NONLINEAR STUDIES - www.nonlinearstudies.com
Vol. 25, No. 1, pp. 173-189, 2018

A mathematically derived definitional/ semantical theory of truth

S. Heikkilä [1⋆]

Department of Mathematical Sciences, University of Oulu
BOX 3000, FIN-90014, Oulu, Finland

⋆ *Corresponding Author*. E-mail: sheikki@cc.oulu.fi

Abstract. Ordinary and transfinite recursion and induction and ZF set theory are used to construct from a fully interpreted object language and from an extra formula a new language. It is fully interpreted under a suitably defined interpretation. This interpretation is equivalent to the interpretation by meanings of sentences if the object language is so interpreted. The added formula provides a truth predicate for the constructed language. The so obtained theory of truth satisfies the norms presented in Hannes Leitgeb's paper 'What Theories of Truth Should be Like (but Cannot be)'.

1 Introduction

Theories of truth are presented for languages. Based on 'Chomsky Definition' (cf. [1]) a language is assumed to be a countably infinite set of well-formed sentences, each of finite length, and constructed out of a finite or a countably infinite set of symbols.

(i) A theory of syntax of a language is formed by symbols, and rules to construct well-formed sentences. Symbols consist of letters, parentheses, commas, dots, constants containing natural numbers, terms containing numerals, and logical symbols $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\rightarrow$ (implies), $\leftrightarrow$ (if and only if), $\forall$ (for all) and $\exists$ (exist). If A and B are (denote) sentences, so are $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \leftrightarrow B$. If $P(x)$ is a formula of a language, and X_P is a set of terms, then P is called a predicate with domain X_P if $P(x)$ is a sentence of that language for each assignment of a term of X_P into x (shortly, for each $x \in X_P$). $\forall x P(x)$ and $\exists x P(x)$ are then sentences of the language. If P has several free variables $x_1, \dots, x_m$, then $P(x_1, \dots, x_m)$ is denoted by $P(x)$, and the sentences $\forall x P(x)$ and $\exists x P(x)$ stand for universal and existential closures $\forall \cdots \forall P(x_1, \dots, x_m)$ and $\exists \cdots \exists P(x_1, \dots, x_m)$. Symbols may contain a lexicon of a first-order predicate logic (cf. [11, Definition II.5.2]).

An interpretation of sentences of a language is also needed.

(ii) A language is called fully interpreted, if every sentence is interpreted either as true or as false, and if the interpretation of those sentences which contain logical symbols satisfy following rules of

[2010] **Mathematics Subject Classification**: 03B10, 03D80, 47H04, 47H10, 97M80
Keywords: language, sentence, meaning, interpretation, Gödel number, truth.

classical logic when A and B denote sentences of the language ('iff' means 'if and only if'): A is true iff $\neg A$ is false, and A is false iff $\neg A$ is true; $A \vee B$ is true iff A or B is true, and false iff A and B are false; $A \wedge B$ is true iff A and B are true, and false iff A or B is false; $A \rightarrow B$ is true iff A is false or B is true, and false iff A is true and B is false; $A \leftrightarrow B$ is true iff A and B are both true or both false, and false iff A is true and B is false or A is false and B is true. If P is a predicate with domain X_P, then $\forall xP(x)$ is true iff $P(x)$ is true for every $x \in X_P$, and false iff $P(x)$ is false for some $x \in X_P$; $\exists xP(x)$ is true iff $P(x)$ is true for some $x \in X_P$, and false iff $P(x)$ is false for every $x \in X_P$.

Any first-order formal language equipped with a consistent theory interpreted by a countable model, and containing natural numbers and numerals, is a fully interpreted language in the above sense. A classical example is the language of arithmetic with its standard interpretation. Another example is the first-order language of set theory, the interpretation being determined by the minimal model constructed in [2] for ZF set theory.

'Truth should be expressed by a predicate' is the first requirement presented in [12] for theories of truth. Many languages, e.g., the above mentioned, don't have such a predicate. Therefore we construct from such a language L, from its predicates, and from sentences induced by an additional formula $T(x)$ a new language $\mathcal{L}$, and choose a fixed Gödel numbering to its sentences. To each proper subset U of the set D of those Gödel numbers we construct other subsets $G(U)$ and $F(U)$ of D.

