Luckraz, Shravan (2014) A note on the relationship between the isotone assumption of the Abian-Brown fixed point theorem and Abian's most basic fixed point theorem. Fixed Point Theory and Applications, 2014 (1). p. 129. ISSN 1687-1812

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A note on the relationship between the isotone assumption of the Abian-Brown fixed point theorem and Abian's most basic fixed point theorem

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Abstract

In a recent paper Xie *et al.* (Fixed Point Theory Appl. 2013:192, 2013) gave several extensions and some applications of the Abian-Brown (AB) fixed point theorem. While the AB fixed point theorem and its extensions (as well as other related fixed point theorems) assume that the mapping is isotone, this note shows that for single-valued finite maps this condition relates to the acyclicity of the map, which in turn relates to Abian's (Nieuw Arch. Wiskd. XVI:184-185, 1968) most basic fixed point theorem for finite sets.

MSC: 46B42; 47H10; 58J20; 91A06; 91A10

1 Introduction

Fixed point theorems play an important part in equilibrium analysis in mathematical sciences. Recently, Li (2014) [1], Xie *et al.* (2013) [2] and Xie *et al.* (2013) [3] extended the celebrated Abian-Brown (AB) fixed point theorem and applied it to some well-known problems in game theory. Since the AB theorem and its extensions have powerful applications, it is important to understand and assess the assumptions behind these theorems. This note looks at one crucial assumption, namely that the mapping is isotone and shows that this assumption is related to another fixed point theorem of Abian (1968) [4] for finite sets.

Li's [1] generalized a whole class of fixed point theorems. Indeed, the first result in this class was given by Knaster and Tarski (1928) [5] and extended by Tarski (1955) [6], Abian and Brown (1961) [7], Fujimoto (1984) [8] and Xie *et al.* (2013) [2]. All these results are based on some isotone-like assumption.[a] Although in some cases this assumption is replaced by some order reversing assumption as in Abian (1979) [9], the monotonicity of the mapping is still crucial to obtain a fixed point. In this note, we attempt to assess this assumption more carefully.

While the existence or lack of existence of fixed points in dynamic systems often depend on the topological or set-theoretic order structure of the mapping, Abian (1968) [4] proved a fixed point theorem for finite sets that does not rely on any such structure.[b] This note relates this theorem of Abian with the assumption of isotone mappings. In order to do so, we first show the connection between isotone mappings and acyclic mappings before showing the relationship between fixed points and the cyclicity of the mapping. In fact, we prove the negation of this relationship by showing a link between the lack of existence of a fixed point and the cyclicity of the mapping.

The finite version of Abian's [4] fixed point theorem states that given a function from a finite set into itself, the set cannot be partitioned into three sets so that the intersection of each of these sets with their image is empty iff the mapping has a fixed point.[c] The main result of this note shows that if the mapping has no fixed point, then all elements of the set are either periodic or eventually periodic with an even prime period iff the set can be partitioned into two sets so that the intersection of each of these sets with their image is empty.

In Section 2, we formally show the connection between cycles and the isotone assumption as well as the link between cycles and a no-fixed point condition. In Section 3 we give the main result of the paper showing that Abian's theorem for finite sets is equivalent the cyclicity of the mapping and in Section 4 we conclude.

2 Periodic maps

Throughout this paper we shall only consider finite sets with at least two elements. Let D be a finite set and let $f : D \to D$ be a function.

Definition 1 The element $d \in D$ is said to be periodic if $d \neq f(d)$ and there exists a positive integer $p > 1$ so that $f^p(d) = d$.

Definition 2 $m \in \mathbb{Z}_+$ is called the prime period of $d \in D$ if d is periodic and m is the least $n \in \mathbb{Z}_+$ so that $f^n(d) = d$.

Definition 3 For any $d \in D$, the sequence $\langle f^i(d) \rangle_{i=0}^{\infty}$ (with the convention that $f^0(d) = d$) is said to be the orbit of d.

