A study of constrained Navier-Stokes equations and related problems

GAURAV DHARIWAL

DOCTOR OF PHILOSOPHY

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Abstract

Fundamental questions in the theory of partial differential equations are that of existence and uniqueness of the solution. In this thesis we address these questions corresponding to two models governing the dynamics of incompressible fluids, both being the modification of classical Navier-Stokes equations: constrained Navier-Stokes equations and tamed Navier-Stokes equations.

The former being Navier-Stokes equations with a constraint on the L^2 norm of the solution considered on a two-dimensional domain with periodic boundary conditions. We prove existence of the unique global-in-time solution in deterministic setting and establish existence of a pathwise unique strong solution under the impact of a stochastic forcing.

The tamed Navier-Stokes equations were introduced by Röckner and Zhang [75], to study the properties of solutions of the 3D Navier-Stokes equations. We use three new ideas to prove the existence of a strong solution and existence of invariant measures: approximating equation on an infinite dimensional space in contrast to classical Faedo-Galerkin approximation; tightness criterion related to the Dubinsky's compactness theorem introduced recently by Brzeźniak and Motyl [23]; and lastly proving the existence of invariant measures based on continuity and compactness in the weak topologies [62].

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To my parents Suneel and Vijayshree Dhariwal

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Author's declaration

I declare that the work in this thesis was carried out in accordance with the requirements of the University's regulations for Research Degree Programmes and I am the sole author. This work was carried out under the supervision of Prof. Zdzisław Brzeźniak, and has not previously been presented for an award at this, or any other, University. All sources are acknowledged and have been listed in the Bibliography.

List of Publications

(1) Z. Brzeźniak, G. Dhariwal and M. Mariani, *2D Constrained Navier-Stokes equations,* Preprint arXiv:1606.08360v2 (Submitted, 2016).

(2) Z. Brzeźniak and G. Dhariwal, *Stochastic Constrained Navier-Stokes Equations on* $\mathbb{T}^2$*,* Preprint arxiv:1701.01385v2 (2017).

(3) Z. Brzeźniak, G. Dhariwal, J. Hussain and M. Mariani *Stochastic and deterministic constrained partial differential equations,* (Submitted, 2017).

(4) Z. Brzeźniak and G. Dhariwal, *Stochastic tamed Navier-Stokes equations on* $\mathbb{R}^3$: *existence, uniqueness of solutions and existence of invariant measures,* (In preparation, 2017).

CHAPTER 1

INTRODUCTION

The time dependent partial differential equations, commonly known as evolution equations play a crucial role in modelling various natural processes mathematically, which are used to study the behaviour of physical entities like wave function of a particle, temperature profile of a system, stocks in a financial market and velocity of a fluid. Well-known examples are Schrödinger equation from quantum mechanics, reaction diffusion equations modelling biological processes and heat flow, Black-Scholes equation from finance and Navier-Stokes equations from fluid mechanics.

At instances these physical processes are subject to external forcing, which mostly is random in nature. Thus, one has to modify the mathematical models accordingly to incorporate this randomness, which in turn gives rise to stochastic partial differential equations (SPDE), providing us with more robust model to study these natural processes.

Though (S)PDE serve the purpose of analysing these physical entities well, they pose quite basic mathematical questions, like global in time existence and uniqueness of the solution, existence of invariant measures. This thesis deals with such questions for constrained Navier-Stokes equations, stochastic constrained Navier-Stokes equations and stochastic tamed Navier-Stokes equations.

1.1 Stochastic and deterministic constrained partial differential equations

In the theory of partial differential equations one often studies equations with constraints on the values of the unknown function. Here primary examples are geometric heat and wave equations where it is required that the solution is a manifold-valued function. Such models have been extensively studied, one could mention Eells-Sampson [41], Struwe and Shatah [83–85] for the deterministic problems; Funaki [44], Carroll [32] and Brzeźniak *et al* [11, 27] for the stochastic problems. If the target manifold is a sphere, one can study a generalisation of the heat flow map, called the Landau-Lifshitz-Gilbert Equations [2, 4, 19, 22]. Recently, different kind of constraints, the nonlocal ones, were investigated by Rybka [81], Caffarelli-Lin [30] and Cagliotti *et al.* [31]. For instance, one imposes the constraint that the L^p norm of the solution remains constant.

It is well understood that how to construct a stochastic or deterministic equation on hypersurfaces of an Euclidean space (or even general Hilbert space) from a given equation on an ambient space, provided the latter is given in terms of smooth functions. To be precise let us describe this procedure.

Suppose that $\big(\mathrm{H}, \langle\cdot,\cdot\rangle\big)$ is a Hilbert space and $\mathcal{M} = \varphi^{-1}(\{1\}) \subset \mathrm{H}$ is a hypersurface for some non-degenerate smooth function $\varphi\colon \mathrm{H} \to [0,\infty)$. Each element $\omega \in \mathcal{M}$ has a tangent space $T_\omega\mathcal{M}$ which can be identified with the closed subspace of H (of co-dimension 1) given by $\ker d_\omega\varphi = \{x \in \mathrm{H}\colon (d_\omega\varphi)(x) = 0\}$, where $\mathrm{d}_\omega\varphi \in \mathscr{L}(\mathrm{H},\mathbb{R})$ is the Fréchet derivative of φ at ω. By the Riesz Lemma there exists a unique element in H, denoted by $(\mathrm{D}\varphi)(\omega)$, such that $(\mathrm{d}_\omega\varphi)(x) = \langle(\mathrm{D}\varphi)(\omega), x\rangle$, $x \in \mathrm{H}$. Since $(\mathrm{D}\varphi)(\omega) \neq 0$, the orthogonal projection $\pi_\omega\colon \mathrm{H} \to T_\omega\mathcal{M}$ is given (with $|\cdot|$ being the norm on H) by the formula

$$\pi_\omega(x) = x - \langle x, \vec{n}(\omega)\rangle\vec{n}(\omega), \quad x \in \mathrm{H}\,, \tag{1.1.1}$$

where

$$\vec{n}(\omega) = \frac{\mathrm{D}\varphi(\omega)}{|\mathrm{D}\varphi(\omega)|}, \quad \omega \in \mathcal{M}\,.$$

Given a vector field $f\colon \mathrm{H} \to \mathrm{H}$ we can consider the “tangent projection” $\hat{f}$ of the restriction of f to $\mathcal{M}$ (which is a “tangent" vector field on $\mathcal{M}$) defined by

$$\hat{f}(\omega) := \pi_\omega(f(\omega)) \in T_\omega\mathcal{M}\,, \quad \omega \in \mathcal{M}\,. \tag{1.1.2}$$

The associated ODE

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t))\,, \quad t \geq 0\,, \tag{1.1.3}$$

takes the following well-known form on $\mathcal{M}$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \hat{f}(x(t))\,, \quad t \geq 0\,. \tag{1.1.4}$$

Note that $\hat{f}$ has a smooth extension to an open neighbourhood of $\mathcal{M}$ and the ODE (1.1.4) is locally well-posed on that neighbourhood. One can then show that given $x_0 \in \mathcal{M}$, the local solution stays on $\mathcal{M}$, by either using local diffeomorphism of some neighbourhood of x_0 in $\mathcal{M}$ (i.e. de facto the Hilbert manifold structure of $\mathcal{M}$) or by showing that $\varphi(x(t)) = 0$ for t in the domain of the solution. If one knows that $\mathcal{M}$ is a compact set (that requires H to be finite dimensional) then we can easily deduce that each solution starting at $x_0 \in \mathscr{M}$ is a global one, i.e. defined on $[0,\infty)$. However, if $\mathscr{M}$ is not compact the solutions may not be global-in-time.

Similar argument can be made for stochastic differential equations, with one small but important difference. Suppose $f_0, f_1, \cdots, f_N$ is a finite collection of vector fields on H and $W = (W(t))$, $t \geq 0$ is an $\mathbb{R}^N$-valued Wiener process, we write $W(t) = (W_j(t))$ defined on some filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$, satisfying the so-called usual conditions [63].

In the whole ambient space we can study stochastic differential equations either in the Itô or Stratonovich form, the latter requiring more regularity assumptions on the vector fields $f_1, \cdots, f_N$:

$$\mathrm{d}x = f_0(x)\,dt + \sum_{i=1}^{N} f_i(x)\,dW_i\,, \tag{1.1.5}$$

or

$$\begin{aligned} \mathrm{d}x &= f_0(x)\,dt + \sum_{i=1}^{N} f_i(x)\circ dW_i \\ &= f_0(x)\,dt + \left[\frac{1}{2}\sum_{i=1}^{N} f_i'(x) f_i(x)\right] dt + \sum_{i=1}^{N} f_i(x)\,dW_i\,, \end{aligned} \tag{1.1.6}$$

where $f_i'(x) = \mathrm{d}_x f_i$, $x \in \mathrm{H}$. On the other hand, it turns out that the correct form of these equations on $\mathscr{M}$ is the Stratonovich one. This fact is related to the Wong-Zakai type theorems, see [10] or the rough paths theory proposed recently by Terry Lyons [59]. With the same notation as before one can consider an equation

$$\begin{aligned} \mathrm{d}x &= \hat{f}_0(x)\,dt + \sum_{j=1}^{N} \hat{f}_j(x)\circ dW_j \\ &= \hat{f}_0(x)\,dt + \sum_{j=1}^{N} \hat{f}_j(x)\,dW_j + \frac{1}{2}\sum_{j=1}^{N} \hat{f}_j'(x)\hat{f}_j(x)\,dt\,. \end{aligned} \tag{1.1.7}$$

The issues of local and global solutions to the above problem can be solved through a similar approach as the one used to answer the analogous questions in the deterministic case, see for instance [15] and references therein.

However, when the vector fields are not smooth or not everywhere defined or both, the situation changes. For instance, let us consider an unbounded, self-adjoint and non-negative operator A on a Hilbert space H. The domain of A, denoted by D(A), is a Hilbert space endowed with the "graph norm" :

$$|x|^2_{\mathrm{D(A)}} = |x|^2 + |\mathrm{A}x|^2, \qquad x \in \mathrm{D(A)}\,.$$

Such an operator A induces a (only densely defined) vector field $f_0(x) = -\mathrm{A}x$. Theory of corresponding deterministic and stochastic problems related to equations (1.1.5) and (1.1.6) is now well developed and understood, see e.g. a monograph [38] by Da Prato and Zabczyk. However, this is not the case for equations (1.1.7) with a vector field $\hat{f}_0$ defined by

$$\hat{f}_0(x) = f_0(x) - \langle f_0(x), \vec{n}(x)\rangle \vec{n}(x)\,, \quad x \in \mathscr{M}\cap \mathrm{D(A)}\,, \tag{1.1.8}$$

in the view of (1.1.1) and (1.1.2). In [48], Hussain studied reaction diffusion equations under a non-local constraint, in both deterministic and stochastic cases. He established the existence of “unique" global “solution" for these equations.

The aim of this thesis is to present a detailed study of such questions in the case of Navier-Stokes equations. We address the questions of existence and uniqueness for two dimensional Navier-Stokes equations with a constraint on the L^2 norm of the solution. Another related problem that we have addressed in this thesis is that of three dimensional tamed Navier-Stokes equations, which can be used to study the properties of the solution (if exist) of three dimensional Navier-Stokes equations. We provide a new approach to prove the existence of a pathwise unique strong solution for the tamed Navier-Stokes equations under the impact of stochastic forcing.

1.2 Thesis layout

Chapter 2 includes all the necessary preliminaries required by the reader to understand the thesis. The majority of this thesis focuses on constrained Navier-Stokes equations. Thus, in Chapter 3 after introducing certain functional spaces and operators we deduce constrained Navier-Stokes equations (CNSE) from Navier-Stokes equations under the constraint of constant energy.

The questions of existence and uniqueness of a global-in-time solution of the deterministic CNSE are considered in Chapter 4. We start by giving the motivation behind studying such a system and stating the main results of the chapter. We prove the global existence of the solutions using the Banach Fixed Point Theorem and a non-explosion principle, i.e. proving the enstrophy (gradient norm) of the solution remains bounded. The periodic boundary conditions play a crucial role in obtaining the boundedness of enstrophy. We show that if the solution starts from the manifold $\mathscr{M}$, then it stays on $\mathscr{M}$. Furthermore we prove the existence of local solutions to CNSE with Dirichlet boundary conditions (as well as invariance of $\mathscr{M}$). In the vanishing viscosity limit, we show that the CNSE converges to the Bardos solution (see [5]) of the Euler equation. We extend our analysis to fractional Sobolev spaces, where using the Banach Fixed Point Theorem, Kuratowski-Zorn Lemma and maximal solutions we prove the global existence of solutions to CNSE. We end the chapter with an informal discussion about the lower bound on the regularity of the initial data corresponding to the well-posedness of CNSE.

The stochastic generalisation of CNSE is studied in Chapter 5. We consider the noise of gradient type in the Stratonovich form. The structure of the noise is such that it is "tangent" to the manifold $\mathscr{M}$. Here we take more classical approach of Faedo-Galerkin approximation to prove the existence of local solutions. We start by showing that each approximating equation has a global solution, which satisfy suitable a'priori estimates. Then, using Aldous condition along a'priori estimates, we prove that the laws of the solutions of these approximating equations are tight on a suitably chosen topological space $\mathscr{Z}_T$. By applications of the Jakubowski-Skorohod and martingale representation theorems we deduce the existence of martingale solutions. We also prove the so-called maximum regularity of solutions and their pathwise uniqueness, which helps us to establish the existence of a strong solution to stochastic constrained Navier-Stokes equations (SCNSE) by invoking Yamada-Watanabe type theorem. We end the chapter by showing that the solution of SCNSE depends continuously on the initial data.

Moving away from constrained equations, in Chapter 6 we shift our focus to tamed Navier-Stokes equations, which were introduced by Röckner and Zhang [75]. In this chapter we study stochastic tamed Navier-Stokes equations on $\mathbb{R}^3$ and reprove the results from [76] using a different approach and in doing so we generalise the L^4 estimate of the solution from $\mathbb{T}^3$ to the whole Euclidean space. We use three new ideas to prove the existence and uniqueness of global solutions and the existence of invariant measures on the whole Euclidean space. Firstly, in contrast to classical Faedo-Galerkin approximation where one uses finite dimensional spaces, we study the truncated equations on infinite dimensional spaces. We prove the existence of global solutions to these truncated equations satisfying suitable a'priori estimates. Secondly, we use a tightness criterion related to Dubinsky's compactness criterion introduced recently by Brzeźniak and Motyl [23]. Finally, we end the chapter by proving the existence of invariant measures using the method based on continuity and compactness in the weak topologies [62].

Chapter 7, the final chapter of the thesis summarises some of the open problems arising from this thesis and also enlists other related problems that will be part of my future research.

CHAPTER 2

PRELIMINARIES

This chapter has been included in the thesis so as to make it self-contained. We hope to provide all the necessary mathematical concepts that the reader might require to understand this thesis. The content of this chapter has been taken from various textbooks which have been aptly listed in the bibliography.

2.1 Hilbert space and orthogonal projection

Let H be a Hilbert space with the norm $|\cdot|_H$ induced by the inner product $\langle\cdot,\cdot\rangle_H$.

Definition 2.1.1. Let $x, y \in H$, then we say x is orthogonal to y if $\langle x, y\rangle_H = 0$. In general, if V is a subspace of H, then so is the set

$$V^{\perp} = \{x \in H : \langle x, y\rangle_H = 0 \ \forall y \in V\} .$$

Lemma 2.1.2. *[50, Theorem 21.4] Let V be a closed subspace of H. Then every $x \in H$ has a unique decomposition*

$$x = y + z \quad \textit{with } \ y \in V,\ z \in V^{\perp} .$$

Theorem 2.1.3 (Projection Theorem)**.** *Let V be a closed subspace of H. Then for every $x \in H$, there exists a unique element $\hat{x} \in V$ (same as y from the previous lemma) such that*

$$|x - \hat{x}|_H = \inf_{\mathrm{v} \in V} |\mathrm{v} - x|_H .$$

Moreover

i) $\hat{x} = x$ *iff* $x \in V$.

ii) $x - \hat{x} \in V^{\perp}$ *and*

$$|x|_H^2 = |\hat{x}|_H^2 + |x - \hat{x}|_H^2 .$$

Corollary 2.1.4. *[50, Corollary 21.5] For every closed subspace V of H there exists a unique linear map*

$$\pi : H \ni x \mapsto \hat{x} \in V ,$$

with

$$\|\pi\| := \sup_{x \in H, x \neq 0} \frac{|\pi x|_H}{|x|_H} = 1 ,$$

$$\pi^2 = \pi \text{ and } \ker \pi = V^{\perp} .$$

Remark 2.1.5. The existence of the element $\hat{x}$ in Corollary 2.1.4 is guaranteed by Theorem 2.1.3. The map π described in Corollary 2.1.4 is called the orthogonal projection of H onto V.

Definition 2.1.6. The sequence $(x_n)_{n\in\mathbb{N}} \subset H$ is said to converge (strongly) to $x \in H$, symbolically $x_n \to x$, if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \colon \forall n \geq N \; |x_n - x|_H < \varepsilon .$$

Definition 2.1.7. A sequence $(x_n)_{n\in\mathbb{N}} \subset H$ is called weakly convergent to $x \in H$, symbolically $x_n \rightharpoonup x$, if

$$\langle x_n, y \rangle_H \to \langle x, y \rangle_H , \quad \text{for all } y \in H .$$

One can easily see that a sequence converging in usual sense also converges weakly, since,

$$|\langle x_n, y \rangle - \langle x, y \rangle| \leq |x_n - x|_H \, |y|_H .$$

Theorem 2.1.8. *[50, Theorem 21.8] Every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset H$ has a weakly convergent subsequence.*

Lemma 2.1.9. *[50, Lemma 21.11] Every weakly convergent sequence $(x_n)_{n\in\mathbb{N}} \subset H$, is bounded.*

Remark 2.1.10. H is not metrizable w.r.t weak convergence. But a closed unit ball in a separable Hilbert space H w.r.t weak convergence is metrizable.

Definition 2.1.11. H is called a separable Hilbert space if it contains a countable dense subset.

Definition 2.1.12. An orthonormal basis of a Hilbert space H is a sequence $\{e_j\}_{j\in\mathbb{N}} \subset H$, such that linear span of $\{e_j\}$ is dense in H and

$$\langle e_j, e_k \rangle_H = \delta_{jk} , \qquad j,k \in \mathbb{N},$$

$$|e_j|_H = 1 , \qquad \forall j \in \mathbb{N}$$

where δ_{jk} is the Kronecker delta.

Theorem 2.1.13. *Every separable Hilbert space H admits an orthonormal basis.*

2.2 Linear operators

Let X and Y be normed spaces with $|\cdot|_X$ and $|\cdot|_Y$ norms respectively.

Definition 2.2.1. A linear operator from X to Y is a map $L\colon X \to Y$ such that for $\alpha, \beta \in \mathbb{R}$

$$L(\alpha x + \beta y) = \alpha L x + \beta L y, \quad x, y \in X.$$

Definition 2.2.2. If $L\colon X \to Y$ is a linear operator, the kernel of L is defined as the pre-image of the null vector in Y i.e.

$$\ker L := \{x \in X : Lx = 0\}.$$

The range of L is the set of all images i.e.

$$L(X) := \{Lx : x \in X\}.$$

Definition 2.2.3. A linear operator $L\colon X \to Y$ is called bounded if there exists a $C > 0$ such that

$$|Lx|_Y \le C|x|_X, \quad x \in X.$$

Theorem 2.2.4. *[50, Theorem 7.18] A linear operator $L\colon X \to Y$ is bounded if and only if it is continuous i.e. if $x_n \to x$, then $Lx_n \to Lx$.*

The set of all bounded linear operators from X into Y is denoted by $\mathscr{L}(X,Y)$. If $X = Y$ then we will write $\mathscr{L}(X)$.

Theorem 2.2.5. *[50, Theorem 7.21] Let Y be a Banach space and X a normed space. Then, $\mathscr{L}(X,Y)$, with the norm*

$$\|L\|_{\mathscr{L}(X,Y)} := \sup_{x \in X, |x|_X = 1} |Lx|_Y, \quad L \in \mathscr{L}(X,Y), \tag{2.2.1}$$

is a Banach space.

Lemma 2.2.6. *Let Z be a real normed space with the norm $|\cdot|_Z$. Assume that $L_1\colon X \to Y$ and $L_2\colon Y \to Z$ are bounded linear operators. Then the composition $L_2 \circ L_1$ is a bounded linear operator from X into Z and*

$$|L_2 \circ L_1|_{\mathscr{L}(X,Z)} \le |L_2|_{\mathscr{L}(Y,Z)} |L_1|_{\mathscr{L}(X,Y)}. \tag{2.2.2}$$

Definition 2.2.7. A sequence of operators $(L_n)_{n\in N} \subset \mathscr{L}(X,Y)$ is said to:

a) converge in operator norm to $L \in \mathscr{L}(X,Y)$, if

$$\|L_n - L\|_{\mathscr{L}(X,Y)} \to 0 \text{ as } n \to \infty.$$

b) strongly converge to $L \in \mathscr{L}(X,Y)$, iff $L_n x$ converges to Lx strongly in Y for each x in X.

Definition 2.2.8. A linear operator $L\colon X \to \mathbb{R}$ is called a linear functional. The space of all bounded linear functionals $f: X \to \mathbb{R}$ is called the dual space of X and is denoted by X^* or X', i.e. $X^* = \mathscr{L}(X, \mathbb{R})$.

Remark 2.2.9. By Theorem 2.2.5, X^* is a Banach space with the norm

$$|L|_{\mathscr{L}(X,\mathbb{R})} = \sup_{x\in X\,:\,|x|_X=1} |Lx|, \quad L \in X^*. \tag{2.2.3}$$

Theorem 2.2.10 (Riesz Representation Theorem)**.** *[50, Theorem 21.6] Let L be a bounded linear functional on the Hilbert space H. Then there exists a uniquely determined* $y \in H$ *with*

$$Lx = \langle x, y\rangle_H, \quad \forall x \in H.$$

Moreover

$$|L|_{\mathscr{L}(H,R)} = |y|_H.$$

Definition 2.2.11. If there exits an injective continuous linear map L from X into Y, then X is said to be embeddable in Y. Such a map L is called the embedding.

Definition 2.2.12. Assume that $X \subseteq Y$. Then X is called continuously embedded in Y if the inclusion map (identity function) $i: X \to Y$ is continuous, i.e. there exists a constant $c > 0$ such that

$$|u|_X \le c|u|_Y \quad u \in X.$$

In this case we denote this embedding symbolically by $X \hookrightarrow Y$ and the map i is called the embedding operator.

Definition 2.2.13. Let X^* be dual and X^{**} be double dual of X respectively. Then we have a canonical map $x \to \widehat{x}$ defined by:

$$\widehat{x}(f) = f(x) \quad f \in X^*,$$

gives an isometric linear isomorphism (embedding) from X into X^{**}. The space X is called reflexive if this map is also surjective.

2.2.1 Closed operators

Definition 2.2.14. $L\colon D(L) \to Y$ is a linear operator, $D(L) \subset X$, then L is called a closed linear operator if its graph

$$G(L) = \{(x, y)\colon x \in D(L), y = Lx\}$$

is closed in the normed space $X \times Y$.

Theorem 2.2.15. *[50] Let L be a linear operator in the Banach space* $(X, |\cdot|_X)$. *Define norm on* $\mathrm{D}(L)$ *by*

$$|x|_{\mathrm{D}(L)} = |x|_X + |Lx|_X, \quad x \in \mathrm{D}(L).$$

Then $(\mathrm{D}(L), |\cdot|_{\mathrm{D}(L)})$ *is a Banach space iff L is closed.*

2.2.2 Adjoint operators

Let H be a Hilbert space with the norm $|\cdot|_H$. Let $L\colon \mathrm{D}(L) \to H$ be a densely defined operator, with $\mathrm{D}(L) \subset H$. Let us denote by $\mathrm{D}(L^*)$ the set,

$$\mathrm{D}(L^*) = \{y \in H\colon \mathrm{D}(L) \ni x \to \langle Lx, y\rangle_H \in \mathbb{R} \text{ is } H\text{-continuous}\}.$$

Note that if L is bounded then $\mathrm{D}(L^*) = H$.

Hence by Riesz Representation Theorem 2.2.10, for every $y \in \mathrm{D}(L^*)$ there exists a unique (uniqueness is guaranteed by the denseness of L) $z \in H$ such that

$$\langle Lx, y\rangle_H = \langle x, z\rangle_H, \quad \forall\, x \in \mathrm{D}(L).$$

For $y \in \mathrm{D}(L^*)$, put $L^* y = z$.

Definition 2.2.16. For a densely defined linear operator $L\colon \mathrm{D}(L) \to H$, with $\mathrm{D}(L) \subset H$, its adjoint is an operator $L^*\colon \mathrm{D}(L^*) \to H$ that satisfies the identity

$$\langle Lx, y\rangle_H = \left\langle x, L^* y\right\rangle_H, \quad x \in \mathrm{D}(L),\ y \in \mathrm{D}(L^*).$$

Definition 2.2.17. A densely defined operator $L\colon D(L) \to H$, with $D(L) \subset H$, is called self-adjoint iff $\mathrm{D}(L) = \mathrm{D}(L^*)$ and

$$\langle Lx, y\rangle_H = \langle x, Ly\rangle_H, \quad x, y \in \mathrm{D}(L).$$

2.2.3 Compact operators

Definition 2.2.18. Let X and Y be normed spaces. A linear operator $L\colon X \to Y$ is called compact if for each bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$, the sequence $(Lx_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y.

Definition 2.2.19. The embedding $X \hookrightarrow Y$ is compact iff the identity map $i : X \to Y$ is compact.

Lemma 2.2.20. *[96, Theorem 8.3] Let X and Y be Banach spaces, then $\mathscr{L}C(X,Y)$, set of all compact operators from X into Y is a closed (and hence complete) subspace of $\mathscr{L}(X,Y)$ with operator norm.*

Definition 2.2.21. Let X and Y be Hilbert spaces. A linear operator $L\colon X \to Y$ is called Hilbert-Schmidt if for every complete orthonormal basis $(e_n)_{n\in\mathbb{N}} \subset X$

$$\|L\|_{HS} := \sum_{j=1}^{\infty} \left|Le_j\right|_Y^2 < \infty. \tag{2.2.4}$$

The space of all Hilbert-Schmidt operators $L\colon X \to Y$ will be denoted by $\mathscr{T}_2(X,Y)$ and the operator norm will be denoted by $\|\cdot\|_2$ or $\|\cdot\|_{\mathscr{T}_2(X,Y)}$ which is equal to $\|\cdot\|_{HS}$ norm defined in (2.2.4).

Theorem 2.2.22. *[96, Theorem 8.7] Hilbert-Schmidt operators are compact.*

2.3 Semigroups

For this section we will assume that X is a Banach space with the norm $|\cdot|_X$. The content of this section is based on [73].

Definition 2.3.1. A function $S : [0,\infty) \ni t \mapsto S(t) \in \mathscr{L}(X)$, which is usually denoted by $S = \{S(t)\}_{t\geq 0}$, is called a semigroup of linear bounded operators on X if

i) $S(0) = I$, where I is the identity operator on X,

ii) for all $t,s \geq 0$,
$$S(t+s) = S(t)S(s),$$
where $S(t)S(s)$ denotes the composition of the operators $S(t)$ and $S(s)$.

Definition 2.3.2. Let S be a semigroup on X. If

$$\lim_{t\to 0} |S(t) - I|_{\mathscr{L}(X)} = 0, \tag{2.3.1}$$

then S is called a uniformly continuous semigroup.

Definition 2.3.3. A semigroup S on X is called a C_0-semigroup (or strongly continuous semigroup) iff for each $x \in X$,

$$\lim_{t\to 0} |S(t)x - x|_X = 0. \tag{2.3.2}$$

Theorem 2.3.4. *If S be a C_0-semigroup on X, then there exist constants $M \geq 1$ and $\beta \geq 0$ such that*

$$|S(t)|_{\mathscr{L}(X)} \leq Me^{\beta t}, \quad t \geq 0. \tag{2.3.3}$$

Corollary 2.3.5. *If S is a C_0-semigroup on X, then for each $x \in X$, the function*

$$S(\cdot)(x) : [0,\infty) \ni t \mapsto S(t)(x) \in X$$

is continuous on $[0,\infty)$.

Definition 2.3.6. Let S be a semigroup on X. A linear operator L defined by

$$\mathrm{D}(L) = \left\{ x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists} \right\} \tag{2.3.4}$$

and

$$Lx = \lim_{t\to 0} \frac{S(t)x - x}{t} = \frac{dS(t)x}{dt}\Big|_{t=0}, \quad x \in \mathrm{D}(L) \tag{2.3.5}$$

is called the infinitesimal generator of the semigroup S.

Theorem 2.3.7. *A linear operator L is the infinitesimal generator of a uniformly continuous semigroup iff L is a bounded linear operator.*

Theorem 2.3.8. *Let S be a C_0-semigroup on X with the infinitesimal generator L. Then for all $t \geq 0$,*

i) for each $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(s)x\, ds = S(t)x, \tag{2.3.6}$$

ii) for each $x \in X$, $\int_0^t S(s)x\, ds \in \mathrm{D}(L)$ and

$$L\left(\int_0^t S(s)x\, ds\right) = S(t)x - x, \tag{2.3.7}$$

iii) for each $x \in \mathrm{D}(L)$, $S(t)x \in \mathrm{D}(L)$ and

$$\frac{d}{dt} S(t)x = LS(t)x = S(t)Lx, \tag{2.3.8}$$

iv) for each $x \in \mathrm{D}(L)$,

$$S(t)x - S(s)x = \int_s^t S(\tau)Lx\, d\tau = \int_s^t LS(\tau)x\, d\tau. \tag{2.3.9}$$

Corollary 2.3.9. *If L is the infinitesimal generator of a C_0 semigroup S on X, then $\mathrm{D}(L)$ is dense in X and L is a closed linear operator.*

Definition 2.3.10. A C_0-semigroup S on X is called uniformly bounded if there exists a constant $M \geq 1$ such that for each $t \geq 0$,

$$|S(t)|_{\mathscr{L}(X)} \leq M. \tag{2.3.10}$$

Definition 2.3.11. A C_0-semigroup S on X is called a contraction semigroup if for each $t \geq 0$,

$$|S(t)|_{\mathscr{L}(X)} \leq 1. \tag{2.3.11}$$

2.4 Deterministic compactness criterion

Let (X, ρ) be a metric space. Consider the set,

$$\mathscr{C}(X) := \{f : X \to \mathbb{R} \text{ is continuous}\},$$

then $\mathscr{C}(X)$ is a Banach space endowed with the norm

$$\|f\|_{\mathscr{C}(X)} := \sup_{x \in X} |f(x)|.$$

We say that a sequence (f_n) converges to f in $\mathscr{C}(X)$ if

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0,$$

i.e. (f_n) converges uniformly to f in X.

Definition 2.4.1. Let (X,ρ) be a metric space. A family of functions $\Lambda \subset \mathscr{C}(X)$ is called equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in \Lambda$,

$$|f(u) - f(\mathrm{v})| < \varepsilon, \text{ for all } u, \mathrm{v} \in X \text{ satisfying } \rho(u,\mathrm{v}) < \delta.$$

The family Λ is called uniformly bounded if there exists a constant $C > 0$ such that

$$|f(u)| \le C, \text{ for all } f \in \Lambda \text{ and for all } u \in X.$$

Definition 2.4.2. If (X,ρ) be a metric space, a subset $K \subset X$ is called precompact (or relatively compact) if closure $\overline{K}$ of K is compact in X.

Theorem 2.4.3 (Arzelà-Ascoli Theorem)**.** *[50, Theorem 5.20] Let (X,ρ) be a compact metric space. For a family of functions $\Lambda \subset \mathscr{C}(X)$ following conditions are equivalent:*

i) Λ is relatively compact,

ii) Λ is equicontinuous and uniformly bounded.

Following is an immediate consequence of the Arzelà-Ascoli theorem.

Corollary 2.4.4. *[50, Corollary 5.21] Let (X,ρ) be a compact metric space. If a sequence of functions $(f_n) \subset \mathscr{C}(X)$ is equicontinuous and uniformly bounded then it contains a uniformly convergent subsequence.*

Now we state the classical compactness criteria due to Dubinsky [94, Theorem IV.4.1] (see also [56]).

Theorem 2.4.5 (Dubinsky Theorem)**.** *Let E_0, E and E_1 be reflexive Banach spaces such that $E_0 \hookrightarrow E \hookrightarrow E_1$ and the embedding $E_0 \hookrightarrow E$ is compact. Let $q \in (1,\infty)$ and let K be a bounded subset in $L^q(0,T;E_0)$ consisting of functions equicontinuous in $\mathscr{C}([0,T];E_1)$. Then K is relatively compact in $L^q(0,T;E) \cap \mathscr{C}([0,T];E_1)$.*

2.5 Random variables

Throughout this section, we assume that X is a separable Banach space with norm $|\cdot|_X$ and Ω is a non-empty set. The contents of this section are based on [38, 79].

Definition 2.5.1. A family $\mathscr{F}$ of subsets of Ω is called a σ-field on Ω if

i) $\Omega \in \mathscr{F}$,

ii) if $A \in \mathscr{F}$, then $\Omega \setminus A \in \mathscr{F}$,

iii) if $A_1, A_2, \ldots$ is a sequence of sets in $\mathscr{F}$, then the countable union $\bigcup_{n\in\mathbb{N}} A_n$ also belongs to $\mathscr{F}$.

The pair $(\Omega,\mathscr{F})$ is called a measurable space.

Definition 2.5.2. Let $(\Omega_1,\mathscr{F})$ and $(\Omega_2,\mathscr{G})$ be two measurable spaces. A map $\xi:\Omega_1\to\Omega_2$ is said to be measurable if for every $A\in\mathscr{G}$,

$$\xi^{-1}(A)=\{\omega\in\Omega_1:\xi(\omega)\in A\}\in\mathscr{F}.$$

Such a measurable map is called a random variable on Ω_1.

Definition 2.5.3. Let $\mathscr{H}$ be a family of subsets of Ω. The smallest σ-field on Ω containing $\mathscr{H}$ is called the σ-field generated by $\mathscr{H}$ and it is denoted by $\sigma(\mathscr{H})$.

Definition 2.5.4. The smallest σ-field containing all closed (or open) subsets of X is called the Borel σ-field of X and it is denoted by $\mathscr{B}(X)$.

Lemma 2.5.5. *[38, Proposition 1.3] Let X^* be the dual space of X. Then $\mathscr{B}(X)$ is the smallest σ-field of X containing all sets of the form*

$$\{x\in X:\varphi(x)\le\alpha\},\quad \varphi\in X^*,\quad \alpha\in\mathbb{R}.$$

Definition 2.5.6. Let $(\Omega,\mathscr{F})$ be a measurable space. A mapping $\xi:\Omega\to X$ is said to be Borel measurable if for each $A\in\mathscr{B}(X)$, $\xi^{-1}(A)\in\mathscr{F}$. Such a Borel measurable map is called an X-valued random variable on Ω.

Definition 2.5.7. Let $(\Omega,\mathscr{F})$ be a measurable space and ξ be an X-valued random variable on Ω. The smallest σ-field $\sigma(\xi)$ containing all sets $\xi^{-1}(A)$, $A\in\mathscr{B}(X)$, is called the σ-field generated by ξ.

Lemma 2.5.8. *[38, Lemma 1.5] Let $(\Omega,\mathscr{F})$ be a measurable space. Assume that ξ and ζ are X-valued random variables on Ω. Then*

i) for $\alpha,\beta\in\mathbb{R}$, $\alpha\xi+\beta\zeta$ is an X-valued random variable on Ω,

ii) the mapping $\Omega\ni\omega\mapsto|\xi(\omega)|_X$ is a real-valued random variable on Ω.

Definition 2.5.9. Let $(\Omega,\mathscr{F})$ be a measurable space. A map $\mu:\mathscr{F}\to\mathbb{R}$ is called a non-negative measure if

i) for all $A\in\mathscr{F}$, $\mu(A)\geqslant 0$,

ii) $\mu(\emptyset)=0$,

iii) for all countable collection $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathscr{F}$,

$$\mu\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}\mu(A_i).$$

The third condition in the definition is called the σ-additivity and the triple $(\Omega,\mathscr{F},\mu)$ is called a measure space. A non-negative measure $\mathbb{P}$ satisfying $\mathbb{P}(\Omega) = 1$ is called a probability measure and the triple $(\Omega,\mathscr{F},\mathbb{P})$ is called a probability space.

Theorem 2.5.10 (Lebesgue's Monotone Convergence Theorem)**.** *[79, Theorem 1.26] Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued measurable functions on Ω, satisfying*

a) $0 \le f_1 \le f_2 \le \ldots \le \infty$

b) $f_n \to f$ *point-wise as* $n \to \infty$.

Then f is measurable and

$$\int_\Omega f_n \, d\mu \to \int_\Omega f \, d\mu \quad \text{as } n \to \infty.$$

Theorem 2.5.11 (Fatou's Lemma)**.** *[79, Lemma 1.28] If $f_n : \Omega \to [0,\infty]$ is measurable for all $n \in \mathbb{N}$, then*

$$\int_\Omega \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int_\Omega f_n \, d\mu,$$
$$\limsup_{n\to\infty} \int_\Omega f_n \, d\mu \le \int_\Omega \left(\limsup_{n\to\infty} f_n\right) d\mu.$$

Theorem 2.5.12. *[79, Theorem 1.33] If $f \in L^1(\mu)$ i.e. $\int_E f \, d\mu < \infty$, then*

$$\left|\int_\Omega f \, d\mu\right| \le \int_\Omega |f| \, d\mu.$$

Theorem 2.5.13 (Lebesgue's Dominated Convergence Theorem)**.** *[79, Theorem 1.34] Suppose $(f_n)_{n\in\mathbb{N}}$ be the sequence of real valued measurable functions on Ω such that $f_n \to f$ point-wise as $n \to \infty$. If there exists a function $g \in L^1(\mu)$ such that $|f_n(\omega)| \le g(\omega)$ for every $\omega \in \Omega$ then $f \in L^1(\mu)$ and*

$$\lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mu = 0,$$
$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu.$$

Theorem 2.5.14 (Vitali Convergence Theorem or Vitali Theorem)**.** *[79, Exercise 6.10 (b)] Let $(\Omega,\mathscr{F},\mu)$ be a measure space. If $\mu(\Omega) < \infty$ and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions on Ω such that*

i) $\{f_n\}$ *is uniformly integrable,*

ii) $f_n \to f$ *pointwise a.e. on* Ω,

iii) $|f(x)| < \infty$ *a.e.*

Then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mu = 0.$$

Definition 2.5.15. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space. A set $\overline{\mathscr{F}}$ defined by

$$\overline{\mathscr{F}} = \{A \subset \Omega : \exists\, B, C \in \mathscr{F}; B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C)\}$$

is a σ-field and is called the completion of $\mathscr{F}$. If $\mathscr{F} = \overline{\mathscr{F}}$, then the probability space $(\Omega,\mathscr{F},\mathbb{P})$ is said to be complete.

Definition 2.5.16. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and ξ be an X-valued random variable on Ω. Then a mapping $L(\xi) : \mathscr{B}(X) \to [0,1]$ defined by

$$L(\xi)(A) = \mathbb{P}\left(\xi^{-1}(A)\right) = \mathbb{P}(\{\omega \in \Omega : \xi(\omega) \in A\}), \quad A \in \mathscr{B}(X)$$

is called the law (or the distribution) of ξ.

Theorem 2.5.17 (Kuratowski Theorem)**.** *[70, Theorem 3.9] Assume that X_1, X_2 are the Polish spaces with their Borel $\sigma-$fields denoted respectively by $\mathscr{B}(X_1), \mathscr{B}(X_2)$. If $\varphi : X_1 \to X_2$ is an injective Borel measurable map then for any $E_1 \in \mathscr{B}(X_1)$, $E_2 := \varphi(E_1) \in \mathscr{B}(X_2)$.*

2.6 Miscellaneous preliminaries

Lemma 2.6.1 (Gagliardo - Nirenberg inequality)**.** *[91] Assume that $r, q \in [1,\infty)$, and $j, m \in \mathbb{Z}$ satisfy $0 \le j < m$. Then for all $\alpha \in \left[\frac{j}{m}, 1\right]$ there exists a constant $C > 0$ such that*

$$\left|D^j u\right|_{L^p(\mathbb{R}^n)} \le C \left|D^m u\right|^{\alpha}_{L^r(\mathbb{R}^n)} |u|^{1-\alpha}_{L^q(\mathbb{R}^n)}, \quad u \in C_0^\infty\left(\mathbb{R}^n\right), \tag{2.6.1}$$

where $\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{d}\right) + (1-\alpha)\frac{1}{q}$. If $m - j - \frac{n}{r}$ is a non-negative integer, then the equality holds only for $\alpha \in \left[\frac{j}{m}, 1\right)$.

The next two results play a pivotal role in this thesis.

Theorem 2.6.2 (Banach Fixed Point Theorem)**.** *[50, Theorem 4.7] Let (X, d) be a complete metric space, $K \subset X$ be a closed subset, $f : K \to K$ be a function that satisfies the inequality, for some $0 \le \alpha < 1$,*

$$d(f(u), f(\mathrm{v})) \le \alpha d(u, \mathrm{v}), \quad \text{for all } u, \mathrm{v} \in X,$$

Then f has uniquely determined fixed point in K i.e. there exists a unique $a \in K$ such that $f(a) = a$.

Lemma 2.6.3. *[88, Lemma III.1.2] Let $T > 0$, V, H be two Hilbert spaces, V^* and H^* be corresponding dual spaces. Assume that $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$, where embeddings are dense too. If a function u belongs to $L^2(0,T;V)$ and its weak derivative u' belongs to $L^2(0,T;V')$, then u is a.e. equal to a continuous function $\mathrm{v} : [0,T] \to \mathrm{H}$ such that the function $[0,T] \ni t \mapsto |\mathrm{v}(t)|^2_{\mathrm{H}} \in \mathbb{R}$ is absolutely continuous and*

$$\frac{d}{dt}|\mathrm{v}(t)|^2_H = 2\left\langle \mathrm{v}', \mathrm{v}\right\rangle_H, \quad \text{for almost all } t \in [0,T].$$

The following lemma is used repeatedly in this thesis. We will later state a generalisation of it for random variables.

Lemma 2.6.4 (Bellman–Gronwall Inequality or Gronwall Lemma). *[86, Section 1.3.6] Suppose $\phi \in L^1[a,b]$ satisfies*

$$\phi(t) \leq f(t) + \beta \int_a^t \phi(s)\,ds\,, \quad a.e.\,,$$

where $f \in L^1[a,b]$ and β is a positive constant, then

$$\phi(t) \leq f(t) + \beta \int_a^t f(s) e^{\beta(t-s)}\,ds, \quad \text{for a.e. } t \in [a,b].$$

In particular if $f(t) = \alpha$ (constant) then

$$\phi(t) \leq \alpha e^{\beta(t-a)}, \quad \text{for a.e. } t \in [a,b]\,.$$

Lemma 2.6.5. *[58, Chapter 2] Let $f \in L^{p_0} \cap L^{p_1}$. Then for $\theta \in (0,1)$ there exists a constant $c > 0$ such that*

$$|f|_{L^{p_\theta}} \leq c |f|_{L^{p_0}}^{1-\theta} |f|_{L^{p_1}}^{\theta}\,, \quad f \in L^{p_0} \cap L^{p_1}\,,$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,.$$

Lemma 2.6.6 (Poincaré Inequality). *[50, Corollary 20.16] Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Then there exists a constant C, depending only on Ω*

$$|u|_{L^2(\Omega)} \leq C |\nabla u|_{L^2(\Omega)} \qquad u \in W_0^{1,2}(\Omega)\,.$$

Theorem 2.6.7 (Plancherel Theorem). *[79, Theorem 9.13] Let $f \in L^2$ and we denote its Fourier transform by $\hat{f}$. Then*

i) for every $f \in L^2$, $|f|_{L^2} = |\hat{f}|_{L^2}$,

ii) the mapping $f \to \hat{f}$ is a Hilbert space isomorphism of L^2 onto L^2.

2.6.1 Kuratowski-Zorn Lemma

This subsection is based on [60, Chapter 1].

Definition 2.6.8. A relation $\preceq$ on Ξ is called **partial order** if $\preceq$ is:

i) reflexive i.e. $x \preceq x$ for all $x \in \Xi$,

ii) antisymmetric i.e. for all $x, y \in \Xi$ if $x \preceq y$ and $y \preceq x$, then $x = y$,

iii) transitive i.e. for all $x, y, z \in \Xi$ if $x \preceq y$ and $y \preceq z$, implies $x \preceq z$.

In this case $(\Xi, \preceq)$ is called partially ordered set (or poset).

Definition 2.6.9. A chain in a poset $(\Xi, \preceq)$ is a subset $B \subseteq \Xi$ such that any two elements in B are comparable.

Definition 2.6.10. Let $(\Xi, \preceq)$ be a poset and $B \subseteq \Xi$ then an element $u \in \Xi$ is called upper bound of B, if $x \preceq u$ for all $x \in B$.

Definition 2.6.11. Let $(\Xi, \preceq)$ be a poset. An element $m \in \Xi$ is called maximal element of Ξ, if there is no element $x \in \Xi$ such that $m \preceq x$ and $m \neq x$.

Now we state the main result of this subsection.

Lemma 2.6.12 (Kuratowski-Zorn Lemma)**.** *If every chain in a poset* $(\Xi, \preceq)$ *has an upper bound in* Ξ*, then* Ξ *contains a maximal element.*

2.7 Stochastic processes and martingale

We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ for this section and assume that H is a separable Hilbert space with the norm $|\cdot|_H$. The content of this section is based on [28, 38].

Definition 2.7.1. An H-valued continuous-time stochastic process $\{\xi_t\}_{t\in\mathbb{T}}$ is a family of H-valued random variables indexed by time t. Moreover, either $\mathbb{T} := [0, T]$ or $\mathbb{T} := [0, \infty)$.

For each $\omega \in \Omega$, the map

$$\xi(\omega) : \mathbb{T} \ni t \to \xi_t(\omega) \in H$$

is called a path (or trajectory) of the process ξ.

We will use the notation ξ instead of $\{\xi(t)\}_{t\in\mathbb{T}}$ for simplicity. And throughout this section we will assume that ξ is an H-valued stochastic process on $\mathbb{T}$ unless specified otherwise.

Definition 2.7.2. A process ξ is called continuous if $\mathbb{P}$-a.s. the trajectories of ξ are continuous on $\mathbb{T}$, i.e. there exists $\bar{\Omega} \in \mathscr{F}$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for each $\omega \in \bar{\Omega}$, the mapping $\mathbb{T} \ni t \mapsto \xi(t, \omega) \in H$ is continuous.

Definition 2.7.3. An H-valued stochastic process ζ on $\mathbb{T}$ is called a modification (or version) of the process ξ if

$$\mathbb{P}(\{w \in \Omega \ : \ \xi(t, w) \neq \zeta(t, w)\}) = 0 \quad \text{for every } t \in \mathbb{T}.$$

Definition 2.7.4. A process ξ is called measurable if the following mapping

$$\xi : [0, T] \times \Omega \to H$$

is $\mathscr{B}([0, T]) \otimes \mathscr{F}$-measurable.

Definition 2.7.5. A process ξ is called stochastically continuous at $t_0 \in [0,T]$ if for every $\epsilon, \delta > 0$, there exists $\rho > 0$ such that for each $t \in [t_0-\rho, t_0+\rho] \cap [0,T]$,

$$\mathbb{P}(\{\omega \in \Omega : |\xi(t,\omega) - \xi(t_0,\omega)|_H \geq \epsilon\}) \leq \delta .$$

If the process ξ is stochastically continuous for each $t_0 \in [0,T]$, then it is said to be stochastically continuous on $[0,T]$.

Lemma 2.7.6. *[38, Proposition 3.2] If a process ξ is stochastically continuous on $[0,T]$, then it has a measurable modification on $[0,T]$.*

Definition 2.7.7. A family $\{\mathscr{F}_t\}_{t\geq 0}$ of σ-fields such that for all t $\mathscr{F}_t \subset \mathscr{F}$, is called a filtration if for any $0 \leq s \leq t < \infty$, $\mathscr{F}_s \subset \mathscr{F}_t$.

From now on, we will assume that $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$ is a filtration.

Definition 2.7.8. A process ξ is said to be adapted to $\mathbb{F}$ if for each $t \in [0,T]$, $\xi(t)$ is $\mathscr{F}_t$-measurable.

Definition 2.7.9. A process ξ is called progressively measurable if for each $t \in [0,T]$, the following mapping

$$\xi : [0,t] \times \Omega \ni (s,w) \mapsto \xi(s,w) \in H$$

is $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$-measurable.

Definition 2.7.10. A subset $P \subseteq [0,\infty) \times \Omega$ is said to be progressively measurable, if the process $\xi_s(\omega) := \mathbb{1}_P(s,\omega)$ is progressively measurable. The σ-field generated by all such subsets P of $[0,\infty) \times \Omega$ is called progressively measurable σ-field.

Remark 2.7.11. If the process ξ is progressively measurable, then it is adapted to $\mathbb{F}$.

Lemma 2.7.12. *Limits of progressively measurable processes are progressively measurable.*

Lemma 2.7.13. *[38, Proposition 3.5] If a process ξ is stochastically continuous on $[0,T]$ and adapted to $\mathbb{F}$, then it has a progressively measurable modification.*

Definition 2.7.14. A random variable $\tau : \Omega \to [0,\infty]$ i.e. a random time, is a stopping time (w.r.t filtration $\mathbb{F}$) if for all $t \in \mathbb{T}$,

$$\{\tau \leq t\} := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathscr{F}_t .$$

Remark 2.7.15.

i) One can easily see that if τ and σ are two stopping times then $\tau \wedge \sigma$, $\tau \vee \sigma$ and $\tau + \sigma$ are also stopping times.

ii) Every stopping time τ is $\mathscr{F}_\tau$-measurable, where

$$\mathscr{F}_\tau = \{B \in \mathscr{F} : B \cap \{\tau \leq t\} \in \mathscr{F}_t, \text{ for all } t \in \mathbb{T}\}$$

is a σ-field.

iii) A r.v ζ is $\mathscr{F}_\tau$-measurable if and only if for all $t \in \mathbb{T}$, $\zeta \mathbb{1}_{\{\tau \le t\}}$ is $\mathscr{F}_t$-measurable.

Lemma 2.7.16. *[74, Proposition 1.1.3] Let ξ be a progressively measurable process, and τ a stopping time. Then $\xi_\tau \mathbb{1}_{\{\tau \le t\}}$ is $\mathscr{F}_\tau$-measurable and the stopped process $\xi_{t\wedge\tau}$ is also progressively measurable.*

Definition 2.7.17. An H-valued process ξ is called an $\mathbb{F}$-martingale if

i) ξ is adapted to $\mathbb{F}$,

ii) for each $t \in [0,T]$, $\mathbb{E}(|\xi(t)|_H) < \infty$,

iii) for each $t, s \in [0,T]$ with $t \ge s$,

$$\mathbb{E}(\xi(t)|\mathscr{F}_s) = \xi(s), \text{ i.e. } \forall A \in \mathscr{F}_s \quad \int_A \xi(t)\, d\mathbb{P} = \int_A \xi(s)\, d\mathbb{P}.$$

Lemma 2.7.18. *[38, Proposition 3.9] Let $\mathscr{M}_T^2$ be the space of all H-valued, continuous and square integrable martingales ξ on $[0,T]$. Then $\mathscr{M}_T^2$ is a Banach space with respect to the following norm*

$$|\xi|_{\mathscr{M}_T^2} = \left(\mathbb{E} \sup_{t\in[0,T]} |\xi(t)|_H^2\right)^{\frac{1}{2}}, \quad \xi \in \mathscr{M}_T^2.$$

Theorem 2.7.19 (Burkholder-Davis-Gundy inequality)**.** *[74, Theorem 1.1.6] Let $1 \le p < \infty$, then for all H-valued continuous martingales M with $M_0 = 0$ and stopping time τ, there exist positive constants c_p and C_p such that*

$$\mathbb{E}\left[\langle M(\tau)\rangle^{p/2}\right] \le \mathbb{E}\left(\sup_{0\le t\le\tau} |M_t|\right)^p \le C_p \mathbb{E}\left[\langle M(\tau)\rangle^{p/2}\right], \tag{2.7.1}$$

where $\langle M\rangle$ denotes the quadratic variation of M.

We will require following generalisation of the Gronwall Lemma [40, Lemma 3.9]:

Lemma 2.7.20 (Generalised Gronwall Lemma)**.** *Let X, Y, I and φ be non-negative processes and Z be a non-negative integrable random variable. Assume that I is non-decreasing and there exist non-negative constants $C, \alpha, \beta, \gamma, \eta$ with the following properties*

$$\int_0^T \varphi(s)\, ds \le C \; a.s., \qquad 2\beta e^C \le 1, \qquad 2\eta e^C \le \alpha, \tag{2.7.2}$$

and such that for $0 \le t \le T$,

$$X(t) + \alpha Y(t) \le Z + \int_0^t \varphi(r) X(r)\, dr + I(t), \quad a.s., \tag{2.7.3}$$

$$\mathbb{E}(I(t)) \le \beta \mathbb{E}(X(t)) + \gamma \int_0^t \mathbb{E}(X(s))\, ds + \eta \mathbb{E}(Y(t)) + \tilde{C}, \tag{2.7.4}$$

where $\tilde{C} > 0$ is a constant. If $X \in L^\infty([0,T]\times\Omega)$, then we have

$$\mathbb{E}[X(t) + \alpha Y(t)] \le 2\exp\left(C + 2t\gamma e^C\right)\left(\mathbb{E}(Z) + \tilde{C}\right), \qquad t \in [0,T]. \tag{2.7.5}$$

2.8 Wiener process and the martingale representation theorem

We assume that H is a separable Hilbert space with the norm $|\cdot|_H$ and $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space. The definitions and results are taken from [28, 38].

Definition 2.8.1. A probability measure μ on $(H, \mathscr{B}(H))$ is called Gaussian if for arbitrary $h \in H$ there exist $m \in \mathbb{R}$, $\sigma \geq 0$, such that

$$\mu\{x \in H : \langle h, x\rangle_H \in A\} = \mathscr{N}(m,\sigma)(A), \quad A \in \mathscr{B}(\mathbb{R}).$$

Definition 2.8.2. An H-valued stochastic process X on $[0,\infty)$ is said to be Gaussian if, for any $n \in N$ and for arbitrary positive numbers, $t_1, t_2, \cdots, t_n$, the H^n-valued random variable $(X(t_1), X(t_2), \cdots, X(t_n))$ is Gaussian.

Definition 2.8.3. A real valued Wiener process (or Brownian motion) is a stochastic process $W(t)$ with values in $\mathbb{R}$ defined for $t \in [0,\infty)$ such that

i) $W(0) = 0$ a.s.,

ii) the sample paths $t \mapsto W(t)$ are a.s. continuous,

iii) for $0 \leq s \leq t < \infty$, $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$,

iv) for any $0 = t_0 \leq t_1 \leq \cdots \leq t_n < \infty$, the increments

$$W(t_1) - W(t_0), \cdots, W(t_n) - W(t_{n-1})$$

are independent.

Definition 2.8.4. We call $W(t) = \big(W^1(t), W^2(t), \cdots, W^d(t)\big)$, a d-dimensional Wiener process if $W^1(t), \cdots, W^d(t)$ are independent $\mathbb{R}$-valued Wiener processes.

Let U be a Hilbert space (can be finite dimensional too) with norm $|\cdot|_U$ and $Q \in \mathscr{L}(U)$ be a symmetric non-negative operator. We also assume that $\mathrm{Tr}\, Q < \infty$. Then there exists a complete orthonormal basis $\{e_k\}$ in U, and a bounded sequence of non-negative real numbers λ_k such that

$$Qe_k = \lambda_k e_k, \quad k = 1, 2, \cdots.$$

Without loss of generality, we may assume that Q is injective.

Definition 2.8.5. A U-valued stochastic process $W(t)$, $t \geq 0$ is called a Q-Wiener process if

i) $W(0) = 0$ a.s.,

ii) W has a.s. continuous trajectories,

iii) W has independent increments,

iv) $\mathscr{L}(W(t)-W(s)) = \mathscr{N}(0,(t-s)Q),\ 0 \le s \le t.$

If a process $W(t), t \in [0,T]$ satisfies $(i)-(iii)$ and (iv) for $t,s \in [0,T]$ then we say that W is a Q-Wiener process on $[0,T]$.

Lemma 2.8.6. *[38, Proposition 4.1] Assume that W is a Q-Wiener process on U, with* $\mathrm{Tr}\, Q < \infty$. *Then the following statements hold*:

i) W is a Gaussian process on U and

$$\mathbb{E}(W(t)) = 0, \qquad \mathrm{Cov}(W(t)) = tQ, \quad t \ge 0. \tag{2.8.1}$$

ii) For arbitrary t,

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \beta_j(t) e_j \quad a.e. \tag{2.8.2}$$

where

$$\beta_j(t) = \frac{1}{\lambda_j} \langle W(t), e_j \rangle_U, \quad j = 1,2,\cdots,$$

are real valued Brownian motions mutually independent on $(\Omega, \mathscr{F}, \mathbb{P})$ *and the series* (2.8.2) *is convergent in* $L^2(\Omega, \mathscr{F}, \mathbb{P})$.

Theorem 2.8.7 (Martingale Representation Theorem). *[38, Theorem 8.2] Assume that* $M \in \mathscr{M}_T^2(H)$ *and*

$$\langle\langle M \rangle\rangle_t = \int_0^t \left(\varphi(s) Q^{1/2}\right)\left(\varphi(s) Q^{1/2}\right)^* ds, \qquad t \in [0,T],$$

where φ *is a predictable* $\mathscr{T}_2(U_0, H)$ *process;* $U_0 = Q^{1/2} U$ *is a Hilbert space endowed with the inner product*

$$\langle u, \mathrm{v} \rangle_{U_0} = \langle Q^{-1/2} u, Q^{-1/2} \mathrm{v} \rangle_U \quad u, \mathrm{v} \in U_0,$$

and Q a given bounded symmetric non-negative operator in U. Then there exists a probability space $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}\right)$, *a filtration* $\{\tilde{\mathscr{F}}_t\}$ *and a Q-Wiener process W, with values in U, defined on* $\left(\Omega \times \tilde{\Omega}, \mathscr{F} \times \tilde{\mathscr{F}}, \mathbb{P} \times \tilde{\mathbb{P}}\right)$ *adapted to* $\{\mathscr{F}_t \times \tilde{\mathscr{F}}_t\}$, *such that*

$$M(t, \omega, \tilde{\omega}) = \int_0^t \varphi(s, \omega, \tilde{\omega})\, dW(s, \omega, \tilde{\omega}), \quad t \in [0,T], (\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}, \tag{2.8.3}$$

where

$$M(t, \omega, \tilde{\omega}) = M(t, \omega), \quad and \quad \varphi(t, \omega, \tilde{\omega}) = \varphi(t, \omega), \quad (\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}. \tag{2.8.4}$$

Next we state a simplified version of an existence theorem for the stochastic differential equation

$$X(t) = \xi + \int_0^t \sigma(X(s))\, dW(s) + \int_0^t b(X(s))\, ds, \tag{2.8.5}$$

where the maps

$$\sigma : \mathbb{R}^d \to \mathcal{T}_2(H, \mathbb{R}^d), \qquad b : \mathbb{R}^d \to \mathbb{R}^d,$$

where H is a real separable, possibly infinite dimensional, Hilbert space, are measurable. Suppose that $\mathcal{U} := (\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ is a filtered probability space and $W = (W(t))_{t\geq 0}$ be an H-cylindrical Wiener process on $\mathcal{U}$.

Theorem 2.8.8. *[1, Theorem 3.1] Assume that the functions σ and b satisfy the following conditions*

(i) For any $R > 0$ there exists a constant $C > 0$ such that

$$\|\sigma(x) - \sigma(y)\|_{\mathcal{T}_2(\mathbb{R}^d;H)} + |b(x) - b(y)|_{\mathbb{R}^d} \leq C|x - y|^2_{\mathbb{R}^d}, \quad |x|_{\mathbb{R}^d}, |y|_{\mathbb{R}^d} \leq R.$$

(ii) There exists a constant $K_1 > 0$ such that

$$\|\sigma(x)\|^2_{\mathcal{T}_2(\mathbb{R}^d;H)} + 2\langle x, b(x)\rangle_{\mathbb{R}^d} \leq K_1(1 + \|x\|^2_{\mathbb{R}^d}), \qquad x \in \mathbb{R}^d.$$

Then, for any $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random variable ξ, there exists a unique global solution $X = (X(t))_{t\geq 0}$ to (2.8.5).

2.9 Tightness and Skorohod Theorem

In this section we take E to be a separable Banach space with the norm $|\cdot|_E$ and let $\mathcal{B}(E)$ be its Borel σ-field. The family of probability measures on $(E, \mathcal{B}(E))$ will be denoted by Λ. The set of all bounded and continuous E-valued functions is denoted by $\mathcal{C}_b(E)$. The content of this section is based on [23, 24] and [38, Chapter 2].

Definition 2.9.1. The family Λ of probability measures on $(E, \mathcal{B}(E))$ is said to be tight if for arbitrary $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset E$ such that

$$\mu(K_\varepsilon) \geq 1 - \varepsilon, \quad \text{for all } \mu \in \Lambda.$$

Definition 2.9.2. A sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $(E, \mathcal{B}(E))$ is said to be weakly convergent to a measure μ if for every $\varphi \in \mathcal{C}_b(E)$ we have

$$\lim_{n\to\infty} \int_E \varphi(x)\mu_n(dx) = \int_E \varphi(x)\,\mu(dx).$$

Definition 2.9.3. The family Λ is said to be compact (respectively relatively compact), if an arbitrary sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of elements from Λ contains a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ weakly convergent to a measure $\mu \in \Lambda$ (respectively to a measure μ on $(E, \mathcal{B}(E))$).

Theorem 2.9.4 (Prokhorov Theorem)**.** *[38, Theorem 2.3] The family Λ of probability measures on $(E, \mathcal{B}(E))$ is relatively compact if and only if it is tight.*

The following theorem, due to Skorohod, links the concept of weak convergence of probability measures with that of almost sure convergence of random variables.

Theorem 2.9.5 (Skorohod Theorem)**.** *[38, Theorem 2.4] For an arbitrary sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $\mathscr{B}(E)$ weakly convergent to a probability measure μ, there exists a probability space $(\Omega,\mathscr{F},\mathbb{P})$ and random variables $X, X_1,\cdots$, such that $\mathscr{L}(X_m)=\mu_m$, $\mathscr{L}(X)=\mu$ and*

$$\lim_{n\to\infty} X_n = X, \quad \mathbb{P}-a.s.$$

We will need the following Jakubowski's generalisation of the Skorohod Theorem, in the form given by Brzeźniak and Ondreját [26, Theorem C.1], see also [49], as we deal with non-metric spaces.

Theorem 2.9.6. *Let $\mathscr{X}$ be a topological space such that there exists a sequence $\{f_m\}_{m\in\mathbb{N}}$ of continuous functions $f_m:\mathscr{X}\to\mathbb{R}$ that separates points of $\mathscr{X}$. Let us denote by $\mathscr{S}$ the σ-algebra generated by the maps $\{f_m\}$. Then*

a) every compact subset of $\mathscr{X}$ is metrizable,

b) if $(\mu_m)_{m\in\mathbb{N}}$ is a tight sequence of probability measures on $(\mathscr{X},\mathscr{S})$, then there exists a subsequence $(m_k)_{k\in\mathbb{N}}$, a probability space $(\Omega,\mathscr{F},\mathbb{P})$ with $\mathscr{X}$-valued Borel measurable variables ξ_k,ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω.

Let $(\mathbb{S},\varrho)$ be a separable and complete metric space.

Definition 2.9.7. Let $u\in\mathscr{C}([0,T];\mathbb{S})$. The modulus of continuity of u on $[0,T]$ is defined by

$$m(u,\delta) := \sup_{s,t\in[0,T],\,|t-s|\le\delta} \varrho(u(t),u(s)), \quad \delta>0.$$

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space with filtration $\mathbb{F}:=(\mathscr{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, see [63], and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous $\mathbb{F}$-adapted $\mathbb{S}$-valued processes.

Definition 2.9.8. We say that the sequence $(X_n)_{n\in\mathbb{N}}$ of $\mathbb{S}$-valued random variables satisfies condition **[T]** iff $\forall\,\varepsilon>0, \forall\,\eta>0, \exists\,\delta>0$:

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left\{m(X_n,\delta)>\eta\right\} \le \varepsilon. \tag{2.9.1}$$

Lemma 2.9.9. *[24, Lemma 2.4] Assume that $(X_n)_{n\in\mathbb{N}}$ satisfies condition* **[T]**. *Let $\mathbb{P}_n$ be the law of X_n on $\mathscr{C}([0,T];\mathbb{S})$, $n\in\mathbb{N}$. Then for every $\varepsilon>0$ there exists a subset $A_\varepsilon\subset\mathscr{C}([0,T];\mathbb{S})$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n(A_\varepsilon) \ge 1-\varepsilon$$

and

$$\lim_{\delta\to 0} \sup_{u\in A_\varepsilon} m(u,\delta) = 0. \tag{2.9.2}$$

Now we recall the Aldous condition [**A**], which is connected with condition [**T**]. This condition allows to investigate the modulus of continuity for the sequence of stochastic processes by means of stopped processes.

Definition 2.9.10. [Aldous condition] A sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition [**A**] iff $\forall\,\varepsilon > 0$, $\forall\,\eta > 0$, $\exists\,\delta > 0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\tau_n \le T$ one has

$$\sup_{n\in\mathbb{N}} \sup_{0\le\theta\le\delta} \mathbb{P}\left\{\varrho(X_n(\tau_n+\theta), X_n(\tau_n)) \ge \eta\right\} \le \varepsilon .$$

Lemma 2.9.11. *[64, Theorem 3.2] Conditions* [**A**] *and* [**T**] *are equivalent.*

CHAPTER 3

CONSTRAINED NAVIER-STOKES EQUATIONS

Incompressible Navier-Stokes equations describe the dynamics of an incompressible viscous fluid. These equations were proposed by C. Navier in 1822 on the basis of a suitable molecular model and were later derived by G. Stokes by means of the theory of continua. A solution to these equations predicts the behaviour of the fluid, in particular, describes the evolution of velocity of the fluid as a function of space and time, given the initial and boundary states. Even though Navier-Stokes equations have variety of applications ranging from aerodynamics to biology, such as modelling the flow of blood in the circulatory system; the basic mathematical question of the existence of a unique global-in-time solution to these parabolic PDEs on a bounded domain in $\mathbb{R}^3$ still remains open. The non-linear convective term poses a lot of problems during the analysis as well as in physical systems by causing physical phenomenons, such as eddy flows and turbulence.

The existence of a unique global-in-time solution to the Navier-Stokes equations on $\mathbb{R}^2$ has been known for a long time. In her seminal paper [54], Ladyzhenskaya proved an inequality to control the non-linear convective term on a bounded domain in $\mathbb{R}^2$, which was later used to prove the global-in-time existence of a unique solution to the Navier-Stokes equations. This trick already fails in the case of a bounded domain in $\mathbb{R}^3$. One can prove the existence of a global-in-time weak solutions [47, 55], also known as Leray solutions on a general bounded domain in $\mathbb{R}^3$.

In this chapter we introduce the constrained Navier-Stokes equations, which are Navier-Stokes equations with a constraint on the L^2-energy of the solution. We assume that the L^2-energy of the solution remains constant and is assumed to be equal to 1. The motivation to study such a constrained problem, is that these equations should be a better approximation to incompressible Euler equations, since for the Euler equations, the energy of (sufficiently smooth) solutions is constant (see [31]).

We end the introduction by giving a brief overview of the chapter: in Section 3.1 we define our functional spaces along with the Stokes operator, for both $\mathbb{R}^2$ and bounded domain with periodic boundary conditions (i.e. a torus). We introduce the bilinear map corresponding to the non-linear convective term along with some of its important properties in Section 3.2. We conclude the chapter by introducing the constraint and corresponding orthogonal projection map which projects the Hilbert manifold $\mathscr{M}$ onto its tangent space, that along with all the functional setting is used to describe the constrained Navier-Stokes equations (CNSE) in Section 3.3.

3.1 Functional setting

Let $\mathscr{O}$ be either a bounded domain in $\mathbb{R}^2$, the full Euclidean space $\mathbb{R}^2$ or the torus $\mathbb{T}^2$. For $p \in [1,\infty]$ and $k \in \mathbb{N}$, the Lebesgue and Sobolev spaces of $\mathbb{R}^2$-valued functions will be denoted by $L^p(\mathscr{O},\mathbb{R}^2)$ and $W^{k,p}(\mathscr{O},\mathbb{R}^2)$ respectively, and often L^p and $W^{k,p}$ whenever the context is understood. The usual scalar product on L^2 is denoted by $\langle u, \mathrm{v}\rangle$ for $u, \mathrm{v} \in L^2$. The associated norm is given by $|u|_{L^2}, u \in L^2$. We also write $W^{k,2}(\mathscr{O},\mathbb{R}^2) := H^k(\mathscr{O})$ and will denote it's norm by $\|\cdot\|_{H^k}$. In particular the scalar product for $H^1(\mathscr{O})$ is given by

$$\langle u, \mathrm{v}\rangle_{H^1} = \langle u, \mathrm{v}\rangle + \langle \nabla u, \nabla \mathrm{v}\rangle, \qquad u, \mathrm{v} \in H^1(\mathscr{O}),$$

and thus the norm is

$$\|u\|_{H^1} = \left[|u|^2_{L^2} + |\nabla u|^2_{L^2}\right]^{1/2}.$$

In the following two subsections we will introduce some additional spaces. The structure of the spaces will depend on the choice of $\mathscr{O}$.

3.1.1 Functional setting for $\mathbb{R}^2$

We consider the whole space $\mathbb{R}^2$. We introduce the following spaces:

$$\begin{aligned} \mathrm{H} &= \left\{u \in L^2(\mathbb{R}^2,\mathbb{R}^2) : \operatorname{div} u = 0\right\}, \\ \mathrm{V} &= H^1 \cap \mathrm{H}. \end{aligned} \tag{3.1.1}$$

We endow H with the scalar product and norm of L^2 and denote it by $\langle u, \mathrm{v}\rangle_{\mathrm{H}}$, $|u|_{\mathrm{H}}$ respectively for $u, \mathrm{v} \in \mathrm{H}$. We equip the space V with the scalar product and norm of H^1 and will denote it by $\langle \cdot, \cdot \rangle_V$ and $\|\cdot\|_{\mathrm{V}}$ respectively.

Let $\Pi : L^2 \to \mathrm{H}$ be the Leray-Helmholtz projection operator [88] which projects vector fields on to the plane of divergence free vector fields. We denote by $\mathrm{A} : \mathrm{D(A)} \to \mathrm{H}$, the Stokes operator which is defined by

$$\begin{aligned} \mathrm{D(A)} &= \mathrm{H} \cap H^2(\mathbb{R}^2), \\ \mathrm{A}u &= -\Pi\,\Delta u, \quad u \in \mathrm{D(A)}. \end{aligned}$$

It is well known that A is a self adjoint non-negative operator in H [33]. Note that Δ and Π commute with each other. Moreover

$$\mathrm{D}((\mathrm{A}+I)^{1/2}) = \mathrm{V} \quad \text{and} \quad \langle \mathrm{A}u, u\rangle = |\nabla u|^2_{L^2}, \ u \in \mathrm{D}(\mathrm{A}).$$

From now onwards we will denote $\mathrm{E} := \mathrm{D}(\mathrm{A})$.

3.1.2 Functional setting for a periodic domain

We denote the bounded domain with periodic boundary conditions by $\mathbb{T}^2$ which can be identified to a two dimensional torus. Let $\mathscr{C}_c^\infty(\mathbb{T}^2, \mathbb{R}^2)$ denote the space of all $\mathbb{R}^2-$valued functions of class $\mathscr{C}^\infty$ with compact supports contained in $\mathbb{T}^2$. We introduce the following spaces:

$$\begin{aligned} \mathcal{V} &= \left\{u \in \mathscr{C}_c^\infty(\mathscr{O}, \mathbb{R}^2) : \operatorname{div} u = 0\right\}, \\ L_0^2 &= \left\{u \in L^2(\mathbb{T}^2, \mathbb{R}^2) : \int_{\mathbb{T}^2} u(x)\,dx = 0\right\}, \\ \mathrm{H} &= \left\{u \in \mathbb{L}_0^2 : \operatorname{div} u = 0\right\}, \\ \mathrm{V} &= H^1 \cap \mathrm{H}. \end{aligned} \tag{3.1.2}$$

We endow H with the scalar product and norm of L^2 and denote it by $\langle u, \mathrm{v}\rangle_{\mathrm{H}}$, $|u|_{\mathrm{H}}$ respectively for $u, \mathrm{v} \in \mathrm{H}$. We equip the space V with the scalar product $\langle \nabla u, \nabla \mathrm{v}\rangle_{L^2}$ and norm $\|u\|_{\mathrm{V}}, u, \mathrm{v} \in \mathrm{V}$. One can show that in the case of $\mathbb{T}^2$, V-norm $\|\cdot\|_{\mathrm{V}}$, and H^1-norm $\|\cdot\|_{H^1}$ are equivalent on V.

As before we denote by $\mathrm{A} : \mathrm{D}(\mathrm{A}) \to \mathrm{H}$, the Stokes operator which is defined by

$$\begin{aligned} \mathrm{D}(\mathrm{A}) &= \mathrm{H} \cap H^2(\mathbb{T}^2), \\ \mathrm{A}u &= -\Pi\,\Delta u, \quad u \in \mathrm{D}(\mathrm{A}). \end{aligned}$$

It is well known that A is a self adjoint positive operator in H [90]. Moreover

$$\mathrm{D}(\mathrm{A}^{1/2}) = \mathrm{V} \quad \text{and} \quad \langle \mathrm{A}u, u\rangle = \|u\|^2_{\mathrm{V}} = |\nabla u|^2_{L^2}, \qquad u \in \mathrm{D}(\mathrm{A}).$$

In the following section we will introduce a tri-linear form corresponding to the non-linear convective term from Navier-Stokes equations, which is well defined for any general domain $\mathscr{O}$ and will state some of its properties.

3.2 Convective term

From now onwards we denote our domain by $\mathscr{O}$ which can be either $\mathbb{R}^2$ or $\mathbb{T}^2$. We introduce a continuous tri-linear form $b : L^p \times W^{1,q} \times L^r \to \mathbb{R}$,

$$b(u, \mathrm{v}, w) = \sum_{i,j=1}^{2} \int_{\mathscr{O}} u^i \frac{\partial \mathrm{v}^j}{\partial x^i} w^j \, dx,$$

where $p,q,r \in [1,\infty]$ satisfies

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \le 1.$$

By the Sobolev Embedding Theorem [91] and the Hölder inequality [91], we obtain the following estimates

$$\begin{aligned} |b(u,v,w)| &\le |u|_{L^4} \|\mathrm{v}\|_{\mathrm{V}} |w|_{L^4}, \qquad u,w \in L^4, \mathrm{v} \in \mathrm{V}, \\ &\le c\|u\|_{\mathrm{V}} \|\mathrm{v}\|_{\mathrm{V}} \|w\|_{\mathrm{V}}, \qquad u,\mathrm{v},w \in \mathrm{V}. \end{aligned} \tag{3.2.1}$$

Hence, we can define a bilinear map $B : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$ such that

$$\langle B(u,\mathrm{v}),\varphi\rangle = b(u,\mathrm{v},\varphi), \quad \text{for } u,\mathrm{v},\varphi \in \mathrm{V},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between V and V'.

Using the following well-known Ladyzhenskaya's inequality on $\mathscr{O} \subset \mathbb{R}^2$ [88] (this is a special case of Gagliardo - Nirenberg inequality):

$$\|u\|_{L^4} \le 2^{1/4} |u|_{L^2}^{1/2} |\nabla u|_{L^2}^{1/2}, \qquad u \in \mathrm{V}, \tag{3.2.2}$$

and the Hölder inequality, we obtain

$$|b(u,\mathrm{v},\varphi)| \le \sqrt{2}\, |u|_{\mathrm{H}}^{\frac{1}{2}} \|u\|_{\mathrm{V}}^{\frac{1}{2}} \|\mathrm{v}\|_{\mathrm{V}}^{\frac{1}{2}} |\mathrm{v}|_{\mathrm{D(A)}}^{\frac{1}{2}} |\varphi|_{\mathrm{H}}, \qquad u \in \mathrm{V}, \mathrm{v} \in \mathrm{D(A)}, \varphi \in \mathrm{H}. \tag{3.2.3}$$

Thus b can be uniquely extended to the tri-linear form (denoted by the same letter)

$$b : \mathrm{V} \times \mathrm{D(A)} \times \mathrm{H} \to \mathbb{R}.$$

We can now also extend the operator B uniquely to a bounded bilinear operator

$$B : \mathrm{V} \times \mathrm{D(A)} \to \mathrm{H}. \tag{3.2.4}$$

The following properties of the tri-linear map b and the bilinear map B are very well established in [88] and Appendix A:

$$\begin{aligned} &b(u,u,u) = 0, && u \in \mathrm{V}, \\ &b(u,w,w) = 0, && u \in \mathrm{V}, w \in H^1, \\ &\langle B(u,u), \mathrm{A}u\rangle_{\mathrm{H}} = 0, && u \in \mathrm{D(A)}. \end{aligned} \tag{3.2.5}$$

Note that the last identity in (3.2.5) holds only in the two cases that we have considered here, i.e. on the whole Euclidean space $\mathbb{R}^2$ and torus $\mathbb{T}^2$.

3.3 NSEs and CNSE

The 2D Navier-Stokes equations (NSE) governing the dynamics of an incompressible viscous fluid are given as following:

$$
\begin{cases}
\dfrac{\partial u(x,t)}{\partial t} - \nu \Delta u(x,t) + (u(x,t)\cdot\nabla)u(x,t) + \nabla p(x,t) = 0\,, \\
\operatorname{div} u(x,t) = 0\,, \\
u(x,0) = u_0(x)\,,
\end{cases}
\tag{3.3.1}
$$

where $x \in \mathscr{O}$ and $t \in [0,T]$ for every $T > 0$; $u : \mathscr{O} \to \mathbb{R}^2$ and $p : \mathscr{O} \to \mathbb{R}$ are velocity and pressure of the fluid respectively. ν is the viscosity of the fluid.

With all the notations as defined in the Sections 3.1 and 3.2, the Navier-Stokes equations (3.3.1) projected on divergence free vector field using the Leray-Helmholtz projection operator is given by

$$
\begin{cases}
\dfrac{du}{dt} + \nu \mathrm{A} u + B(u,u) = 0\,, \\
u(0) = u_0\,.
\end{cases}
\tag{3.3.2}
$$

Let us denote the set of divergence free $\mathbb{R}^2$-valued functions with unit L^2 norm, as following

$$\mathscr{M} = \{u \in \mathrm{H} : |u|_{\mathrm{H}} = 1\}\,.$$

Then the tangent space at u is defined as the following closed subspace of H,

$$T_u\mathscr{M} = \{\mathrm{v} \in \mathrm{H} : \langle \mathrm{v}, u\rangle_{\mathrm{H}} = 0\}\,, \qquad u \in \mathscr{M}\,.$$

A linear map $\pi_u : \mathrm{H} \to T_u\mathscr{M}$ defined by

$$\pi_u(\mathrm{v}) = \mathrm{v} - \langle \mathrm{v}, u\rangle_{\mathrm{H}}\, u\,,$$

is the orthogonal projection from H onto $T_u\mathscr{M}$.