In Appendix it is shown that there is the smallest subset U of D which is consistent, i.e., for no sentence A of $\mathcal{L}$ the Gödel numbers of both A and $\neg A$ are in U, and for which $U = G(U)$. The sentences of $\mathcal{L}$ whose Gödel numbers are in $G(U)$ or in $F(U)$ and the symbols of L form a language $\mathcal{L}^0$ that contains L. A sentence of $\mathcal{L}^0$ is interpreted as true iff its Gödel number is in $G(U)$, and as false iff its Gödel number is in $F(U)$. In this interpretation $\mathcal{L}^0$ is shown to be fully interpreted, if the object language L is fully interpreted. If L is fully interpreted by meanings of its sentences, so is $\mathcal{L}^0$, and this interpretation is proved to be equivalent to that defined above. Moreover, the interpretations of L and $\mathcal{L}^0$ are compatible in L, and T is a truth predicate of $\mathcal{L}^0$ when its domain X_T is the set of numerals of Gödel numbers of all sentences of $\mathcal{L}^0$. This provides a theory of truth for $\mathcal{L}^0$. That theory is shown to satisfy all the norms presented in [12] for truth theories.

Ordinary and transfinite recursions and inductions and ZF set theory for sets of natural numbers are main tools in constructions and proofs. As for these tools see, e.g., [11].

2 Recursive constructions

Let L be a fully interpreted language without a truth predicate. Construct a language $\mathcal{L}_0$ as follows: Its base language is formed by L, an extra formula $T(x)$ and its assignments when x goes through all numerals. Fix a Gödel numbering to the base language. The Gödel number of a sentence (denoted by) A is denoted by $\#A$, and the numeral of $\#A$ by $\lceil A \rceil$. If P is a predicate of L with domain X_P, then $P(x)$ is a sentence of L for each $x \in X_P$, and $\lceil P(x) \rceil$ is the numeral of its Gödel number. Thus $T(\lceil P(x) \rceil)$ is a sentence of $\mathcal{L}_0$ for each $x \in X_P$, whence $T(\lceil P(\cdot) \rceil)$ is a predicate of $\mathcal{L}_0$ with domain X_P. The construction of $\mathcal{L}_0$ is completed by adding to it sentences $\forall xT(x)$, $\exists xT(x)$, $\forall xT(\lceil T(x) \rceil)$ and $\exists xT(\lceil T(x) \rceil)$, and sentences $\forall xT(\lceil P(x) \rceil)$ and $\exists xT(\lceil P(x) \rceil)$ for every predicate P of L.

When a language $\mathcal{L}_n$, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, is defined, let $\mathcal{L}_{n+1}$ be the language which is formed by the sentences A, $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \leftrightarrow B$, where A and B go through all sentences of $\mathcal{L}_n$. The language $\mathcal{L}$ is defined as the union of languages $\mathcal{L}_n$, $n \in \mathbb{N}_0$. Extend the Gödel numbering of the base language to $\mathcal{L}$, and denote by D the set of those Gödel numbers. If A is a sentence of $\mathcal{L}$, denote by '$x = \lceil A \rceil$' the assignment of the numeral $\lceil A \rceil$ of the Gödel number $\#A$ of A into x.

Denote by $\mathcal{P}$ the set of all predicates of L. Divide $\mathcal{P}$ into three disjoint subsets.

$$\begin{cases} \mathcal{P}_1 = \{P \in \mathcal{P} : P(x) \text{ is a true sentence of } L \text{ for every } x \in X_P\}, \\ \mathcal{P}_2 = \{P \in \mathcal{P} : P(x) \text{ is a false sentence of } L \text{ for every } x \in X_P\}, \\ \mathcal{P}_3 = \{P \in \mathcal{P} : P(x) \text{ is a true sentence of } L \text{ for some but not for all } x \in X_P\}. \end{cases} \tag{2.1}$$

Given a proper subset U of D, construct new subsets $G(U)$ and $F(U)$ of D as follows. Define

$$\begin{cases} D_1(U) = \{\#T(x) \colon x = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#A \text{ is in } U\}, \\ D_2(U) = \{\#[\neg T(x)] \colon x = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#[\neg A] \text{ is in } U\}, \\ D_1 = \{\#[\neg \forall x T(x)], \#[\exists x T(x)], \#[\neg(\forall x T(\lceil T(x) \rceil))], \#[\exists x T(\lceil T(x) \rceil)]\}, \\ D_2 = \{\#[\forall x T(\lceil P(x) \rceil)], \#[\exists x T(\lceil P(x) \rceil)] : P \in \mathcal{P}_1\}, \\ D_3 = \{\#[\neg(\forall x T(\lceil P(x) \rceil))], \#[\neg(\exists x T(\lceil P(x) \rceil))] : P \in \mathcal{P}_2\}, \\ D_4 = \{\#[\neg(\forall x T(\lceil P(x) \rceil))], \#[\exists x T(\lceil P(x) \rceil)] : P \in \mathcal{P}_3\}. \end{cases} \tag{2.2}$$