Definition 4 $d \in D$ is said to be eventually periodic if it not periodic itself, but some term in its orbit is periodic.

The following lemma demonstrates the link between the isotone assumption of the AB theorem and the cyclicity of the mapping for the finite case. In the lemma we will assume that the set is partially ordered and that the mapping is isotone. Moreover, we also assume that each point is comparable to its image through the binary relation. These assumptions and the lemma will not be used anywhere else in the note. Its only purpose is to relate the cyclicity of the map to the isotone assumption used in the AB class of fixed point theorems. At this stage, it is important to draw the attention of the reader on the distinction between the partial order as a binary relation and the mapping as a binary relation. In this note, when we refer to cycles, we refer to the cyclicity of the mapping.[d]

Lemma 1 *Let D be a finite set that is partially ordered by $\preceq$ and let $f : D \to D$ be an isotone mapping, that is, for any $x, y \in D$, $x \preceq y$ implies $f(x) \preceq f(y)$. Moreover, assume that for each $d \in D$ either $d \preceq f(d)$ or $f(d) \preceq d$. Then D does not contain any periodic element.*

Proof We prove by contradiction. Suppose there exists such an element d and consider the orbit of d given by $s = \langle f^i(d) \rangle_{i=0}^{\infty}$. Then s has an initial segment given by $\langle d, f(d), f^2(d), \dots, f^n(d) \rangle$, where $d \neq f(d) \neq f^2(d) \neq \cdots \neq f^{n-1}(d)$ and n is the prime period of d. Since d relates to $f(d)$ through the binary relation, one can assume without loss of generality that $d \preceq f(d)$. Then by the isotone assumption, we have $f(d) \preceq f^2(d)$. Using this argument, it can be shown by induction that $f^i(d) \preceq f^{i+1}(d)$ for $i = 0, 1, 2, \ldots, n-1$, where $f^i(d)$ is a component of s. By the transitivity of $\preceq$ we obtain $d \preceq f(d) \preceq d = f^n(d)$. By the anti-symmetry of $\preceq$, we then obtain $d = f(d)$, which contradicts the fact that d is a periodic point. □

Lemma 2 *Let D be a finite set and let $f: D \to D$ be a mapping that has no fixed point, then there exists some $d \in D$ such that d is periodic.*

Proof We prove by contradiction. We first observe that if there exists no periodic d in D, then no elements of D can be eventually periodic under mapping f. Now, using this observation together the fact that f has no fixed point, we conclude that each $d \in D$ has an orbit of the following form $\langle d, f(d), f^2(d), \ldots, f^n(d), \ldots \rangle$, where $d \neq f(d) \neq f^2(d) \neq \cdots \neq f^n(d)$. Thus, we have $f^n(d) = e$ for some $e \in D$, where $e \neq d \neq f(d) \neq f^2(d) \neq \cdots \neq f^n(d)$. But since this argument holds for any $n \in \mathbb{Z}_+$, $\mathrm{Card}(D)$ cannot be finite, which gives a contradiction. □

Lemma 3 *Let D be a finite set and let $f: D \to D$ be a mapping that has no fixed point. Suppose that $d \in D$ is not a periodic element, then it is eventually periodic.*

Proof We prove by contradiction. Suppose some $d \in D$ is neither periodic nor eventually periodic. Then d has an orbit of the following form: $\langle d, f(d), f^2(d), \ldots, f^n(d), \ldots \rangle$, where $d \neq f(d) \neq f^2(d) \neq \cdots \neq f^n(d)$. Thus, we have $f^n(d) = e$ for some $e \in D$, where $e \neq d \neq f(d) \neq f^2(d) \neq \cdots \neq f^n(d)$. But since this argument holds for any $n \in \mathbb{Z}_+$, $\mathrm{Card}(D)$ cannot be finite, which gives a contradiction. □

Definition 5 We say that orbit s is permutation isomorphic to orbit s' iff s' is an order-preserving (in the natural order) permutation of s.