Remark 3.3.1. It follows from (3.2.5) that

$$B(u,u) \in T_u\mathscr{M}, \quad u \in \mathscr{M} \cap \mathrm{D}(\mathrm{A})\,.$$

In particular,

$$\pi_u(B(u,u)) = B(u,u), \quad u \in \mathscr{M} \cap \mathrm{D}(\mathrm{A})\,.$$

Let

$$F(u) = \nu \mathrm{A} u + B(u,u)$$

and $\hat{F}(u)$ be the projection of $F(u)$ on the tangent space $T_u\mathcal{M}$, then for $u \in \mathrm{D(A)}$,

$$\begin{aligned}\hat{F}(u) &= \pi_u(F(u)) = F(u) - \langle F(u), u\rangle_{\mathrm{H}}\, u \\ &= \nu \mathrm{A}u + B(u,u) - \langle \nu\mathrm{A}u + B(u,u), u\rangle_{\mathrm{H}}\, u \\ &= \nu\mathrm{A}u - \nu\langle \mathrm{A}u, u\rangle_{\mathrm{H}}\, u + B(u,u) - \langle B(u,u), u\rangle_{\mathrm{H}}\, u \\ &= \nu\mathrm{A}u - \nu|\nabla u|_{L^2}^2\, u + B(u,u).\end{aligned}$$

The last equality follows heuristically from Remark 3.3.1.

Thus by projecting NSEs (3.3.2) on the tangent space $T_u\mathcal{M}$, we obtain our constrained Navier-Stokes equations (CNSE) which is given by

$$\begin{cases} \dfrac{du}{dt} + \nu\mathrm{A}u - \nu|\nabla u|_{L^2}^2\, u + B(u,u) = 0, \\ u(0) = u_0. \end{cases} \tag{3.3.3}$$

The majority of this thesis is dedicated to the study of the constrained Navier-Stokes equations (3.3.3) under the impact of external forcing, both deterministic and stochastic. Even though in Chapter 4 the analysis has been carried out in the absence of any deterministic external forcing (i.e. assuming external force is identically zero) one can easily generalise the results obtained there for non-zero deterministic external forcing, under suitable assumptions. We also show that the solution of CNSE (3.3.3) converge to the unique solution (Bardos solution, see [5]) of the Euler equations (formally obtained by putting $\nu = 0$ in (3.3.1)) in inviscid limit ($\nu \searrow 0$) with appropriate assumptions on the initial data.

In Chapter 5 we shift our focus to the stochastic generalisation of (3.3.3) where we assume that the stochastic forcing is tangent to the manifold $\mathcal{M}$, enabling the solution to stay on the manifold. The analysis is carried out using classical tools from the theory of partial differential equations, like Faedo-Galerkin approximations and compactness.

CHAPTER 4

DETERMINISTIC CONSTRAINED NAVIER-STOKES EQUATIONS ON A 2D TORUS

We study 2D Navier-Stokes equations with a constraint on L^2 energy of the solution. We prove the existence and uniqueness of a global solution for the constrained Navier-Stokes equations on $\mathbb{R}^2$ and $\mathbb{T}^2$, by a fixed point argument. We also show that the solution of constrained Navier-Stokes equations converges to the solution of Euler equations as viscosity ν vanishes.

4.1 Introduction

The motivation behind this chapter is threefold. Firstly Caglioti *et.al.* in [31] studied the well-posedness and asymptotic behaviour of two dimensional Navier-Stokes equations in the vorticity form with two constraints: constant energy $E(\omega)$ and moment of inertia $I(\omega)$

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = \nu\Delta\omega - \nu\,\mathrm{div}\left[\omega\nabla\left(b\psi + a\frac{|x|^2}{2}\right)\right],$$

which can be rewritten as

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = \nu\,\mathrm{div}\left[\omega\nabla\left(\log\omega - b\psi - a\frac{|x|^2}{2}\right)\right], \tag{4.1.1}$$

where $\omega = \mathrm{Curl}(u)$, $a = a(\omega)$ and $b = b(\omega)$ are the Lagrange multipliers associated to those constraints and

$$E(\omega) = \int_{\mathbb{R}^2}\psi\omega\,dx, \qquad I(\omega) = \int_{\mathbb{R}^2}|x|^2\omega\,dx, \quad \psi = -\Delta^{-1}\omega.$$

They were able to show the existence of a unique classical global-in-time solution to (4.1.1) for a family of initial data [31, Theorem 5]. They also showed that the solution to (4.1.1) converges, as time tends to $+\infty$, to the unique solution of an associated microcanonical variational problem [31, Theorem 8].

Secondly, Rybka [81] and Caffarelli & Lin [30] study the linear heat equation with constraints. Rybka studied heat flow on a manifold $\mathscr{M}$ given by

$$\mathscr{M} = \left\{ u \in L^2(\Omega) \cap \mathscr{C}(\Omega) : \int_\Omega u^k(x)\,dx = C_k,\, k = 1,\dots,N \right\},$$

where Ω denotes a connected bounded region in $\mathbb{R}^2$ with smooth boundary. He proved [81, Theorem 2.5] the existence of the unique global solution for the projected heat equation

$$\begin{cases} \frac{du}{dt} = \Delta u - \sum_{k=1}^N \lambda_k u^{k-1} & \text{in } \Omega \subset \mathbb{R}^2, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, & u(0,x) = u_0, \end{cases} \tag{4.1.2}$$

where $\lambda_k = \lambda_k(u)$ are such that u_t is orthogonal to Span$\{u^{k-1}\}$, for a more regular initial data. He also showed that the solutions to (4.1.2) converge to a steady state as time tends to $+\infty$.
On the other hand Caffarelli and Lin initially establish the existence and uniqueness of a global, energy-conserving solution to the heat equation [30, Theorem 1.1]. They were then able to extend these results to more general family of singularly perturbed systems of non-local parabolic equations [30, Theorem 3.1]. Their main result was to prove the strong convergence of the solutions of these perturbed systems to some weak-solutions of the limiting constrained non-local heat flows of maps into a singular space.

Finally, these equations should be a better approximation of the Euler equations (for small viscosity), since for the Euler equations, the energy of (sufficiently smooth) solutions is constant (see [31]).

In this chapter we consider a problem which links the aforementioned works. We consider Navier-Stokes equations as in [31], but subject to the same energy constraint as in [30, 81]. Contrary to [31] we prove global-in-time existence of the solution but only on a torus, namely in the periodic case. Surprisingly our proof of global existence does not hold for a general bounded domain, although the local existence holds. We also prove our result of global existence of the solution for $\mathbb{R}^2$. We additionally show that, in vanishing viscosity limit, the solution of the constrained equation (4.1.3) below, converges to the Bardos solution (see [5]) of the Euler equation (formally obtained setting $\nu = 0$).

We are interested in the Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = -\nu \mathrm{A}u + \nu |\nabla u|^2_{L^2} u - B(u,u), \\ u(0) = u_0, \end{cases} \tag{4.1.3}$$

where $u \in \mathrm{H}$, and H is a space of divergence free, mean zero vector fields on a torus, see (3.1.2) for the precise definition.

The above problem has a local *maximal* solution for each $u_0 \in \mathrm{V} \cap \mathscr{M}$, where V is defined in (3.1.2) and

$$\mathscr{M} = \{u \in \mathrm{H} : |u|_{\mathrm{H}} = 1\}.$$

Moreover $u(t) \in \mathcal{M}$ for all times t. This result is true, both for constrained Navier-Stokes equations on a bounded domain or with periodic boundary conditions (i.e. on a torus). In a more geometrical fashion, equation (4.1.3) can be also written as

$$\frac{du}{dt} = -\nabla_{\mathcal{M}}\mathcal{E}(u) - B(u,u),$$

where $\mathcal{E}(u) = \frac{1}{2}|\nabla u|^2_{L^2}, u \in \mathcal{M}$ and $\nabla_{\mathcal{M}}\mathcal{E}(u)$ is the gradient of $\mathcal{E}$ with respect to H-norm projected onto $T_u\mathcal{M}$. The remarkable feature of this is that on a torus $\nabla_{\mathcal{M}}\mathcal{E}(u)$ and $B(u,u)$ are orthogonal in H. This orthogonality holds for the Navier-Stokes without constraint too, i.e. on a torus $\nabla\mathcal{E}(u)$ is orthogonal to $B(u,u)$ in H. The fact that this constraint preserves the orthogonality somehow makes it a natural constraint.

Hence, at least in a heuristic way

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u(t)) &= \left\langle \nabla_{\mathcal{M}}\mathcal{E}(u(t)), \frac{du}{dt} \right\rangle_{\mathrm{H}} \\ &= \langle \nabla_{\mathcal{M}}\mathcal{E}(u(t)), -\nabla_{\mathcal{M}}\mathcal{E}(u(t)) - B(u,u)\rangle_{\mathrm{H}} \\ &= -|\nabla_{\mathcal{M}}\mathcal{E}(u(t))|^2_{\mathrm{H}},\end{aligned}$$

so that $\mathcal{E}(u(t))$ is decreasing and thus the $H^{1,2}$ norm of the solution remains bounded.

Next we state the two main results of this chapter on a torus.

Let us denote

$$X_T = \mathcal{C}([0,T];\mathrm{V}) \cap L^2(0,T;\mathrm{E}).$$

Theorem 4.1.1. *Let $\nu > 0$ be fixed. Let $u_0 \in \mathrm{V} \cap \mathcal{M}$. Then there exists a global and locally unique solution u of* (4.1.3) *such that $u \in X_T$ for each $T > 0$.*

The space X_T with more details and the precise definition of the solution of (4.1.3) will be given in Section 4.2. Theorem 4.1.1 will be proved in steps in Sections 4.2 and 4.3.

Theorem 4.1.2. *Let $u_0, u_0^\nu \in \mathrm{V} \cap \mathcal{M}$ and u^ν be the solution of* (4.1.3) *(existence and uniqueness of u^ν follows from Theorem 4.1.1). Assume that $u_0^\nu \to u_0$ in V as $\nu \searrow 0$, and that* $\mathrm{Curl}(u_0^\nu)$, $\mathrm{Curl}(u_0)$ *stays uniformly bounded in $L^\infty(\mathbb{T}^2)$. Then for each $T > 0$, u^ν converges in $\mathcal{C}([0,T];L^2(\mathbb{T}^2))$ to the unique solution u of the limiting equation (namely* (4.1.3) *with $\nu = 0$).*

We end the introduction with a brief description of the content of the chapter. In Section 4.2, a precise definition of the solution to problem (4.1.3) is given, and local existence and uniqueness are proved, together with some basic properties of the solution. In Section 4.3, global existence is established. After proving Theorem 4.1.2 in Section 4.4, we study CNSE (4.1.3) in fractional Sobolev spaces and establish the existence of a unique solution for much more regular initial data in Section 4.5. We end the chapter by presenting a formal discussion regarding the lower bound on the regularity of the initial data so as to have the existence of a solution to problem (4.1.3).

4.2 Local solution : Existence and Uniqueness

In this section we will establish the existence of a local solution to

$$\begin{cases} \dfrac{du}{dt} + \mathrm{A}u - |\nabla u|_{L^2}^2 u + B(u,u) = 0\,, \\ u(0) = u_0 \in \mathrm{V} \cap \mathscr{M}\,, \end{cases} \tag{4.2.1}$$

by using the Banach fixed point theorem. We obtain certain estimates for non-linear terms of (4.2.1) using results from Chapter 3. After obtaining these estimates we construct a globally Lipschitz map. Some ideas in the Subsection 4.2.1 are based on [29].

In what follows we put E := D(A) and V, H are spaces as defined in Section 3.1.

Lemma 4.2.1. *Let* $G_1 : \mathrm{V} \to \mathrm{H}$ *be defined by*

$$G_1(u) = |\nabla u|_{L^2}^2 u, \quad u \in \mathrm{V}.$$

Then, there exists $C > 0$ *such that for* $u_1, u_2 \in \mathrm{V}$*,*

$$|G_1(u_1) - G_1(u_2)|_{\mathrm{H}} \le C \|u_1 - u_2\|_{\mathrm{V}} \big[\|u_1\|_{\mathrm{V}} + \|u_2\|_{\mathrm{V}}\big]^2. \tag{4.2.2}$$

Proof. Let us consider $u_1, u_2 \in \mathrm{V}$, then

$$\begin{aligned} |G_1(u_1) - G_1(u_2)|_{\mathrm{H}} &= \big||\nabla u_1|_{L^2}^2 u_1 - |\nabla u_2|_{L^2}^2 u_2\big|_{\mathrm{H}} \\ &= \big||\nabla u_1|_{L^2}^2 u_1 - |\nabla u_1|_{L^2}^2 u_2 + |\nabla u_1|_{L^2}^2 u_2 - |\nabla u_2|_{L^2}^2 u_2\big|_{\mathrm{H}} \\ &= \big||\nabla u_1|_{L^2}^2 (u_1 - u_2) + (|\nabla u_1|_{L^2}^2 - |\nabla u_2|_{L^2}^2) u_2\big|_{\mathrm{H}} \\ &\le |\nabla u_1|_{L^2}^2 |u_1 - u_2|_{\mathrm{H}} + \big[|\nabla u_1|_{L^2} + |\nabla u_2|_{L^2}\big]\big[|\nabla u_1|_{L^2} - |\nabla u_2|_{L^2}\big] |u_2|_{\mathrm{H}} \\ &\le |\nabla u_1|_{L^2}^2 |u_1 - u_2|_{\mathrm{H}} + \big[|\nabla u_1|_{L^2} + |\nabla u_2|_{L^2}\big] |\nabla(u_1 - u_2)|_{L^2} |u_2|_{\mathrm{H}} \\ &\le C\big[|\nabla u_1|_{L^2}^2 \|u_1 - u_2\|_{\mathrm{V}} + \big[|\nabla u_1|_{L^2} + |\nabla u_2|_{L^2}\big] |\nabla(u_1 - u_2)|_{L^2} \|u_2\|_{\mathrm{V}}\big] \\ &\le C \|u_1 - u_2\|_{\mathrm{V}} \big[|\nabla u_1|_{L^2}^2 + |\nabla u_2|_{L^2} \|u_2\|_{\mathrm{V}} + |\nabla u_1|_{L^2} \|u_2\|_{\mathrm{V}}\big]\,, \end{aligned}$$

where we have repeatedly used the fact that V is continuously embedded in H. Thus, we obtain (4.2.2). ■

Lemma 4.2.2. *Let* $G_2 : \mathrm{E} \to \mathrm{H}$ *be defined by*

$$G_2(u) = B(u,u), \qquad u \in \mathrm{E}.$$

Then, there exists $\tilde{C} > 0$ *such that for* $u_1, u_2 \in \mathrm{E}$*,*

$$|G_2(u_1) - G_2(u_2)|_{\mathrm{H}} \le \tilde{C}\Big[\|u_1\|_{\mathrm{V}}^{1/2} |u_1|_{\mathrm{E}}^{1/2} \|u_1 - u_2\|_{\mathrm{V}} + \|u_2\|_{\mathrm{V}} \|u_1 - u_2\|_{\mathrm{V}}^{1/2} |u_1 - u_2|_{\mathrm{E}}^{1/2}\Big]. \tag{4.2.3}$$

Proof. Let us take $u_1, u_2 \in \mathrm{E}$, then

$$\begin{aligned}
|G_2(u_1) - G_2(u_2)|_{\mathrm{H}} &= |B(u_1,u_1) - B(u_2,u_2)|_{\mathrm{H}} \\
&= |B(u_1,u_1) - B(u_2,u_1) + B(u_2,u_1) - B(u_2,u_2)|_{\mathrm{H}} \\
&= |B(u_1-u_2,u_1) + B(u_2,u_1-u_2)|_{\mathrm{H}} \\
&= |\Pi[(u_1-u_2)\cdot\nabla u_1] + \Pi[u_2\cdot\nabla(u_1-u_2)]|_{\mathrm{H}} \\
&\le |(u_1-u_2)\cdot\nabla u_1|_{\mathrm{H}} + |u_2\cdot\nabla(u_1-u_2)|_{\mathrm{H}} \\
&\le |u_1-u_2|_{L^4(\mathscr{O})}|\nabla u_1|_{L^4(\mathscr{O})} + |u_2|_{L^4(\mathscr{O})}|\nabla(u_1-u_2)|_{L^4(\mathscr{O})}.
\end{aligned}$$

Now using the Ladyzhenkaya's inequality (3.2.2) and the embedding of V in H, we obtain

$$\begin{aligned}
|G_2(u_1) - G_2(u_2)|_{\mathrm{H}} &\le \sqrt{2}\,|u_1-u_2|_{\mathrm{H}}^{1/2}|\nabla(u_1-u_2)|_{\mathrm{H}}^{1/2}|\nabla u_1|_{\mathrm{H}}^{1/2}|\nabla^2 u_1|_{\mathrm{H}}^{1/2} \\
&\quad + \sqrt{2}\,|u_2|_{\mathrm{H}}^{1/2}|\nabla u_2|_{\mathrm{H}}^{1/2}|\nabla(u_1-u_2)|_{\mathrm{H}}^{1/2}|\nabla^2(u_1-u_2)|_{\mathrm{H}}^{1/2} \\
&\le \sqrt{2}C\Big[\|u_1-u_2\|_{\mathrm{V}}\|u_1\|_{\mathrm{V}}^{1/2}|u_1|_{\mathrm{E}}^{1/2} \\
&\qquad + \|u_2\|_{\mathrm{V}}\|u_1-u_2\|_{\mathrm{V}}^{1/2}|u_1-u_2|_{\mathrm{E}}^{1/2}\Big].
\end{aligned}$$

Thus, we obtain the inequality (4.2.3). ■

4.2.1 Construction of a globally Lipschitz map

Let $\theta : \mathbb{R}_+ \to [0,1]$ be a C_0^∞ non-increasing function such that

$$\min_{x\in\mathbb{R}_+}\theta'(x) \ge -1, \quad \theta(x) = 1 \text{ iff } x\in[0,1] \text{ and } \theta(x) = 0 \text{ iff } x \in [3,\infty)$$

and for $n \ge 1$ set $\theta_n(\cdot) = \theta(\frac{\cdot}{n})$. Observe that if $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function, then for every $x, y \in \mathbb{R}_+$,

$$\theta_n(x)h(x) \le h(3n), \ |\theta_n(x) - \theta_n(y)| \le 3n|x-y|.$$

Let us fix $T > 0$. We will first construct a solution on $[0,T]$. For that, set

$$X_T = \mathscr{C}([0,T];\mathrm{V}) \cap L^2(0,T;\mathrm{E}),$$

with norm

$$|u|_{X_T}^2 = \sup_{t\in[0,T]}\|u(t)\|_{\mathrm{V}}^2 + \int_0^T |u(t)|_{\mathrm{E}}^2\,dt.$$

Let us define $G : \mathrm{E} \to \mathrm{H}$ as

$$G(u) := G_1(u) - G_2(u) = |\nabla u|_{L^2}^2 u - B(u,u). \tag{4.2.4}$$

Lemma 4.2.3. *Suppose* $G : \mathrm{E} \to \mathrm{H}$ *is a map defined in* (4.2.4)*. Define a map*

$$\Phi_{n,T} : X_T \to L^2(0,T;\mathrm{H})$$

by

$$\Phi_{n,T}(u)(t) = \theta_n(|u|_{X_t}) G(u(t)), \qquad t \in [0,T]. \tag{4.2.5}$$

Then $\Phi_{n,T}$ *is globally Lipschitz and moreover, for any* $u_1, u_2 \in X_T$

$$|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} \le K(n,T)|u_1 - u_2|_{X_T} T^{\frac{1}{4}}, \tag{4.2.6}$$

where

$$K(n,T) = 3n\left(27n^3T^{1/4} + 9n^2 + 12nT^{1/4} + 2\right),$$

depends on n and T only.

Proof. Assume that $u_1, u_2 \in X_T$. Set

$$\tau_i = \inf\{t \in [0,T]; |u_i|_{X_t} \ge 3n\}, \qquad i = 1,2.$$

Without loss of generality assume that $\tau_1 \le \tau_2$. Consider

$$\begin{aligned}|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} &= \left[\int_0^T |\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|^2_{\mathrm{H}}\, dt\right]^{\frac{1}{2}} \\ &= \left[\int_0^T \left|\theta_n(|u_1|_{X_t})G(u_1) - \theta_n(|u_2|_{X_t})G(u_2)\right|^2_{\mathrm{H}} dt\right]^{\frac{1}{2}},\end{aligned}$$

for $i = 1,2$ $\theta_n(|u_i|_{X_t}) = 0$ for $t \ge \tau_2$, thus we have

$$\begin{aligned}|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} &= \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})G(u_1) - \theta_n(|u_2|_{X_t})G(u_2)\right|^2_{\mathrm{H}} dt\right]^{\frac{1}{2}} \\ &= \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})[G_1(u_1) - G_2(u_1)] - \theta_n(|u_2|_{X_t})[G_1(u_2) - G_2(u_2)]\right|^2_{\mathrm{H}} dt\right]^{\frac{1}{2}} \\ &= \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})G_1(u_1) - \theta_n(|u_1|_{X_t})G_1(u_2) + \theta_n(|u_1|_{X_t})G_1(u_2) - \theta_n(|u_2|_{X_t})G_1(u_2)\right.\right. \\ &\quad \left.\left. + \theta_n(|u_1|_{X_t})G_2(u_2) - \theta_n(|u_1|_{X_t})G_2(u_1) + \theta_n(|u_2|_{X_t})G_2(u_2) - \theta_n(|u_1|_{X_t})G_2(u_2)\right|^2_{\mathrm{H}} dt\right]^{\frac{1}{2}}.\end{aligned}$$

Using the Minkowski inequality we get,

$$\begin{aligned}&|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} \\ &\le \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})[G_1(u_1) - G_1(u_2)]\right|^2_{\mathrm{H}} dt\right]^{\frac{1}{2}} + \left[\int_0^{\tau_2} \left|\left[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right] G_1(u_2)\right|^2_{\mathrm{H}} dt\right]^{\frac{1}{2}} \\ &\quad + \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})[G_2(u_2) - G_2(u_1)]\right|^2_{\mathrm{H}} dt\right]^{\frac{1}{2}} + \left[\int_0^{\tau_2} \left|\left[\theta_n(|u_2|_{X_t}) - \theta_n(|u_1|_{X_t})\right] G_2(u_2)\right|^2_{\mathrm{H}} dt\right]^{\frac{1}{2}}.\end{aligned}$$

Set

$$A_1 = \left[\int_0^{\tau_2} \left| \left[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t}) \right] G_1(u_2) \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}},$$
$$A_2 = \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t}) [G_1(u_1) - G_1(u_2)] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}},$$
$$A_3 = \left[\int_0^{\tau_2} \left| \left[\theta_n(|u_2|_{X_t}) - \theta_n(|u_1|_{X_t}) \right] G_2(u_2) \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}},$$
$$A_4 = \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t}) [G_2(u_2) - G_2(u_1)] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}},$$

and hence

$$|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} \le A_1 + A_2 + A_3 + A_4. \tag{4.2.7}$$

Since θ_n is a Lipschitz function with Lipschitz constant $3n$, we obtain

$$A_1^2 = \int_0^{\tau_2} \left| \left[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t}) \right] G_1(u_2) \right|_{\mathrm{H}}^2 dt \le 9n^2 \int_0^{\tau_2} \left| |u_1|_{X_t} - |u_2|_{X_t} \right|_{\mathrm{H}}^2 |G_1(u_2)|_{\mathrm{H}}^2 \, dt.$$

Again, using the Minkowski inequality, we get

$$\begin{aligned} A_1^2 &\le 9n^2 \int_0^{\tau_2} |u_1 - u_2|_{X_t}^2 \, |G_1(u_2)|_{\mathrm{H}}^2 \, dt \\ &\le 9n^2 \, |u_1 - u_2|_{X_T}^2 \int_0^{\tau_2} |G_1(u_2)|_{\mathrm{H}}^2 \, dt. \end{aligned} \tag{4.2.8}$$

Now consider $\int_0^{\tau_2} |G_1(u_2)|_{\mathrm{H}}^2 \, dt$. Using (4.2.2) we get

$$\int_0^{\tau_2} |G_1(u_2)|_{\mathrm{H}}^2 \, dt \le C \int_0^{\tau_2} \|u_2(t)\|_{\mathrm{V}}^6 \, dt \le C \left[\sup_{t \in [0,\tau_2]} \|u_2(t)\|_{\mathrm{V}}^2 \right]^3 \tau_2 .$$

Since

$$|u_2|_{X_{\tau_2}}^2 = \sup_{t \in [0,\tau_2]} \|u_2(t)\|_{\mathrm{V}}^2 + \int_0^{\tau_2} |u_2(t)|_{\mathrm{E}}^2 \, dt,$$

thus

$$\sup_{t \in [0,\tau_2]} \|u_2(t)\|_{\mathrm{V}}^2 \le |u_2|_{X_{\tau_2}}^2,$$

and using

$$|u_2|_{X_{\tau_2}} \le 3n,$$

we get

$$\int_0^{\tau_2} |G_1(u_2)|_{\mathrm{H}}^2 \, dt \le C \left[\sup_{t \in [0,\tau_2]} \|u_2(t)\|_{\mathrm{V}}^2 \right]^3 \tau_2 \le C |u_2|_{X_{\tau_2}}^6 \, \tau_2 \le C (3n)^6 \tau_2 .$$

Hence, the inequality (4.2.8) takes the form

$$A_1^2 \le 9n^2 C \, |u_1 - u_2|_{X_T}^2 (3n)^6 \, \tau_2 ,$$

from where we deduce

$$A_1 \le (3n)^4 C\,|u_1 - u_2|_{X_T}\,\tau_2^{\frac{1}{2}}. \tag{4.2.9}$$

Similarly, since $\theta_n(|u_1|_{X_t}) = 0$ for $t \ge \tau_1$ and $\tau_1 \le \tau_2$, we have

$$A_2 = \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})[G_1(u_1) - G_1(u_2)]\right|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}} = \left[\int_0^{\tau_1} \left|\theta_n(|u_1|_{X_t})[G_1(u_1) - G_1(u_2)]\right|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}}.$$

Since $\theta_n(|u_1|_{X_t}) \le 1$ for $t \in [0,\tau_1)$ and using (4.2.2), we have

$$\begin{aligned}
A_2^2 &\le \int_0^{\tau_1} |G_1(u_1) - G_1(u_2)|_{\mathrm{H}}^2\, dt \le C\int_0^{\tau_1} \|u_1(t) - u_2(t)\|_{\mathrm{V}}^2 [\|u_1(t)\|_{\mathrm{V}} + \|u_2(t)\|_{\mathrm{V}}]^4\, dt\\
&\le C \sup_{t\in[0,\tau_1]} \|u_1(t) - u_2(t)\|_{\mathrm{V}}^2 \int_0^{\tau_1} [\|u_1(t)\|_{\mathrm{V}} + \|u_2(t)\|_{\mathrm{V}}]^4\, dt\\
&\le C|u_1 - u_2|_{X_T}^2 \sup_{t\in[0,\tau_1]} [\|u_1(t)\|_{\mathrm{V}} + \|u_2(t)\|_{\mathrm{V}}]^4 \int_0^{\tau_1} dt\\
&\le C|u_1 - u_2|_{X_T}^2 \left[|u_1|_{X_{\tau_1}} + |u_2|_{X_{\tau_1}}\right]^4 \tau_1.
\end{aligned}$$

Since $|u_i|_{X_{\tau_i}} \le 3n, i = 1,2$, we get,

$$A_2^2 \le C|u_1 - u_2|_{X_T}^2 \left[|u_1|_{X_{\tau_1}} + |u_2|_{X_{\tau_1}}\right]^4 \tau_1 \le (6n)^4 C|u_1 - u_2|_{X_T}^2 \tau_1.$$

Thus

$$A_2 \le (6n)^2 C|u_1 - u_2|_{X_T} \tau_1^{\frac{1}{2}}. \tag{4.2.10}$$

Now we consider

$$A_3^2 = \int_0^{\tau_2} \left|\left[\theta_n(|u_2|_{X_t}) - \theta_n(|u_1|_{X_t})\right] G_2(u_2)\right|_{\mathrm{H}}^2 dt.$$

Since θ_n is a Lipschitz function with Lipschitz constant $3n$, we obtain

$$A_3^2 \le 9n^2 \int_0^{\tau_2} \left||u_2|_{X_t} - |u_1|_{X_t}\right|_{\mathrm{H}}^2 \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt.$$

Since

$$\left||u_2|_{X_t} - |u_1|_{X_t}\right|_{\mathrm{H}} \le \left|u_1 - u_2\right|_{X_t},$$

we get

$$\begin{aligned}
A_3^2 &\le 9n^2 \int_0^{\tau_2} \left|u_1 - u_2\right|_{X_t}^2 \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt\\
&\le 9n^2 \left|u_1 - u_2\right|_{X_T}^2 \int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt. \qquad (4.2.11)
\end{aligned}$$

Now consider $\int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt$. Using (4.2.3) we get

$$\int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt \le \tilde{C} \int_0^{\tau_2} \|u_2(t)\|_{\mathrm{V}}^3 |u_2(t)|_{\mathrm{E}}\, dt \le \tilde{C}\left[\sup_{t\in[0,\tau_2]} \|u_2(t)\|_{\mathrm{V}}^2\right]^{\frac{3}{2}} \int_0^{\tau_2} |u_2(t)|_{\mathrm{E}}\, dt.$$

We apply the Hölder inequality to obtain

$$\int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt \le \tilde{C}|u_2|^3_{X_{\tau_2}} \left[\int_0^{\tau_2} |u_2(t)|^2_{\mathrm{E}}\, dt\right]^{\frac{1}{2}} \left[\int_0^{\tau_2} dt\right]^{\frac{1}{2}}.$$

Now since $\int_0^{\tau_2} |u_2|^2_{\mathrm{E}}\, dt \le |u_2|^2_{X_{\tau_2}}$ and $|u_2|_{X_{\tau_2}} \le 3n$,

$$\int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt \le \tilde{C}|u_2|^3_{X_{\tau_2}} |u_2|_{X_{\tau_2}} \tau_2^{\frac{1}{2}} \le \tilde{C}(3n)^4 \tau_2^{\frac{1}{2}}.$$

Hence, the inequality (4.2.11) takes the form

$$A_3^2 \le 9n^2 \tilde{C}|u_1-u_2|^2_{X_T}(3n)^4 \tau_2^{\frac{1}{2}},$$

from where we deduce

$$A_3 \le (3n)^3 \tilde{C}|u_1-u_2|_{X_T} \tau_2^{\frac{1}{4}}. \tag{4.2.12}$$

Since $\theta_n(|u_1|_{X_t}) = 0$ for $t > \tau_1$ and $\tau_1 < \tau_2$, we have

$$A_4 = \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})[G_2(u_2)-G_2(u_1)]\right|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}} = \left[\int_0^{\tau_1} \left|\theta_n(|u_1|_{X_t})[G_2(u_2)-G_2(u_1)]\right|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}}.$$

Since $\theta_n(|u_1|_{X_t}) \le 1$ for $t \in [0,\tau_1]$ and using (4.2.3), we have

$$\begin{aligned}
A_4 &\le \left[\int_0^{\tau_1} \left|G_2(u_2)-G_2(u_1)\right|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}} \\
&\le \tilde{C}\left[\int_0^{\tau_1} \left[\|u_1\|_{\mathrm{V}}^{1/2}|u_1|_{\mathrm{E}}^{1/2}\|u_1-u_2\|_{\mathrm{V}} + \|u_1-u_2\|_{\mathrm{V}}^{1/2}|u_1-u_2|_{\mathrm{E}}^{1/2}\|u_2\|_{\mathrm{V}}\right]^2 dt\right]^{\frac{1}{2}}.
\end{aligned}$$

Now by the Minkowski inequality,

$$\begin{aligned}
A_4 &\le \tilde{C}\left[\left[\int_0^{\tau_1} |u_1|_{\mathrm{E}}\|u_1-u_2\|^2_{\mathrm{V}}\|u_1\|_{\mathrm{V}}\, dt\right]^{\frac{1}{2}} + \left[\int_0^{\tau_1} \|u_2\|^2_{\mathrm{V}}|u_1-u_2|_{\mathrm{E}}^{1/2}\|u_1-u_2\|_{\mathrm{V}}\, dt\right]^{\frac{1}{2}}\right] \\
&\le \tilde{C}\left[\sup_{t\in[0,\tau_1]} \|u_1-u_2\|^2_{\mathrm{V}} \left[\sup_{t\in[0,\tau_1} \|u_1\|^2_{\mathrm{V}}\right]^{\frac{1}{2}} \int_0^{\tau_1} |u_1|_{\mathrm{E}}\, dt\right]^{\frac{1}{2}} \\
&\quad + \tilde{C}\left[\sup_{t\in[0,\tau_1]} \|u_2\|^2_{\mathrm{V}} \left[\sup_{t\in[0,\tau_1]} \|u_1-u_2\|^2_{\mathrm{V}}\right]^{\frac{1}{2}} \int_0^{\tau_1} |u_1-u_2|_{\mathrm{E}}\, dt\right]^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\sup_{t\in[0,\tau_1]} \|u_i\|^2_{\mathrm{V}} \le |u_i|^2_{X_{\tau_1}}, \qquad \int_0^{\tau_1} |u_1|^2_{\mathrm{E}}\, dt \le |u_1|^2_{X_{\tau_1}},$$

and by using the Hölder inequality, we obtain

$$\begin{aligned}
A_4 &\le \tilde{C}\left[|u_1-u_2|^2_{X_T}|u_1|_{X_{\tau_1}} \left[\int_0^{\tau_1} |u_1|^2_{\mathrm{E}} dt\right]^{\frac{1}{2}} \left[\int_0^{\tau_1} dt\right]^{\frac{1}{2}}\right]^{\frac{1}{2}} \\
&\quad + \tilde{C}\left[|u_1-u_2|_{X_T}|u_2|^2_{X_{\tau_1}} \left[\int_0^{\tau_1} |u_1-u_2|^2_{\mathrm{E}} dt\right]^{\frac{1}{2}} \left[\int_0^{\tau_1} dt\right]^{\frac{1}{2}}\right]^{\frac{1}{2}} \\
&\le \tilde{C}\left[|u_1-u_2|^2_{X_T}|u_1|^2_{X_{\tau_1}} \tau_1^{\frac{1}{2}}\right]^{\frac{1}{2}} + \tilde{C}\left[|u_1-u_2|^2_{X_T}|u_2|^2_{X_{\tau_1}} \tau_1^{\frac{1}{2}}\right]^{\frac{1}{2}}.
\end{aligned}$$

For $i = 1,2$, $|u_i|_{X_{\tau_1}} \le 3n$, thus

$$A_4 \le 6n\tilde{C}|u_1 - u_2|_{X_T}\tau_1^{\frac{1}{4}}. \tag{4.2.13}$$

Now using (4.2.9), (4.2.11), (4.2.12) and (4.2.13) in (4.2.7), we obtain

$$\begin{aligned}
&|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} \\
&\le (3n)^4 C|u_1 - u_2|_{X_T}\tau_2^{\frac{1}{2}} + (6n)^2 C|u_1 - u_2|_{X_T}\tau_1^{\frac{1}{2}} + (3n)^3\tilde{C}|u_1 - u_2|_{X_T}\tau_2^{\frac{1}{4}} + 6n\tilde{C}|u_1 - u_2|_{X_T}\tau_1^{\frac{1}{4}} \\
&\le (3n)^4 C|u_1 - u_2|_{X_T}T^{\frac{1}{2}} + (6n)^2 C|u_1 - u_2|_{X_T}T^{\frac{1}{2}} + (3n)^3\tilde{C}|u_1 - u_2|_{X_T}T^{\frac{1}{4}} + 6n\tilde{C}|u_1 - u_2|_{X_T}T^{\frac{1}{4}} \\
&= K(n,T)|u_1 - u_2|_{X_T}T^{\frac{1}{4}},
\end{aligned}$$

where

$$K(n,T) = 3n\left(27n^3T^{1/4} + 9n^2 + 12nT^{1/4} + 2\right),$$

is a constant which depends only on n and T. Thus we have proved that $\Phi_{n,T}$ is a Lipschitz function and satisfies (4.2.6). ■

4.2.2 Definition of a solution

Let us recall that $\mathrm{E} \hookrightarrow \mathrm{V} \hookrightarrow \mathrm{H}$. $S = (S(t))_{t\ge 0}$ is the Stokes semigroup. The following are well-known [39, 88, 90]

A1. For every $T > 0$ and $f \in L^2(0,T;\mathrm{H})$ a function $u = S * f$, defined by

$$u(t) = \int_0^T S(t-r)f(r)\,dr, \qquad t \in [0,T],$$

belongs to X_T and

$$|u|_{X_T} \le C_1|f|_{L^2(0,T;\mathrm{H})}. \tag{4.2.14}$$

A2. For every $T > 0$ and $u_0 \in \mathrm{V}$ a function $u = Su_0$ defined by

$$u(t) = S(t)u_0,$$

belongs to X_T and

$$|u|_{X_T} \le C_2\|u_0\|_{\mathrm{V}}. \tag{4.2.15}$$

Definition 4.2.4.

- A solution of (4.2.1) on $[0,T]$, $T \in [0,\infty)$ is a function $u \in X_T$ satisfying

$$u(t) = S(t)u_0 + \int_0^t S(t-r)G(u(r))\,dr, \qquad \forall\ t \in [0,T],$$

where $G : \mathrm{E} \to \mathrm{H}$ is defined by

$$G(u) = |\nabla u|_{L^2}^2 u - B(u,u), \qquad u \in \mathrm{E}.$$

- Let $\tau \in [0,\infty]$. A function $u \in \mathscr{C}([0,\tau),\mathrm{V})$ is a solution to (4.2.1) on $[0,\tau)$ iff $\forall\ T < \tau$, $u|_{[0,T]} \in X_T$ is a solution of (4.2.1) on $[0,T]$.

4.2.3 Local existence

Lemma 4.2.5. *Let $K(n,T)$ be as introduced in Lemma 4.2.3. Consider a map $\Psi_{n,T} : X_T \to X_T$ defined by*

$$\Psi_{n,T}(u) = S u_0 + S * \Phi_{n,T}(u), \quad u \in X_T .$$

Then for every $u_0 \in \mathrm{V}$, there exists a constant $C_1 > 0$ such that

$$|\Psi_{n,T}(u_1) - \Psi_{n,T}(u_2)|_{X_T} \le C_1 K(n,T)|u_1 - u_2|_{X_T} T^{\frac{1}{4}}, \quad u_1, u_2 \in X_T . \tag{4.2.16}$$

Moreover, $\forall\ \varepsilon \in (0,1)\ \exists\ T_0 = T_0(n,\varepsilon)$ such that $\Psi_{n,T}$ is an ε-contraction for $T \le T_0$.

Proof. The map $\Psi_{n,T}$ is evidently well defined. Now for any $u_1, u_2 \in X_T$

$$\begin{aligned} |\Psi_{n,T}(u_1) - \Psi_{n,T}(u_2)|_{X_T} &= \left| S(t)u_0 + S * \Phi_{n,T}(u_1) - S(t)u_0 - S * \Phi_{n,T}(u_2) \right|_{X_T} \\ &= \left| S * (\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)) \right|_{X_T}, \end{aligned}$$

then by treating $S * (\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2))$ as u and $\left[\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)\right] \in L^2(0,T;\mathrm{H})$ as f in inequality (4.2.14) and using Lemma 4.2.3 we get

$$\begin{aligned} |\Psi_{n,T}(u_1) - \Psi_{n,T}(u_2)|_{X_T} &\le C_1 |\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} \\ &\le C_1 K(n,T)|u_1 - u_2|_{X_T} T^{\frac{1}{4}}, \end{aligned}$$

which shows that $\Psi_{n,T}$ is globally Lipschitz and satisfies (4.2.16).
Let us fix $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$. Since the constant C_1 is independent of T, we can find a $T_0 = T_0(n,\varepsilon)$ such that

$$C_1 K(n,T_0) T_0^{\frac{1}{4}} = \varepsilon,$$

and thus $\Psi_{n,T}$ is an ε-contraction for $T \le T_0$. ■

Let $\varepsilon \in (0,1)$ then from Lemma 4.2.5, $\Psi_{n,T}$ is an ε-contraction for $T = T_0(n,\varepsilon)$ and thus by Banach Fixed Point Theorem there exists a unique $u^n \in X_T$ [1] s.t.

$$u^n = \Psi_{n,T}(u^n).$$

This implies that

$$u^n(t) = [\Psi_{n,T}(u^n)](t), \qquad t \in [0,T_0].$$

Let us define

$$\tau_n = \inf\left\{t \in [0,T_0] : |u^n|_{X_t} \ge n\right\}.$$

Remark 4.2.6. If $|u^n|_{X_t} < n$ for each $t \in [0,T_0^n]$ then $\tau_n = T_0^n$.

[1] In fact u^n should have been denoted by $u^{n,T}$ but we have refrained from this.

Theorem 4.2.7. *Let $R > 0$ be given then $\exists\, T_* = T_*(R)$ such that for every $u_0 \in \mathrm{V}$ with $\|u_0\|_{\mathrm{V}} \le R$, there exists a unique local solution $u : [0, T_*] \to \mathrm{V}$ of* (4.2.1).

Proof. Let $R > 0$ and fix $\varepsilon \in (0,1)$. Let us choose[2] $n = \lfloor \frac{C_2 R}{1-\varepsilon} \rfloor + 1$ where C_2 is as defined in (4.2.15). Now for these fixed n and ε, $\exists\, T_0(n,\varepsilon)$ such that $\Psi_{n,T}$ is an ε-contraction for all $T \le T_0$. In particular, it is true for $T = T_0$ and hence by Banach Fixed Point Theorem $\exists !\, u^n \in X_{T_0}$ such that

$$u^n = \Psi_{n,T}(u^n).$$

Note that we have

$$\begin{aligned} |u^n|_{X_{T_0}} &= |\Psi_{n,T}(u^n)|_{X_{T_0}} = |Su_0 + S * \Phi_{n,T}(u^n)|_{X_{T_0}} \\ &\le |Su_0|_{X_{T_0}} + |S * \Phi_{n,T}(u^n)|_{X_{T_0}}. \end{aligned}$$

Now from (4.2.15) and Lemma 4.2.5, we have

$$|u^n|_{X_{T_0}} \le C_2 \|u_0\|_{\mathrm{V}} + \varepsilon |u^n|_{X_{T_0}}.$$

Since $\|u_0\|_{\mathrm{V}} \le R$, hence on rearranging we get

$$(1-\varepsilon)|u^n|_{X_{T_0}} \le C_2 R,$$

and so

$$|u^n|_{X_{T_0}} \le \frac{C_2 R}{1-\varepsilon} \le n.$$

Now since $t \mapsto |\cdot|_{X_t}$ is an increasing function, the following holds

$$|u^n|_{X_t} \le n, \qquad \forall\, t \in [0, T_0].$$

In particular, $|u^n|_{X_{T_0}} \le n$, i.e. $|u^n|_{X_{T_0}}$ is finite and thus $u^n \in X_{T_0}$. This implies

$$\theta_n(|u^n|_{X_t}) = 1, \quad t \in [0, T_0].$$

Thus for $t \in [0, T_0]$

$$u^n(t) = S(t)u_0 + \int_0^t S(t-r) G(u^n(r))\, dr.$$

So u^n on $[0, T_*(R)]$, where $T_* = T_0(n,\varepsilon)$, solves (4.2.1) and T_* depends only on R. Thus, we have proved the existence of a unique local solution of (4.2.1) for every initial data $u_0 \in \mathrm{V}$, and this unique solution is denoted by u. ■

[2] $\lfloor M \rfloor$ denotes the largest integer less than or equal to M.

4.2.4 The local solution stays on the manifold $\mathscr{M}$

Lemma 4.2.8. *If u is the solution of* (4.2.1) *on* $[0,\tau)$ *then* $u' \in L^2(0,T;\mathrm{H})$*, for every* $T<\tau$*, i.e.* $u' \in L^2_{loc}([0,\tau);\mathrm{H})$*. Moreover,* $u \in L^2(0,T;\mathrm{D(A)})$.

Proof. Let us fix $T<\tau$. Since u is the solution of (4.2.1) on $[0,\tau)$ it satisfies

$$\frac{du}{dt} = -Au + |\nabla u|^2_{L^2} u - B(u,u). \tag{4.2.17}$$

We will show that RHS of (4.2.17) belongs to $L^2(0,T;\mathrm{H})$ and hence $u' \in L^2(0,T;\mathrm{H})$.
Since $u \in L^2(0,T;\mathrm{E})$, $Au \in L^2(0,T;\mathrm{H})$. From (4.2.2) we have

$$\begin{aligned}\int_0^T \left| |\nabla u(t)|^2_{L^2} u(t) \right|^2_{\mathrm{H}} dt &\le \int_0^T C^2 \|u(t)\|^6_{\mathrm{V}} dt \le C^2 \sup_{t\in[0,T]} \|u(t)\|^6_{\mathrm{V}} \int_0^T dt \\ &\le C^2 T \left[\sup_{t\in[0,T]} \|u(t)\|^2_{\mathrm{V}} \right]^3 \le C^2 T |u|^6_{X_T} < \infty,\end{aligned}$$

thus we have shown that $|\nabla u|^2_{L^2} u \in L^2(0,T;\mathrm{H})$.
From (4.2.3), we have

$$\begin{aligned}\int_0^T \left| B(u(t),u(t)) \right|^2_{\mathrm{H}} dt &\le \tilde{C}^2 \int_0^T \|u(t)\|^3_{\mathrm{V}} |u(t)|_{\mathrm{E}} dt \le \tilde{C}^2 \sup_{t\in[0,T]} \|u(t)\|^3_{\mathrm{V}} \int_0^T |u(t)|_{\mathrm{E}} dt \\ &\le \tilde{C}^2 \left[\sup_{t\in[0,T]} \|u(t)\|^2_{\mathrm{V}} \right]^{\frac{3}{2}} \left[\int_0^T |u(t)|^2_{\mathrm{E}} dt \right]^{\frac{1}{2}} \left[\int_0^T dt \right]^{\frac{1}{2}} \\ &\le \tilde{C}^2 |u|^3_{X_T} |u|_{X_T} T^{\frac{1}{2}} < \infty.\end{aligned}$$

Thus the convective term from Navier-Stokes also belongs to $L^2(0,T;\mathrm{H})$ and hence RHS of (4.2.17) belongs to $L^2(0,T;\mathrm{H})$ which implies that $u' \in L^2(0,T;\mathrm{H})$ for all $T<\tau$. The second conclusion can be inferred from property **A1**. ■

Let us recall that the inner product $\langle \cdot,\cdot \rangle_{\mathrm{V}}$ was defined in Section 3.1 for $\mathbb{R}^2$ as well as $\mathbb{T}^2$.

Remark 4.2.9. In the framework of Lemma 2.6.3, we can identify v with u and so we get

$$\frac{1}{2}|u(t)|^2_{\mathrm{H}} = \frac{1}{2}|u_0|^2 + \int_0^t \langle u'(s), u(s) \rangle_{\mathrm{H}} ds, \qquad \text{for a.e. } t \in [0,\tau). \tag{4.2.18}$$

Moreover, from Theorem 4.2.7 and Lemma 4.2.8

$$\frac{1}{2}\|u(t)\|^2_{\mathrm{V}} = \frac{1}{2}\|u_0\|^2_{\mathrm{V}} + \int_0^t \langle u'(s), u(s) \rangle_{\mathrm{V}} ds, \qquad \text{for a.e. } t \in [0,\tau). \tag{4.2.19}$$

Theorem 4.2.10. *If* $\tau \in [0,\infty]$, $u_0 \in \mathscr{M} \cap \mathrm{V}$ *and u is a solution to* (4.2.1) *on* $[0,\tau)$ *then* $u(t) \in \mathscr{M}$ *for all* $t \in [0,\tau)$.

Proof. Let u be the solution to (4.2.1) and $u_0 \in \mathcal{M} \cap \mathrm{V}$. Let us define $\phi(t) = |u(t)|^2_{\mathrm{H}} - 1$. Then ϕ is absolutely continuous and by Remark 4.2.9 and (4.2.1) we have a.e. on $[0,\tau)$

$$\begin{aligned}\frac{d}{dt}\phi(t) = \frac{d}{dt}[|u(t)|^2_{\mathrm{H}} - 1] &= 2\langle u'(t), u(t)\rangle_{\mathrm{H}} \\ &= 2\langle -Au(t) + |\nabla u(t)|^2_{L^2}\, u(t) - B(u(t), u(t)), u(t)\rangle_{\mathrm{H}} \\ &= -2\langle Au(t), u(t)\rangle_{\mathrm{H}} + 2|\nabla u(t)|^2_{L^2}\langle u(t), u(t)\rangle_{\mathrm{H}} \\ &= -2|\nabla u(t)|^2_{L^2} + 2|\nabla u(t)|^2_{L^2}|u(t)|^2 \\ &= 2|\nabla u(t)|^2_{L^2}(|u(t)|^2_{\mathrm{H}} - 1) = |\nabla u(t)|^2_{L^2}\,\phi(t).\end{aligned}$$

This on integration gives

$$\phi(t) = \phi(0)\exp\left[\int_0^t |\nabla u(s)|^2_{L^2}\, ds\right], \qquad t \in [0,\tau).$$

Since $u_0 \in \mathcal{M}, \phi(0) = 0$ and also as $u \in X_T$ is the solution of (4.2.1),

$$\int_0^t |\nabla u(s)|^2_{L^2}\, ds \le \int_0^t \|u(s)\|^2_{\mathrm{V}}\, ds < \infty, \qquad t \in [0,\tau).$$

Hence, we infer that $|u(t)|^2_{\mathrm{H}} = 1$ for every $t \in [0,\tau)$. Thus, $u(t) \in \mathcal{M}$ for every $t \in [0,\tau)$. ■

Corollary 4.2.11. *Let the initial data $u_0 \in \mathcal{M} \cap \mathrm{V}$ and u is the solution to* (4.2.1) *on* $[0,\tau)$*. Then $u'(t)$ is orthogonal to $u(t)$ in* H *for almost all $t \in [0,\tau)$.*

Remark 4.2.12. We can also prove Theorem 4.2.7 and Theorem 4.2.10 for any general bounded domain. Thus, establishing the existence of a local solution to (4.2.1) for any general bounded domain and $\mathbb{R}^2$.

4.3 Global solution: Existence and Uniqueness

The main result of this section is the proof of Theorem 4.1.1, i.e. we will show that the local solution obtained in Theorem 4.2.7 is indeed a global one. Lemma A.1 and the Remark 4.3.1 play crucial role in proving the global existence of the solution. We first show that the enstrophy (gradient norm) of the solution remains bounded (see Lemma 4.3.2) and then use stitching argument to extend our solution from $[0,T]$, $T < \infty$ on to the whole real line.

We recall the orthogonality property of the Stokes-operator in the following remark.

Remark 4.3.1. Let $u \in \mathrm{D(A)}$, then

$$\langle B(u,u), \mathrm{A}u\rangle_{\mathrm{H}} = 0, \qquad \forall\ u \in \mathrm{D(A)},$$

on a bounded domain with periodic boundary conditions (i.e. on a torus) [90] or on $\mathbb{R}^2$.

We define the energy of our system by

$$\mathscr{E}(u) = \frac{1}{2}|\nabla u|^2_{L^2}, \quad u \in \mathrm{V}.$$

Then, heuristically, for $u \in \mathrm{V} \cap \mathscr{M}$,

$$\begin{aligned}\nabla_{\mathscr{M}}\mathscr{E}(u) &= \Pi_u(\nabla\mathscr{E}) = \Pi_u(\mathrm{A}u)\\ &= \mathrm{A}u - |\nabla u|^2_{L^2}\, u.\end{aligned}$$

Thus, for $u \in \mathscr{M}$

$$\begin{aligned}|\nabla_{\mathscr{M}}\mathscr{E}(u)|^2_{\mathrm{H}} &= |\mathrm{A}u|^2_{\mathrm{H}} + |\nabla u|^4_{L^2}|u|^2_{\mathrm{H}} - 2|\nabla u|^2_{L^2}\langle \mathrm{A}u, u\rangle_{\mathrm{H}}\\ &= |u|^2_{\mathrm{E}} + |\nabla u|^4_{L^2} - 2|\nabla u|^4_{L^2} = |u|^2_{\mathrm{E}} - |\nabla u|^4_{L^2}.\end{aligned} \tag{4.3.1}$$

In particular, the R.H.S. of (4.3.1) is ≥ 0.

Lemma 4.3.2. *Let $u_0 \in \mathrm{V}$ and u be the local solution of* (4.2.1) *on* $[0,\tau)$*, then*

$$\sup_{s\in[0,\tau)} \|u(s)\|_{\mathrm{V}} \leq \|u_0\|_{\mathrm{V}}.$$

Proof. Let u be the solution of (4.2.1). Then, from (4.2.1), Remark 4.2.9 and Corollary 4.2.11, for any $t \in [0,\tau)$ we have

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{\mathrm{V}} &= \frac{1}{2}\|u_0\|^2_{\mathrm{V}} + \int_0^t \langle u'(s), u(s)\rangle_{\mathrm{V}}\, ds\\ &= \frac{1}{2}\|u_0\|^2_{\mathrm{V}} + \int_0^t \langle u'(s), u(s)\rangle_{\mathrm{H}}\, ds + \int_0^t \langle u'(s), \mathrm{A}u(s)\rangle_{\mathrm{H}}\, ds\\ &= \frac{1}{2}\|u_0\|^2_{\mathrm{V}} + \int_0^t \langle -\mathrm{A}u(s) + |\nabla u(s)|^2_{L^2}\, u(s) - B(u(s),u(s)), \mathrm{A}u(s)\rangle_{\mathrm{H}}\, ds\\ &= \frac{1}{2}\|u_0\|^2_{\mathrm{V}} + \int_0^t \left[-\langle \mathrm{A}u(s), \mathrm{A}u(s)\rangle_{\mathrm{H}} + |\nabla u(s)|^2_{L^2}\langle u(s), \mathrm{A}u(s)\rangle_{\mathrm{H}}\right] ds\\ &\quad - \int_0^t \langle B(u(s),u(s)), \mathrm{A}u(s)\rangle_{\mathrm{H}}\, ds\\ &= \frac{1}{2}\|u_0\|^2_{\mathrm{V}} + \int_0^t \left[-|u(s)|^2_{\mathrm{E}} + |\nabla u(s)|^4_{L^2}\right] ds.\end{aligned}$$

Now from Theorem 4.2.10 we know that $u(t) \in \mathscr{M}$ for every $t \in [0,\tau)$ and hence by using (4.3.1) we obtain,

$$\frac{1}{2}\|u(t)\|^2_{\mathrm{V}} = \frac{1}{2}\|u_0\|^2_{\mathrm{V}} - \int_0^t \left|[\nabla_{\mathscr{M}}\mathscr{E}(u)](s)\right|^2_{\mathrm{H}} ds,$$

and thus

$$\frac{1}{2}\|u(t)\|^2_{\mathrm{V}} + \int_0^t \left|[\nabla_{\mathscr{M}}\mathscr{E}(u)](s)\right|^2_{\mathrm{H}} ds = \frac{1}{2}\|u_0\|^2_{\mathrm{V}}.$$

Hence, we have shown that

$$\|u(t)\|_{\mathrm{V}} \leq \|u_0\|_{\mathrm{V}}, \qquad t \in [0,\tau).$$

■

Remark 4.3.3. The boundedness of enstrophy (the square of the gradient norm) of the solution, as proved in the Lemma 4.3.2, will play a crucial role in proving the existence of a global-in-time solution to problem (4.2.1). Note that, in the proof of the Lemma 4.3.2, the orthogonality of the Stokes operator A to the convective term $B(u,u)$ in H was essential, which as far as we know, holds only on $\mathbb{R}^2$ and on bounded domains with periodic boundary conditions (i.e. on a torus). This is the reason we were unable to prove the existence of a global-in-time solution to problem (4.2.1) on any general bounded domain.

Lemma 4.3.4. *Let* $0 \le a < b < c < \infty$ *and* $u \in X_{[a,b]}, \mathrm{v} \in X_{[b,c]}$*, such that* $u(b^-) = \mathrm{v}(b^+)$*. Then* $z \in X_{[a,c)}$ *where,*

$$z(t) = \begin{cases} u(t), & t \in [a,b), \\ \mathrm{v}(t), & t \in [b,c). \end{cases}$$

Proof. Let us take $0 \le a < b < c < \infty$ and $u \in X_{[a,b]}, \mathrm{v} \in X_{[b,c]}$, such that $u(b^-) = \mathrm{v}(b^+)$. Then for any $0 \le t_1 < t_2 < \infty$, using the definition of the norm $|\cdot|_{X_{[t_1,t_2]}}$, we have

$$\begin{aligned} |z|^2_{X_{[a,c]}} &= \sup_{t\in[a,c]} \|z(t)\|^2_{\mathrm{V}} + \int_a^c |z(t)|^2_{\mathrm{E}}\, dt \\ &\le \sup_{t\in[a,b]} \|z(t)\|^2_{\mathrm{V}} + \sup_{t\in[b,c]} \|z(t)\|^2_{\mathrm{V}} + \int_a^b |z(t)|^2_{\mathrm{E}}\, dt + \int_b^c |z(t)|^2_{\mathrm{E}}\, dt. \end{aligned}$$

Now by the definition of z we have

$$\begin{aligned} |z|^2_{X_{[a,c]}} &\le \sup_{t\in[a,b]} \|u(t)\|^2_{\mathrm{V}} + \sup_{t\in[b,c]} \|\mathrm{v}(t)\|^2_{\mathrm{V}} + \int_a^b |u(t)|^2_{\mathrm{E}}\, dt + \int_b^c |\mathrm{v}(t)|^2_{\mathrm{E}}\, dt \\ &= \sup_{t\in[a,b]} \|u(t)\|^2_{\mathrm{V}} + \int_a^b |u(t)|^2_{\mathrm{E}}\, dt + \sup_{t\in[b,c]} \|\mathrm{v}(t)\|^2_{\mathrm{V}} + \int_b^c |\mathrm{v}(t)|^2_{\mathrm{E}}\, dt \\ &= |u|^2_{X_{[a,b]}} + |\mathrm{v}|^2_{X_{[b,c]}}. \end{aligned}$$

Now since $u \in X_{[a,b]}$ and $\mathrm{v} \in X_{[b,c]}$ we have $|z|_{X_{[a,c]}} < \infty$, and thus, $z \in X_{[a,c]}$. ∎

We will use the following lemma to prove the main result about existence of the global solution.

Lemma 4.3.5. *Let* τ *be finite and the initial data* $u_0 \in \mathrm{V} \cap \mathscr{M}$*. If* $u : [0,\tau] \to \mathrm{V}$ *is the solution of* (4.2.1) *on* $[0,\tau]$ *and* $\mathrm{v} : [\tau, 2\tau] \to \mathrm{V}$ *is the solution of* (4.2.1) *on* $[\tau, 2\tau]$ *such that* $u(\tau^-) = \mathrm{v}(\tau^+)$*, then* $z : [0,2\tau] \to \mathrm{V}$ *defined as*

$$z(t) = \begin{cases} u(t), & t \in [0,\tau], \\ \mathrm{v}(t), & t \in [\tau, 2\tau], \end{cases}$$

is the solution of (4.2.1) *on* $[0,2\tau]$ *and* $z \in X_{[0,2\tau]}$.

Proof. Since u is the solution of (4.2.1) on $[0,\tau]$ then $u \in X_{[0,\tau]}$ and similarly $\mathrm{v} \in X_{[\tau,2\tau]} := \mathscr{C}([\tau,2\tau];\mathrm{V}) \cap L^2(\tau,2\tau;\mathrm{E})$. Thus by Lemma 4.3.4 and the definition of z, $z \in X_{[0,2\tau]}$. Now we are left to show that $z : [0,2\tau] \to \mathrm{V}$ defined as

$$z(t) = \begin{cases} u(t), & t \in [0,\tau], \\ \mathrm{v}(t), & t \in [\tau,2\tau], \end{cases}$$

is the solution of (4.2.1) on $[0,2\tau]$. In order to achieve this we will have to show that z satisfies (4.3.2) for every $t \in [0,2\tau]$.

$$z(t) = S(t)z(0) + \int_0^t S(t-r)G(z(r))\,dr. \tag{4.3.2}$$

For $t \in [0,\tau]$, z satisfies (4.3.2), since $z(t) = u(t)$, $\forall\ t \in [0,\tau]$ and u is the solution of (4.2.1) on $[0,\tau]$.

For $t \in [\tau,2\tau]$, $z(t) = \mathrm{v}(t)$ and since v is the solution to (4.2.1) on $[\tau,2\tau]$,

$$z(t) = \mathrm{v}(t) = S(t-\tau)\mathrm{v}(\tau) + \int_\tau^t S(t-r)G(\mathrm{v}(r))\,dr.$$

Now because of continuity of u and v, $\mathrm{v}(\tau) = u(\tau)$,

$$z(t) = S(t-\tau)\Big[S(\tau)u_0 + \int_0^\tau S(\tau-r)G(u(r))\,dr\Big] + \int_\tau^t S(t-r)G(\mathrm{v}(r))\,dr.$$

Now using the definition of z we obtain,

$$\begin{aligned} z(t) &= S(t)z(0) + \int_0^\tau S(t-r)G(z(r))\,dr + \int_\tau^t S(t-r)G(z(r))\,dr \\ &= S(t)z(0) + \int_0^t S(t-r)G(z(r))\,dr. \end{aligned}$$

Thus z satisfies (4.3.2) on $[0,2\tau]$ and hence z is a solution to (4.2.1) on $[0,2\tau]$. ■

Proof of Theorem 4.1.1 Let us take $u_0 \in \mathrm{V}$. Put $R = \|u_0\|_\mathrm{V}$. By Theorem 4.2.7 there exists a $T > 0$ such that there exists a unique function $u : [0,T] \to \mathrm{V}$ which solves (4.2.1) on $[0,T]$ and $u \in X_T$. Also by Lemma 4.3.2 $\|u(T)\|_\mathrm{V} \le R$ thus again by Theorem 4.2.7 there exists a unique function $\mathrm{v} : [T,2T] \to \mathrm{V}$ which solves (4.2.1) on $[T,2T]$ and $\mathrm{v} \in X_{[T,2T]}$. Now if we define a new function $z : [0,2T] \to \mathrm{V}$ as

$$z(t) = \begin{cases} u(t), & t \in [0,T], \\ \mathrm{v}(t), & t \in [T,2T], \end{cases}$$

then by Lemma 4.3.5, z is also a solution of (4.2.1) and $z \in X_{2T}$. Moreover $\|z(2T)\|_\mathrm{V} \le R$. We can keep doing this and extend our solution further and hence obtaining a global solution of (4.2.1) still denoted by u such that $u \in X_T$ for every $T < \infty$. Each bit of the solution is unique on the respective domain and hence when we glue two unique bits we get a unique extension and thus obtain a unique global solution due to its construction. ■

4.4 Convergence to the Euler equation, i.e. the inviscous limit

In this section we are concerned with the convergence of the solution of the constrained Navier-Stokes equations, namely

$$\begin{cases} \frac{du}{dt} + \nu A u - \nu |\nabla u|^2_{L^2} u + B(u,u) = 0\,, \\ u(0) = u_0^\nu \in \mathrm{V} \cap \mathscr{M}\,, \end{cases} \tag{4.4.1}$$

as ν vanishes on a torus.

The curl of a vector field u is defined by

$$\mathrm{Curl}(u) := D_1 u_2 - D_2 u_1\,. \tag{4.4.2}$$

We will prove Theorem 4.1.2 after several preliminary results.

Remark 4.4.1. Curl is a linear isomorphism between V and $L^2_0(\mathbb{T}^2)$, where

$$L^2_0(\mathbb{T}^2) := \left\{ \omega \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} \omega(x) dx = 0 \right\}.$$

Moreover for $u \in V$ and some universal constants $C > 0$, $C_p > 0$

$$|\Delta u|_{L^2(\mathbb{T}^2)} \le C |\nabla \mathrm{Curl}(u)|_{L^2(\mathbb{T}^2)}\,, \tag{4.4.3}$$

$$\|\nabla u\|_{L^p(\mathbb{T}^2)} \le C_p \|\mathrm{Curl}(u)\|_{L^\infty(\mathbb{T}^2)}\,. \tag{4.4.4}$$

This remark is proved in Appendix B.

Hereafter u^ν is the solution to (4.4.1), and $\omega^\nu(t,x) := \mathrm{Curl}(u^\nu(t))(x)$. In particular, due to Remark 4.4.1 and Theorem 4.2.10, $\omega^\nu \in \mathscr{C}([0,T]; L^2_0(\mathbb{T}^2)) \cap L^2(0,T; H^1(\mathbb{T}^2))$. It is then easy to check that ω^ν is a weak solution to

$$\begin{cases} \frac{d\omega^\nu}{dt} + \nabla \cdot (u^\nu \omega^\nu) = \nu \Delta \omega^\nu + \nu |\nabla u^\nu|^2_{L^2} \omega^\nu\,, \\ \omega^\nu(0) = \omega_0^\nu := \mathrm{Curl}(u_0^\nu) \in L^2_0(\mathbb{T}^2). \end{cases} \tag{4.4.5}$$

Proposition 4.4.2. *Let us fix $T > 0$, and assume that $\omega_0^\nu \in L^\infty(\mathbb{T}^2)$. Then*

$$\sup_{t \in [0,T]} \|\omega^\nu(t)\|_{L^\infty(\mathbb{T}^2)} \le \|\omega_0^\nu\|_{L^\infty(\mathbb{T}^2)} \exp\left(\nu \|u_0^\nu\|^2_{\mathrm{V}} T\right), \tag{4.4.6}$$

$$\nu \int_0^T |\nabla \omega^\nu(t)|^2_{L^2(\mathbb{T}^2)} dt \le \tfrac{1}{2} |\omega_0^\nu|^2_{L^2(\mathbb{T}^2)} + \nu T \|u_0^\nu\|^2_{\mathrm{V}} \|\omega_0^\nu\|^2_{L^\infty(\mathbb{T}^2)} \exp\left(2\nu \|u_0^\nu\|^2_{\mathrm{V}} T\right). \tag{4.4.7}$$

Proof. Take $h \in C^2(\mathbb{R})$, convex, with bounded second derivative. Then, since $\omega \in \mathscr{C}([0,T];L^2_0(\mathbb{T}^2))$

$$\begin{aligned}
&\langle h(\omega^\nu(t)),\mathbf{1}\rangle - \langle h(\omega^\nu_0),\mathbf{1}\rangle \\
&= \nu\int_0^t \left[-\langle h''(\omega^\nu(s)),|\nabla\omega^\nu(s)|^2_{L^2}\rangle + |\nabla u^\nu(s)|^2_{L^2}\langle h'(\omega^\nu(s)),\omega^\nu(s)\rangle\right] ds \\
&\le \nu\int_0^t |\nabla u^\nu(s)|^2_{L^2}\langle h'(\omega^\nu(s)),\omega^\nu(s)\rangle\, ds.
\end{aligned} \tag{4.4.8}$$

For $p \ge 2, R > 0$, take

$$h(w) \equiv h_{p,R}(w) := \begin{cases} |w|^p, & \text{if } |w| \le R, \\ R^p + pR^{p-1}(|w|-R) + \frac{p(p-1)}{2}R^{p-2}(|w|-R)^2, & \text{if } |w| > R. \end{cases} \tag{4.4.9}$$

Then $|h'(w)w| \le p\,h(w)$ and, by Lemma 4.3.2 $\forall\, s \in [0,t]$, $\|u^\nu(s)\|^2_{\mathrm{V}} \le \|u^\nu_0\|^2_{\mathrm{V}}$

$$\langle h(\omega^\nu(t)),\mathbf{1}\rangle \le \langle h(\omega^\nu_0),\mathbf{1}\rangle + \nu p\int_0^t \|u^\nu_0\|^2_{\mathrm{V}}\langle h(\omega^\nu(s)),\mathbf{1}\rangle\, ds. \tag{4.4.10}$$

By the Gronwall Lemma

$$\langle h(\omega^\nu(t)),\mathbf{1}\rangle \le \langle h(\omega^\nu_0),\mathbf{1}\rangle \exp\left(\nu p\|u^\nu_0\|^2_{\mathrm{V}} t\right), \quad t \in [0,T]. \tag{4.4.11}$$

Since

$$\|\omega^\nu\|_{L^\infty} = \sup_{p,R}\langle h_{p,R}(\omega^\nu),\mathbf{1}\rangle^{1/p}, \tag{4.4.12}$$

we get (4.4.6).

On the other hand, from the first equality in (4.4.8), taking now $h(w) = w^2/2$

$$\begin{aligned}
\tfrac{1}{2}|\omega^\nu(T)|^2_{L^2(\mathbb{T}^2)} + \nu\int_0^T |\nabla\omega^\nu(t)|^2_{L^2(\mathbb{T}^2)}\,dt &= \tfrac{1}{2}|\omega^\nu_0|^2_{L^2(\mathbb{T}^2)} + \nu\int_0^T |\nabla u^\nu(t)|^2_{L^2}|\omega^\nu(t)|^2_{L^2(\mathbb{T}^2)}\,dt \\
&\le \tfrac{1}{2}|\omega^\nu_0|^2_{L^2(\mathbb{T}^2)} + \nu T\|u^\nu_0\|^2_{\mathrm{V}}\|\omega^\nu_0\|^2_{L^\infty(\mathbb{T}^2)}e^{2\nu T\|u^\nu_0\|^2_{\mathrm{V}}},
\end{aligned}$$

where in the last line we used (4.4.6). Hence (4.4.7). ■

Proposition 4.4.3. *For each $\varphi \in \mathrm{H}^2(\mathbb{T}^2)$, and $\nu > 0$*

$$\langle \omega^\nu(t) - \omega^\nu(s),\varphi\rangle \le (t-s)\left(|\omega^\nu|_{L^\infty([0,T]\times\mathbb{T}^2)} + 2\nu\|u^\nu_0\|_{\mathrm{V}}(1+\|u^\nu_0\|^2_{\mathrm{V}})\right)|\varphi|_{H^2(\mathbb{T}^2)}. \tag{4.4.13}$$

Proposition 4.4.4. *Suppose that, uniformly in ν, u^ν_0 is bounded in* V *and* $\mathrm{Curl}(u^\nu_0)$ *is bounded in $L^\infty(\mathbb{T}^2)$. Then the sequence u^ν is precompact in $\mathscr{C}([0,T];L^2(\mathbb{T}^2))$.*

Proof. Let us take and fix $\varphi \in H^2(\mathbb{T}^2)$. Also fix $0 \le s < t \le T$. Then from the equation (4.4.5) and $\|u^\nu(t)\|^2_{\mathrm{V}} \le \|u^\nu_0\|^2_{\mathrm{V}}$ we get,

$$\left|\langle u^\nu(t) - u^\nu(s),\varphi\rangle\right| \le \nu\left|\int_s^t \langle\Delta u^\nu,\varphi\rangle\,dr\right| + \nu\|u^\nu_0\|^2_{\mathrm{V}}\int_s^t \left|\langle u^\nu,\varphi\rangle\right|dr + \left|\int_s^t \langle u^\nu\nabla u^\nu,\varphi\rangle\,dr\right|. \tag{4.4.14}$$

By (4.4.3), (4.4.7) and the hypotheses on the initial data, the first term in the R.H.S. is bounded by $C_T|\varphi|_{L^2}(t-s)^{1/2}$ for some constant C_T independent on ν. The second term in the R.H.S. of (4.4.14) easily enjoys the same bound. As for the third term in the R.H.S., for any $p>2$, $|u|_{L^\infty} \leq C_p(|u|_{L^2} + \|\nabla u\|_{L^p})$, so that from (4.4.4) and (4.4.6), this term is still bounded by $C_T|\varphi|_{L^2}(t-s)^{1/2}$.

Therefore, since u_0^ν is bounded uniformly in $L^2(\mathbb{T}^2)$ by Poincaré inequality, it follows that u^ν is equibounded and equicontinuous in $L^2(\mathbb{T}^2)$ and, by Arzelà-Ascoli theorem (see Theorem 2.4.3), precompact in $\mathscr{C}([0,T];L^2(\mathbb{T}^2))$. ■

Proof of Theorem 4.1.2 Fix $T>0$. Using Propositions 4.4.3 - 4.4.4, from each subsequence we can extract a further subsequence such that $\omega^\nu \to \omega$ in $\mathscr{C}([0,T];H^{-2}(\mathbb{T}^2))$ and weakly in $L^\infty([0,T]\times\mathbb{T}^2)$, $u^\nu \to u$ weakly in $L^\infty([0,T];\mathrm{V})$ and in $\mathscr{C}([0,T];L^2(\mathbb{T}^2))$. It is immediate to check that $\omega = \mathrm{Curl}(u)$.

Notice that $\omega_0^\nu := \mathrm{Curl}(u_0^\nu)$ converges weakly in $L^\infty(\mathbb{T}^2)$ to $\omega_0 := \mathrm{Curl}(u_0)$. Passing to the limit in the weak formulation of the equation one then has, for each $\varphi \in C^2([0,T]\times\mathbb{T}^2)$

$$\langle\omega(t),\varphi(t)\rangle - \langle\omega_0,\varphi(0)\rangle - \int_0^t \langle\omega(s),\partial_s\varphi(s)\rangle\,ds - \int_0^t \langle u(s)\,\omega(s),\nabla\varphi(s)\rangle\,ds = 0, \tag{4.4.15}$$

and $\omega(0)=\omega_0$. Recalling that $\omega = \mathrm{Curl}(u)$

$$\langle u(t),\nabla^\perp\varphi(t)\rangle - \langle u_0,\nabla^\perp\varphi(0)\rangle - \int_0^t \langle u(s),\partial_s\nabla^\perp\varphi(s)\rangle\,ds - \int_0^t \langle u(s)\cdot\nabla u(s),\nabla^\perp\varphi(s)\rangle\,ds = 0. \tag{4.4.16}$$

Since $\langle u\,\omega,\nabla\varphi\rangle = \langle u\cdot\nabla u,\nabla^\perp\varphi\rangle$ holds.

By Bardos uniqueness theorem [5, 37], we conclude that $u^\nu \to u$. ■

4.5 CNSE in the fractional Sobolev spaces

We study 2D Navier-Stokes equations with a constraint on L^2 energy of the solution in fractional Sobolev spaces. In Theorem 4.1.1 we proved the existence of a unique global-in-time solution for the constrained Navier-Stokes equations (4.1.3) on $\mathbb{R}^2$ and $\mathbb{T}^2$ with initial data $u_0 \in \mathrm{V}$ (see Section 3.1). In this section we consider more regular initial data, $u_0 \in \hat{\mathrm{V}} = \mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}})$, $\alpha \in (1,3/2)\cup\{2\}$ (see below for details), and prove the existence of a unique global-in-time solution for the constrained Navier-Stokes equations on a general bounded domain under condition (4.5.21). In particular, we show it for a bounded periodic domain.

Our proof heavily relies on the Lemma 4.5.2, which holds true only for $\gamma \in (0,\frac{1}{2})$. This restriction on γ, is the essential reason for not having an existence of the solution in the case of $\alpha \in [\frac{3}{2},2)$, included in this thesis.

We are interested in the initial value problem

$$\begin{cases} \dfrac{du}{dt} = -\nu\mathrm{A}u + \nu|\nabla u|_{L^2}^2 u - B(u,u), \quad \text{on } \mathscr{O}, \\ u(0) = u_0, \quad \text{on } \mathscr{O}, \\ u\cdot\mathbf{n} = 0 \quad \text{on } \Gamma, \end{cases} \tag{4.5.1}$$

where $u \in \mathrm{H}$, and H is a space of divergence free vector fields, see (4.5.2) below for a precise definition. Γ is the smooth boundary of the bounded and simply connected domain $\mathscr{O}$. $\mathbf{n}$ is the unit outward normal to Γ.

The above problem has a local *maximal* solution for each $u_0 \in \hat{\mathrm{V}} \cap \mathscr{M}$, where $\hat{\mathrm{V}}$ is defined in (4.5.3) and

$$\mathscr{M} = \{u \in \mathrm{H} : |u|_{\mathrm{H}} = 1\}.$$

As in Theorem 4.2.10, the solution u of (4.5.4) stays on the manifold $\mathscr{M}$ for all times t. We use a different approach to prove the existence of a global solution compared to the proof of Theorem 4.1.1. We first show existence of a local solution using Banach fixed point theorem and then use these local solutions to construct a maximal solution. Next using the maximality of the solution we show that either the solution is a global one or $\hat{\mathrm{V}}$-norm blows up in finite time, Lemma 4.5.10. Instead of using the geometric structure of (4.5.1) we use Lemma 2.6.5 and the Gronwall Lemma to obtain the bound on $\hat{\mathrm{V}}$-norm of the solution, Lemma 4.5.12 and Remark 4.5.13. Invoking contradiction by the use of Lemmas 4.5.10 and 4.5.12, we infer the existence of a global-in-time solution.

Let us fix $T > 0$ and set

$$X_T = \mathscr{C}([0,T];\hat{\mathrm{V}}) \cap L^2(0,T;\hat{\mathrm{E}}).$$

The following theorem holds true for both, bounded domain with Dirichlet boundary conditions and bounded domain with periodic boundary conditions, i.e. on a torus.

Theorem 4.5.1. *Let $\alpha \in \left(1, \frac{3}{2}\right) \cup \{2\}$ and $u_0 \in \hat{\mathrm{V}} \cap \mathscr{M}$. If the function β defined in* (4.5.21) *belongs to $L^1([0,T])$ for every $T > 0$, then there exists a global and locally unique solution u of* (4.5.1)*, such that for every $T > 0$, $u \in X_T$.*

In Subsection 4.5.1, the space X_T with more details along with a precise definition of the solution is given, and existence of a local solution is proved, together with some basic properties of the solution. In Subsection 4.5.2, the maximal solution is defined and it's existence is proved. Finally, in Subsection 4.5.3, we define the function β and establish the existence of the global solution.

Let $\mathscr{O}$ be a bounded simply connected domain in $\mathbb{R}^2$ with sufficiently regular boundary Γ. We introduce the following spaces:

$$\begin{aligned} \mathrm{H} &= \left\{u \in L^2(\mathscr{O},\mathbb{R}^2) : \operatorname{div} u = 0, u \cdot \mathbf{n} = 0 \text{ on } \Gamma\right\}, \\ \mathrm{V} &= H_0^1 \cap \mathrm{H}, \end{aligned} \tag{4.5.2}$$

where $\mathbf{n}$ is the outward normal to Γ.

We endow H with the scalar product and norm of L^2 and denote it by $\langle u, \mathrm{v}\rangle_{\mathrm{H}}$, $|u|_{\mathrm{H}}$ respectively for $u, \mathrm{v} \in \mathrm{H}$. We equip the space V with the scalar product and norm of H^1 and will denote it by $\langle\cdot,\cdot\rangle_{\mathrm{V}}$ and $\|\cdot\|_{\mathrm{V}}$ respectively.

Let us recall the Stokes operator $\mathrm{A} : \mathrm{D}(\mathrm{A}) \to \mathrm{H}$, which is defined by

$$\mathrm{D}(\mathrm{A}) = \mathrm{H} \cap H^2(\mathbb{R}^2),$$
$$\mathrm{A}u = -\Pi \Delta u, \quad u \in \mathrm{D}(\mathrm{A}).$$

It is well known that A is a self adjoint positive operator in H [93, Chapter 4]. Thus, the fractional powers A^α exist for $\alpha \in \mathbb{R}$, and

$$\mathrm{D}(\mathrm{A}^\alpha) = [\mathrm{H}, \mathrm{D}(\mathrm{A})]_\alpha,$$

where $[\cdot,\cdot]_\alpha$ is the complex interpolation functor of order α (see [58, Chapters 2 & 4]). The norms in the space $\mathrm{D}(\mathrm{A}^\alpha)$ are equivalent to the norms in the space $H^{2\alpha}$. We will be using the following spaces

$$\begin{aligned} \hat{\mathrm{V}} &:= \mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}}), \\ \hat{\mathrm{H}} &:= \mathrm{D}(\mathrm{A}^{\frac{\alpha-1}{2}}), \\ \hat{\mathrm{E}} &:= \mathrm{D}(\mathrm{A}^{\frac{\alpha+1}{2}}). \end{aligned} \tag{4.5.3}$$

Moreover, for $\alpha > 1$, we have the following identities (see [92])

$$\mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}}) = H^\alpha(\mathscr{O}) \cap \mathrm{V},$$
$$\mathrm{D}(\mathrm{A}^{\frac{\alpha-1}{2}}) = H^{\alpha-1}(\mathscr{O}) \cap \mathrm{H},$$
$$\mathrm{D}(\mathrm{A}^{\frac{\alpha+1}{2}}) = H^{\alpha+1}(\mathscr{O}) \cap \mathrm{V}.$$

4.5.1 Local solution : Existence and Uniqueness

In this subsection we will establish the existence of a local solution to the problem (we have taken $\nu = 1$)

$$\begin{cases} \dfrac{du}{dt} + \mathrm{A}u - |\nabla u|_{L^2}^2 u + B(u,u) = 0, \\ u(0) = u_0, \end{cases} \tag{4.5.4}$$

with $u_0 \in \hat{\mathrm{V}} \cap \mathscr{M}$, by using the Banach fixed point theorem. We follow the same methodology as we did while establishing the existence of a local solution to the problem (4.2.1).