Subsets $G_n(U)$, $n \in \mathbb{N}_0$, of D are defined recursively as follows.

$$G_0(U) = \begin{cases} W = \{\#A : A \text{ is a true sentence of } L\} \text{ if } U = \emptyset \text{ (the empty set)}, \\ W \cup D_1(U) \cup D_2(U) \cup D_1 \cup D_2 \cup D_3 \cup D_4 \text{ if } \emptyset \subset U \subset D. \end{cases} \tag{2.3}$$

Let A and B denote sentences of $\mathcal{L}$. When $n \in \mathbb{N}_0$, and $G_n(U)$ is defined, define

$$\begin{cases} G_n^0(U) = \{\#[\neg(\neg A)] : \#A \text{ is in } G_n(U)\}, \\ G_n^1(U) = \{\#[A \vee B] : \#A \text{ or } \#B \text{ is in } G_n(U)\}, \\ G_n^2(U) = \{\#[A \wedge B] : \#A \text{ and } \#B \text{ are in } G_n(U)\}, \\ G_n^3(U) = \{\#[A \to B] : \#[\neg A] \text{ or } \#B \text{ is in } G_n(U)\}, \\ G_n^4(U) = \{\#[A \leftrightarrow B] : \text{both } \#A \text{ and } \#B \text{ or both } \#[\neg A] \text{ and } \#[\neg B] \text{ are in } G_n(U)\}, \\ G_n^5(U) = \{\#[\neg(A \vee B)] : \#[\neg A] \text{ and } \#[\neg B] \text{ are in } G_n(U)\}, \\ G_n^6(U) = \{\#[\neg(A \wedge B)] : \#[\neg A] \text{ or } \#[\neg B] \text{ is in } G_n(U)\}, \\ G_n^7(U) = \{\#[\neg(A \to B)] : \#A \text{ and } \#[\neg B] \text{ are in } G_n(U)\}, \\ G_n^8(U) = \{\#[\neg(A \leftrightarrow B)] : \#A \text{ and } \#[\neg B], \text{ or } \#[\neg A] \text{ and } \#B \text{ are in } G_n(U)\}, \end{cases} \tag{2.4}$$

and

$$G_{n+1}(U) = G_n(U) \cup \bigcup_{k=0}^{8} G_n^k(U). \tag{2.5}$$

The above constructions imply that $G_n(U) \subseteq G_{n+1}(U) \subset D$ and $G_n^k(U) \subseteq G_{n+1}^k(U)$ for all $n \in \mathbb{N}_0$ and $k = 0, \ldots, 8$. Define subsets $G(U)$ and $F(U)$ of D by

$$G(U) = \bigcup_{n=0}^{\infty} G_n(U), \quad \text{and } F(U) = \{\#A : \#[\neg A] \in G(U)\}. \tag{2.6}$$

3 A language, its interpretations and their properties

Let L, $\mathcal{L}$ and D be as in Section 2, and let $G(U)$ and $F(U)$, $U \subset D$, be defined by (2.6). Recall that a subset U of D is consistent if there is no sentence A in $\mathcal{L}$ such that both $\#A$ and $\#[\neg A]$ are in U. The existence of the smallest consistent subset U of D which satisfies $U = G(U)$ is proved in Appendix by a transfinite recursion method.

Definition 3.1. Let U be the smallest consistent subset of D which satisfies $U = G(U)$. Denote by $\mathcal{L}^0$ a language which is the set of all those sentences of $\mathcal{L}$ whose Gödel numbers are in $G(U)$ or in $F(U)$.

(I) A theory of syntax for $\mathcal{L}^0$ is determined by that of the object language L, by an extra formula $T(x)$ and its assignments added to L to construct a base language, and by construction of sentences of $\mathcal{L}^0$ using ordinary and transfinite recursion methods.

3.1 Definitional interpretation

An interpretation to sentences of $\mathcal{L}^0$ is defined as follows.