Lemma 4 *Let D be a finite set and let $f: D \to D$ be a mapping that has no fixed point. Suppose that the mapping has k distinct non-permutation isomorphic orbits. Then there exists a partition of D into k cells such that each cell contains all periodic points of exactly one orbit and the eventually periodic points that lead to that orbit.*

Proof See the Appendix. □

3 Fixed point theorems for finite maps

Theorem 1 (Abian (1968) [4]) *Let D be a finite set and let $f: D \to D$ be a mapping. Then f has a fixed point iff $\nexists A, B$ and C so that collection $\{A, B, C\}$ is a partition of D and $A \cap f(A) = B \cap f(B) = C \cap f(C) = \emptyset$.*

Proof See [4]. □

Theorem 2 *Let D be a finite set and let $f: D \to D$ be a mapping that has no fixed point. Then the following statements are equivalent.*

(i) *D can be partitioned into two sets A and B so that $A \cap f(A) = B \cap f(B) = \emptyset$.*

(ii) *Each $d \in D$ is either a periodic element with an even prime period or an eventually periodic element that leads to a periodic element whose prime period is even.*

Proof See the Appendix. □

Corollary 1 *Let D be a finite set and let $f : D \to D$ be a mapping that has no fixed point. Then the following statements are equivalent.*

(i) *D cannot be partitioned into two sets A and B so that $A \cap f(A) = B \cap f(B) = \emptyset$ but can be partitioned into three sets A, B and C so that collection $\{A, B, C\}$ is a partition of D and $A \cap f(A) = B \cap f(B) = C \cap f(C) = \emptyset$.*

(ii) *Each $d \in D$ is either a periodic element or an eventually periodic element whose prime period is odd.*

The proof is omitted as it is similar to the proof of Theorem 2.

Corollary 2 *Let D be a finite set and let $f : D \to D$ be a mapping. Then f has a fixed point iff D contains at least one element that is neither periodic nor eventually periodic under f.*

Proof The (if) part follows from Lemma 3 and the (only if) part follows from the fact that if x is a fixed point of f, then it is not periodic by Definition 1. □

4 Conclusion

We have shown in Theorem 2 that Abian's theorem relates to the idea of cycles and in Lemma 1 we have demonstrated that for finite maps acyclicity is implied by the isotone assumption as long each element relates to its image via the binary relation. Therefore, we have shown how the isotone assumption relates to the idea of cycles and hence to fixed point theorems through the finite version of Abian's theorem. Corollary 1 has shown that Abian theorem can be broken into two cases: (i) All periodic points have even prime periods and (ii) the negation of (i). Corollary 2 has shown that in order to find the fixed point of a finite set, it suffices to look for an element that it not periodic. These results can help researchers in the mathematical sciences who are trying to understand the assumptions underpinning this class of fixed point theorems and apply them to a broader class of problems.

Appendix

Proof of Lemma 4 Since from Lemma 3 we know that if some $d \in D$ is not periodic, then it is eventually periodic, we need to consider only two types of elements of D; the periodic elements and the eventually periodic elements.