The following lemma [12] along with the Gagliardo-Nirenberg inequality (4.5.6) plays a crucial role in obtaining the bounds on the non-linear terms of (4.5.4)

Lemma 4.5.2. *Assume that $\gamma \in (0, \frac{1}{2})$. Then for any $s \in (1,2]$ there exists a constant $C > 0$ such that*

$$\|B(u,\mathrm{v})\|_{D(\mathrm{A}^{\frac{\gamma}{2}})} \le C\|u\|_{D(\mathrm{A}^{\frac{s}{2}})}\|\mathrm{v}\|_{D(\mathrm{A}^{\frac{\gamma+1}{2}})}, \qquad u,\mathrm{v} \in \mathrm{D}(\mathrm{A}).$$

In particular,

$$\|u\nabla \mathrm{v}\|_{H^\gamma} \le C\|u\|_{H^s}\|\mathrm{v}\|_{H^{\gamma+1}}, \qquad u,\mathrm{v} \in H^2. \tag{4.5.5}$$

Let $u \in H^s$ for any $s \in (1,2]$ and $\mathrm{v} \in H^1$ then there exists a positive constant C depending on s such that

$$\|u\mathrm{v}\|_{H^1} \le C\|u\|_{H^s}\|\mathrm{v}\|_{H^1}, \qquad u \in H^s, \mathrm{v} \in H^1. \tag{4.5.6}$$

In what follows we assume that D(A), V, H, $\hat{\mathrm{E}}$, $\hat{\mathrm{V}}$ and $\hat{\mathrm{H}}$ are spaces as defined above. The next lemma establishing the estimates on the non-linear term arising from the constraint can be proved in the similar way as Lemma 4.2.1. Thus, we will state the lemma without the proof.

Lemma 4.5.3. *Let $G_1 : \hat{\mathrm{V}} \to \hat{\mathrm{H}}$ be defined by*

$$G_1(u) = |\nabla u|^2_{L^2} u, \qquad u \in \hat{\mathrm{V}}.$$

Then there exists $C > 0$ such that for $u_1, u_2 \in \hat{\mathrm{V}}$,

$$|G_1(u_1) - G_1(u_2)|_{\hat{\mathrm{H}}} \le C\|u_1 - u_2\|_{\hat{\mathrm{V}}}\left[\|u_1\|_{\hat{\mathrm{V}}} + \|u_2\|_{\hat{\mathrm{V}}}\right]^2. \tag{4.5.7}$$

Lemma 4.5.4. *Let $\alpha \in \left(1, \frac{3}{2}\right) \cup \{2\}$ and $G_2 : \hat{\mathrm{E}} \to \hat{\mathrm{H}}$ be defined by*

$$G_2(u) = B(u,u), \qquad u \in \hat{\mathrm{E}}.$$

Then there exists $\tilde{C} > 0$ such that for $u_1, u_2 \in \hat{\mathrm{E}}$,

$$|G_2(u_1) - G_2(u_2)|_{\hat{\mathrm{H}}} \le \tilde{C}\|u_1 - u_2\|_{\hat{\mathrm{V}}}\left(\|u_1\|_{\hat{\mathrm{V}}} + \|u_2\|_{\hat{\mathrm{V}}}\right). \tag{4.5.8}$$

Proof. Let us take $u_1, u_2 \in \hat{\mathrm{E}}$, then

$$\begin{aligned}
|G_2(u_1) - G_2(u_2)|_{\hat{\mathrm{H}}} &= |B(u_1,u_1) - B(u_2,u_2)|_{\hat{\mathrm{H}}} \\
&= |B(u_1,u_1) - B(u_2,u_1) + B(u_2,u_1) - B(u_2,u_2)|_{\hat{\mathrm{H}}} \\
&= |B(u_1 - u_2, u_1) + B(u_2, u_1 - u_2)|_{\hat{\mathrm{H}}} \\
&= |\Pi[(u_1 - u_2)\cdot\nabla u_1] + \Pi[u_2 \cdot \nabla(u_1 - u_2)]|_{\hat{\mathrm{H}}} \\
&\le \|(u_1 - u_2)\cdot\nabla u_1\|_{H^{\alpha-1}} + \|u_2\cdot\nabla(u_1 - u_2)\|_{H^{\alpha-1}}.
\end{aligned}$$

Now for $\alpha \in (1, \frac{3}{2})$, we use (4.5.5) with $\gamma = \alpha - 1 \in (0, \frac{1}{2})$, and for $\alpha = 2$ we use (4.5.6). Since, (4.5.5) and (4.5.6) hold for any $s \in (1,2]$; we choose $s = \alpha$, and hence we obtain

$$\begin{aligned}
\|(u_1 - u_2)\cdot\nabla u_1\|_{H^{\alpha-1}} + \|u_2\nabla\cdot(u_1 - u_2)\|_{H^{\alpha-1}} &\le C\|u_1 - u_2\|_{H^\alpha}\|u_1\|_{H^\alpha} + C\|u_2\|_{H^\alpha}\|u_1 - u_2\|_{H^\alpha} \\
&\le C\|u_1 - u_2\|_{H^\alpha}(\|u_1\|_{H^\alpha} + \|u_2\|_{H^\alpha}) \\
&\le C\|u_1 - u_2\|_{\hat{\mathrm{V}}}(\|u_1\|_{\hat{\mathrm{V}}} + \|u_2\|_{\hat{\mathrm{V}}}). \qquad (4.5.9)
\end{aligned}$$

In case of $\alpha = 2$ the above inequalities take the form

$$\begin{aligned}
\|(u_1 - u_2)\cdot\nabla u_1\|_{H^1} + \|u_2\nabla\cdot(u_1 - u_2)\|_{H^1} &\le C\|u_1 - u_2\|_{H^2}\|\nabla u_1\|_{H^1} + C\|u_2\|_{H^2}\|\nabla(u_1 - u_2)\|_{H^1} \\
&\le C\|u_1 - u_2\|_{H^2}(\|u_1\|_{H^2} + \|u_2\|_{H^2}) \\
&\le C\|u_1 - u_2\|_{\hat{\mathrm{V}}}(\|u_1\|_{\hat{\mathrm{V}}} + \|u_2\|_{\hat{\mathrm{V}}}), \qquad (4.5.10)
\end{aligned}$$

where the last inequality holds since for $\alpha = 2$, $D(A^{\frac{\alpha}{2}}) = H^2 \cap V$.

Thus for $\alpha \in (1, \frac{3}{2}) \cup \{2\}$, from (4.5.9) and (4.5.10), we obtain

$$|G_2(u_1) - G_2(u_2)|_{\hat{H}} \le \tilde{C} \|u_1 - u_2\|_{\hat{V}} \left(\|u_1\|_{\hat{V}} + \|u_2\|_{\hat{V}} \right).$$

■

Let us recall that $\hat{E} \hookrightarrow \hat{V} \hookrightarrow \hat{H}$. Let $S = (S(t))_{t \ge 0}$ be the semigroup on $\hat{H}$ generated by the Stokes operator. Then the following are well-known [39, 88, 90]:

A1. For every $T > 0$ and $f \in L^2(0,T;\hat{H})$ a function $u = S * f$, defined by

$$u(t) = \int_0^T S(t-r) f(r)\, dr \qquad t \in [0,T],$$

belongs to $X_T := \mathscr{C}([0,T];\hat{V}) \cap L^2(0,T;\hat{E})$ and

$$|u|_{X_T} \le C_1 |f|_{L^2(0,T;\hat{H})}, \tag{4.5.11}$$

where $|u|^2_{X_T} := \sup_{t \in [0,T]} \|u(t)\|^2_{\hat{V}} + \int_0^T |u(t)|^2_{\hat{E}}\, dt$.

A2. For every $T > 0$ and $u_0 \in \hat{V}$ a function $u = S u_0$ defined by

$$u(t) = S(t) u_0,$$

belongs to X_T and

$$|u|_{X_T} \le C_2 \|u_0\|_{\hat{V}}. \tag{4.5.12}$$

Definition 4.5.5.
- A solution of (4.5.4) on $[0,T]$, $T \in [0,\infty)$ is a function $u \in X_T$ satisfying

$$u(t) = S(t) u_0 + \int_0^t S(t-r) G(u(r))\, dr \qquad t \in [0,T],$$

where $G : \hat{E} \to \hat{H}$ is defined by

$$G(u) = |\nabla u|^2_{L^2} u - B(u,u), \qquad u \in \hat{E}.$$

- Let $\tau \in [0,\infty]$. A function $u \in \mathscr{C}([0,\tau),\hat{V})$ is a solution to (4.5.4) on $[0,\tau)$ iff $\forall\ T < \tau$, $u|_{[0,T]} \in X_T$ is a solution of (4.5.4) on $[0,T]$.

Now we state the main result of this section, which can be proved as Theorem 4.2.7 using Lemmas 4.5.3 - 4.5.4.

Theorem 4.5.6. *Let $R > 0$ be given then $\exists\ T_* = T_*(R)$ such that for every $u_0 \in \hat{V}$ with $\|u_0\|_{\hat{V}} \le R$ there exists a unique local solution $u : [0,T_*] \to \hat{V}$ of* (4.5.4).

The following theorem states that the local solution of (4.5.4) stays on the manifold $\mathcal{M}$.

Theorem 4.5.7. *If $\tau \in [0,\infty]$, $u_0 \in \mathcal{M} \cap \hat{V}$ and u is a solution to* (4.5.4) *on* $[0,\tau)$ *then $u(t) \in \mathcal{M}$ for all $t \in [0,\tau)$.*

The proof of Theorem 4.5.7 is essentially the same as of Theorem 4.2.10 and hence, we have skipped it here.

4.5.2 Maximal solution

Let $\mathscr{G}$ be a set of all local solutions, whose existence was established in Theorem 4.5.6. Let $u^1, u^2 \in \mathscr{G}$ defined on $[0,\tau_1)$ and $[0,\tau_2)$ respectively. We define an order " $\preceq$ " on $\mathscr{G}$ by

$$u^1 \preceq u^2 \text{ iff } \tau_1 \leq \tau_2 \text{ and } u^2|_{[0,\tau_1)} = u^1.$$

Lemma 4.5.8. *If $\mathscr{G}$ and $\preceq$ are as described above. Then, $\mathscr{G}$ has a maximal element.*

Proof. In order to show that $\mathscr{G}$ has a maximal element we will prove that $\{\mathscr{G},\preceq\}$ is a partially ordered set (poset) and every chain in $\mathscr{G}$ has an upper bound in $\mathscr{G}$.

Claim $\{\mathscr{G},\preceq\}$ is a poset.

Let $u^1, u^2, u^3 \in \mathscr{G}$ s.t. u^i is a solution of (4.5.4) on $[0,\tau_i),\ i = 1,2,3$.

(a) $\preceq$ is reflexive.

$u^1 \preceq u^1$ since $\tau_1 = \tau_1$ and $u^1|_{[0,\tau_1)} = u^1$. Hence $\preceq$ is reflexive.

(b) $\preceq$ is anti-symmetric.

Let $u^1 \preceq u^2$ and $u^2 \preceq u^1$.

$$u^1 \preceq u^2 \implies \tau_1 \leq \tau_2 \text{ and } u^2|_{[0,\tau_1)} = u^1. \tag{4.5.13}$$

$$u^2 \preceq u^1 \implies \tau_2 \leq \tau_1 \text{ and } u^1|_{[0,\tau_2)} = u^2. \tag{4.5.14}$$

(4.5.13) and (4.5.14) $\implies \tau_1 = \tau_2$ and

$$u^1 = u^2|_{[0,\tau_1)} = u^2|_{[0,\tau_2)} = u^2.$$

Hence $\preceq$ is anti-symmetric.

(c) $\preceq$ is transitive.

Let $u^1 \preceq u^2$ and $u^2 \preceq u^3$.

$$u^1 \preceq u^2 \implies \tau_1 \leq \tau_2 \text{ and } u^2|_{[0,\tau_1)} = u^1, \tag{4.5.15}$$

$$u^2 \preceq u^3 \implies \tau_2 \leq \tau_3 \text{ and } u^3|_{[0,\tau_2)} = u^2, \tag{4.5.16}$$

(4.5.15) and (4.5.16) $\implies \tau_1 \leq \tau_3$ and

$$u^3|_{[0,\tau_1)} = u^2|_{[0,\tau_1)} = u^1.$$

Thus $u^1 \preceq u^3$. Hence $\preceq$ is transitive.

Thus from (a), (b) and (c), we conclude that $\{\mathscr{G},\preceq\}$ is a poset.

Let $u^{i_1} \preceq u^{i_2} \preceq u^{i_3} \preceq \cdots$ be a chain in $\mathscr{G}$, where each u^{i_n} is a solution of (4.5.4) on $[0,\tau_{i_n})$ and $\tau_{i_1} \leq \tau_{i_2} \leq \tau_{i_3} \leq \cdots$. Also, let $\tau = \sup_{i_n, n\in\mathbb{N}} \tau_{i_n}$ and we define $u : [0,\tau) \to \hat{V}$ by

$$u(t) = u^{i_n}(t) \ \text{ if } t \in [0,\tau_{i_n}).$$

Since $[0,\tau_{i_n}) \subset [0,\tau)$ for every $n \in \mathbb{N}$, u is the upper bound of the chain. Since each of u^{i_n} is a solution of (4.5.4), $u|_{[0,\tau)} \in X_T$ and from the definition of u it is clear that u solves (4.5.4) on $[0,\tau)$, thus $u \in \mathscr{G}$.

Hence every chain in $\mathscr{G}$ has an upper bound in $\mathscr{G}$. Thus by the Kuratowski-Zorn Lemma (see Lemma 2.6.12) $\mathscr{G}$ has at least one maximal element. ■

Definition 4.5.9. We say that a function u is a maximal solution on $[0,\tau)$ if u is a solution to (4.5.4) on $[0,\tau)$ and if there exists another function $\mathrm{v} \in X_T$ satisfying (4.5.4) on $[0,\tau_{\mathrm{v}})$ such that $u \preceq \mathrm{v}$ then $\tau_{\mathrm{v}} = \tau$ and $u(t) = \mathrm{v}(t)\ \forall\ t \in [0,\tau)$.

Lemma 4.5.10. *Suppose u is a local maximal solution of* (4.5.4) *on* $[0,\tau)$ *and* $\tau < \infty$. *Then*

$$\forall\ R > 0,\ \exists\ \delta > 0 : \|u(t)\|_{\hat{\mathrm{V}}} > R \ \textit{if}\ t \in (\tau - \delta, \tau).$$

Proof. We will prove the proposition by contradiction, i.e. we assume that $\exists\ R > 0$ such that $\forall\ \delta > 0,\ \exists\ t_\delta \in (\tau-\delta,\tau) : \|u(t_\delta)\|_{\hat{\mathrm{V}}} < R$.

Let us choose $\delta > 0$ such that $\delta < \frac{T_*}{2} \wedge \tau$, where T_* will be chosen later in the proof. Then for this δ, there exists a $t_\delta \in (\tau-\delta,\tau)$ such that $\|u(t_\delta)\|_{\hat{\mathrm{V}}} < R$.

Now since $\|u(t_\delta)\|_{\hat{\mathrm{V}}} < R$, then by Theorem 4.5.6, there exists $T_* = T_*(R)$ such that there exists a unique solution v of (4.5.4) on $[t_\delta, t_\delta + T_*]$ and $\mathrm{v}(t_\delta) = u(t_\delta)$. So on the common domain $\mathrm{v} = u$, i.e.

$$\mathrm{v}(t) = u(t), \qquad t \in [t_\delta, \tau).$$

Now let us define $z : [0,\hat{\tau}] \to \hat{\mathrm{V}}$ as follows:

$$z(t) = \begin{cases} u(t), & t \in [0, t_\delta] \\ \mathrm{v}(t), & t \in [t_\delta, \hat{\tau}], \end{cases}$$

where $\hat{\tau} := t_\delta + T_*$.

Claim $u \preceq z$ and $u \neq z$.

Since $[0,\tau] \subsetneq [0,\hat{\tau}]$, i.e. the domain of z is bigger than that of u and thus by the definition of z, $u \neq z$.

Now we need to show that z is a solution of (4.5.4). It is clear from Lemma 4.3.4 that $z \in X_{\hat{\tau}}$. Now it remains to show that z satisfies (4.5.17) for $t \in [0,\hat{\tau}]$.

$$z(t) = S(t)z(0) + \int_0^t S(t-r)G(z(r))\,dr. \tag{4.5.17}$$

For $t \in [0,t_\delta]$, z satisfies (4.5.17), since $z(t) = u(t),\ \forall\ t \in [0,t_\delta]$.

For $t \in [t_\delta,\hat{\tau}]$, $z(t) = \mathrm{v}(t)$ and since v is the solution of (4.5.4) on $[t_\delta,\hat{\tau}]$,

$$z(t) = \mathrm{v}(t) = S(t-t_\delta)\mathrm{v}(t_\delta) + \int_{t_\delta}^t S(t-r)G(\mathrm{v}(r))\,dr.$$

Now since $\mathrm{v}(t_\delta) = u(t_\delta)$,

$$z(t) = S(t-t_\delta)\Big[S(t_\delta)u_0 + \int_0^{t_\delta} S(t_\delta - r)G(u(r))\,dr\Big] + \int_{t_\delta}^t S(t-r)G(\mathrm{v}(r))\,dr$$

Now using the definition of z we obtain,

$$\begin{aligned} z(t) &= S(t)z(0) + \int_0^{t_\delta} S(t-r)G(z(r))\,dr + \int_{t_\delta}^t S(t-r)G(z(r))\,dr \\ &= S(t)z(0) + \int_0^t S(t-r)G(z(r))\,dr. \end{aligned}$$

Thus z satisfies (4.5.17) on $[0,\hat{\tau}]$ and hence z is a solution to (4.5.4) on $[0,\hat{\tau}]$.

Hence, $u \preceq z$, but u is the maximal solution and thus we have the contradiction. This implies $\lim_{t\to\tau^-} \|u(t)\|_{\hat{\mathrm{V}}} = \infty$. ■

4.5.3 Global solution: Existence and Uniqueness

In this subsection we will prove the existence of a global solution of (4.5.4) with $u_0 \in \mathcal{M} \cap \hat{\mathrm{V}}$ for both, bounded domain with Dirichlet boundary conditions (see Theorem 4.5.14) and bounded domain with periodic boundary conditions, i.e. on a torus (see Corollary 4.5.15). We use the stitching argument to extend our solution from $[0,T]$, $T < \infty$ on to the whole real line.

Let us recall the functional spaces that we will be using

$$\begin{aligned} \hat{\mathrm{V}} &:= \mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}}), \\ \hat{\mathrm{H}} &:= \mathrm{D}(\mathrm{A}^{\frac{\alpha-1}{2}}), \\ \hat{\mathrm{E}} &:= \mathrm{D}(\mathrm{A}^{\frac{\alpha+1}{2}}). \end{aligned}$$

Lemma 4.5.11. *Let $\alpha \in (1, \frac{3}{2})$, $\theta \in (0,1)$. Then for every $\varepsilon > 0$ there exists a constant $C > 0$, such that*

$$|\langle B(u,u), \mathrm{A}^\alpha u\rangle_{\mathrm{H}}| \le \varepsilon |u|^2_{\hat{\mathrm{E}}} + \frac{C}{\varepsilon}\|u\|^2_{\hat{\mathrm{V}}}\|u\|^{2-2\theta}_{\mathrm{V}}|u|^{2\theta}_{\mathrm{E}}, \qquad u \in \mathrm{D}(\mathrm{A}^\alpha). \tag{4.5.18}$$

Moreover for $\alpha = 2$, for every $\varepsilon > 0$ there exists a constant $\tilde{C} > 0$, such that

$$|\langle B(u,u), \mathrm{A}^2 u\rangle_{\mathrm{H}}| \le \varepsilon |u|^2_{\hat{\mathrm{E}}} + \frac{\tilde{C}}{\varepsilon}|u|^4_{\mathrm{E}}, \qquad u \in \mathrm{D}(\mathrm{A}^2). \tag{4.5.19}$$

Proof. Let $\alpha \in (1,3/2)$ and $u \in \mathrm{D}(\mathrm{A}^\alpha)$, then

$$\begin{aligned} |\langle B(u,u), \mathrm{A}^\alpha u\rangle_{\mathrm{H}}| &= |\langle \mathrm{A}^{\frac{\alpha-1}{2}} B(u,u), \mathrm{A}^{\frac{\alpha+1}{2}} u\rangle_{\mathrm{H}}| \\ &\le |\mathrm{A}^{\frac{\alpha-1}{2}} B(u,u)|_{\mathrm{H}} |\mathrm{A}^{\frac{\alpha+1}{2}} u|_{\mathrm{H}}. \end{aligned}$$

Now using Lemma 4.5.2 with $\gamma = \alpha - 1$, Interpolation Theorem and the Young inequality we get for $u \in \mathrm{D}(\mathrm{A}^\alpha)$

$$\begin{aligned} |\langle B(u,u), \mathrm{A}^\alpha u\rangle_{\mathrm{H}}| &\le C\|u\|_{H^s}\|u\|_{\hat{\mathrm{V}}}|u|_{\hat{\mathrm{E}}} \\ &\le C\|u\|^{1-\theta}_{H^1}\|u\|^{\theta}_{H^2}\|u\|_{\hat{\mathrm{V}}}|u|_{\hat{\mathrm{E}}} \\ &\le \varepsilon |u|^2_{\hat{\mathrm{E}}} + \frac{C}{\varepsilon}\|u\|^2_{\hat{\mathrm{V}}}\|u\|^{2-2\theta}_{\mathrm{V}}|u|^{2\theta}_{\mathrm{E}}, \end{aligned}$$

where $\theta(s) = s - 1 \in (0,1]$.

Let $\alpha = 2$, then for $u \in D(A^2)$ using Cauchy-Schwarz inequality we get

$$|\langle B(u,u), A^2 u\rangle_H| = |\langle A^{\frac{1}{2}} B(u,u), A^{\frac{3}{2}} u\rangle_H \le \|B(u,u)\|_V |u|_{\hat{E}},$$

where $\hat{E} = D(A^{\frac{3}{2}})$. Using (4.5.10) and the Young inequality, we obtain

$$|\langle B(u,u), A^2 u\rangle_H| \le \tilde{C}|u|_E^2 |u|_{\hat{E}} \le \varepsilon |u|_{\hat{E}}^2 + \frac{\tilde{C}}{\varepsilon}|u|_E^4.$$

Now since for $\alpha = 2$, $\hat{V} = E$, (4.5.19) can be rewritten as

$$|\langle B(u,u), A^2 u\rangle_H| \le \varepsilon |u|_{\hat{E}}^2 + \frac{\tilde{C}}{\varepsilon}|u|_{\hat{V}}^4. \tag{4.5.20}$$

■

Lemma 4.5.12. *Let $\alpha \in (1, \frac{3}{2})$ and u be a maximal solution of* (4.5.4) *on* $[0,\tau)$ *with $u_0 \in \mathcal{M} \cap \hat{V}$. If $\varepsilon \in (0,1]$, $\theta \in (0,1)$, then there exists a $C > 0$ such that*

$$\|u(t)\|_{\hat{V}}^2 \le \|u_0\|_{\hat{V}}^2 + \|u_0\|_{\hat{V}}^2 \int_0^t \beta(s) e^{\int_s^t \beta(r)\,dr}\,ds, \quad t \in [0,\tau),$$

where

$$\beta(s) = 2|\nabla u(s)|_{L^2}^2 + \frac{2C}{\varepsilon}\|u(s)\|_{\hat{V}}^{2-2\theta} |u(s)|_{\hat{E}}^{2\theta}, \quad s \in [0,\tau). \tag{4.5.21}$$

Proof. Let $\alpha \in (1, \frac{3}{2})$, $u_0 \in \mathcal{M} \cap V$ and u be the maximal solution of (4.5.4) on $[0,\tau)$. Then using (4.5.4), we have

$$\begin{aligned}
\frac{1}{2}\|u(t)\|_{\hat{V}}^2 &= \frac{1}{2}\|u_0\|_{\hat{V}}^2 + \int_0^t \langle u'(s), A^\alpha u(s)\rangle_H\,ds \\
&= \frac{1}{2}\|u_0\|_{\hat{V}}^2 + \int_0^t \langle |\nabla u(s)|_{L^2}^2 u(s) - Au(s) - B(u(s),u(s)), A^\alpha u(s)\rangle_H\,ds \\
&= \frac{1}{2}\|u_0\|_{\hat{V}}^2 + \int_0^t \langle |\nabla u(s)|_{L^2}^2 u(s), A^\alpha u(s)\rangle_H\,ds - \int_0^t \langle Au(s), A^\alpha u(s)\rangle_H\,ds \\
&\quad - \int_0^t \langle B(u(s),u(s)), A^\alpha u(s)\rangle_H\,ds \\
&= \frac{1}{2}\|u_0\|_{\hat{V}}^2 + \int_0^t |\nabla u(s)|_{L^2}^2 \langle u(s), A^\alpha u(s)\rangle_H\,ds - \int_0^t \langle A^{\frac{\alpha+1}{2}} u(s), A^{\frac{\alpha+1}{2}} u(s)\rangle_H\,ds \\
&\quad - \int_0^t \langle B(u(s),u(s)), A^\alpha u(s)\rangle_H\,ds.
\end{aligned}$$

On rearranging we get

$$\frac{1}{2}\|u(t)\|_{\hat{V}}^2 + \int_0^t |u(s)|_{\hat{E}}^2\,ds = \frac{1}{2}\|u_0\|_{\hat{V}}^2 + \int_0^t |\nabla u(s)|_{L^2}^2 \|u(s)\|_{\hat{V}}^2\,ds - \int_0^t \langle B(u(s),u(s)), A^\alpha u(s)\rangle_H\,ds.$$

Using the Cauchy-Schwarz inequality, the Young inequality and Lemma 4.5.11 for every $\varepsilon \in (0,1]$ there exists a $C > 0$ such that for every $\theta \in (0,1)$

$$\frac{1}{2}\|u(t)\|_{\hat{\mathrm{V}}}^2 + \int_0^t |u(s)|_{\hat{\mathrm{E}}}^2\,ds \le \frac{1}{2}\|u_0\|_{\hat{\mathrm{V}}}^2 + \int_0^t |\nabla u(s)|^2\|u(s)\|_{\hat{\mathrm{V}}}^2\,ds + \int_0^t |\langle B(u(s),u(s)), \mathrm{A}^\alpha u(s)\rangle_{\mathrm{H}}|\,ds$$
$$\le \frac{1}{2}\|u_0\|_{\hat{\mathrm{V}}}^2 + \int_0^t |\nabla u(s)|_{L^2}^2\|u(s)\|_{\hat{\mathrm{V}}}^2\,ds + \int_0^t \left[\varepsilon|u(s)|_{\hat{\mathrm{E}}}^2 + \frac{C}{\varepsilon}\|u(s)\|_{\hat{\mathrm{V}}}^2\|u(s)\|_{\mathrm{V}}^{2-2\theta}|u(s)|_{\mathrm{E}}^{2\theta}\right]ds.$$

Thus on rearranging we get

$$\frac{1}{2}\|u(t)\|_{\hat{\mathrm{V}}}^2 + (1-\varepsilon)\int_0^t |u(s)|_{\hat{\mathrm{E}}}^2\,ds \le \frac{1}{2}\|u_0\|_{\hat{\mathrm{V}}}^2 + \int_0^t \left[|\nabla u(s)|_{L^2}^2 + \frac{C}{\varepsilon}\|u(s)\|_{\mathrm{V}}^{2-2\theta}|u(s)|_{\mathrm{E}}^{2\theta}\right]\|u(s)\|_{\hat{\mathrm{V}}}^2\,ds.$$

In particular

$$\frac{1}{2}\|u(t)\|_{\hat{\mathrm{V}}}^2 \le \frac{1}{2}\|u_0\|_{\hat{\mathrm{V}}}^2 + \int_0^t \left[|\nabla u(s)|_{L^2}^2 + \frac{C}{\varepsilon}\|u(s)\|_{\mathrm{V}}^{2-2\theta}|u(s)|_{\mathrm{E}}^{2\theta}\right]\|u(s)\|_{\hat{\mathrm{V}}}^2\,ds. \tag{4.5.22}$$

Now we apply the Gronwall Lemma in integral form with

$$\beta(s) = 2|\nabla u(s)|_{L^2}^2 + \frac{2C}{\varepsilon}\|u(s)\|_{\mathrm{V}}^{2-2\theta}|u(s)|_{\mathrm{E}}^{2\theta}, \quad s \in [0,t],$$

to obtain

$$\|u(t)\|_{\hat{\mathrm{V}}}^2 \le \|u_0\|_{\hat{\mathrm{V}}}^2 + \|u_0\|_{\hat{\mathrm{V}}}^2\int_0^t \beta(s)e^{\int_s^t \beta(r)\,dr}\,ds. \tag{4.5.23}$$

■

Remark 4.5.13. Now for $\alpha = 2$, using (4.5.19), equation (4.5.22) transforms to

$$\frac{1}{2}|u(t)|_{\mathrm{E}}^2 \le \frac{1}{2}|u_0|_{\mathrm{E}}^2 + \int_0^t \left[|\nabla u(s)|_{L^2}^2 + \frac{C}{\varepsilon}|u(s)|_{\mathrm{E}}^2\right]|u(s)|_{\mathrm{E}}^2\,ds. \tag{4.5.24}$$

Thus, for every $\varepsilon \in (0,1]$, Lemma 4.5.12 holds true for

$$\beta(s) = 2|\nabla u(s)|_{L^2}^2 + \frac{2C}{\varepsilon}|u(s)|_{\mathrm{E}}^2, \quad s \in [0,\tau).$$

Theorem 4.5.14. *Let u be a local maximal solution of* (4.5.4) *on $[0,\tau)$ with $\tau < \infty$, $u_0 \in \mathscr{M} \cap \hat{\mathrm{V}}$. If the function β defined in* (4.5.21) *belongs to $L^1([0,\tau))$, then $\tau = \infty$.*

Proof. We will prove the theorem by contradiction. Assume $\tau < \infty$, then by Lemma 4.5.10

$$\lim_{t\to\tau^-}\|u(t)\|_{\hat{\mathrm{V}}}^2 = \infty.$$

But on the other hand, if $\beta \in L^1([0,\tau))$ then by Lemma 4.5.12 there exists some $K > 0$ such that

$$sup_{t\in[0,\tau)}\|u(t)\|_{\hat{\mathrm{V}}}^2 < K,$$

which is a contradiction. Thus we infer that $\tau = \infty$. ■

Corollary 4.5.15. *Let $\mathscr{O}$ be a bounded periodic domain, $\alpha \in (1, \frac{3}{2}) \cup \{2\}$ and $u_0 \in \mathscr{M} \cap \mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}})$. Then there exists a unique global solution u of* (4.5.4) *such that for every $T > 0$, $u \in X_T$.*

Proof. Let $u_0 \in \mathscr{M} \cap \mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}})$. It is enough to show that for $\tau < \infty$, the function β defined by (4.5.21) belongs to $L^1([0,\tau)$, in case of a periodic bounded domain. Now by Theorem 4.1.1, for every $\tau > 0$, $u \in \mathscr{C}([0,\tau];\mathrm{V}) \cap L^2(0,\tau;\mathrm{E})$ and thus β belongs to $L^1([0,\tau))$. Hence from Theorem 4.5.14, we have a unique global solution of (4.5.4). ■

4.6 Lower bound on the regularity of the initial data

This section is dedicated to finding a lower bound on the regularity of the initial data u_0 such that problem (4.5.4) has a local solution. The analysis carried out in this section is on a formal level and some of the details need to be verified hence, remains an open problem.

In the previous sections we have established the existence of a local solution for any general bounded domain $\mathscr{O} \subset \mathbb{R}^2$ and hence we have focussed on such a case here too. We are interested in the existence of a local solution of the following problem:

$$\begin{cases} \dfrac{du}{dt} + \mathrm{A}u = |\nabla u|^2_{L^2} u - B(u,u), \\ u(0) = u_0 \in \hat{\mathrm{V}} \cap \mathscr{M}, \end{cases} \tag{4.6.1}$$

where the Stokes operator A, the bilinear map B, the manifold $\mathscr{M}$ and the functional space $\hat{\mathrm{V}} = \mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}})$ are understood as in the previous section. After establishing the existence for $\alpha \in [1, \frac{3}{2}) \cup \{2\}$ we are specifically interested in $\alpha \in (0,1)$.

If the initial data $u_0 \in \hat{\mathrm{V}}$, then according to the maximal regularity principle [39], the solution u of the parabolic equation (4.6.1) should belong to $X_T := \mathscr{C}([0,T];\hat{\mathrm{V}}) \cap L^2(0,T;\hat{\mathrm{E}})$, which in turn is possible only if the R.H.S. of equation (4.6.1) belongs to $L^2(0,T;\hat{\mathrm{H}})$.

Since we are interested in finding a lower bound on α we will take a bottom to top approach, i.e. we will assume that u, the solution of (4.6.1) belongs to X_T and then obtain bounds on α for which each term in the R.H.S. of (4.6.1) belongs to $L^2(0,T;\hat{\mathrm{H}})$.

Lemma 4.6.1. *Let $\alpha \in (0,1)$ and $u \in \hat{\mathrm{E}}$. Then, there exists a constant $C > 0$ such that*

$$|B(u,u)|_{\hat{\mathrm{H}}} \le C \|u\|_{\hat{\mathrm{V}}} |u|_{\hat{\mathrm{E}}}. \tag{4.6.2}$$

In particular, if $u \in X_T$ then for every $\alpha \in (0,1)$, $B(u,u) \in L^2(0,T;\hat{\mathrm{H}})$.

Proof. Let $\alpha > 0$ and $\varphi \in H^{1-\alpha}$, then

$$\begin{aligned} |b(u,u,\varphi)| = |\langle B(u,u),\varphi\rangle| &= \left| \int_{\mathscr{O}} (u(x)\cdot\nabla) u(x) \varphi(x)\, dx \right|, \\ &\le |u|_{L^{\frac{2}{1-\alpha}}} |\nabla u|_{L^2} |\varphi|_{L^{\frac{2}{\alpha}}}, \end{aligned} \tag{4.6.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{\alpha-1}$ and $H^{1-\alpha}$ and we have used the Hölder inequality to obtain the last relation.

Using the Sobolev embedding theorem for a bounded domain in $\mathbb{R}^2$, we have

$$H^{1-\alpha} \subset L^{\frac{2}{\alpha}} \quad \text{and} \quad H^{\alpha} \subset L^{\frac{2}{1-\alpha}},$$

where $\alpha \in (0,1)$. Thus (4.6.3) can be rewritten as

$$\begin{aligned} |b(u,u,\varphi)| &\le C\|u\|_{H^\alpha}|\nabla u|_{L^2}|\varphi|_{H^{1-\alpha}} \\ &\le C\|u\|_{H^\alpha}\|u\|_{\mathrm{V}}|\varphi|_{H^{1-\alpha}}, \end{aligned}$$

where $C>0$ is a generic constant. Since $\varphi \in H^{1-\alpha}$ and $\hat{\mathrm{E}} \subset \mathrm{V}$, we have

$$|B(u,u)|_{\hat{\mathrm{H}}} \le C\|u\|_{\hat{\mathrm{V}}}|u|_{\hat{\mathrm{E}}}, \tag{4.6.4}$$

where we have used the definition of functional spaces from (4.5.3).

Using (4.6.4) and the Hölder inequality, we get

$$\begin{aligned} \int_0^T |B(u(t),u(t))|^2_{\hat{\mathrm{H}}}\,dt &\le C\int_0^T |u(t)|^2_{\hat{\mathrm{E}}}\|u(t)\|^2_{\hat{\mathrm{V}}}\,dt \\ &\le C \sup_{t\in[0,T]} \|u(t)\|^2_{\hat{\mathrm{V}}}\int_0^T |u(t)|^2_{\hat{\mathrm{E}}}\,dt \\ &\le C|u|^2_{X_T}|u|^2_{X_T}. \end{aligned}$$

Since $u \in X_T$, we infer that $B(u,u) \in L^2(0,T;\hat{\mathrm{H}})$. ∎

Lemma 4.6.2. *Let $u \in X_T$, then for $\alpha \in \left[\frac{1}{2},1\right)$, $|\nabla u|^2_{L^2}u \in L^2(0,T;\hat{\mathrm{H}})$.*

Proof. Let $u \in X_T$. Note that for $\alpha \in (0,1)$ we have the following inclusion of functional spaces $\hat{\mathrm{V}} \subset \mathrm{H} \subset \hat{\mathrm{H}}$. Then by the Hölder inequality, we have

$$\begin{aligned} \Big||\nabla u|^2_{L^2}u\Big|^2_{L^2(0,T;\hat{H})} &= \int_0^T |\nabla u(t)|^4_{L^2}|u(t)|^2_{\hat{\mathrm{H}}}\,dt \\ &\le \sup_{t\in[0,T]} \|u(t)\|^2_{\hat{\mathrm{V}}}\int_0^T |\nabla u(t)|^4_{L^2}\,dt. \end{aligned} \tag{4.6.5}$$

Using the interpolation between $\hat{\mathrm{V}}$ and $\hat{\mathrm{H}}$, for every $\mathrm{v} \in \hat{\mathrm{V}}$ there exists a constant $C>0$ such that

$$|\mathrm{v}|_{\mathrm{H}} \le C|\mathrm{v}|^{\alpha}_{\hat{\mathrm{H}}}\|\mathrm{v}\|^{1-\alpha}_{\hat{\mathrm{V}}}.$$

Therefore on using the above relation in (4.6.5), we obtain

$$\begin{aligned} &\Big||\nabla u|^2_{L^2}u\Big|^2_{L^2(0,T;\hat{H})} \le C\sup_{t\in[0,T]}\|u(t)\|^2_{\hat{\mathrm{V}}}\int_0^T |\nabla u(t)|^{4\alpha}_{\hat{\mathrm{H}}}\|\nabla u(t)\|^{4(1-\alpha)}_{\hat{\mathrm{V}}}\,dt \\ &\le C|u|^2_{X_T}\int_0^T \|u(t)\|^{4\alpha}_{\hat{\mathrm{V}}}|u(t)|^{4(1-\alpha)}_{\hat{\mathrm{E}}}\,dt \le C|u|^2_{X_T}\sup_{t\in[0,T]}\|u(t)\|^{4\alpha}_{\hat{\mathrm{V}}}\int_0^T |u(t)|^{4(1-\alpha)}_{\hat{\mathrm{E}}}\,dt \\ &\le C|u|^2_{X_T}|u|^{4\alpha}_{X_T}\left[\int_0^T |u(t)|^2_{\hat{\mathrm{E}}}\,dt\right]^{2(1-\alpha)}\left[\int_0^T dt\right]^{2\alpha-1} \le C|u|^{2(1+2\alpha)}_{X_T}|u|^{4(1-\alpha)}_{X_T}T^{2\alpha-1}. \end{aligned}$$

Thus, we infer that for $\alpha \in \left[\frac{1}{2},1\right)$, $|\nabla u|^2_{L^2}u \in L^2(0,T;\hat{\mathrm{H}})$. ∎

Hence, from Lemmas 4.6.1 - 4.6.2 and maximal regularity principle we can conclude that the minimum regularity required for initial data u_0 such that the problem (4.6.1) has a local solution is $D(A^{1/4})$.

As mentioned at the beginning of this section, the analysis carried out here is at a formal level, thus one needs to make sure that this holds rigorously too.

CHAPTER 5

STOCHASTIC CONSTRAINED NAVIER-STOKES EQUATIONS

Stochastic constrained Navier-Stokes equations are a generalisation of constrained Navier-Stokes equations which were introduced in Chapter 3. In this chapter we study constrained 2-dimensional Navier-Stokes equations driven by a multiplicative Gaussian noise in the Stratonovich form. In the deterministic case (see Chapter 4) we showed the existence of the global solution on a two dimensional torus and hence we concentrate on such a case here. We prove the existence of a martingale solution and later using Schmalfuss idea [82] we show the pathwise uniqueness of the solutions. We also establish the existence of a strong solution using results from Ondreját [68].

5.1 Introduction

In this chapter we consider the stochastic Navier-Stokes equations

$$du + [(u\cdot\nabla)u - \nu\Delta u + \nabla p]\,dt = \sum_{j=1}^{m}(c_j\cdot\nabla)u \circ dW_j(t), \quad t\in[0,\infty) \tag{5.1.1}$$

in $\mathcal{O} = [0,2\pi]^2$ with periodic boundary conditions and with the incompressibility condition

$$\operatorname{div} u = 0.$$

This problem can be seen as a problem on a two-dimensional torus $\mathbb{T}^2$ what we will assume to be our case. Here $u:[0,\infty)\times\mathcal{O}\to\mathbb{R}^2$ and $p:[0,\infty)\times\mathcal{O}\to\mathbb{R}$ represent the velocity and the pressure of the fluid. Furthermore $\sum_{j=1}^{m}(c_j\cdot\nabla)u\circ dW_j(t)$ stands for the random forcing, where c_j, $j=1,\cdots,m$, are divergence free $\mathbb{R}^2$-valued vectors (so that the corresponding transport operators $\tilde{C}_j u := (c_j\cdot\nabla)u$ are skew symmetric in $L^2(\mathbb{T}^2,\mathbb{R}^2)$) and W_j, $j=1,\dots,m$ are independent $\mathbb{R}-$valued standard Brownian Motions.

The above problem projected on $\mathrm{H}\cap\mathscr{M}$ can be written in an abstract form as the following initial value problem

$$\begin{cases} du(t)+\nu \mathrm{A}u(t)\,dt+B(u(t))\,dt=\nu|\nabla u(t)|^2_{L^2}u(t)\,dt+\sum_{j=1}^m C_j u(t)\circ dW_j(t),\ t\in[0,T], \\ u(0)=u_0, \end{cases} \tag{5.1.2}$$

where H is the space of square integrable, divergence free and mean zero vector fields on $\mathscr{O}$ and

$$\mathscr{M}=\{u\in \mathrm{H}:|u|_{\mathrm{H}}=1\}.$$

Here A and B are appropriate maps corresponding to the Laplacian and the nonlinear term, respectively in the Navier-Stokes equations, see Chapter 3 and $C_j=\Pi(\tilde{C}_j)$, where $\Pi:L^2(\mathbb{T}^2,\mathbb{R}^2)\to\mathrm{H}$ is the Leray-Helmholtz projection operator [88] that projects the square integrable vector fields onto the divergence free vector field.

We prove the existence and uniqueness of a strong solution. The construction of a solution is based on the classical Faedo-Galerkin approximation, i.e.

$$\begin{cases} du_n(t)=-\left[P_n\mathrm{A}u_n(t)+P_nB(u_n(t))-|\nabla u_n(t)|^2_{L^2}u_n(t)\right]dt \\ \qquad\qquad +\sum_{j=1}^m P_nC_ju_n(t)\circ dW_j(t),\quad t\in[0,T], \\ u_n(0)=\dfrac{P_nu_0}{|P_nu_0|} \end{cases} \tag{5.1.3}$$

given in Section 5.5. Let us point out that without the normalisation of the initial condition in the above problem (5.1.3), the solution may not be a global one, even in the deterministic case. The crucial point is to prove suitable uniform a'priori estimates on the sequence u_n. We will prove that the following estimates hold

$$\sup_{n\ge1}\mathbb{E}\left[\int_0^T|u_n(s)|^2_{\mathrm{D(A)}}\,ds\right]<\infty,$$

and

$$\sup_{n\ge1}\mathbb{E}\left(\sup_{0\le s\le T}\|u_n(s)\|^{2p}_{\mathrm{V}}\right)<\infty,$$

for $p\in[1,1+\frac{1}{K_c^2})$, where D(A) is the domain of the Stokes operator and $\mathrm{V}=\mathrm{D}(\mathrm{A}^{1/2})$, see Section 3.1 for precise definitions and the positive constant K_c is defined in (5.3.1).

In Theorem 5.3.3 we prove the existence of a martingale solution using the tightness criterion in the topological space $\mathcal{Z}_T=\mathscr{C}([0,T];\mathrm{H})\cap L^2_{\mathrm{w}}(0,T;D(\mathrm{A}))\cap L^2(0,T;\mathrm{V})\cap\mathscr{C}([0,T];\mathrm{V}_{\mathrm{w}})$ showing that the trajectories of the solution lie in $\mathscr{C}([0,T];\mathrm{V}_{\mathrm{w}})$ but later on in Lemma 5.3.5 we show that in fact the trajectories lie in $\mathscr{C}([0,T];\mathrm{V})$.

This chapter is an extension of Chapter 4 from the deterministic to a stochastic setting. More information and motivation can also be found therein. Let us recall that already in the deterministic setting, we have been able to prove the global existence of solutions for CNSE only on a bounded domain with periodic boundary conditions and this is why we have concentrated here on such a case. A similar problem for stochastic heat equation with polynomial drift but with a different type of noise has recently been a subject of a PhD thesis by Javed Hussain [48]. It's remarkable that in that case the result holds for Dirichlet boundary conditions as well.

We consider the noise of gradient type in the Stratonovich form (5.1.1). The structure of noise is such that it is tangent to the manifold $\mathcal{M}$ just like the non-linear part from Navier-Stokes and hence, there is no contribution to the equation (5.1.2) because of the constraint. In the deterministic setting (see Chapter 4) we proved the existence of a global solution by proving the existence of a local solution using the Banach Fixed Point Theorem and no explosion principle, i.e enstrophy (V-norm) of the solution remains bounded. We can't take the similar approach in the stochastic setting as one can't prove the existence of a local solution using the Banach Fixed Point Theorem and hence we switch to more classical approach of proving the existence of a solution using the Faedo-Galerkin approximation.

We consider the Faedo-Galerkin approximation (5.1.3) of (5.1.2). We prove that each approximating equation has a global solution. One can show that for every $n \in \mathbb{N}$ global solution to (5.1.3) exist for all domains, in particular for Dirichlet boundary conditions. But in order to obtain a'priori estimates, Lemma 5.5.4, we need to consider the Navier-Stokes Equations (NSE) on a two dimensional torus $\mathbb{T}^2$ (i.e. the NSEs with periodic boundary conditions).

In order to prove that the laws of the solution of these approximating equations are tight on $\mathcal{Z}_T$ (defined in (5.4.3)), apart from a'priori estimates we also need the Aldous condition, Definition 2.9.10. After proving that the laws are tight in Lemma 5.5.5, by the application of Jakubowski-Skorohod Theorem and the martingale representation theorem we prove Theorem 5.3.3. The chapter is organised in the following way:

Stochastic constrained Navier-Stokes equations (SCNSE) are introduced in Section 5.2. The definitions of a martingale solution and strong solution and all the important results of this chapter are given in Section 5.3. Section 5.4 contains the well-known and already established results regarding compactness. In Section 5.5 we establish certain estimates on the way to prove Theorem 5.3.3. Existence and uniqueness of a strong solution using the results from Ondreját [68] is proved in Section 5.6. In Section 5.7, we prove the continuous dependence of the solution of (5.1.2) on the initial data. We conclude the chapter by showing that the semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{B}_b(\mathrm{V})$ (defined by (5.8.1)) generated by the solution of SCNSE are sequentially weakly Feller in V.

5.2 Stochastic constrained Navier-Stokes equations

The 2D Navier-Stokes equations driven by multiplicative Gaussian noise in the Stratonovich form are given by:

$$\begin{cases} du(t) + [(u(t)\cdot\nabla)u(t) - \nu\Delta u(t) + \nabla p(t)]\,dt = \sum_{j=1}^m \left[\left(c_j\cdot\nabla\right)u(t)\right]\circ dW_j(t), \quad t\in[0,\infty), \\ \operatorname{div} u(\cdot,t) = 0,\ t\in[0,\infty), \\ u(x,0) = u_0(x),\ x\in\mathscr{O}, \end{cases} \tag{5.2.1}$$

where $u:[0,\infty)\times\mathscr{O}\to\mathbb{R}^2$ and $p:[0,\infty)\times\mathscr{O}\to\mathbb{R}$ are velocity and pressure of the fluid respectively. ν is the viscosity of the fluid (with no loss of generality, ν will be taken equal to 1 for the rest of the article). Here we assume that c_j are divergence free $\mathbb{R}^2$-valued vectors, W_j are $\mathbb{R}-$valued i.i.d. standard Brownian motions and $\circ$ denotes the Stratonovich form. Note that the operators $\tilde{C}_j$, $j\in\{1,\dots,m\}$, defined by $\tilde{C}_j u := \left(c_j\cdot\nabla\right)u$, for $u\in \mathrm{V}$ are skew-symmetric on $L^2(\mathbb{T}^2,\mathbb{R}^2)$, i.e. $\tilde{C}_j^* = -\tilde{C}_j$, where $\tilde{C}_j^*$ denotes the adjoint of $\tilde{C}_j$ on $L^2(\mathbb{T}^2,\mathbb{R}^2)$.

We will be frequently using the following short-cut notation

$$Cu\circ dW(t) = \sum_{j=1}^m C_j u(t)\circ dW_j(t),$$

where $C_j = \Pi(\tilde{C}_j)$ and Π is the Leray-Helmholtz projection operator, as introduced in Chapter 3.

With all the notations as defined above and in Chapter 3, the stochastic Navier-Stokes equations (5.2.1) projected on divergence free vector field is given by

$$\begin{cases} du(t) + [\mathrm{A}u(t) + B(u(t))]\,dt = Cu(t)\circ dW(t), \\ u(0) = u_0. \end{cases} \tag{5.2.2}$$

Let us recall the orthogonal projection map $\pi_u : \mathrm{H}\to T_u\mathscr{M}$, which is given by

$$\pi_u(\mathrm{v}) = \mathrm{v} - \langle \mathrm{v}, u\rangle_{\mathrm{H}}\, u, \qquad \text{for } u\in\mathscr{M},$$

where $T_u\mathscr{M}$ is the tangent space corresponding to the manifold $\mathscr{M}$ as introduced earlier in Chapter 3.

Since for every $j\in\{1,\dots,m\}$, $C_j^* = -C_j$ in H we infer that

$$\langle C_j u, u\rangle_{\mathrm{H}} = 0, \quad u\in\mathrm{V},\ \ j\in\{1,\dots,m\}. \tag{5.2.3}$$

In particular, if $u\in \mathrm{V}\cap\mathscr{M}$, then $C_j u\in T_u\mathscr{M}$ for every $j\in\{1,\dots,m\}$ and hence won't produce any correction terms when projected onto the tangent space $T_u\mathscr{M}$, which is shown explicitly below.

Let

$$F(u) = \mathrm{A}u + B(u,u) - Cu\circ dW(t)$$

and $\hat{F}(u)$ be the projection of $F(u)$ onto the tangent space $T_u\mathcal{M}$, then

$$\begin{aligned}\hat{F}(u) &= \pi_u(F(u)) = F(u) - \langle F(u), u\rangle_{\mathrm{H}}\, u \\ &= \mathrm{A}u + B(u) - Cu \circ dW - \langle \mathrm{A}u + B(u) - Cu \circ dW, u\rangle_{\mathrm{H}}\, u \\ &= \mathrm{A}u - \langle \mathrm{A}u, u\rangle_{\mathrm{H}}\, u + B(u) - \langle B(u), u\rangle_{\mathrm{H}}\, u - Cu \circ dW + \langle Cu, u\rangle_{\mathrm{H}} u \circ dW \\ &= \mathrm{A}u - |\nabla u|^2_{L^2}\, u + B(u) - Cu \circ dW.\end{aligned}$$

The last equality follows from (5.2.3) and the identity that $\langle B(u), u\rangle_{\mathrm{H}} = 0$.

Thus by projecting NSEs (5.2.2) onto the tangent space $T_u\mathcal{M}$, we obtain the following stochastic constrained Navier-Stokes equations (SCNSE)

$$\begin{cases} du(t) + [\mathrm{A}u(t) + B(u(t))]\,dt = |\nabla u(t)|^2_{L^2} u(t)\,dt + Cu(t) \circ dW(t), \\ u(0) = u_0. \end{cases} \tag{5.2.4}$$

5.3 Assumptions, definitions and results

From now on we will assume that c_j are constant vector fields. Whether our results are true in a more general setting is an open problem.

Assumptions. *We assume that*

(A.1) *Vectors $c_1, \dots, c_m$ belong to $\mathbb{R}^2$ such that $K_c^2 < 1$ where*

$$K_c := \max_{j\in\{1,\cdots,m\}} |c_j|_{\mathbb{R}^2}, \tag{5.3.1}$$

$|\cdot|_{\mathbb{R}^2}$ is the Euclidean norm in $\mathbb{R}^2$.

(A.2) *$u_0 \in \mathrm{V} \cap \mathcal{M}$.*

Definition 5.3.1. We say that problem (5.2.4) has a strong solution iff for every stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ and every $\mathbb{R}^m-$ valued $\mathbb{F}-$Wiener process $W = (W(t))_{t\geq 0}$, there exists a $\mathbb{F}-$progressively measurable process $u : [0,T] \times \Omega \to \mathrm{D(A)}$ with $\mathbb{P}$-a.e. paths

$$u(\cdot, \omega) \in \mathscr{C}([0,T]; \mathrm{V}) \cap L^2(0,T; \mathrm{D(A)}),$$

such that for all $t \in [0,T]$ and all $\mathrm{v} \in \mathrm{V}$ $\mathbb{P}$-a.s.

$$\begin{aligned}&\langle u(t), \mathrm{v}\rangle - \langle u_0, \mathrm{v}\rangle + \int_0^t \langle \mathrm{A}u(s), \mathrm{v}\rangle\, ds + \int_0^t \langle B(u(s)), \mathrm{v}\rangle\, ds \\ &= \int_0^t |\nabla u(s)|^2_{L^2} \langle u(s), \mathrm{v}\rangle\, ds + \frac{1}{2}\int_0^t \sum_{j=1}^m \langle C_j^2 u(s), \mathrm{v}\rangle\, ds + \int_0^t \sum_{j=1}^m \langle C_j u(s), \mathrm{v}\rangle\, d\hat{W}_j(s).\end{aligned} \tag{5.3.2}$$

Definition 5.3.2. We say that there exists a martingale solution of (5.2.4) iff there exist

- a stochastic basis $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$,

- an $\mathbb{R}^m$−valued $\hat{\mathbb{F}}$−Wiener process $\hat{W}$,
- and a $\hat{\mathbb{F}}$−progressively measurable process $u : [0,T] \times \hat{\Omega} \to \mathrm{D(A)}$ with $\hat{\mathbb{P}}$-a.e. paths

$$u(\cdot,\omega) \in \mathscr{C}([0,T];\mathrm{V}_w) \cap L^2(0,T;\mathrm{D(A)}),$$

such that for all $t \in [0,T]$ and all $\mathrm{v} \in \mathrm{V}$ the identity (5.3.2) holds $\hat{\mathbb{P}}$-a.s.

Next we state some important results of this chapter which will be proved in further sections.

Theorem 5.3.3. *Let assumptions **(A.1) - (A.2)** be satisfied. Then there exists a martingale solution* $(\hat{\Omega},\hat{\mathscr{F}},\hat{\mathbb{F}},\hat{\mathbb{P}},\hat{W},u)$ *of problem* (5.2.4) *such that*

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} \|u(t)\|^2_{\mathrm{V}} + \int_0^T |u(t)|^2_{\mathrm{D(A)}}\,dt\right] < \infty. \tag{5.3.3}$$

Remark 5.3.4. The solution obtained in the above theorem is weak in probabilistic sense and strong in PDE sense.

The next lemma shows that almost all the trajectories of the solution obtained in Theorem 5.3.3 are almost everywhere equal to a continuous V-valued function defined on $[0,T]$.

Lemma 5.3.5. *Assume that the assumptions **(A.1) - (A.2)** are satisfied. Let* $(\hat{\Omega},\hat{\mathscr{F}},\hat{\mathbb{F}},\hat{\mathbb{P}},\hat{W},u)$ *be a martingale solution of* (5.2.4) *such that*

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} \|u(t)\|^2_{\mathrm{V}} + \int_0^T |u(s)|^2_{\mathrm{D(A)}}\,ds\right] < \infty. \tag{5.3.4}$$

Then for $\hat{\mathbb{P}}$ almost all $\omega \in \hat{\Omega}$ the trajectory $u(\cdot,\omega)$ is almost everywhere equal to a continuous V−*valued function defined on* $[0,T]$*. Moreover, the following equality in* H *holds for every $t \in [0,T]$,* $\hat{\mathbb{P}}$−*a.s.*

$$\begin{aligned} u(t) = u_0 &- \int_0^t \left[\mathrm{A}u(s) + B(u(s)) - |\nabla u(s)|^2_{L^2}\, u(s)\right] ds \\ &+ \frac{1}{2}\int_0^t \sum_{j=1}^m C_j^2 u(s)\,ds + \int_0^t \sum_{j=1}^m C_j u(s)\,d\hat{W}(s). \end{aligned} \tag{5.3.5}$$

Definition 5.3.6. Let $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P},W,u^i)$, $i = 1,2$ be the martingale solutions of (5.2.4) with $u^i(0) = u_0$, $i = 1,2$. Then we say that the solutions are pathwise unique if for all $t \in [0,T]$, $\mathbb{P}$−a.s. $u^1(t) = u^2(t)$.

In Lemma 5.6.1 we will show that the pathwise uniqueness property for our problem holds. This will enable us to deduce the following theorem that summarises the main result of this chapter:

Theorem 5.3.7. *For every $u_0 \in \mathrm{V}$ there exists a pathwise unique strong solution u of stochastic constrained Navier-Stokes equations* (5.2.4) *such that*

$$\mathbb{E}\left[\int_0^T |u(t)|^2_{\mathrm{D(A)}}\,dt + \sup_{t\in[0,T]} \|u(t)\|^2_{\mathrm{V}}\right] < \infty. \tag{5.3.6}$$

Remark 5.3.8. The solution of (5.2.4) obtained in previous theorem is strong in both probabilistic and PDE sense.

5.4 Compactness

Let us consider the following functional spaces for fixed $T > 0$:
$\mathscr{C}([0,T];\mathrm{H}) :=$ the space of continuous functions $u : [0,T] \to \mathrm{H}$ with the topology $\mathscr{T}_1$ induced by the norm $|u|_{\mathscr{C}([0,T];\mathrm{H})} := \sup_{t\in[0,T]} |u(t)|_{\mathrm{H}}$,
$L^2_{\mathrm{w}}(0,T;\mathrm{D(A)}) :=$ the space $L^2(0,T;\mathrm{D(A)})$ with the weak topology $\mathscr{T}_2$,
$L^2(0,T;\mathrm{V}) :=$ the space of measurable functions $u : [0,T] \to \mathrm{V}$ such that

$$|u|_{L^2(0,T;\mathrm{V})} = \left(\int_0^T \|u(t)\|^2_{\mathrm{V}}\,dt\right)^{\frac{1}{2}} < \infty,$$

with the topology $\mathscr{T}_3$ induced by the norm $|u|_{L^2(0,T;\mathrm{V})}$.
Let V_w denote the Hilbert space V endowed with the weak topology.
$\mathscr{C}([0,T];\mathrm{V}_w) :=$ the space of weakly continuous functions $u : [0,T] \to \mathrm{V}$ endowed with the weakest topology $\mathscr{T}_4$ such that for all $h \in \mathrm{V}$ the mappings

$$\mathscr{C}([0,T];\mathrm{V}_w) \ni u \to \langle u(\cdot), h\rangle_{\mathrm{V}} \in \mathscr{C}([0,T];\mathbb{R})$$

are continuous. In particular, $u_n \to u$ in $\mathscr{C}([0,T];\mathrm{V}_w)$ iff for all $h \in \mathrm{V}$:

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\langle u_n(t) - u(t), h\rangle_{\mathrm{V}}| = 0.$$

Consider the ball

$$\mathbb{B} := \{x \in \mathrm{V} : \|x\|_{\mathrm{V}} \le r\}.$$

Let q be the metric compatible with the weak topology on $\mathbb{B}$. Let us consider the following subspace of the space $\mathscr{C}([0,T];\mathrm{V}_w)$

$$\begin{aligned} \mathscr{C}([0,T];\mathbb{B}_w) = &\text{ the space of weakly continuous functions } u : [0,T] \to \mathrm{V} \\ &\text{such that } \sup_{t\in[0,T]} \|u(t)\|_{\mathrm{V}} \le r. \end{aligned} \tag{5.4.1}$$

The space $\mathscr{C}([0,T];\mathbb{B}_w)$ is metrizable (see [9, 23]) with metric

$$\varrho(u,\mathrm{v}) = \sup_{t\in[0,T]} q(u(t),\mathrm{v}(t)). \tag{5.4.2}$$

Since by the Banach-Alaoglu theorem [80] $\mathbb{B}_w$ is compact, $(\mathscr{C}([0,T];\mathbb{B}_w),\varrho)$ is a complete metric space.

The following lemma [24, Lemma 2.1] says that any sequence $(u_n)_{n\in\mathbb{N}}\subset\mathscr{C}([0,T];\mathbb{B})$ convergent in $\mathscr{C}([0,T];\mathrm{H})$ is also convergent in the space $\mathscr{C}([0,T];\mathbb{B}_w)$.

Lemma 5.4.1. *Let $u_n:[0,T]\to\mathrm{V}, n\in\mathbb{N}$ be functions such that*

(i) $\sup_{n\in\mathbb{N}}\sup_{s\in[0,T]}\|u_n(s)\|_{\mathrm{V}}\le r$,

(ii) $u_n\to u$ *in* $\mathscr{C}([0,T];\mathrm{H})$.

Then $u,u_n\in\mathscr{C}([0,T];\mathbb{B}_w)$ and $u_n\to u$ in $\mathscr{C}([0,T];\mathbb{B}_w)$ as $n\to\infty$.

Let

$$\mathcal{Z}_T=\mathscr{C}([0,T];\mathrm{H})\cap L^2_{\mathrm{w}}(0,T;\mathrm{D(A)})\cap L^2(0,T;\mathrm{V})\cap\mathscr{C}([0,T];\mathrm{V}_w),\tag{5.4.3}$$

and let $\mathscr{T}$ be the supremum of the corresponding topologies.

Now we formulate the compactness criterion analogous to the result due to Mikulevicus and Rozowskii [65], Brzeźniak and Motyl [24] for the space $\mathcal{Z}_T$.

Lemma 5.4.2. *Let $\mathcal{Z}_T$, $\mathscr{T}$ be as defined in* (5.4.3). *Then a set $\mathscr{K}\subset\mathcal{Z}_T$ is $\mathscr{T}-$relatively compact if the following three conditions hold*

(a) $\sup_{u\in\mathscr{K}}\sup_{s\in[0,T]}\|u(s)\|_{\mathrm{V}}<\infty$,

(b) $\sup_{u\in\mathscr{K}}\int_0^T|u(s)|^2_{\mathrm{D(A)}}\,ds<\infty$, *i.e. $\mathscr{K}$ is bounded in $L^2(0,T;\mathrm{D(A)})$,*

(c) $\lim_{\delta\to0}\sup_{u\in\mathscr{K}}\sup_{\substack{s,t\in[0,T]\\|t-s|\le\delta}}|u(t)-u(s)|_{\mathrm{H}}=0$.

Proof. Let $\mathscr{K}$ be a subset of $\mathcal{Z}_T$. Because of the assumption (a) we may consider the metric space $\mathscr{C}([0,T];\mathbb{B}_w)\subset\mathscr{C}([0,T];\mathrm{V}_w)$ defined by (5.4.1) and (5.4.2) with $r=\sup_{u\in\mathscr{K}}\sup_{s\in[0,T]}\|u(s)\|_{\mathrm{V}}$. Because of the assumption (b) the restriction to $\mathscr{K}$ of the weak topology in $L^2(0,T;\mathrm{D(A)})$ is metrizable. Since the restrictions to $\mathscr{K}$ of the four topologies considered in $\mathcal{Z}_T$ are metrizable, compactness of a subset of $\mathcal{Z}_T$ is equivalent to its sequential compactness.

Let (u_n) be a sequence in $\mathscr{K}$. By the Banach-Alaoglu Theorem [80], condition (b) yields that $\bar{\mathscr{K}}$ is compact in $L^2_w(0,T;\mathrm{D(A)})$. Condition (c) implies that the functions u_n are equicontinuous in $\mathscr{C}([0,T],\mathrm{H})$. Since the embeddings $\mathrm{D(A)}\hookrightarrow\mathrm{V}\hookrightarrow\mathrm{H}$ are continuous and the embedding $\mathrm{D(A)}\hookrightarrow\mathrm{V}$ is compact, then Dubinsky Theorem (see Theorem 2.4.5) with conditions (b) and (c) imply that $\mathscr{K}$ is compact in $L^2(0,T;\mathrm{V})\cap\mathscr{C}([0,T];\mathrm{H})$. Hence in particular, there exists a subsequence, still denoted by (u_n), convergent in H. Therefore by Lemma 5.4.1 (u_n) is convergent in $\mathscr{C}([0,T];\mathbb{B}_w)$. This completes the proof of the lemma. ■

5.4.1 Tightness

Using Section 2.9 and the compactness criterion from Lemma 5.4.2 we obtain the following corollary which we will use to prove tightness of the laws defined by the Galerkin approximations.

Corollary 5.4.3 (Tightness criterion). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous $\mathbb{F}$-adapted* H-*valued processes such that*

(a) there exists a constant $C_1 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{s\in[0,T]} \|X_n(s)\|_{\mathrm{V}}^2\right] \le C_1,$$

(b) there exists a constant $C_2 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T |X_n(s)|_{\mathrm{D(A)}}^2\,ds\right] \le C_2,$$

(c) $(X_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition **[A]** *in* H.

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on $\mathcal{Z}_T$. Then for every $\varepsilon > 0$ there exists a compact subset K_ε of $\mathcal{Z}_T$ such that

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n(K_\varepsilon) \ge 1-\varepsilon.$$

Proof. Let $\varepsilon > 0$. By the Chebyshev inequality and (a), we infer that for any $n\in\mathbb{N}$ and any $r > 0$

$$\tilde{\mathbb{P}}_n\Big(\sup_{s\in[0,T]} \|X_n(s)\|_{\mathrm{V}}^2 > r\Big) \le \frac{\tilde{\mathbb{E}}_n\big[\sup_{s\in[0,T]} \|X_n(s)\|_{\mathrm{V}}^2\big]}{r} \le \frac{C_1}{r}.$$

Let R_1 be such that $\frac{C_1}{R_1} \le \frac{\varepsilon}{3}$. Then

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n\Big(\sup_{s\in[0,T]} \|X_n(s)\|_{\mathrm{V}}^2 > R_1\Big) \le \frac{\varepsilon}{3}.$$

Let $B_1 := \{u\in\mathcal{Z}_T : \sup_{s\in[0,T]} \|u(s)\|_{\mathrm{V}}^2 \le R_1\}$.

By the Chebyshev inequality and (b), we infer that for any $n\in\mathbb{N}$ and any $r > 0$

$$\tilde{\mathbb{P}}_n\big(|X_n|_{L^2(0,T;\mathrm{D(A)})} > r\big) \le \frac{\tilde{\mathbb{E}}_n\big[|X_n|_{L^2(0,T;\mathrm{D(A)})}^2\big]}{r^2} \le \frac{C_2}{r^2}.$$

Let R_2 be such that $\frac{C_2}{R_2^2} \le \frac{\varepsilon}{3}$. Then

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n\big(|X_n|_{L^2(0,T;\mathrm{D(A)})} > R_2\big) \le \frac{\varepsilon}{3}.$$

Let $B_2 := \{u\in\mathcal{Z}_T : |u|_{L^2(0,T;\mathrm{D(A)})} \le R_2\}$.

By Lemmas 2.9.9 and 2.9.11 there exists a subset $A_{\frac{\varepsilon}{3}} \subset \mathscr{C}([0,T],\mathrm{H})$ such that $\tilde{\mathbb{P}}_n\big(A_{\frac{\varepsilon}{3}}\big) \ge 1-\frac{\varepsilon}{3}$ and

$$\lim_{\delta\to 0} \sup_{u\in A_{\frac{\varepsilon}{3}}} \sup_{\substack{s,t\in[0,T]\\ |t-s|\le\delta}} |u(t)-u(s)|_{\mathrm{H}} = 0.$$

It is sufficient to define K_ε as the closure of the set $B_1\cap B_2\cap A_{\frac{\varepsilon}{3}}$ in $\mathcal{Z}_T$. By Lemma 5.4.2, K_ε is compact in $\mathcal{Z}_T$. The proof is thus complete. ■

5.4.2 The Skorohod Theorem

Let us recall the Jakubowski's generalisation of the Skorohod Theorem as given by Brzeźniak and Ondreját [26, Theorem C.1], see also [49].

Theorem 5.4.4. *Let $\mathscr{X}$ be a topological space such that there exists a sequence $\{f_m\}_{m\in\mathbb{N}}$ of continuous functions $f_m : \mathscr{X} \to \mathbb{R}$ that separates points of $\mathscr{X}$. Let us denote by $\mathscr{S}$ the σ-algebra generated by the maps $\{f_m\}$. Then*

(a) every compact subset of $\mathscr{X}$ is metrizable,

(b) if $(\mu_m)_{m\in\mathbb{N}}$ is a tight sequence of probability measures on $(\mathscr{X},\mathscr{S})$, then there exists a subsequence $(m_k)_{k\in\mathbb{N}}$, a probability space $(\Omega,\mathscr{F},\mathbb{P})$ with $\mathscr{X}$-valued Borel measurable variables ξ_k,ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω. Moreover, the law of ξ is a Radon measure.

Lemma 5.4.5. *The topological space $\mathcal{Z}_T$ satisfies the assumptions of Theorem 5.4.4.*

Proof. We want to prove that on each space appearing in the definition (5.4.3) of the space $\mathcal{Z}_T$ there exists a countable set of continuous real-valued functions separating points.

Since the spaces $\mathscr{C}([0,T];\mathrm{H})$ and $L^2(0,T;\mathrm{V})$ are separable, metrizable and complete, this condition is satisfied, see [3], exposé 8.

For the space $L^2_w(0,T;\mathrm{D(A)})$ it is sufficient to put

$$f_m(u) := \int_0^T \langle u(t), \mathrm{v}_m(t)\rangle_{\mathrm{D(A)}}\, dt \in \mathbb{R}, \qquad u \in L^2_w(0,T;\mathrm{D(A)}), \quad m \in \mathbb{N},$$

where $\{\mathrm{v}_m, m\in\mathbb{N}\}$ is a dense subset of $L^2(0,T;\mathrm{D(A)})$.

Let us consider the space $\mathscr{C}([0,T];\mathrm{V}_w)$. Let $\{h_m,\, m\in\mathbb{N}\}$ be any dense subset of H and let $\mathbb{Q}_T$ be the set of rational numbers belonging to the interval $[0,T]$. Then the family $\{f_{m,t},\, m\in\mathbb{N},\, t\in\mathbb{Q}_T\}$ defined by

$$f_{m,t}(u) := \langle u(t), h_m\rangle_{\mathrm{V}} \in \mathbb{R}, \qquad u \in \mathscr{C}([0,T];\mathrm{V}_w), \quad m\in\mathbb{N}, \quad t\in\mathbb{Q}_T$$

consists of continuous functions separating points in $\mathscr{C}([0,T];\mathrm{V}_w)$, thus concluding the proof of the lemma. ■

Using Theorem 5.4.4 and Lemma 5.4.5, we obtain the following corollary which we will apply to construct a martingale solution to the stochastic constrained Navier-Stokes equations (5.2.4).

Corollary 5.4.6. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{Z}_T$-valued random variables such that their laws $\mathscr{L}(\eta_n)$ on $(\mathcal{Z}_T,\mathscr{T})$ form a tight sequence of probability measures. Then there exists a subsequence (n_k), a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ and $\mathcal{Z}_T$-valued random variables $\tilde{\eta}$, $\tilde{\eta}_k, k\in\mathbb{N}$ such that the variables η_k and $\tilde{\eta}_k$ have the same laws on $\mathcal{Z}_T$ and $\tilde{\eta}_k$ converges to $\tilde{\eta}$ almost surely on $\tilde{\Omega}$.*

5.5 Faedo-Galerkin approximation and existence of a martingale solution

As mentioned in the introduction, the proof of the existence of a martingale solution is based on the Faedo-Galerkin approximation. In this section we first talk about the basic ingredients required for the approximation and then obtain the a'priori estimates, which we later use in the Subsection 5.5.1 to prove the tightness of laws induced by the solutions of the approximating equations (5.5.2).
Let $\{e_i\}_{i=1}^{\infty}$ be the orthonormal basis in H composed of eigenvectors of A. Let $\mathrm{H}_n := \mathrm{span}\{e_1, \dots, e_n\}$ be the subspace with the norm inherited from H, then $P_n : \mathrm{H} \to \mathrm{H}_n$ given by

$$P_n u := \sum_{i=1}^{n} \langle u, e_i \rangle_{\mathrm{H}} e_i, \quad u \in \mathrm{H}, \tag{5.5.1}$$

is the orthogonal projection onto H_n.

Let us consider the classical Faedo-Galerkin approximation of (5.2.4) in the space H_n:

$$\begin{cases} du_n(t) = -\left[P_n A u_n(t) + P_n B(u_n(t)) + |\nabla u_n(t)|_{L^2}^2 u_n(t)\right] dt \\ \qquad + \sum_{j=1}^{m} P_n C_j u_n(t) \circ dW_j(t), \qquad t \in [0,T], \\ u_n(0) = \frac{P_n u_0}{|P_n u_0|}. \end{cases} \tag{5.5.2}$$

Using the idea from [48] and the Banach Fixed Point Theorem we can show that the SDE (5.5.2) has a local maximal solution up to some stopping time $\tau \le T$. In the following lemma we show that this local solution stays on the manifold $\mathcal{M}$ if we start from the manifold, i.e. if the initial data $u_n(0) \in \mathcal{M}$ then $u_n(t) \in \mathcal{M}$ for every $t \in [0, \tau)$.

Lemma 5.5.1. *Let $u_0 \in \mathrm{V} \cap \mathcal{M}$ then the solution of* (5.5.2) *stays on the manifold $\mathcal{M}$, i.e. for all $t \in [0,\tau), u_n(t) \in \mathcal{M}$.*

Proof. Let u_n be the solution of (5.5.2). Then applying Itô formula to the function $|x|_{\mathrm{H}}^2$ and the process u_n along (5.5.2), (3.2.5) and assumption **(A.1)**, we get

$$\begin{aligned} \frac{1}{2} d|u_n(t)|_{\mathrm{H}}^2 &= \langle u_n(t), -P_n \mathrm{A} u_n(t) - P_n B(u_n(t)) + |\nabla u_n(t)|_{L^2}^2 u_n(t) \rangle_{\mathrm{H}} \, dt \\ &\quad + \frac{1}{2} \sum_{j=1}^{m} \langle u_n(t), (P_n C_j)^2 u_n(t) \rangle_{\mathrm{H}} \, dt + \frac{1}{2} \sum_{j=1}^{m} \langle P_n C_j u_n(t), P_n C_j u_n(t) \rangle_{\mathrm{H}} \, dt \\ &\quad + \sum_{j=1}^{m} \langle u_n(t), P_n C_j u_n(t) dW_j(t) \rangle_{\mathrm{H}} \\ &= -\|u_n(t)\|_{\mathrm{V}}^2 dt + |\nabla u_n(t)|_{L^2}^2 |u_n(t)|_{\mathrm{H}}^2 dt + \frac{1}{2} \sum_{j=1}^{m} \langle C_j^* u_n(t), C_j u_n(t) \rangle_{\mathrm{H}} \, dt \\ &\quad + \frac{1}{2} \sum_{j=1}^{m} |C_j u_n(t)|_{\mathrm{H}}^2 \, dt \\ &= \|u_n(t)\|_{\mathrm{V}}^2 \left[|u_n(t)|_{\mathrm{H}}^2 - 1\right] dt + \frac{1}{2} \sum_{j=1}^{m} \left[|C_j u_n(t)|_{\mathrm{H}}^2 - |C_j u_n(t)|_{\mathrm{H}}^2\right] dt. \end{aligned}$$

Thus, we have

$$d\left[|u_n(t)|^2_{\mathrm{H}}-1\right]=2\|u_n(t)\|^2_{\mathrm{V}}\left[|u_n(t)|^2_{\mathrm{H}}-1\right]dt.$$

Integrating on both sides from 0 to t, we obtain

$$|u_n(t)|^2-1=\left[|u_n(0)|^2_{\mathrm{H}}-1\right]\exp\left[2\int_0^t\|u_n(s)\|^2_{\mathrm{V}}\,ds\right].$$

Now since $|u_n(0)|_{\mathrm{H}}=1$ and $\int_0^t\|u_n(s)\|^2_{\mathrm{V}}ds<\infty$, we get $|u_n(t)|_{\mathrm{H}}=1$ for all $t\in[0,\tau)$, i.e $u_n(t)\in\mathcal{M}$ for every $t\in[0,\tau)$. ■

Since on the finite dimensional space H_n the H and V norm are equivalent, we can infer from the previous lemma that the V-norm of the solution stays bounded. Hence using this non-explosion result as in the case of deterministic setting (see Chapter 4) we can prove the following lemma:

Lemma 5.5.2. *For each $n\in\mathbb{N}$, there exists a continuous H_n-valued global solution u_n of* (5.5.2)*. Moreover for every $T>0$, for any $q\in[2,\infty)$*

$$\mathbb{E}\left[\int_0^T|u_n(s)|^q_{\mathrm{H}}\,ds\right]<\infty.$$

We will require the following lemma to obtain a'priori bounds.