(II) A sentence of $\mathcal{L}^0$ is interpreted as true iff its Gödel number is in $G(U)$, and as false iff its Gödel number is in $F(U)$.

Lemma 3.1. Let the language $\mathcal{L}^0$ be defined by Definition 3.1 and interpreted by (II). Then a sentence of L is true (resp. false) in the interpretation of L iff it is true (resp. false) in the interpretation (II).

Proof. Let A denote a sentence of L. A is true in the interpretation (II) iff $\#A$ is in $G(U)$ iff (by the construction of $G(U)$) $\#A$ is in W iff A is true in the interpretation of L. A is false in the interpretation (II) iff $\#A$ is in $F(U)$ iff (by (2.6)) $\#[\neg A]$ is in $G(U)$ iff ($\neg A$ is a sentence of L) $\#[\neg A]$ is in W iff $\neg A$ is true in the interpretation of L iff (by negation rule) A is false in the interpretation of L.

Proposition 3.1. Let $\mathcal{L}^0$ be defined by Definition 3.1. Then the interpretation rules given in (ii) of Introduction are satisfied in the interpretation (II). T is a predicate of $\mathcal{L}^0$ when its domain is defined by

$$X_T = \{x : x = \ulcorner A \urcorner, \text{ where } A \text{ is a sentence of } \mathcal{L}^0\}. \tag{3.1}$$

Proof. We shall first derive the following auxiliary rule.

(t0) If A is a sentence of $\mathcal{L}^0$, then $\neg(\neg A)$ is true iff A is true.

To prove (t0), assume first that $\neg(\neg A)$ is true. Then $\#[\neg(\neg A)]$ is in $G(U)$. By (2.6) $\#[\neg(\neg A)]$ is in $G_n(U)$ for some $n \in \mathbb{N}_0$. If $\#[\neg(\neg A)]$ is in $G_0(U)$ then $\#[\neg(\neg A)]$ is in W, so that sentence $\neg(\neg A)$ is true in the interpretation of L. This holds by the negation rule iff $\neg A$ is false in the interpretation of L iff A is true in the interpretation of L. Thus $\#A$ is in $W \subset G(U)$, whence A is true. Assume next that the least of those n for which $\#[\neg(\neg A)]$ is in $G_n(U)$ is > 0. The definition of $G_n(U)$ implies that if $\#[\neg(\neg A)]$ is in $G_n(U)$, then $\#[\neg(\neg A)]$ is in $G^0_{n-1}(U)$, so that $\#A$ is in $G_{n-1}(U)$, and hence in $G(U)$, i.e., A is true.

Thus A is true if $\neg(\neg A)$ is true.

Conversely, assume that A is true. Then $\#A$ is in $G(U)$, so that $\#A$ is in $G_n(U)$ for some $n \in \mathbb{N}_0$. Thus $\#[\neg(\neg A)]$ is in $G^0_n(U)$, and hence in $G_{n+1}(U)$. Consequently, $\#[\neg(\neg A)]$ is in $G(U)$, whence $\neg(\neg A)$ is true. This concludes the proof of (t0).

Rule (t0) is applied next to prove

(t1) Negation rule: A is true iff $\neg A$ is false, and A is false iff $\neg A$ is true.

Let A be a sentence of $\mathcal{L}^0$. Then A is true iff (by (t0)) $\neg(\neg A)$ is true iff #[$\neg(\neg A)$] is in $G(U)$ iff (by (2.6)) #[$\neg A$] is in $F(U)$ iff $\neg A$ is false.

A is false iff #A is in $F(U)$ iff (by (2.6)) #[$\neg A$] is in $G(U)$ iff $\neg A$ is true. Thus (t1) is satisfied.

Next we shall prove

(t2) Disjunction rule: $A \vee B$ is true iff A or B is true. $A \vee B$ false iff A and B are false.

Let A and B be sentences of $\mathcal{L}^0$. If A or B is true, i.e., #A or #B is in $G(U)$, there is by (2.6) an $n \in \mathbb{N}_0$ such that #A or #B is in $G_n(U)$. Thus #[$A \vee B$] is in $G_n^1(U)$, and hence in $G(U)$, so that $A \vee B$ is true.

Conversely, assume that $A \vee B$ is true, or equivalently, #[$A \vee B$] is in $G(U)$. Then there is by (2.6) an $n \in \mathbb{N}_0$ such that #[$A \vee B$] is in $G_n(U)$. Assume first that $n = 0$. If #[$A \vee B$] is in $G_0(U)$, it is in W. Thus $A \vee B$ is true in the interpretation of L. Because L is fully interpreted, then A or B is true in the interpretation of L, and hence also in the interpretation (II) by Lemma 3.1.

