C^1(\mathbb{R}_+)$. Therefore, $W^{(2)}$ is absolutely continuous on $\mathbb{R}_+$ (with a bounded density) and differentiable on $\mathbb{R}_+ \setminus \{1\}$ but not at $y = 1$, since $\overline{\overline{\Pi}}(1^-) = -1$

and $\overline{\overline{\overline{\Pi}}}(1^+) = 0$. By (5.12),

$$-\left(\mathcal{W}_n^{(2)}(y) + \lambda_n \frac{W^{(2)}(y)}{y}\right) = 2\frac{\mathcal{W}_n(y)}{y} + \frac{\lambda_n}{y}\Bigg(\int_0^{\frac{1}{2}y} W^{(2)}(y-r)\mathcal{W}_n(r)dz$$

$$+ \int_0^{\frac{1}{2}y} \mathcal{W}_n^{(1)}(y-r)W^{(1)}(r)dr + W^{(1)}(\tfrac{y}{2})\mathcal{W}_n(\tfrac{y}{2})\Bigg)$$

and thus by invoking Lemma 2.4 for the first integral, $y \mapsto y\mathcal{W}_n^{(2)}(y) + \lambda_n W^{(2)}(y) \in C^1(\mathbb{R}_+)$. Following the proof of Theorem 1.1(4b) in Section 6.1, we deduce that the first infinite sum on the right hand side of (7.9) is in $C^1(\mathbb{R}_+)$. But the second infinite sum is not differentiable at $y = 1$ and thus $y \mapsto p_t(x, y) \notin C^3(\mathbb{R}_+)$.

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