Lemma 5.5.3. *Let $c\in\mathbb{R}^2$ and let $\mathbf{c}:\mathbb{T}^2\to\mathbb{R}^2$ be the corresponding constant vector field. Put, for $u\in H^{1,2}(\mathbb{T}^2,\mathbb{R}^2)$*

$$\tilde{C}u=\mathbf{c}\cdot\nabla u \qquad \text{and} \qquad Cu=\Pi(\tilde{C}u).$$

If the vector field $u\in H^{2,2}(\mathbb{T}^2,\mathbb{R}^2)$ is divergence free, then $\tilde{C}u$ is divergence free as well and in particular,

$$\mathrm{A}Cu-C\mathrm{A}u=0, \qquad u\in H^{3,2}(\mathbb{T}^2,\mathbb{R}^2). \tag{5.5.3}$$

Proof. Let $c=(c_1,c_2)$ then $\tilde{C}u=(c_1D_1+c_2D_2)u$. We have

$$\begin{aligned}\operatorname{div}(\tilde{C}u)&=D_1\big((c_1D_1+c_2D_2)u_1\big)+D_2\big((c_1D_1+c_2D_2)u_2\big)\\&=c_1D_1D_1u_1+c_2D_1D_2u_1+c_1D_2D_1u_2+c_2D_2D_2u_2\\&=c_1D_1\big(D_1u_1+D_2u_2\big)+c_2D_2\big(D_1u_1+D_2u_2\big)\\&=\big(c_1D_1+c_2D_2\big)(\operatorname{div}u)=0,\end{aligned}$$

where we used that vector c is constant and u is divergence free respectively. In order to establish the equality (5.5.3) we start by considering $\mathrm{A}Cu-C\mathrm{A}u$. Since $\mathrm{A}u$ is divergence free, from first part we have $\Pi(\tilde{C}\mathrm{A}u)=\tilde{C}\mathrm{A}u$ and so $C\mathrm{A}u=\tilde{C}\mathrm{A}u$. Thus

$$\begin{aligned}\mathrm{A}Cu-C\mathrm{A}u&=-\Delta\big((c_1D_1+c_2D_2)u\big)-\big(c_1D_1+c_2D_2\big)\big(-\Delta u\big)\\&=-[c_1\Delta D_1u+c_2\Delta D_2u]+[c_1\Delta D_1u+c_2\Delta D_2u]=0,\end{aligned}$$

since c is a constant vector, completing the proof. ■

Lemma 5.5.4. *Let $T>0$ and u_n be the solution of* (5.5.2). *Then under the assumptions **(A.1)** - **(A.2)**, for all $\rho>0$ and $p\in[1,1+\frac{1}{K_c^2})$, there exist positive constants $C_1(p,\rho)$, $C_2(p,\rho)$ and $C_3(\rho)$ such that if $\|u_0\|_{\mathrm{V}}\le\rho$, then*

$$\sup_{n\ge1}\mathbb{E}\left(\sup_{r\in[0,T]}\|u_n(r)\|_{\mathrm{V}}^{2p}\right)\le C_1(p,\rho) \tag{5.5.4}$$

$$\sup_{n\ge1}\mathbb{E}\int_0^T\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds\le C_2(p,\rho), \tag{5.5.5}$$

and

$$\sup_{n\ge1}\mathbb{E}\int_0^T|u_n(s)|_{\mathrm{D(A)}}^2\,ds\le C_3(\rho). \tag{5.5.6}$$

Proof. Let $u_n(t)$ be the solution of (5.5.2) then applying the Itô formula to $\phi(x)=\|x\|_{\mathrm{V}}^2$ and the process $u_n(t)$, we get

$$\begin{aligned}
d\|u_n(t)\|_{\mathrm{V}}^2&=2\langle\mathrm{A}u_n(t),-P_n\mathrm{A}u_n(t)-P_nB(u_n(t),u_n(t))+|\nabla u_n(t)|_{L^2}^2u_n(t)\rangle_{\mathrm{H}}dt\\
&\quad+2\times\frac12\sum_{j=1}^m\langle\mathrm{A}u_n(t),(P_nC_j)^2u_n(t)\rangle_{\mathrm{H}}dt+2\times\frac12\sum_{j=1}^m\langle\mathrm{A}P_nC_ju_n(t),P_nC_ju_n(t)\rangle_{\mathrm{H}}dt\\
&\quad+2\sum_{j=1}^m\langle\mathrm{A}u_n(t),P_nC_ju_n(t)dW_j(t)\rangle_{\mathrm{H}}.
\end{aligned}$$

Now since $\langle|\nabla u_n(t)|_{L^2}^2u_n(t),\mathrm{A}u_n(t)-|\nabla u_n(t)|_{L^2}^2u_n(t)\rangle=0$, using (3.2.5), we have

$$\begin{aligned}
d\|u_n(t)\|_{\mathrm{V}}^2&=-2\langle\mathrm{A}u_n(t)-|\nabla u_n(t)|_{L^2}^2u_n(t),\mathrm{A}u_n(t)-|\nabla u_n(t)|^2u_n(t)\rangle_{\mathrm{H}}dt\\
&\quad+2\langle|\nabla u_n(t)|^2u_n(t),\mathrm{A}u_n(t)-|\nabla u_n(t)|_{L^2}^2u_n(t)\rangle_{\mathrm{H}}dt\\
&\quad-2\langle\mathrm{A}u_n(t),B(u_n(t),u_n(t))\rangle_{\mathrm{H}}dt+\sum_{j=1}^m\langle\mathrm{A}u_n(t),C_j^2u_n(t)\rangle_{\mathrm{H}}dt\\
&\quad+\sum_{j=1}^m\langle\mathrm{A}C_ju_n(t),C_ju_n(t)\rangle_{\mathrm{H}}dt+2\sum_{j=1}^m\langle\mathrm{A}u_n(t),C_ju_n(t)dW_j(t)\rangle_{\mathrm{H}}\\
&=-2|\mathrm{A}u_n(t)-|\nabla u_n(t)|_{L^2}^2u_n(t)|_{\mathrm{H}}^2dt+\sum_{j=1}^m\langle\mathrm{A}C_ju_n(t)-C_j\mathrm{A}u_n(t),C_ju_n(t)\rangle_{\mathrm{H}}dt\\
&\quad+2\sum_{j=1}^m\langle\mathrm{A}u_n(t),C_ju_n(t)dW_j(t)\rangle_{\mathrm{H}}.
\end{aligned}$$

By Assumption **(A.1)** and Lemma 5.5.3 we have $\mathrm{A}C_ju-C_j\mathrm{A}u=0$ for every $j\in\{1,\cdots,m\}$. Thus, integrating on both sides we get

$$\begin{aligned}
&\|u_n(t)\|_{\mathrm{V}}^2+2\int_0^t|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2ds\\
&=\|u_n(0)\|_{\mathrm{V}}^2+2\sum_{j=1}^m\int_0^t\langle\mathrm{A}u_n(s),C_ju_n(s)dW_j(s)\rangle_{\mathrm{H}}\\
&\le\|u(0)\|_{\mathrm{V}}^2+2\sum_{j=1}^m\int_0^t\langle\mathrm{A}u_n(s),C_ju_n(s)dW_j(s)\rangle_{\mathrm{H}}.
\end{aligned} \tag{5.5.7}$$

By Lemma 5.5.2, we infer that the process

$$\mu_n(t) = \sum_{j=1}^{m} \int_0^t \langle \mathrm{A}u_n(s), C_j u_n(s)\,dW_j(s)\rangle_{\mathrm{H}}, \quad t \in [0,T]$$

is a $\mathbb{R}$-valued $\mathbb{F}$-martingale and that $\mathbb{E}[\mu_n(t)] = 0$ for $t \in [0,T]$. Thus

$$\mathbb{E}\|u_n(t)\|_{\mathrm{V}}^2 + 2\mathbb{E}\int_0^t |\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2\,ds \le \mathbb{E}\|u(0)\|_{\mathrm{V}}^2, \quad t \in [0,T]. \tag{5.5.8}$$

Hence

$$\sup_{n\ge 1}\sup_{t\in[0,T]} \mathbb{E}\|u_n(t)\|_{\mathrm{V}}^2 \le \mathbb{E}\|u(0)\|_{\mathrm{V}}^2. \tag{5.5.9}$$

Note that using (5.5.9) in (5.5.8), we also have the following estimate

$$\sup_{n\ge 1}\mathbb{E}\int_0^T |\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2\,ds \le \mathbb{E}\|u(0)\|_{\mathrm{V}}^2. \tag{5.5.10}$$

Let $\xi(t) = \|u_n(t)\|_{\mathrm{V}}^2$, $t \in [0,T]$ and $\phi(x) = x^p$, for some fixed $p \in [1,\infty)$. Using the Itô formula and (5.5.7), we obtain

$$\begin{aligned}\|u_n(t)\|_{\mathrm{V}}^{2p} &= \|u_n(0)\|_{\mathrm{V}}^{2p} - 2p\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2 ds \\ &\quad + 2p(p-1)\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-2)}\langle \mathrm{A}u_n(s), C_j u_n(s)\rangle_{\mathrm{H}}^2\,ds \\ &\quad + 2p\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle \mathrm{A}u_n(s), C_j u_n(s)\,dW_j(s)\rangle_{\mathrm{H}}.\end{aligned} \tag{5.5.11}$$

Since C is skew symmetric, $\langle Cu_n(s), u_n(s)\rangle = 0$ and thus, we get

$$\begin{aligned}&\|u_n(t)\|_{\mathrm{V}}^{2p} + 2p\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2 ds \\ &\quad = \|u_n(0)\|_{\mathrm{V}}^{2p} + 2p\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle \mathrm{A}u_n(s), C_j u_n(s)\,dW_j(s)\rangle_{\mathrm{H}} \\ &\quad + 2p(p-1)\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-2)}\langle \mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s)\rangle_{\mathrm{H}}^2\,ds.\end{aligned}$$

Using the Hölder inequality we have

$$\begin{aligned}&\|u_n(t)\|_{\mathrm{V}}^{2p} + 2p\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2 ds \\ &\quad \le \|u_n(0)\|_{\mathrm{V}}^{2p} + 2p\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle \mathrm{A}u_n(s), C_j u_n(s)\,dW_j(s)\rangle_{\mathrm{H}} \\ &\quad + 2p(p-1)\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-2)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2 |C_j u_n(s)|_{\mathrm{H}}^2\,ds.\end{aligned}$$

On rearranging we get

$$\begin{aligned}\|u_n(t)\|_{\mathrm{V}}^{2p} + 2p\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2 ds \\ \leq \|u_n(0)\|_{\mathrm{V}}^{2p} + 2p\sum_{j=1}^m \int_0^t \|u_n(s)\|^{2(p-1)}\langle \mathrm{A}u_n(s), C_j u_n(s)dW_j(s)\rangle_{\mathrm{H}} \\ + 2p(p-1)K_c^2\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2\, ds,\end{aligned}$$

where K_c is the positive constant defined in equality (5.3.1).

For $p \in [1, 1 + \frac{1}{K_c^2})$, $K_p = 2p\left[1 - K_c^2(p-1)\right] > 0$, thus

$$\begin{aligned}\|u_n(t)\|_{\mathrm{V}}^{2p} + K_p\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2 ds \\ \leq \|u_n(0)\|_{\mathrm{V}}^{2p} + 2p\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle \mathrm{A}u_n(s), C_j u_n(s)dW_j(s)\rangle_{\mathrm{H}}.\end{aligned} \tag{5.5.12}$$

Using Lemma 5.5.2 we infer that the process

$$\eta_n(t) = \sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle \mathrm{A}u_n(s), C_j u_n(s)dW_j(s)\rangle_{\mathrm{H}}, \quad t \in [0,T],$$

is a martingale and $\mathbb{E}[\eta_n(t)] = 0$. Thus

$$\mathbb{E}\|u_n(t)\|_{\mathrm{V}}^{2p} + K_p\mathbb{E}\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2\, ds \leq \mathbb{E}\|u_n(0)\|_{\mathrm{V}}^{2p}. \tag{5.5.13}$$

In particular

$$\sup_{n\geq 1}\sup_{t\in[0,T]} \mathbb{E}\|u_n(t)\|^{2p} \leq \mathbb{E}\|u_0\|_{\mathrm{V}}^{2p}. \tag{5.5.14}$$

Note that using (5.5.14) in (5.5.13), we also have the following estimate,

$$\sup_{n\geq 1}\mathbb{E}\int_0^T \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2\, ds \leq \frac{1}{K_p}\mathbb{E}\|u_0\|_{\mathrm{V}}^{2p}. \tag{5.5.15}$$

In order to prove (5.5.4) we start from (5.5.11),

$$\begin{aligned}\|u_n(t)\|_{\mathrm{V}}^{2p} = \|u_n(0)\|_{\mathrm{V}}^{2p} - 2p\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2 ds \\ + 2p(p-1)\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-2)}\langle \mathrm{A}u_n(s), C_j u_n(s)\rangle_{\mathrm{H}}^2\, ds \\ + 2p\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle \mathrm{A}u_n(s), C_j u_n(s)dW_j(s)\rangle_{\mathrm{H}}.\end{aligned}$$

Since for every $j \in \{1, \cdots, m\}$, $\langle C_j u_n(s), u_n(s)\rangle_{\mathrm{H}} = 0$, hence

$$\begin{aligned}\|u_n(t)\|_{\mathrm{V}}^{2p} + 2p\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_{\mathrm{H}}^2\, ds = \|u_n(0)\|_{\mathrm{V}}^{2p} \\ + 2p(p-1)\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-2)}\langle \mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s)\rangle_{\mathrm{H}}^2\, ds \\ + 2p\sum_{j=1}^m \int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle \mathrm{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s)dW_j(s)\rangle_{\mathrm{H}}.\end{aligned}$$

Taking the mathematical expectation and using the Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\|u_n(r)\|_{\mathrm{V}}^{2p}+2p\mathbb{E}\sup_{r\in[0,t]}\int_0^r\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2ds\leq\mathbb{E}\|u_n(0)\|_{\mathrm{V}}^{2p}\\
&\quad+2p(p-1)K_c^2\mathbb{E}\sup_{r\in[0,t]}\left[\int_0^r\|u_n(s)\|^{2(p-2)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2|\nabla u_n(s)|_{L^2}^2\,ds\right]\\
&\quad+2p\mathbb{E}\sup_{r\in[0,t]}\left[\sum_{j=1}^m\int_0^t\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s),C_ju_n(s)dW_j(s)\rangle_{\mathrm{H}}\right].
\end{aligned}\tag{5.5.16}$$

Using the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left|\sum_{j=1}^m\int_0^r\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s),C_ju_n(s)dW_j(s)\rangle_{\mathrm{H}}\right|\\
&\quad\leq 3\mathbb{E}\left|\sum_{j=1}^m\int_0^t\|u_n(s)\|_{\mathrm{V}}^{4(p-1)}\langle\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s),C_ju_n(s)\rangle_{\mathrm{H}}^2ds\right|^{1/2}\\
&\quad\leq 3\mathbb{E}\left|\sum_{j=1}^m\int_0^t\|u_n(s)\|_{\mathrm{V}}^{4(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2|C_ju_n(s)|_{\mathrm{H}}^2\,ds\right|^{1/2}\\
&\quad\leq 3\mathbb{E}K_c\left[\int_0^t\|u_n(s)\|_{\mathrm{V}}^{2p}\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds\right]^{1/2}.
\end{aligned}$$

Using the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left|\sum_{j=1}^m\int_0^r\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s),C_ju_n(s)dW_j(s)\rangle_{\mathrm{H}}\right|\\
&\leq 3\mathbb{E}\left[K_c\left(\sup_{r\in[0,t]}\|u_n(r)\|_{\mathrm{V}}^{2p}\right)^{1/2}\left(\int_0^t\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds\right)^{1/2}\right]\\
&\leq 3\mathbb{E}\left[\varepsilon\sup_{r\in[0,t]}\|u_n(r)\|_{\mathrm{V}}^{2p}+\frac{K_c^2}{4\varepsilon}\int_0^t\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds\right].
\end{aligned}$$

Thus using this in (5.5.16), we get

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\|u_n(r)\|_{\mathrm{V}}^{2p}+2\mathbb{E}\sup_{r\in[0,t]}\int_0^r\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds\\
&\quad\leq\mathbb{E}\|u_n(0)\|_{\mathrm{V}}^{2p}+2p(p-1)K_c^2\mathbb{E}\sup_{r\in[0,t]}\int_0^r\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds\\
&\quad+\frac{3pK_c^2}{2\varepsilon}\mathbb{E}\int_0^t\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2ds.
\end{aligned}\tag{5.5.17}$$

Hence for $\varepsilon=\frac{1}{12p}$, Eq. (5.5.17) reduces to

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\|u_n(r)\|_{\mathrm{V}}^{2p}+4\mathbb{E}\sup_{r\in[0,t]}\int_0^r\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds\leq 2\mathbb{E}\|u_n(0)\|_{\mathrm{V}}^{2p}\\
&\quad+4p(p-1)K_c^2\mathbb{E}\sup_{r\in[0,t]}\int_0^r\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds\\
&\quad+36p^2K_c^2\mathbb{E}\int_0^t\|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)-|\nabla u_n(s)|_{L^2}^2u_n(s)|_{\mathrm{H}}^2\,ds.
\end{aligned}$$

Since $\int_0^r |\mathrm{A}u_n(s) - |\nabla u_n(s)|^2_{L^2} u_n(s)|^2_{\mathrm{H}}\, ds$ is an increasing function, we have

$$\mathbb{E}\sup_{r\in[0,t]} \|u_n(r)\|_{\mathrm{V}}^{2p} + 4\mathbb{E}\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} |\mathrm{A}u_n(s) - |\nabla u_n(s)|^2_{L^2} u_n(s)|^2_{\mathrm{H}}\, ds$$
$$\leq 2\mathbb{E}\|u_n(0)\|_{\mathrm{V}}^{2p} + 4pK_c^2[10p-1]\mathbb{E}\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} |\mathrm{A}u_n(s) - |\nabla u_n(s)|^2_{L^2} u_n(s)|^2_{\mathrm{H}}\, ds.$$

In particular

$$\mathbb{E}\sup_{r\in[0,t]} \|u_n(r)\|_{\mathrm{V}}^{2p} \leq 4pK_c^2[10p-1]\mathbb{E}\int_0^t \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} |\mathrm{A}u_n(s) - |\nabla u_n(s)|^2_{L^2} u_n(s)|^2_{\mathrm{H}}\, ds$$
$$+ 2\mathbb{E}\|u_n(0)\|_{\mathrm{V}}^{2p}.$$

Since $\mathbb{E}\|u_n(0)\|_{\mathrm{V}}^{2p} \leq \mathbb{E}\|u_0\|_{\mathrm{V}}^{2p}$ and using (5.5.15), for $p \in [1, 1+\frac{1}{K_c^2})$

$$\mathbb{E}\int_0^T \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} |\mathrm{A}u_n(s) - |\nabla u_n(s)|^2_{L^2} u_n(s)|^2_{\mathrm{H}}\, ds$$

is uniformly bounded in n, thus

$$\sup_{n\geq 1}\mathbb{E}\sup_{r\in[0,T]} \|u_n(r)\|_{\mathrm{V}}^{2p} \leq C_1(p,\rho).$$

Now we will establish (5.5.6). Note that

$$\mathbb{E}\int_0^T |u_n(s)|^2_{\mathrm{D(A)}} ds = \mathbb{E}\int_0^T |\mathrm{A}u_n(s) - |\nabla u_n(s)|^2_{L^2} u_n(s)|^2_{\mathrm{H}}\, ds + \mathbb{E}\int_0^T \|u_n(s)\|^4\, ds.$$

Using (5.5.4) for $p = 2$ and (5.5.5) for $p = 1$, we get

$$\sup_{n\geq 1}\mathbb{E}\int_0^T |u_n(s)|^2_{\mathrm{D(A)}} ds \leq C_2(1,\rho) + C_1(2,\rho)T =: C_3(\rho).$$

■

5.5.1 Tightness of the laws of approximating solutions

In this subsection using the a'priori estimates from the Lemma 5.5.4 and the Corollary 5.4.3 we will prove that for every $n \in \mathbb{N}$ the measures $\mathscr{L}(u_n)$ on $(\mathcal{Z}_T, \mathscr{T})$ defined by the solutions of the stochastic ODE (5.5.2) are tight. The following is the main result of this subsection.

Lemma 5.5.5. *The set of measures $\{\mathscr{L}(u_n), n \in \mathbb{N}\}$ is tight on $(\mathcal{Z}_T, \mathscr{T})$.*

Proof. We apply Corollary 5.4.3. According to the a'priori estimates (5.5.4) (for $p = 1$) and (5.5.6), conditions (a) and (b) of Corollary 5.4.3 are satisfied. Thus it is sufficient to prove that the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition **[A]** in H. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times such that $0 \leq \tau_n \leq T$. By (5.5.2), for $t \in [0,T]$ we have

$$u_n(t) = u_n(0) - \int_0^t P_n \mathrm{A}u_n(s)\,ds - \int_0^t P_n B(u_n(s))\,ds + \int_0^t |\nabla u_n(s)|^2_{L^2} u_n(s)\,ds$$
$$+ \frac{1}{2}\int_0^t (P_n C)^2 u_n(s)\,ds + \int_0^t P_n C u_n(s)\,dW(s)$$
$$:= J_1^n + J_2^n(t) + J_3^n(t) + J_4^n(t) + J_5^n(t) + J_6^n(t), \quad t \in [0,T].$$

Let $\theta > 0$. We start by estimating each term in the R.H.S. of the above equality.

Ad. J_2^n. Since $\mathrm{A} : \mathrm{D(A)} \to \mathrm{H}$ is a bounded linear map, then by the Hölder inequality and estimate (5.5.6), we have the following inequalities

$$\mathbb{E}\left[|J_2^n(\tau_n+\theta) - J_2^n(\tau_n)|_{\mathrm{H}}\right] = \mathbb{E}\left|\int_{\tau_n}^{\tau_n+\theta} P_n \mathrm{A} u_n(s)\, ds\right|_{\mathrm{H}} \le c\mathbb{E}\int_{\tau_n}^{\tau_n+\theta} |\mathrm{A} u_n(s)|_{\mathrm{H}}\, ds$$

$$\le c\mathbb{E}\int_{\tau_n}^{\tau_n+\theta} |u_n(s)|_{\mathrm{D(A)}}\, ds \le c\theta^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T |u_n(s)|^2_{D(\mathrm{A})} ds\right]\right)^{\frac{1}{2}} \le cC_3^{\frac{1}{2}} \cdot \theta^{\frac{1}{2}} =: c_2 \cdot \theta^{\frac{1}{2}}. \tag{5.5.18}$$

Ad. J_3^n. Since $B : \mathrm{V} \times \mathrm{V} \to \mathrm{H}$ is bilinear and continuous, then using (3.2.3), the Cauchy-Schwarz inequality, estimates (5.5.4) (for $p = 1$) and (5.5.6), we have the following estimates

$$\mathbb{E}[|J_3^n(\tau_n+\theta) - J_3^n(\tau_n)|_{\mathrm{H}}] = \mathbb{E}\left|\int_{\tau_n}^{\tau_n+\theta} P_n B(u_n(s)) ds\right|_{\mathrm{H}} \le c\mathbb{E}\int_{\tau_n}^{\tau_n+\theta} |B(u_n(s), u_n(s))|_{\mathrm{H}} ds$$

$$\le c\mathbb{E}\int_{\tau_n}^{\tau_n+\theta} |u_n(s)|_{\mathrm{H}}^{\frac{1}{2}} \|u_n(s)\|_{\mathrm{V}} |u_n(s)|^{\frac{1}{2}}_{\mathrm{D(A)}}\, ds \le c\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta} \|u_n(s)\|_{\mathrm{V}}^{\frac{3}{2}} |u_n(s)|^{1/2}_{\mathrm{D(A)}}\, ds\right]$$

$$\le c\mathbb{E}\left(\left[\int_{\tau_n}^{\tau_n+\theta} \|u_n(s)\|^2_{\mathrm{V}} ds\right]^{\frac{3}{4}} \left[\int_{\tau_n}^{\tau_n+\theta} |u_n(s)|^2_{\mathrm{D(A)}} ds\right]^{\frac{1}{4}}\right)$$

$$\le c\theta^{\frac{3}{4}}\left[\mathbb{E}\sup_{s\in[0,T]} \|u_n(s)\|^2_{\mathrm{V}}\right]^{\frac{3}{4}} \left[\mathbb{E}\int_0^T |u_n(s)|^2_{\mathrm{D(A)}} ds\right]^{\frac{1}{4}} \le cC_1(1)^{\frac{3}{4}} C_3^{\frac{1}{4}} \cdot \theta^{\frac{3}{4}} =: c_3 \cdot \theta^{\frac{3}{4}}. \tag{5.5.19}$$

Ad. J_4^n. Using Lemma 5.5.1 and estimate (5.5.4) (for $p = 1$), we have

$$\mathbb{E}[|J_4^n(\tau_n+\theta) - J_4^n(\tau_n)|_{\mathrm{H}}] = \mathbb{E}\left|\int_{\tau_n}^{\tau_n+\theta} |\nabla u_n(s)|^2_{L^2} u_n(s)\, ds\right|_{\mathrm{H}}$$

$$\le \mathbb{E}\int_{\tau_n}^{\tau_n+\theta} |\nabla u_n(s)|^2_{L^2} |u_n(s)|_{\mathrm{H}}\, ds \le \mathbb{E}\sup_{s\in[0,T]} \|u_n(s)\|^2_{\mathrm{V}} \theta \le C_1(1) \cdot \theta =: c_4 \cdot \theta. \tag{5.5.20}$$

Ad. J_5^n. Since C is linear and continuous, then using the Cauchy-Schwarz inequality, assumption **(A.1)** and (5.5.6), we have the following

$$\mathbb{E}[|J_5^n(\tau_n+\theta) - J_5^n(\tau_n)|_{\mathrm{H}}] = \mathbb{E}\left|\frac{1}{2}\sum_{j=1}^m \int_{\tau_n}^{\tau_n+\theta} (P_n C_j)^2 u_n(s) ds\right|_{\mathrm{H}}$$

$$\le \frac{1}{2} c\mathbb{E}\left(\sum_{j=1}^m \int_{\tau_n}^{\tau_n+\theta} |C_j^2 u_n(s)|_{\mathrm{H}} ds\right) \le \frac{1}{2} cK_c^2 \mathbb{E}\int_{\tau_n}^{\tau_n+\theta} |u_n(s)|_{\mathrm{D(A)}}\, ds$$

$$\le \frac{1}{2} cK_c^2 \left[\mathbb{E}\int_0^T |u_n(s)|^2_{\mathrm{D(A)}} ds\right]^{\frac{1}{2}} \theta^{\frac{1}{2}} \le \frac{cK_c^2}{2} C_3^{\frac{1}{2}} \cdot \theta^{\frac{1}{2}} =: c_5 \cdot \theta^{\frac{1}{2}}. \tag{5.5.21}$$

Ad. J_6^n. Using the Itô isometry, assumption **(A.1)** and estimate (5.5.4) (for $p = 1$), we obtain the following

$$\mathbb{E}\left[|J_6^n(\tau_n+\theta) - J_6^n(\tau_n)|^2_{\mathrm{H}}\right] = \mathbb{E}\left|\int_{\tau_n}^{\tau_n+\theta} P_n C u_n(s) dW(s)\right|^2_{\mathrm{H}} \le c\mathbb{E}\int_{\tau_n}^{\tau_n+\theta} |Cu_n(s)|^2_{\mathrm{H}} ds$$

$$\le cK_c^2 \mathbb{E}\int_{\tau_n}^{\tau_n+\theta} \|u_n(s)\|^2_{\mathrm{V}} ds \le cK_c^2 \mathbb{E}\sup_{s\in[0,T]} \|u_n(s)\|^2_{\mathrm{V}} \theta \le cK_c^2 C_1(1) \cdot \theta =: c_6 \cdot \theta. \tag{5.5.22}$$

Let us fix $\kappa > 0$ and $\varepsilon > 0$. By the Chebyshev's inequality and estimates (5.5.18) - (5.5.21), we obtain

$$\mathbb{P}(\{|J_i^n(\tau_n+\theta)-J_i^n(\tau_n)|_{\mathrm{H}} \geq \kappa\}) \leq \frac{1}{\kappa}\mathbb{E}\left[|J_i^n(\tau_n+\theta)-J_i^n(\tau_n)|_{\mathrm{H}}\right] \leq \frac{c_i\theta}{\kappa}; \quad n \in \mathbb{N},$$

where $i = 1, \cdots, 5$. Let $\delta_i = \dfrac{\kappa}{c_i}\varepsilon$. Then

$$\sup_{n\in\mathbb{N}} \sup_{0\leq\theta\leq\delta_i} \mathbb{P}(\{|J_i^n(\tau_n+\theta)-J_i^n(\tau_n)|_{\mathrm{H}} \geq \kappa\}) \leq \varepsilon, \quad i = 1\ldots 5.$$

By the Chebyshev inequality and (5.5.22), we have

$$\begin{aligned}\mathbb{P}(\{|J_6^n(\tau_n+\theta)-J_6^n(\tau_n)|_{\mathrm{H}} \geq \kappa\}) &\leq \frac{1}{\kappa^2}\mathbb{E}\left[|J_i^n(\tau_n+\theta)-J_i^n(\tau_n)|_{\mathrm{H}}^2\right] \\ &\leq \frac{c_6\theta}{\kappa^2}, \quad n \in \mathbb{N}.\end{aligned}$$

Let $\delta_6 = \dfrac{\kappa^2}{C_6}\varepsilon$. Then

$$\sup_{n\in\mathbb{N}} \sup_{0\leq\theta\leq\delta_6} \mathbb{P}(\{|J_6^n(\tau_n+\theta)-J_6^n(\tau_n)|_{\mathrm{H}} \geq \kappa\}) \leq \varepsilon.$$

Since [**A**] holds for each term J_i^n, $i = 1, 2, \cdots, 6$; we infer that it holds also for (u_n). Therefore we can conclude the proof of the lemma by invoking Corollary 5.4.3. ■

5.5.2 Proof of Theorem 5.3.3

We prove certain pointwise convergence in Lemma 5.5.8 which is later used to construct a continuous H−valued martingale. Martingale representation theorem then guarantees the existence of a martingale solution of problem (5.2.4), proving Theorem 5.3.3.

By Lemma 5.5.5 the set of measures $\{\mathscr{L}(u_n), n \in \mathbb{N}\}$ is tight on the space $(\mathcal{Z}_T, \mathscr{T})$, defined by (5.4.3). Hence by Corollary 5.4.6 there exist a subsequence $(n_k)_{k\in\mathbb{N}}$, a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and, on this space, $\mathcal{Z}_T$-valued random variables $\tilde{u}, \tilde{u}_{n_k}, k \geq 1$ such that

(5.5.23) $$\tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ and } \tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathcal{Z}_T, \quad \tilde{\mathbb{P}}-\text{a.s.}$$

$\tilde{u}_{n_k} \to \tilde{u}$ in $\mathcal{Z}_T$, $\tilde{\mathbb{P}}-$a.s. precisely means that

$$\begin{aligned}&\tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathscr{C}([0,T];\mathrm{H}),\\ &\tilde{u}_{n_k} \rightharpoonup \tilde{u} \text{ in } L^2(0,T;\mathrm{D(A)}),\\ &\tilde{u}_{n_k} \to \tilde{u} \text{ in } L^2(0,T;\mathrm{V}),\\ &\tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathscr{C}([0,T];\mathrm{V}_w).\end{aligned}$$

Let us denote the subsequence $(\tilde{u}_{n_k})$ again by $(\tilde{u}_n)_{n\in\mathbb{N}}$.

Since $u_n \in \mathscr{C}([0,T];\mathrm{H}_n)$, $\mathbb{P}$-a.s. and $\mathscr{C}([0,T];\mathrm{H}_n)$ is a Borel subset of $\mathscr{C}([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V})$ and also $\tilde{u}_n$, u_n have the same laws on $\mathcal{Z}_T$ we can make the following inferences

$$\mathscr{L}(\tilde{u}_n)(\mathscr{C}([0,T];\mathrm{H}_n) = 1, \quad n \geq 1,$$
$$|\tilde{u}_n(t)|_{\mathrm{H}} = |u_n(t)|_{\mathrm{H}}, \quad \text{a.s.}$$

Also from (5.5.23) $\tilde{u}_n \to \tilde{u}$ in $\mathscr{C}([0,T];\mathrm{H})$ and by Lemma 5.5.1 $u_n(t) \in \mathcal{M}$ for every $t \in [0,T]$. Therefore we can conclude that

$$\tilde{u}(t) \in \mathcal{M}, \quad t \in [0,T]. \tag{5.5.24}$$

Moreover by (5.5.4) and (5.5.6), for $p \in [1, 1+\frac{1}{K_c^2})$

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\left(\sup_{0\leq s\leq T} \|\tilde{u}_n(s)\|_{\mathrm{V}}^{2p}\right) \leq C_1(p), \tag{5.5.25}$$

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\left[\int_0^T |\tilde{u}_n(s)|_{\mathrm{D(A)}}^2\,ds\right] \leq C_3. \tag{5.5.26}$$

By inequality (5.5.26) we infer that the sequence $(\tilde{u}_n)$ contains a subsequence, still denoted by $(\tilde{u}_n)$ convergent weakly in the space $L^2([0,T]\times\tilde{\Omega};\mathrm{D(A)})$. Since by (5.5.23) $\tilde{u}_n \to \tilde{u}$ in $\mathcal{Z}_T$ $\tilde{\mathbb{P}}$−a.s., we conclude that $\tilde{u} \in L^2([0,T]\times\tilde{\Omega};\mathrm{D(A)})$, i.e.

$$\tilde{\mathbb{E}}\left[\int_0^T |\tilde{u}(s)|_{\mathrm{D(A)}}^2\,ds\right] < \infty. \tag{5.5.27}$$

Similarly by inequality (5.5.25) we can choose a subsequence of $(\tilde{u}_n)$ convergent weak star in the space $L^2(\tilde{\Omega};L^\infty(0,T;\mathrm{V}))$ and, using (5.5.23), we infer that

$$\tilde{\mathbb{E}}\left(\sup_{0\leq s\leq T} \|\tilde{u}(s)\|_{\mathrm{V}}^2\right) < \infty. \tag{5.5.28}$$

For each $n \geq 1$, let us consider a process $\tilde{M}_n$ with trajectories in $\mathscr{C}([0,T];\mathrm{H}_n)$, in particular in $\mathscr{C}([0,T];\mathrm{H})$ defined by

$$\begin{aligned}\tilde{M}_n(t) &= \tilde{u}_n(t) - P_n\tilde{u}(0) + \int_0^t P_n\mathrm{A}\tilde{u}_n(s)\,ds + \int_0^t P_nB(\tilde{u}_n(s))\,ds \\ &\quad - \int_0^t |\nabla\tilde{u}_n(s)|^2\tilde{u}_n(s)\,ds - \frac{1}{2}\sum_{j=1}^m \int_0^t (P_nC_j)^2\tilde{u}_n(s)\,ds, \qquad t\in[0,T].\end{aligned} \tag{5.5.29}$$

Lemma 5.5.6. *$\tilde{M}_n$ is a square integrable martingale with respect to the filtration $\tilde{\mathbb{F}}_n = (\tilde{\mathscr{F}}_{n,t})$, where $\tilde{\mathscr{F}}_{n,t} = \sigma\{\tilde{u}_n(s), s\leq t\}$, with the quadratic variation*

$$\langle\langle\tilde{M}_n\rangle\rangle_t = \int_0^t \sum_{j=1}^m |P_nC_j\tilde{u}_n(s)|_{\mathrm{H}}^2\,ds. \tag{5.5.30}$$

Proof. Indeed since $\tilde{u}_n$ and u_n have the same laws, for all $s,t \in [0,T], s\leq t$, for all bounded continuous functions h on $\mathscr{C}([0,s];\mathrm{H})$, and all $\psi,\zeta \in \mathrm{H}$, we have

$$\tilde{\mathbb{E}}\left[\langle\tilde{M}_n(t) - \tilde{M}_n(s), \psi\rangle_{\mathrm{H}}\, h(\tilde{u}_{n|[0,s]})\right] = 0 \tag{5.5.31}$$

and

$$\tilde{\mathbb{E}}\Big[\Big(\langle\tilde{M}_n(t),\psi\rangle_{\mathrm{H}}\langle\tilde{M}_n(t),\zeta\rangle_{\mathrm{H}}-\langle\tilde{M}_n(s),\psi\rangle_{\mathrm{H}}\langle\tilde{M}_n(s),\zeta\rangle_{\mathrm{H}} \\ -\sum_{j=1}^{m}\int_s^t\big\langle\big(C_j\tilde{u}_n(\sigma)\big)^*P_n\psi,\big(C_j\tilde{u}_n(\sigma)\big)^*P_n\zeta\big\rangle_{\mathbb{R}}\,d\sigma\Big)\cdot h(\tilde{u}_{n|[0,s]})\Big]=0. \tag{5.5.32}$$

■

Lemma 5.5.7. *Let us define a process $\tilde{M}$ for $t\in[0,T]$ by*

$$\tilde{M}(t)=\tilde{u}(t)-\tilde{u}(0)+\int_0^t \mathrm{A}\tilde{u}(s)\,ds+\int_0^t B(\tilde{u}(s))\,ds \tag{5.5.33}$$

$$-\int_0^t|\nabla\tilde{u}(s)|_{L^2}^2\tilde{u}(s)\,ds-\frac{1}{2}\sum_{j=1}^{m}\int_0^t C_j^2\tilde{u}(s)\,ds. \tag{5.5.34}$$

Then $\tilde{M}$ is an H*−valued continuous process.*

Proof. Since $\tilde{u}\in\mathscr{C}([0,T];\mathrm{V})$ we just need to show that each of the remaining four terms on the RHS of (5.5.33) are H−valued and well defined.

Using the Cauchy-Schwarz inequality repeatedly and by (5.5.27) we have the following inequalities

$$\tilde{\mathbb{E}}\int_0^T|\mathrm{A}\tilde{u}(s)|_{\mathrm{H}}\,ds\le T^{1/2}\left(\tilde{\mathbb{E}}\int_0^T|\tilde{u}(s)|_{\mathrm{D(A)}}^2\,ds\right)^{1/2}<\infty.$$

Using (3.2.3), the Hölder inequality, (5.5.24) and the estimates (5.5.27) and (5.5.28) we obtain the following:

$$\begin{aligned}\tilde{\mathbb{E}}\int_0^T|B(\tilde{u}(s))|_{\mathrm{H}}\,ds&\le 2\tilde{\mathbb{E}}\int_0^T|\tilde{u}(s)|_{\mathrm{H}}^{1/2}|\nabla\tilde{u}(s)|_{L^2}|\tilde{u}(s)|_{\mathrm{D(A)}}^{1/2}\,ds\\ &\le 2\tilde{\mathbb{E}}\left[\left(\int_0^T\|\tilde{u}(s)\|_{\mathrm{V}}^{4/3}\,ds\right)^{3/4}\left(\int_0^T|\tilde{u}(s)|_{\mathrm{D(A)}}^2\,ds\right)^{1/4}\right]\\ &\le 2T^{3/4}\left(\tilde{\mathbb{E}}\sup_{s\in[0,T]}\|\tilde{u}(s)\|_{\mathrm{V}}^{4/3}\right)^{3/4}\left(\tilde{\mathbb{E}}\int_0^T|\tilde{u}(s)|_{\mathrm{D(A)}}^2\,ds\right)^{1/4}<\infty.\end{aligned}$$

Using the Hölder inequality, (5.5.24) and inequality (5.5.28) we have

$$\tilde{\mathbb{E}}\int_0^T|\nabla\tilde{u}(s)|_{L^2}^2|\tilde{u}(s)|_{\mathrm{H}}\,ds\le\tilde{\mathbb{E}}\int_0^T\|\tilde{u}(s)\|_{\mathrm{V}}^2\,ds\le\tilde{\mathbb{E}}\left(\sup_{s\in[0,T]}\|\tilde{u}(s)\|_{\mathrm{V}}^2\right)T<\infty.$$

Now we are left to deal with the last term on the RHS. Using assumption **(A.1)** and the estimate (5.5.27), we have the following inequalities for every $j\in\{1,\cdots,m\}$,

$$\tilde{\mathbb{E}}\int_0^T|C_j^2\tilde{u}(s)|_{\mathrm{H}}\le K_cT^{1/2}\left(\tilde{\mathbb{E}}\int_0^T|\tilde{u}(s)|_{\mathrm{D(A)}}^2\,ds\right)^{1/2}<\infty.$$

This concludes the proof of the lemma. ■

Lemma 5.5.8. *For all $s,t\in[0,T]$ such that $s\le t$, we have :*

(a) $\lim_{n\to\infty}\langle\tilde{u}_n(t),P_n\psi\rangle_{\mathrm{H}}=\langle\tilde{u}(t),\psi\rangle_{\mathrm{H}}$, $\tilde{\mathbb{P}}$-*a.s.* $\psi\in\mathrm{H}$,

(b) $\lim_{n\to\infty}\int_s^t\langle\mathrm{A}\tilde{u}_n(\sigma),P_n\psi\rangle_{\mathrm{H}}\,d\sigma=\int_s^t\langle\mathrm{A}\tilde{u}(\sigma),\psi\rangle_{\mathrm{H}}\,d\sigma$, $\tilde{\mathbb{P}}$-*a.s.* $\psi\in\mathrm{H}$,

(c) $\lim_{n\to\infty}\int_s^t\langle B(\tilde{u}_n(\sigma),\tilde{u}_n(\sigma)),P_n\psi\rangle_{\mathrm{H}}\,d\sigma=\int_s^t\langle B(\tilde{u}(\sigma),\tilde{u}(\sigma)),\psi\rangle_{\mathrm{H}}d\sigma$, $\tilde{\mathbb{P}}$-*a.s.* $\psi\in\mathrm{V}$,

(d) $\lim_{n\to\infty}\int_s^t|\nabla\tilde{u}_n(\sigma)|^2_{L^2}\langle\tilde{u}_n(\sigma),P_n\psi\rangle_{\mathrm{H}}\,d\sigma=\int_s^t|\nabla\tilde{u}(\sigma)|^2_{L^2}\langle\tilde{u}(\sigma),\psi\rangle_{\mathrm{H}}\,d\sigma$, $\tilde{\mathbb{P}}$-*a.s.* $\psi\in\mathrm{H}$,

(e) $\lim_{n\to\infty}\langle\int_s^t C_j^2\tilde{u}_n(\sigma),P_n\psi\rangle_{\mathrm{H}}\,d\sigma=\int_s^t\langle C_j^2\tilde{u}(\sigma),\psi\rangle_{\mathrm{H}}\,d\sigma$, $\tilde{\mathbb{P}}$-*a.s.* $\psi\in\mathrm{H}$.

Proof. Let us fix $s,t\in[0,T]$, $s\le t$. By (5.5.23) we know that

$$\tilde{u}_n\to\tilde{u}\ in\ \mathscr{C}([0,T];\mathrm{H})\cap L^2_w(0,T;\mathrm{D(A)})\cap L^2(0,T;\mathrm{V})\cap\mathscr{C}([0,T];\mathrm{V}_w),\quad\tilde{\mathbb{P}}\text{-a.s.}\tag{5.5.35}$$

Let $\psi\in\mathrm{H}$. Since $\tilde{u}_n\to\tilde{u}$ in $\mathscr{C}([0,T];\mathrm{H})$ $\tilde{\mathbb{P}}$-a.s. and $P_n\psi\to\psi$ in H, we have

$$\begin{aligned}&\lim_{n\to\infty}\langle\tilde{u}_n(t),P_n\psi\rangle_{\mathrm{H}}-\langle\tilde{u}(t),\psi\rangle_{\mathrm{H}}\\&\qquad=\lim_{n\to\infty}\langle\tilde{u}_n(t)-\tilde{u}(t),P_n\psi\rangle_{\mathrm{H}}+\lim_{n\to\infty}\langle\tilde{u}(t),P_n\psi-\psi\rangle_{\mathrm{H}}=0\quad\tilde{\mathbb{P}}\text{-a.s.}\end{aligned}$$

Thus we infer that assertion (a) holds.

Let $\psi\in\mathrm{H}$, then

$$\begin{aligned}&\int_s^t\langle\mathrm{A}\tilde{u}_n(\sigma),P_n\psi\rangle_{\mathrm{H}}\,d\sigma-\int_s^t\langle\mathrm{A}\tilde{u}(\sigma),\psi\rangle_{\mathrm{H}}\,d\sigma\\&=\int_s^t\langle\mathrm{A}\tilde{u}_n(\sigma)-\mathrm{A}\tilde{u}(\sigma),\psi\rangle_{\mathrm{H}}\,d\sigma+\int_s^t\langle\mathrm{A}\tilde{u}_n(\sigma),P_n\psi-\psi\rangle_{\mathrm{H}}\,d\sigma\\&\le\int_s^t\langle\tilde{u}_n(\sigma)-\tilde{u}(\sigma),\mathrm{A}^{-1}\psi\rangle_{\mathrm{D(A)}}\,d\sigma+\int_s^t|\tilde{u}_n(\sigma)|_{\mathrm{D(A)}}|P_n\psi-\psi|_{\mathrm{H}}\,d\sigma\\&\le\int_s^t\langle\tilde{u}_n(\sigma)-\tilde{u}(\sigma),\mathrm{A}^{-1}\psi\rangle_{\mathrm{D(A)}}\,d\sigma+|P_n\psi-\psi|_{\mathrm{H}}|\tilde{u}_n|_{L^2(0,T;\mathrm{D(A)})}T^{1/2}.\end{aligned}$$

By (5.5.35) $\tilde{u}_n\to\tilde{u}$ weakly in $L^2(0,T;\mathrm{D(A)})$ $\tilde{\mathbb{P}}$-a.s., $\tilde{u}_n$ is a uniformly bounded sequence in $L^2(0,T;\mathrm{D(A)})$ and $P_n\psi\to\psi$ in H. Hence we have $\tilde{\mathbb{P}}$-a.s.

$$\lim_{n\to\infty}\int_s^t\langle\tilde{u}_n(\sigma)-\tilde{u}(\sigma),\mathrm{A}^{-1}\psi\rangle_{\mathrm{D(A)}}\,d\sigma\to0,$$

and

$$\lim_{n\to\infty}|P_n\psi-\psi|_{\mathrm{H}}\to0.$$

Thus, we have shown that assertion (b) is true.

We will now prove assertion (c). Let $\psi\in\mathrm{V}$. Then we have the following estimates:

$$\begin{aligned}&\int_s^t\langle B(\tilde{u}_n(\sigma)),P_n\psi\rangle_{\mathrm{H}}-\int_s^t\langle B(\tilde{u}(\sigma)),\psi\rangle_{\mathrm{H}}\,d\sigma\\&=\int_s^t\langle B(\tilde{u}_n(\sigma))-B(\tilde{u}(\sigma)),\psi\rangle_{\mathrm{H}}\,d\sigma+\int_s^t\langle B(\tilde{u}_n(\sigma)),P_n\psi-\psi\rangle_{\mathrm{H}}\,d\sigma\\&=\int_s^t\big[b(\tilde{u}_n(\sigma),\tilde{u}_n(\sigma),\psi)-b(\tilde{u}(\sigma),\tilde{u}(\sigma),\psi)\big]\,d\sigma+\int_s^t\langle B(\tilde{u}_n(\sigma)),P_n\psi-\psi\rangle_{\mathrm{H}}\,d\sigma.\end{aligned}$$

Using (3.2.1), we get

$$\begin{aligned}
&\int_s^t \langle B(\tilde{u}_n(\sigma)), P_n\psi\rangle_{\mathrm{H}} - \int_s^t \langle B(\tilde{u}(\sigma)), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&= \int_s^t b(\tilde{u}_n(\sigma) - \tilde{u}(\sigma), \tilde{u}_n(\sigma), \psi)\, d\sigma + \int_s^t b(\tilde{u}(\sigma), \tilde{u}_n(\sigma) - \tilde{u}(\sigma), \psi)\, d\sigma \\
&\quad + \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n\psi - \psi\rangle_{\mathrm{H}}\, d\sigma \\
&\le \int_s^t \|\tilde{u}_n(\sigma) - \tilde{u}(\sigma)\|_{\mathrm{V}} \|\tilde{u}_n(\sigma)\|_{\mathrm{V}} \|\psi\|_{\mathrm{V}}\, d\sigma + \int_s^t \|\tilde{u}(\sigma)\|_{\mathrm{V}} \|\tilde{u}_n(\sigma) - \tilde{u}(\sigma)\|_{\mathrm{V}} \|\psi\|_{\mathrm{V}}\, d\sigma \\
&\quad + \int_s^t \|\tilde{u}_n(\sigma)\|_{\mathrm{V}}^2 \|P_n\psi - \psi\|_{\mathrm{V}}\, d\sigma.
\end{aligned}$$

Now since by (5.5.35) $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;\mathrm{V})$ strongly, the sequence $(\tilde{u}_n)$ is uniformly bounded in $L^2(0,T;\mathrm{V})$. Thus using the Cauchy-Schwarz inequality and the convergence of $P_n\psi \to \psi$ in V, we have $\tilde{\mathbb{P}}-$a.s.

$$\begin{aligned}
&\lim_{n\to\infty} \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n\psi\rangle_{\mathrm{H}} - \int_s^t \langle B(\tilde{u}(\sigma)), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&\le \lim_{n\to\infty} |\tilde{u}_n - \tilde{u}|_{L^2(0,T;\mathrm{V})} \Big[|\tilde{u}_n|_{L^2(0,T;\mathrm{V})} \\
&+ |\tilde{u}|_{L^2(0,T;\mathrm{V})} \Big] \|\psi\|_{\mathrm{V}} + \lim_{n\to\infty} |\tilde{u}_n|^2_{L^2(0,T;V)} \|P_n\psi - \psi\|_{\mathrm{V}} \to 0.
\end{aligned}$$

Next we deal with (d). Let $\psi \in \mathrm{H}$, then

$$\begin{aligned}
&\int_s^t |\nabla \tilde{u}_n(\sigma)|^2_{L^2} \langle \tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma - \int_s^t |\nabla \tilde{u}(\sigma)|^2_{L^2} \langle \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&= \int_s^t \left[|\nabla \tilde{u}_n(\sigma)|^2_{L^2} - |\nabla \tilde{u}(\sigma)|^2_{L^2} \right] \langle \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma + \int_s^t |\nabla \tilde{u}_n(\sigma)|^2_{L^2} \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&\quad + \int_s^t |\nabla \tilde{u}_n(\sigma)|^2_{L^2} \langle \tilde{u}_n(\sigma), P_n\psi - \psi\rangle_{\mathrm{H}}\, d\sigma \\
&= \int_s^t \left[|\nabla \tilde{u}_n(\sigma)|_{L^2} - |\nabla \tilde{u}(\sigma)|_{L^2} \right] \left[|\nabla \tilde{u}_n(\sigma)|_{L^2} + |\nabla \tilde{u}(\sigma)|_{L^2} \right] \langle \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&\quad + \int_s^t |\nabla \tilde{u}_n(\sigma)|^2_{L^2} \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma + \int_s^t |\nabla \tilde{u}_n(\sigma)|^2_{L^2} \langle \tilde{u}_n(\sigma), P_n\psi - \psi\rangle_{\mathrm{H}}\, d\sigma.
\end{aligned}$$

Thus, by the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\int_s^t |\nabla \tilde{u}_n(\sigma)|^2_{L^2} \langle \tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma - \int_s^t |\nabla \tilde{u}(\sigma)|^2_{L^2} \langle \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&\le \int_s^t [\|\tilde{u}_n(\sigma) - \tilde{u}(\sigma)\|_{\mathrm{V}}][\|\tilde{u}_n(\sigma)\|_{\mathrm{V}} + \|\tilde{u}(\sigma)\|_{\mathrm{V}}] |\tilde{u}(\sigma)|_{\mathrm{H}} |\psi|_{\mathrm{H}}\, d\sigma \\
&\quad + \int_s^t \|\tilde{u}_n(\sigma)\|^2_{\mathrm{V}} |\tilde{u}_n(\sigma) - \tilde{u}(\sigma)|_{\mathrm{H}} |\psi|_{\mathrm{H}}\, d\sigma + \int_s^t \|\tilde{u}_n(\sigma)\|^2_{\mathrm{V}} |\tilde{u}_n(\sigma)|_{\mathrm{H}} |P_n\psi - \psi|_{\mathrm{H}}\, d\sigma
\end{aligned}$$

By (5.5.35) $\tilde{u}_n \to \tilde{u}$ strongly in $\mathscr{C}([0,T];\mathrm{H}) \cap L^2(0,T;\mathrm{V})$, in particular $\tilde{u} \in L^2(0,T;\mathrm{V})$, the sequence $(\tilde{u}_n)$ is uniformly bounded in $L^2(0,T;\mathrm{V})$ and $P_n\psi \to \psi$ in H. Thus, we have $\tilde{\mathbb{P}}$−a.s.

$$\begin{aligned}
&\lim_{n\to\infty} \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma - \int_s^t |\nabla \tilde{u}(\sigma)|_{L^2}^2 \langle \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&\le \lim_{n\to\infty} \left[|\tilde{u}_n|_{L^2(0,T;V)} + |\tilde{u}|_{L^2(0,T;V)} \right] |\tilde{u}_n|_{L^\infty(0,T;H)} |\tilde{u}_n - \tilde{u}|_{L^2(0,T;V)} |\psi|_{\mathrm{H}} \\
&\quad + \lim_{n\to\infty} |\tilde{u}_n|_{L^2(0,T;V)}^2 |\tilde{u}_n - \tilde{u}|_{L^\infty(0,T;H)} |\psi|_{\mathrm{H}} + \lim_{n\to\infty} |\tilde{u}_n|_{L^2(0,T;V)}^2 |\tilde{u}_n|_{L^\infty(0,T;H)} |P_n\psi - \psi|_{\mathrm{H}} \to 0.
\end{aligned}$$

Hence we infer that assertion (d) holds.

Now we are left to show that (e) holds. Let $\psi \in \mathrm{H}$, then

$$\begin{aligned}
&\int_s^t \langle C^2 \tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma - \int_s^t \langle C^2 \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&= \int_s^t \langle C^2(\tilde{u}_n(\sigma) - \tilde{u}(\sigma)), \psi\rangle_{\mathrm{H}}\, d\sigma + \int_s^t \langle C^2 \tilde{u}_n(\sigma), P_n\psi - \psi\rangle_{\mathrm{H}}\, d\sigma \\
&\le \int_s^t \langle C^2 A^{-1} \mathrm{A}(\tilde{u}_n(\sigma) - \tilde{u}(\sigma)), \psi\rangle_{\mathrm{H}}\, d\sigma + K_c^2 \int_s^t |\tilde{u}_n(\sigma)|_{\mathrm{D(A)}} |P_n\psi - \psi|_{\mathrm{H}}\, d\sigma,
\end{aligned}$$

where K_c is defined in (5.3.1). Since $(\tilde{u}_n)$ is a uniformly bounded sequence in $L^2(0,T;\mathrm{D(A)})$ and $C^2\mathrm{A}^{-1}$ is a bounded operator thus by (5.5.35), we have $\tilde{\mathbb{P}}$-a.s.

$$\begin{aligned}
&\lim_{n\to\infty} \int_s^t \langle C^2 \tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma - \int_s^t \langle C^2 \tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&\le \lim_{n\to\infty} \int_s^t \langle \mathrm{A}(\tilde{u}_n(\sigma) - \tilde{u}(\sigma)), (C^2\mathrm{A}^{-1})^*\psi\rangle_{\mathrm{H}}\, d\sigma + \lim_{n\to\infty} K_c^2 |\tilde{u}|_{L^2(0,T;\mathrm{D(A)})} |P_n\psi - \psi|_{\mathrm{H}} T^{1/2} \\
&= \lim_{n\to\infty} \int_s^t \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), \mathrm{A}^{-1}(C^2\mathrm{A}^{-1})^*\psi\rangle_{\mathrm{D(A)}}\, d\sigma + \lim_{n\to\infty} K_c^2 |\tilde{u}|_{L^2(0,T;\mathrm{D(A)})} |P_n\psi - \psi|_{\mathrm{H}} T^{\frac{1}{2}} \to 0,
\end{aligned}$$

where to establish the convergence we have used that $P_n\psi \to \psi$ in H.

This completes the proof of Lemma 5.5.8. ■

Let h be a bounded continuous function on $\mathscr{C}([0,T];\mathrm{H})$ and $\tilde{\mathbb{F}} = (\tilde{\mathscr{F}}_t) = \sigma\{\tilde{u}(s), s \le t\}$ be the filtration of sigma fields generated by the process $\tilde{u}$.

Lemma 5.5.9. *For all $s,t \in [0,T]$, such that $s \le t$ and all $\psi \in \mathrm{V}$:*

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\left[\langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi\rangle h(\tilde{u}_{n|[0,s]})\right] = \tilde{\mathbb{E}}\left[\langle \tilde{M}(t) - \tilde{M}(s), \psi\rangle h(\tilde{u}_{|[0,s]})\right], \quad h \in \mathscr{C}([0,T];\mathrm{H}), \tag{5.5.36}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between V *and* V′.

Proof. Let us fix $s,t \in [0,T], s \le t$ and $\psi \in \mathrm{V}$. By (5.5.29), we have

$$\begin{aligned}
\langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi\rangle &= \langle \tilde{u}_n(t), P_n\psi\rangle_H - \langle \tilde{u}_n(s), P_n\psi\rangle_{\mathrm{H}} + \int_s^t \langle \mathrm{A}\tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma \\
&+ \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n\psi\rangle\, d\sigma - \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma \\
&- \frac{1}{2} \int_s^t \langle C^2 \tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma.
\end{aligned}$$

By Lemma 5.5.8, we infer that

$$\lim_{n\to\infty}\langle \tilde{M}_n(t)-\tilde{M}_n(s),\psi\rangle = \langle \tilde{M}(t)-\tilde{M}(s),\psi\rangle, \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{5.5.37}$$

In order to prove (5.5.36) we first observe that since $\tilde{u}_n \to \tilde{u}$ in $\mathcal{Z}_T$, in particular in $\mathscr{C}([0,T];\mathrm{H})$ and h is a bounded continuous function on $\mathscr{C}([0,T];\mathrm{H})$, we get

$$\lim_{n\to\infty} h(\tilde{u}_{n|[0,s]}) = h(\tilde{u}_{|[0,s]}) \quad \tilde{\mathbb{P}}-a.s. \tag{5.5.38}$$

and

$$\sup_{n\in\mathbb{N}} |h(\tilde{u}_{n|[0,s]})|_{L^\infty} < \infty. \tag{5.5.39}$$

Let us define a sequence of $\mathbb{R}-$valued random variables:

$$f_n(\omega) := \left[\langle \tilde{M}_n(t,\omega),\psi\rangle - \langle \tilde{M}_n(s,\omega),\psi\rangle\right] h(\tilde{u}_{n|[0,s]}), \quad \omega\in\tilde{\Omega}.$$

We will prove that the functions $\{f_n\}_{n\in\mathbb{N}}$ are uniformly integrable in order to apply the Vitali theorem later on. We claim that

$$\sup_{n\ge 1}\tilde{\mathbb{E}}[|f_n|^2] < \infty. \tag{5.5.40}$$

By the Cauchy-Schwarz inequality and the embedding $\mathrm{V}' \hookrightarrow \mathrm{H}$, for each $n\in\mathbb{N}$ there exists a positive constant c such that

$$\tilde{\mathbb{E}}[|f_n|^2] \le 2c|h|^2_{L^\infty}|\psi|^2_{\mathrm{V}}\tilde{\mathbb{E}}\left[|\tilde{M}_n(t)|^2_{\mathrm{H}} + |\tilde{M}_n(s)|^2_{\mathrm{H}}\right]. \tag{5.5.41}$$

Since $\tilde{M}_n$ is a continuous martingale with quadratic variation defined in (5.5.30), by the Burkholder-Davis-Gundy inequality we obtain

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{M}_n(t)|^2_{\mathrm{H}}\right] \le c\tilde{\mathbb{E}}\left[\sum_{j=1}^m\int_0^T |P_nC_j\tilde{u}_n(\sigma)|^2_{\mathrm{H}}\,d\sigma\right]. \tag{5.5.42}$$

Since $P_n:\mathrm{H}\to\mathrm{H}$ is a contraction then by assumption **(A.1)** and (5.5.25) for $p=1$, we have

$$\begin{aligned}\tilde{\mathbb{E}}\left[\sum_{j=1}^m\int_0^T |P_nC_j\tilde{u}_n(\sigma)|^2_{\mathrm{H}}\,d\sigma\right] &\le \tilde{\mathbb{E}}\left[mK_c^2\int_0^T \|\tilde{u}_n(\sigma)\|^2_{\mathrm{V}}\,d\sigma\right]\\ &\le mK_c^2\tilde{\mathbb{E}}\left[\sup_{\sigma\in[0,T]}\|\tilde{u}_n(\sigma)\|^2_{\mathrm{V}}\right]T < \infty.\end{aligned} \tag{5.5.43}$$

Then, by (5.5.41) and (5.5.43) we see that (5.5.40) holds. Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable and by (5.5.37) it is $\tilde{\mathbb{P}}-$a.s. point-wise convergent, then application of the Vitali Theorem (see Theorem 2.5.14) completes the proof of the lemma. ■

From Lemma 5.5.6 and Lemma 5.5.9 we have the following corollary.

Corollary 5.5.10. *For all* $s,t \in [0,T]$ *such that* $s \le t$:

$$\mathbb{E}\big(\tilde{M}(t) - \tilde{M}(s)\big|\tilde{\mathscr{F}}_t\big) = 0\,.$$

Lemma 5.5.11. *For all* $s,t \in [0,T]$ *such that* $s \le t$ *and all* $\psi,\zeta \in \mathrm{V}$:

$$\begin{aligned}\lim_{n\to\infty}\tilde{\mathbb{E}}\Big[\Big(\langle \tilde{M}_n(t),\psi\rangle\langle \tilde{M}_n(t),\zeta\rangle - \langle \tilde{M}_n(s),\psi\rangle\langle \tilde{M}_n(s),\zeta\rangle\Big)h(\tilde{u}_{n|[0,s]})\Big]\\ = \tilde{\mathbb{E}}\Big[\Big(\langle \tilde{M}(t),\psi\rangle\langle \tilde{M}(t),\zeta\rangle - \langle \tilde{M}(s),\psi\rangle\langle \tilde{M}(s),\zeta\rangle\Big)h(\tilde{u}_{|[0,s]})\Big], \quad h \in \mathscr{C}([0,T];\mathrm{H}),\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ *denotes the dual pairing between* V *and* V′.

Proof. Let us fix $s,t \in [0,T]$ such that $s \le t$ and $\psi,\zeta \in \mathrm{V}$ and define the random variables f_n and f by

$$\begin{aligned}f_n(\omega) &:= \Big(\langle \tilde{M}_n(t,\omega),\psi\rangle\langle \tilde{M}_n(t,\omega),\zeta\rangle - \langle \tilde{M}_n(s,\omega),\psi\rangle\langle \tilde{M}_n(s,\omega),\zeta\rangle\Big)h(\tilde{u}_{n|[0,s]}(\omega)),\\ f(\omega) &:= \Big(\langle \tilde{M}(t,\omega),\psi\rangle\langle \tilde{M}(t,\omega),\zeta\rangle - \langle \tilde{M}(s,\omega),\psi\rangle\langle \tilde{M}(s,\omega),\zeta\rangle\Big)h(\tilde{u}_{|[0,s]}(\omega)), \quad \omega \in \tilde{\Omega}.\end{aligned}$$

By (5.5.37) and (5.5.38) we infer that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$, for $\tilde{\mathbb{P}}$ almost all $\omega \in \tilde{\Omega}$. We will prove that the functions $\{f_n\}_{n\in\mathbb{N}}$ are uniformly integrable. We claim that for some $r > 1$,

$$\sup_{n\ge 1}\tilde{\mathbb{E}}\left[|f_n|^r\right] < \infty. \tag{5.5.44}$$

For each $n \in \mathbb{N}$, as before we have

$$\tilde{\mathbb{E}}\left[|f_n|^r\right] \le C\|h\|^r_{L^\infty}\|\psi\|^r_{\mathrm{V}}\|\zeta\|^r_{\mathrm{V}}\tilde{\mathbb{E}}\left[|\tilde{M}_n(t)|^{2r} + |\tilde{M}_n(s)|^{2r}\right]. \tag{5.5.45}$$

Since $\tilde{M}_n$ is a continuous martingale with quadratic variation defined in (5.5.29), by the Burkholder-Davis-Gundy inequality we obtain

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{M}_n(t)|^{2r}\right] \le c\tilde{\mathbb{E}}\left[\sum_{j=1}^m\int_0^T |P_nC_j\tilde{u}_n(\sigma)|^2_{\mathrm{H}}\,d\sigma\right]^r. \tag{5.5.46}$$

Since $P_n : \mathrm{H} \to \mathrm{H}$ is a contraction, by assumption **(A.1)**, we have

$$\begin{aligned}\tilde{\mathbb{E}}\left[\sum_{j=1}^m\int_0^T |P_nC_j\tilde{u}_n(\sigma)|^2_{\mathrm{H}}\,d\sigma\right]^r &\le \tilde{\mathbb{E}}\left[mK_c^2\int_0^T \|\tilde{u}_n(\sigma)\|^2_{\mathrm{V}}\,d\sigma\right]^r\\ &\le (mK_c^2)^r\,\tilde{\mathbb{E}}\left(\sup_{\sigma\in[0,T]}\|\tilde{u}_n(\sigma)\|^{2r}_{\mathrm{V}}\right)T^r.\end{aligned} \tag{5.5.47}$$

Thus for $r \in (1, 1+\frac{1}{K_c^2})$, by (5.5.45), (5.5.46), (5.5.47) and (5.5.25) we infer that condition (5.5.44) holds. By the Vitali theorem

$$\lim_{n\to\infty}\tilde{\mathbb{E}}[f_n] = \tilde{\mathbb{E}}[f]. \tag{5.5.48}$$

The proof of the lemma is thus complete. ■

Lemma 5.5.12 (Convergence of quadratic variations). *For any $s,t \in [0,T]$ and $\psi,\zeta \in \mathrm{V}$, for all $h \in \mathscr{C}([0,T];\mathrm{H})$ we have*

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\left[\left(\sum_{j=1}^m \int_s^t \left\langle \left(C_j \tilde{u}_n(\sigma)\right)^* P_n\psi, \left(C_j\tilde{u}_n(\sigma)\right)^* P_n\zeta\right\rangle_{\mathbb{R}} d\sigma\right)\cdot h(\tilde{u}_{n|[0,s]})\right]$$
$$= \tilde{\mathbb{E}}\left[\left(\sum_{j=1}^m \int_s^t \left\langle \left(C_j \tilde{u}(\sigma)\right)^* \psi, \left(C_j\tilde{u}(\sigma)\right)^* \zeta\right\rangle_{\mathbb{R}} d\sigma\right)\cdot h(\tilde{u}_{|[0,s]})\right].$$

Proof. Let us fix $\psi,\zeta \in \mathrm{V}$ and define a sequence of random variables by

$$f_n(\omega) := \left(\sum_{j=1}^m \int_s^t \left\langle \left(C_j \tilde{u}_n(\sigma,\omega)\right)^* P_n\psi, \left(C_j\tilde{u}_n(\sigma,\omega)\right)^* P_n\zeta\right\rangle_{\mathbb{R}} d\sigma\right)\cdot h(\tilde{u}_{n|[0,s]}), \quad \omega \in \tilde{\Omega}.$$

We will prove that these random variables are uniformly integrable and convergent $\tilde{\mathbb{P}}-$a.s. to some random variable f. In order to do that we will show that for some $r>1$,

$$\sup_{n\geq 1} \tilde{\mathbb{E}}|f_n|^r < \infty. \tag{5.5.49}$$

Since $P_n : \mathrm{H} \to \mathrm{H}$ is a contraction, by the Cauchy-Schwarz inequality, and assumption **(A.1)** there exists a positive constant c such that

$$\left|\left(C_j\tilde{u}_n(\sigma,\omega)\right)^* P_n\psi\right|_{\mathbb{R}} \leq \left|\left(C_j\tilde{u}_n(\sigma,\omega)\right)^*\right|_{\mathscr{L}(\mathrm{H};\mathbb{R})} |P_n\psi|_{\mathrm{H}} \leq |C_j\tilde{u}_n(\sigma,\omega)|_{\mathscr{L}(\mathbb{R};\mathrm{H})}|\psi|_{\mathrm{H}}$$
$$\leq K_c \,\|\tilde{u}_n(\sigma,\omega)\|_{\mathrm{V}}|\psi|_{\mathrm{H}}, \qquad j \in \{1,\cdots,m\},$$

where $\mathscr{L}(X,Y)$ denotes the operator norm of the linear operators from X to Y. Thus using the Hölder inequality, we obtain

$$\tilde{\mathbb{E}}|f_n|^r = \tilde{\mathbb{E}}\left|\left(\sum_{j=1}^m \int_s^t \left\langle \left(C_j \tilde{u}_n(\sigma)\right)^* P_n\psi, (C\tilde{u}_n(\sigma))^* P_n\zeta\right\rangle_{\mathbb{R}} d\sigma\right)\cdot h(\tilde{u}_{n|[0,s]})\right|^r$$
$$\leq |h|^r_{L^\infty}\tilde{\mathbb{E}}\left(\sum_{j=1}^m \int_s^t \left|\left(C_j \tilde{u}_n(\sigma)\right)^* P_n\psi\right|_{\mathbb{R}}\cdot \left|\left(C_j\tilde{u}_n(\sigma)\right)^* P_n\zeta\right|_{\mathbb{R}} d\sigma\right)^r$$
$$\leq (mK_c^2)^r\, |h|^r_{L^\infty}|\psi|^r_{\mathrm{H}}|\zeta|^r_{\mathrm{H}}\tilde{\mathbb{E}}\left(\int_s^t \|\tilde{u}_n(\sigma)\|^2_{\mathrm{V}}\, d\sigma\right)^r$$
$$\leq (mK_c^2)^r\, |h|^r_{L^\infty}|\psi|^r_{\mathrm{H}}|\zeta|^r_{\mathrm{H}}\tilde{\mathbb{E}}\left(\sup_{\sigma\in[0,T]} \|\tilde{u}_n(\sigma)\|^{2r}_{\mathrm{V}}\right) T^r. \tag{5.5.50}$$

Therefore using (5.5.50) and (5.5.25) we infer that (5.5.49) holds for every $r \in (1, 1+\frac{1}{K_c^2})$.

Now for pointwise convergence we will show that for a fix $\omega \in \tilde{\Omega}$,

$$\lim_{n\to\infty}\int_s^t \sum_{j=1}^m \left\langle \left(C_j \tilde{u}_n(\sigma,\omega)\right)^* P_n\psi, \left(C_j\tilde{u}_n(\sigma,\omega)\right)^* P_n\zeta\right\rangle_{\mathbb{R}} d\sigma \tag{5.5.51}$$
$$= \int_s^t \sum_{j=1}^m \left\langle \left(C_j \tilde{u}(\sigma,\omega)\right)^* \psi, \left(C_j\tilde{u}(\sigma,\omega)\right)^* \zeta\right\rangle_{\mathbb{R}} d\sigma.$$

Let us fix $\omega \in \tilde{\Omega}$ such that

(i) $\tilde{u}_n(\cdot,\omega) \to \tilde{u}(\cdot,\omega)$ in $L^2(0,T;\mathrm{V})$,

(ii) and the sequence $(\tilde{u}_n(\cdot,\omega))_{n\geq 1}$ is uniformly bounded in $L^2(0,T;\mathrm{V})$.

Note that to prove (5.5.51), it is sufficient to prove that

$$\big(C_j\tilde{u}_n(\sigma,\omega)\big)^* P_n\psi \to \big(C_j\tilde{u}(\sigma,\omega)\big)^*\psi \quad \text{in } L^2(s,t;\mathbb{R}), \tag{5.5.52}$$

for every $j \in \{1,\cdots,m\}$. Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\int_s^t \left|\big(C_j\tilde{u}_n(\sigma,\omega)\big)^* P_n\psi - \big(C_j\tilde{u}(\sigma,\omega)\big)^*\psi\right|^2_{\mathbb{R}}\, d\sigma \\
&\leq \int_s^t \Big(\left|\big(C_j\tilde{u}_n(\sigma,\omega)\big)^*\left(P_n\psi-\psi\right)\right|_{\mathbb{R}} + \left|\big(C_j\tilde{u}_n(\sigma,\omega)-C_j\tilde{u}(\sigma,\omega)\big)^*\psi\right|_{\mathbb{R}}\Big)^2 d\sigma \\
&\leq 2\int_s^t \left|C_j\tilde{u}_n(\sigma,\omega)\right|^2_{\mathscr{L}(\mathbb{R};\mathrm{H})}\left|P_n\psi-\psi\right|^2_{\mathrm{H}}\, d\sigma + 2\int_s^t \left|C_j\tilde{u}_n(\sigma,\omega)-C_j\tilde{u}(\sigma,\omega)\right|^2_{\mathscr{L}(\mathbb{R};\mathrm{H})}|\psi|^2_{\mathrm{H}}\, d\sigma \\
&=: I_n^1(t) + I_n^2(t).
\end{aligned}$$

We will deal with each of the terms individually. We start with $I_n^1(t)$. Since

$$\lim_{n\to\infty} |P_n\psi-\psi|_{\mathrm{H}} = 0, \quad \psi \in \mathrm{V},$$

and by assumption **(A.1)**, (ii) there exists a positive constant K such that

$$\sup_{n\geq 1}\int_s^t |C\tilde{u}_n(\sigma,\omega)|^2_{\mathscr{L}(\mathbb{R};\mathrm{H})}\, d\sigma \leq K_c^2 \sup_{n\geq 1}\int_s^t \|\tilde{u}_n(\sigma,\omega)\|^2_{\mathrm{V}}\, d\sigma \leq K.$$

Thus we infer that

$$\lim_{n\to\infty} I_n^1(t) = 0.$$

Next we consider $I_n^2(t)$. Using assumption **(A.1)** and (i) we can show that for every $j \in \{1,\cdots,m\}$,

$$\begin{aligned}
&\lim_{n\to\infty}\int_s^t \left|C_j\tilde{u}_n(\sigma,\omega)-C_j\tilde{u}(\sigma,\omega)\right|^2_{\mathscr{L}(\mathbb{R};\mathrm{H})}|\psi|^2_{\mathrm{H}}\, d\sigma \\
&\qquad \leq \lim_{n\to\infty} |\psi|^2_{\mathrm{H}} K_c^2 \int_s^t \|\tilde{u}_n(\sigma,\omega)-\tilde{u}(\sigma,\omega)\|^2_{\mathrm{V}}\, d\sigma = 0.
\end{aligned}$$

Hence, we have proved (5.5.52), finishing the proof of lemma. ■

By Lemma 5.5.9 we can pass to the limit in (5.5.31). By Lemmas 5.5.11 and 5.5.12 we can pass to the limit in (5.5.32) as well. After passing to the limits we infer that for all $\psi,\zeta \in \mathrm{V}$ and $h \in \mathscr{C}([0,T];\mathrm{H})$:

$$\tilde{\mathbb{E}}\left[\langle \tilde{M}(t)-\tilde{M}(s),\psi\rangle h(\tilde{u}_{|[0,s]})\right] = 0, \tag{5.5.53}$$

and

$$\tilde{\mathbb{E}}\Big[\Big(\langle\tilde{M}(t),\psi\rangle\langle\tilde{M}(t),\zeta\rangle - \langle\tilde{M}(s),\psi\rangle\langle\tilde{M}(s),\zeta\rangle \\ -\sum_{j=1}^{m}\int_s^t \big\langle\big(C_j\tilde{u}(\sigma)\big)^*\psi,\big(C_j\tilde{u}(\sigma)\big)^*\zeta\big\rangle_{\mathbb{R}}\,d\sigma\Big)\cdot h(\tilde{u}_{|[0,s]})\Big] = 0. \tag{5.5.54}$$

From the two previous lemmas and Lemma 5.5.6, we infer the following corollary.

Corollary 5.5.13. *For* $t \in [0,T]$

$$\langle\langle\tilde{M}\rangle\rangle_t = \int_0^t \sum_{j=1}^{m} \big|C_j\tilde{u}(s)\big|_{\mathrm{H}}^2\,ds, \qquad t\in[0,T].$$

Theorem 5.3.3 proof continued. Now we apply the idea analogous to that used by Da Prato and Zabczyk, see [38, Section 8.3]. By Lemma 5.5.7 and Corollary 5.5.10, we infer that $\tilde{M}(t)$, $t\in[0,T]$ is an H-valued continuous square integrable martingale with respect to the filtration $\tilde{\mathbb{F}} = (\tilde{\mathscr{F}}_t)_{t\geq 0}$. Moreover, by Corollary 5.5.13 the quadratic variation of $\tilde{M}$ is given by

$$\langle\langle\tilde{M}\rangle\rangle_t = \int_0^t \sum_{j=1}^{m} \big|C_j\tilde{u}(s)\big|_{\mathrm{H}}^2\,ds, \qquad t\in[0,T].$$

Therefore by the martingale representation theorem (see Theorem 2.8.7), there exist

- a stochastic basis $(\tilde{\tilde{\Omega}},\tilde{\tilde{\mathscr{F}}},\tilde{\tilde{\mathscr{F}}}_{t\geq 0},\tilde{\tilde{\mathbb{P}}})$,
- a $\mathbb{R}^m$−valued $\tilde{\tilde{\mathbb{F}}}$−Wiener process $\tilde{\tilde{W}}(t)$ defined on this basis,
- and a progressively measurable process $\tilde{\tilde{u}}(t)$ such that for all $t\in[0,T]$ and $\mathrm{v}\in\mathrm{V}$:

$$\langle\tilde{\tilde{u}}(t),\mathrm{v}\rangle - \langle\tilde{\tilde{u}}(0),\mathrm{v}\rangle + \int_0^t\langle \mathrm{A}\tilde{\tilde{u}}(s),\mathrm{v}\rangle\,ds + \int_0^t \langle B(\tilde{\tilde{u}}(s)),\mathrm{v}\rangle\,ds \\ = \int_0^t |\nabla\tilde{\tilde{u}}(s)|_{L^2}^2\langle\tilde{\tilde{u}}(s),\mathrm{v}\rangle\,ds + \frac{1}{2}\int_0^t\sum_{j=1}^{m}\langle C_j^2\tilde{\tilde{u}}(s),\mathrm{v}\rangle\,ds + \int_0^t\sum_{j=1}^{m}\langle C_j\tilde{\tilde{u}}(s),\mathrm{v}\rangle\,d\tilde{\tilde{W}}(s).$$

Thus the conditions from Definition 5.3.2 hold with $(\hat{\Omega},\hat{\mathscr{F}},\{\hat{\mathscr{F}}_t\}_{t\geq 0},\hat{\mathbb{P}}) = (\tilde{\tilde{\Omega}},\tilde{\tilde{\mathscr{F}}},\{\tilde{\tilde{\mathscr{F}}}_t\}_{t\geq 0},\tilde{\tilde{\mathbb{P}}})$, $\hat{W} = \tilde{\tilde{W}}$ and $\hat{u} = \tilde{\tilde{u}}$. Hence the proof of Theorem 5.3.3 is complete.

5.6 Pathwise uniqueness and strong solution

In this section we will show that the solutions of (5.2.4) are pathwise unique and that problem (5.2.4) has a strong solution in PDE as well as in probabilistic sense. In the previous section we showed that paths of martingale solution u of (5.2.4) belong to $\mathscr{C}([0,T];\mathrm{V}_w)\cap L^2(0,T;\mathrm{D(A)})$. We start by proving Lemma 5.3.5, i.e. showing that the trajectories of the solution $u\in\mathscr{C}([0,T];\mathrm{V})\cap L^2(0,T;\mathrm{D(A)})$.

Proof of Lemma 5.3.5 u is a martingale solution of (5.2.4) thus, $u \in \mathscr{C}([0,T];\mathrm{V}_w) \cap L^2(0,T;\mathrm{D(A)})$ $\hat{\mathbb{P}}-$a.s. We start by showing that RHS of (5.3.5) makes sense. In order to do so we will show that each term on the RHS is well defined.

Firstly we consider the non-linear term arising from Navier-Stokes. Using (3.2.3), the Hölder inequality, (5.5.24) and the estimate (5.3.4), we have the following bounds :

$$\begin{aligned}\hat{\mathbb{E}}\int_0^T |B(u(s))|_{\mathrm{H}}^2\,ds &\le 2\hat{\mathbb{E}}\int_0^T |u(s)|_{\mathrm{H}}|\nabla u(s)|_{L^2}^2|u(s)|_{\mathrm{D(A)}}\,ds \\ &\le 2T^{1/2}\left(\hat{\mathbb{E}} sup_{s\in[0,T]}\|u(s)\|_{\mathrm{V}}^4\right)^{1/2}\left(\hat{\mathbb{E}}\int_0^T |u(s)|_{\mathrm{D(A)}}^2\,ds\right)^{1/2} < \infty.\end{aligned}$$

Using (5.5.24), the Hölder inequality, (5.5.23), estimates (5.5.25) and (5.3.4) we have the following inequalities for the non-linear term generated from the projection of the Stokes operator,

$$\hat{\mathbb{E}}\int_0^T \left||\nabla u(s)|_{L^2}^2 u(s)\right|_{\mathrm{H}}^2 ds = \hat{\mathbb{E}}\int_0^T |\nabla u(s)|_{L^2}^4\,ds \le T\left(\hat{\mathbb{E}}\sup_{s\in[0,T]}\|u(s)\|_{\mathrm{V}}^4\right) < \infty.$$

Next we deal with the correction term arising from the Stratonovich integral. Using assumption **(A.1)** and (5.3.4), for every $j \in \{1,\cdots,m\}$ we have

$$\hat{\mathbb{E}}\int_0^T |C_j^2 u(s)|_{\mathrm{H}}^2 \le K_c^4\hat{\mathbb{E}}\int_0^T |u(s)|_{\mathrm{D(A)}}^2\,ds < \infty,$$

where K_c is defined in (5.3.1).

We are left to show that the Itô integral belongs to $L^2(\Omega\times[0,T];\mathrm{V})$. Due to Itô isometry it is enough to show that for every $j \in \{1,\cdots,m\}$

$$\hat{\mathbb{E}}\int_0^T \|C_j u(s)\|_{\mathrm{V}}^2\,ds < \infty. \tag{5.6.1}$$

Using assumption **(A.1)** and (5.3.4), we have

$$\hat{\mathbb{E}}\int_0^T \|C_j u(s)\|_{\mathrm{V}}^2\,ds \le K_c\hat{\mathbb{E}}\int_0^T |u(s)|_{\mathrm{D(A)}}^2\,ds < \infty.$$

Thus we have shown that each term in (5.3.5) is well defined. Now we will show that the equality holds.