Assume next that the least of those n for which #[$A \vee B$] is in $G_n(U)$ is > 0. Then #[$A \vee B$] is in $G_{n-1}^1(U)$, so that #A or #B is in $G_{n-1}(U)$, and hence in $G(U)$, i.e., A or B is true.

Consequently, $A \vee B$ is true iff A or B is true.

It follows from (2.6) that

(a) #[$A \vee B$] is in $F(U)$ iff #[$\neg(A \vee B)$] is in $G(U)$.

If #[$\neg(A \vee B)$] is in $G(U)$, there is by (2.6) an $n \in \mathbb{N}_0$ such that #[$\neg(A \vee B)$] is in $G_n(U)$. Assume that $n = 0$. If #[$\neg(A \vee B)$] is in $G_0(U)$, it is in W. Then $\neg(A \vee B)$ is true and $A \vee B$ is false in the interpretation of L, so that A and B are false and $\neg A$ and $\neg B$ are true in the interpretation of L, i.e., #[$\neg A$] and #[$\neg B$] are in W, and hence in $G(U)$.

Assume next that the least of those n for which #[$\neg(A \vee B)$] is in $G_n(U)$ is > 0. Then #[$\neg(A \vee B)$] is in $G_{n-1}^5(U)$, so that #[$\neg A$] and #[$\neg B$] are in $G_{n-1}(U)$, and hence in $G(U)$. Thus, #[$\neg A$] and #[$\neg B$] are in $G(U)$ if #[$\neg(A \vee B)$] is in $G(U)$.

Conversely, if #[$\neg A$] and #[$\neg B$] are in $G(U)$, there exist by (2.6) $n_1, n_2 \in \mathbb{N}_0$ such that #[$\neg A$] is in $G_{n_1}(U)$ and #[$\neg B$] is in $G_{n_2}(U)$. Denoting $n = \max\{n_1, n_2\}$, then both #[$\neg A$] and #[$\neg B$] are in $G_n(U)$. This result and the definition of $G_n^5(U)$ imply that #[$\neg(A \vee B)$] is in $G_n^5(U)$, and hence in $G(U)$. Consequently,

(b) #[$\neg(A \vee B)$] is in $G(U)$ iff #[$\neg A$] and #[$\neg B$] are in $G(U)$ iff (by (2.6)) #A and #B are in $F(U)$.

Thus, by (a) and (b), #[$A \vee B$] is in $F(U)$ iff #A and #B are in $F(U)$. But this means that $A \vee B$ is false iff A and B are false. This concludes the proof of (t2).

The proofs of the following rules are similar to the above proof of (t2).

(t3) Conjunction rule: $A \wedge B$ is true iff A and B are true. $A \wedge B$ is false iff A or B is false.

(t4) Implication rule: $A \to B$ is true iff A is false or B is true. $A \to B$ is false iff A is true and B is false.

(t5) Biconditionality rule: $A \leftrightarrow B$ is true iff A and B are both true or both false. $A \leftrightarrow B$ is false iff A is true and B is false or A is false and B is true.

Interpretation (II), the fact that $U = G(U)$ and the definition (3.1) of X_T imply that $T(x)$ is for each $x \in X_T$ a sentence, true or false, of $\mathcal{L}^0$. Because U is nonempty, then #[$\exists x T(x)$] is in $G_0(U)$ by (2.2) and (2.3), and hence in $G(U)$ by (2.6). Thus $\exists x T(x)$ is by (II) a true sentence of $\mathcal{L}^0$. If #A_1 is in $G(U) = U$, then #$T(x)$ is by (2.2) in $D_1(U)$, and hence in $G(U)$ when $x = \lceil A_1 \rceil$. This result, (3.1) and (II) imply that $T(x)$ is true when $x = \lceil A_1 \rceil$, and hence for some $x \in X_T$.