Let d be a periodic element such that $f^n(d) = d$ and let $\langle f^i(d)\rangle_{i=0}^{\infty}$ be its orbit, with the convention that $f^0(d) = d$. First, observe that by Definition 1, for each $i < n$, $f^i(d)$ is also a periodic element. Therefore, $f^m(f^i(d)) = f^i(d)$ for some m. Let $\langle f^k(f^i(d))\rangle_{k=0}^{\infty}$, be the orbit of $f^i(d)$. Then using Definition 1 again, we can infer that $\langle f^k(f^i(d))\rangle_{k=0}^{\infty}$ must be a permutation of $\langle f^i(d)\rangle_{i=0}^{\infty}$. Hence, for each i, the orbit of $f^i(d)$ traverses each term of the orbit of d and *vice versa* (that is, the orbit of d traverses each term of the orbit of $f^i(d)$). We now claim that if $\exists e \in D$ so that $e \neq d$ and e is a periodic element satisfying $f^p(e) = e$ for some p, with orbit $\langle f^i(e)\rangle_{i=0}^{\infty}$, then either $\langle f^i(e)\rangle_{i=0}^{\infty}$ is a permutation of $\langle f^i(d)\rangle_{i=0}^{\infty}$ or the two orbits have no terms in common. To see this, suppose $\langle f^i(e)\rangle_{i=0}^{\infty}$ has a term x in $\langle f^i(d)\rangle_{i=0}^{\infty}$ and also a term y that is not in $\langle f^i(d)\rangle_{i=0}^{\infty}$, then there will exist some i, such that the orbit of $f^i(d)$ does not traverse each term of the orbit of d, which would contradict the previous argument. Therefore if the mapping f induces k orbits on D, then each of these k orbits will have initial segments that can be represented as k disjoint subsets of D. A similar argument can be used to show that if d is an eventually periodic element and some $e \in D$ is either periodic or eventually periodic then either the two orbits have no terms in common or they share all periodic points in common.[e] Hence, each eventually periodic element either leads to the same or to a different orbit. As a result, the k orbits of D in addition to their eventually periodic elements can still be represented as k disjoint subsets of D. But since D contains only periodic and eventually periodic elements, the collection of such subsets will form a partition of D. □

Proof of Theorem 2 We first show that (ii) implies (i). Let d be a periodic point. We first construct a bijection from the set of all terms of an initial segment of the orbit of d into a finite index set. Consider the following initial segment of the orbit of d:

$$\left\langle d,f(d),f^2(d),\dots,f^{n-1}(d)\right\rangle.$$

We know from Lemma 2 that such an element exists. Since $d \neq f(d) \neq f^2(d) \neq \cdots \neq f^{n-1}(d)$, there exists a bijection from $\{d,f(d),f^2(d),\dots,f^{n-1}(d)\}$ to $\{0,1,2,\dots,n-1\}$. Let

$$g:\left\{d,f(d),f^2(d),\dots,f^{n-1}(d)\right\}\to\{0,1,2,\dots,n-1\}$$

be a bijection so that $g(f^i(d)) = i$ for each $i \in \{0,1,2,\dots,n-1\}$. Let collection $\{A,B\}$ be a partition of $\{d,f(d),f^2(d),\dots,f^{n-1}(d)\}$ so that A and B are defined as follows:

$$A=\left\{g^{-1}(1),g^{-1}(3),\dots,g^{-1}(2i+1),\dots\right\},$$
$$B=\left\{g^{-1}(0),g^{-1}(2),\dots,g^{-1}(2i),\dots\right\},$$

where $i \in \{0,1,2,\dots,n-1\}$. Hence, $a \in \{d,f(d),f^2(d),\dots,f^{n-1}(d)\}$ is in A iff $g(a)$ is odd and $b \in \{d,f(d),f^2(d),\dots,f^{n-1}(d)\}$ is in B iff $g(b)$ is even. We now show that if n is even, then $A \cap f(A) = B \cap f(B) = \emptyset$. Note that for any adjacent terms x, y (where y succeeds x in the sequence ordering) in $\langle d,f(d),f^2(d),\dots,f^{n-1}(d)\rangle$, $y = f(x)$. Therefore since the mapping has no fixed point we must have $d \neq f(d) \neq f^2(d) \neq \cdots \neq f^{n-1}(d)$. Thus, for $A \cap f(A) = B \cap f(B) = \emptyset$ to be true, we need the following condition. x, y are adjacent in $\langle d,f(d),f^2(d),\dots,f^{n-1}(d)\rangle$ iff $x \in A$ and $y \in B$ or $y \in A$ and $x \in B$. Therefore, since $g^{-1}(0) = d = f^n(d)$ is in B we only need to make sure that $g^{-1}(n-1) \in A$. But this is true since $n-1$ being odd guarantees that $g^{-1}(n-1) \in A$.