Since u is a martingale solution of (5.2.4), for every $\mathrm{v} \in \mathrm{V}$ and $t \in [0,T]$ it satisfies the equality (5.3.2), i.e. $\hat{\mathbb{P}}-$a.s.

$$\begin{aligned}&\langle u(t),\mathrm{v}\rangle - \langle u_0,\mathrm{v}\rangle + \int_0^t \langle Au(s),\mathrm{v}\rangle\,ds + \int_0^t \langle B(u(s)),\mathrm{v}\rangle\,ds \\ &= \int_0^t |\nabla u(s)|_{L^2}^2\langle u(s),\mathrm{v}\rangle\,ds + \frac{1}{2}\int_0^t\sum_{j=1}^m\langle C_j^2 u(s),\mathrm{v}\rangle\,ds + \int_0^t\sum_{j=1}^m\langle C_j u(s),\mathrm{v}\rangle\,d\hat{W}_j(s).\end{aligned}$$

Note that the above equation holds true for every $\mathrm{v} \in \mathscr{V}$(as defined in (3.1.2)) and hence (5.3.5) holds in the distribution sense. But since $\mathscr{V}$ is dense in V, equality (5.3.5) holds true almost everywhere, which justifies Remark 5.3.4.

We use [71, Lemma 4.1] to prove the first part of the lemma. We work with the $\mathrm{D(A)} \subset \mathrm{V} \subset \mathrm{H}$ space triple. Let us rewrite (5.3.5) in the following form

$$u(t) = u_0 + \int_0^t g(s)\,ds + N(t),$$

where g contains all the deterministic terms and N corresponds to the noise term. We have shown that $g \in L^2(\Omega; L^2(0,T;\mathrm{H}))$ and $N \in L^2(\Omega; L^2(0,T;\mathrm{V}))$. Thus from [71, Lemma 4.1] we infer that $u \in L^2(\Omega; \mathscr{C}([0,T];\mathrm{V}))$. This concludes the proof of lemma. ∎

In the following lemma we will prove that the solutions of (5.2.4) are pathwise unique. The proof uses the Schmalfuss idea of application of the Itô formula for appropriate function (see [82]).

Lemma 5.6.1. *Assume that the assumptions* ***(A.1) - (A.2)*** *are satisfied. If u_1, u_2 are two martingale solutions of* (5.2.4) *defined on the same filtered probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ then $\hat{\mathbb{P}}$−a.s. for all $t \in [0,T]$, $u_1(t) = u_2(t)$.*

Proof. Let us denote the difference of the two solutions by $U := u_1 - u_2$. Then U satisfies the following equation

$$\begin{aligned} dU(t) + [AU(t) + B(u_2(t)) - B(u_1(t))]\,dt &= \left[|\nabla u_1(t)|^2_{L^2} u_1(t) - |\nabla u_2(t)|^2_{L^2} u_2(t)\right] dt \\ &\quad + \sum_{j=1}^m C_j U(t) \circ dW_j(t), \qquad t \in [0,T]. \end{aligned} \tag{5.6.2}$$

Let us define the stopping time

$$\tau_N : = T \wedge \inf\{t \in [0,T] : \|u_1(t)\|^2_{\mathrm{V}} \vee \|u_2(t)\|^2_{\mathrm{V}} > N\}, \quad N \in \mathbb{N}. \tag{5.6.3}$$

Since $\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} \|u_i(t)\|^2_{\mathrm{V}}\right] < \infty$ $\hat{\mathbb{P}}$-a.s. for $i = 1,2$, $\lim_{N\to\infty} \tau_N = T$.

We apply the Itô formula to the function

$$F(t,x) = e^{-r(t)}|x|^2_{\mathrm{H}}, \quad t \in [0,T],\ x \in \mathrm{V}$$

where $r(t)$, $t \in [0,T]$, is a real valued $\mathscr{C}^1$-class function which will be defined precisely later in the proof.

Since

$$\frac{\partial F}{\partial t} = -r'(t) e^{-r(t)}|x|^2_{\mathrm{H}}, \quad \frac{\partial F}{\partial x}(\cdot) = 2e^{-r(t)}\langle x, \cdot\rangle_{\mathrm{H}},$$

we obtain for all $t \in [0,T]$

$$\begin{aligned} e^{-r(t\wedge\tau_N)}|U(t\wedge\tau_N)|^2_{\mathrm{H}} &= \int_0^{t\wedge\tau_N} e^{-r(s)}\left(-r'(s)|U(s)|^2_{\mathrm{H}} + 2\langle -\mathrm{A}U(s) + B(u_1(s)) - B(u_2(s)), U(s)\rangle_{\mathrm{H}}\right) ds \\ &\quad + \int_0^{t\wedge\tau_N} e^{-r(s)}\left(2\langle |\nabla u_1(s)|^2_{L^2} u_1(s) - |\nabla u_2(s)|^2_{L^2} u_2(s), U(s)\rangle_{\mathrm{H}} + \sum_{j=1}^m \langle C_j^2 U(s), U(s)\rangle_{\mathrm{H}}\right) ds \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau_N} \sum_{j=1}^m Tr\left[C_j U(s) \frac{\partial^2 F}{\partial x^2}(C_j U(s))^*\right] ds + 2\int_0^{t\wedge\tau_N} e^{-r(s)} \sum_{j=1}^m \langle C_j U(s), U(s)\rangle_{\mathrm{H}} dW(s). \end{aligned}$$

Thus using the assumption **(A.1)**, we obtain the following simplified expression

$$
\begin{aligned}
e^{-r(t\wedge\tau_N)}|U(t\wedge\tau_N)|_{\mathrm{H}}^2 \le& \int_0^{t\wedge\tau_N} e^{-r(s)}\left(-r'(s)|U(s)|_{\mathrm{H}}^2 - 2\|U(s)\|_{\mathrm{V}}^2 - 2b(U(s),u_1(s),U(s))\right) ds \\
&+2\int_0^{t\wedge\tau_N} e^{-r(s)}\left((|\nabla u_1(s)|_{L^2}^2 - |\nabla u_2(s)|_{L^2}^2)\langle u_1(s),U(s)\rangle_{\mathrm{H}} + |\nabla u_2(s)|_{L^2}^2|U(s)|_{\mathrm{H}}^2\right) ds \\
&+\int_0^{t\wedge\tau_N} e^{-r(s)}\sum_{j=1}^m \left(\langle C_j^2U(s),U(s)\rangle_{\mathrm{H}} + \frac{1}{2}\times 2\langle C_jU(s),C_jU(s)\rangle_{\mathrm{H}}\right) ds.
\end{aligned}
$$

Using (3.2.1) and the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
e^{-r(t\wedge\tau_N)}&|U(t\wedge\tau_N)|_{\mathrm{H}}^2 + 2\int_0^{t\wedge\tau_N} e^{-r(s)}\|U(s)\|_{\mathrm{V}}^2\, ds \\
&\le \int_0^{t\wedge\tau_N} e^{-r(s)}\left(-r'(s)|U(s)|_{\mathrm{H}}^2 + 4|U(s)|_{\mathrm{H}}\|U(s)\|_{\mathrm{V}}\|u_1(s)\|_{\mathrm{V}}\right) ds \\
&\quad +2\int_0^{t\wedge\tau_N} e^{-r(s)}\|U(s)\|_{\mathrm{V}}\Big(|\nabla u_1(s)|_{L^2} + |\nabla u_2(s)|_{L^2}\Big)|u_1(s)|_{\mathrm{H}}|U(s)|_{\mathrm{H}}\, ds \\
&\quad +2\int_0^{t\wedge\tau_N} e^{-r(s)}|\nabla u_2(t)|_{L^2}^2|U(s)|_{\mathrm{H}}^2\, ds.
\end{aligned}
$$

Using the Young inequality, we obtain

$$
\begin{aligned}
e^{-r(t\wedge\tau_N)}|U(t\wedge\tau_N)|_{\mathrm{H}}^2 + 2\int_0^{t\wedge\tau_N} e^{-r(s)}\|U(s)\|_{\mathrm{V}}^2\, ds \le& \int_0^{t\wedge\tau_N} e^{-r(s)}\left[-r'(s) + 8\|u_1(s)\|_{\mathrm{V}}^2\right]|U(s)|_{\mathrm{H}}^2\, ds \\
&+2\int_0^{t\wedge\tau_N} e^{-r(s)}\big(|\nabla u_1(s)|_{L^2} + |\nabla u_2(s)|_{L^2}\big)^2|u_1(s)|_{\mathrm{H}}^2|U(s)|_{\mathrm{H}}^2\, ds \\
&+\int_0^{t\wedge\tau_N} e^{-r(s)}\|U(s)\|_{\mathrm{V}}^2\, ds.
\end{aligned} \tag{5.6.4}
$$

Now choosing

$$
r(t) := \int_0^t \left[8\|u_1(s)\|_{\mathrm{V}}^2 + 2\big(|\nabla u_1(s)|_{L^2} + |\nabla u_2(s)|_{L^2}\big)^2|u_1(s)|_{\mathrm{H}}^2\right] ds,
$$

inequality (5.6.4) reduces to

$$
e^{-r(t\wedge\tau_N)}|U(t\wedge\tau_N)|_{\mathrm{H}}^2 + \int_0^{t\wedge\tau_N} e^{-r(s)}\|U(s)\|_{\mathrm{V}}^2\, ds \le 0.
$$

In particular

$$
\sup_{t\in[0,T]}\left[e^{-r(t\wedge\tau_N)}|U(t\wedge\tau_N)|_{\mathrm{H}}^2\right] = 0. \tag{5.6.5}
$$

Note that since u_1 and u_2 are the martingale solutions of (5.2.4) satisfying the estimates (5.5.4) and (5.5.6) and because of the Lemma 5.5.1, r is well defined for all $t \in [0,T]$.

Since $\hat{\mathbb{P}}$−a.s. $\lim_{N\to\infty}\tau_N = T$ and $\hat{\mathbb{E}}[r(T)] < \infty$, thus from (5.6.5) we infer that $\hat{\mathbb{P}}$−a.s. for all $t \in [0,T]$, $U(t) = 0$. The proof of the lemma is thus complete. ■

Definition 5.6.2. Let $(\Omega^i,\mathscr{F}^i,\mathbb{F}^i,\mathbb{P}^i,W^i,u^i)$, $i=1,2$ be the martingale solutions of (5.2.4) with $u^i(0)=u_0$, $i=1,2$. Then we say that the solutions are unique in law if

$$\mathrm{Law}_{\mathbb{P}^1}(u^1)=\mathrm{Law}_{\mathbb{P}^2}(u^2) \text{ on } \mathscr{C}([0,\infty);\mathrm{V}_w)\cap L^2([0,\infty);\mathrm{D(A)}),$$

where $\mathrm{Law}_{\mathbb{P}^i}(u^i)$, $i=1,2$ are by definition probability measures on $\mathscr{C}([0,\infty);\mathrm{V}_w)\cap L^2([0,\infty);\mathrm{D(A)})$.

Corollary 5.6.3. *Assume that assumptions **(A.1) - (A.2)** are satisfied. Then*

(1) There exists a pathwise unique strong solution of (5.2.4).

(2) Moreover, if $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P},W,u)$ *is a strong solution of* (5.2.4) *then for* $\mathbb{P}-$*almost all* $\omega\in\Omega$ *the trajectory* $u(\cdot,\omega)$ *is equal almost everywhere to a continuous* V$-$*valued function defined on* $[0,T]$.

(3) The martingale solution of (5.2.4) *are unique in law.*

Proof. By Theorem 5.3.3 there exists a martingale solution and in the Lemma 5.6.1 we showed it is pathwise unique, thus assertion (1) follows from [68, Theorem 2]. Assertion (2) is a direct consequence of Lemma 5.3.5. Assertion (3) follows from [68, Theorems 2,11]. ■

Using Theorem 5.3.3, Lemma 5.6.1 and Corollary 5.6.3 one can infer Theorem 5.3.7.

5.7 The continuous dependence of solutions on the initial data

This section deals with the continuous dependence of martingale solutions of (5.2.4) on the initial data. Roughly speaking, we will show that if $(u_{0,n})_{n\in\mathbb{N}}\subset \mathrm{V}\cap\mathscr{M}$ is a sequence of initial conditions approaching in V topology to $u_0\in\mathrm{V}\cap\mathscr{M}$, then the sequence $(u_n)_{n\in\mathbb{N}}$ of martingale solutions of (5.2.4) corresponding to initial data $(u_{0,n})$, satisfying inequalities (5.5.4) – (5.5.6), on a changed probability basis, converges to a martingale solution with the initial condition u_0. Note that existence of such solutions u_n, $n\in\mathbb{N}$, is guaranteed by Theorem 5.3.3. Let us recall that for a fixed $T>0$,

$$\mathcal{Z}_T=\mathscr{C}([0,T];\mathrm{H})\cap L^2_w(0,T;\mathrm{D(A)})\cap L^2(0,T;\mathrm{V})\cap\mathscr{C}([0,T];\mathrm{V}_w).$$

The following auxiliary result which is needed in the proof of Theorem 5.7.7, cannot be deduced directly from the Kuratowski Theorem.

Lemma 5.7.1. *Assume that* $T>0$. *Then the following sets* $\mathscr{C}([0,T];\mathrm{V})\cap\mathcal{Z}_T$ *and* $L^2(0,T;\mathrm{D(A)})\cap\mathcal{Z}_T$ *are Borel subsets of* $\mathcal{Z}_T$.

Proof. First of all $\mathscr{C}([0,T];\mathrm{V})\subset\mathscr{C}([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V})$. Secondly, $\mathscr{C}([0,T];\mathrm{V})$ and $\mathscr{C}([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V})$ are Polish spaces. And finally, since V is continuously embedded in H, the map

$$i:\mathscr{C}([0,T];\mathrm{V})\to\mathscr{C}([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V}),$$

is continuous and hence Borel. Thus by application of the Kuratowski Theorem (see Theorem 2.5.17) $\mathscr{C}([0,T];\mathrm{V})$ is a Borel subset of $\mathscr{C}([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V})$. Therefore by Lemma C.1 $\mathscr{C}([0,T];\mathrm{V})\cap\mathcal{Z}_T$ is a Borel subset of $\mathscr{C}([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V})\cap\mathcal{Z}_T$ which is equal to $\mathcal{Z}_T$.

Similarly we can show that $L^2(0,T;\mathrm{D(A)})\cap\mathcal{Z}_T$ is a Borel subset of $\mathcal{Z}_T$. $L^2(0,T;\mathrm{D(A)})\hookrightarrow L^2(0,T;\mathrm{V})$ and both are Polish spaces thus by application of the Kuratowski Theorem, $L^2(0,T;\mathrm{D(A)})$ is a Borel subset of $L^2(0,T;\mathrm{V})$. Finally, we can conclude the proof of theorem by Lemma C.1. ■

5.7.1 Tightness criterion and the Jakubowski-Skorohod Theorem

One of the main tools in this section is the tightness criterion in the space $\mathcal{Z}_T$. We will use a slight generalization of the criterion stated in Corollary 5.4.3. Namely, we will consider the sequence of stochastic processes defined on their own probability spaces in contrast to one common probability space. Let $(\Omega_n,\mathscr{F}_n,\mathbb{F}_n,\mathbb{P}_n)$, $n\in\mathbb{N}$, be a sequence of probability spaces with the filtration $\mathbb{F}_n=(\mathscr{F}_{n,t})_{t\geq 0}$.

Corollary 5.7.2. *(Tightness criterion) Assume that $(X_n)_{n\in\mathbb{N}}$ is a sequence of continuous $\mathbb{F}_n$-adapted* H*-valued processes defined on Ω_n such that*

$$\sup_{n\in\mathbb{N}}\mathbb{E}_n\big[\sup_{s\in[0,T]}\|X_n(s)\|^2\big]<\infty, \tag{5.7.1}$$

$$\sup_{n\in\mathbb{N}}\mathbb{E}_n\Big[\int_0^T|X_n(s)|^2_{\mathrm{D(A)}}\,ds\Big]<\infty, \tag{5.7.2}$$

(a) *and for every $\varepsilon>0$ and for every $\eta>0$ there exists $\delta>0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued $\mathbb{F}_n$-stopping times one has*

$$\sup_{n\in\mathbb{N}}\sup_{0\leq\theta\leq\delta}\mathbb{P}_n\big\{|X_n(\tau_n+\theta)-X_n(\tau_n)|_{\mathrm{H}}\geq\eta\big\}\leq\varepsilon. \tag{5.7.3}$$

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on the Borel σ-field $\mathscr{B}(\mathcal{Z}_T)$. Then for every $\varepsilon>0$ there exists a compact subset K_ε of $\mathcal{Z}_T$ such that

$$\sup_{n\in\mathbb{N}}\tilde{\mathbb{P}}_n(K_\varepsilon)\geq 1-\varepsilon.$$

The proof of Corollary 5.7.2 is essentially same as the proof of Corollary 5.4.3.

If the sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition **(a)** then we say that it satisfies the Aldous condition **[A]** in H on [0,T]. If it satisfies condition **(a)** for each $T>0$, we say that it satisfies the Aldous condition **[A]** in H (see Definition 2.9.10).

Below we will formulate a sufficient condition for the Aldous condition. This idea has been used in the proof of Lemma 5.5.5, but has not been formulated in such a way.

Lemma 5.7.3. *Assume that Y is a separable Banach space, $\sigma\in(0,1]$ and that $(u_n)_{n\in\mathbb{N}}$ is a sequence of continuous $\mathbb{F}_n$-adapted Y-valued processes indexed by $[0,T]$ for some $T>0$, such that*

(a′) *there exists $C>0$ such that for every $\theta>0$ and for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued $\mathbb{F}_n$-stopping times one has*

$$\mathbb{E}_n\left[|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y\right]\le C\theta^{\sigma}. \tag{5.7.4}$$

Then the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition **[A]** *in Y on $[0,T]$.*

Proof. Let us fix $\eta>0$ and $\varepsilon>0$. By the Chebyshev inequality and the estimate (5.7.4) we obtain

$$\mathbb{P}_n\big(\big\{|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y\ge\eta\big\}\big)\le\frac{1}{\eta}\mathbb{E}_n\left[|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y\right]\le\frac{C\cdot\theta^{\sigma}}{\eta},\qquad n\in\mathbb{N}.$$

Let us choose $\delta:=\left[\frac{\eta\cdot\varepsilon}{C}\right]^{\frac{1}{\sigma}}$. Then we have

$$\sup_{n\in\mathbb{N}}\sup_{1\le\theta\le\delta}\mathbb{P}_n\big\{|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y\ge\eta\big\}\le\varepsilon.$$

This completes the proof. ■

We restate the version of the Skorohod Theorem that we stated in Theorem 5.4.4 in a slightly different way.

Theorem 5.7.4. *Let $(\mathscr{X},\tau)$ be a topological space such that there exists a sequence (f_m) of continuous functions $f_m:\mathscr{X}\to\mathbb{R}$ that separates points of $\mathscr{X}$. Let (X_n) be a sequence of $\mathscr{X}$-valued Borel random variables. Suppose that for every $\varepsilon>0$ there exists a compact subset $K_\varepsilon\subset\mathscr{X}$ such that*

$$\sup_{n\in\mathbb{N}}\mathbb{P}(\{X_n\in K_\varepsilon\})>1-\varepsilon.$$

Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$, a sequence $(Y_k)_{k\in\mathbb{N}}$ of $\mathscr{X}$-valued Borel random variables and an $\mathscr{X}$-valued Borel random variable Y defined on some probability space $(\Omega,\mathscr{F},\mathbb{P})$ such that

$$\mathscr{L}(X_{n_k})=\mathscr{L}(Y_k),\qquad k=1,2,\dots$$

and for all $\omega\in\Omega$:

$$Y_k(\omega)\xrightarrow{\tau}Y(\omega)\quad\text{as } k\to\infty.$$

Note that the sequence (f_m) defines another, weaker topology on $\mathscr{X}$. However, this topology restricted to σ-compact subsets of $\mathscr{X}$ is equivalent to the original topology τ. Let us emphasize that thanks to the assumption on the tightness of the set of laws $\{\mathscr{L}(X_n),n\in\mathbb{N}\}$ on the space $\mathscr{X}$ the maps Y and Y_k, $k\in\mathbb{N}$, in Theorem 5.7.4 are measurable with respect to the Borel σ-field in the space $\mathscr{X}$.

In Lemma 5.4.5 we have already shown that the topological space $\mathscr{Z}_T$ satisfies the assumptions of Theorem 5.7.4.

5.7.2 The continuous dependence

We prove the following result related to the continuous dependence on the deterministic initial condition.

Theorem 5.7.5. *Let $T > 0$. Assume that $\left(u_{0,n}\right)_{n\in\mathbb{N}}$ is a $\mathrm{V}\cap\mathscr{M}$-valued sequence bounded in V and*

$$\left(\hat{\Omega}_n, \hat{\mathscr{F}}_n, \hat{\mathbb{F}}_n, \hat{\mathbb{P}}_n, \hat{W}_n, u_n\right)$$

be a martingale solution of problem (5.2.4) *with the initial data $u_{0,n}$ and satisfying inequalities* (5.5.4) – (5.5.6)*. Then, the set of Borel measures $\{\mathscr{L}(u_n), n \in \mathbb{N}\}$ is tight on the space $(\mathcal{Z}_T, \mathcal{T})$.*

Proof. Let us fix $T > 0$. Let $\left(u_{0,n}\right)_{n\in\mathbb{N}}$ be a $\mathrm{V}\cap\mathscr{M}$-valued sequence. Let

$$\left(\hat{\Omega}_n, \hat{\mathscr{F}}_n, \hat{\mathbb{F}}_n, \hat{\mathbb{P}}_n, \hat{W}_n, u_n\right)$$

be the martingale solution of problem (5.2.4) with the initial data u_0^n and satisfying inequalities (5.5.4) – (5.5.6). Such a solution exists by Theorem 5.3.3.

To show that the set of measures $\{\mathscr{L}(u_n), n \in \mathbb{N}\}$ are tight on the space $(\mathcal{Z}_T, \mathcal{T})$, we argue as in the proof of Lemma 5.5.3 using Corollary 5.7.2. We first observe that due to estimates (5.5.4) (with $p = 1$) and (5.5.6), conditions (5.7.1) and (5.7.2) of Corollary 5.7.2 are satisfied. Thus, we are left to prove condition (**a**), i.e. the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition **[A]**. By Lemma 5.7.3 it is sufficient to prove the condition (**a**′).

Note that we have to choose our steps very carefully as we no longer treat strong solutions to an SDE in a finite dimensional Hilbert space but instead a strong solution to an SPDE in an infinite dimensional Hilbert space.

Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times taking values in $[0, T]$. Since each process satisfies equation (5.3.2), by Lemma 5.3.5 we have

$$\begin{aligned} u_n(t) = u_{0,n} &- \int_0^t \mathrm{A}u_n(s)\,ds - \int_0^t B\big(u_n(s)\big)\,ds + \int_0^t |\nabla u_n(s)|_{L^2}^2 u_n(s)\,ds \\ &+ \frac{1}{2}\int_0^t C^2 u_n(s)\,ds + \int_0^t C u_n(s)\,dW(s) \\ =: J_1^n &+ J_2^n(t) + J_3^n(t) + J_4^n(t) + J_5^n(t) + J_6^n(t), \qquad t \in [0, T], \end{aligned}$$

where the above equality is understood in the space V. Let us choose $\theta > 0$. It is sufficient to show that each sequence J_i^n of processes, $i = 1, \cdots, 6$ satisfies the sufficient condition (**a**′) from Lemma 5.7.3. Now the rest of the proof is identical to that of Lemma 5.5.3. ■

Remark 5.7.6. It is easy to be convinced that u_n take values in $\mathcal{Z}_T$ but it's not so obvious to see that in fact u_n are Borel measurable functions. Indeed, this is so because our construction of the martingale solution is based on the Jakubowski-Skorohod Theorem, see Theorem 5.7.4 for details.

The main result about the continuous dependence of the solutions of the stochastic constrained Navier-Stokes equations on the initial state is expressed in the following theorem.

Theorem 5.7.7. *Assume that* $\big(u_{0,n}\big)_{n\in\mathbb{N}}$ *is a* $\mathrm{V}\cap\mathcal{M}$*-valued sequence that is convergent weakly to* $u_0\in\mathrm{V}\cap\mathcal{M}$*. Let*

$$\big(\hat{\Omega}_n,\hat{\mathscr{F}}_n,\hat{\mathbb{F}}_n,\hat{\mathbb{P}}_n\hat{W}_n,u_n\big)$$

be a martingale solution of problem (5.2.4) *on* $[0,\infty)$ *with the initial data* u_0^n *and satisfying inequalities* (5.5.4) – (5.5.6)*. Then for every* $T>0$ *there exist*

- *a subsequence* $(n_k)_k$*,*
- *a stochastic basis* $\big(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{F}},\tilde{\mathbb{P}}\big)$*,*
- *a* $\mathbb{R}^m$*−valued* $\tilde{\mathbb{F}}$*−Wiener process* $\tilde{W}$
- *and* $\tilde{\mathbb{F}}$*-progressively measurable processes* $\tilde{u}$*,* $\big(\tilde{u}_{n_k}\big)_{k\ge1}$ *(defined on this basis) with laws supported in* $\mathcal{Z}_T$ *such that*

  (5.7.5) $\qquad \tilde{u}_{n_k}$ *has the same law as* u_{n_k} *on* $\mathcal{Z}_T$ *and* $\tilde{u}_{n_k}\to\tilde{u}$ *in* $\mathcal{Z}_T$*,* $\quad\tilde{\mathbb{P}}$ *- a.s.*

  and the system

  $$\big(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{F}},\tilde{\mathbb{P}},\tilde{W},\tilde{u}\big)$$

  is a martingale solution to problem (5.2.4) *on the interval* $[0,T]$ *with the initial data* u_0*. In particular, for all* $t\in[0,T]$ *and all* $\mathrm{v}\in\mathrm{V}$

  $$\begin{aligned}\langle\tilde{u}(t),v\rangle_{\mathrm{V}}-\langle\tilde{u}(0),\mathrm{v}\rangle_{\mathrm{V}}+\int_0^t\langle\mathrm{A}\tilde{u}(s),\mathrm{v}\rangle_{\mathrm{V}}\,ds+\int_0^t\langle B(\tilde{u}(s)),\mathrm{v}\rangle_{\mathrm{V}}\,ds\\ =\int_0^t\langle|\nabla\tilde{u}(s)|_{\mathrm{H}}^2\tilde{u}(s),\mathrm{v}\rangle_{\mathrm{V}}\,ds+\Big\langle\int_0^t C\tilde{u}(s)\circ d\tilde{W}(s),\mathrm{v}\Big\rangle_{\mathrm{V}},\quad\tilde{\mathbb{P}}\text{-a.s.}\end{aligned}$$

  Moreover, the process $\tilde{u}$ *satisfies the following inequality for every* $p\in[1,1+\frac{1}{K_c^2})$

  $$\tilde{\mathbb{E}}\left[\sup_{s\in[0,T]}\|\tilde{u}(s)\|_{\mathrm{V}}^{2p}+\int_0^T|\tilde{u}(s)|_{\mathrm{D(A)}}^2\,ds\right]<\infty. \tag{5.7.6}$$

Proof. Since the product topological space $\mathcal{Z}_T\times\mathscr{C}([0,T],\mathbb{R}^m)$ satisfies the assumptions of Theorem 5.7.4, by applying it together with Theorem 5.7.5, there exists a subsequence (n_k), a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ and $\mathcal{Z}_T\times\mathscr{C}([0,T],\mathbb{R}^m)$-valued Borel random variables $\big(\tilde{u},\tilde{W}\big),\big(\tilde{u}_k,\tilde{W}_k\big)$, $k\in\mathbb{N}$ such that $\tilde{W}$ and $\tilde{W}_k$, $k\in\mathbb{N}$ are $\mathbb{R}^m$-valued Wiener processes such that

(5.7.7) $\qquad$ the laws on $\mathscr{B}(\mathcal{Z}_T\times\mathscr{C}([0,T],\mathbb{R}^m))$ of (u_{n_k},W) and $(\tilde{u}_k,\tilde{W}_k)$ are equal.

where $\mathscr{B}(\mathcal{Z}_T\times\mathscr{C}([0,T],\mathbb{R}^m))$ is the Borel σ-algebra on $\mathcal{Z}_T\times\mathscr{C}([0,T],\mathbb{R}^m)$, and

(5.7.8) $\qquad \big(\tilde{u}_k,\tilde{W}_k\big)$ converges to $\big(\tilde{u},\tilde{W}\big)$ in $\mathcal{Z}_T\times\mathscr{C}([0,T],\mathbb{R}^m)$ $\quad\tilde{\mathbb{P}}$-almost surely on $\tilde{\Omega}$.

Note that since $\mathscr{B}(\mathcal{Z}_T \times \mathscr{C}([0,T],\mathbb{R}^m)) \subset \mathscr{B}(\mathcal{Z}_T) \times \mathscr{B}(\mathscr{C}([0,T],\mathbb{R}^m))$, the function u is $\mathcal{Z}_T$ Borel random variable.

Define a corresponding sequence of filtrations by

$$\tilde{\mathbb{F}}_k = (\tilde{\mathscr{F}}_k(t))_{t\geq 0}, \text{ where } \tilde{\mathscr{F}}_k(t) = \sigma\big(\{(\tilde{u}_k(s), \tilde{W}_k(s)),\ s \leq t\}\big),\ t \in [0,T]. \tag{5.7.9}$$

To conclude the proof, we need to show that the random variable $\tilde{u}$ gives rise to a martingale solution. The proof of this claim is very similar to the proof of Theorem 2.3 in [66]. Let us denote the subsequence $(\tilde{u}_{n_k})_k$ again by $(\tilde{u}_n)_n$.

The few differences are:

(i) The finite dimensional space H_n is replaced by the whole space H. But now since the space $\mathscr{C}([0,T];\mathrm{H})$ is a Borel subset of $\mathcal{Z}_T$ and $\tilde{u}_n$ and u_n have the same laws on $\mathcal{Z}_T$, we infer that

$$\tilde{u}_n \in \mathscr{C}([0,T];\mathrm{H}) \qquad n \geq 1,\ \tilde{\mathbb{P}}\text{-a.s.}$$

(ii) The operator P_n has to be replaced by the identity. But this is rather a simplification.

In addition to point (i) above, we have that for every $p \in [1, 1+\frac{1}{K_c^2})$

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\Big(\sup_{0\leq s\leq T} \|\tilde{u}_n(s)\|_{\mathrm{V}}{}^{2p}\Big) \leq C_1(p), \tag{5.7.10}$$

Similarly,

$$\tilde{u}_n \in L^2(0,T;\mathrm{D(A)}) \qquad n \geq 1,\ \mathbb{P}\text{-a.s.}$$

and

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\Big[\int_0^T |\tilde{u}_n(s)|^2_{\mathrm{D(A)}}\, ds\Big] \leq C_2. \tag{5.7.11}$$

By inequality (5.7.11) we infer that the sequence $(\tilde{u}_n)$ contains a subsequence, still denoted by $(\tilde{u}_n)$, convergent weakly in the space $L^2([0,T]\times\tilde{\Omega};\mathrm{D(A)})$. Since by (5.7.8) $\tilde{\mathbb{P}}$-a.s. $\tilde{u}_n \to \tilde{u}$ in $\mathcal{Z}_T$, we conclude that $\tilde{u} \in L^2([0,T]\times\tilde{\Omega};\mathrm{D(A)})$, i.e.

$$\tilde{\mathbb{E}}\Big[\int_0^T |\tilde{u}(s)|^2_{\mathrm{D(A)}}\, ds\Big] < \infty. \tag{5.7.12}$$

Similarly, by inequality (5.7.10) we can choose a subsequence of $(\tilde{u}_n)$ convergent weak star in the space $L^p(\tilde{\Omega};L^\infty(0,T;\mathrm{V}))$ and, using (5.7.8), infer that

$$\tilde{\mathbb{E}}\Big[\sup_{0\leq s\leq T} \|\tilde{u}(s)\|_{\mathrm{V}}{}^{2p}\Big] < \infty. \tag{5.7.13}$$

The remaining proof will be done in two steps.

Step 1. Let us fix $T > 0$. We will first prove the following Lemma.

Lemma 5.7.8. *For all $t \in (0,T]$ and $\varphi \in \mathrm{V}$*

(a) $\lim_{n\to\infty}\tilde{\mathbb{E}}\left[\int_0^T \left|\langle \tilde{u}_n(t)-\tilde{u}(t),\varphi\rangle_{\mathrm{H}}\right|^2 dt\right]=0,$

(b) $\lim_{n\to\infty}\tilde{\mathbb{E}}\left[\left|\langle \tilde{u}_n(0)-\tilde{u}(0),\varphi\rangle_{\mathrm{H}}\right|^2\right]=0,$

(c) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\int_0^t \langle \mathrm{A}\tilde{u}_n(s)-\mathrm{A}\tilde{u}(s),\varphi\rangle\, ds\big|\, dt\big]=0,$

(d) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\int_0^t \langle B(\tilde{u}_n(s))-B(\tilde{u}(s)),\varphi\rangle\, ds\big|\, dt\big]=0,$

(e) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\int_0^t \langle |\nabla\tilde{u}_n(s)|^2_{L^2}\tilde{u}_n(s)-|\nabla\tilde{u}(s)|^2_{L^2}\tilde{u}(s),\varphi\rangle\, ds\big|\, dt\big]=0,$

(f) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\int_0^t \langle C^2\tilde{u}_n(s)-C^2\tilde{u}(s),\varphi\rangle\, ds\big|\, dt\big]=0,$

(g) $\lim_{n\to\infty}\tilde{\mathbb{E}}\big[\int_0^T\big|\langle\int_0^t [C\tilde{u}_n(s)-C\tilde{u}(s)]\,d\tilde{W}(s),\varphi\rangle\big|^2 dt\big]=0.$

Proof. Let us fix $\varphi\in\mathrm{V}$.

Ad (a). Since by (5.7.8) $\tilde{u}_n\to\tilde{u}$ in $\mathscr{C}([0,T];\mathrm{H})$ $\tilde{\mathbb{P}}$-a.s., $\langle\tilde{u}_n(\cdot),\varphi\rangle_{\mathrm{H}}\to\langle\tilde{u}(\cdot),\varphi\rangle_{\mathrm{H}}$ in $\mathscr{C}([0,T];\mathbb{R})$, $\tilde{\mathbb{P}}$-a.s. Hence, in particular, for all $t\in[0,T]$

$$\lim_{n\to\infty}\langle\tilde{u}_n(t),\varphi\rangle_{\mathrm{H}}=\langle\tilde{u}(t),\varphi\rangle_{\mathrm{H}},\qquad \tilde{\mathbb{P}}\text{-a.s.}$$

Since $\tilde{u}_n(t)\in\mathscr{M}$ for all $t\in[0,T]$, $\sup_{t\in[0,T]}|\tilde{u}_n(t)|^2_{\mathrm{H}}<\infty$, $\tilde{\mathbb{P}}$-a.s., using the dominated convergence theorem we infer that

$$\lim_{n\to\infty}\int_0^T|\langle\tilde{u}_n(t)-\tilde{u}(t),\varphi\rangle_{\mathrm{H}}|^2\,dt=0\qquad\tilde{\mathbb{P}}\text{-a.s.}\,. \tag{5.7.14}$$

Since $\tilde{u}_n(t),\tilde{u}(t)\in\mathscr{M}$ for all $t\in[0,T]$, by the Hölder inequality for every $n\in\mathbb{N}$ and $r\in[1,\infty)$

$$\tilde{\mathbb{E}}\Big[\Big|\int_0^T|\tilde{u}_n(t)-\tilde{u}(t)|^2_{\mathrm{H}}\,dt\Big|^r\Big]\le c\tilde{\mathbb{E}}\Big[\int_0^T\big(|\tilde{u}_n(t)|^{2r}_{\mathrm{H}}+|\tilde{u}(t)|^{2r}_{\mathrm{H}}\big)\,dt\Big]=2cT, \tag{5.7.15}$$

where c is some positive constant. To conclude the proof of assertion (a) it is sufficient to use (5.7.14), (5.7.15) and the Vitali Theorem.

Ad (b). Since by (5.7.8) $\tilde{u}_n\to\tilde{u}$ in $\mathscr{C}(0,T;\mathrm{H})$ $\tilde{\mathbb{P}}$-a.s. and $\tilde{u}$ is continuous at $t=0$, we infer that $\langle\tilde{u}_n(0),\varphi\rangle_{\mathrm{H}}\to\langle\tilde{u}(0),\varphi\rangle_{\mathrm{H}}$, $\tilde{\mathbb{P}}$-a.s. Now, assertion (b) follows from the Vitali Theorem.

Ad (c). Since by (5.7.8) $\tilde{u}_n\to\tilde{u}$ in $L^2_w(0,T;\mathrm{D(A)})$, $\tilde{\mathbb{P}}$-a.s., we infer that $\tilde{\mathbb{P}}$-a.s.

$$\begin{aligned}\lim_{n\to\infty}\int_0^t\langle\mathrm{A}\tilde{u}_n(s),\varphi\rangle\,ds&=\lim_{n\to\infty}\int_0^t\langle\tilde{u}_n(s),\mathrm{A}^{-1}\varphi\rangle_{\mathrm{D(A)}}\,ds\\&=\int_0^t\langle\tilde{u}(s),\mathrm{A}^{-1}\varphi\rangle_{\mathrm{D(A)}}\,ds=\int_0^t\langle\mathrm{A}\tilde{u}(s),\varphi\rangle\,ds\end{aligned} \tag{5.7.16}$$

By the Hölder inequality and estimate (5.7.11) we infer that for all $t\in[0,T]$ and $n\in\mathbb{N}$

$$\tilde{\mathbb{E}}\Big[\Big|\int_0^t\langle\mathrm{A}\tilde{u}_n(s),\varphi\rangle\,ds\Big|^2\Big]\le c\,|\varphi|^2_{\mathrm{H}}\,\tilde{\mathbb{E}}\Big[\int_0^T|\tilde{u}_n(s)|^2_{\mathrm{D(A)}}\,ds\Big]\le\tilde{c}C_2, \tag{5.7.17}$$

where $c,\tilde{c}>0$ are some constants. By (5.7.16), (5.7.17) and the Vitali Theorem we conclude that for all $t\in[0,T]$

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\Big[\Big|\int_0^t\langle\mathrm{A}\tilde{u}_n(s)-\mathrm{A}\tilde{u}(s),\varphi\rangle\,ds\Big|\Big]=0.$$

Assertion (c) follows now from (5.7.11) and the dominated convergence theorem.

Ad (d). By (5.7.8), $\tilde{u}_n \to \tilde{u}$ in $\mathscr{C}([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V})$, $\tilde{\mathbb{P}}$-a.s. and since $\tilde{u}_n(t)\in\mathscr{M}$ for all $t\in[0,T]$ hence by (3.2.1) we infer that $\tilde{\mathbb{P}}$-a.s. for all $t\in[0,T]$ and $\varphi\in\mathrm{V}$

$$\begin{aligned}
&\lim_{n\to\infty}\left[\int_0^t \langle B(\tilde{u}_n(s)),\varphi\rangle_{\mathrm{H}}\,ds - \int_0^t \langle B(\tilde{u}(s)),\varphi\rangle_{\mathrm{H}}\,ds\right]\\
&= \lim_{n\to\infty}\int_0^t \left[b(\tilde{u}_n(s),\tilde{u}_n(s),\varphi) - b(\tilde{u}(s),\tilde{u}(s),\varphi)\right]\,ds\\
&= \lim_{n\to\infty}\left[\int_0^t b(\tilde{u}_n(s)-\tilde{u}(s),\tilde{u}_n(s),\varphi)\,ds + \int_0^t b(\tilde{u}(s),\tilde{u}_n(s)-\tilde{u}(s),\varphi)\,ds\right]\\
&\le \lim_{n\to\infty}\int_0^t (\|\tilde{u}_n(s)\|_{\mathrm{V}}\| + \|\tilde{u}(s)\|_{\mathrm{V}})\,\|\tilde{u}_n(s)-\tilde{u}(s)\|_{\mathrm{V}}\|\varphi\|_{\mathrm{V}}\,ds\\
&\le \lim_{n\to\infty}\left(|\tilde{u}_n|_{L^2(0,T;\mathrm{V})} + |\tilde{u}|_{L^2(0,T;\mathrm{V})}|\right)|\tilde{u}_n-\tilde{u}|_{L^2(0,T;\mathrm{V})}\|\varphi\|_{\mathrm{V}} = 0. \qquad (5.7.18)
\end{aligned}$$

Using the Hölder inequality, (3.2.1), and the estimate (5.7.10) we infer that for all $t\in[0,T]$, $r\in\left(1,1+\frac{1}{K_c^2}\right)$ and $n\in\mathbb{N}$ the following inequalities hold

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\left|\int_0^t \langle B(\tilde{u}_n(s)),\varphi\rangle\,ds\right|^r\right] \le \tilde{\mathbb{E}}\left[\left(\int_0^t |B(\tilde{u}_n(s))|_{\mathrm{V}'}\|\varphi\|_{\mathrm{V}}\,ds\right)^r\right]\\
&\le (c\|\varphi\|_{\mathrm{V}})^r\,\tilde{\mathbb{E}}\left(\int_0^t \|\tilde{u}_n(s)\|_{\mathrm{V}}{}^{2r}\,ds\right)\\
&\le \tilde{c}\tilde{\mathbb{E}}\big[\sup_{s\in[0,T]}\|\tilde{u}_n(s)\|_{\mathrm{V}}{}^{2r}\big] \le \tilde{C}C_1(r). \qquad (5.7.19)
\end{aligned}$$

By (5.7.18), (5.7.19) and the Vitali Theorem we obtain for all $t\in[0,T]$

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\left[\left|\int_0^t \langle B(\tilde{u}_n(s))-B(\tilde{u}(s)),\varphi\rangle\,ds\right|\right] = 0. \qquad (5.7.20)$$

Hence by (5.7.20) and the dominated convergence theorem, we infer that assertion (d) holds.

Ad (e). By (5.7.8), $\tilde{u}_n \to \tilde{u}$ in $\mathscr{C}([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V})$, $\tilde{\mathbb{P}}$-a.s. and since $\tilde{u}_n(t)\in\mathscr{M}$ for all $t\in[0,T]$ hence we infer that $\tilde{\mathbb{P}}$-a.s. for all $t\in[0,T]$

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t |\nabla\tilde{u}_n(s)|_{L^2}^2\langle\tilde{u}_n(s),\varphi\rangle_{\mathrm{H}}\,ds - \int_0^t |\nabla\tilde{u}(s)|_{L^2}^2\langle\tilde{u}(s),\varphi\rangle_{\mathrm{H}}\,ds\\
&= \lim_{n\to\infty}\int_0^t \left[|\nabla\tilde{u}_n(s)|_{L^2}^2 - |\nabla\tilde{u}(s)|_{L^2}^2\right]\langle\tilde{u}_n(s),\varphi\rangle_{\mathrm{H}}\,ds + \int_0^t |\nabla\tilde{u}(s)|_{L^2}^2\langle\tilde{u}_n(s)-\tilde{u}(s),\varphi\rangle_{\mathrm{H}}\,ds\\
&= \lim_{n\to\infty}\int_0^t \left[|\nabla\tilde{u}_n(s)|_{L^2} - |\nabla\tilde{u}(s)|_{L^2}\right]\left[|\nabla\tilde{u}_n(s)|_{L^2} + |\nabla\tilde{u}(s)|_{L^2}\right]\langle\tilde{u}_n(s),\varphi\rangle_{\mathrm{H}}\,ds\\
&\quad + \lim_{n\to\infty}\int_0^t |\nabla\tilde{u}(s)|_{L^2}^2\langle\tilde{u}_n(s)-\tilde{u}(s),\varphi\rangle_{\mathrm{H}}\,ds\\
&\le \lim_{n\to\infty}\int_0^t (\|\tilde{u}_n(s)-\tilde{u}(s)\|_{\mathrm{V}})(\|\tilde{u}_n(s)\|_{\mathrm{V}}\|\tilde{u}(s)\|_{\mathrm{V}})\,|\tilde{u}_n(s)|_{\mathrm{H}}|\varphi|_{\mathrm{H}}\,ds\\
&\quad + \lim_{n\to\infty}\int_0^t \|\tilde{u}(s)\|^2|\tilde{u}_n(s)-\tilde{u}(s)|_{\mathrm{H}}|\varphi|_{\mathrm{H}}\,ds\\
&\le \lim_{n\to\infty}\tilde{c_1}\left[|\tilde{u}_n|_{L^2(0,T;\mathrm{V})} + |\tilde{u}|_{L^2(0,T;\mathrm{V})}\right]|\tilde{u}|_{L^\infty(0,T;\mathrm{H})}|\tilde{u}_n-\tilde{u}|_{L^2(0,T;\mathrm{V})}\\
&\quad + \lim_{n\to\infty}\tilde{c_2}|\tilde{u}(s)|_{L^2(0,T;\mathrm{V})}^2|\tilde{u}_n-\tilde{u}|_{L^\infty(0,T;\mathrm{H})} = 0. \qquad (5.7.21)
\end{aligned}$$

Using the Hölder inequality, (5.7.10) and the fact that $u_n(t) \in \mathscr{M}$ for all $t \in [0,T]$ we infer that for all $t \in [0,T]$, $r \in \left(1, 1+\frac{1}{K_c^2}\right)$ and $n \in \mathbb{N}$ the following inequalities hold

$$\begin{aligned}
\tilde{\mathbb{E}}\Big[\Big|\int_0^t |\nabla \tilde{u}_n(s)|^2_{L^2} \langle \tilde{u}_n(s), \varphi\rangle_{\mathrm{H}}\, ds\Big|^r\Big] &\le \tilde{\mathbb{E}}\Big[\Big(\int_0^t \|\tilde{u}_n(s)\|^2_{\mathrm{V}} |\tilde{u}_n(s)|_{\mathrm{H}} |\varphi|_{\mathrm{H}}\, ds\Big)^r\Big] \\
&\le (c|\varphi|_{\mathrm{H}})^r\, \tilde{\mathbb{E}}\Big[\Big(\int_0^t |\tilde{u}_n(s)|_{\mathrm{H}}^{\frac{r}{r-1}}\Big)^{r-1} \int_0^t \|\tilde{u}_n(s)\|_{\mathrm{V}}{}^{2r}\, ds\Big] \\
&\le \tilde{c} t^{r-1} \tilde{\mathbb{E}}\big[\sup_{s\in[0,T]} \|\tilde{u}_n(s)\|_{\mathrm{V}}{}^{2r}\big] \le \tilde{C} C_1(r). \qquad (5.7.22)
\end{aligned}$$

By (5.7.21), (5.7.22) and the Vitali Theorem we obtain for all $t \in [0,T]$

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\left[\left|\int_0^t \left\langle |\nabla \tilde{u}_n(s)|^2_{\mathrm{H}} \tilde{u}_n(s) - |\nabla \tilde{u}(s)|^2_{\mathrm{H}} \tilde{u}(s), \varphi\right\rangle_{\mathrm{H}} ds\right|\right] = 0. \qquad (5.7.23)$$

Hence by (5.7.23) and the dominated convergence theorem, we infer that assertion (e) holds.

Ad (f). Since by (5.7.8), $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;\mathrm{D(A)})$, $\tilde{\mathbb{P}}$-a.s., using (5.7.11) we infer that $\tilde{\mathbb{P}}$-a.s.

$$\begin{aligned}
\lim_{n\to\infty} \int_0^t \langle C^2 \tilde{u}_n(s) - C^2 \tilde{u}(s), \varphi\rangle_{\mathrm{H}}\, ds &= \lim_{n\to\infty} \int_0^t \langle C^2(\tilde{u}_n(s) - \tilde{u}(s)), \varphi\rangle_{\mathrm{H}}\, ds \\
&= \lim_{n\to\infty} \int_0^t \langle C^2 \mathrm{A}^{-1} \mathrm{A}(\tilde{u}_n(s) - \tilde{u}(s)), \varphi\rangle_{\mathrm{H}}\, ds.
\end{aligned}$$

Now since $C^2\mathrm{A}^{-1}$ is a bounded operator

$$\begin{aligned}
\lim_{n\to\infty} \int_0^t \langle C^2 \tilde{u}_n(s) - C^2 \tilde{u}(s), \varphi\rangle_{\mathrm{H}}\, ds &= \lim_{n\to\infty} \int_0^t \langle \mathrm{A}(\tilde{u}_n(s) - \tilde{u}(s)), (C^2\mathrm{A}^{-1})^* \varphi\rangle_{\mathrm{H}}\, ds \\
&= \lim_{n\to\infty} \int_0^t \langle \tilde{u}_n(s) - \tilde{u}(s), \mathrm{A}^{-1}(C^2\mathrm{A}^{-1})^* \varphi\rangle_{\mathrm{D(A)}}\, ds = 0. \qquad (5.7.24)
\end{aligned}$$

By the Hölder inequality and estimate (5.7.11) we infer that for all $t \in [0,T]$ and $n \in \mathbb{N}$

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\left|\int_0^t \langle C^2 \tilde{u}_n(s), \varphi\rangle_{\mathrm{H}}\, ds\right|^2\right] &= \tilde{\mathbb{E}}\left[\left|\int_0^t \langle \tilde{u}_n(s), \mathrm{A}^{-1}(C^2 A^{-1})^* \varphi\rangle_{\mathrm{D(A)}}\, ds\right|^2\right] \\
&\le \tilde{c} |\varphi|^2_{\mathrm{H}} \tilde{\mathbb{E}}\left[\int_0^t |\tilde{u}_n(s)|^2_{\mathrm{D(A)}}\, ds\right] \le \tilde{c} C_2, \qquad (5.7.25)
\end{aligned}$$

where $\tilde{c} > 0$ is some constant. By (5.7.24), (5.7.25) and the Vitali theorem we conclude that for all $t \in [0,T]$

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\left[\left|\int_0^t \langle C^2 \tilde{u}_n(s) - C^2 \tilde{u}(s), \varphi\rangle_{\mathrm{H}}\, ds\right|\right] = 0.$$

Assertion (f) follows from (5.7.11) and the dominated convergence theorem.

Ad (g) Since by (5.7.8) $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;\mathrm{V})$, $\tilde{\mathbb{P}}$-a.s., we infer that for all $t \in [0,T]$ and $\varphi \in \mathrm{H}$

$$\begin{aligned}
\lim_{n\to\infty} \int_0^t |\langle C\tilde{u}_n(s) - C\tilde{u}(s), \varphi\rangle|^2_{\mathcal{T}_2(\mathbb{R}^m,\mathbb{R})}\, ds &\le \lim_{n\to\infty} |\varphi|^2_{\mathrm{H}} \int_0^t |C\tilde{u}_n(s) - C\tilde{u}(s)|^2_{\mathrm{H}}\, ds \\
&\le \lim_{n\to\infty} K_c^2 |\varphi|^2_{\mathrm{H}} \int_0^t \|\tilde{u}_n(s) - \tilde{u}(s)\|^2_{\mathrm{V}}\, ds = 0, \qquad (5.7.26)
\end{aligned}$$

where $c>0$ is some constant.

By the uniform estimates (5.7.10) and (5.7.13) we obtain the following inequalities for every $t\in[0,T]$, $r\in(1,1+\frac{1}{K_c^2})$ and $n\in\mathbb{N}$

$$\begin{aligned}\tilde{\mathbb{E}}\left[\left|\int_0^t |\langle C\tilde{u}_n(s)-C\tilde{u}(s),\varphi\rangle|^2_{\mathcal{T}_2(\mathbb{R}^m;\mathbb{R})}\,ds\right|^r\right] &\le c\tilde{\mathbb{E}}\left[|\varphi|^{2r}_{\mathrm{H}}\int_0^t \left[|C\tilde{u}_n(s)|^{2r}_{\mathrm{H}}+|C\tilde{u}(s)|^{2r}_{\mathrm{H}}\right]ds\right]\\ &\le \tilde{c}\tilde{\mathbb{E}}\left[\sup_{s\in[0,T]}\|\tilde{u}_n(s)\|^{2r}_{\mathrm{V}}+\sup_{s\in[0,T]}\|\tilde{u}(s)\|^{2r}_{\mathrm{V}}\right]\le 2\tilde{c}C_1(r),\end{aligned}\tag{5.7.27}$$

where $c,\tilde{c}$ are some positive constants. Using the Vitali theorem, by (5.7.26) and (5.7.27) we infer that for all $\varphi\in\mathrm{H}$

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\left[\int_0^t |\langle C\tilde{u}_n(s)-C\tilde{u}(s),\varphi\rangle|^2_{\mathcal{T}_2(\mathbb{R}^m,\mathbb{R})}\,ds\right]=0.\tag{5.7.28}$$

Hence, by the properties of the Itô integral we infer that for all $t\in[0,T]$ and $\varphi\in\mathrm{H}$

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\left[\left|\left\langle\int_0^t [C\tilde{u}_n(s)-C\tilde{u}(s)]\,d\tilde{W}(s),\varphi\right\rangle\right|^2\right]=0.\tag{5.7.29}$$

By the Itô isometry, and estimates (5.7.10), (5.7.13) we have for all $\varphi\in\mathrm{H}$, $t\in[0,T]$ and $n\in\mathbb{N}$

$$\begin{aligned}&\tilde{\mathbb{E}}\big[\big|\big\langle\int_0^t [C\tilde{u}_n(s)-C\tilde{u}(s)]d\tilde{W}(s),\varphi\big\rangle\big|^2\big]\\ &=\tilde{\mathbb{E}}\big[\int_0^t |\langle C\tilde{u}_n(s)-C\tilde{u}(s),\varphi\rangle|^2_{\mathcal{T}_2(\mathbb{R}^m;\mathbb{R})}ds\big]\\ &\le\tilde{\mathbb{E}}\big[|\varphi|^2_{\mathrm{H}}\int_0^t |C\tilde{u}_n(s)-C\tilde{u}(s)|^2_{\mathrm{H}}ds\big]\\ &\le c\tilde{\mathbb{E}}\big[\sup_{s\in[0,T]}\|\tilde{u}_n(s)\|^2_{\mathrm{V}}+\sup_{s\in[0,T]}\|\tilde{u}(s)\|^2_{\mathrm{V}}\big]\le 2cC_1(1),\end{aligned}\tag{5.7.30}$$

where $c>0$ is some constant. Thus by (5.7.29), (5.7.30) and the Lebesgue Dominated Convergence theorem we infer that for all $\varphi\in\mathrm{H}$

$$\lim_{n\to\infty}\int_0^T\tilde{\mathbb{E}}\left[\left|\left\langle\int_0^t [C\tilde{u}_n(s)-C\tilde{u}(s)]\,d\tilde{W}(s),\varphi\right\rangle\right|^2\right]=0.\tag{5.7.31}$$

□

As a direct consequence of Lemma 5.7.8 we get the following corollary which we precede by introducing some auxiliary notation. Analogously to [20] and [66], let us denote

$$\begin{aligned}\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi)(t):=\langle\tilde{u}_n(0),\varphi\rangle_{\mathrm{H}}&-\int_0^t\langle \mathrm{A}\tilde{u}_n(s),\varphi\rangle ds-\int_0^t\langle B(\tilde{u}_n(s)),\varphi\rangle ds\\ &+\int_0^t\langle|\nabla\tilde{u}_n(s)|^2_{L^2}\tilde{u}_n(s),\varphi\rangle\,ds+\langle\int_0^t C\tilde{u}_n(s)\circ d\tilde{W}_n(s),\varphi\rangle,\quad t\in[0,T],\end{aligned}\tag{5.7.32}$$

and

$$\begin{aligned}\Lambda(\tilde{u},\tilde{W},\varphi)(t):=\langle\tilde{u}(0),\varphi\rangle_{\mathrm{H}}&-\int_0^t\langle \mathrm{A}\tilde{u}(s),\varphi\rangle ds-\int_0^t\langle B(\tilde{u}(s)),\varphi\rangle ds\\ &+\int_0^t\langle|\nabla\tilde{u}(s)|^2_{L^2}\tilde{u}(s),\varphi\rangle\,ds+\langle\int_0^t C\tilde{u}(s)\circ d\tilde{W}(s),\varphi\rangle,\quad t\in[0,T].\end{aligned}\tag{5.7.33}$$

Corollary 5.7.9. *For every* $\varphi \in \mathrm{V}$*,*

$$\lim_{n\to\infty} |\langle \tilde{u}_n(\cdot),\varphi\rangle_{\mathrm{H}} - \langle \tilde{u}(\cdot),\varphi\rangle_{\mathrm{H}}|_{L^2([0,T]\times\tilde{\Omega})} = 0 \tag{5.7.34}$$

and

$$\lim_{n\to\infty} |\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi) - \Lambda(\tilde{u},\tilde{W},\varphi)|_{L^1([0,T]\times\tilde{\Omega})} = 0. \tag{5.7.35}$$

Proof. Assertion (5.7.34) follows from the equality

$$|\langle \tilde{u}_n(\cdot),\varphi\rangle_{\mathrm{H}} - \langle \tilde{u}(\cdot),\varphi\rangle_{\mathrm{H}}|^2_{L^2([0,T]\times\tilde{\Omega})} = \tilde{\mathbb{E}}\Big[\int_0^T |\langle \tilde{u}_n(t)-\tilde{u}(t),\varphi\rangle_{\mathrm{H}}|^2\,dt\Big]$$

and Lemma 5.7.8 (a). Let us move to the proof of assertion (5.7.35). Note that by the Fubini theorem, we have

$$\begin{aligned}&|\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi) - \Lambda(\tilde{u},\tilde{W},\varphi)|_{L^1([0,T]\times\tilde{\Omega})}\\ &= \int_0^T \tilde{\mathbb{E}}\big[|\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi)(t) - \Lambda(\tilde{u},\tilde{W},\varphi)(t)|\,\big]dt.\end{aligned}$$

To conclude the proof of Corollary 5.7.9 it is sufficient to note that by Lemma 5.7.8 (b) – (g), each term on the right hand side of (5.7.32) tends at least in $L^1([0,T]\times\tilde{\Omega})$ to the corresponding term in (5.7.33). □

Step 2. Since u_n is a solution of the stochastic constrained Navier-Stokes equations (5.2.4), for all $t\in[0,T]$ and $\varphi\in\mathrm{V}$

$$\langle u_n(t),\varphi\rangle_{\mathrm{H}} = \Lambda_n(u_n,W,\varphi)(t), \qquad \mathbb{P}\text{-a.s.}$$

In particular,

$$\int_0^T \mathbb{E}\big[|\langle u_n(t),\varphi\rangle_{\mathrm{H}} - \Lambda_n(u_n,W,\varphi)(t)|\,\big]\,dt = 0.$$

Since $\mathscr{L}(u_n,W) = \mathscr{L}(\tilde{u}_n,\tilde{W}_n)$,

$$\int_0^T \tilde{\mathbb{E}}\big[|\langle \tilde{u}_n(t),\varphi\rangle_{\mathrm{H}} - \Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi)(t)|\,\big]\,dt = 0.$$

Moreover, by (5.7.34) and (5.7.35)

$$\int_0^T \tilde{\mathbb{E}}\big[|\langle \tilde{u}(t),\varphi\rangle_{\mathrm{H}} - \Lambda(\tilde{u},\tilde{W},\varphi)(t)|\,\big]\,dt = 0.$$

Hence for Lebesgue-almost all $t\in[0,T]$ and $\tilde{\mathbb{P}}$-almost all $\omega\in\tilde{\Omega}$

$$\langle \tilde{u}(t),\varphi\rangle_{\mathrm{H}} - \Lambda(\tilde{u},\tilde{W},\varphi)(t) = 0,$$

i.e. for Lebesgue-almost all $t\in[0,T]$ and $\tilde{\mathbb{P}}$-almost all $\omega\in\tilde{\Omega}$

$$\begin{aligned}&\langle \tilde{u}(t),\varphi\rangle_{\mathrm{H}} + \int_0^t \langle \mathrm{A}\tilde{u}(s),\varphi\rangle\,ds + \int_0^t \langle B(\tilde{u}(s)),\varphi\rangle\,ds\\ &= \langle \tilde{u}(0),\varphi\rangle_{\mathrm{H}} + \int_0^t \langle |\nabla u(s)|^2_{L^2} u(s),\varphi\rangle\,ds + \langle \int_0^t Cu(s)\circ d\tilde{W}(s),\varphi\rangle.\end{aligned} \tag{5.7.36}$$

Putting $\tilde{\mathscr{U}} := (\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}})$, we infer that the system $(\tilde{\mathscr{U}}, \tilde{W}, \tilde{u})$ is a martingale solution of equation (5.2.4). By (5.7.12) and (5.7.13) the process $\tilde{u}$ satisfies inequality (5.7.6). The proof of Theorem 5.7.7 is thus complete. ■

5.8 Sequentially weak Feller property

In this section we show that the family $\{T_t\}_{t\geq 0}$ defined by formula (5.8.1) is sequentially weakly Feller. We show that the weak convergence of the solutions of SCNSE in V is sufficient to establish the sequentially weak Feller property of $\{T_t\}_{t\geq 0}$. This property, along with some a'priori estimates (e.g. boundedness in probability) implies existence of an invariant measure, see [62] for a generalised result and [25] for a particular case, i.e. NSEs in two dimensional unbounded domains. However, so far we have been unable to find such a'priori bounds.

Let us fix a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$ and an $\mathbb{R}^m$-valued standard Wiener process W on this stochastic basis. By $u(t, u_0)$ we denote the pathwise unique strong solution to equation (5.2.4), defined on the above stochastic basis (which exists by Theorem 5.3.7). For any bounded Borel function $\varphi \in \mathscr{B}_b(\mathrm{V})$, $t \geq 0$, we define a function $T_t\varphi \colon \mathrm{V} \to \mathbb{R}$ by

$$(T_t\varphi)(u_0) := \mathbb{E}\left[\varphi(u(t, u_0))\right], \quad u_0 \in \mathrm{V}. \tag{5.8.1}$$

It follows from Lemma 5.6.1 and Ondrejat [69] (see also [17]) that $T_t\varphi \in \mathscr{B}_b(\mathrm{V})$ and $\{T_t\}_{t\geq 0}$ is a semigroup on $\mathscr{B}_b(\mathrm{V})$. Moreover, $\{T_t\}_{t\geq 0}$ is a Feller semigroup, i.e. T_t maps $C_b(\mathrm{V})$ into itself.

We also have a different version of the Feller property, which is proved in the following theorem.

Theorem 5.8.1. *The semigroup $\{T_t\}_{t\geq 0}$ is sequentially weakly Feller, i.e., if $\varphi : \mathrm{V} \to \mathbb{R}$ is a bounded sequentially weakly continuous function, then for $t > 0$, $T_t\varphi : \mathrm{V} \to \mathbb{R}$ is also a bounded sequentially weakly continuous function. In particular, if $\xi_n \to \xi$ weakly in* V *then,*

$$T_t\varphi(\xi_n) \to T_t\varphi(\xi). \tag{5.8.2}$$

Proof. Let us choose and fix $0 < t \leq T, \xi \in \mathrm{V}$ and $\varphi : \mathrm{V} \to \mathbb{R}$ be a bounded weakly continuous function. Need to show that $T_t\varphi$ is sequentially weakly Feller in V. For this aim let us choose an V-valued sequence (ξ_n) weakly convergent in V to ξ. Since the function $T_t\varphi : \mathrm{V} \to \mathbb{R}$ is bounded, we only need to prove (5.8.2).

Let $u_n(\cdot) = u(\cdot, \xi_n)$ be the strong solution of (5.2.4) on $[0, T]$ with the initial data ξ_n and let $u(\cdot) = u(\cdot, \xi)$ be the strong solution of (5.2.4) with the initial data ξ on the same stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W)$, which exist by Theorem 5.3.7. By Theorem 5.7.7, about the continuous dependence on the initial data, there exist

- a subsequence $(n_k)_k$,

- a stochastic basis $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$, where $\tilde{\mathbb{F}} = \{\tilde{\mathscr{F}}_s\}_{s\in[0,T]}$,
- an $\mathbb{R}^m$-valued $\tilde{\mathbb{F}}$-Wiener process $\tilde{W}$,
- and progressively measurable processes $\tilde{u}(s)$, $(\tilde{u}_{n_k}(s))_{k\geq 1}$, $s \in [0,T]$ (defined on this basis) with laws supported in $\mathcal{Z}_T$ such that

$$\tilde{u}_{n_k} \textit{ has the same law as } u_{n_k} \textit{ on } \mathcal{Z}_T \textit{ and } \tilde{u}_{n_k} \to \tilde{u} \textit{ in } \mathcal{Z}_T, \quad \tilde{\mathbb{P}}\text{ - a.s.} \tag{5.8.3}$$

and the system

$$(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u}) \tag{5.8.4}$$

is a martingale solution to (5.2.4) on the interval $[0,T]$ with the initial data ξ.

In particular, by (5.8.3), $\tilde{\mathbb{P}}$-almost surely[1]

$$\tilde{u}_{n_k}(t) \to \tilde{u}(t) \text{ weakly in V}.$$

Since the function $\varphi : \mathrm{V} \to \mathbb{R}$ is sequentially weakly continuous, we infer that $\tilde{\mathbb{P}}$-a.s.,

$$\varphi(\tilde{u}_{n_k}(t)) \to \varphi(\tilde{u}(t)) \text{ in } \mathbb{R}.$$

Since the function φ is also bounded, by the Lebesgue dominated convergence theorem we infer that

$$\lim_{k\to\infty} \tilde{\mathbb{E}}\left[\varphi(\tilde{u}_{n_k}(t))\right] = \tilde{\mathbb{E}}\left[\varphi(\tilde{u}(t))\right]. \tag{5.8.5}$$

From the equality of laws of $\tilde{u}_{n_k}$ and u_{n_k}, $k \in \mathbb{N}$, on the space $\mathcal{Z}_T$ we infer that $\tilde{u}_{n_k}$ and u_{n_k} have the same laws on V_w and so

$$\tilde{\mathbb{E}}\left[\varphi(\tilde{u}_{n_k}(t))\right] = \mathbb{E}\left[\varphi(u_{n_k}(t))\right]. \tag{5.8.6}$$

On the other hand, R.H.S. of (5.8.6) is equal by (5.8.1), to $T_t\varphi(\xi_{n_k})$.

Since u, by assumption, is a martingale solution of (5.2.4) with the initial data ξ and by the above, $\tilde{u}$ is also a martingale solution with the initial data ξ. Thus, by Corollary 5.6.3, we infer that

$$\text{the processes } u \text{ and } \tilde{u} \text{ have the same law on the space } \mathcal{Z}_T.$$

Hence

$$\tilde{\mathbb{E}}\left[\varphi(\tilde{u}(t))\right] = \mathbb{E}\left[\varphi(u(t))\right]. \tag{5.8.7}$$

As before, the R.H.S. of (5.8.7) is equal by (5.8.1), to $T_t\varphi(\xi)$.

[1] Let us observe that it would be sufficient to have strong convergence below. But we have been unable to get such a stronger result. The power of our method lies in the fact that the weak convergence is sufficient for our purposes.

Thus by (5.8.5), (5.8.6) and (5.8.7), we infer

$$\lim_{k\to\infty} T_t\varphi(\xi_{n_k}) = T_t\varphi(\xi).$$

Using the subsequence argument, we can conclude that the whole sequence $(T_t\varphi(\xi_n))_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} T_t\varphi(\xi_n) = T_t\varphi(\xi).$$

■

CHAPTER 6

STOCHASTIC TAMED NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3$

The tamed Navier-Stokes equations on $\mathbb{R}^3$ were introduced by Röckner and Zhang [75], where they proved the existence and uniqueness of a smooth solution to tamed 3D Navier-Stokes equations in the whole space. Later on in [76] they proved the existence of a unique strong solution to stochastic tamed 3D Navier-Stokes equations in the whole space and for the periodic boundary case using a result from Stroock and Varadhan [87]. In this chapter we reprove their results for a slightly simplified system using a self-contained approach. We generalise Röckner and Zhang result corresponding to estimate on L^4-norm of the solution from torus to the Euclidean space $\mathbb{R}^3$. We also establish the existence of an invariant measure on $\mathbb{R}^3$ for time homogeneous damped tamed 3D Navier-Stokes equations, given by (6.6.1).

6.1 Introduction

We are interested in the study of the stochastic tamed Navier-Stokes equations (NSE) on $\mathbb{R}^3$ which were introduced by Röckner and Zhang [76]. We consider the following stochastic tamed NSEs with viscosity ν (assumed to be positive), on $\mathbb{R}^3$:

$$\begin{aligned} du(t,x) &= \left[\nu\Delta u(t,x) - (u(t,x)\cdot\nabla)u(t,x) - \nabla p(t,x) - g(|u(t,x)|^2)u(t) + f(x,u(t,x))\right] dt \\ &\quad + \sum_{j=1}^{\infty} \left[(\sigma_j(t,x)\cdot\nabla)u(t) + \nabla\tilde{p}_j(t,x)\right] dW_t^j, \qquad (t,x)\in[0,T]\times\mathbb{R}^3, \end{aligned} \tag{6.1.1}$$

subject to the incompressibility condition

$$\operatorname{div} u(t,x) = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^3, \tag{6.1.2}$$

and the initial condition

$$u(0,x) = u_0(x), \quad x\in\mathbb{R}^3, \tag{6.1.3}$$

where $p(t,x)$ and $\tilde{p}_k(t,x)$ are unknown scalar functions, and the taming function $g:\mathbb{R}_+\to\mathbb{R}_+$ is smooth and satisfies for some $N\in\mathbb{N}$

$$\begin{cases} g(r)=0, & \text{if } r\le N,\\ g(r)=(r-N)/\nu, & \text{if } r\ge N+1,\\ 0\le g'(r)\le 2/(\nu\wedge 1), & r\in[N,N+1]. \end{cases} \tag{6.1.4}$$

$\{W_t^j; t\ge 0, j=1,2,\dots\}$ is a sequence of independent one-dimensional standard $\mathbb{F}=(\mathscr{F}_t)_{t\ge 0}$-Brownian Motions on the complete filtered probability space $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$. The stochastic integral is understood as Itô integral. The arguments of the coefficients are given as follows:

$$\mathbb{R}^3\times\mathbb{R}^3\ni(x,u)\mapsto f(x,u)\in\mathbb{R}^3$$
$$\mathbb{R}_+\times\mathbb{R}^3\ni(t,x)\mapsto\sigma(t,x)\in\mathbb{R}^3\times\ell^2,$$

where ℓ^2 is the Hilbert space consisting of all sequences of square summable numbers with standard norm $\|\cdot\|_{\ell^2}$. In the following f and σ are always assumed to be measurable with respect to all their variables.

In classical Navier-Stokes equations on $\mathbb{R}^3$ with $u_0\in\mathrm{V}$ (see Section 6.2) there is only existence of local solution [88]. The addition of tamed term enables to prove the global existence [75]. The non-explosion of the solution is due to the tamed term. Röckner and Zhang [76] proved the existence of a martingale solution to (6.1.1) (for more generalised noise) in the absence of compact Sobolev embeddings. They use the localization method to prove tightness, a method introduced by Stroock and Varadhan [87]. In this chapter we present a self-contained proof of the same. In order to prove the existence of a martingale solution they use the Faedo-Galerkin approximation with the non-classical finite dimensional space $H_n^1=\mathrm{span}\{e_i, i=1\cdots n\}$ where $\mathscr{E}=\{e_i\}_{i\in\mathbb{N}}\subset\mathscr{V}$ (see Section 6.2) is the orthonormal basis of H^1. They also require that in the case of periodic boundary conditions $\mathscr{E}$ is an orthogonal basis of H^0 which was essential in obtaining the L^4-estimate of the solution. We generalise this result to the Euclidean space $\mathbb{R}^3$. Another reason for them to choose periodic boundary conditions was the compactness of $H^2\hookrightarrow H^1$ embedding, which along with the L^4-estimate of the solution was crucial in establishing the existence of invariant measures. We do not require this embedding and hence are able to obtain the existence of invariant measures for time homogeneous damped tamed Navier-Stokes equations (6.6.1) on $\mathbb{R}^3$.

In the present chapter we prove the existence of a unique strong solution to the stochastic tamed 3D Navier-Stokes equations (6.1.1) under some natural assumptions (**A1**) - (**A2**) on f and σ (see Section 6.2). To prove the existence of strong solution we use the Yamada-Watanabe theorem [95] which states that the existence of martingale solutions plus pathwise uniqueness implies the existence of a unique strong solution. In order to establish the existence of martingale solutions, instead of using the standard Faedo-Galerkin approximations we use a different approach motivated from [42] and [61]. We study a truncated SPDE on an infinite dimensional space H_n, defined in the Section 6.4 and then use the tightness criterion, the Jakubowski-Skorohod Theorem and the martingale representaion theorem to prove the existence of martingale solutions. The essential reason, for us to incorporate this approximation scheme was the non-commutativity of gradient operator (∇) with the standard Faedo-Galerkin projection operator P_n [13, Section 5]. The commutativity is essential for us to obtain a'priori bounds. We also prove the existence of invariant measures, Theorem 6.6.1, for time homogeneous damped tamed Navier-Stokes equations (6.6.1) under the assumptions $(\mathbf{A1})'$ - $(\mathbf{A3})'$ (see Section 6.6). We use the technique (Theorem 6.6.4) of Maslowski and Seidler [62] working with weak topologies to establish the existence of invariant measures. We show the two conditions of Theorem 6.6.4, boundedness in probability and sequentially weak Feller property are satisfied for the semigroup $(T_t)_{t\geq 0}$, defined by (6.6.2). In contrast to Röckner and Zhang [76], a'priori bound on L^4-norm of the solution plays an essential role in the existence of martingale solutions and not in the existence of invariant measures.

This chapter is organised as follows: in Section 6.2, we recall some standard notations and results and set the assumptions on f and σ. We also establish certain estimates on the tamed term which we use later in Sections 6.4 and 6.5. In Section 6.3, we establish the tightness criterion and state Skorohod's theorem which we use along with a'priori estimates obtained in the Section 6.5 to prove the existence of a martingale solution and path-wise uniqueness of the solution. In Section 6.4 we introduce our truncated SPDE and describe the approximation scheme motivated from [42, 61], along with all the machinery required. Finally in Section 6.6 we establish the existence of an invariant measure for time homogeneous damped tamed 3D Navier-Stokes equations (6.6.1).

6.2 Functional setting

6.2.1 Notations

Let $C_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$ denote the set of all smooth functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ with compact supports. For $p\geq 1$, let $L^p(\mathbb{R}^3,\mathbb{R}^3)$ be the vector valued L^p-space in which the norm is denoted by $\|\cdot\|_{L^p}$. If $p=2$, then $L^2(\mathbb{R}^3,\mathbb{R}^3)$ is a Hilbert space with the scalar product given by

$$\langle u,\mathrm{v}\rangle_{L^2} := \int_{\mathbb{R}^3} u(x)\cdot \mathrm{v}(x)\,dx, \quad u,\mathrm{v}\in L^2(\mathbb{R}^3,\mathbb{R}^3).$$

Let $H^1(\mathbb{R}^3,\mathbb{R}^3)$ stand for the Sobolev space of all $u\in L^2(\mathbb{R}^3,\mathbb{R}^3)$ for which there exist weak derivatives $D_i u\in L^2(\mathbb{R}^3,\mathbb{R}^3)$, $i=1,\dots,3$. It is a Hilbert space with the scalar product given by

$$\langle u,\mathrm{v}\rangle_{H^1} := \langle u,\mathrm{v}\rangle_{L^2} + ((u,\mathrm{v})), \quad u,\mathrm{v}\in H^1(\mathbb{R}^3,\mathbb{R}^3),$$

where

$$((u,\mathrm{v})) := \langle \nabla u,\nabla \mathrm{v}\rangle_{L^2} = \sum_{i=1}^{3}\int_{\mathbb{R}^3}\frac{\partial u}{\partial x_i}\cdot\frac{\partial \mathrm{v}}{\partial x_i}\,dx, \quad u,\mathrm{v}\in H^1(\mathbb{R}^3,\mathbb{R}^3). \tag{6.2.1}$$

Let

$$\begin{aligned}
\mathcal{V} &:= \left\{u \in C_0^\infty(\mathbb{R}^3,\mathbb{R}^3) : \operatorname{div} u = 0\right\},\\
\mathrm{H} &:= \text{the closure of } \mathcal{V} \text{ in } L^2(\mathbb{R}^3,\mathbb{R}^3),\\
\mathrm{V} &:= \text{the closure of } \mathcal{V} \text{ in } H^1(\mathbb{R}^3,\mathbb{R}^3),\\
\mathrm{D(A)} &:= \mathrm{H} \cap H^2(\mathbb{R}^3,\mathbb{R}^3).
\end{aligned}$$

On H we consider the scalar product and the norm inherited from $L^2(\mathbb{R}^3,\mathbb{R}^3)$ and denote them by $\langle\cdot,\cdot\rangle_{\mathrm{H}}$ and $\|\cdot\|_{\mathrm{H}}$ respectively, i.e.

$$\langle u,\mathrm{v}\rangle_{\mathrm{H}} := \langle u,\mathrm{v}\rangle_{L^2}, \qquad |u|_{\mathrm{H}} := |u|_{L^2}, \quad u,\mathrm{v} \in \mathrm{H}.$$

On V we consider the scalar product and norm inherited from $H^1(\mathbb{R}^3,\mathbb{R}^3)$, i.e.

$$\langle u,\mathrm{v}\rangle_{\mathrm{V}} := \langle u,\mathrm{v}\rangle_{L^2} + ((u,\mathrm{v})), \qquad \|u\|_{\mathrm{V}}^2 := |u|_{\mathrm{H}}^2 + |\nabla u|_{L^2}^2, \quad u,\mathrm{v} \in \mathrm{V}, \tag{6.2.2}$$

where $((\cdot,\cdot))$ is defined in (6.2.1). D(A) is a Hilbert space under the graph norm

$$|u|_{\mathrm{D(A)}}^2 := |u|_{\mathrm{H}}^2 + |\mathrm{A}u|_{L^2}^2, \qquad u \in \mathrm{D(A)},$$

where the inner product is given by

$$\langle u,\mathrm{v}\rangle_{\mathrm{D(A)}} := \langle u,\mathrm{v}\rangle_{\mathrm{H}} + \langle \mathrm{A}u,\mathrm{A}\mathrm{v}\rangle_{L^2}, \qquad u,\mathrm{v} \in \mathrm{D(A)}.$$

6.2.2 Some operators

Let us recall the tri-linear form $b : L^p \times W^{1,q} \times L^r \to \mathbb{R}$ which was introduced earlier in Chapter 3

$$b(u,w,\mathrm{v}) = \int_{\mathbb{R}^3} (u\cdot\nabla w)\mathrm{v}\,dx, \qquad u \in L^p,\, w \in W^{1,q},\, \mathrm{v} \in L^r, \tag{6.2.3}$$

where $p,q,r \in [1,\infty]$, satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1.$$

We will recall the fundamental properties of the form b which are valid in unbounded domains.

By the Sobolev embedding theorem and Hölder inequality, we obtain the following estimates

$$\begin{aligned}
|b(u,w,\mathrm{v})| &\le \|u\|_{L^4}\|w\|_{\mathrm{V}}\|\mathrm{v}\|_{L^4}, \quad u,\mathrm{v} \in L^4, w \in \mathrm{V} && (6.2.4)\\
&\le c\|u\|_{\mathrm{V}}\|w\|_{\mathrm{V}}\|\mathrm{v}\|_{\mathrm{V}}, \quad u,\mathrm{v},w \in \mathrm{V} && (6.2.5)
\end{aligned}$$

for some positive constant c. Thus the form b is continuous on V. Moreover, if we define a bilinear map B by $B(u,w) := b(u,w,\cdot)$ then by inequality (6.2.5) we infer that $B(u,w) \in \mathrm{V}'$ for all $u,w \in \mathrm{V}$ and that the following inequality holds

$$\|B(u,w)\|_{\mathrm{V}'} \le c\|u\|_{\mathrm{V}}\|w\|_{\mathrm{V}}, \quad u,w \in \mathrm{V}. \tag{6.2.6}$$

Moreover, the mapping $B : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$ is bilinear and continuous.

Let us, for any $s > 0$, define the following standard scale of Hilbert spaces (see Rudin [80] for the definition of $H^s(\mathbb{R}^3,\mathbb{R}^3)$ space)

$$\mathrm{V}_s := \text{the closure of } \mathcal{V} \text{ in } H^s(\mathbb{R}^3,\mathbb{R}^3).$$

If $s > \frac{d}{2} + 1$, then by the Sobolev Embedding theorem

$$H^{s-1}(\mathbb{R}^3,\mathbb{R}^3) \hookrightarrow C_b(\mathbb{R}^3,\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3,\mathbb{R}^3). \tag{6.2.7}$$

Here $C_b(\mathbb{R}^3,\mathbb{R}^3)$ denotes the space of continuous and bounded $\mathbb{R}^3-$valued functions defined on $\mathbb{R}^3$. If $u,w \in \mathrm{V}$ and $\mathrm{v} \in \mathrm{V}_s$ with $s > \frac{d}{2} + 1$ then

$$|b(u,w,\mathrm{v})| = |b(u,\mathrm{v},w)| \le |u|_{L^2}|w|_{L^2}\|\nabla \mathrm{v}\|_{L^\infty} \le c|u|_{L^2}|w|_{L^2}\|\mathrm{v}\|_{\mathrm{V}_s}$$

for some constant $c > 0$. Thus b can be uniquely extended to the tri-linear form (denoted by the same letter)

$$b : \mathrm{H} \times \mathrm{H} \times \mathrm{V}_s \to \mathbb{R}$$

and

$$|b(u,w,\mathrm{v})| \le c|u|_{\mathrm{H}}|w|_{\mathrm{H}}\|\mathrm{v}\|_{\mathrm{V}_s}, \quad u,w \in \mathrm{H}, \mathrm{v} \in \mathrm{V}_s.$$

At the same time, the operator B can be uniquely extended to a bounded bilinear operator

$$B : \mathrm{H} \times \mathrm{H} \to \mathrm{V}_s'.$$

In particular, it satisfies the following estimate

$$\|B(u,w)\|_{\mathrm{V}_s'} \le c|u|_{\mathrm{H}}|w|_{\mathrm{H}}, \quad u,w \in \mathrm{H}. \tag{6.2.8}$$

We will also use the notation, $B(u) := B(u,u)$.

Let us assume that $s > 1$. It is clear that V_s is dense in V and the embedding $j_s : \mathrm{V}_s \hookrightarrow \mathrm{V}$ is continuous. Then there exists [23, Lemma C.1] a Hilbert space U such that $\mathrm{U} \subset \mathrm{V}_s$, U is dense in V_s and

$$\text{the natural embedding } i_s : \mathrm{U} \hookrightarrow \mathrm{V}_s \text{ is compact}.$$

The following Gagliardo-Nirenberg interpolation inequality will be used frequently. Let $q \in [1,\infty]$ and $m \in \mathbb{N}$. If

$$\frac{1}{q} = \frac{1}{2} - \frac{m\alpha}{3}, \quad 0 \le \alpha \le 1,$$

then for any $u \in H^m$ there exists a constant $C_{m,q}$ depending on m and q such that

$$\|u\|_{L^q} \le C_{m,q}\|u\|_{H^m}^{\alpha}|u|_{L^2}^{1-\alpha} \tag{6.2.9}$$

Let Π be the orthogonal projection from $L^2(\mathbb{R}^3,\mathbb{R}^3)$ to H, famously known as the Leray-Helmholtz projection [88]. For any $u \in \mathrm{H}$ and $\mathrm{v} \in L^2(\mathbb{R}^3,\mathbb{R}^3)$, we have

$$\langle u,\mathrm{v}\rangle_{\mathrm{H}} := \langle u,\Pi\mathrm{v}\rangle_{\mathrm{H}} = \langle u,\mathrm{v}\rangle_{L^2}.$$

The Stokes operator $\mathrm{A}\colon \mathrm{D(A)} \to \mathrm{H}$, is given by

$$\mathrm{A}u = -\Pi(\Delta u), \qquad u \in \mathrm{D(A)}.$$

The bilinear map $B\colon \mathrm{H}\times\mathrm{H} \to \mathrm{H}$ will be given by

$$B(u,\mathrm{v}) = \Pi((u\cdot\nabla)\mathrm{v}), \qquad u,\mathrm{v} \in \mathrm{H}.$$

6.2.3 Assumptions

We now introduce the assumptions on the coefficients f and σ:

(A1) A function $f:\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$ is of C^1 class and for any $T>0$ there exist a constant $C_{T,f}>0$ such that for any $x\in\mathbb{R}^3, u\in\mathbb{R}^3$,

$$|\partial_{x^j} f(x,u)|^2 + |f(x,u)|^2 \le C_{T,f}\cdot(1+|u|^2), \quad j=1,2,3,$$
$$|\partial_{u^j} f(x,u)| \le C_{T,f}.$$

(A2) A measurable function $\sigma:[0,\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$ of C^1 class with respect to the x-variable and for any $T>0$ there exist a constant $C_{\sigma,T}>0$ such that for all $t\in[0,T]$, $x\in\mathbb{R}^3$

$$\|\partial_{x^j}\sigma(t,x)\|_{\ell^2} \le C_{\sigma,T}, \quad j=1,2,3$$

and, for all $t\in[0,\infty)$, $x\in\mathbb{R}^3$

$$\|\sigma(t,x)\|^2_{\ell^2} \le \frac{1}{4}. \tag{6.2.10}$$

Below for the sake of simplicity the variable "x" in the coefficients will be dropped.

Define, for $k\in\mathbb{N}$, $G_j:[0,T]\times\mathrm{H}\to\mathrm{H}$ by

$$G_j(t,u) := \Pi[(\sigma_j(t)\cdot\nabla)u], \quad t\in[0,T], u\in\mathrm{H}. \tag{6.2.11}$$

Then a function $G:\mathrm{H}\to\mathscr{T}_2(\ell^2;\mathrm{H})$ is defined by

$$G(u)(k) = \sum_{j=1}^{\infty} k_j G_j(u), \quad u\in\mathrm{H}. \tag{6.2.12}$$

Let $\{e_j\}_{j=1}^{\infty}$ be the orthonormal basis of ℓ^2 then we see that (6.2.12) implies

$$G(u)(e_j) = G_j(u).$$

For simplicity we will assume that $\nu = 1$. In particular, the function g defined by (6.1.4) will from now on be given by

$$\begin{cases} g(r) = 0, & \text{if } r \le N, \\ g(r) = (r - N), & \text{if } r \ge N+1, \\ 0 \le g'(r) \le 2, & r \in [N, N+1]. \end{cases} \tag{6.2.13}$$

Observe that the function g defined in this way satisfies

$$|g(r)| \le r, \qquad r \ge 0, \tag{6.2.14}$$

and

$$|g(r) - g(r')| \le 2|r - r'|, \qquad r, r' \ge 0. \tag{6.2.15}$$

We are interested in proving the existence of solutions to (6.1.1) - (6.1.3). In particular, we want to prove the existence of divergence free vector fields u and scalar pressure p satisfying (6.1.1) and (6.1.3). Thus we project equation (6.1.1) using the orthogonal projection operator Π on the space H of the L^2−valued, divergence free vector fields. On projecting, we obtain the following abstract stochastic evolution equation:

$$\begin{cases} du(t) = \left[-Au(t) - B(u(t)) - \Pi[g(|u(t)|^2)u(t)] + \Pi f(u(t))\right] dt + \sum_{j=1}^{\infty} G_j(t, u(t))\, dW_j(t), \\ u(0) = u_0, \end{cases} \tag{6.2.16}$$

where we assume that $u_0 \in \mathrm{V}$ and $W(t) = (W_j(t))_{j=1}^{\infty}$ is a cylindrical Wiener process on ℓ^2 and $\{W^j(t), t \ge 0, j \in \mathbb{N}\}$ is an infinite sequence of independent standard Brownian motions. We will repeatedly use the following notation

$$G(t,u)\, dW(t) = \sum_{j=1}^{\infty} G_j(t,u)\, dW_j(t).$$

We will need the following lemma in Section 6.4 to obtain the a'priori estimates.

Lemma 6.2.1. *i) For any* $u \in \mathrm{D(A)}$

$$|\langle B(u), u\rangle_{\mathrm{V}}| \le \frac{1}{2}|u|^2_{\mathrm{D(A)}} + \frac{1}{2}\big\||u| \cdot |\nabla u|\big\|^2_{L^2} \tag{6.2.17}$$

ii) If $u \in \mathrm{H}$, *then*

$$\begin{cases} ((-g(|u|^2)u, u)) \le C_N |\nabla u|^2_{L^2} - 2\big\||u| \cdot |\nabla u|\big\|^2_{L^2}, \\ \langle -g(|u|^2)u, u\rangle_{\mathrm{H}} \le -\|u\|^4_{L^4} + C_N |u|^2_{\mathrm{H}}, \end{cases} \tag{6.2.18}$$

where the semi-inner product $((\cdot,\cdot))$ *is defined in* (6.2.1).

iii) For any $u \in \mathrm{D(A)}$,

$$\|G(t,u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})} \leq \frac{1}{4}|\nabla u|^2_{L^2}, \tag{6.2.19}$$

$$\|G(t,u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{V})} \leq \frac{1}{2}|\mathrm{A}u|^2_{L^2} + C_{\sigma,T}|\nabla u|^2_{L^2}. \tag{6.2.20}$$

Proof. Let $u \in \mathrm{D(A)}$. Since $\langle B(u),u\rangle_{\mathrm{H}} = 0$, using the Cauchy-Schwartz and Young inequality we get

$$\begin{aligned}|\langle B(u),u\rangle_{\mathrm{V}}| &= |\langle B(u),(I-\Delta)u\rangle_{\mathrm{H}}| \leq |-\Delta u|_{L^2}|(u\cdot\nabla)u|_{L^2} \\ &\leq \frac{1}{2}|-\Delta u|^2_{L^2} + \frac{1}{2}|(u\cdot\nabla)u|^2_{L^2} \leq \frac{1}{2}|u|^2_{\mathrm{D(A)}} + \frac{1}{2}\big||u|\cdot|\nabla u|\big|^2_{L^2}.\end{aligned}$$

Let us introduce a function $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ such that $g(r) = r - \phi(r)$ which in particular satisfies

$$\phi(r) = \begin{cases} r, & r \leq N, \\ N, & r \geq N+1. \end{cases}$$

Since $\phi'(r) = 1 - g'(r)$, there exists a constant $C_N > 0$ such that $|\phi'(r)| \leq \tilde{C}_N$ for every $r \geq 0$. Moreover

$$\phi'(r) = \begin{cases} 1, & r \leq N, \\ 0, & r \geq N+1. \end{cases}$$

Hence, we infer that $|\phi'(r)\cdot r|$ is bounded by some positive constant C_N.

Let $u \in \mathrm{D(A)}$. Using the definitions of g and of semi-norm $((\cdot,\cdot))$, we get

$$\begin{aligned}((-g(|u|^2)u,u)) &= -\langle g(|u|^2)u, -\Delta u\rangle_{L^2} \\ &= -\int_{\mathbb{R}^3} g(|u(x)|^2)u(x)\cdot(-\Delta u(x))\,dx \\ &= -\int_{\mathbb{R}^3} |u(x)|^2 u(x)(-\Delta u(x))\,dx + \int_{\mathbb{R}^3} \phi(|u(x)|^2)u(x)(-\Delta u(x))\,dx.\end{aligned}$$

Thus, on integration by parts we get

$$\begin{aligned}((-g(|u|^2)u,u)) = &-\left[\int_{\mathbb{R}^3}|u(x)|^2\cdot|\nabla u(x)|^2\,dx + 2\int_{\mathbb{R}^3}|u(x)|^2\cdot|\nabla u(x)|^2\,dx\right] \\ &+ \int_{\mathbb{R}^3}\phi(|u(x)|^2)\cdot|\nabla u(x)|^2\,dx + \sum_{j,k=1}^{3}\int_{\mathbb{R}^3} D_k\left(\phi(|u(x)|^2)\right)u_j(x)\cdot D_k u_j(x)\,dx.\end{aligned} \tag{6.2.21}$$

Using the bound on $|\phi'(r)\cdot r|$, we obtain

$$\begin{aligned}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} D_k\left(\phi(|u(x)|^2)\right)u_j(x)\cdot D_k u_j(x)\,dx &= 2\sum_{k=1}^{3}\int_{\mathbb{R}^3}\phi'(|u(x)|^2)\langle u(x), D_k u(x)\rangle^2_{\mathbb{R}^3}\,dx \\ &\leq 2\int_{\mathbb{R}^3}|\phi'(|u(x)|^2)|\cdot|u(x)|^2|\nabla u(x)|^2\,dx \leq C_N|\nabla u|^2_{L^2}.\end{aligned} \tag{6.2.22}$$

Since $g(r) \geq 0$, $|\phi(r)| \leq r$ for all $r \geq 0$. Thus using (6.2.22) in (6.2.21), we obtain

$$\begin{aligned}((-g(|u|^2)u,u)) &\leq -3\big||u|\cdot|\nabla u|\big|^2_{L^2} + C_N|\nabla u|^2_{L^2} + \big||u|\cdot|\nabla u|\big|^2_{L^2} \\ &= C_N|\nabla u|^2_{L^2} - 2\big||u|\cdot|\nabla u|\big|^2_{L^2}.\end{aligned}$$

Now to prove the second inequality, we take the similar approach. Let $u \in \mathrm{H}$, then

$$\langle -g(|u|^2)u,u\rangle_{\mathrm{H}} = -\int_{\mathbb{R}^3} |u(x)|^2|u(x)|^2\,dx + \int_{\mathbb{R}^3} \phi(|u(x)|^2)|u(x)|^2\,dx.$$

By the definition of ϕ there exists a constant $C_N > 0$ such that $|\phi(r)| \leq C_N$ for all $r > 0$, thus

$$\langle -g(|u|^2)u,u\rangle_{\mathrm{H}} \leq -\|u\|^4_{L^4} + C_N|u|^2_{\mathrm{H}}.$$

This completes the proof of part (ii).

Now for (iii), by (**A1**) and (**A2**) we have

$$\begin{aligned}\|G(t,u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})} &= \sum_{j=1}^{\infty}|G_j(t,u)|^2_{\mathrm{H}} = \sum_{j=1}^{\infty}\int_{\mathbb{R}^3}|G_j(t,x,u(x))|^2\,dx \\ &\leq \int_{\mathbb{R}^3}\|\sigma(t,x)\|^2_{\ell^2}|\nabla u(x)|^2\,dx \leq \sup_{x\in\mathbb{R}^3}\|\sigma(t,x)\|^2_{\ell^2}|\nabla u|^2_{L^2} \leq \frac{1}{4}|\nabla u|^2_{L^2}.\end{aligned}$$

Secondly, noting that

$$\|G(t,u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{V})} = \|G(t,u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})} + \|\nabla G(t,u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})}$$

and

$$\partial_{x^j}G_k(t,u) = \Pi\partial_{x^j}[(\sigma_k(t,x)\cdot\nabla)u] = \Pi\left[(\partial_{x^j}\sigma_k(t,x)\cdot\nabla)u + (\sigma_k(t,x)\cdot\nabla)\partial_{x^j}u\right].$$

Thus

$$\begin{aligned}\|\nabla G(t,u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})} &= \sum_{k=1}^{\infty}\left|\sum_{j=1}^{3}\Pi\left[(\partial_{x^j}\sigma_k(t,x)\cdot\nabla)u + (\sigma_k(t,x)\cdot\nabla)\partial_{x^j}u\right]\right|^2_{\mathrm{H}} \\ &\leq 2\int_{\mathbb{R}^3}\sum_{j=1}^{3}\|\partial_{x^j}\sigma(t,x)\|^2_{\ell^2}|\nabla u(x)|^2\,dx + 2\int_{\mathbb{R}^3}\|\sigma(t,x)\|^2_{\ell^2}|\Delta u(x)|^2\,dx.\end{aligned}$$

Hence, by assumptions (**A1**), (**A2**) and (6.2.19), we have

$$\|G(t,u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{V})} \leq \frac{1}{2}|\mathrm{A}u|^2_{L^2} + C_{T,\sigma}|\nabla u|^2_{L^2}.$$

■

Remark 6.2.2. On a purely heuristic level, by the application of Itô Lemma to the function $|u|^2_{\mathrm{H}}$ and a solution u to (6.1.1), using Lemma 6.2.1 one obtains the following inequality

$$\begin{aligned}\frac{1}{2}d|u(t)|^2_{\mathrm{H}} &= \langle u(t), -\mathrm{A}u(t) - B(u(t)) - g(|u(t)|^2)u(t) + f(u(t))\rangle_{\mathrm{H}} \\ &\quad + \langle u(t), G(s,u(t))\,dW_t\rangle_{\mathrm{H}} + \frac{1}{2}\|G_n(s,u_n(s))\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})} \\ &\leq -\frac{7}{8}|\nabla u(t)|^2_{L^2} - \|u(t)\|^4_{L^4} + C_{N,f}|u(t)|^2_{\mathrm{H}} + \langle u(t), G(s,u(t))\,dW_t\rangle_{\mathrm{H}},\end{aligned} \tag{6.2.23}$$

which could lead to a'priori estimates that can be used further to prove the existence of the solution.

6.3 Compactness

Let $(\mathcal{O}_R)_{R\in\mathbb{N}}$ be a sequence of bounded open subsets of $\mathbb{R}^3$ with regular boundaries $\partial\mathcal{O}_R$ such that $\mathcal{O}_R \subset \mathcal{O}_{R+1}$. Let us consider the following functional spaces:
$\mathscr{C}([0,T];\mathrm{U}')$:= the space of continuous functions $u:[0,T]\to \mathrm{U}'$ with the topology $\mathscr{T}_1$ induced by the norm $|u|_{\mathscr{C}([0,T];\mathrm{U}')} := \sup_{t\in[0,T]}|u(t)|_{\mathrm{U}'}$,
$L^2_w(0,T;\mathrm{D(A)})$:= the space $L^2(0,T;\mathrm{D(A)})$ with the weak topology $\mathscr{T}_2$,
$L^2(0,T;\mathrm{H}_{\mathrm{loc}})$:= the space of measurable functions $u:[0,T]\to \mathrm{H}$ such that for all $R\in\mathbb{N}$

$$q_{T,R}(u) := \|u\|_{L^2(0,T;H_{\mathcal{O}_R})} = \left(\int_0^T\int_{\mathcal{O}_R}|u(t,x)|^2\,dx\,dt\right)^{1/2} < \infty, \tag{6.3.1}$$

with the topology $\mathscr{T}_3$ induced by the semi-norms $(q_{T,R})_{R\in\mathbb{N}}$.

The following lemma is inspired by the classical Dubinsky Theorem (see Theorem 2.4.5) and the compactness result due to Mikulevicus and Rozovskii [65, Lemma 2.7].

Lemma 6.3.1. *Let*

$$\tilde{\mathcal{Z}}_T := \mathscr{C}([0,T];\mathrm{U}')\cap L^2_w(0,T;\mathrm{D(A)})\cap L^2(0,T;\mathrm{H}_{loc}) \tag{6.3.2}$$

and let $\tilde{\mathscr{T}}$ be the supremum of the corresponding topologies. Then a set $\mathscr{K}\subset\tilde{\mathcal{Z}}_T$ is $\tilde{\mathscr{T}}$-relatively compact if the following two conditions hold :

i) $\sup_{u\in\mathscr{K}}\int_0^T|u(s)|^2_{\mathrm{D(A)}}\,ds<\infty$, *i.e. $\mathscr{K}$ is bounded in $L^2(0,T;\mathrm{D(A)})$,*

ii) $\lim_{\delta\to0}\sup_{u\in\mathscr{K}}\sup_{\substack{s,t\in[0,T]\\|t-s|\le\delta}}|u(t)-u(s)|_{\mathrm{U}'}=0.$

The above lemma can be proved by modifying the proof of [23, Lemma 3.1], see also [94, Theorem IV.4.1].

Let V_w denote the Hilbert space V endowed with the weak topology.
$\mathscr{C}([0,T];\mathrm{V}_w)$:= the space of weakly continuous functions $u:[0,T]\to\mathrm{V}$ endowed with the weakest topology $\mathscr{T}_4$ such that for all $h\in\mathrm{V}$ the mappings

$$\mathscr{C}([0,T];\mathrm{V}_w)\ni u\to\langle u(\cdot),h\rangle_{\mathrm{V}}\in\mathscr{C}([0,T];\mathbb{R})$$

are continuous. In particular $u_n\to u$ in $\mathscr{C}([0,T];\mathrm{V}_w)$ iff for all $h\in\mathrm{V}$:

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|\langle u_n(t)-u(t),h\rangle_{\mathrm{V}}|=0.$$

Consider the ball

$$\mathbb{B} := \{x\in\mathrm{V}:\|x\|_{\mathrm{V}}\le r\}.$$

Let q be the metric compatible with the weak topology on $\mathbb{B}$. Let us recall the following subspace of the space $\mathscr{C}([0,T];\mathrm{V}_w)$

$$\begin{aligned} \mathscr{C}([0,T];\mathbb{B}_w) := & \text{ the space of weakly continuous functions } u:[0,T]\to \mathrm{V} \\ & \text{such that } \sup_{t\in[0,T]} \|u(t)\|_{\mathrm{V}} \le r. \end{aligned} \tag{6.3.3}$$

The space $\mathscr{C}([0,T];\mathbb{B}_w)$ is metrizable with metric

$$\varrho(u,\mathrm{v}) = \sup_{t\in[0,T]} q(u(t),\mathrm{v}(t)). \tag{6.3.4}$$

Since by the Banach-Alaoglu theorem [80], the set $\mathbb{B}_w$ is compact, $(\mathscr{C}([0,T];\mathbb{B}_w),\varrho)$ is a complete metric space.

The following lemma says that any sequence $(u_n)_{n\in\mathbb{N}}\subset\mathscr{C}([0,T];\mathbb{B})$ convergent in $\mathscr{C}([0,T];\mathrm{H})$ is also convergent in the space $\mathscr{C}([0,T];\mathbb{B}_w)$. The proof of the lemma is similar to the proof of [24, Lemma 2.1].

Lemma 6.3.2. *Let $u_n:[0,T]\to\mathrm{V}, n\in\mathbb{N}$ be functions such that*

(i) $\sup_{n\in\mathbb{N}}\sup_{s\in[0,T]}\|u_n(s)\|_{\mathrm{V}}\le r$,

(ii) $u_n\to u$ *in* $\mathscr{C}([0,T];\mathrm{H})$.

Then $u,u_n\in\mathscr{C}([0,T];\mathbb{B}_w)$ and $u_n\to u$ in $\mathscr{C}([0,T];\mathbb{B}_w)$ as $n\to\infty$.

Let

$$\mathcal{Z}_T = \mathscr{C}([0,T];\mathrm{U}')\cap L^2_w(0,T;\mathrm{D(A)})\cap L^2(0,T;\mathrm{H_{loc}})\cap\mathscr{C}([0,T];\mathrm{V}_w), \tag{6.3.5}$$

and let $\mathscr{T}$ be the supremum of the corresponding topologies.

Now we formulate the compactness criterion analogous to the result due to Mikulevicus and Rozowskii [65], Brzeźniak and Motyl [23, Lemma 3.3] for the space $\mathcal{Z}_T$.

Lemma 6.3.3. *Let $(\mathcal{Z}_T,\mathscr{T})$ be as defined in (6.3.5). Then a set $\mathscr{K}\subset\mathcal{Z}_T$ is $\mathscr{T}-$relatively compact if the following three conditions hold*

(a) $\sup_{u\in\mathscr{K}}\sup_{s\in[0,T]}\|u(s)\|_{\mathrm{V}}<\infty$,

(b) $\sup_{u\in\mathscr{K}}\int_0^T |u(s)|^2_{\mathrm{D(A)}}\,ds<\infty$, *i.e. $\mathscr{K}$ is bounded in $L^2(0,T;\mathrm{D(A)})$,*

(c) $\lim_{\delta\to0}\sup_{u\in\mathscr{K}}\sup_{\substack{s,t\in[0,T]\\ |t-s|\le\delta}}|u(t)-u(s)|_{\mathrm{H}}=0$.

Proof. Let us notice that $\mathcal{Z}_T=\tilde{\mathcal{Z}}_T\cap\mathscr{C}([0,T];\mathrm{V}_w)$, where $\tilde{\mathcal{Z}}_T$ is defined by (6.3.2). Let $\mathscr{K}$ be a subset of $\mathcal{Z}_T$. Because of the assumption (a) we may consider the metric space $\mathscr{C}([0,T];\mathbb{B}_w)\subset\mathscr{C}([0,T];\mathrm{V}_w)$ defined by (6.3.3) and (6.3.4) with $r=\sup_{u\in\mathscr{K}}\sup_{s\in[0,T]}\|u(s)\|_{\mathrm{V}}$. Because of the assumption (b) the restriction to $\mathscr{K}$ of the weak topology in $L^2(0,T;\mathrm{D(A)})$ is metrizable. Since the restrictions to $\mathscr{K}$ of the four topologies considered in $\mathcal{Z}_T$ are metrizable, compactness of a subset of $\mathcal{Z}_T$ is equivalent to it's sequential compactness.

Let (u_n) be a sequence in $\mathscr{K}$. By Lemma 6.3.1, the boundedness of the set $\mathscr{K}$ in $L^2(0,T;\mathrm{D(A)})$ and assumption (c) imply that $\mathscr{K}$ is compact in $\tilde{\mathcal{Z}}_T$. Since the embeddings $\mathrm{D(A)} \hookrightarrow \mathrm{V} \hookrightarrow \mathrm{H}$ are continuous and the embedding $\mathrm{D(A)} \hookrightarrow \mathrm{V}$ is compact, by Dubinsky Theorem 2.4.5 assumptions (b) and (c) imply that $\mathscr{K}$ is relatively compact in $L^2(0,T;\mathrm{V}) \cap \mathscr{C}([0,T];\mathrm{H})$. Hence in particular, there exists a subsequence, still denoted by (u_n), convergent in H. Therefore by Lemma 6.3.2 and assumption (a), (u_n) is convergent in $\mathscr{C}([0,T];\mathbb{B}_w)$. This completes the proof of the lemma. ■

6.3.1 Tightness

Using Section 2.9 and the compactness criterion from Lemma 6.3.3 we obtain the following corollary which we will use to prove tightness of the laws defined by the truncated SPDE (6.4.26).

Corollary 6.3.4 (Tightness criterion). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous $\mathbb{F}$-adapted* H-*valued processes such that*

(a) there exists a constant $C_1 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{s\in[0,T]} \|X_n(s)\|_{\mathrm{V}}^2\right] \le C_1,$$

(b) there exists a constant $C_2 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T |X_n(s)|_{\mathrm{D(A)}}^2\, ds\right] \le C_2,$$

(c) $(X_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition **[A]** *in* H.

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on $\mathcal{Z}_T$. Then for every $\varepsilon > 0$, $\exists$ a compact subset K_ε of $\mathcal{Z}_T$ such that

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n(K_\varepsilon) \ge 1 - \varepsilon.$$

Proof. Let $\varepsilon > 0$. By the Chebyshev inequality and (a), we infer that for any $n \in \mathbb{N}$ and any $r > 0$

$$\tilde{\mathbb{P}}_n\Big(\sup_{s\in[0,T]} \|X_n(s)\|_{\mathrm{V}}^2 > r\Big) \le \frac{\tilde{\mathbb{E}}_n\left[\sup_{s\in[0,T]} \|X_n(s)\|_{\mathrm{V}}^2\right]}{r} \le \frac{C_1}{r}.$$

Let R_1 be such that $\frac{C_1}{R_1} \le \frac{\varepsilon}{3}$. Then

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n\Big(\sup_{s\in[0,T]} \|X_n(s)\|_{\mathrm{V}}^2 > R_1\Big) \le \frac{\varepsilon}{3}.$$

Let $B_1 := \{u \in \mathcal{Z}_T : \sup_{s\in[0,T]} \|u(s)\|_{\mathrm{V}}^2 \le R_1\}$.

By the Chebyshev inequality and (b), we infer that for any $n \in \mathbb{N}$ and any $r > 0$

$$\tilde{\mathbb{P}}_n\big(|X_n|_{L^2(0,T;\mathrm{D(A)})} > r\big) \leq \frac{\tilde{\mathbb{E}}_n\big[|X_n|^2_{L^2(0,T;\mathrm{D(A)})}\big]}{r^2} \leq \frac{C_2}{r^2}.$$

Let R_2 be such that $\frac{C_2}{R_2^2} \leq \frac{\varepsilon}{3}$. Then

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n\big(|X_n|_{L^2(0,T;\mathrm{D(A)})} > R_2\big) \leq \frac{\varepsilon}{3}.$$

Let $B_2 := \{u \in \mathcal{Z} : |u|_{L^2(0,T;\mathrm{D(A)})} \leq R_2\}$.

By Lemmas 2.9.9, 2.9.11 there exists a subset $A_{\frac{\varepsilon}{3}} \subset \mathscr{C}([0,T],\mathrm{H})$ such that $\tilde{\mathbb{P}}_n\big(A_{\frac{\varepsilon}{3}}\big) \geq 1 - \frac{\varepsilon}{3}$ and

$$\lim_{\delta\to 0} \sup_{u\in A_{\frac{\varepsilon}{3}}} \sup_{\substack{s,t\in[0,T]\\ |t-s|\leq\delta}} |u(t) - u(s)|_{\mathrm{H}} = 0.$$

It is sufficient to define K_ε as the closure of the set $B_1 \cap B_2 \cap A_{\frac{\varepsilon}{3}}$ in $\mathcal{Z}_T$. By Lemma 6.3.3, K_ε is compact in $\mathcal{Z}_T$. The proof is thus complete. ∎

6.3.2 The Skorohod Theorem

Let us recall the Jakubowski's generalisation of the Skorohod Theorem as given by Brzeźniak and Ondreját [26, Theorem C.1], see also [49].

Theorem 6.3.5. *Let $\mathcal{X}$ be a topological space such that there exists a sequence $\{f_m\}_{m\in\mathbb{N}}$ of continuous functions $f_m : \mathcal{X} \to \mathbb{R}$ that separates points of $\mathcal{X}$. Let us denote by $\mathcal{S}$ the σ-algebra generated by the maps $\{f_m\}$. Then*

(a) every compact subset of $\mathcal{X}$ is metrizable,

(b) if $(\mu_m)_{m\in\mathbb{N}}$ is a tight sequence of probability measures on $(\mathcal{X},\mathcal{S})$, then there exists a subsequence $(m_k)_{k\in\mathbb{N}}$, a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with $\mathcal{X}$-valued Borel measurable variables ξ_k, ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω. Moreover, the law of ξ is a Radon measure.

Using Theorem 6.3.5, we obtain the following corollary which we will apply to construct a martingale solution to the tamed Navier-Stokes equations.

Corollary 6.3.6. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{Z}_T$-valued random variables such that their laws $\mathcal{L}(\eta_n)$ on $(\mathcal{Z}_T,\mathcal{T})$ form a tight sequence of probability measures. Then there exists a subsequence (n_k), a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and $\mathcal{Z}_T$-valued random variables $\tilde{\eta}$, $\tilde{\eta}_k, k \in \mathbb{N}$ such that the variables η_k and $\tilde{\eta}_k$ have the same laws on $\mathcal{Z}_T$ and $\tilde{\eta}_k$ converges to $\tilde{\eta}$ almost surely on $\tilde{\Omega}$.*

Proof. It is sufficient to prove that on each space appearing in the definition (6.3.5) of the space $\mathcal{Z}_T$, there exists a countable set of continuous real-valued functions separating points.

Since the spaces $\mathscr{C}([0,T];\mathrm{U}')$ and $L^2(0,T;\mathrm{H}_{\mathrm{loc}})$ are separable, metrizable and complete, this condition is satisfied, see [3], exposé 8.

For the space $L^2_w(0,T;\mathrm{D(A)})$ it is sufficient to put

$$f_m(u) := \int_0^T \langle u(t), \mathrm{v}_m(t)\rangle_{\mathrm{D(A)}}\, dt \in \mathbb{R}, \qquad u \in L^2_w(0,T;\mathrm{D(A)}), \quad m \in \mathbb{N},$$

where $\{\mathrm{v}_m, m \in \mathbb{N}\}$ is a dense subset of $L^2(0,T;\mathrm{D(A)})$.

Let us consider the space $\mathscr{C}([0,T];\mathrm{V}_w)$. Let $\{h_m,\, m \in \mathbb{N}\}$ be any dense subset of H and let $\mathbb{Q}_T$ be the set of rational numbers belonging to the interval $[0,T]$. Then the family $\{f_{m,t},\, m \in \mathbb{N},\, t \in \mathbb{Q}_T\}$ defined by

$$f_{m,t}(u) := \langle u(t), h_m\rangle_{\mathrm{V}} \in \mathbb{R}, \qquad u \in \mathscr{C}([0,T];\mathrm{V}_w), \quad m \in \mathbb{N}, \quad t \in \mathbb{Q}_T$$

consists of continuous functions separating points in $\mathscr{C}([0,T];\mathrm{V}_w)$. The statement of the corollary follows from Theorem 6.3.5, concluding the proof. ■

6.3.3 Martingale and strong solution

We end this section by giving the definitions of a martingale and strong solution to (6.2.16).

Definition 6.3.7. We say that there exists a martingale solution of (6.2.16) iff there exist

- a stochastic basis $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$,
- a cylindrical Wiener process $\hat{W}(t) = \big(\hat{W}_j(t)\big)_{j=1}^{\infty}$ on ℓ^2, where $\big\{\hat{W}_j(t), t \ge 0, j \in \mathbb{N}\big\}$ is an infinite sequence of independent standard $(\hat{\mathscr{F}}_t)$–Brownian motions,
- and a progressively measurable process $u : [0,T] \times \hat{\Omega} \to \mathrm{D(A)}$ with $\hat{\mathbb{P}}$-a.e. paths

  $$u(\cdot,\omega) \in \mathscr{C}([0,T];\mathrm{V}_w) \cap L^2(0,T;\mathrm{D(A)}),$$

  such that for all $t \in [0,T]$ and all $\mathrm{v} \in \mathscr{V}$ $\hat{\mathbb{P}}$-a.s.

$$\begin{aligned} &\langle u(t),\mathrm{v}\rangle + \int_0^t \langle Au(s),\mathrm{v}\rangle\, ds + \int_0^t \langle B(u(s)),\mathrm{v}\rangle\, ds + \int_0^t \langle g(|u(s)|^2)u(s),\mathrm{v}\rangle\, ds \\ &= \langle u_0,\mathrm{v}\rangle + \int_0^t \langle f(u(s)),\mathrm{v}\rangle\, ds + \left\langle \int_0^t G(s,u(s))\, dW(s),\mathrm{v}\right\rangle. \end{aligned} \tag{6.3.6}$$

Definition 6.3.8. We say that problem (6.2.16) has a strong solution iff for every stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ and every cylindrical Wiener process $W(t) = \big(W_j(t)\big)_{j=1}^{\infty}$ on ℓ^2 there exists a progressively measurable process $u : [0,T] \times \Omega \to \mathrm{D(A)}$ with $\mathbb{P}$-a.e. paths

$$u(\cdot,\omega) \in \mathscr{C}([0,T];\mathrm{V}) \cap L^2(0,T;\mathrm{D(A)}),$$

such that for all $t \in [0,T]$ and all $\mathrm{v} \in \mathscr{V}$ (6.3.6) holds $\mathbb{P}$-a.s.

6.4 Truncated SPDE

We will be using the following notations and spaces repeatedly in this section.

$$B_n := \left\{x \in \mathbb{R}^3 : |x| \le n\right\} \subset \mathbb{R}^3,\ n \in \mathbb{N}.$$

We will use $\mathscr{F}(u)$ and $\hat{u}$ interchangeably to denote the Fourier transform of u. The inverse Fourier transform will be given by $\mathscr{F}^{-1}$.

We define H_n as the subspace of H

$$\mathrm{H}_n := \{u \in \mathrm{H} : \operatorname{supp}(\hat{u}) \subset B_n\}.$$

The norm on H_n is inherited from H and will be denoted by $\|\cdot\|_{\mathrm{H}_n}$.

Let

$$P_n : \mathrm{H} \to \mathrm{H}_n,$$

be the orthogonal projection i.e. $\forall\, u \in \mathrm{H}$, $u - P_n u \perp \mathrm{H}_n$ and

$$y = P_n u \Leftrightarrow y \in \mathrm{H}_n \text{ and } u - y \perp \mathrm{H}_n.$$

One can show that P_n is given by

$$P_n u = \mathscr{F}^{-1}(\mathbb{1}_{B_n}\hat{u}). \tag{6.4.1}$$

Let us recall that $\mathrm{D(A)} := \mathrm{H} \cap H^{2,2}$ and the Stokes operator is given by

$$\mathrm{A}u = -\Pi(\Delta u), \qquad u \in \mathrm{D(A)},$$

and D(A) is a Hilbert space under the graph norm

$$|u|^2_{\mathrm{D(A)}} := |u|^2_{\mathrm{H}} + |\mathrm{A}u|^2_{\mathrm{H}}.$$

Lemma 6.4.1. *Let P_n be the orthogonal projection given by* (6.4.1)*, then $P_n : \mathrm{V} \to \mathrm{V}$ is a contraction.*

Proof. Let $u \in \mathrm{V}$, then by the definition of P_n and V

$$\begin{aligned}\|P_n u\|_{\mathrm{V}} &= \left[\int_{\mathbb{R}^3}(1+|\xi|^2)|\mathscr{F}(P_n u)(\xi)|^2\,d\xi\right]^{1/2} = \left[\int_{\mathbb{R}^3}(1+|\xi|^2)|\mathbb{1}_{B_n}(\xi)\hat{u}(\xi)|^2\,d\xi\right]^{1/2}\\ &= \left[\int_{|\xi|\le n}(1+|\xi|^2)|\hat{u}(\xi)|^2\,d\xi\right]^{1/2} \le \left[\int_{\mathbb{R}^3}(1+|\xi|^2)|\hat{u}(\xi)|^2\,d\xi\right]^{1/2} = \|u\|_{\mathrm{V}}.\end{aligned}$$

Thus we have shown that

$$\|P_n u\|_{\mathrm{V}} \le \|u\|_{\mathrm{V}}.$$

■

Lemma 6.4.2. *If $u \in \mathrm{D(A)}$ then $\Delta u \in \mathrm{H}$. In particular, if $u \in \mathrm{D(A)}$ then $\mathrm{A}u = -\Delta u$.*

Proof. Since $u \in \mathrm{D(A)}$, it is clear that $\Delta u \in L^2$. Thus we are left to show that $\mathrm{div}(\Delta u) = 0$ in the weak sense. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$, then using the definition of div and Δ, we get

$$\begin{aligned}\langle \mathrm{div}(\Delta u) \,|\, \varphi \rangle &= -\langle \Delta u \,|\, \nabla \varphi \rangle \\ &= -\langle u \,|\, \Delta(\nabla \varphi) \rangle \\ &= \langle \mathrm{div}\, u \,|\, \Delta \varphi \rangle = 0.\end{aligned}$$

By definition $\mathrm{A}u = -\Pi(\Delta u)$ but since $\Delta u \in \mathrm{H}$, and $\Pi : L^2 \to \mathrm{H}$ is an orthogonal projection, $\Pi(\Delta u) = \Delta u$ and hence,

$$\mathrm{A}u = -\Delta u, \qquad u \in \mathrm{D(A)}. \tag{6.4.2}$$

■

Lemma 6.4.3. $\mathrm{H}_n \subset \mathrm{D(A)}$ *and*

$$P_n(\mathrm{A}u) = \mathrm{A}u, \qquad u \in \mathrm{H}_n. \tag{6.4.3}$$

Proof. We start with proving the first statement. Let $u \in \mathrm{H}_n$. By definition

$$\mathrm{D(A)} = \{u \in \mathrm{H} : u \in H^{2,2}\} = \left\{ u \in \mathrm{H} : \int_{\mathbb{R}^3} \left(1 + |\xi|^2\right)^2 |\hat{u}(\xi)|^2 \, d\xi < \infty \right\}.$$

Since $u \in \mathrm{H}_n, \mathrm{supp}(\hat{u}) \subset B_n$,

$$\begin{aligned}\int_{\mathbb{R}^3} \left(1 + |\xi|^2\right)^2 |\hat{u}(\xi)|^2 \, d\xi &= \int_{|\xi| \le n} \left(1 + |\xi|^2\right)^2 |\hat{u}(\xi)|^2 \, d\xi \le (1 + n^2)^2 \int_{|\xi| \le n} |\hat{u}(\xi)|^2 \, d\xi \\ &= (1 + n^2)^2 \int_{\mathbb{R}^3} |\hat{u}(\xi)|^2 \, d\xi = (1 + n^2)^2 \|u\|_{\mathrm{H}_n}^2 < \infty.\end{aligned}$$

Thus we have proved that $u \in \mathrm{D(A)}$ and hence $\mathrm{H}_n \subset \mathrm{D(A)}$. Moreover we showed that there exists a constant $C_n > 0$, depending on n such that

$$|u|_{\mathrm{D(A)}} \le C_n \|u\|_{\mathrm{H}_n}, \qquad u \in \mathrm{H}_n. \tag{6.4.4}$$

Now in order to establish the equality (6.4.3), we just need to show that $\mathrm{A}u \in \mathrm{H}_n$. Since $u \in \mathrm{H}_n$, $u \in \mathrm{D(A)}$. Hence, Lemma 6.4.1 implies $\mathrm{A}u = -\Delta u$. We are left to show that $\mathrm{supp}(\mathscr{F}(\mathrm{A}u)) \subset B_n$. Using the definition of $\mathrm{A}u$, we get following equalities

$$\mathscr{F}(\mathrm{A}u)(\xi) = -\mathscr{F}(\Delta u)(\xi) = -|\xi|^2 \hat{u}(\xi).$$

Thus

$$\mathrm{supp}(\mathscr{F}(\mathrm{A}u)) \subset \mathrm{supp}(|\cdot|^2) \cap \mathrm{supp}(\hat{u}) \subset B_n.$$

Hence $\mathrm{A}u \in \mathrm{H}_n$. Since $P_n : \mathrm{H} \to \mathrm{H}_n$ is an orthogonal projection, we infer that

$$P_n(\mathrm{A}u) = \mathrm{A}u.$$

■

Lemma 6.4.4. $\mathrm{A}_n := \mathrm{A}|_{\mathrm{H}_n} : \mathrm{H}_n \to \mathrm{H}_n$, *is linear and bounded.*

Proof. In Lemma 6.4.2 we showed that A_n is well defined and it's straightforward to show it is linear. We are left to show that it is bounded. Let $u \in \mathrm{H}_n$, then by the Plancherel Theorem (see Theorem 2.6.7) and the definition of H_n

$$\|\mathrm{A}_n u\|_{\mathrm{H}_n} = |-\Delta u|_{L^2} = \left[\int_{\mathbb{R}^3} |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi\right]^{1/2} = \left[\int_{|\xi|\le n} |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi\right]^{1/2}$$
$$\le \left[n^2 \int_{|\xi|\le n} |\hat{u}(\xi)|^2 \, d\xi\right]^{1/2} = \left[n^2 \int_{\mathbb{R}^3} |\hat{u}(\xi)|^2 \, d\xi\right]^{1/2} = n\|u\|_{\mathrm{H}_n}.$$

Thus

$$\|A_n u\|_{\mathrm{H}_n} \le n\|u\|_{\mathrm{H}_n}. \tag{6.4.5}$$

■

Lemma 6.4.5. *The map B_n defined by*

$$B_n : \mathrm{H}_n \times \mathrm{H}_n \ni (u, \mathrm{v}) \mapsto P_n(B(u,\mathrm{v})) \in \mathrm{H}_n \tag{6.4.6}$$

is well defined and Lipschitz on balls. Moreover

$$\langle B_n(u), u\rangle_{\mathrm{H}} = 0, \qquad u \in \mathrm{H}_n, \tag{6.4.7}$$

$$|((B_n(u), u))| \le \frac{1}{2}|u|^2_{\mathrm{D(A)}} + \frac{1}{2}\big||u|\cdot|\nabla u|\big|^2_{L^2}, \quad u \in \mathrm{H}_n, \tag{6.4.8}$$

where $B_n(u) := B_n(u,u)$ and $((\cdot,\cdot))$ is defined in (6.2.1)*.*

Proof. We will show that $\forall\, u, \mathrm{v} \in \mathrm{H}_n$, $B(u,\mathrm{v}) \in \mathrm{H}$. Since $u, \mathrm{v} \in \mathrm{H}_n$, $u, \mathrm{v} \in \mathrm{D(A)}$. Thus, by the Hölder inequality

$$|B(u,\mathrm{v})|_{\mathrm{H}} = |\Pi(u\cdot\nabla \mathrm{v})|_{\mathrm{H}} \le |u\cdot\nabla \mathrm{v}|_{L^2} \le \|u\|_{L^\infty}|\nabla \mathrm{v}|_{L^2}.$$

From (6.2.7), $H^{s,2} \hookrightarrow L^\infty$ for every $s > \frac{d}{2}$. Therefore, there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty} \le C\|u\|_{H^{s,2}}.$$

In particular it holds true for $s = 2$. Thus, we have

$$|B(u,\mathrm{v})|_{\mathrm{H}} \le C\|u\|_{H^2}\|\mathrm{v}\|_{H^1}.$$

Now by (6.4.4) and (6.4.15)

$$|B(u,v)|_{\mathrm{H}} \le K_n\|u\|_{\mathrm{H}_n}\|\mathrm{v}\|_{\mathrm{H}_n} < \infty. \tag{6.4.9}$$

Hence $B(u,\mathrm{v}) \in \mathrm{H}$, which implies $B_n(u,\mathrm{v}) \in \mathrm{H}_n$ and is well defined.

Let $u, \mathrm{v} \in \mathbb{B}_R$, where

$$\mathbb{B}_R := \left\{u \in \mathrm{H}_n : \|u\|_{\mathrm{H}_n} \le R\right\}. \tag{6.4.10}$$

Then, as before

$$\begin{aligned}\|B_n(u) - B_n(\mathrm{v})\|_{\mathrm{H}_n} &\le |B(u) - B(\mathrm{v})|_{\mathrm{H}} \le |u \cdot \nabla u - \mathrm{v} \cdot \nabla \mathrm{v}|_{L^2} \\ &\le |(u - \mathrm{v}) \cdot \nabla u|_{L^2} + |\mathrm{v} \cdot \nabla (u - \mathrm{v})|_{L^2} \\ &\le \|u - \mathrm{v}\|_{L^\infty} |\nabla u|_{L^2} + \|\mathrm{v}\|_{L^\infty} |\nabla (u - \mathrm{v})|_{L^2} \\ &\le \|u - \mathrm{v}\|_{H^2} \|u\|_{H^1} + \|\mathrm{v}\|_{H^2} \|u - \mathrm{v}\|_{H^1}.\end{aligned}$$

Since $u, \mathrm{v} \in \mathbb{B}_R$, using (6.4.4) and (6.4.15), we get

$$\|B_n(u) - B_n(\mathrm{v})\|_{\mathrm{H}_n} \le C_{n,R} \|u - \mathrm{v}\|_{\mathrm{H}_n}, \quad u, \mathrm{v} \in \mathbb{B}_R. \tag{6.4.11}$$

Since $u \in \mathrm{H}_n$ and P_n is the orthogonal projection on H,

$$\langle B_n(u), u \rangle_{\mathrm{H}} = \langle P_n(B(u,u)), u \rangle_{\mathrm{H}} = \langle B(u,u), P_n u \rangle_{\mathrm{H}} = \langle B(u,u), u \rangle_{\mathrm{H}} = 0.$$

Also by using the definition of $((\cdot,\cdot))$ and the Cauchy-Schwartz inequality we get

$$\begin{aligned}|((B_n(u), u))| &= |\langle B_n(u), -\Delta u \rangle_{\mathrm{H}}| = |\langle B(u,u), -P_n(\Delta u) \rangle_{\mathrm{H}}| = |\langle B(u,u), -\Delta u \rangle_{\mathrm{H}}| \\ &\le |B(u,u)|_{\mathrm{H}} |(-\Delta u)|_{\mathrm{H}} \le \frac{1}{2} |u|^2_{\mathrm{D(A)}} + \frac{1}{2} \big| |u| \cdot |\nabla u| \big|^2_{L^2}.\end{aligned}$$

■

Lemma 6.4.6. *The map g_n defined by*

$$g_n : \mathrm{H}_n \ni u \mapsto P_n \left[\Pi(g(|u|^2) u)\right] \in \mathrm{H}_n, \tag{6.4.12}$$

is well defined and Lipschitz on balls. Moreover

$$\begin{cases} ((-g_n(u), u)) \le C_N |\nabla u|^2_{L^2} - 2\big| |u| \cdot |\nabla u| \big|^2_{L^2}, & u \in \mathrm{H}_n, \\ \langle -g_n(u), u \rangle_{\mathrm{H}} \le -\|u\|^4_{L^4} + C_N |u|^2_{\mathrm{H}}, & u \in \mathrm{H}_n. \end{cases} \tag{6.4.13}$$

Proof. Let $u \in \mathrm{H}_n$, then by the definition of g (6.2.13), the estimate (6.2.14) and the embedding of $H^1 \hookrightarrow L^6$, we have

$$\begin{aligned}\|g_n(u)\|_{\mathrm{H}_n} &= \left\|P_n \left[\Pi(g(|u|^2) u)\right]\right\|_{\mathrm{H}_n} \le |\Pi(g(|u|^2) u)|_{\mathrm{H}} \le |g(|u|^2) u|_{L^2} \\ &= \left[\int_{\mathbb{R}^3} \left|g(|u(x)|^2)\right|^2 |u(x)|^2 \, dx\right]^{1/2} \le \left[\int_{\mathbb{R}^3} |u(x)|^6 \, dx\right]^{1/2} = \|u\|^3_{L^6} \\ &\le C \|u\|^3_{H^1} = C \left[\int_{\mathbb{R}^3} (1 + |\xi|^2) |\hat{u}(\xi)|^2 \, d\xi\right]^{3/2} = C \left[\int_{|\xi| \le n} (1 + |\xi|^2) |\hat{u}(\xi)|^2 \, d\xi\right]^{3/2} \\ &\le C(1 + n^2)^{3/2} \left[\int_{|\xi| \le n} |\hat{u}(\xi)|^2 \, d\xi\right]^{3/2} = C(1 + n^2)^{3/2} \left[\int_{\mathbb{R}^3} |\hat{u}(\xi)|^2 \, d\xi\right]^{3/2} \\ &= C(1 + n^2)^{3/2} |u|^3_{L^2} = C_n \|u\|^3_{\mathrm{H}_n} < \infty.\end{aligned} \tag{6.4.14}$$

Therefore $g_n : \mathrm{H}_n \to \mathrm{H}_n$ is well defined. From above we can also infer that there exists a constant $C_n > 0$ depending on n such that

$$\|u\|_{H^1} \leq C_n \|u\|_{\mathrm{H}_n}, \qquad u \in \mathrm{H}_n. \tag{6.4.15}$$

Let $u, \mathrm{v} \in \mathbb{B}_R$, then as before using (6.2.13), we have

$$\begin{aligned}
&\|g_n(u) - g_n(\mathrm{v})\|_{\mathrm{H}_n} \leq |\Pi(g(|u|^2)u) - \Pi(g(|\mathrm{v}|^2)\mathrm{v})|_{\mathrm{H}} \leq |g(|u|^2)u - g(|\mathrm{v}|^2)\mathrm{v}|_{L^2} \\
&\quad \leq |g(|u|^2)(u-\mathrm{v})|_{L^2} + |\left(g(|u|^2) - g(|\mathrm{v}|^2)\right)\mathrm{v}|_{L^2} \\
&\quad \leq \left[\int_{\mathbb{R}^3} |u(x)|^4 |u(x) - \mathrm{v}(x)|^2 \, dx\right]^{1/2} + 4\left[\int_{\mathbb{R}^3} |u(x) - \mathrm{v}(x)|^2 \left[|u(x)|^2 + |\mathrm{v}(x)|^2\right] |\mathrm{v}(x)|^2 \, dx\right]^{1/2}.
\end{aligned}$$

Since $H^1 \hookrightarrow L^6$, we obtain

$$\begin{aligned}
\|g_n(u) - g_n(\mathrm{v})\|_{\mathrm{H}_n} &\leq \left[\int_{\mathbb{R}^3} |u(x)|^6 \, dx\right]^{1/3} \left[\int_{\mathbb{R}^3} |u(x) - \mathrm{v}(x)|^6 \, dx\right]^{1/6} \\
&\quad + 4\left[\int_{\mathbb{R}^3} |u(x) - \mathrm{v}(x)|^6 \, dx\right]^{1/6} \left[\left[\int_{\mathbb{R}^3} |u(x)|^6 \, dx\right]^{1/3} + \left[\int_{\mathbb{R}^3} |\mathrm{v}(x)|^6 \, dx\right]^{1/3}\right]^{1/2} \left[\int_{\mathbb{R}^3} |\mathrm{v}(x)|^6 \, dx\right]^{1/6} \\
&= \left[\|u\|_{L^6}^2 \|u - \mathrm{v}\|_{L^6} + 4\|u - \mathrm{v}\|_{L^6} \left(\|u\|_{L^6}^2 + \|\mathrm{v}\|_{L^6}^2\right)^{1/2} \|\mathrm{v}\|_{L^6}\right] \\
&\leq C\|u - \mathrm{v}\|_{H^1} \left[\|u\|_{H^1}^2 + 4\left(\|u\|_{H^1}^2 + \|\mathrm{v}\|_{H^1}^2\right)^{1/2} \|\mathrm{v}\|_{H^1}\right].
\end{aligned}$$

Since $u, \mathrm{v} \in \mathbb{B}_R$, using (6.4.15), we get

$$\begin{aligned}
\|g_n(u) - g_n(\mathrm{v})\|_{\mathrm{H}_n} &\leq \hat{C}_n \|u - \mathrm{v}\|_{\mathrm{H}_n} \left[\|u\|_{\mathrm{H}_n}^2 + 4\left(\|u\|_{\mathrm{H}_n}^2 + \|\mathrm{v}\|_{\mathrm{H}_n}^2\right)^{1/2} \|\mathrm{v}\|_{\mathrm{H}_n}\right] \\
&\leq C_{n,R} \|u - \mathrm{v}\|_{\mathrm{H}_n}.
\end{aligned} \tag{6.4.16}$$

Let $u \in \mathrm{H}_n$, then using Lemmas 6.4.2 and 6.4.3, the definitions of g_n and $((\cdot,\cdot))$ we get

$$\begin{aligned}
((-g_n(u), u)) &= -\langle g_n(u), -\Delta u\rangle_{\mathrm{H}} = -\langle \Pi(g(|u|^2)u, -P_n(\Delta u)\rangle_{\mathrm{H}} \\
&= -\langle g(|u|^2)u, -\Pi(\Delta u)\rangle_{\mathrm{H}} = -\langle g(|u|^2)u, -\Delta u\rangle_{L^2}.
\end{aligned}$$

Also note that for $u \in \mathrm{H}_n$

$$\langle -g_n(u), u\rangle_{\mathrm{H}} = -\langle \Pi(g(|u|^2))u, P_n u\rangle_{\mathrm{H}} = -\langle g(|u|^2)u, \Pi(u)\rangle_{L^2} = -\langle g(|u|^2)u, u\rangle_{L^2}.$$

Hence the inequalities (6.4.13) can be established with the help of the above two relations and Lemma 6.2.1 (ii). This completes proof of the lemma. ∎

Lemma 6.4.7. *Let us assume that the function f satisfies the assumption* (**A1**). *Then the map*

$$f_n : \mathrm{H}_n \ni u \mapsto P_n[\Pi(f(u))] \in \mathrm{H}_n \tag{6.4.17}$$

is well defined and Lipschitz.

Proof. Let $u \in \mathrm{H}_n$, then by the assumption (**A1**),

$$\|f_n(u)\|_{\mathrm{H}_n} \le |\Pi(f(u))|_{\mathrm{H}} \le |f(u)|_{L^2} \le C_f |u|_{L^2} = C_f \|u\|_{\mathrm{H}_n} < \infty.$$

Therefore $f_n : \mathrm{H}_n \to \mathrm{H}_n$ is well defined. Let $u, \mathrm{v} \in \mathrm{H}_n$, then

$$\begin{aligned} \|f_n(u) - f_n(\mathrm{v})\|_{\mathrm{H}_n} &\le |\Pi f(u) - \Pi f(\mathrm{v})|_{\mathrm{H}} \le |f(u) - f(\mathrm{v})|_{L^2} \\ &\le C_f |u - \mathrm{v}|_{L^2} = C_f \|u - \mathrm{v}\|_{\mathrm{H}_n}. \end{aligned} \tag{6.4.18}$$

■

Lemma 6.4.8. *Let σ satisfy the assumption* (**A2**)*. Then the map*

$$G_n : \mathrm{H}_n \ni u \mapsto P_n \circ (G(u)) \in \mathcal{T}_2(\ell^2; \mathrm{H}_n) \tag{6.4.19}$$

is well defined and Lipschitz.

Proof. Let $u \in \mathrm{H}_n$, then

$$\begin{aligned} \|G_n(u)\|_{\mathcal{T}_2(\ell^2;\mathrm{H}_n)} &\le \|(G(u))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})} \le \left[\int_{\mathbb{R}^3} \|\sigma(x)\|_{\ell^2}^2 |\nabla u(x)|^2 \, dx \right]^{1/2} \\ &\le \left[\sup_{x \in \mathbb{R}^3} \|\sigma(x)\|_{\ell^2}^2 \right]^{1/2} |\nabla u|_{L^2} \le \frac{1}{2} \|u\|_{H^1}. \end{aligned}$$

Using (6.4.15), we infer

$$\|G_n(u)\|_{\mathcal{T}_2(\ell^2;\mathrm{H}_n)} \le C_n \|u\|_{\mathrm{H}_n} < \infty. \tag{6.4.20}$$

Thus $G_n : \mathrm{H}_n \to \mathcal{T}_2(\ell^2; \mathrm{H}_n)$ is well defined. Let $u, \mathrm{v} \in \mathrm{H}_n$, then

$$\begin{aligned} \|G_n(u) - G_n(\mathrm{v})\|_{\mathcal{T}_2(\ell^2;\mathrm{H}_n)} &\le \|G(u) - G(\mathrm{v})\|_{\mathcal{T}_2(\ell^2;\mathrm{H})} \\ &\le \left[\int_{\mathbb{R}^3} \sum_{j=1}^{\infty} |\sigma_j(x)|^2 |\nabla (u - \mathrm{v})(x)|^2 \, dx \right]^{1/2} \\ &= \left[\int_{\mathbb{R}^3} \|\sigma(x)\|_{\ell^2}^2 |\nabla (u - \mathrm{v})(x)|^2 \, dx \right]^{1/2} \\ &\le \left(\sup_{x \in \mathbb{R}^3} \|\sigma(x)\|_{\ell^2}^2 \right)^{1/2} |\nabla (u - \mathrm{v})|_{L^2} \le \frac{1}{2} \|u - \mathrm{v}\|_{H^1}. \end{aligned}$$

Using (6.4.15), we infer

$$\|G_n(u) - G_n(\mathrm{v})\|_{\mathcal{T}_2(\ell^2;\mathrm{H}_n)} \le C_n \|u - \mathrm{v}\|_{\mathrm{H}_n}. \tag{6.4.21}$$

■

Proposition 6.4.9. *L^2,H^1 and* D(A)*-norms on H_n are equivalent.*

Proof. Let $u \in \mathrm{H}_n$, then using the Plancherel Theorem

$$|u|_{L^2} = \left[\int_{\mathbb{R}^3} |\hat{u}(\xi)|^2\, d\xi\right]^{1/2} = \left[\int_{|\xi|\le n} |\hat{u}(\xi)|^2\, d\xi\right]^{1/2} = \|u\|_{\mathrm{H}_n}. \tag{6.4.22}$$

Thus if $u \in \mathrm{H}_n$ then L^2 and H_n have equal norms. The equivalence of H^1 and H_n norms is established from (6.4.15). Using (6.4.5) and (6.4.22) we can establish equivalence of D(A) and H_n norms. ■

As discussed earlier in the introduction instead of using standard Galerkin approximation of SPDE on the finite dimensional space we will look at the truncated SPDE on an infinite dimensional space H_n. We will establish the existence of a unique global solution to the truncated SPDE and obtain a'priori estimates in order to prove the tightness of measures on a suitable space.

In order to study the truncated SPDE on H_n we project the SPDE (6.2.16) on H_n using P_n. The projected SPDE on H_n is given by

$$\begin{cases} du_n(t) = -[\mathrm{A}_n u_n(t) + B_n(u_n(t)) + g_n(u_n(t)) - f_n(u_n(t))]\, dt + G_n(u_n(t))dW(t), \\ u_n(0) = P_n(u_0), \end{cases} \tag{6.4.23}$$

where $u_n \in \mathrm{H}_n$, $u_0 \in \mathrm{V}$ and other operators B_n, g_n, f_n and G_n are as defined in Lemmas 6.4.4 - 6.4.8.

Lemma 6.4.10. *Let us define $F : \mathrm{H}_n \to \mathbb{R}$ by*

$$F(u) := \|G_n(u)\|_{\mathcal{T}_2(\ell^2;\mathrm{H}_n)} + 2\langle u, -\mathrm{A}_n u - B_n(u) - g_n(u) + f_n(u)\rangle_{\mathrm{H}}, \quad u \in \mathrm{H}_n. \tag{6.4.24}$$

Then for every $u \in \mathrm{H}_n$ there exists $K_1 > 0$ such that

$$F(u) \le K_1(1 + \|u\|^2_{\mathrm{H}_n}). \tag{6.4.25}$$

Proof. From the definition of A_n, B_n, g_n and f_n, we have

$$\begin{aligned} &\|G_n(u)\|_{\mathcal{T}_2(\ell^2;\mathrm{H}_n)} + 2\langle u, -\mathrm{A}_n u - B_n(u) - g_n(u) + f_n(u)\rangle_{\mathrm{H}} \\ &\quad = \|G_n(u)\|_{\mathcal{T}_2(\ell^2;\mathrm{H}_n)} + 2\left\langle u, -\Pi(\Delta u) - P_n(B(u)) - P_n[\Pi(g(|u|^2)u)] + P_n[\Pi(f(u))]\right\rangle_{\mathrm{H}}. \end{aligned}$$

Since $u \in \mathrm{H}_n$, using Lemma 6.4.8, we get

$$\begin{aligned} F(u) &\le \frac{1}{4}\|u\|^2_{\mathrm{H}_n} - 2|\nabla u|^2_{L^2} - 2\langle u, B(u)\rangle_{\mathrm{H}} - 2\langle u, g(|u|^2)u\rangle_{\mathrm{H}} + 2\langle u, f(u)\rangle_{\mathrm{H}} \\ &\le \frac{1}{4}\|u\|^2_{\mathrm{H}_n} - 2|\nabla u|^2_{L^2} - 2\left|\sqrt{g(|u|^2)}\,|u|\right|^2_{L^2} + C_f\|u\|^2_{\mathrm{H}_n} \end{aligned}$$

$$F(u) + 2|\nabla u|^2_{L^2} + 2\left|\sqrt{g(|u|^2)}\,|u|\right|^2_{L^2} \le \frac{1}{4}\|u\|^2_{\mathrm{H}_n} + C_f\|u\|^2_{\mathrm{H}_n} \le K_1(1 + \|u\|^2_{\mathrm{H}_n}),$$

for appropriately chosen K_1. Thus, in particular

$$F(u) \le K_1(1 + \|u\|^2_{\mathrm{H}_n}).$$

■

We will need the following theorem to prove Theorem 6.4.12. We have modified it in the way (compared to the statement in Theorem 2.8.8, see also [1, Theorem 3.1]) we will use it.

Theorem 6.4.11. *Let X be a separable, possibly infinite dimensional, Hilbert space. Assume that σ and b satisfy the following conditions*

(i) For any $R>0$ there exists a constant $C>0$ such that

$$\|\sigma(u)-\sigma(\mathrm{v})\|_{\mathcal{T}_2(\ell^2;X)}+\|b(u)-b(\mathrm{v})\|_X \le C\|u-\mathrm{v}\|_X^2, \quad \|u\|_X,\|\mathrm{v}\|_X \le R.$$

(ii) There exists a constant $K_1>0$ such that

$$\|\sigma(u)\|^2_{\mathcal{T}_2(\ell^2;X)}+2\langle u,b(u)\rangle_{L^2} \le K_1(1+\|u\|_X^2), \qquad u\in X.$$

Then for any $X-$valued ξ, there exists a unique global solution $u=(u(t))_{t\ge 0}$ to

$$u(t)=\xi+\int_0^t \sigma(u(s))\,dW(s)+\int_0^t b(u(s))\,ds.$$

Theorem 6.4.12. *Let the assumptions* **(A1)** *and* **(A2)** *hold. Then for every $u_0\in \mathrm{V}$ there exists a unique global solution $u_n=(u_n(t))_{t\ge 0}$ to*

(6.4.26)
$$\begin{cases} u_n(t)+\int_0^t [\mathrm{A}_n u_n(s)+B_n(u_n(s))+g_n(u_n(s))]\,ds=\int_0^t f_n(u_n(s))\,ds+\int_0^t G_n(u_n(s))\,dW(s),\\ u_n(0)=P_n u_0. \end{cases}$$

Proof. The proof is direct application of Theorem 6.4.11. Using Lemmas 6.4.4 - 6.4.8, we can show that the condition (i) of Theorem 6.4.11 is satisfied. In Lemma 6.4.10 we proved that the condition (ii) is satisfied. Thus we have the existence of a unique global solution u_n to (6.4.26). ■

By Lemma 6.4.4 the map A_n is linear and bounded on H_n and thus, $\mathrm{A}_n=\mathrm{A}$ on H_n.

6.5 Existence of solution

6.5.1 A'priori estimates

In this subsection we will obtain certain a'priori estimates for the solution u_n of (6.4.26). We will use these a'priori estimates in Lemma 6.5.3 to prove the tightness of measures on the space $\mathcal{Z}_T$, defined in (6.3.5). We will also establish certain higher order estimates which will be required to prove the convergence of non-linear terms in later sections.

Let us fix $T>0$. For any $R>0$, define the stopping time

$$\tau_R^n := \inf\{t\in[0,T] : \|u_n(t)\|_{\mathrm{V}} \ge R\}, \tag{6.5.1}$$

where u_n is the solution of (6.4.26). By the definition of martingale solution one can infer that for every $n\ge 1$, $\tau_R^n \nearrow \infty$ as $R\nearrow\infty$.

Lemma 6.5.1. *Let u_n be the solution of* (6.4.26). *For all $\rho > 0$ there exist positive constants $C_1(\rho), C_2(\rho)$ such that if $\|u_0\|_{\mathrm{V}} \le \rho$, then*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left(\sup_{t\in[0,T]} \|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^2\right) \le C_1(\rho), \tag{6.5.2}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^{T\wedge\tau_R^n} |u_n(t)|_{\mathrm{D(A)}}^2\,dt \le C_2(\rho), \tag{6.5.3}$$

Moreover, for every $\delta > 0$ there exists a constant $C(\delta) > 0$ such that if $|u_0|_{\mathrm{H}} \le \delta$, then

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^{T\wedge\tau_R^n} \|u_n(s)\|_{L^4}^4 \le C_3(\delta). \tag{6.5.4}$$

Proof. Let u_n be the solution of (6.4.26) then applying the Itô formula to $\phi(x) = |x|_{\mathrm{H}}^2$ and the process u_n, we get

$$\begin{aligned} |u_n(t\wedge\tau_R^n)|_{\mathrm{H}}^2 &= |P_n u_0|_{\mathrm{H}}^2 + 2\int_0^{t\wedge\tau_R^n} \langle u_n(s), -\mathrm{A}u_n(s) - B_n(u_n(s)) - g_n(u_n(s))\rangle_{\mathrm{H}}\,ds \\ &\quad + 2\int_0^{t\wedge\tau_R^n} \langle u_n(s), f_n(u_n(s))\rangle_{\mathrm{H}}\,ds + 2\int_0^{t\wedge\tau_R^n} \langle u_n(s), G_n(s,u_n(s))\,dW_s\rangle_{\mathrm{H}} \\ &\quad + \int_0^{t\wedge\tau_R^n} \|G_n(s,u_n(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2\,ds. \end{aligned} \tag{6.5.5}$$

Using Lemma 6.2.1, assumptions (**A1**) and (**A2**), boundedness of P_n in H, the Cauchy-Schwarz and the Young inequality, we get

$$\begin{aligned} |u_n(t\wedge\tau_R^n)|_{\mathrm{H}}^2 &\le |u_0|_{\mathrm{H}}^2 - 2\int_0^{t\wedge\tau_R^n} |\nabla u_n(s)|_{L^2}^2\,ds - 2\int_0^{t\wedge\tau_R^n} \langle g(|u_n(s)|^2)u_n(s), u_n(s)\rangle_{\mathrm{H}}\,ds \\ &\quad + 2C_f\int_0^{t\wedge\tau_R^n} |u_n(s)|_{\mathrm{H}}^2\,ds + 2\int_0^{t\wedge\tau_R^n} \langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{H}} + \frac{1}{4}\int_0^{t\wedge\tau_R^n} |\nabla u_n(s)|_{L^2}^2\,ds. \end{aligned} \tag{6.5.6}$$

Since $u_n \in \mathrm{H}_n$, we have the following identities

$$\begin{aligned} \langle g_n(u_n), u_n\rangle_{\mathrm{H}} &= \langle \Pi\left(g(|u_n|^2)u_n\right), P_n u_n\rangle_{\mathrm{H}} = \langle g(|u_n|^2)u_n, \Pi u_n\rangle_{L^2} \\ &= \langle g(|u_n|^2)u_n, u_n\rangle_{L^2}. \end{aligned}$$

Thus, using the second part of the inequality (6.4.13), we get

$$\begin{aligned} |u_n(t\wedge\tau_R^n)|_{\mathrm{H}}^2 &\le |u_0|_{\mathrm{H}}^2 - 2\int_0^{t\wedge\tau_R^n} |\nabla u_n(s)|_{L^2}^2\,ds - 2\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{L^4}^4\,ds + 2C_N\int_0^{t\wedge\tau_R^n} |u_n(s)|_{\mathrm{H}}^2\,ds \\ &\quad + 2C_f\int_0^{t\wedge\tau_R^n} |u_n(s)|_{\mathrm{H}}^2\,ds + 2\int_0^{t\wedge\tau_R^n} \langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{H}} \\ &\quad + \frac{1}{4}\int_0^{t\wedge\tau_R^n} |\nabla u_n(s)|_{L^2}^2\,ds. \end{aligned}$$

On rearranging we have

$$\begin{aligned} &|u_n(t\wedge\tau_R^n)|_{\mathrm{H}}^2 + \frac{7}{4}\int_0^{t\wedge\tau_R^n} |\nabla u_n(s)|_{L^2}^2\,ds + 2\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{L^4}^4\,ds \\ &\quad \le |u_0|_{\mathrm{H}}^2 + C_{f,N}\int_0^{t\wedge\tau_R^n} |u_n(s)|_{\mathrm{H}}^2\,ds + 2\int_0^{t\wedge\tau_R^n} \langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{H}}. \end{aligned} \tag{6.5.7}$$

Now since the process $\mu_n(t \wedge t \wedge \tau_R^n)$, $t \in [0,T]$

$$\mu_n(t \wedge \tau_R^n) = \int_0^{t\wedge\tau_R^n} \langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{H}}, \quad t \in [0,T]$$

is a $\mathbb{F}$-martingale, as by Lemma 6.2.1 and (6.5.1) we have the following inequalities

$$\begin{aligned}\mathbb{E}\int_0^{t\wedge\tau_R^n} \left|\langle u_n(s), G(s,u_n(s))\rangle_{\mathrm{H}}\right|^2 ds &\le \mathbb{E}\int_0^{t\wedge\tau_R^n} |u_n(s)|_{\mathrm{H}}^2 \|G(s,u_n(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2\, ds \\ &\le \frac{1}{4}\mathbb{E}\int_0^{t\wedge\tau_R^n} |u_n(s)|_{\mathrm{H}}^2 |\nabla u_n(s)|_{L^2}^2\, ds < \infty,\end{aligned}$$

where to establish the last inequality we have used the equivalences of norm from Proposition 6.4.9. Thus $\mathbb{E}[\mu_n(t)] = 0$.

Using Lemma 2.7.20 for the following three processes:

$$X(t) = |u_n(t\wedge\tau_R^n)|_{\mathrm{H}}^2, \qquad Y(t) = \frac{7}{4}\int_0^{t\wedge\tau_R^n} |\nabla u_n(s)|_{L^2}^2\, ds + 2\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{L^4}^4\, ds,$$

and

$$I(t) = 2\mu_n(t) = 2\int_0^{t\wedge\tau_R^n} \langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{H}},$$

we see that from (6.5.7), condition (2.7.3) is satisfied for $\alpha = 1$, $Z = |u_0|_{\mathrm{H}}^2$ and $\phi(r) = C_{f,N}$. Since $\mathbb{E}(I(t)) = 0$, condition (2.7.4) is satisfied and hence all inequalities for the parameters (see (2.7.2)) are trivially satisfied. Thus if $|u_0|_{\mathrm{H}} \le \delta$, we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|u_n(t\wedge\tau_R^n)|_{\mathrm{H}}^2 + \frac{7}{4}\int_0^{t\wedge\tau_R^n} |\nabla u_n(s)|_{L^2}^2\, ds + 2\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{L^4}^4\, ds\right] \le C_T(\delta). \tag{6.5.8}$$

In particular

$$\sup_{n\in\mathbb{N}}\left(\sup_{t\in[0,T]} \mathbb{E}|u_n(t\wedge\tau_R^n)|_{\mathrm{H}}^2\right) \le C_T(\delta). \tag{6.5.9}$$

Hence, using (6.5.8) and (6.5.9) we infer that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^{T\wedge\tau_R^n} \|u_n(s)\|_{L^4}^4 \le \tilde{C}_T(|u_0|_{\mathrm{H}}^2) =: C_3(\delta). \tag{6.5.10}$$

Since we are interested in the estimates involving V norm of u. We apply the Itô formula to $\phi(x) = |\nabla x|_{L^2}^2$ and the process $u_n(t)$, obtaining

$$\begin{aligned}|\nabla u_n(t\wedge\tau_R^n)|_{L^2}^2 = |\nabla(P_n u_0)|_{L^2}^2 &+ 2\int_0^{t\wedge\tau_R^n} ((u_n(s), -\mathrm{A}u_n(s) - B_n(u_n(s)) - g_n(u_n(s))))\, ds \\ &+ 2\int_0^{t\wedge\tau_R^n} ((u_n(s), f_n(u_n(s))))\, ds + 2\int_0^{t\wedge\tau_R^n} ((u_n(s), G_n(s,u_n(s))\,dW_s)) \\ &+ \int_0^{t\wedge\tau_R^n} \|\nabla(G_n(s,u_n(s)))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2\, ds,\end{aligned} \tag{6.5.11}$$

where $((\cdot,\cdot))$ is as defined in (6.2.1). Using Lemma 6.2.1, assumptions (**A1**) and (**A2**), boundedness of P_n in H, estimates (6.4.8), (6.4.13), the Cauchy-Schwarz and the Young inequality, we get

$$
\begin{aligned}
|\nabla u_n(t\wedge\tau_R^n)|_{L^2}^2 &\le |\nabla u_0|_{L^2}^2 - 2\int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2\,ds + \int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2 \\
&\quad + \int_0^{t\wedge\tau_R^n}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\,ds + 2C_N\int_0^{t\wedge\tau_R^n}|\nabla u_n(s)|_{L^2}^2\,ds \\
&\quad - 4\int_0^{t\wedge\tau_R^n}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\,ds + \int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}|f(u_n(s))|_{L^2}\,ds \\
&\quad + \frac{1}{2}\int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2\,ds + C_{T,\sigma}\int_0^{t\wedge\tau_R^n}|\nabla u_n(s)|_{L^2}^2\,ds \\
&\quad + 2\int_0^{t\wedge\tau_R^n}((u_n(s),G(s,u_n(s))\,dW_s)) \\
&\le |\nabla u_0|_{L^2}^2 - \frac{1}{2}\int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2\,ds - 3\int_0^{t\wedge\tau_R^n}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\,ds \\
&\quad + C_{T,\sigma,N}\int_0^{t\wedge\tau_R^n}|\nabla u_n(s)|_{L^2}^2\,ds + \frac{1}{4}\int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2\,ds + C_f\int_0^{t\wedge\tau_R^n}|u_n(s)|_{\mathrm{H}}^2\,ds \\
&\quad + 2\int_0^{t\wedge\tau_R^n}((u_n(s),G(s,u_n(s))\,dW_s)).
\end{aligned}
$$

On rearranging we have

$$
\begin{aligned}
&|\nabla u_n(t\wedge\tau_R^n)|_{L^2}^2 + \frac{1}{4}\int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2\,ds + 3\int_0^{t\wedge\tau_R^n}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\,ds \\
&\le |\nabla u_0|_{L^2}^2 + C_{T,\sigma,N}\int_0^{t\wedge\tau_R^n}|\nabla u_n(s)|_{L^2}^2\,ds + C_f\int_0^{t\wedge\tau_R^n}|u_n(s)|_{\mathrm{H}}^2\,ds \\
&\quad + 2\int_0^{t\wedge\tau_R^n}((u_n(s),G(s,u_n(s))\,dW_s)). \qquad (6.5.12)
\end{aligned}
$$

Now since the process $\mu_n(t\wedge t\wedge\tau_R^n)$, $t\in[0,T]$

$$
\mu_n(t\wedge\tau_R^n) = \int_0^{t\wedge\tau_R^n}((u_n(s),G(s,u_n(s))\,dW_s)), \quad t\in[0,T]
$$

is a $\mathbb{F}$-martingale, as by Lemma 6.2.1 and (6.5.1) we have the following inequalities

$$
\begin{aligned}
\mathbb{E}\int_0^{t\wedge\tau_R^n}\big|((u_n(s),G(s,u_n(s))))\big|^2\,ds &\le \mathbb{E}\int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2\|G(s,u_n(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2\,ds \\
&\le \frac{1}{4}\mathbb{E}\int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2|\nabla u_n(s)|_{L^2}^2\,ds < \infty,
\end{aligned}
$$

where to establish the last inequality we have used the equivalences of norm from Proposition 6.4.9. Thus $\mathbb{E}[\mu_n(t)] = 0$.

Again as before, by applying Lemma 2.7.20 for

$$
X(t) = |\nabla u_n(t\wedge\tau_R^n)|_{L^2}^2, \qquad Y(t) = \frac{1}{4}\int_0^{t\wedge\tau_R^n}|\mathrm{A}u_n(s)|_{L^2}^2\,ds + 3\int_0^{t\wedge\tau_R^n}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\,ds,
$$

$$
I(t) = 2\mu_n(t) = 2\int_0^{t\wedge\tau_R^n}((u_n(s),G(s,u_n(s))\,dW_s)),
$$

the inequalities (2.7.3) and (2.7.4) are satisfied. Thus from (6.5.9) and (6.5.11), if $\|u_0\|_{\mathrm{V}} \le \rho$, then

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|\nabla u_n(t\wedge\tau_R^n)|_{L^2}^2 + \frac{1}{4}\int_0^{t\wedge\tau_R^n} |\mathrm{A}u_n(s)|_{L^2}^2\,ds + 3\int_0^{t\wedge\tau_R^n} \big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\,ds \right] \le C_T(\rho). \tag{6.5.13}$$

In particular

$$\sup_{n\in\mathbb{N}} \left(\sup_{t\in[0,T]} \mathbb{E}|\nabla u_n(t\wedge\tau_R^n)|_{L^2}^2 \right) \le C_T(\rho). \tag{6.5.14}$$

From (6.5.14) and (6.5.13), we have the following estimate

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^{T\wedge\tau_R^n} |\mathrm{A}u_n(t)|_{L^2}^2\,dt \le C_T(\rho). \tag{6.5.15}$$

Note that $|u|_{\mathrm{D(A)}}^2 := |u|_{L^2}^2 + |\mathrm{A}u|_{L^2}^2$. Thus from (6.5.9) and (6.5.15) we can infer (6.5.3). On combining (6.5.9) and (6.5.14), we get

$$\sup_{n\in\mathbb{N}} \left(\sup_{t\in[0,T]} \mathbb{E}\|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^2 \right) \le C_T(\rho). \tag{6.5.16}$$

Using the Burkholer-Davis-Gundy inequality, the definition of $\langle\cdot,\cdot\rangle_{\mathrm{V}}$ and the Young's inequality, for every $\varepsilon > 0$ we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\int_0^{t\wedge\tau_R^n} \langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{V}} \\
&= \mathbb{E}\left[\int_0^{T\wedge\tau_R^n} \left|\langle u_n(s), G(s,u_n(s))\rangle_{\mathrm{H}} + \big(\big(u_n(s), G(s,u_n(s))\big)\big)\right|^2 ds\right]^{1/2} \\
&\le \mathbb{E}\left[\int_0^{T\wedge\tau_R^n} \|G(s,u_n(s))\|_{\mathscr{T}_2(\ell^2;\mathrm{H})}^2 |u_n(s)|_{\mathrm{D(A)}}^2\,ds\right]^{1/2} \\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{s\in[0,T]} |\nabla u_n(s\wedge\tau_R^n)|_{L^2}^2 \int_0^{T\wedge\tau_R^n} |u_n(s)|_{\mathrm{D(A)}}^2\,ds\right]^{1/2} \\
&\le \varepsilon\mathbb{E}\sup_{s\in[0,T]} |\nabla u_n(s\wedge\tau_R^n)|_{L^2}^2 + C_\varepsilon\mathbb{E}\int_0^{T\wedge\tau_R^n} |u_n(s)|_{\mathrm{D(A)}}^2\,ds.
\end{aligned} \tag{6.5.17}$$

On combining (6.5.7) and (6.5.12), then using (6.5.3), (6.5.9), (6.5.14), (6.5.17) and Lemma 2.7.20, we can infer (6.5.2). Thus the proof of the lemma is complete. ■

In the next lemma we will use the estimates from Lemma 6.5.1 to establish higher order estimates.