Hence, the set of all elements of an orbit can be partitioned into A and B such that $A \cap f(A) = B \cap f(B) = \emptyset$ as long as n is even.

Now, let us consider the eventually periodic elements. For each eventually periodic element h in D, there exists a periodic element d in D so that h is term of some sequence of the following form:

$$p=\left\langle h,f(h),f^2(h),\dots,f^{k-1}(h),d,f(d),f^2(d),\dots,f^{n-1}(d),f^n(d),\dots\right\rangle,$$

where $d = f^k(h)$ for some k and $f^n(d) = d$ for some $n > k$. Observe that the set of all terms of p can still be partitioned into the same two sets A and B such that $A \cap f(A) = B \cap f(B) = \emptyset$ as long for any adjacent terms x, y in p, either $x \in A$ and $y \in B$ or $y \in A$ and $x \in B$. Since h is eventually periodic we have $h \neq f(h) \neq f^2(h) \neq \cdots \neq f^{k-1}(h) \neq d$. Therefore, as long as n is even the same sets A and B that were chosen to partition the set of all terms of the orbit of d can be used to partition the set of all terms of p.

Now from Lemma 3, we know that all elements of D under mapping f are either periodic or eventually periodic and from Lemma 4 we know that D can be partitioned in such a way that each cell contains all the periodic and eventually periodic elements of an orbit. Therefore if each orbit (together with its eventually periodic elements) can be partitioned into two sets satisfying the desired conditions, the whole set D can also be partitioned into two sets satisfying the desired condition. From the previous arguments we know that this is possible if n is even.

We now show that (i) implies (ii). We prove by contradiction. Suppose D can be partitioned into two sets A and B so that $A \cap f(A) = B \cap f(B) = \emptyset$ and suppose that the prime period of some periodic element d is odd.[f] Then consider the following initial segment of the orbit of d,

$$\langle d, f(d), f^2(d), \ldots, f^{n-1}(d), f^n(d) \rangle.$$

Note that for any adjacent terms x, y in $\langle d, f(d), f^2(d), \ldots, f^{n-1}(d), f^n(d)\rangle$, $y = f(x)$. Therefore either $x \in A$ and $y \in B$ or $y \in A$ and $x \in B$. Now since $f^n(d) = d$, d and $f^{n-1}(d)$ cannot be in the same set. Also, $f^{j-1}(d)$ and $f^j(d)$ cannot be in the same set for all $j < n$. Hence, d and $f^{n-1}(d)$ can be separated iff $(n-1) + 1$ is even, which implies that n cannot be odd. □

Competing interests

The author declares that he has no competing interests.

Acknowledgements

The author would like to thank Jeff Kline for introducing him to this topic.

Endnotes

[a] For set-valued maps, the corresponding assumption is that the map is order-increasing upward.

[b] In fact, Abian (1968) [10] also gave the necessary and sufficient conditions for a fixed point to exist without any finiteness or countability assumption. In this note we only use the finite version of Abian's theorem to demonstrate how the cyclicity of the map relates to the isotone assumption.

[c] While it may not always be easy to verify the latter condition, one question that one might ask is under what conditions would two sets be enough to guarantee no fixed point. This note also gives an answer to this question.

[d] It can also be verified that extending our definition of cycles to the partial order, the binary relation will be acyclical.

[e] If the orbit of d reached some periodic point d' which is not encountered by the orbit of e and yet the orbits of d and e shared some point x in common, then since d' is reachable from x, by the definition of a function d' must be encountered by the orbit of e, which will lead to a contradiction.

[f] From Lemma 3 we know that since the mapping has no fixed point, each element of d is either periodic or eventually periodic. Therefore we only need to prove that n is even.

Received: 7 February 2014 Accepted: 22 May 2014 Published: 29 May 2014

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10.1186/1687-1812-2014-129
Cite this article as: Luckraz: A note on the relationship between the isotone assumption of the Abian-Brown fixed point theorem and Abian's most basic fixed point theorem. *Fixed Point Theory and Applications* 2014, 2014:129