Lemma 6.5.2. *Let τ_R^n be as defined in* (6.5.1). *For all $\rho > 0$ and $p \in [1,3]$ there exist positive constants $C_1(p,\rho)$, $C_2(p,\rho)$ such that if $\|u_0\|_{\mathrm{V}} \le \rho$, then*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left(\sup_{t\in[0,T]} \|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p} \right) \le C_1(p,\rho), \tag{6.5.18}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^{T\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} |\mathrm{A}u_n(s)|_{L^2}^2\,ds \le C_2(p,\rho). \tag{6.5.19}$$

Proof. Let $p \in [1,3]$. Then by using the Itô formula for $\xi(t) = \|u_n(t)\|_{\mathrm{V}}^2$, $\phi(x) = x^p$, equations (6.5.5), (6.5.11) and the definition of $\|\cdot\|_{\mathrm{V}}$, we obtain

$$
\begin{aligned}
\|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p} &= \|u_n(0)\|_{\mathrm{V}}^{2p} - 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\left(|\nabla u_n(s)|_{L^2}^2 + |\mathrm{A}u_n(s)|_{L^2}^2\right) ds \\
&- 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), B_n(u_n(s))\rangle_{\mathrm{V}}\, ds - 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), g_n(u_n(s))\rangle_{\mathrm{V}}\, ds \\
&- 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), f_n(u_n(s))\rangle_{\mathrm{V}}\, ds + p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\|G_n(s,u_n(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{V})}^2\, ds \\
&+ 2p(p-1)\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-2)}\langle u_n(s), G_n(s,u_n(s))\rangle_{\mathrm{V}}^2\, ds
\end{aligned}
$$

(6.5.20)

$$
+ 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), G_n(s,u_n(s))dW_s\rangle_{\mathrm{V}}.
$$

Using Lemma 6.2.1, the definition of g (6.2.13), boundedness of P_n in V and assumption (**A1**), we can simplify (6.5.20)

$$
\begin{aligned}
\|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p} \le \|u_n(0)\|_{\mathrm{V}}^{2p} &- 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\left(|\nabla u_n(s)|_{L^2}^2 + |\mathrm{A}u_n(s)|_{L^2}^2\right) ds \\
&+ 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\left[\frac{1}{2}|\mathrm{A}u_n(s)|_{L^2}^2 + \frac{1}{2}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\right] ds \\
&- 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), |u_n(s)|^2 u_n(s) - N u_n(s)\rangle_{\mathrm{V}}\, ds \\
&+ 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\left(C_f|u_n(s)|_{\mathrm{H}}^2 + \frac{1}{4}|\mathrm{A}u_n(s)|_{L^2}^2\right) ds \\
&+ p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\left[\frac{1}{2}|\mathrm{A}u_n(s)|_{L^2}^2 + C_{T,\sigma}|\nabla u_n(s)|_{L^2}^2\right] ds \\
&+ 2p(p-1)\int_0^{t\wedge\tau_R^n} \|u_n(s)\|^{2(p-2)}\|G(s,u_n(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{V})}^2\|u_n(s)\|_{\mathrm{V}}^2\, ds \\
&+ 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{V}}.
\end{aligned}
$$

On rearranging we get

$$
\begin{aligned}
\|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p} &\le \|u_n(0)\|_{\mathrm{V}}^{2p} - \frac{p}{2}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)|_{L^2}^2\, ds \\
&- 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|^{2(p-1)}|\nabla u_n(s)|_{L^2}^2\, ds + p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\, ds \\
&- 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\big||\sqrt{g(|u(s)|^2)}|\cdot|u(s)|\big|_{L^2}^2\, ds + C_N\cdot 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\nabla u_n(s)|_{L^2}^2\, ds \\
&- 4p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\, ds + 2pC_f\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|u_n(s)|_{\mathrm{H}}^2\, ds \\
&+ pC_{T,\sigma}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\nabla u_n(s)|_{L^2}^2\, ds + 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{V}} \\
&+ 2p(p-1)\int_0^{t\wedge\tau_R^n} \|u_n(s)\|^{2(p-1)}\left[\frac{1}{4}|\mathrm{A}u_n(s)|_{L^2}^2 + C_{T,\sigma}|\nabla u_n(s)|_{L^2}^2\right] ds
\end{aligned}
$$

which on further simplification yields

$$
\begin{aligned}
&\|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p} + \frac{p(3-p)}{2}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)|_{L^2}^2\,ds \\
&+3p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\,ds + 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\big||\sqrt{g(|u(s)|^2)}|\cdot|u(s)|\big|_{L^2}^2\,ds \\
&\le \|u_n(0)\|_{\mathrm{V}}^{2p} + C_{T,\sigma,N,p}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\nabla u_n(s)|_{L^2}^2\,ds + C_f\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|u_n(s)|_{\mathrm{H}}^2\,ds
\end{aligned}
$$

(6.5.21)

$$
+2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{V}}.
$$

As before we will show that the process $\mu_n(t\wedge t\wedge\tau_R^n)$, $t\in[0,T]$

$$
\mu_n(t\wedge\tau_R^n) = \int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{V}}, \quad t\in[0,T]
$$

is a $\mathbb{F}$-martingale. By Lemma 6.2.1 and (6.5.1) we have the following inequalities

$$
\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{4(p-1)}\big|\langle u_n(s), G(s,u_n(s))\rangle_{\mathrm{V}}\big|^2\,ds \\
&\quad\le \mathbb{E}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{4(p-1)}|u_n(s)|_{\mathrm{D(A)}}^2\|G(s,u_n(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2\,ds \\
&\quad\le \frac{1}{4}\mathbb{E}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{4(p-1)}|u_n(s)|_{\mathrm{D(A)}}^2|\nabla u_n(s)|_{L^2}^2\,ds < \infty,
\end{aligned}
$$

where the finiteness of the integral follows from Proposition 6.4.9. Hence $\mathbb{E}[\mu_n(t)] = 0$.

Since $|u_n(s)|_{\mathrm{H}} \le \|u_n(s)\|_{\mathrm{V}}$ and $|\nabla u_n(s)|_{L^2} \le \|u_n(s)\|_{\mathrm{V}}$ on applying the generalised version of the Gronwall Lemma (Lemma 2.7.20) for

$$
X(t) = \|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p}, \qquad I(t) = 2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\langle u_n(s), G(s,u_n(s))\,dW_s\rangle_{\mathrm{V}},
$$

and

$$
\begin{aligned}
Y(t) = &\frac{p(3-p)}{2}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)|_{L^2}^2\,ds + 3p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\big||u_n(s)|\cdot|\nabla u_n(s)|\big|_{L^2}^2\,ds \\
&+2p\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}\big||\sqrt{g(|u(s)|^2)}|\cdot|u(s)|\big|_{L^2}^2\,ds,
\end{aligned}
$$

we have

$$
\sup_{n\in\mathbb{N}}\left(\sup_{t\in[0,T]}\mathbb{E}\,\|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p}\right) \le C_{T,p}\|u_0\|_{\mathrm{V}}^{2p}, \quad p\in[1,3]. \tag{6.5.22}
$$

Using (6.5.22) in (6.5.20), we also obtain

$$
\sup_{n\in\mathbb{N}}\mathbb{E}\int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)}|\mathrm{A}u_n(s)|_{L^2}^2\,ds \le C_{T,p}\|u_0\|_{\mathrm{V}}^{2p} := C_2(p,\rho), \quad p\in[1,3]. \tag{6.5.23}
$$

Now we are left to show the estimate (6.5.18). Using the Burkholder- Davis- Gundy inequality, Lemma 6.2.1 and the Young inequality for every $\varepsilon > 0$, we get

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \int_0^{t\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} \langle u_n(s), G(s,u_n(s)) dW_s\rangle_{\mathrm{V}} \\
&\le \mathbb{E}\left[\int_0^{T\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{4(p-1)} \left|\langle u_n(s), G(s,u_n(s))\rangle_{\mathrm{V}}\right|^2 ds\right]^{1/2} \\
&\le \mathbb{E}\left[\int_0^{T\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{4(p-1)} \|G(s,u_n(s))\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})} |u_n(s)|^2_{\mathrm{D(A)}}\, ds\right]^{1/2} \\
&\le \frac{1}{4}\mathbb{E}\left[\int_0^{T\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{4(p-1)} |\nabla u_n(s)|^2_{L^2} |u_n(s)|^2_{\mathrm{D(A)}}\, ds\right]^{1/2} \\
&\le \frac{1}{4}\mathbb{E}\left[\int_0^{T\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2p} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} |u_n(s)|^2_{\mathrm{D(A)}}\, ds\right]^{1/2} \\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p} \int_0^{T\wedge\tau_R^n} \|u_n(s)\|_{\mathrm{V}}^{2(p-1)} |u_n(s)|^2_{\mathrm{D(A)}}\, ds\right]^{1/2} \\
&\le \varepsilon\mathbb{E} \sup_{t\in[0,T]} \|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p} + C_\varepsilon \mathbb{E}\int_0^{T\wedge\tau_R^n} \|u_n(s)\|^{2(p-1)} |u_n(s)|^2_{\mathrm{D(A)}}\, ds. \qquad (6.5.24)
\end{aligned}$$

Thus from (6.5.21) and using (6.5.23), (6.5.24) and Lemma 2.7.20, we have

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p}\right) \le C_{T,p}\left(\mathbb{E}\|u_0\|_{\mathrm{V}}^{2p}\right) + \varepsilon\mathbb{E}\left(\sup_{t\in[0,T]} \|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p}\right) + C_{T,p,\varepsilon}.$$

Choosing ε small enough we get

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left(\sup_{t\in[0,T]} \|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^{2p}\right) \le C_{T,p}\left(\|u_0\|_{\mathrm{V}}^{2p}\right) := C_1(p,\rho), \quad p\in[1,3]. \qquad (6.5.25)$$

■

6.5.2 Tightness of measures

For each $n\in\mathbb{N}$, the solution u_n of the truncated equation (6.4.26) defines a measure $\mathscr{L}(u_n)$ on $(\mathcal{Z}_T, \mathscr{T})$, defined in (6.3.5). In this subsection we will prove that this sequence of measures defined on $\mathcal{Z}_T$ is tight.

Lemma 6.5.3. *The set of measures* $\{\mathscr{L}(u_n), n\in\mathbb{N}\}$ *is tight on* $(\mathcal{Z}_T, \mathscr{T})$.

Proof. We recall the definition of stopping time, τ_R^n

$$\tau_R^n := \inf\{t\in[0,T] : \|u_n(t)\|_{\mathrm{V}} \ge R\}.$$

We will use Corollary 6.3.4 to prove the tightness of measures. According to estimates (6.5.2) and (6.5.3), conditions (a) and (b) are satisfied. Thus it is sufficient to prove that the sequence $(u_n)_{n\in\mathbb{N}}$

satisfies the Aldous condition [**A**] in H. By (6.4.26), for $t \in [0, T \wedge \tau_R^n]$ we have

$$\begin{aligned} u_n(t) = u_n(0) - \int_0^t \mathrm{A}_n u_n(s)\,ds - \int_0^t B_n(u_n(s))\,ds - \int_0^t g_n(u_n(s))\,ds + \int_0^t f_n(u_n(s))\,ds \\ + \int_0^t G_n(s, u_n(s))\,dW(s) \\ := J_1^n + J_2^n(t) + J_3^n(t) + J_4^n(t) + J_5^n(t) + J_6^n(t), \quad t \in [0, T \wedge \tau_R^n]. \end{aligned}$$

Let $s, t \in [0, T], s < t$ and $\theta := t - s$. First we will establish estimates for each term of the above equality.

Ad. J_2^n. Since $\mathrm{A} : \mathrm{D(A)} \to \mathrm{H}$ is a bounded linear map, then by the Hölder inequality and estimate (6.5.3), we have the following inequalities

$$\begin{aligned} \mathbb{E}\left[|J_2^n(t \wedge \tau_R^n) - J_2^n(s \wedge \tau_R^n)|_{\mathrm{H}}\right] = \mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} \mathrm{A}_n u_n(s)\,ds\right|_{\mathrm{H}} \le \mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} |\mathrm{A}u_n(s)|_{\mathrm{H}}\,ds \\ \le c\theta^{\frac12}\left(\mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} |u_n(s)|^2_{\mathrm{D(A)}}\,ds\right)^{\frac12} \le c(C_2(R))^{\frac12}\cdot\theta := c_2\cdot\theta. \end{aligned} \tag{6.5.26}$$

Ad. J_3^n. $B : \mathrm{D(A)} \times \mathrm{V} \to \mathrm{H}$ is bilinear and continuous and $P_n : \mathrm{H} \to \mathrm{H}$ is bounded then by Lemma 6.4.5, the Cauchy-Schwarz inequality and the estimates (6.5.2), (6.5.3) we have

$$\begin{aligned} \mathbb{E}\left[|J_3^n(t \wedge \tau_R^n) - J_3^n(s \wedge \tau_R^n)|_{\mathrm{H}}\right] = \mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} P_n B(u_n(s))\,ds\right|_{\mathrm{H}} \le \mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} |P_n B(u_n(s), u_n(s))|_{\mathrm{H}}\,ds \\ \le \mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} \|B\|\cdot|u_n(s)|_{\mathrm{D(A)}}\|u_n(s)\|_{\mathrm{V}}\,ds \\ \le \|B\|\cdot\mathbb{E}\left(\left[\sup_{t\in[0,T]}\|u_n(t\wedge\tau_R^n)\|^2_{\mathrm{V}}\right]^{1/2}\cdot\theta^{1/2}\left[\int_0^{t\wedge\tau_R^n}|u_n(s)|^2_{\mathrm{D(A)}}\,ds\right]^{1/2}\right) \\ \le (C_1(R))^{1/2}(C_2(R))^{1/2}\cdot\theta^{1/2} := c_3\cdot\theta^{1/2}. \end{aligned} \tag{6.5.27}$$

Ad. J_4^n. Since $H^1 \hookrightarrow L^6$, then by the definition of g and estimate (6.5.18) (for $p = 2$), we have

$$\begin{aligned} \mathbb{E}\left[|J_4^n(t \wedge \tau_R^n) - J_4^n(s \wedge \tau_R^n)|_{\mathrm{H}}\right] = \mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} g_n(u_n(s))\,ds\right|_{\mathrm{H}} \le \mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} |P_n(\Pi g(|u_n(s)|^2)u_n(s))|_{\mathrm{H}}\,ds \\ \le \mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} |g(|u_n(s)|^2)u_n(s)|_{L^2}\,ds \le \mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}\left(\int_{\mathbb{R}^3}|u_n(s,x)|^6\,dx\right)^{1/2} ds \\ = \mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}\|u_n(s)\|^3_{L^6}ds \le C\,\mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}\|u_n(s)\|^3_{\mathrm{V}}\,ds \le C\left[\mathbb{E}\left(\sup_{t\in[0,T]}\|u_n(t\wedge\tau_R^n)\|^4_{\mathrm{V}}\right)\right]^{3/4}\theta \\ \le C\cdot(C_1(2,R))^{3/4}\cdot\theta := c_4\cdot\theta. \end{aligned} \tag{6.5.28}$$

Ad. J_5^n. Using the assumption **H1**, (6.5.9) and the Cauchy-Schwarz inequality, we obtain the following inequalities

$$\mathbb{E}\left[|J_5^n(t\wedge\tau_R^n)-J_5^n(s\wedge\tau_R^n)|_{\mathrm{H}}\right]=\mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}P_n(\Pi f(u_n(s))ds)\right|_{\mathrm{H}}\le\mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}|f(u_n(s))|_{\mathrm{H}}ds$$
$$\le\mathbb{E}\left(\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}|f(u_n(s))|_{\mathrm{H}}^2ds\right)^{1/2}\cdot\theta^{1/2}\le C_f\left(\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}|u_n(s)|_{\mathrm{H}}^2ds\right)^{1/2}\theta^{\frac{1}{2}}$$
$$\le C_f(C(R))^{1/2}\theta:=c_5\cdot\theta. \tag{6.5.29}$$

Ad. J_6^n. Using the Itô isometry, Lemma 6.2.1 and (6.5.2), we obtain the following

$$\mathbb{E}\left[|J_6^n(t\wedge\tau_R^n)-J_6^n(s\wedge\tau_R^n)|_{\mathrm{H}}^2\right]=\mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}G_n(s,u_n(s))dW(s)\right|_{\mathrm{H}}^2$$
$$\le c\,\mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}\|G(s,u_n(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2ds\le\frac{c}{4}\mathbb{E}\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}|\nabla u_n(s)|_{L^2}^2ds$$
$$\le\frac{c}{4}\mathbb{E}\left(\sup_{t\in[0,T]}\|u_n(t\wedge\tau_R^n)\|_{\mathrm{V}}^2\right)\theta\le\frac{1}{4}C_1(R)\cdot\theta:=c_6\cdot\theta. \tag{6.5.30}$$

Let us fix $\kappa>0$ and $\varepsilon>0$. By the Chebyshev's inequality and estimates (6.5.26) - (6.5.29), we obtain

$$\mathbb{P}(\{|J_i^n(t\wedge\tau_R^n)-J_i^n(s\wedge\tau_R^n)|_{\mathrm{H}}\ge\kappa\})\le\frac{1}{\kappa}\mathbb{E}\left[|J_i^n(t\wedge\tau_R^n)-J_i^n(s\wedge\tau_R^n)|_{\mathrm{H}}\right]\le\frac{c_i\theta}{\kappa};\quad n\in\mathbb{N},$$

where $i=1,\dots,5$. Let $\delta_i=\dfrac{\kappa}{c_i}\varepsilon$. Then

$$\sup_{n\in\mathbb{N}}\sup_{0\le\theta\le\delta_i}\mathbb{P}(\{|J_i^n(t\wedge\tau_R^n)-J_i^n(s\wedge\tau_R^n)|_{\mathrm{H}}\ge\kappa\})\le\varepsilon,\quad i=1\dots5.$$

By the Chebyshev inequality and (6.5.30), we have

$$\mathbb{P}(\{|J_6^n(t\wedge\tau_R^n)-J_6^n(s\wedge\tau_R^n)|_{\mathrm{H}}\ge\kappa\})\le\frac{1}{\kappa^2}\mathbb{E}\left[|J_6^n(t\wedge\tau_R^n)-J_6^n(s\wedge\tau_R^n)|_{\mathrm{H}}^2\right]\le\frac{c_6\theta}{\kappa^2};\quad n\in\mathbb{N}$$

Let $\delta_6=\dfrac{\kappa^2}{c_6}\varepsilon$. Then

$$\sup_{n\in\mathbb{N}}\sup_{0\le\theta\le\delta_6}\mathbb{P}(\{|J_6^n(t\wedge\tau_R^n)-J_6^n(s\wedge\tau_R^n)|_{\mathrm{H}}\ge\kappa\})\le\varepsilon.$$

Since [**A**] holds for each term J_i^n, $i=1,2,\dots,6$; we infer that it holds also for $(u_n)_{n\in\mathbb{N}}$. Thus the proof of lemma can be concluded by invoking Corollary 6.3.4. ■

Now we will state the main theorem of this section.

Theorem 6.5.4. *Let assumptions* (**H1**) *and* (**H2**) *be satisfied. Then there exists a martingale solution* $(\hat{\Omega},\hat{\mathscr{F}},\hat{\mathbb{F}},\hat{\mathbb{P}},u)$ *of problem* (6.2.16) *such that*

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|u(t)\|_{\mathrm{V}}^2+\int_0^T|u(t)|_{\mathrm{D(A)}}^2dt\right]<\infty. \tag{6.5.31}$$

In the following subsection we will prove Theorem 6.5.4 in several steps.

6.5.3 Proof of Theorem 6.5.4

By Lemma 6.5.3 the set of measures $\{\mathscr{L}(u_n), n \in \mathbb{N}\}$ is tight on the space $(\mathcal{Z}_T, \mathscr{T})$ defined by (6.3.5). Hence by Corollary 6.3.6 there exist a subsequence $(n_k)_{k\in\mathbb{N}}$, a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and, on this space, $\mathcal{Z}_T$-valued random variables $\tilde{u}, \tilde{u}_{n_k}, k \geq 1$ such that

$$\tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ and } \tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathcal{Z}_T, \quad \tilde{\mathbb{P}} - \text{a.s.} \tag{6.5.32}$$

$\tilde{u}_{n_k} \to \tilde{u}$ in $\mathcal{Z}_T, \tilde{\mathbb{P}} -$ a.s. precisely means that

$$\begin{aligned} &\tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathscr{C}([0,T];\mathrm{U}'),\\ &\tilde{u}_{n_k} \rightharpoonup \tilde{u} \text{ in } L^2(0,T;\mathrm{D(A)}),\\ &\tilde{u}_{n_k} \to \tilde{u} \text{ in } L^2(0,T;\mathrm{H_{loc}}),\\ &\tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathscr{C}([0,T];\mathrm{V}_w). \end{aligned}$$

Let us denote the subsequence $(\tilde{u}_{n_k})$ again by $(\tilde{u}_n)_{n\in\mathbb{N}}$.

The following auxiliary result which is needed in the proof of Theorem 6.5.4, cannot be deduced directly from the Kuratowski Theorem (see Theorem 2.5.17).

Lemma 6.5.5. *Let $T > 0$ and $\mathcal{Z}_T$ be as defined in* (6.3.5)*. Then the following three sets $\mathscr{C}([0,T];\mathrm{V}) \cap \mathcal{Z}_T$, $\mathscr{C}([0,T];\mathrm{H}_n) \cap \mathcal{Z}_T$ and $L^2(0,T;\mathrm{D(A)}) \cap \mathcal{Z}_T$ are Borel subsets of $\mathcal{Z}_T$.*

In order to prove Lemma 6.5.5 we will need the following space:

$$L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^3) = \left\{u : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3 : \int_0^T \int_{|x|\leq R} |u(x,t)|^2\, dx\, dt < \infty, \forall R > 0\right\}.$$

$L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^3)$ is complete under the family of semi-norms

$$\rho_R := \left[\int_0^T \int_{|x|\leq R} |u(x,t)|^2\, dx\, dt\right]^{1/2}.$$

In particular it's a Frechét space with the metric

$$d(u,v) = \sum_{n\geq 1} \frac{1}{2^n} \frac{\rho_n(u-v)}{1+\rho_n(u-v)}.$$

Remark 6.5.6. $L^2(0,T;\mathrm{H_{loc}}) \subset L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^3)$ and we can define the open sets in $L^2(0,T;\mathrm{H_{loc}})$ by restricting the metric d to $L^2(0,T;\mathrm{H_{loc}})$. Hence $L^2(0,T;\mathrm{H_{loc}})$ is a topological space with the trace topology from $L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^3)$.

Let us define a new topological space:

$$\tilde{\mathcal{Z}}_T := \mathscr{C}([0,T];\mathrm{U}') \cap L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^3) \cap L^2_w(0,T;\mathrm{D(A)}) \cap \mathscr{C}([0,T];\mathrm{V}_w).$$

Note that $\tilde{\mathcal{Z}}_T$ and $\mathcal{Z}_T$ are same as a set. Because $L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)\cap L^2_w(0,T;\mathrm{D(A)})$ and $L^2(0,T;\mathrm{H}_{\text{loc}})\cap L^2_w(0,T;\mathrm{D(A)})$ are same as a set. $L^2(0,T;\mathrm{H}_{\text{loc}})\subset L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$ and the only extra elements in $L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$ are the ones which are locally square integrable but have non-zero divergence. But the intersection of $L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$ with $L^2_w(0,T;\mathrm{D(A)})$ eliminates those elements as the divergence free condition is imposed by the second set.

By Remark 6.5.6 and Lemma C.2, $\tilde{\mathcal{Z}}_T$ and $\mathcal{Z}_T$ have the same topologies. Thus we will prove Lemma 6.5.5 for $\tilde{\mathcal{Z}}_T$ instead of $\mathcal{Z}_T$.

Proof of Lemma 6.5.5 First of all $\mathscr{C}([0,T];\mathrm{V})\subset\mathscr{C}([0,T];\mathrm{U}')\cap L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$. Secondly, $\mathscr{C}([0,T];\mathrm{V})$ and $\mathscr{C}([0,T];\mathrm{U}')\cap L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$ are Polish spaces. And finally, since V is continuously embedded in U', the map

$$i:\mathscr{C}([0,T];\mathrm{V})\to\mathscr{C}([0,T];\mathrm{U}')\cap L^2_{\text{loc}}([0,T]\times\mathbb{R}^3),$$

is continuous and hence Borel. Thus by application of the Kuratowski Theorem, $\mathscr{C}([0,T];\mathrm{V})$ is a Borel subset of $\mathscr{C}([0,T];\mathrm{U}')\cap L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$. Therefore by Lemma C.1, $\mathscr{C}([0,T];\mathrm{V})\cap\tilde{\mathcal{Z}}_T$ is a Borel subset of $\mathscr{C}([0,T];\mathrm{U}')\cap L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)\cap\tilde{\mathcal{Z}}_T$ which is equal to $\tilde{\mathcal{Z}}_T$. We can show in the same way in the case of $\mathscr{C}([0,T];\mathrm{H}_n)\cap\mathcal{Z}_T$.

Similarly we can show that $L^2(0,T;\mathrm{D(A)})\cap\tilde{\mathcal{Z}}_T$ is a Borel subset of $\tilde{\mathcal{Z}}_T$. $L^2(0,T;\mathrm{D(A)})\hookrightarrow L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$ and both are Polish spaces thus by application of the Kuratowski Theorem, $L^2(0,T;\mathrm{D(A)})$ is a Borel subset of $L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$. Finally, we can conclude the proof of theorem by Lemma C.1. ■

By Lemma 6.5.5, $\mathscr{C}([0,T];\mathrm{H}_n)$ is a Borel subset of $\mathscr{C}([0,T];\mathrm{U}')\cap L^2(0,T;\mathrm{H}_{\text{loc}})$. Since $u_n\in\mathscr{C}([0,T];\mathrm{H}_n)$, $\mathbb{P}$-a.s., and $\tilde{u}_n$, u_n have the same laws on $\mathcal{Z}_T$, thus

$$\mathscr{L}(\tilde{u}_n)(\mathscr{C}([0,T];\mathrm{H}_n)=1,\quad n\in\mathbb{N}. \tag{6.5.33}$$

Since $\mathscr{C}([0,T];\mathrm{V})\cap\mathcal{Z}_T$ and $L^2(0,T;\mathrm{D(A)})\cap\mathcal{Z}_T$ are Borel subsets of $\mathcal{Z}_T$ (Lemma 6.5.5) and $\tilde{u}_n$ and u_n have the same laws on $\mathcal{Z}_T$; from (6.5.18) and (6.5.3), we have for $p\in[1,3]$

$$\sup_{n\in\mathbb{N}}\tilde{\mathbb{E}}\left(\sup_{0\le s\le T}\|\tilde{u}_n(s)\|^{2p}_{\mathrm{V}}\right)\le C_1(p), \tag{6.5.34}$$

$$\sup_{n\in\mathbb{N}}\tilde{\mathbb{E}}\left[\int_0^T|\tilde{u}_n(s)|^2_{\mathrm{D(A)}}\,ds\right]\le C_2(\|u_0\|^2_{\mathrm{V}}). \tag{6.5.35}$$

Since $\mathscr{C}([0,T];\mathrm{H}_n)$ is continuously embedded in $L^4(0,T;L^4)$ and $\tilde{u}_n$, u_n have same law μ on $\mathscr{C}([0,T];\mathrm{H}_n)$, we have

$$\begin{aligned}
\tilde{\mathbb{E}}\int_0^T\|\tilde{u}_n(s)\|^4_{L^4}\,ds &= \int_{\tilde{\Omega}}\left[\int_0^T\|\tilde{u}_n(s,\omega)\|^4_{L^4}\,ds\right]d\tilde{\mathbb{P}}(\omega)=\int_{L^4(0,T;L^4)}\left[\int_0^T\|y\|^4_{L^4}\,ds\right]d\mu(y)\\
&=\int_{\mathscr{C}([0,T];\mathrm{H}_n)}\left[\int_0^T\|y\|^4_{L^4}\,ds\right]d\mu(y)=\int_{L^4(0,T;L^4)}\left[\int_0^T\|y\|^4_{L^4}\,ds\right]d\mu(y)\\
&=\int_{\Omega}\left[\int_0^T\|u_n(s,\omega)\|^4_{L^4}\,ds\right]d\mathbb{P}(\omega)=\mathbb{E}\int_0^T\|u_n(s)\|^4_{L^4}\,ds.
\end{aligned}$$

Thus, by estimate (6.5.4) we infer

$$\sup_{n\in N}\tilde{\mathbb{E}}\int_0^T \|\tilde{u}_n(s)\|_{L^4}^4\,dt \le C_3(|u_0|_{\mathrm{H}}^2). \tag{6.5.36}$$

By inequality (6.5.34) we infer that the sequence $(\tilde{u}_n)$ contains a subsequence, still denoted by $(\tilde{u}_n)$ convergent weakly in the space $L^2([0,T]\times\tilde{\Omega};\mathrm{D(A)})$. Since by (6.5.32) $\tilde{\mathbb{P}}$-a.s $\tilde{u}_n \to \tilde{u}$ in $\mathcal{Z}_T$, we conclude that $\tilde{u}\in L^2([0,T]\times\tilde{\Omega};\mathrm{D(A)})$, i.e.

$$\tilde{\mathbb{E}}\left[\int_0^T |\tilde{u}(s)|_{\mathrm{D(A)}}^2\,ds\right] < \infty. \tag{6.5.37}$$

Similarly by inequality (6.5.34) for $p=1$ we can choose a subsequence of $(\tilde{u}_n)$ convergent weak star in the space $L^2(\tilde{\Omega};L^\infty(0,T;\mathrm{V}))$ and using convergences (6.5.32), we infer that

$$\tilde{\mathbb{E}}\left(\sup_{0\le s\le T}\|\tilde{u}(s)\|_{\mathrm{V}}^2\right) < \infty. \tag{6.5.38}$$

For each $n\ge 1$, let us consider a process $\tilde{M}_n$ with trajectories in $\mathscr{C}([0,T];\mathrm{H}_n)$, in particular in $\mathscr{C}([0,T];\mathrm{H})$, defined by

$$\begin{aligned}\tilde{M}_n(t) = \tilde{u}_n(t) - P_n\tilde{u}(0) + \int_0^t \mathrm{A}\tilde{u}_n(s)\,ds + \int_0^t B_n(\tilde{u}_n(s))\,ds + \int_0^t g_n(\tilde{u}_n(s))\,ds \\ - \int_0^t f_n(\tilde{u}_n(s))\,ds, \qquad t\in[0,T].\end{aligned} \tag{6.5.39}$$

Lemma 6.5.7. *$\tilde{M}_n$ is a square integrable martingale with respect to the filtration $\tilde{\mathbb{F}}_n = (\tilde{\mathscr{F}}_{n,t})$, where $\tilde{\mathscr{F}}_{n,t} = \sigma\{\tilde{u}_n(s), s\le t\}$, with the quadratic variation*

$$\langle\langle \tilde{M}_n\rangle\rangle_t = \int_0^t \|G_n(s,\tilde{u}_n(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2\,ds. \tag{6.5.40}$$

Proof. Indeed since $\tilde{u}_n$ and u_n have the same laws, for all $s,t\in[0,T], s\le t$, then for all bounded continuous functions h on $\mathscr{C}([0,s];\mathrm{V}_w)$, and all $\psi,\zeta\in\mathrm{V}_\gamma$ (for $\gamma>\frac{d}{2}$), we have

$$\tilde{\mathbb{E}}\left[\langle \tilde{M}_n(t)-\tilde{M}_n(s),\psi\rangle h(\tilde{u}_{n|[0,s]})\right] = 0 \tag{6.5.41}$$

and

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\Big(\langle\tilde{M}_n(t),\psi\rangle\langle\tilde{M}_n(t),\zeta\rangle - \langle\tilde{M}_n(s),\psi\rangle\langle\tilde{M}_n(s),\zeta\rangle \\ - \int_s^t \langle (G(\sigma,\tilde{u}_n(\sigma)))^* P_n\psi, (G(\sigma,\tilde{u}_n(\sigma)))^* P_n\zeta\rangle_{\ell^2}\,d\sigma\Big)\cdot h(\tilde{u}_{n|[0,s]})\Big] = 0.\end{aligned} \tag{6.5.42}$$

■

Lemma 6.5.8. *Let us define a process $\tilde{M}$ for $t\in[0,T]$ by*

$$\begin{aligned}\tilde{M}(t) = \tilde{u}(t) - \tilde{u}(0) + \int_0^t \mathrm{A}\tilde{u}(s)\,ds + \int_0^t B(\tilde{u}(s))\,ds + \int_0^t \Pi(g(|\tilde{u}(s)|^2)\tilde{u}(s))\,ds \\ - \int_0^t \Pi f(\tilde{u}(s))\,ds.\end{aligned} \tag{6.5.43}$$

Then $\tilde{M}$ is an H*−valued continuous process.*

Proof. Since $\tilde{u} \in \mathscr{C}([0,T];\mathrm{V})$ we just need to show that each of the remaining terms on the RHS of (6.5.43) are H−valued a.s. and well-defined.

Using the Cauchy-Schwarz inequality repeatedly and (6.5.37) we have the following inequalities

$$\tilde{\mathbb{E}}\int_0^T |\mathrm{A}\tilde{u}(s)|_{\mathrm{H}}\,ds \le T^{1/2}\left(\tilde{\mathbb{E}}\int_0^T |\tilde{u}(s)|^2_{\mathrm{D(A)}}\,ds\right)^{1/2} < \infty.$$

Since $H^{k,p} \hookrightarrow L^\infty$ for every $k > d/p$, hence there exists a $C > 0$ such that $\|u\|_{L^\infty} \le C\|u\|_{H^{2,2}}$ for every $u \in H^{2,2}$. Thus by the Cauchy-Schwarz inequality, (6.5.37) and (6.5.38) we obtain the following estimate

$$\begin{aligned}
&\tilde{\mathbb{E}}\int_0^T |B(\tilde{u}(s))|_{\mathrm{H}}\,ds \le T^{1/2}\tilde{\mathbb{E}}\left(\int_0^T |\tilde{u}(s)\cdot\nabla\tilde{u}(s)|^2_{L^2}\,ds\right)^{1/2} \\
&\le T^{1/2}\tilde{\mathbb{E}}\left(\int_0^T \|\tilde{u}(s)\|^2_{L^\infty}|\nabla\tilde{u}(s)|^2_{L^2}\,ds\right)^{1/2} \le T^{1/2}C\,\tilde{\mathbb{E}}\left(\int_0^T |\tilde{u}(s)|^2_{\mathrm{D(A)}}\|\tilde{u}(s)\|^2_{\mathrm{V}}\,ds\right)^{1/2} \\
&\le T^{1/2}C\left[\tilde{\mathbb{E}}\sup_{s\in[0,T]}\|\tilde{u}(s)\|^2_{\mathrm{V}}\right]^{1/2}\left[\tilde{\mathbb{E}}\int_0^T |\tilde{u}(s)|^2_{\mathrm{D(A)}}\,ds\right]^{1/2} < \infty.
\end{aligned}$$

We know that for $d = 3$, $H^{1,2} \hookrightarrow L^6$, thus using (6.2.14), convergences (6.5.32) and (6.5.34), we get

$$\begin{aligned}
&\tilde{\mathbb{E}}\int_0^T \left|\Pi g(|\tilde{u}(s)|^2)\tilde{u}(s)\right|_{\mathrm{H}}\,ds \le \tilde{\mathbb{E}}\int_0^T |g(|\tilde{u}(s)|^2)\tilde{u}(s)|_{L^2}\,ds \le \tilde{\mathbb{E}}\int_0^T \|\tilde{u}(s)\|^3_{L^6}\,ds \\
&\qquad \le C\,\tilde{\mathbb{E}}\int_0^T \|\tilde{u}(s)\|^3_{\mathrm{V}}\,ds \le C\left(\tilde{\mathbb{E}}\sup_{s\in[0,T]}\|\tilde{u}(s)\|^4_{\mathrm{V}}\right)^{3/4} T < \infty.
\end{aligned}$$

Using the assumptions (**A1**) and (6.5.38) we can show that

$$\begin{aligned}
&\tilde{\mathbb{E}}\int_0^T |\Pi f(\tilde{u}(s))|_{\mathrm{H}}\,ds \le \tilde{\mathbb{E}}\int_0^T |f(\tilde{u}(s))|_{L^2}\,ds \le T^{1/2}\,\tilde{\mathbb{E}}\left(\int_0^T |f(\tilde{u}(s))|^2_{L^2}\,ds\right)^{1/2} \\
&\qquad \le T^{1/2}C_f\left(\tilde{\mathbb{E}}\int_0^T |\tilde{u}(s)|^2_{\mathrm{H}}\,ds\right)^{1/2} < \infty.
\end{aligned}$$

This concludes the proof of the lemma. ■

Lemma 6.5.9. *Let us fix $\gamma > \frac{d}{2}$. If $u \in L^2(0,T;\mathrm{H}) \cap L^4(0,T;L^4)$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(0,T;\mathrm{H}) \cap L^4(0,T;L^4)$ such that $u_n \to u$ in $L^2(0,T;\mathrm{H}_{loc})$, then for all $r,t \in [0,T]$ and all $\psi \in \mathrm{V}_\gamma$:*

$$\lim_{n\to\infty}\int_r^t \langle g(|u_n(s)|^2)u_n(s),\psi\rangle\,ds = \int_r^t \langle g(|u(s)|^2)u(s),\psi\rangle\,ds. \tag{6.5.44}$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V_γ and V'_γ.

Proof. We will prove the lemma in two steps.

Step I

Let us fix $\gamma > \frac{d}{2}$ and $r,t \in [0,T]$. Assume first that $\psi \in \mathcal{V}$. Then there exists a $R > 0$ such that $\operatorname{supp}(\psi)$ is a compact subset of $\mathcal{O}_R$. There exists a constant $C \geq 0$ such that

$$|\langle g(|u|^2)u, \psi\rangle| = \left|\int_{\mathcal{O}_R} g(|u(x)|^2)u(x)\psi(x)\,dx\right| \leq |g(|u|^2)|_{L^2(\mathcal{O}_R)}|u|^2_{L^2(\mathcal{O}_R)}\|\psi\|_{L^\infty(\mathcal{O}_R)}$$

$$\leq \left||u|^2\right|_{L^2(\mathcal{O}_R)}|u|_{L^2(\mathcal{O}_R)}\|\psi\|_{L^\infty} \leq C\|u\|^2_{L^4}|u|_{L^2(\mathcal{O}_R)}\|\psi\|_{\mathrm{V}_\gamma}, \qquad u \in \mathrm{H}\cap L^4, \tag{6.5.45}$$

where we used (6.2.7) to establish the last inequality. We have

$$g(|u_n|^2)u_n - g(|u|^2)u = g(|u_n|^2)(u_n - u) + \left[g(|u_n|^2) - g(|u|^2)\right]u.$$

Thus using the estimate (6.5.45), the Hölder inequality, (6.2.15) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\int_r^t \langle g(|u_n(s)|^2)u_n(s), \psi\rangle\,ds - \int_r^t \langle g_n(|u(s)|^2)u(s), \psi\rangle\,ds\right| \\
&\leq \left|\int_r^t \langle g(|u_n(s)|^2)(u_n(s) - u(s)), \psi\rangle\,ds\right| + \left|\int_r^t \left\langle \left(g(|u_n(s)|^2) - g(|u(s)|^2)\right)u(s), \psi\right\rangle\,ds\right| \\
&\leq C\int_r^t \|u_n(s)\|^2_{L^4}|u_n(s) - u(s)|_{L^2(\mathcal{O}_R)}\|\psi\|_{\mathrm{V}_\gamma}\,ds \\
&\quad + 2\int_r^t \left|\langle |u_n(s) - u(s)|\,(|u_n(s)| + |u(s)|)\,u(s), \psi\rangle\right|\,ds \\
&\leq C\|\psi\|_{\mathrm{V}_\gamma}\int_r^t \|u_n(s)\|^2_{L^4}|u_n(s) - u(s)|_{L^2(\mathcal{O}_R)}\,ds \\
&\quad + 2C\,\|\psi\|_{\mathrm{V}_\gamma}\int_s^t |u_n(s) - u(s)|_{L^2(\mathcal{O}_R)}\left|u(s)[|u_n(s)| + |u(s)|]\right|_{L^2(\mathcal{O}_R)}\,ds.
\end{aligned}$$

Thus by the Hölder inequality, we get

$$\begin{aligned}
&\left|\int_r^t \langle g(|u_n(s)|^2)u_n(s), \psi\rangle\,ds - \int_r^t \langle g_n(|u(s)|^2)u(s), \psi\rangle\,ds\right| \\
&\leq C\|\psi\|_{\mathrm{V}_\gamma}\Bigg[|u_n|^2_{L^4(0,T;L^4)}|u_n - u|_{L^2(0,T;L^2(\mathcal{O}_R))} \\
&\qquad + 2|u_n - u|_{L^2(0,T;L^2(\mathcal{O}_R))}\left[\int_r^t \left|u(s)[|u_n(s)| + |u(s)|]\right|^2_{L^2(\mathcal{O}_R)}\,ds\right]^{1/2}\Bigg] \\
&\leq C\left[|u_n|^2_{L^4(0,T;L^4)} + 2|u|^2_{L^4(0,T;L^4)}\left(|u_n|^2_{L^4(0,T;L^4)} + |u|^2_{L^4(0,T;L^4)}\right)^{1/2}\right]|u_n - u|_{L^2(0,T;L^2(\mathcal{O}_R))}\|\psi\|_{\mathrm{V}_\gamma}.
\end{aligned}$$

Since $u_n \to u$ in $L^2(0,T;\mathrm{H}_{\mathrm{loc}})$ we infer that (6.5.44) holds for every $\psi \in \mathcal{V}$.

Step II

Let $\psi \in \mathrm{V}_\gamma$ and $\varepsilon > 0$. Then there exists a $\psi_\varepsilon \in \mathcal{V}$ such that $\|\psi_\varepsilon - \psi\|_{\mathrm{V}_\gamma} < \varepsilon$. Hence, we get

$$\begin{aligned}
&\left|\langle g(|u_n|^2)u_n - g(|u|^2)u, \psi\rangle\right| \\
&\leq \left|\langle g(|u_n|^2)u_n - g(|u|^2)u, \psi_\varepsilon\rangle\right| + \left|\langle g(|u_n|^2)u_n - g(|u|^2)u, \psi - \psi_\varepsilon\rangle\right| \\
&\leq \left|\langle g(|u_n|^2)u_n - g(|u|^2)u, \psi_\varepsilon\rangle\right| + \left[\|g(|u_n|^2)u_n\|_{\mathrm{V}'_\gamma} + \|g(|u|^2)u\|_{\mathrm{V}'_\gamma}\right]\|\psi - \psi_\varepsilon\|_{\mathrm{V}_\gamma}.
\end{aligned} \tag{6.5.46}$$

Since $\mathcal{V}$ is dense in V_γ, (6.5.45) holds for all $\psi \in \mathrm{V}_\gamma$. In particular, there exists a constant $C > 0$ such that

$$\|g(|u|^2)u\|_{\mathrm{V}'_\gamma} \le C\|u\|^2_{L^4}|u|_{\mathrm{H}}, \qquad u \in \mathrm{H}\cap L^4. \tag{6.5.47}$$

Using (6.5.46), (6.5.47) and the Cauchy-Schwarz inequality we have following inequalities

$$\begin{aligned}
&\left|\int_r^t \langle g(|u_n(s)|^2)u_n(s) - g(|u(s)|^2)u(s), \psi\rangle\, ds\right| \\
&\le \varepsilon C \int_r^t \left(\|u_n(s)\|^2_{L^4}|u_n(s)|_{\mathrm{H}} + \|u(s)\|^2_{L^4}|u(s)|_{\mathrm{H}}\right) ds \\
&\quad + \left|\int_r^t \langle g(|u_n(s)|^2)u_n(s) - g(|u(s)|^2)u(s), \psi_\varepsilon\rangle\, ds\right| \\
&\le \varepsilon C \left[\|u_n\|^2_{L^4(0,T;L^4)}\|u_n\|_{L^2(0,T;\mathrm{H})} + \|u\|^2_{L^4(0,T;L^4)}\|u\|_{L^2(0,T;\mathrm{H})}\right] \\
&\quad + \left|\int_r^t \langle g(|u_n(s)|^2)u_n(s) - g(|u(s)|^2)u(s), \psi_\varepsilon\rangle\, ds\right|.
\end{aligned}$$

Hence by **Step I** and the assumptions on u, u_n there exists a $M > 0$ such that

$$\limsup_{n\to\infty}\left|\int_r^t \langle g(|u_n(s)|^2)u_n(s) - g(|u(s)|^2)u(s), \psi\rangle\, ds\right| \le M\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we conclude the proof. ■

Corollary 6.5.10. *Let us fix $\gamma > \frac{d}{2}$. If $u \in L^2(0,T;\mathrm{H})\cap L^4(0,T;L^4)$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(0,T;\mathrm{H})\cap L^4(0,T;L^4)$ such that $u_n \to u$ in $L^2(0,T;\mathrm{H}_{loc})$, then for all $r,t \in [0,T]$ and all $\psi \in \mathrm{V}_\gamma$:*

$$\lim_{n\to\infty}\int_r^t \langle g(|u_n(s)|^2)u_n(s), P_n\psi\rangle\, ds = \int_r^t \langle g(|u(s)|^2)u(s), \psi\rangle\, ds. \tag{6.5.48}$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V_γ and V'_γ.

Proof. Let us fix $\gamma > \frac{d}{2}$ and take $r,t \in [0,T]$ and $\psi \in \mathrm{V}_\gamma$. We have

$$\begin{aligned}
\int_r^t \langle g(|u_n(s)|^2)u_n(s), P_n\psi\rangle\, ds &= \int_r^t \langle g(|u_n(s)|^2)u_n(s), P_n\psi - \psi\rangle\, ds + \int_r^t \langle g(|u_n(s)|^2)u_n(s), \psi\rangle\, ds \\
&:= I_1(n) + I_2(n).
\end{aligned}$$

We will consider each of these integrals individually. Using the estimate from (6.5.47), we have

$$\begin{aligned}
|I_1(n)| &\le \int_r^t \|g(|u_n(s)|^2)u_n(s)\|_{\mathrm{V}'_\gamma}\|P_n\psi - \psi\|_{\mathrm{V}_\gamma}\, ds \\
&\le \|P_n\psi - \psi\|_{\mathrm{V}_\gamma}\int_r^t \|u_n(s)\|^2_{L^4}|u_n(s)|_{\mathrm{H}}\, ds.
\end{aligned}$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;\mathrm{H})\cap L^4(0,T;L^4)$ and $P_n\psi \to \psi$ in V_γ, we infer

$$\lim_{n\to\infty} I_1(n) = 0.$$

By Lemma 6.5.9, we conclude

$$\lim_{n\to\infty} I_2(n) = \int_r^t \langle g(|u(s)|^2)u(s), \psi\rangle\, ds.$$

■

Lemma 6.5.11. *For all $s,t \in [0,T]$ such that $s \le t$ and $\gamma > \frac{d}{2}$:*

(a) $\lim_{n\to\infty}\langle \tilde{u}_n(t), P_n\psi\rangle = \langle \tilde{u}(t), \psi\rangle$, *$\tilde{\mathbb{P}}$-a.s.,* $\psi \in \mathrm{V}$,

(b) $\lim_{n\to\infty}\int_s^t \langle \mathrm{A}\tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, ds = \int_s^t \langle \mathrm{A}\tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma$, *$\tilde{\mathbb{P}}$-a.s.,* $\psi \in \mathrm{H}$,

(c) $\lim_{n\to\infty}\int_s^t \langle B(\tilde{u}_n(\sigma)), P_n\psi\rangle\, d\sigma = \int_s^t \langle B(\tilde{u}(\sigma)), \psi\rangle\, d\sigma$, *$\tilde{\mathbb{P}}$-a.s.,* $\psi \in \mathrm{V}_\gamma$,

(d) $\lim_{n\to\infty}\int_s^t \langle g(|\tilde{u}_n(\sigma)|^2)\tilde{u}_n(\sigma), P_n\psi\rangle\, d\sigma = \int_s^t \langle g(|\tilde{u}(\sigma)|^2)\tilde{u}(\sigma), \psi\rangle\, d\sigma$, *$\tilde{\mathbb{P}}$-a.s.,* $\psi \in \mathrm{V}_\gamma$,

(e) $\lim_{n\to\infty}\int_s^t \langle f(\tilde{u}_n(\sigma)), P_n\psi\rangle\, d\sigma = \int_s^t \langle f(\tilde{u}(\sigma)), \psi\rangle\, d\sigma$, *$\tilde{\mathbb{P}}$-a.s.,* $\psi \in \mathrm{V}_\gamma$,

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between appropriate spaces.

Proof. Let us fix $s,t \in [0,T]$, $s \le t$ and $\gamma > \frac{d}{2}$. By (6.5.32) we know that

(6.5.49) $$\tilde{u}_n \to \tilde{u} \textit{ in } \mathscr{C}([0,T];\mathrm{U}') \cap L^2_w(0,T;\mathrm{D(A)}) \cap L^2(0,T;\mathrm{H_{loc}}) \cap \mathscr{C}([0,T];V_w), \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Let $\psi \in \mathrm{V}$. Since $\tilde{u}_n \to \tilde{u}$ in $\mathscr{C}([0,T];\mathrm{V}_w)$ $\tilde{\mathbb{P}}$-a.s., from (6.5.34) $\tilde{u}_n$ is uniformly bounded in $\mathscr{C}([0,T];\mathrm{V}_w)$ and $P_n\psi \to \psi$ in V, thus

$$\lim_{n\to\infty}\langle \tilde{u}_n(t), P_n\psi\rangle - \langle \tilde{u}(t), \psi\rangle = \lim_{n\to\infty}\langle \tilde{u}_n(t) - \tilde{u}(t), \psi\rangle + \lim_{n\to\infty}\langle \tilde{u}_n(t), P_n\psi - \psi\rangle = 0 \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Hence, we infer that assertion (a) holds.

Let $\psi \in \mathrm{H}$. Since by (6.5.49) $\tilde{u}_n \to \tilde{u}$ in $L^2_w(0,T;\mathrm{D(A)})$ $\tilde{\mathbb{P}}$-a.s., from (6.5.35) $\tilde{u}_n$ is uniformly bounded in $L^2_w(0,T;\mathrm{D(A)})$ and $P_n\psi \to \psi$ in H. Thus, we have $\tilde{\mathbb{P}}$−a.s.,

$$\begin{aligned}
&\lim_{n\to\infty}\int_s^t \langle \mathrm{A}\tilde{u}_n(\sigma), P_n\psi\rangle_{\mathrm{H}}\, d\sigma - \int_s^t \langle \mathrm{A}\tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma \\
&= \lim_{n\to\infty}\int_s^t \langle \mathrm{A}\tilde{u}_n(\sigma) - \mathrm{A}\tilde{u}(\sigma), \psi\rangle_{\mathrm{H}}\, d\sigma + \lim_{n\to\infty}\int_s^t \langle \mathrm{A}\tilde{u}_n(\sigma), P_n\psi - \psi\rangle_{\mathrm{H}}\, d\sigma \to 0.
\end{aligned}$$

Hence, we have shown that assertion (b) is true.

For every $\psi \in \mathrm{V}_\gamma$ assertion (c) follows directly from [23, Lemma B.1] and a modification of Corollary 6.5.10.

By (6.5.49) $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;\mathrm{H_{loc}})$. From Lemma 6.5.1, convergences (6.5.49) and estimate (6.5.36) the sequence $(\tilde{u}_n)$ is bounded in $L^2(0,T;\mathrm{H}) \cap L^4(0,T;L^4)$ and $\tilde{u} \in L^2(0,T;\mathrm{H}) \cap L^4(0,T;L^4)$. Thus, using Corollary 6.5.10 we infer that (d) holds for every $\psi \in \mathrm{V}_\gamma$.

Now we are left to deal with (e). Let $\psi \in \mathrm{V}_\gamma$,

$$\int_s^t \langle f(\tilde{u}_n(\sigma)), P_n\psi\rangle\, ds - \int_0^t \langle f(\tilde{u}(\sigma), \psi\rangle\, d\sigma$$
$$= \int_0^t \langle f(\tilde{u}_n(\sigma)) - f(\tilde{u}(\sigma)), \psi\rangle\, d\sigma + \int_0^t \langle f(\tilde{u}_n(\sigma)), P_n\psi - \psi\rangle\, d\sigma$$

Since $\mathrm{V}_\gamma \hookrightarrow \mathrm{H}$, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\int_s^t \langle f(\tilde{u}_n(\sigma)), P_n\psi\rangle\, ds - \int_0^t \langle f(\tilde{u}(\sigma), \psi\rangle\, d\sigma \\
&\leq \int_0^t \langle f(\tilde{u}_n(\sigma)) - f(\tilde{u}(\sigma)), \psi\rangle\, d\sigma + \int_s^t \|f(u_n(s))\|_{\mathrm{V}'_\gamma} \|P_n\psi - \psi\|_{\mathrm{V}_\gamma}\, d\sigma \\
&\leq \int_0^t \langle f(\tilde{u}_n(\sigma)) - f(\tilde{u}(\sigma)), \psi\rangle\, d\sigma + C_f \|P_n\psi - \psi\|_{\mathrm{V}_\gamma} \int_s^t \|\tilde{u}_n(\sigma)\|_{\mathrm{H}}\, d\sigma \\
&:= I_1(n) + I_2(n).
\end{aligned}$$

Since $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;\mathrm{H}_{\mathrm{loc}})$ and $\tilde{u}_n$ is a bounded sequence in $L^2(0,T;\mathrm{H})$, $I_1(n)$ can be shown to converge to zero as $n \to \infty$ following the methodology of Lemma 6.5.9 and Corollary 6.5.10. Since $P_n\psi \to \psi$ in V_γ, $I_2(n) \to 0$ as $n \to \infty$. This completes the proof of Lemma 6.5.11. ∎

The proofs of Lemmas 6.5.12, 6.5.15 and 6.5.17 follow the similar methodology as that of Lemmas 5.6 - 5.8 [23] and Lemmas 5.5.9 - 5.5.12.

Let h be the bounded continuous function on $\mathscr{C}([0,T];\mathrm{U}')$ and $\tilde{\mathbb{F}} = (\tilde{\mathscr{F}}_t) = \sigma\{\tilde{u}(s), s \leq t\}$ be the filtration of sigma fields generated by the process $\tilde{u}$.

Lemma 6.5.12. *For all $s,t \in [0,T]$, such that $s \leq t$ and all $\psi \in \mathrm{V}_\gamma$:*

(6.5.50) $$\lim_{n\to\infty} \tilde{\mathbb{E}}\left[\langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi\rangle h(\tilde{u}_{n|[0,s]})\right] = \tilde{\mathbb{E}}\left[\langle \tilde{M}(t) - \tilde{M}(s), \psi\rangle h(\tilde{u}_{|[0,s]})\right].$$

Proof. Let us fix $s,t \in [0,T], s \leq t$ and $\psi \in \mathrm{V}_\gamma$. By identity (6.5.39) we have

$$\begin{aligned}
\langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi\rangle &= \langle \tilde{u}_n(t) - \tilde{u}_n(s), P_n\psi\rangle + \int_s^t \langle \mathrm{A}\tilde{u}_n(\sigma), P_n\psi\rangle\, d\sigma \\
&+ \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n\psi\rangle\, d\sigma + \int_s^t \langle g(|\tilde{u}_n(\sigma)|^2)\tilde{u}_n(\sigma), P_n\psi\rangle\, d\sigma - \int_s^t \langle f(\tilde{u}_n(\sigma)), P_n\psi\rangle\, d\sigma.
\end{aligned}$$

By Lemma 6.5.11, we infer that

(6.5.51) $$\lim_{n\to\infty} \langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi\rangle = \langle \tilde{M}(t) - \tilde{M}(s), \psi\rangle, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

In order to prove (6.5.50) we first observe that since $\tilde{u}_n \to \tilde{u}$ in $\mathscr{Z}_T$, in particular in $\mathscr{C}([0,T];\mathrm{U}')$ and h is a bounded continuous function on $\mathscr{C}([0,T];\mathrm{U}')$, we get

(6.5.52) $$\lim_{n\to\infty} h(\tilde{u}_{n|[0,s]}) = h(\tilde{u}_{|[0,s]}),$$

and

$$\sup_{n\in\mathbb{N}} \|h(\tilde{u}_{n|[0,s]})\|_{L^\infty} < \infty. \tag{6.5.53}$$

Let us define a sequence of $\mathbb{R}-$valued random variables :

$$f_n(\omega) := \left[\langle \tilde{M}_n(t,\omega),\psi\rangle - \langle \tilde{M}_n(s,\omega),\psi\rangle\right] h(\tilde{u}_{n|[0,s]}), \quad \omega \in \tilde{\Omega}.$$

We will prove that the functions $\{f_n\}_{n\in\mathbb{N}}$ are uniformly integrable in order to apply the Vitali theorem. We claim that

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}[|f_n|^2] < \infty. \tag{6.5.54}$$

Since $\mathrm{H} \hookrightarrow \mathrm{V}'_\gamma$ then by the Cauchy-Schwarz inequality, for each $n \in \mathbb{N}$ we have

$$\tilde{\mathbb{E}}[|f_n|^2] \le 2c\|h\circ \tilde{u}_n\|^2_{L^\infty} |\psi|^2_{\mathrm{V}_\gamma} \tilde{\mathbb{E}}\left[|\tilde{M}_n(t)|^2_H + |\tilde{M}_n(s)|^2_H\right]. \tag{6.5.55}$$

Since $\tilde{M}_n$ is a continuous martingale with quadratic variation defined in (6.5.40), by the Burkholder-Davis-Gundy inequality we obtain

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]} |\tilde{M}_n(t)|^2_{\mathrm{H}}\right] \le c\tilde{\mathbb{E}}\left[\int_0^T \|G_n(\sigma,\tilde{u}_n(\sigma))\|^2_{\mathscr{T}_2(\ell^2;\mathrm{H})}\, d\sigma\right] \tag{6.5.56}$$

Since $P_n : \mathrm{H} \to \mathrm{H}$ is a contraction and by Lemma 6.2.1, (6.5.18) for $p = 1$, we have

$$\begin{aligned} \tilde{\mathbb{E}}\left[\int_0^T \|G_n(\sigma,\tilde{u}_n(\sigma))\|^2_{\mathscr{T}_2(\ell^2;\mathrm{H})}\, d\sigma\right] &\le \tilde{\mathbb{E}}\left[\int_0^T \|G(\sigma,\tilde{u}_n(\sigma))\|^2_{\mathscr{T}_2(\ell^2;\mathrm{H})}\, d\sigma\right] \\ \le \tilde{\mathbb{E}}\left[\int_0^T \frac{1}{4}|\nabla \tilde{u}_n(\sigma)|^2_{L^2}\, d\sigma\right] &\le \tilde{\mathbb{E}}\left[\sup_{\sigma\in[0,T]} \|\tilde{u}_n(\sigma)\|^2_{\mathrm{V}}\right] T < \infty. \end{aligned} \tag{6.5.57}$$

Then by (6.5.55) and (6.5.57) we see that (6.5.54) holds. Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable and by (6.5.51) it is $\tilde{\mathbb{P}}-$a.s. point-wise convergent, then application of the Vitali Theorem completes the proof of the lemma. ■

Remark 6.5.13. Using the Burkholder-Davis-Gundy inequality we have proved a stronger claim (6.5.56) than what we needed.

From Lemma 6.5.7 and Lemma 6.5.12 we have the following corollary.

Corollary 6.5.14. *For all $s,t \in [0,T]$ such that $s \le t$:*

$$\mathbb{E}\left(\tilde{M}(t) - \tilde{M}(s)\middle|\tilde{\mathscr{F}}_t\right) = 0.$$

Lemma 6.5.15. *For all $s,t \in [0,T]$ such that $s \le t$ and all $\psi,\zeta \in \mathrm{V}_\gamma$:*

$$\begin{aligned} \lim_{n\to\infty} \tilde{\mathbb{E}}&\Big[\Big(\langle \tilde{M}_n(t),\psi\rangle\langle \tilde{M}_n(t),\zeta\rangle - \langle \tilde{M}_n(s),\psi\rangle\langle \tilde{M}_n(s),\zeta\rangle\Big) h(\tilde{u}_{n|[0,s]})\Big] \\ &= \tilde{\mathbb{E}}\Big[\Big(\langle \tilde{M}(t),\psi\rangle\langle \tilde{M}(t),\zeta\rangle - \langle \tilde{M}(s),\psi\rangle\langle \tilde{M}(s),\zeta\rangle\Big) h(\tilde{u}_{|[0,s]})\Big], \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the appropriate dual pairing.

Proof. Let us fix $s,t \in [0,T]$ such that $s \le t$ and $\psi, \zeta \in \mathrm{V}_\gamma$ and define $\mathbb{R}-$valued random variables f_n and f by

$$f_n(\omega) := \Big(\langle \tilde{M}_n(t,\omega),\psi\rangle\langle \tilde{M}_n(t,\omega),\zeta\rangle - \langle \tilde{M}_n(s,\omega),\psi\rangle\langle \tilde{M}_n(s,\omega),\zeta\rangle\Big) h(\tilde{u}_{n|[0,s]}(\omega)),$$
$$f(\omega) := \Big(\langle \tilde{M}(t,\omega),\psi\rangle\langle \tilde{M}(t,\omega),\zeta\rangle - \langle \tilde{M}(s,\omega),\psi\rangle\langle \tilde{M}(s,\omega),\zeta\rangle\Big) h(\tilde{u}_{|[0,s]}(\omega)), \quad \omega \in \tilde{\Omega}.$$

By Lemma 6.5.11 or more precisely by (6.5.51) and (6.5.52) we infer that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$, for $\tilde{\mathbb{P}}$ almost all $\omega \in \tilde{\Omega}$. We will prove that the functions $\{f_n\}_{n\in\mathbb{N}}$ are uniformly integrable. We claim that for some $r > 1$,

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\left[|f_n|^r\right] < \infty. \tag{6.5.58}$$

For each $n \in \mathbb{N}$ as before we have

$$\tilde{\mathbb{E}}\left[|f_n|^r\right] \le C\|h\circ\tilde{u}_n\|^r_{L^\infty}\|\psi\|^r_{\mathrm{V}_\gamma}\|\zeta\|^r_{\mathrm{V}_\gamma}\tilde{\mathbb{E}}\left[|\tilde{M}_n(t)|^{2r}_{\mathrm{H}} + |\tilde{M}_n(s)|^{2r}_{\mathrm{H}}\right]. \tag{6.5.59}$$

Since $\tilde{M}_n$ is a continuous martingale with quadratic variation defined in (6.5.40), by the Burkholder-Davis-Gundy inequality we obtain

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}|\tilde{M}_n(t)|^{2r}_{\mathrm{H}}\right] \le c\tilde{\mathbb{E}}\left[\int_0^T \|G_n(\sigma,\tilde{u}_n(\sigma))\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})}\,d\sigma\right]^r. \tag{6.5.60}$$

Since $P_n : \mathrm{H} \to \mathrm{H}$ is a contraction and by Lemma 6.2.1 we have

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\int_0^T \|G_n(\sigma,\tilde{u}_n(\sigma))\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})}\,d\sigma\right]^r &\le \tilde{\mathbb{E}}\left[\int_0^T \|G(\sigma,\tilde{u}_n(\sigma))\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})}\,d\sigma\right]^r \\
&\le \tilde{\mathbb{E}}\left[\int_0^T \frac{1}{4}|\nabla\tilde{u}_n(\sigma)|^2_{L^2}\,d\sigma\right]^r \le C_r T^{r-1}\tilde{\mathbb{E}}\left[\int_0^T \|\tilde{u}_n(\sigma)\|^{2r}_{\mathrm{V}}\,d\sigma\right] \\
&\le C_r\tilde{\mathbb{E}}\left[\sup_{\sigma\in[0,T]}\|\tilde{u}_n(\sigma)\|^{2r}_{\mathrm{V}}\right]T^r.
\end{aligned} \tag{6.5.61}$$

Thus, if $r \in [1,3]$ then, by (6.5.18), (6.5.53) and (6.5.59) - (6.5.61) we infer that (6.5.58) holds. Hence by application of the Vitali theorem

$$\lim_{n\to\infty}\tilde{\mathbb{E}}[f_n] = \tilde{\mathbb{E}}[f]. \tag{6.5.62}$$

■

We will be using the following notations in the following lemmas. $\mathrm{V}'(\mathcal{O}_R)$ is the dual space to $\mathrm{V}(\mathcal{O}_R)$, where

$$\mathrm{V}(\mathcal{O}_R) := \text{the closure of } \mathcal{V}(\mathcal{O}_R) \text{ in } H^1(\mathcal{O}_R,\mathbb{R}^3),$$

where $\mathcal{V}(\mathcal{O}_R)$ denotes the space of all divergence free vector fields of class $\mathcal{C}^\infty$ with compact supports contained in $\mathcal{O}_R$. We recall that $\mathrm{H}_{\mathcal{O}_R}$ is the space of restrictions to the subset $\mathcal{O}_R$ of elements of the space H i.e.,

$$\mathrm{H}_{\mathcal{O}_R} := \{u_{|\mathcal{O}_R} : u \in \mathrm{H}\},$$

with the scalar product defined by

$$\langle u, \mathrm{v}\rangle_{\mathrm{H}_{\mathcal{O}_R}} := \int_{\mathcal{O}_R} u(x)\mathrm{v}(x)\,dx, \quad u, \mathrm{v} \in \mathrm{H}_{\mathcal{O}_R}.$$

Lemma 6.5.16. *The map* $G\colon \mathrm{H}_{\mathcal{O}_R} \to \mathcal{T}_2(\ell^2; \mathrm{V}'(\mathcal{O}_R))$ *given by* (6.2.12) *is well defined and there exists some constant* $C_R > 0$ *such that*

$$\|G(u)\|_{\mathcal{T}_2(\ell^2;\mathrm{V}'(\mathcal{O}_R))} \le C_R \|u\|_{\mathrm{H}_{\mathcal{O}_R}}, \quad u \in \mathrm{H}. \tag{6.5.63}$$

Moreover, for every $\psi \in \mathcal{V}$ *the mapping* $\mathrm{H} \ni u \mapsto \langle G(u), \psi\rangle \in \ell^2$ *is continuous, if in the space* H *we consider the Fréchet topology inherited from the space* $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$.

Proof. Let $\sigma = (\sigma^1, \dots, \sigma^d)\colon \overline{\mathcal{O}} \to \mathbb{R}^d$ and fix $R > 0$. Let $u \in \mathcal{V}(\mathcal{O}_R)$. Then

$$\sum_{j=1}^d \frac{\partial}{\partial x_j}(\sigma^j u) = \sum_{j=1}^d \Big(\frac{\partial \sigma^j}{\partial x_j} u + \sigma^j \frac{\partial u}{\partial x_j}\Big) = (\mathrm{div}\sigma)u + \sum_{j=1}^d \sigma^j \frac{\partial u}{\partial x_j}. \tag{6.5.64}$$

Let $\mathrm{v} \in \mathcal{V}(\mathcal{O}_R)$. Since v on the boundary $\partial\mathcal{O}_R$ is equal to zero, thus using the integration by parts formula, we obtain for $\mathrm{v} \in \mathcal{V}(\mathcal{O}_R)$

$$\begin{aligned}\int_{\mathcal{O}} \Big(\sum_{j=1}^d \sigma^j \frac{\partial u}{\partial x_j}\Big)\mathrm{v}\,dx &= \sum_{j=1}^d \int_{\mathcal{O}} \frac{\partial}{\partial x_j}(\sigma^j u)\mathrm{v}\,dx - \int_{\mathcal{O}} (\mathrm{div}\sigma)u\,\mathrm{v}\,dx \\ &= -\sum_{j=1}^d \int_{\mathcal{O}} (\sigma^j u)\frac{\partial \mathrm{v}}{\partial x_j}\,dx - \int_{\mathcal{O}} (\mathrm{div}\sigma)u\,\mathrm{v}\,dx.\end{aligned}$$

Using the Hölder inequality, we obtain

$$\Big|\int_{\mathcal{O}_R} \Big(\sum_{j=1}^d \sigma^j \frac{\partial u}{\partial x_j}\Big)\mathrm{v}\,dx\Big| \le \|\sigma\|_{L^\infty} |u|_{\mathrm{H}_{\mathcal{O}_R}} \|\mathrm{v}\|_{\mathrm{V}(\mathcal{O}_R)} + \|\mathrm{div}\sigma\|_{L^\infty} |u|_{\mathrm{H}_{\mathcal{O}_R}} \|\mathrm{v}\|_{\mathrm{V}(\mathcal{O}_R)} \tag{6.5.65}$$

Therefore, if we define a linear functional $\hat{B}_R$ by

$$\hat{B}_R \mathrm{v} := \int_{\mathcal{O}_R} \Big(\sum_{j=1}^d \sigma^j \frac{\partial u}{\partial x_j}\Big)\mathrm{v}\,dx, \qquad \mathrm{v} \in \mathcal{V}(\mathcal{O}_R),$$

we infer that it is bounded in the norm of the space $\mathrm{V}(\mathcal{O}_R)$. Thus it can be uniquely extended to a linear bounded functional (denoted also by $\hat{B}_R$) on $\mathrm{V}(\mathcal{O}_R)$. Moreover, by estimate (6.5.65) we have the following inequality

$$\|\hat{B}_R\|_{\mathrm{V}'(\mathcal{O}_R)} \le \big(\|\sigma\|_{L^\infty} + \|\mathrm{div}\sigma\|_{L^\infty}\big)|u|_{\mathrm{H}_{\mathcal{O}_R}}$$

or equivalently

$$\|(\sigma \cdot \nabla)u\|_{\mathrm{V}'(\mathcal{O}_R)} \le \big(\|\sigma\|_{L^\infty} + \|\mathrm{div}\sigma\|_{L^\infty}\big) \cdot |u|_{\mathrm{H}_{\mathcal{O}_R}}. \tag{6.5.66}$$

Since by equality (6.2.12), $G(u)(e_j) = \Pi\big[(\sigma_j \cdot \nabla)u\big]$, for some orthonormal basis $\{e_j\}_{j=1}^\infty$ of ℓ^2, we get by estimate (6.5.66)

$$\|G(u)\|_{\mathcal{T}_2(\ell^2;\mathrm{V}'(\mathcal{O}_R))} = \Bigg[\sum_{j=1}^\infty \|G(u)(e_j)\|^2_{\mathrm{V}'(\mathcal{O}_R)}\Bigg]^{1/2} \le \big(\|\sigma\|_{\ell^2} + \|\mathrm{div}\sigma\|_{\ell^2}\big) \cdot |u|_{\mathrm{H}_{\mathcal{O}_R}}.$$

Therefore, using the assumption (**A2**), $G(u) \in \mathcal{T}_2(\ell^2, \mathrm{V}'(\mathcal{O}_R))$ and

$$\|G(u)\|_{\mathcal{T}_2(\ell^2,\mathrm{V}'(\mathcal{O}_R))} \le C_R \cdot |u|_{\mathrm{H}_{\mathcal{O}_R}}.$$

By estimate (6.5.63) and the continuity of the embedding $\mathcal{T}_2(\ell^2, \mathrm{V}'(\mathcal{O}_R)) \hookrightarrow \mathcal{L}(\ell^2, V'(\mathcal{O}_R))$, we obtain

$$\|G(u)y\|_{\mathrm{V}'(\mathcal{O}_R)} \le C(R)|u|_{\mathrm{H}_{\mathcal{O}_R}} \|y\|_{\ell^2}, \qquad u \in \mathrm{H}, \quad y \in \ell^2$$

for some constant $C(R) > 0$. Thus, for any $\psi \in \mathrm{V}(\mathcal{O}_R)$

$$|(G(u)y)\psi| \le C(R)|u|_{\mathrm{H}_{\mathcal{O}_R}} \|y\|_{\ell^2} \|\psi\|_{\mathrm{V}(\mathcal{O}_R)}, \qquad u \in \mathrm{H}, \quad y \in \ell^2. \tag{6.5.67}$$

Now we identify ${}_{\mathrm{V}'}\langle G(\cdot), \psi\rangle_{\mathrm{V}}$ with the mapping $\psi^{**}G \colon \mathrm{H} \to (\ell^2)'$ defined by

$$(\psi^{**}G(u))y := (G(u)y)\psi \in \mathbb{R}, \qquad u \in \mathrm{H}, \quad y \in \ell^2.$$

Thus, from the inequality (6.5.67), we infer that

$$\|\psi^{**}G(u)\|_{\ell^2} \le C(R)\|\psi\|_{\mathrm{V}}|u|_{\mathrm{H}_{\mathcal{O}_R}}. \tag{6.5.68}$$

Therefore, if we fix $\psi \in \mathcal{V}$ then, there exists $R_0 > 0$ such that $\operatorname{supp}\psi$ is a compact subset of $\mathcal{O}_{R_0}$. Since G is linear, estimate (6.5.68) with $R := R_0$ yields that the mapping

$$L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3) \supset \mathrm{H} \ni u \mapsto \psi^{**}G(u) \in \ell^2$$

is continuous in the Fréchet topology inherited on the space H from the space $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, concluding the proof of the lemma. ■

Lemma 6.5.17 (Convergence of quadratic variations)**.** *For any $s, t \in [0.T]$ and $\psi, \zeta \in \mathrm{V}_\gamma$, we have*

$$\begin{aligned}\lim_{n\to\infty} \tilde{\mathbb{E}}\left[\left(\int_s^t \langle (G(\sigma, \tilde{u}_n(\sigma)))^* P_n\psi, (G(\sigma, \tilde{u}_n(\sigma)))^* P_n\zeta\rangle_{\ell^2}\, d\sigma\right) \cdot h(\tilde{u}_{n|[0,s]})\right] \\ = \tilde{\mathbb{E}}\left[\left(\int_s^t \langle (G(\sigma, \tilde{u}(\sigma)))^*\, \psi, (G(\sigma, \tilde{u}(\sigma))^*\, \zeta\rangle_{\ell^2}\, d\sigma\right) \cdot h(\tilde{u}_{|[0,s]})\right],\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\ell^2}$ is the inner product in ℓ^2.

Proof. Let us fix $\psi, \zeta \in \mathrm{V}_\gamma$ and define a sequence of random variables by

$$f_n(\omega) := \left(\int_s^t \langle (G(\sigma, \tilde{u}_n(\sigma, \omega)))^* P_n\psi, (G(\sigma, \tilde{u}_n(\sigma, \omega)))^* P_n\zeta\rangle_{\ell^2}\, d\sigma\right) \cdot h(\tilde{u}_{n|[0,s]}), \quad \omega \in \tilde{\Omega}.$$

We will prove that the functions are uniformly integrable and convergent $\tilde{\mathbb{P}}$−a.s. We will prove that for some $r > 1$,

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}[|f_n|^r] < \infty. \tag{6.5.69}$$

For some $c > 0$ we have the following inequalities

$$|(G(\sigma,\tilde{u}_n(\sigma,\omega)))^* P_n\psi|_{\ell^2} \le \|G(\sigma,\tilde{u}_n(\sigma,\omega))\|_{\mathscr{L}(\ell^2;\mathrm{H})}|P_n\psi|_{\mathrm{H}} \le \frac{1}{2}|\nabla\tilde{u}_n(\sigma,\omega)|_{L^2}|\psi|_{\mathrm{H}},$$

and thus

$$\begin{aligned}\mathbb{E}|f_n|^r &= \mathbb{E}\left|\left(\int_s^t \langle (G(\sigma,\tilde{u}_n(\sigma)))^* P_n\psi,(G(\sigma,\tilde{u}_n(\sigma)))^* P_n\zeta\rangle_{\ell^2} d\sigma\right)\cdot h(\tilde{u}_{n|[0,s]})\right|^r \\ &\le \|h\circ\tilde{u}_n\|_{L^\infty}^r \mathbb{E}\left(\int_s^t |(G(\sigma,\tilde{u}_n(\sigma,\omega)))^* P_n\psi|_{\ell^2}|(G(\sigma,\tilde{u}_n(\sigma,\omega)))^* P_n\zeta|_{\ell^2}\, d\sigma\right)^r \\ &\le c^4\|h\circ\tilde{u}_n\|_{L^\infty}^r|\psi|_{\mathrm{H}}^r|\zeta|_{\mathrm{H}}^r\mathbb{E}\left(\int_s^t \|\tilde{u}_n(\sigma,\omega)\|_{\mathrm{V}}^2\, d\sigma\right)^r.\end{aligned}$$

Using the Hölder inequality, we get

$$\mathbb{E}\left(\int_s^t \|\tilde{u}_n(\sigma,\omega)\|_{\mathrm{V}}^2 d\sigma\right)^r \le (t-s)^{r-1}\mathbb{E}\int_s^t \|\tilde{u}_n(\sigma,\omega)\|_{\mathrm{V}}^{2r}\, d\sigma \le C\mathbb{E}\left(\sup_{\sigma\in[0,T]}\|\tilde{u}_n(\sigma,\omega)\|_{\mathrm{V}}^{2r}\right)$$

for some $C > 0$. Thus

$$\mathbb{E}|f_n|^r \le \tilde{C}\mathbb{E}\left(\sup_{\sigma\in[0,T]}\|\tilde{u}_n(\sigma,\omega)\|_{\mathrm{V}}^{2r}\right)$$

for some $\tilde{C} > 0$. Hence by (6.5.18) for $r \in (1,3]$

$$\sup_{n\ge1}\tilde{\mathbb{E}}|f_n|^r \le \tilde{C}\sup_{n\ge1}\tilde{\mathbb{E}}\left[\sup_{\sigma\in[0,T]}\|\tilde{u}_n(\sigma,\omega)\|_{\mathrm{V}}^{2r}\right] \le \tilde{C}C_1(r) < \infty,$$

inferring (6.5.69).

Pointwise convergence : Next, we have to prove the following pointwise convergence for a fix $\omega \in \tilde{\Omega}$, i.e. we will show that for a fix $\omega \in \tilde{\Omega}$

$$\begin{aligned}&\lim_{n\to\infty}\int_s^t \left\langle (G(\sigma,\tilde{u}_n(\sigma,\omega)))^* P_n\psi,(G(\sigma,\tilde{u}_n(\sigma,\omega)))^* P_n\zeta\right\rangle_{\ell^2} d\sigma \\ &\quad= \int_s^t \left\langle (G(\sigma,\tilde{u}(\sigma,\omega)))^*\, \psi,(G(\sigma,\tilde{u}(\sigma,\omega))^*\, \zeta\right\rangle_{\ell^2} d\sigma.\end{aligned} \tag{6.5.70}$$

Let us fix $\omega \in \tilde{\Omega}$ such that

(i) $\tilde{u}_n(\cdot,\omega) \to \tilde{u}(\cdot,\omega)$ in $L^2(0,T,H_{loc})$,

(ii) $\tilde{u}(\cdot,\omega) \in L^2(0,T;H)$ and the sequence $(\tilde{u}_n(\cdot,\omega))_{n\ge1}$ is bounded in $\mathscr{C}([0,T];\mathrm{V})$.

Notice that, in order to prove (6.5.70) it is sufficient to prove that

$$G(\cdot,\tilde{u}_n(\cdot,\omega))^* P_n\psi \to G(\cdot,\tilde{u}(\cdot,\omega))^*\psi \quad \text{in} \quad L^2(s,t;\ell^2). \tag{6.5.71}$$

We have

$$\begin{aligned}
&\int_s^t \big\|G(\sigma,\tilde u_n(\sigma,\omega))^* P_n\psi - G(\sigma,\tilde u(\sigma,\omega))^*\psi\big\|_{\ell^2}^2\,d\sigma\\
&\le \int_s^t \Big(\big\|G(\sigma,\tilde u_n(\sigma,\omega))^*(P_n\psi-\psi)\big\|_{\ell^2} + \big\|G(\sigma,\tilde u_n(\sigma,\omega))^*\psi - G(\sigma,\tilde u(\sigma,\omega))^*\psi\big\|_{\ell^2}\Big)^2 d\sigma\\
&\le 2\int_s^t \big\|G(\sigma,\tilde u_n(\sigma,\omega))^*\big\|_{\mathscr{L}(\mathrm{H},\ell^2)}^2\cdot |P_n\psi-\psi|_{\mathrm{H}}^2\,d\sigma + 2\int_s^t \big\|G(\sigma,\tilde u_n(\sigma,\omega))^*\psi - G(\sigma,\tilde u(\sigma,\omega))^*\psi\big\|_{\ell^2}^2\,d\sigma
\end{aligned}$$

(6.5.72)

$$=: 2\{I_1(n)+I_2(n)\}.$$

Let us consider the term $I_1(n)$. Since $\psi\in \mathrm{V}_\gamma$, we have

$$\lim_{n\to\infty} |P_n\psi-\psi|_{\mathrm{H}} = 0.$$

By Lemma 6.2.1, the continuity of the embedding $\mathscr{T}_2(\ell^2,\mathrm{H})\hookrightarrow \mathscr{L}(\ell^2,\mathrm{H})$ and (ii), we infer that

$$\int_s^t \big\|G(\sigma,\tilde u_n(\sigma,\omega))^*\big\|_{\mathscr{L}(\mathrm{H},\ell^2)}^2\,d\sigma \le C\int_s^t |\nabla\tilde u_n(\sigma,\omega)|_{L^2}^2\,d\sigma \le \tilde C T\sup_{n\ge 1}\|\tilde u_n(\omega)\|_{\mathscr{C}([0,T];\mathrm{V})} \le K$$

for some constant $K>0$. Thus

$$\lim_{n\to\infty} I_1(n) = \lim_{n\to\infty}\int_s^t \big\|G(\sigma,\tilde u_n(\sigma,\omega))^*\big\|_{\mathscr{L}(\mathrm{H},\ell^2)}^2\cdot |P_n\psi-\psi|_{\mathrm{H}}^2\,d\sigma = 0.$$

Let us move to the term $I_2(n)$ in (6.5.72). We will prove that for every $\psi\in\mathrm{V}_\gamma$ the term $I_2(n)$ tends to zero as $n\to\infty$. Assume first that $\psi\in\mathscr{V}$. Then there exists $R>0$ such that $\operatorname{supp}\psi$ is a compact subset of $\mathscr{O}_R$. Since $\tilde u_n(\cdot,\omega)\to\tilde u(\cdot,\omega)$ in $L^2(0,T;H_{\mathrm{loc}})$, then in particular

$$\lim_{n\to\infty} q_{T,R}\big(\tilde u_n(\cdot,\omega)-\tilde u(\cdot,\omega)\big) = 0,$$

where $q_{T,R}$ is the seminorm defined by (6.3.1). In other words, $\tilde u_n(\cdot,\omega)\to\tilde u(\cdot,\omega)$ in $L^2(0,T;H_{\mathscr{O}_R})$. Therefore there exists a subsequence $(\tilde u_{n_k}(\cdot,\omega))_k$ such that

$$\tilde u_{n_k}(\sigma,\omega)\to\tilde u(\sigma,\omega) \quad \text{in } H_{\mathscr{O}_R} \text{ for almost all } \sigma\in[0,T] \text{ as } k\to\infty.$$

Hence by Lemma 6.5.16

$$G\big(\sigma,\tilde u_{n_k}(\sigma,\omega)\big)^*\psi \to G\big(\sigma,\tilde u(\sigma,\omega)\big)^*\psi \quad \text{in } \ell^2 \text{ for almost all } \sigma\in[0,T] \text{ as } k\to\infty.$$

In conclusion, by the Vitali Theorem

$$\lim_{k\to\infty}\int_s^t \|G\big(\tilde u_{n_k}(\sigma,\omega)\big)^*\psi - G\big(\tilde u(\sigma,\omega)\big)^*\psi\|_{\ell^2}^2\,d\sigma = 0 \qquad \text{for } \psi\in\mathscr{V}.$$

Repeating the above reasoning for all subsequences, we infer that from every subsequence of the sequence $\big(G\big(\sigma,\tilde u_n(\sigma,\omega)\big)^*\psi\big)_n$ we can choose the subsequence convergent in $L^2(s,t;\ell^2)$ to the same limit. Thus the whole sequence $\big(G\big(\sigma,\tilde u_n(\sigma,\omega)\big)^*\psi\big)_n$ is convergent to $G\big(\sigma,\tilde u(\sigma,\omega)\big)^*\psi$ in $L^2(s,t;\ell^2)$. At the same time

$$\lim_{n\to\infty} I_2(n)=0 \qquad \text{for every } \psi\in\mathcal{V}.$$

If $\psi\in \mathrm{V}_\gamma$ then for every $\varepsilon>0$ we can find $\psi_\varepsilon\in\mathcal{V}$ such that $\|\psi-\psi_\varepsilon\|_{\mathrm{V}_\gamma}<\varepsilon$. By the continuity of embeddings $\mathcal{T}_2(\ell^2,\mathrm{H})\hookrightarrow\mathcal{L}(\ell^2,\mathrm{H})\hookrightarrow\mathcal{L}(\ell^2,\mathrm{V}'_\gamma)$, Lemma 6.2.1 and (ii), we obtain

$$\begin{aligned}
&\int_s^t \big\|G(\sigma,\tilde u_n(\sigma,\omega))^*\psi - G(\sigma,\tilde u(\sigma,\omega))^*\psi\big\|^2_{\ell^2}\,d\sigma\\
&\le 2\int_s^t \big\|[G(\sigma,\tilde u_n(\sigma,\omega))^* - G(\sigma,\tilde u(\sigma,\omega))^*](\psi-\psi_\varepsilon)\big\|^2_{\ell^2}\,d\sigma\\
&\qquad + 2\int_s^t \big\|[G(\sigma,\tilde u_n(\sigma,\omega))^* - G(\tilde u(\sigma,\omega))^*]\psi_\varepsilon\big\|^2_{\ell^2}\,d\sigma\\
&\le 4\int_s^t \big[\|G(\sigma,\tilde u_n(\sigma,\omega))\|^2_{\mathcal{L}(\ell^2,\mathrm{V}'_\gamma)} + \|G(\sigma,\tilde u(\sigma,\omega))\|^2_{\mathcal{L}(\ell^2,\mathrm{V}'_\gamma)}\big]\,\|\psi-\psi_\varepsilon\|^2_{\mathrm{V}_\gamma}\,d\sigma\\
&\qquad + 2\int_s^t \big\|[G(\sigma,\tilde u_n(\sigma,\omega))^* - G(\sigma,\tilde u(\sigma,\omega))^*]\psi_\varepsilon\big\|^2_{\ell^2}\,d\sigma\\
&\le c\,T\big(\|\tilde u_n(\cdot,\omega)\|^2_{\mathcal{C}(0,T;\mathrm{V})} + \|\tilde u(\cdot,\omega)\|^2_{\mathcal{C}(0,T;\mathrm{V})}\big)\cdot\varepsilon^2 + 2\int_s^t \big\|[G(\sigma,\tilde u_n(\sigma,\omega))^* - G(\sigma,\tilde u(\sigma,\omega))^*]\psi_\varepsilon\big\|^2_{\ell^2}\,d\sigma\\
&\le C\varepsilon^2 + 2\int_s^t \big\|[G(\sigma,\tilde u_n(\sigma,\omega))^* - G(\sigma,\tilde u(\sigma,\omega))^*]\psi_\varepsilon\big\|^2_{\ell^2}\,d\sigma,
\end{aligned}$$

for some positive constants c and C. Passing to the upper limit as $n\to\infty$, we infer that

$$\limsup_{n\to\infty}\int_s^t \big\|G(\sigma,\tilde u_n(\sigma,\omega))^*\psi - G(\sigma,\tilde u(\sigma,\omega))^*\psi\big\|^2_{\ell^2}\,d\sigma \le C\varepsilon^2.$$

In conclusion, we proved that

$$\lim_{n\to\infty}\int_s^t \big\|G(\sigma,\tilde u_n(\sigma,\omega))^*\psi - G(\sigma,\tilde u(\sigma,\omega))^*\psi\big\|^2_{\ell^2}\,d\sigma = 0$$

which completes the proof of (6.5.71). Thus, by (6.5.69), convergence (6.5.70) and Vitali Theorem, we conclude the proof of Lemma 6.5.17. ■

By Lemma 6.5.12 we can pass to the limit in (6.5.41). By Lemmas 6.5.15 and 6.5.17 we can pass to the limit in (6.5.42) as well. After passing to the limits we infer that for all $\psi,\zeta\in\mathrm{V}_\gamma$ and all bounded continuous functions h on $\mathcal{C}([0,T];\mathrm{U}')$:

$$\tilde{\mathbb{E}}\big[\langle\tilde M(t)-\tilde M(s),\psi\rangle h(\tilde u_{|[0,s]})\big]=0, \tag{6.5.73}$$

and

$$\begin{aligned}
\tilde{\mathbb{E}}\Big[\Big(&\langle\tilde M(t),\psi\rangle\langle\tilde M(t),\zeta\rangle - \langle\tilde M(s),\psi\rangle\langle\tilde M(s),\zeta\rangle\\
&-\int_s^t \langle (G(r,\tilde u(r)))^*\psi,(G(r,\tilde u(r)))^*\zeta\rangle_{\ell^2}\,dr\Big)\cdot h(\tilde u_{|[0,s]})\Big]=0,
\end{aligned}\tag{6.5.74}$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing between V'_γ and V_γ.

From Lemma 6.5.7, Lemma 6.5.15 and Lemma 6.5.17, we infer the following corollary.

Corollary 6.5.18. *For* $t \in [0,T]$

$$\langle\langle \tilde{M} \rangle\rangle_t = \int_0^t \|G(s,\tilde{u}(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}\, ds, \qquad t \in [0,T].$$

Continuation of the proof of Theorem 6.5.4. Now we apply the idea analogous to Chapter 5, see also [38, Section 8.3]. By Lemma 6.5.8 and Corollary 6.5.14, we infer that $\tilde{M}(t), t \in [0,T]$ is an H-valued continuous square integrable martingale with respect to the filtration $\tilde{\mathbb{F}} = (\tilde{\mathscr{F}}_t)$. Moreover, by Corollary 6.5.18 the quadratic variation of $\tilde{M}$ is given by

$$\langle\langle \tilde{M} \rangle\rangle_t = \int_0^t \|G(s,\tilde{u}(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}\, ds \qquad t \in [0,T]. \tag{6.5.75}$$

Therefore by the martingale representation theorem, there exist

- a stochastic basis $(\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathscr{F}}}, \tilde{\tilde{\mathbb{F}}}, \tilde{\tilde{\mathbb{P}}})$,
- a cylindrical Wiener process $\tilde{\tilde{W}}(t)$ on ℓ^2,
- and a progressively measurable process $\tilde{\tilde{u}}(t)$ such that for all $t \in [0,T]$ and all $\mathrm{v} \in \mathscr{V}$:

$$\begin{aligned}\langle \tilde{\tilde{u}}(t), \mathrm{v}\rangle - \langle \tilde{\tilde{u}}(0), \mathrm{v}\rangle + \int_0^t \langle \mathrm{A}\tilde{\tilde{u}}(s), \mathrm{v}\rangle\, ds + \int_0^t \langle B(\tilde{\tilde{u}}(s), \tilde{\tilde{u}}(s)), \mathrm{v}\rangle\, ds + \int_0^t \langle g(|\tilde{\tilde{u}}(s)|^2)\tilde{\tilde{u}}(s), \mathrm{v}\rangle\, ds \\ = \int_0^t \langle f(\tilde{\tilde{u}}(s)), \mathrm{v}\rangle\, ds + \left\langle \int_0^t G(s,\tilde{\tilde{u}}(s))\, d\tilde{\tilde{W}}(s), \mathrm{v} \right\rangle.\end{aligned}$$

Thus the conditions from Definition 6.3.7 hold with $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}) = (\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathscr{F}}}, \tilde{\tilde{\mathbb{F}}}, \tilde{\tilde{\mathbb{P}}})$, $\hat{W} = \tilde{\tilde{W}}$ and $\hat{u} = \tilde{\tilde{u}}$. The proof of Theorem 6.5.4 is thus complete.

6.5.4 Uniqueness and strong solutions

In this subsection we will show that the solutions of (6.2.16) are pathwise unique and that the martingale solution of (6.2.16) is the strong solution. Let us recall the definition of pathwise unique solutions.

Definition 6.5.19. Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, u^i), i = 1,2$ be the martingale solutions of (6.2.16) with $u^i(0) = u_0, i = 1,2$. Then we say that the solutions are pathwise unique if $\mathbb{P}$−a.s. for all $t \in [0,T]$, $u^1(t) = u^2(t)$.

Theorem 6.5.20. *Assume that the assumptions* **(A1)** *and* **(A2)** *are satisfied. If* u_1, u_2 *are two solutions of* (6.2.16) *defined on the same filtered probability space* $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ *then* $\hat{\mathbb{P}}$*−a.s. for all* $t \in [0,T]$*,* $u_1(t) = u_2(t)$*.*

The theorem has been proved in [76, Theorem 3.7].

Theorem 6.5.21. *Assume that assumptions* **(A1)** *and* **(A2)** *are satisfied. Then there exists a path-wise unique strong solution* $u \in \mathscr{C}([0,T];\mathrm{V}) \cap L^2(0,T;\mathrm{D(A)})$ *of* (6.2.16) *such that*

$$\sup_{t \in [0,T]} \|u(t)\|_{\mathrm{V}}^2 + \int_0^T |u(t)|_{\mathrm{D(A)}}^2\, dt < \infty.$$

Proof. Since by Theorem 6.5.4 there exists a martingale solution and by Theorem 6.5.20 it is pathwise unique, the existence of strong solution follows from [68, Theorem 2] or by the Yamada-Watanabe Theorem [95]. ∎

6.6 Invariant measures

In the following, we consider time homogeneous damped tamed NSEs, i.e. the coefficients f, σ are independent of t and furthermore $f \in \mathrm{H}$ is not dependent on u. The time homogeneous damped tamed NSEs in abstract form are given by

$$\begin{cases} du(t) = \left[-\mathrm{A}_\alpha u(t) - B(u(t)) - \Pi[g(|u(t)|^2)u(t)] + \Pi f\right] dt + \sum_{j=1}^{\infty} G_j(u(t))\, dW_j(t), \\ u(0) = u_0 \in \mathrm{V}, \end{cases} \tag{6.6.1}$$

where $A_\alpha = \alpha I - \nu\Delta$ for some $\alpha \in \mathbb{R}$ and $\nu > 0$ is the viscosity. The operator B and the cylindrical Wiener process $W = (W_j)_{j=1}^{\infty}$ on ℓ^2, is same as defined in Section 6.2 and G_j are as defined in (6.2.11).

Let $\mathscr{B}_b(\mathrm{V})$ denote the set of all bounded and Borel measurable functions on V. For any $\varphi \in \mathscr{B}_b(\mathrm{V})$, $t \geq 0$, we define a function $T_t\varphi : \mathrm{V} \to \mathbb{R}$ by

$$T_t\varphi(\mathrm{v}) := \mathbb{E}\left(\varphi(u(t;\mathrm{v}))\right), \quad \mathrm{v} \in \mathrm{V}. \tag{6.6.2}$$

It follows from Theorem 6.6.2 and Ondrejat [69] (see also [17]) that $T_t\varphi \in \mathscr{B}_b(\mathrm{V})$ and $\{T_t\}_{t\geq 0}$ is a semigroup on $\mathscr{B}_b(\mathrm{V})$. Also since this unique solution to (6.6.1) has $a.e.$ path in $\mathscr{C}([0,T];\mathrm{V})$, it is also a Markov semigroup (see [69, Theorem 27]). Moreover, $\{T_t\}_{t\geq 0}$ is a Feller semigroup, i.e. T_t maps $C_b(\mathrm{V})$ into itself.

Next we state the main result of this section, regarding invariant measures:

Theorem 6.6.1. *Let for every $\alpha > 0$, the assumptions* $(\mathbf{A1})' - (\mathbf{A3})'$ *be satisfied. Then there exists an invariant measure $\mu \in \mathscr{P}(\mathrm{V})$ of the semigroup $(T_t)_{t\geq 0}$ defined by* (6.6.2)*, such that for any $t \geq 0$ and $\varphi \in \mathrm{SC}_b(\mathrm{V}_w)$*

$$\int_{\mathrm{V}} T_t\varphi(u)\mu(du) = \int_{\mathrm{V}} \varphi(u)\mu(du).$$

If T_t is sequentially weakly Feller Markov semigroup then for every $\varphi \in \mathrm{SC}_b(\mathrm{V}_w)$, $T_t\varphi \in \mathrm{SC}_b(\mathrm{V}_w) \subset B_b(\mathrm{V})$ (see [17, 62] for the definitions and inclusions of the spaces); therefore the integral on LHS in Theorem 6.6.1 makes sense.

Now we list the assumptions that we make on the coefficients f and σ along with a coercivity type assumption, see [72].

Assumptions. *(A1)′ The function $f : \mathbb{R}^3 \to \mathbb{R}^3$, is time independent and* H*-valued.*

*(**A2**)′* *A measurable function* $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ *of* C^1 *class with respect to the x-variable and for all* $x \in \mathbb{R}^3$ *there exists a constant* $C_\sigma > 0$ *such that*

$$\|\partial_{x^j}\sigma(x)\|_{\ell^2} \le C_\sigma, \quad j = 1,2,3$$

and, for all $x \in \mathbb{R}^3$,

$$\|\sigma(x)\|_{\ell^2}^2 \le \frac{1}{4}.$$

*(**A3**)′* *there exists a* $\delta > 0$ *such that*

$$2\nu|\nabla u|_{L^2}^2 - \|G(u)\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})} \ge 2\delta|\nabla u|^2_{L^2}.$$

The following theorem regarding the existence of a pathwise unique strong solution to the time homogeneous damped tamed NSEs (6.6.1) can be proved by modifying the proofs of Theorem 6.5.4 and Theorem 6.5.20 to incorporate the extra linear term αu.

Theorem 6.6.2. *Assume that assumptions* (**A1**)′ *and* (**A2**)′ *are satisfied. Then for every* $u_0 \in \mathrm{V}$*, there exists a path-wise unique strong solution* u *of* (6.6.1) *for every* $T > 0$ *such that* $u \in \mathscr{C}([0,T];\mathrm{V}) \cap L^2(0,T;\mathrm{D(A)})$*,* $\mathbb{P}$*-a.s.*

For the fixed initial data $u_0 = \mathrm{v} \in \mathrm{V}$ we denote the (pathwise) unique solution of (6.6.1), whose existence is proved in Theorem 6.6.2 by $u(t;\mathrm{v})$. Then $\{u(t;\mathrm{v}) : \mathrm{v} \in \mathrm{V}, t \ge 0\}$ forms a strong Markov process with state space V. We have the following result:

Lemma 6.6.3. *For* $\mathrm{v}, \mathrm{v}' \in \mathrm{V}$ *and* $R > 0$*, define*

$$\tau_R^{\mathrm{v}} := \inf\{t \in [0,T] : \|u(t;\mathrm{v})\|_{\mathrm{V}} > R\},$$

and

$$\tau_R^{\mathrm{v},\mathrm{v}'} := \tau_R^{\mathrm{v}} \wedge \tau_R^{\mathrm{v}'}.$$

Suppose that assumptions (**A1**)′ *and* (**A2**)′ *hold, then*

$$\mathbb{E}\|u(t\wedge\tau_R^{\mathrm{v},\mathrm{v}'};\mathrm{v}) - u(t\wedge\tau_R^{\mathrm{v},\mathrm{v}'};\mathrm{v}')\|^2_{\mathrm{V}} \le C_{t,R}\cdot\|\mathrm{v}-\mathrm{v}'\|^2_{\mathrm{V}}.$$

Proof. Let $u(t) := u(t;\mathrm{v})$, $\tilde{u}(t) := u(t;\mathrm{v}')$, and

$$w(t) := u(t) - \tilde{u}(t).$$

Set $t_R := \tau_R^{\mathrm{v},\mathrm{v}'} \wedge t$. By Itô Lemma, we have

$$\begin{aligned}
\|w(t)\|^2_{\mathrm{V}} &= \|w(0)\|^2_{\mathrm{V}} - 2\int_0^{t_R}\langle \mathrm{A}_\alpha w(s), w(s)\rangle_{\mathrm{V}}\,ds - 2\int_0^{t_R}\langle B(u(s)) - B(\tilde{u}(s)), w(s)\rangle_{\mathrm{V}}\,ds \\
&\quad - 2\int_0^{t_R}\langle g(|u(s)|^2)u(s) - g(|\tilde{u}(s)|^2)\tilde{u}(s), w(s)\rangle_{\mathrm{V}}\,ds + \int_0^{t_R}\|G(u(s)) - G(\tilde{u}(s))\|^2_{\mathcal{T}_2(\ell^2;\mathrm{V})}\,ds \\
&\quad + 2\int_0^{t_R}\langle\sum_{j=1}^{\infty}\big(G_j(u(s)) - G_j(\tilde{u}(s))\big)\,dW_j(s), w(s)\rangle_{\mathrm{V}} \\
&:= \|w(0)\|^2_{\mathrm{V}} + I_1(t_R) + I_2(t_R) + I_3(t_R) + I_4(t_R) + I_5(t_R).
\end{aligned}$$

Now we deal with each term individually. From the definition of A_α, we have

$$I_1(t_R) = -2\int_0^{t_R} \langle \mathrm{A}_\alpha w(s), w(s)\rangle_{\mathrm{V}}\, ds = -2\alpha \int_0^{t_R} \|w(s)\|_{\mathrm{V}}^2 - 2\nu\int_0^{t_R} |w(s)|_{H^2}^2 + 2\nu\int_0^{t_R} |w(s)|_{\mathrm{H}}^2\, ds.$$

By definition of the operator B and the Hölder inequality, we get

$$\begin{aligned}
|I_2(t_R)| &= 2\left|\int_0^{t_R} \langle B(u(s)) - B(\tilde{u}(s)), w(s)\rangle_{\mathrm{V}}\, ds\right| \\
&= 2\left|\int_0^{t_R} \langle B(w(s), u(s)), w(s)\rangle_{\mathrm{V}}\, ds + \int_0^{t} \langle B(\tilde{u}(s), w(s)), w(s)\rangle_{\mathrm{V}}\, ds\right| \\
&\le 2\int_0^{t_R} |w(s)\cdot\nabla u(s)|_{L^2}\left(|w(s)|_{\mathrm{H}} + |\mathrm{A}w(s)|_{L^2}\right) ds \\
&\quad + 2\int_0^{t_R} |\tilde{u}(s)\cdot\nabla w(s)|_{L^2}\left(|w(s)|_{\mathrm{H}} + |\mathrm{A}w(s)|_{L^2}\right) ds.
\end{aligned}$$

Using Lemma 6.2.1, the Gagliardo-Nirenberg inequality (6.2.9), the Hölder inequality, the Sobolev embedding theorem ($H^1 \hookrightarrow L^6$) and the Young inequality, we obtain the following estimate on the second term,

$$\begin{aligned}
|I_2(t_R)| &\le C\int_0^{t_R} |w(s)\cdot\nabla u(s)|_{L^2}^2\, ds + C_\nu\int_0^{t_R} |\tilde{u}(s)\cdot\nabla w(s)|_{L^2}^2\, ds + \frac{\nu}{2}\int_0^{t_R} |w(s)|_{\mathrm{D(A)}}^2\, ds \\
&\le C\int_0^{t_R} \|w(s)\|_{L^\infty}^2 |\nabla u(s)|_{L^2}^2 + C_\nu\int_0^{t_R} \|u(s)\|_{L^6}^2 \|\nabla w(s)\|_{L^3}^2\, ds + \frac{\nu}{2}\int_0^{t_R} |w(s)|_{\mathrm{D(A)}}^2\, ds \\
&\le C_R\int_0^{t_R} \|w(s)\|_{H^2}^{3/2} |w(s)|_{\mathrm{H}}^{1/2}\, ds + C_{R,\nu}\int_0^{t_R} \|\nabla w(s)\|_{H^1} |\nabla w(s)|_{L^2}\, ds \\
&\quad + \frac{\nu}{2}\int_0^{t_R} |w(s)|_{\mathrm{D(A)}}^2\, ds \\
&\le C_{R,\nu}\int_0^{t_R} \|w(s)\|_{\mathrm{V}}^2\, ds + \frac{\nu}{2}\int_0^{t_R} \|w(s)\|_{H^2}^2\, ds + \frac{\nu}{2}\int_0^{t_R} |w(s)|_{\mathrm{D(A)}}^2\, ds.
\end{aligned}$$

Since $|g(r) - g(r')| \le |r - r'|$, the Hölder inequality and the Young inequality gives

$$\begin{aligned}
|I_3(t_R)| &= 2\left|\int_0^{t_R} \langle g(|u(s)|^2)u(s) - g(|\tilde{u}(s)|^2)\tilde{u}(s), w(s)\rangle_{\mathrm{V}}\, ds\right| \\
&\le 4\int_0^{t_R} \langle |w(s)|(|u(s)| + |\tilde{u}(s)|)\tilde{u}(s) + |u(s)|^2 w(s), w(s)\rangle_{\mathrm{V}}\, ds \\
&\le C_\nu\int_0^{t_R} |w(s)\cdot(|u(s)| + |\tilde{u}(s)|)^2|_{\mathrm{H}}^2 + \frac{\nu}{4}\int_0^{t_R} |w(s)|_{\mathrm{D(A)}}^2\, ds.
\end{aligned}$$

Using the Hölder inequality, the Sobolev embedding theorem and the Gagliardo-Nirenberg inequality (6.2.9) we get the following estimates,

$$\begin{aligned}
|I_3(t_R)| &\le C_\nu\int_0^{t_R} \|w(s)\|_{L^6}^2 \big\||u(s)| + |\tilde{u}(s)|\big\|_{L^6}^4\, ds + \frac{\nu}{4}\int_0^{t_R} |w(s)|_{\mathrm{D(A)}}^2\, ds \\
&\le C_{R,\nu}\int_0^{t_R} \|w(s)\|_{H^2} |w(s)|_{\mathrm{H}}\, ds + \frac{\nu}{4}\int_0^{t_R} |w(s)|_{\mathrm{D(A)}}^2\, ds \\
&\le C_{R,\nu}\int_0^{t_R} \|w(s)\|_{\mathrm{V}}^2\, ds + \frac{\nu}{4}\int_0^{t_R} \|w(s)\|_{H^2}^2\, ds + \frac{\nu}{4}\int_0^{t_R} |w(s)|_{\mathrm{D(A)}}^2\, ds.
\end{aligned}$$

Using assumption $(\mathbf{A2})'$ and Lemma 6.2.1 we get the following estimate on I_4

$$|I_4(t_R)| = \left|\int_0^{t_R} \|G(u(s)) - G(\tilde{u}(s))\|^2_{\mathcal{T}_2(\ell^2;\mathrm{V})} ds\right| \le \frac{1}{2}\int_0^{t_R} |\mathrm{A}w(s)|^2_{\mathrm{H}} ds + C_\sigma \int_0^{t_R} |\nabla w(s)|^2_{L^2} ds.$$

Since $\mathbb{E}(I_5(t_R)) = 0$, by combining all the above estimates and using $|u|_{\mathrm{D(A)}} \le \|u\|_{H^2}$, we get

$$\begin{aligned}\mathbb{E}\|w(t\wedge\tau_R^{\mathrm{v},\mathrm{v}'})\|^2_{\mathrm{V}} &\le \|w(0)\|^2_{\mathrm{V}} + C_{R,\alpha,\nu}\int_0^{t_R} \mathbb{E}\,\|w(s)\|^2_{\mathrm{V}} ds \\ &\le \|\mathrm{v}-\mathrm{v}'\|^2_{\mathrm{V}} + C_{R,\alpha,\nu}\int_0^{t} \mathbb{E}\,\|w(s\wedge\tau_R^{\mathrm{v},\mathrm{v}'})\|^2_{\mathrm{V}} ds.\end{aligned}$$

The desired estimate follows from the application of generalised Gronwall Lemma. ■

For a metric space $\mathbb{U}$, we use $\mathscr{P}(\mathbb{U})$ to denote the total of all probability measures on $\mathbb{U}$. We will use the following theorem from Maslowski-Seidler [62] to prove the existence of invariant measures.

Theorem 6.6.4. *Assume that*

(i) the semigroup $\{T_t\}_{t\ge 0}$, defined by (6.6.2) *is sequentially weakly Feller in* V,

(ii) for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_{T\ge 1}\frac{1}{T}\int_0^T \mathbb{P}(\{\|u(t;u_0)\|_{\mathrm{V}} > R\})\,dt < \varepsilon.$$

Then there exists at least one invariant measure for (6.6.1).

6.6.1 Boundedness in probability

Lemma 6.6.5. *Let $u_0 \in \mathrm{V}$. Then, under the assumptions of Theorem 6.6.1, for every $\varepsilon > 0$, there exists $R > 0$ such that*

$$\sup_{T\ge 1}\frac{1}{T}\int_0^T \mathbb{P}(\{\|u(t;u_0)\|_{\mathrm{V}} > R\})\,dt < \varepsilon. \tag{6.6.3}$$

Proof. Using the Itô lemma for the function $|x|^2_{\mathrm{H}}$ and the process $u(t)$, we have

$$\begin{aligned}\frac{1}{2}|u(t)|^2_{\mathrm{H}} &= \frac{1}{2}|u_0|^2_{\mathrm{H}} + \int_0^t \langle -\mathrm{A}_\alpha u(s) - B(u(s)) - \Pi(g(|u(s)|^2)u(s)) + \Pi f, u(s)\rangle_{\mathrm{H}}\, ds \\ &\quad + \int_0^t \sum_{j=1}^\infty \langle G_j(u(s))dW_j(s), u(s)\rangle_{\mathrm{H}} + \frac{1}{2}\int_0^t \|G(u(s))\|^2_{\mathcal{T}_2(\ell^2;\mathrm{H})} ds.\end{aligned} \tag{6.6.4}$$

Now we deal with each term individually.

$$\langle \mathrm{A}_\alpha u, u\rangle_{\mathrm{H}} = \alpha\|u\|^2_{\mathrm{H}} + \nu\|\nabla u\|^2_{L^2}. \tag{6.6.5}$$

$$\langle B(u), u\rangle_{\mathrm{H}} = 0. \tag{6.6.6}$$

$$\langle \Pi g(|u|^2)u, u\rangle_{\mathrm{H}} = \left\| \sqrt{g(|u|^2)} \,|\cdot|u|\right\|_{L^2}^2. \tag{6.6.7}$$

Using the assumptions on f, for some $\beta > 0$ we obtain the following estimate:

$$\langle \Pi f, u\rangle_{\mathrm{H}} \le \|f\|_{\mathrm{V}'}\|u\|_{\mathrm{V}} \le \frac{1}{4\beta}\|f\|_{\mathrm{V}'}^2 + \beta\|u\|_{\mathrm{V}}^2. \tag{6.6.8}$$

Since u is the solution of (6.6.1) and satisfies the estimates (6.4.25) (Theorem 6.5.4 and Theorem 6.5.21 hold for the tamed NSEs, but we can also prove similar theorems for the damped tamed NSEs too), we can show that the process

$$M(t) = \int_0^t \langle u(s), \sum_{j=1}^{\infty} G_j(u(s))dW_j(s)\rangle_{\mathrm{H}},$$

is a $\mathbb{F}$-martingale. Thus taking expectation in (6.6.4) and using the estimates (6.6.5) - (6.6.8), we infer

$$\begin{aligned}\frac{1}{2}\mathbb{E}|u(t)|_{\mathrm{H}}^2 \le \frac{1}{2}|u_0|_{\mathrm{H}}^2 &- \alpha\mathbb{E}\int_0^t |u(s)|_{\mathrm{H}}^2 ds - \nu\mathbb{E}\int_0^t |\nabla u(s)|_{L^2}^2 ds \\ &- \mathbb{E}\int_0^t \left\||\sqrt{g(|u(s)|^2)}\,|\cdot|u(s)|\right\|_{L^2}^2 ds + \frac{1}{4\beta}\mathbb{E}\int_0^t \|f\|_{\mathrm{V}'}^2 ds + \beta\mathbb{E}\int_0^t \|u(s)\|_{\mathrm{V}}^2 ds \\ &+ \frac{1}{2}\int_0^t \|G(u(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2 ds.\end{aligned}$$

On rearranging, we get

$$\begin{aligned}\frac{1}{2}\mathbb{E}|u(t)|_{\mathrm{H}}^2 &+ \frac{1}{2}\mathbb{E}\int_0^t \left(2\nu|\nabla u(s)|_{L^2}^2 - \|G(u(s))\|_{\mathcal{T}_2(\ell^2;\mathrm{H})}^2\right) ds + \mathbb{E}\int_0^t \left\||\sqrt{g(|u(s)|^2)}\,|\cdot|u(s)|\right\|_{L^2}^2 ds \\ &\le \frac{1}{2}|u_0|_{\mathrm{H}}^2 + \frac{1}{4\beta}T\|f\|_{\mathrm{V}'}^2 + \beta\mathbb{E}\int_0^t |\nabla u(s)|_{L^2}^2 ds + (\beta-\alpha)\mathbb{E}\int_0^t |u(s)|_{\mathrm{H}}^2 ds.\end{aligned} \tag{6.6.9}$$

Now using the assumption $(\mathbf{A3})'$ in (6.6.9), we obtain

$$\begin{aligned}\frac{1}{2}\mathbb{E}|u(t)|_{\mathrm{H}}^2 &+ (\delta-\beta)\mathbb{E}\int_0^t |\nabla u(s)|_{L^2}^2 ds + (\alpha-\beta)\mathbb{E}\int_0^t |u(s)|_{\mathrm{H}}^2 ds \\ &+ \mathbb{E}\int_0^t \left\||\sqrt{g(|u(s)|^2)}\,|\cdot|u(s)|\right\|_{L^2}^2 ds \le \frac{1}{2}|u_0|_{\mathrm{H}}^2 + \frac{1}{4\beta}T\|f\|_{\mathrm{V}'}^2.\end{aligned}$$

Choosing $\beta \le \frac{1}{2}\min\{\delta,\alpha\}$ yields

$$\begin{aligned}\frac{1}{2}\mathbb{E}|u(t)|_{\mathrm{H}}^2 &+ \frac{\delta}{2}\mathbb{E}\int_0^t |\nabla u(s)|_{L^2}^2 ds + \frac{\alpha}{2}\mathbb{E}\int_0^t |u(s)|_{\mathrm{H}}^2 ds + \mathbb{E}\int_0^t \left\||\sqrt{g(|u(s)|^2)}\,|\cdot|u(s)|\right\|_{L^2}^2 ds \\ &\le \frac{1}{2}|u_0|_{\mathrm{H}}^2 + \frac{1}{4\beta}T\|f\|_{\mathrm{V}'}^2.\end{aligned}$$

Therefore for $\gamma = \frac{1}{2}\min\{\alpha,\delta\}$,

$$\frac{1}{2}\mathbb{E}|u(t)|_{\mathrm{H}}^2 + \gamma\mathbb{E}\int_0^t \|u(s)\|_{\mathrm{V}}^2 ds + \mathbb{E}\int_0^t \left\||\sqrt{g(|u(s)|^2)}\,|\cdot|u(s)|\right\|_{L^2}^2 ds \le \frac{1}{2}|u_0|_{\mathrm{H}}^2 + \frac{1}{4\beta}T\|f\|_{\mathrm{V}'}^2.$$

Thus for any $T > 0$, we infer that

$$\frac{1}{T}\int_0^T \mathbb{E}\,\|u(s)\|_{\mathrm{V}}^2\,ds \le \frac{1}{2\gamma T}|u_0|_{\mathrm{H}}^2 + \frac{1}{4\gamma\beta}\|f\|_{\mathrm{V}'}^2. \tag{6.6.10}$$

Using the Chebyshev inequality and inequality (6.6.10), we infer that for every $T \ge 0$

$$\begin{aligned}\frac{1}{T}\int_0^T \mathbb{P}(\{\|u(t,u_0)\|_{\mathrm{V}} > R\})\,dt &\le \frac{1}{TR^2}\int_0^T \mathbb{E}\,\|u(t)\|_{\mathrm{V}}^2\,dt \\ &\le \frac{1}{R^2}\left[\frac{1}{2\gamma T}|u_0|_{\mathrm{H}}^2 + \frac{1}{4\gamma\beta}\|f\|_{\mathrm{V}'}^2\right].\end{aligned}$$

Now for sufficiently large $R > 0$ depending on $\varepsilon, |u_0|_{\mathrm{H}}$ and $\|f\|_{\mathrm{V}'}$ the assertion follows. ■

6.6.2 Sequentially weak Feller property

We are left to verify the assumption (i) of Theorem 6.6.4, i.e. the Markov semigroup $\{T_t\}_{t\ge 0}$ is sequentially weakly Feller in V. In other words we want to show that for any $t > 0$ and any bounded and weakly continuous function $\varphi : \mathrm{V} \to \mathbb{R}$, if $\xi_n \to \xi$ weakly in V, then

$$T_t\varphi(\xi_n) \to T_t\varphi(\xi). \tag{6.6.11}$$

In Theorem 5.7.7, we proved for stochastic constrained Navier-Stokes equations that the martingale solution of SCNSE continuously depends on the initial data. We have a similar result for time homogeneous damped tamed NSEs, which can be proved analogously, see also [25, Theorem 4.11].

Theorem 6.6.6. *Assume that $(u_{0,n})_{n=1}^{\infty}$ is a V−valued sequence that is convergent weakly to $u_0 \in \mathrm{V}$. Let*

$$(\Omega_n, \mathscr{F}_n, \mathbb{F}_n, \mathbb{P}_n, W_n, u_n)$$

be a martingale solution of problem (6.6.1) *on* $[0,\infty)$ *with the initial data $u_{0,n}$. Then for every $T > 0$ there exist*

- *a subsequence $(n_k)_k$,*
- *a stochastic basis $\big(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}\big)$,*
- *a cylindrical Wiener process $\tilde{W}(t) = \big(\tilde{W}^j(t)\big)_{j=1}^{\infty}$ on ℓ^2,*
- *and $\tilde{\mathbb{F}}$-progressively measurable processes $\tilde{u}$, $\big(\tilde{u}_{n_k}\big)_{k\ge 1}$ (defined on this basis) with laws supported in $\mathscr{Z}_T$ (see* (6.3.5)*) such that*

$$\tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ on } \mathscr{Z}_T \text{ and } \tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathscr{Z}_T, \quad \tilde{\mathbb{P}}\text{ - a.s.} \tag{6.6.12}$$

and the system

$$\big(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u}\big)$$

is a martingale solution to problem (6.6.1) *on the interval* $[0,T]$ *with the initial data* u_0. *In particular, for all* $t \in [0,T]$ *and all* $\mathrm{v} \in \mathcal{V}$

$$\langle \tilde{u}(t), \mathrm{v}\rangle + \int_0^t \langle \mathrm{A}_\alpha \tilde{u}(s), \mathrm{v}\rangle\, ds + \int_0^t \langle B(\tilde{u}(s)), \mathrm{v}\rangle\, ds + \int_0^t \langle g(|\tilde{u}(s)|^2)\tilde{u}(s), \mathrm{v}\rangle\, ds$$
$$= \langle \tilde{u}(0), \mathrm{v}\rangle_{\mathrm{V}} + \int_0^t \langle f, \mathrm{v}\rangle\, ds + \langle \int_0^t \sum_{j=1}^{\infty} G_j(s, \tilde{u}(s))\, dW_j(s), \mathrm{v}\rangle, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Moreover, the process $\tilde{u}$ *satisfies the following inequality*

$$\tilde{\mathbb{E}}\left[\sup_{s\in[0,T]} \|\tilde{u}(s)\|_{\mathrm{V}}^2 + \int_0^T |\tilde{u}(s)|^2_{\mathrm{D(A)}}\, ds\right] < \infty. \tag{6.6.13}$$

We will need the uniqueness in law of solutions of (6.6.1). We define the uniqueness in law here:

Definition 6.6.7. Let $(\Omega^i, \mathscr{F}^i, \mathbb{F}^i, \mathbb{P}^i, W^i, u^i), i = 1,2$ be the martingale solutions of (6.6.1) with $u^i(0) = u_0, i = 1,2$. Then we say that the solutions are unique in law if

$$\mathrm{Law}_{\mathbb{P}^1}(u^1) = \mathrm{Law}_{\mathbb{P}^2}(u^2) \,\mathrm{on}\, \mathscr{C}([0,\infty); \mathrm{V}_w) \cap L^2([0,\infty); \mathrm{D(A)}),$$

where $\mathrm{Law}_{\mathbb{P}^i}(u^i), i = 1,2$ are by definition probability measures on $\mathscr{C}([0,\infty); \mathrm{V}_w) \cap L^2([0,\infty); \mathrm{D(A)})$.

Lemma 6.6.8. *Assume that assumptions* **(A1)**$'$ – **(A2)**$'$ *are satisfied. Then the martingale solution of* (6.6.1) *are unique in law.*

The proof of the above lemma is the direct application of Theorems 2 and 11 of [68] once we have proved the existence of a pathwise unique martingale solution of (6.6.1), which follows from Theorem 6.6.2.

Lemma 6.6.9. *The semigroup* $\{T_t\}_{t\geq 0}$ *is sequentially weakly Feller in* V.

Proof. Let us choose and fix $0 < t \leq T, \xi \in \mathrm{V}$ and $\varphi : \mathrm{V} \to \mathbb{R}$ be a bounded weakly continuous function. Need to show that $T_t\varphi$ is sequentially weakly Feller in V. For this aim let us choose an V-valued sequence (ξ_n) weakly convergent in V to ξ. Since the function $T_t\varphi : \mathrm{V} \to \mathbb{R}$ is bounded, we only need to prove (6.6.11).

Let $u_n(\cdot) = u(\cdot, \xi_n)$ be a strong solution of (6.6.1) on $[0,T]$ with the initial data ξ_n and let $u(\cdot) = u(\cdot, \xi)$ be a strong solution of (6.6.1) with the initial data ξ. We assume these processes are defined on the stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W)$. By Theorem 6.6.6 there exist

- a subsequence $(n_k)_k$,
- a stochastic basis $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, where $\tilde{\mathbb{F}} = \{\tilde{\mathscr{F}}_s\}_{s\in[0,T]}$,
- a cylindrical Wiener process $\tilde{W}(t) = \left(\tilde{W}^j(t)\right)_{j=1}^{\infty}$ on ℓ^2,

- and progressively measurable processes $\tilde{u}(s), (\tilde{u}_{n_k}(s))_{k\geq 1}, s \in [0,T]$ (defined on this basis) with laws supported in $\mathcal{Z}_T$ such that

$$\tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ on } \mathcal{Z}_T \text{ and } \tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathcal{Z}_T, \quad \tilde{\mathbb{P}} - a.s. \tag{6.6.14}$$

and the system

$$(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u}) \tag{6.6.15}$$

is a martingale solution to (6.6.1) on the interval $[0,T]$ with the initial data ξ.

In particular, by (6.6.14), $\tilde{\mathbb{P}}$-almost surely

$$\tilde{u}_{n_k}(t) \to \tilde{u}(t) \text{ weakly in V.}$$

Since the function $\varphi : \mathrm{V} \to \mathbb{R}$ is sequentially weakly continuous, we infer that $\tilde{\mathbb{P}}$-a.s.,

$$\varphi(\tilde{u}_{n_k}(t)) \to \varphi(\tilde{u}(t)) \text{ in } \mathbb{R}.$$

Since the function φ is also bounded, by the Lebesgue dominated convergence theorem we infer that

$$\lim_{k\to\infty} \tilde{\mathbb{E}}\left[\varphi(\tilde{u}_{n_k}(t))\right] = \tilde{\mathbb{E}}\left[\varphi(\tilde{u}(t))\right]. \tag{6.6.16}$$

From the equality of laws of $\tilde{u}_{n_k}$ and u_{n_k}, $k \in \mathbb{N}$, on the space $\mathcal{Z}_T$ we infer that $\tilde{u}_{n_k}$ and u_{n_k} have the same laws on V_w and so

$$\tilde{\mathbb{E}}\left[\varphi(\tilde{u}_{n_k}(t))\right] = \mathbb{E}\left[\varphi(u_{n_k}(t))\right]. \tag{6.6.17}$$

On the other hand, R.H.S. of (6.6.17) is equal by (6.6.2), to $T_t\varphi(\xi_{n_k})$.

Since u, by assumption, is a martingale solution of (6.6.1) with the initial data ξ and by the above $\tilde{u}$ is also a solution of (6.6.1) with the initial data ξ. Thus, by Lemma 6.6.8, we infer that

$$\text{the processes } \; u \; \text{ and } \; \tilde{u} \; \text{ have the same law on the space } \; \mathcal{Z}_T.$$

Hence

$$\tilde{\mathbb{E}}\left[\varphi(\tilde{u}(t))\right] = \mathbb{E}\left[\varphi(u(t))\right]. \tag{6.6.18}$$

As before, the R.H.S. of (6.6.18) is equal by (6.6.2), to $T_t\varphi(\xi)$.

Thus by equations (6.6.16), (6.6.17) and (6.6.18), we infer

$$\lim_{k\to\infty} T_t\varphi(\xi_{n_k}) = T_t\varphi(\xi).$$

Using the subsequence argument, we can deduce that the whole sequence $(T_t\varphi(\xi_n))_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} T_t\varphi(\xi_n) = T_t\varphi(\xi).$$

■

Thus the existence of an invariant measure is established by using Theorem 6.6.4, Lemmas 6.6.5 and 6.6.9; completing the proof of Theorem 6.6.1.

CHAPTER 7

Open problems and future directions

In the final chapter of this thesis, we formulate some of the open problems that arise from the work carried out here and will also list down other related problems that I plan to explore in the future.

7.1 Open problems

7.1.1 Constrained Navier-Stokes equations on a general bounded domain

In Chapter 4, we proved the existence of the global solution to the CNSE

$$\begin{cases} \dfrac{du}{dt} + \mathrm{A}u - |\nabla u|_{L^2}^2 u + B(u,u) = 0, \\ u(0) = u_0 \in \mathrm{V} \cap \mathscr{M}, \end{cases} \tag{7.1.1}$$

on a two dimensional bounded domain with periodic boundary conditions. We were also able to establish the existence of local solutions to (7.1.1) as well as invariance of the manifold $\mathscr{M}$, with Dirichlet boundary conditions. Our approach of proving boundedness of the enstrophy using the gradient type structure of the equation obviously fails, since the orthogonal property (3.2.5) is not satisfied in this case. Thus, it will be interesting to see if one can prove the existence of a global solution to (7.1.1) on a general bounded domain, possibly by obtaining suitable bounds on the gradient norm of the solution.

7.1.2 Lower bound on the regularity of the initial data

At the end of Chapter 4 we provided a formal analysis to show that for the CNSE

$$\begin{cases} \dfrac{du}{dt} + \mathrm{A}u - |\nabla u|^2_{L^2} u + B(u,u) = 0, \\ u(0) = u_0 \in \hat{\mathrm{V}} \cap \mathscr{M}, \end{cases} \tag{7.1.2}$$

where $\hat{\mathrm{V}} = \mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}})$ is as in (4.5.3), to be well-posed we need α to be at least $\frac{1}{2}$. It's important to note that we consider these equations on a bounded domain with the periodic boundary conditions. Since, the formal analysis indicates that the initial data u_0 at least needs to be in $\mathrm{D}(\mathrm{A}^{1/4})$ for (7.1.2) to be well posed, one should prove the same rigorously. Another reason to explore this problem is to compare it with the two dimensional Navier-Stokes equations, where the well-posedness results for $u_0 \in \mathrm{H}$ and V (see [88]), for more regular initial data with a "compatibility condition" on the boundary of the domain (see [89]) and for $u_0 \in \hat{\mathrm{V}} = \mathrm{D}(\mathrm{A}^{\frac{\alpha}{2}})$, $\alpha \in \left(1, \frac{3}{2}\right)$ (see [12, Proposition 3.3]) are known.

7.1.3 Stochastic constrained Navier-Stokes equations on $\mathbb{R}^2$

In Chapter 5 we proved the existence of a strong pathwise unique solution to the SCNSE

$$\begin{cases} du(t) + [\mathrm{A}u(t) + B(u(t))]\,dt = |\nabla u(t)|^2_{L^2} u(t)\,dt + Cu(t) \circ dW(t), \\ u(0) = u_0, \end{cases} \tag{7.1.3}$$

on a two dimensional bounded domain with periodic boundary conditions, i.e. on a torus. We believe that an analogous result holds also in the Euclidean space $\mathbb{R}^2$, but so far we have not addressed this problem (even the existence of a martingale solution). Here we briefly describe a possible approach to the problem (inspired from Chapter 6 and [23]).

Step I

In the spirit of Chapter 6, one considers truncated equations on infinite dimensional space rather than Faedo-Galerkin approximations on finite dimensional space. The non-commutativity of the classical Faedo-Galerkin projection operator with the gradient operator (∇) on the whole Euclidean space, as mentioned in Chapter 6, is the motivation towards studying truncated SPDEs instead of approximated finite dimensional SDEs.

Step II

The second major difference between our method and the approach taken in the case of a torus is in the choice of the space $\mathcal{Z}_T$. We will have to prove the tightness of the laws of the solutions of the truncated SPDEs on a different space instead of the one used in the case of a torus, see Eq. 5.4.3. Due to the lack of compactness of embedding $H^1 \hookrightarrow L^2$ on $\mathbb{R}^3$, we will use L^2_{loc} space to deduce strong convergence of approximating solutions. We will also require an auxiliary space U, as in Chapter 6. In contrast to [23] we will not use this space to prove convergence, but just to establish the Aldous condition, see Definition 2.9.10. Once we have the tightness of the laws, the rest of the proof of the existence of a martingale solution should be analogous to the existence proof for stochastic tamed Navier-Stokes equations, see Theorem 6.5.4.

7.1.4 Existence of invariant measures for stochastic constrained Navier-Stokes equations

In Theorem 5.8.1, we proved that the family of semigroups generated by the solution of (7.1.3) are sequentially weakly Feller, which along with the other necessary condition corresponding to boundedness in probability implies the existence of invariant measures. The proof will essentially use Theorem 6.6.4 (see [62]), but we haven't been able to verify the second condition and this remains for now an open problem.

7.1.5 Stochastic tamed Navier-Stokes equations: Invariant measures

Röckner and Zhang proved the existence of a unique invariant measure for the time homogeneous stochastic tamed Navier-Stokes equations on $\mathbb{T}^3$ [76]. We established the existence of invariant measures for the time homogeneous damped tamed Navier-Stokes equations

$$\begin{cases} du(t) = \left[-\mathrm{A}_\alpha u(t) - B(u(t)) - \Pi[g(|u(t)|^2)u(t)] + \Pi f\right] dt + \sum_{j=1}^{\infty} G_j(u(t))\, dW_t^j, \\ u(0) = u_0 \in \mathrm{V}, \end{cases} \tag{7.1.4}$$

where $A_\alpha = \alpha I - \nu\Delta$ for some $\alpha \in \mathbb{R}$ and $\nu > 0$, on the whole Euclidean space. The question of uniqueness of invariant measure remains open.

Another interesting problem in this direction could be to prove the existence of invariant measures for the time homogeneous tamed Navier-Stokes equations on a general bounded domain after establishing the existence of solutions.

7.2 Possible future research directions

7.2.1 Stochastic 2D viscous shallow water equations

Bresch and Desjardins [8] studied a two dimensional viscous shallow water model with friction term in a bounded domain with periodic boundary conditions. They proved the existence of global weak solutions. To the best of my knowledge, this problem has not been studied in a stochastic setting. Thus, I would like to study the well-posedness of the following SDE which will be a good starting point to understand the dynamics of such a model under random forcing.

$$\begin{cases} \partial_t h + \mathrm{div}(h\,u) = 0, \\ d(h\,u) + \Big[\mathrm{div}(h\,u \otimes u) + \dfrac{(h\,u)^\perp}{R_0} + r_0 u + r_1 h\,|u|\,u \\ \qquad\qquad -\kappa h \nabla\Delta h + \dfrac{h\nabla h}{\mathrm{Fr}^2} - \nu \mathrm{div}(h\nabla u)\Big] dt = h\,f\,dW, \end{cases} \tag{7.2.1}$$

supplemented with initial conditions

$$h|_{t=0} = h_0, \qquad (hu)|_{t=0} = m_0. \tag{7.2.2}$$

In the equation (7.2.1) u denotes the horizontal mean velocity field, h the depth variation, $\mathrm{Fr} > 0$ the Froude number, $R_0 > 0$ the Rossby number and $\kappa \geq 0$ the capillary coefficient. The terms $r_0 u$ and $r_1 h|u|u$ correspond to the drag terms and $W = (W^1, W^2)$ is a two dimensional Wiener process where W^1 and W^2 are real-valued independent Brownian motions defined on a suitable probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$.

Even though the motivations behind the following two problems are completely different, they are related in the sense that both provide an approximation scheme for CNSE in deterministic and stochastic setting respectively. The motivation for studying such a constrained problem (7.2.5) is that these equations should be a better approximation of the Euler equations (for small viscosity) since, for the Euler equations, the energy of (sufficiently smooth) solutions is constant (see [31]).

7.2.2 Slightly compressible approximation of constrained Navier-Stokes equations

Rubinstein *et.al.* [78] constructed an asymptotic solution for small ε to the following reaction-diffusion problem on $\Omega \subset \mathbb{R}^m$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \Delta u - \varepsilon^{-1} \nabla V(u) \\ u(x,0,\varepsilon) = g(x), \qquad \partial_n u = 0 \text{ on } \partial\Omega. \end{cases} \tag{7.2.3}$$

They showed that at each $x \in \Omega$, u tends quickly to a minimum of $V(u)$. Motivated by their work, I would like to show that asymptotically (as $\varepsilon \to 0$) the solution to

$$\frac{\partial u^\varepsilon}{\partial t} = \nu \Delta u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \varepsilon^{-1} \nabla_{L^2} \varphi(u^\varepsilon), \tag{7.2.4}$$

on $\mathbb{T}^2$, where

$$\varphi : \mathbb{R}^2 \ni u \to \frac{1}{4}(|u|_{\mathrm{H}}^2 - 1)^2 \in \mathbb{R},$$

converges to the solution of Navier-Stokes equations whose L^2-norm is conserved and equal to 1:

$$\frac{\partial u}{\partial t} = \nu \Delta u + u \cdot \nabla u + \nu |\nabla u|_{L^2}^2 u. \tag{7.2.5}$$

We have already established the existence of the global solution to (7.2.5) on $\mathbb{T}^2$ in Chapter 4.

7.2.3 Stochastic hyperbolic constrained Navier-Stokes equations

The motion of a particle of mass μ under the impact of a random forcing $b(q)+\sigma(q)\dot{W}$ with the damping coefficient proportional to the speed is described according to the Newton law

$$\mu\ddot{q}_t^\mu = b(q_t^\mu)+\sigma(q_t^\mu)\dot{W}_t - \alpha\dot{q}_t^\mu, \quad q_0^\mu = q \in \mathbb{R}^n, \quad \dot{q}_0^\mu = p \in \mathbb{R}^n. \tag{7.2.6}$$

In practice, the dynamics of the position q_t is of interest and thus in principal we are working with a system involving twice as many variables $(q_t, \dot{q}_t)$. Valid approximations that reduce the state space are crucial for both theoretical and computational applications.

Smoluchowski showed that for Brownian particle under an external field, there exists a Markov process which under certain circumstances is a good approximation to the position of the Ornstein-Uhlenbeck process (governs the dynamics of a Brownian particle). Developing on the same, it has been shown [67] that for very small μ, q_t^μ can be approximated by the solution of the first order equation

$$\dot{q}_t = b(q_t)+\sigma(q_t)\dot{W}_t, \quad q_0 = q \in \mathbb{R}^N. \tag{7.2.7}$$

In short, this small mass limit or the Smoluchowski-Kramers approximation of the system (7.2.6) reduces the state space from $(q_t, \dot{q}_t)$ to q_t cutting the dimension of state space to half.

Apart from it's application in reducing the computational cost significantly Smoluchowski-Kramers approximation is also used by several mathematicians and physicists to study fundamental mathematical notions, like invariant measures and large time behaviour. In particular, Cerrai and Freidlin [34, 35, 43] have shown that the solution of damped wave equations perturbed by stochastic forcing converges to the solution of corresponding stochastic heat equation. They also established relations between stationary distributions and large deviations of a general class of SPDEs and their limiting equations. Existence of such convergences make it possible to analyse the simpler equation (first order) in order to understand the large time behaviour and other asymptotics of second order equation.

Brenier *et.al.* [7] showed that the solution of a damped wave equation converges to the solution of Navier-Stokes equations on a two dimensional torus in small mass limit, which could be seen as a deterministic version of Smoluchowski-Kramers approximation. I would like to consider a similar problem in the context of stochastic hyperbolic CNSEs. To be precise, I would like to show that as $\mu \to 0$ the solution of

$$\mu\frac{\partial^2 u^\mu}{\partial t^2} + \frac{\partial u^\mu}{\partial t} = \nu\Delta u^\mu + u^\mu \cdot \nabla u^\mu + \nu|\nabla u^\mu|_{L^2}^2 u^\mu + g(\cdot, u^\mu)\frac{\partial W}{\partial t}, \tag{7.2.8}$$

converges to the solution of SCNSE with the same external forcing. This would also enable us to address the questions of stationary distributions and large deviations for the stochastic hyperbolic CNSE.

APPENDIX A

ORTHOGONALITY OF BILINEAR MAP TO THE STOKES OPERATOR

In this appendix we show that the bilinear map $B : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$, defined in Chapter 3 is orthogonal to the Stokes operator A in H on $\mathbb{R}^2$. This in fact holds true for any bounded domain with periodic boundary conditions and the proof for that can be found in [90]. The proof of the following lemma is motivated from the same.

Lemma A.1. *Let $x \in \mathbb{R}^2$ and $u \in \mathrm{D(A)}$, then*

$$\langle B(u,u), \mathrm{A}u\rangle_{\mathrm{H}} = 0, \ \ \forall\, u \in \mathrm{D(A)}. \tag{A.0.1}$$

Proof. Let $u \in \mathrm{D(A)}$ then, by the definition of $B(u,\mathrm{v})$ and $\mathrm{A}u$,

$$\begin{aligned}\langle B(u,u), \mathrm{A}u\rangle_{\mathrm{H}} &= \int_{\mathscr{O}} (u(x)\cdot\nabla)u(x)\cdot \mathrm{A}u(x)\,dx \\ &= \sum_{i,j,k=1}^{2} \int_{\mathscr{O}} (u_i D_i u_j)(-\Delta u_j)\,dx \\ &= -\sum_{i,j,k=1}^{2} \int_{\mathscr{O}} u_i D_i u_j D_k^2 u_j\,dx.\end{aligned}$$

Now by integration by parts and the Stokes formula

$$\begin{aligned}\langle B(u,u), \mathrm{A}u\rangle_{\mathrm{H}} &= -\left(\sum_{i,j,k=1}^{2} u_i D_i u_j D_k u_j\right)\Big|_{\partial\mathscr{O}} + \sum_{i,j,k=1}^{2} \int_{\mathscr{O}} D_k(u_i D_i u_j) D_k u_j\,dx \\ &= \sum_{i,j,k=1}^{2} \int_{\mathscr{O}} D_k u_i D_i u_j D_k u_j\,dx + \sum_{i,j,k=1}^{2} \int_{\mathscr{O}} u_i D_{k\,i} u_j D_k u_j\,dx.\end{aligned}$$

Now we will show that each of the terms in RHS will vanish. We will consider the first term and show that it vanishes.

$$\begin{aligned}\sum_{i,j,k=1}^{2} D_k u_i D_i u_j D_k u_j &= (D_1u_1)^3 + D_1u_2D_2u_1D_1u_1 + D_1u_1(D_1u_2)^2 + (D_1u_2)^2D_2u_2 \\ &\quad + (D_2u_1)^2D_1u_1 + D_2u_2(D_2u_1)^2 + D_2u_1D_1u_2D_2u_2 + (D_2u_2)^3 \\ &= (D_1u_1 + D_2u_2)\left[(D_1u_1)^2 + (D_2u_2)^2 - D_1u_1D_2u_2\right] \\ &\quad + D_1u_2D_2u_1(D_1u_1 + D_2u_2) + (D_1u_2)^2(D_1u_1 + D_2u_2) \\ &\quad + (D_2u_1)^2(D_1u_1 + D_2u_2).\end{aligned}$$

Now since $\nabla \cdot u = D_1u_1 + D_2u_2 = 0$, the first term vanishes identically.

The second term vanishes because

$$\begin{aligned}2\sum_{i,j,k=1}^{2}\int_{\mathcal{O}} u_i D_{ki} u_j D_k u_j \, dx &= \sum_{i,j,k=1}^{2}\int_{\mathcal{O}} u_i D_i (D_k u_j)^2 \, dx \\ &= \left(\sum_{i,j,k=1}^{2} u_i (D_k u_j)^2\right)\Bigg|_{\partial\mathcal{O}} - \sum_{i,j,k=1}^{2}\int_{\mathcal{O}} D_i u_i (D_k u_j)^2 \, dx \\ &= -\sum_{j,k=1}^{2}\int_{\mathcal{O}} (\nabla \cdot u)(D_k u_j)^2 \, dx \;\; = 0.\end{aligned}$$

Thus we have shown that for every $u \in \mathrm{D(A)}$, $\langle B(u,u), \mathrm{A}u\rangle_{\mathrm{H}} = 0$. ■

APPENDIX B

SOME RESULTS IN THE SUPPORT OF SECTION 4.4

This appendix is dedicated to the proof of Remark 4.4.1, which plays a crucial role in proving the convergence of solution of Constrained Navier-Stokes to the solution of Euler equations in the inviscid limit, Theorem 4.1.2.

Remark B.1. *If* $\nabla \cdot u = 0$ *and* $\mathrm{Curl}(u) = 0$*, then u is constant by Hodge decomposition. In particular, if* $u \in \mathrm{V}$ *and* $\mathrm{Curl}(u) = 0$*, then* $u = 0$.

Proof of Remark 4.4.1. We want to show that Curl is a linear isomorphism between V and $L^2_0(\mathbb{T}^2)$. It is clear that the map

$$\mathrm{Curl} : \mathrm{V} \ni u \mapsto \omega = \mathrm{Curl}(u) \in L^2_0(\mathbb{T}^2),$$

is linear and continuous. Hence in order to prove the Remark 4.4.1 it is sufficient to find a continuous linear map

$$\Lambda : L^2_0(\mathbb{T}^2) \to \mathrm{V}, \tag{B.0.1}$$

such that,

$$\mathrm{Curl} \circ \Lambda = \mathrm{id} \text{ on } L^2_0(\mathbb{T}^2), \tag{B.0.2}$$

$$\Lambda \circ \mathrm{Curl} = \mathrm{id} \text{ on } \mathrm{V}. \tag{B.0.3}$$

Let $\omega \in L^2_0(\mathbb{T}^2)$ then by elliptic regularity [46] (applies also for $p \neq 2$) there exists a unique $\psi \in L^2_0(\mathbb{T}^2) \cap H^2(\mathbb{T}^2)$ such that

$$\Delta \psi = \omega, \tag{B.0.4}$$

and the map

$$L^2_0 \ni \omega \mapsto \psi \in L^2_0 \cap H^2,$$

is bounded. Let us put $u = -\nabla^\perp \psi$, i.e.

$$u = (-D_2\psi, D_1\psi). \tag{B.0.5}$$

Then $u \in H^1(\mathbb{T}^2)$ and $\nabla \cdot u = 0$ in the weak sense. Thus $u \in \mathrm{V}$. Using all of this we define the bounded linear map $\Lambda : L_0^2(\mathbb{T}^2) \ni \omega \mapsto u \in \mathrm{V}$. Now we are left to check that (B.0.2) and (B.0.3) holds for this Λ.

Let us take $\omega \in L_0^2(\mathbb{T}^2)$ and put $u := \Lambda(\omega) \in \mathrm{V}$. Now considering LHS of (B.0.2),

$$\begin{aligned}(\mathrm{Curl} \circ \Lambda)(\omega) = \mathrm{Curl}(u) &= D_1 u_2 - D_2 u_1 \\ &= D_1 D_1 \psi - (-D_2 D_2 \psi) = \Delta \psi = \omega,\end{aligned}$$

where we have used the definitions of ψ and u from (B.0.4) and (B.0.5). Hence we have established (B.0.2).

Now we take $\mathrm{v} \in \mathrm{V}$ and put $\omega = \mathrm{Curl}(\mathrm{v}) \in L_0^2(\mathbb{T}^2)$. Define $\psi \in L_0^2(\mathbb{T}^2) \cap H^2(\mathbb{T}^2)$ by

$$\Delta \psi = \omega. \tag{B.0.6}$$

Observe that in view of (4.4.2) we have

$$\Delta \varphi = \mathrm{Curl}(-D_2\varphi, D_1\varphi), \quad \varphi \in H^2(\mathbb{T}^2).$$

Thus by (B.0.6) and the definition of u from (B.0.5) we obtain

$$\mathrm{Curl}(u) = \mathrm{Curl}(\mathrm{v}),$$

where $u = -\nabla^\perp \psi \in \mathrm{V}$. Therefore using Remark B.1 $u = \mathrm{v}$, thus proving that Curl is a linear isomorphism between V and $L_0^2(\mathbb{T}^2)$. It is straightforward to show (4.4.3). Thus we are left to prove (4.4.4).

Let us fix $p \in (1,\infty)$ and take $u \in H^{1,p}(\mathbb{T}^2)$. Denote $\omega = \mathrm{Curl}(u) \in L_0^p(\mathbb{T}^2)$. From the first part of the proof there exists a bounded linear map $\Lambda : L_0^p(\mathbb{T}^2) \to H^{1,p}(\mathbb{T}^2)$

$$\Lambda : L_0^p \ni \omega \mapsto u \in H^{1,p},$$

such that

$$\mathrm{Curl} \circ \Lambda = \mathrm{id} \text{ on } L_0^p(\mathbb{T}^2).$$

In particular, there exists a $C_p' > 0$,

$$\|\Lambda \omega\|_{H^{1,p}(\mathbb{T}^2)} \le C_p' \|\omega\|_{L^p(\mathbb{T}^2)}, \quad \omega \in L_0^p(\mathbb{T}^2).$$

Hence

$$\|\nabla \Lambda \omega\|_{L^p(\mathbb{T}^2)} \le C_p' \|\omega\|_{L^p(\mathbb{T}^2)}, \quad \omega \in L_0^p(\mathbb{T}^2). \tag{B.0.7}$$

Taking now $u \in H^{1,p}(\mathbb{T}^2)$. Putting $\omega = \text{Curl}(u)$ so that $\Lambda\omega = u$ from (B.0.7) we infer (B.0.8),

$$\|\nabla u\|_{L^p(\mathbb{T}^2)} \leq C_p \|\omega\|_{L^p(\mathbb{T}^2)}. \tag{B.0.8}$$

Now since $\|\omega\|_{L^p(\mathbb{T}^2)} \leq \|\omega\|_{L^\infty(\mathbb{T}^2)}$ for every p, we can establish (4.4.4). ■

APPENDIX C

Kuratowski Theorem

The main objective of this appendix is to establish the preliminaries that are required to prove Lemmas 5.7.1 and 6.5.5. The proof of lemmas heavily rely on the Kuratowski Theorem [70, Theorem 3.9], which we recall below for the sake of completeness.

Theorem C.1. *Assume that X_1, X_2 are the Polish spaces with their Borel $\sigma-$fields denoted respectively by $\mathscr{B}(X_1), \mathscr{B}(X_2)$. If $\varphi\colon X_1 \to X_2$ is an injective Borel measurable map then for any $E_1 \in \mathscr{B}(X_1)$, $E_2 := \varphi(E_1) \in \mathscr{B}(X_2)$.*

Next two lemmas are the main results of this appendix.

Lemma C.1. *Let X_1, X_2 and Z be topological spaces such that X_1 is a Borel subset of X_2. Then $X_1 \cap Z$ is a Borel subset of $X_2 \cap Z$, where $X_2 \cap Z$ is a topological space too, with the topology given by*

$$\tau(X_2 \cap Z) = \{A \cap B : A \in \tau(X_2), B \in \tau(Z)\}. \tag{C.0.1}$$

Proof. Since the Borel $\sigma-$filed on $X_2 \cap Z$ is the smallest $\sigma-$field generated by $\tau(X_2 \cap Z)$, i.e. $\mathscr{B}(X_2 \cap Z) = \sigma(\tau(X_2 \cap Z))$, in order to prove the lemma it is enough to show that $\forall\, Y \in \mathscr{B}(X_1)$

$$Y \cap Z \in \mathscr{B}(X_2 \cap Z). \tag{C.0.2}$$

Firstly, we show that (C.0.2) holds for all $Y \in \tau(X_1)$. Since $X_1 \in \mathscr{B}(X_2)$, $X_1 \subset X_2$ and has trace topology from X_2, i.e $\forall\, Y \in \tau(X_1)$ there exists a $C \in \tau(X_2)$ such that

$$Y = C \cap X_1 .$$

As $X_1 \in \mathscr{B}(X_2)$ there exists a countable collection $\{K_i\}_{i\in\mathbb{N}}$ of open subsets of X_2 such that

$$X_1 = \bigcup_{i\in\mathbb{N}} K_i .$$

Therefore,

$$Y \cap Z = C \cap X_1 \cap Z = C \cap \left(\bigcup_{i \in \mathbb{N}} K_i \right) \cap Z = \bigcup_{i \in \mathbb{N}} (C \cap K_i) \cap Z \,.$$

Since $C \in \tau(X_2)$, for every $i \in \mathbb{N}$, $C \cap K_i$ is open in X_2 and there exists a collection $\{B_j\}_{j \in \mathbb{N}} \in \tau(X_2)$ such that

$$\bigcup_{i \in \mathbb{N}} (C \cap K_i) = \bigcup_{j \in \mathbb{N}} B_j \,.$$

Thus

$$Y \cap Z = \bigcup_{j \in \mathbb{N}} \left(B_j \cap Z \right) \,,$$

and for every $j \in \mathbb{N}$, $B_j \cap Z \in \mathscr{B}(X_2 \cap Z)$. Since $\mathscr{B}(X_2 \cap Z)$ is a $\sigma-$field, the countable union also belongs to $\mathscr{B}(X_2 \cap Z)$, proving (C.0.2) for every $Y \in \tau(X_1)$.

Secondly, we implement the method of good sets to prove (C.0.2) for a larger class of subsets of X_1. Let

$$\mathscr{G} = \{A \subset X_1 : A \cap Z \in \mathscr{B}(X_2 \cap Z)\} \,.$$

Claim: $\mathscr{G}$ is a $\sigma-$field.

i) $X_1 \in \mathscr{G}$ since $X_1 \subset X_1$ and $X_1 \in \tau(X_1)$ by the definition of topology.

ii) Let $A \in \mathscr{G}$. We want to show that $A^c := X_1 \setminus A \in \mathscr{G}$, i.e. $A^c \subset X_1$ and $A^c \cap Z \in \mathscr{B}(X_2 \cap Z)$. Since $A \in \mathscr{G}$, $A \subset X_1$ and $A \cap Z \in \mathscr{B}(X_2 \cap Z)$. Clearly $A^c = X_1 \setminus A \subset X_1$.
Since $A \cap Z \in \mathscr{B}(X_2 \cap Z)$, then by the definition of $\sigma-$field

$$^c(A \cap Z) := (X_2 \cap Z) \setminus (A \cap Z) \in \mathscr{B}(X_2 \cap Z).$$

We have the following set relations

$$\begin{aligned} ^c(A \cap Z) &= {}^cA \cup {}^cZ = [(X_2 \cap Z) \setminus A] \cup [(X_2 \cap Z) \setminus Z] \\ &= [(X_2 \setminus A) \cap Z] \cup \emptyset = (X_2 \setminus A) \cap Z \\ &= \left[A^c \cup (X_2 \setminus X_1)\right] \cap Z \\ &= \left(A^c \cap Z\right) \cup [(X_2 \setminus X_1) \cap Z] \\ &= \left(A^c \cap Z\right) \cup {}^cX_1 \,. \end{aligned}$$

Now in the above identity $^c(A \cap Z)$, $^cX_1$ belongs to $\mathscr{B}(X_2 \cap Z)$ and hence $A^c \cap Z \in \mathscr{B}(X_2 \cap Z)$, inferring $A^c \in \mathscr{G}$.

iii) Let $\{A_i\}_{i \in \mathbb{N}} \in \mathscr{G}$. Then $A_i \subset X_1$ for every $i \in \mathbb{N}$ hence

$$\bigcup_{i \in \mathbb{N}} A_i \subset X_1.$$

Also, the following holds

$$\left(\bigcup_{i \in \mathbb{N}} A_i \right) \cap Z = \bigcup_{i \in \mathbb{N}} (A_i \cap Z).$$

Since $A_i \in \mathscr{G}$, $A_i \cap Z \in \mathscr{B}(X_2 \cap Z)$ and as $\mathscr{B}(X_2 \cap Z)$ is a $\sigma-$field

$$\bigcup_{i \in \mathbb{N}} (A_i \cap Z) \in \mathscr{B}(X_2 \cap Z).$$

From $i) - iii)$ we can infer that $\mathscr{G}$ is a $\sigma-$field. We have already shown that $\tau(X_1) \subset \mathscr{G}$ thus

$$\mathscr{B}(X_1) = \sigma(\tau(X_1)) \subset \mathscr{G}.$$

Therefore, we have shown that for every $Y \in \mathscr{B}(X_1)$, $Y \cap Z \in \mathscr{B}(X_2 \cap Z)$. ■

Lemma C.2. *Let X_1, X_2, Y be topological spaces such that $X_1 \subset X_2$, X_1 has trace topology from X_2 and $X_1 \cap Y = X_2 \cap Y$ then*

$$\tau(X_1 \cap Y) = \tau(X_2 \cap Y).$$

Proof. The topologies of $X_1 \cap Y$ and $X_2 \cap Y$ denoted by $\tau(X_1 \cap Y)$ and $\tau(X_2 \cap Y)$ respectively are given by

$$\tau(X_1 \cap Y) = \text{generated by } \{A \cap B : A \in \tau(X_1), B \in \tau(Y)\},$$
$$\tau(X_2 \cap Y) = \text{generated by } \{C \cap B : C \in \tau(X_2), B \in \tau(Y)\}.$$

Since X_1 has a trace topology from X_2, for every $A \in \tau(X_1)$ there exists a $C \in \tau(X_2)$ such that $A = C \cap X_1$. Thus

$$\tau(X_1 \cap Y) = \text{generated by } \{C \cap X_1 \cap B : C \in \tau(X_2), B \in \tau(Y)\}.$$

Thus all we are left to show is $C \cap X_1 \cap B = C \cap B$ for every $C \in \tau(X_2)$ and $B \in \tau(Y)$. Since $X_1 \cap Y = X_2 \cap Y$, we have the following set relations

$$\begin{aligned} C \cap X_1 \cap B &= (C \cap X_1) \cap (Y \cap B) = (C \cap X_1 \cap Y) \cap B \\ &= (C \cap X_2 \cap Y) \cap B = (C \cap X_2) \cap (Y \cap B) = C \cap B. \end{aligned}$$

■

APPENDIX D

CONVERGENCE OF P_n

Here we present various convergence results for the orthogonal projection P_n as given by (6.4.1) (for more details see Section 6.4) as $n \to \infty$.

Lemma D.1. *Let $\gamma > \frac{d}{2}$ and $P_n : \mathrm{H} \to \mathrm{H}_n$ be the orthogonal projection. Then as $n \to \infty$*

(i) $P_n\psi \to \psi$ *in* H *for* $\psi \in \mathrm{H}$,

(ii) $P_n\psi \to \psi$ *in* V *for* $\psi \in \mathrm{V}$,

(iii) $P_n\psi \to \psi$ *in* V_γ *for* $\psi \in \mathrm{V}_\gamma$.

Proof. Let $\psi \in \mathrm{H}$, then by (6.4.1) and Plancherel Theorem, we have

$$|P_n\psi - \psi|_{\mathrm{H}}^2 = \int_{\mathbb{R}^3} |\mathscr{F}(P_n\psi)(\xi) - \hat{\psi}(\xi)|^2\, d\xi = \int_{\mathbb{R}^3} |\mathbb{1}_{B_n}(\xi)\hat{\psi}(\xi) - \hat{\psi}(\xi)|^2\, d\xi = \int_{|\xi|>n} |\hat{\psi}(\xi)|^2\, d\xi.$$

Now since $\psi \in \mathrm{H}$ using Lebesgue dominated convergence theorem it can be shown that

$$\lim_{n\to\infty} \int_{|\xi|>n} |\hat{\psi}(\xi)|^2\, d\xi = 0,$$

which infers (i).

Let $\psi \in \mathrm{V}$, then by (6.4.1) and the definition of $\mathrm{V}-$norm we get

$$\begin{aligned}\|P_n\psi - \psi\|_{\mathrm{V}}^2 &= \int_{\mathbb{R}^3} (1 + |\xi|^2) \left|\mathscr{F}(P_n\psi)(\xi) - \hat{\psi}(\xi)\right|^2 d\xi = \int_{\mathbb{R}^3} (1 + |\xi|^2) \left|\mathbb{1}_{B_n}(\xi)\hat{\psi}(\xi) - \hat{\psi}(\xi)\right|^2 d\xi \\ &= \int_{|\xi|>n} (1 + |\xi|^2) |\hat{\psi}(\xi)|^2\, d\xi.\end{aligned}$$

Again using the Lebesgue dominated convergence theorem and the fact that $\psi \in \mathrm{V}$, we can show that

$$\lim_{n\to\infty} \int_{|\xi|>n} (1 + |\xi|^2) |\hat{\psi}(\xi)|^2\, d\xi = 0,$$

thus proving (ii).

Let $\psi \in \mathrm{V}_\gamma$, then by (6.4.1) and the definition of V_γ−norm we get

$$\begin{aligned}\|P_n\psi - \psi\|^2_{\mathrm{V}_\gamma} &= \int_{\mathbb{R}^3} (1+|\xi|^2)^\gamma \left|\mathscr{F}(P_n\psi)(\xi) - \hat{\psi}(\xi)\right|^2 d\xi = \int_{\mathbb{R}^3} (1+|\xi|^2)^\gamma \left|\mathbb{1}_{B_n}(\xi)\hat{\psi}(\xi) - \hat{\psi}(\xi)\right|^2 d\xi \\ &= \int_{|\xi|>n} (1+|\xi|^2)^\gamma \, |\hat{\psi}(\xi)|^2 \, d\xi.\end{aligned}$$

Similarly as before it can be shown that

$$\lim_{n\to\infty} \int_{|\xi|>n} (1+|\xi|^2)^\gamma \, |\hat{\psi}(\xi)|^2 \, d\xi = 0,$$

as $\psi \in \mathrm{V}_\gamma$, which concludes the proof. ■

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