Since $\#[\exists x T(x)]$ is in $G(U) = U$, and since U is consistent, then $\#[\neg\exists x T(x)]$ is not in $U = G(U)$. This implies by (2.6) that $\#[\exists x T(x)]$ is not in $F(U)$, i.e., by (II), $\exists x T(x)$ is not false. As shown above, $T(x)$ is true for $x = \ulcorner A_1 \urcorner$. Since $\# A_1$ is in U which is consistent, then $\#[\neg A_1]$ is not in U. Thus it follows from (2.2) that $\#[\neg T(x)]$ is not in $D_2(U)$, and hence not in $G(U)$ for $x = \ulcorner A_1 \urcorner$. The above results mean by (2.6) and (II) that $T(x)$ is not false when $x = \ulcorner A_1 \urcorner$, and hence not false for every $x \in X_T$.

The above proof shows that T satisfies the following rule.

(t6) $\exists x T(x)$ is true iff $T(x)$ is true for some $x \in X_T$. $\exists x T(x)$ is false iff $T(x)$ is false for every $x \in X_T$.

$\forall x T(x)$ is false, because $\#[\neg\forall x T(x)]$ is in $G_0(U)$, and hence in $G(U)$, so that $\#[\forall x T(x)]$ is in $F(U)$ by (2.6). If $\#[\neg A_2]$ is in $G(U) = U$, then $\#[\neg T(x)]$ is by (2.2) in $D_2(U)$ when $x = \ulcorner A_2 \urcorner$. Thus $\#[\neg T(x)]$ is in $G(U) = U$, so that $\neg T(x)$ is true, and hence $T(x)$ is false when $x = \ulcorner A_2 \urcorner$, and hence is false for some $x \in X_T$.

Because $\#[\neg\forall x T(x)]$ is in $G(U) = U$ and since U is consistent, then $\#[\forall x T(x)]$ is not in $U = G(U)$, whence $\forall x T(x)$ is not true.

It is shown above that $T(x)$ is false when $x = \ulcorner A_2 \urcorner$. Because $\#[\neg A_2]$ is in U and U is consistent, then $\# A_2$ is not in U. This implies that $\# T(x)$ is not in $D_1(U)$, and hence not in $G(U) = U$, i.e., $T(x)$ is not true when $x = \ulcorner A_2 \urcorner$, and hence not true for every $x \in X_T$.

The above results imply that T satisfies the following rule.

(t7) $\forall x T(x)$ is true iff $T(x)$ is true for every $x \in X_T$. $\forall x T(x)$ is false iff $T(x)$ is false for some $x \in X_T$.

It remains to show that the following properties hold.

(tt6) $\exists x T(\ulcorner T(x) \urcorner)$ is true iff $T(\ulcorner T(x) \urcorner)$ is true for some $x \in X_T$. $\exists x T(\ulcorner T(x) \urcorner)$ is false iff $T(\ulcorner T(x) \urcorner)$ is false for every $x \in X_T$;

(tt7) $\forall x T(\ulcorner T(x) \urcorner)$ is true iff $T(\ulcorner T(x) \urcorner)$ is true for every $x \in X_T$. $\forall x T(\ulcorner T(x) \urcorner)$ is false iff $T(\ulcorner T(x) \urcorner)$ is false for some $x \in X_T$;

and if P is a predicate of L with domain X_P, then

(tp6) $\exists x T(\ulcorner P(x) \urcorner)$ is true iff $T(\ulcorner P(x) \urcorner)$ is true for some $x \in X_P$. $\exists x T(\ulcorner P(x) \urcorner)$ false iff $T(\ulcorner P(x) \urcorner)$ is false for every $x \in X_P$;

(tp7) $\forall x T(\ulcorner P(x) \urcorner)$ is true iff $T(\ulcorner P(x) \urcorner)$ is true for every $x \in X_P$. $\forall x T(\ulcorner P(x) \urcorner)$ false iff $T(\ulcorner P(x) \urcorner)$ is false for some $x \in X_P$.

To begin with properties (tp6) and (tp7), consider first the case when $P \in \mathcal{P}_1$. Then $P(x)$ is a true sentence of L for every $x \in X_P$. Since U is nonempty, then $\#[\exists x T(\ulcorner P(x) \urcorner)]$ and $\#[\forall x T(\ulcorner P(x) \urcorner)]$ are in $G_0(U)$ by (2.2) and (2.3), and hence in $G(U)$. Thus $\exists x T(\ulcorner P(x) \urcorner)$ and $\forall x T(\ulcorner P(x) \urcorner)$ are by (II) true sentences of $\mathcal{L}^0$.

As shown above $\#[\exists x T(\ulcorner P(x) \urcorner)]$ and $\#[\forall x T(\ulcorner P(x) \urcorner)]$ are in $G(U) = U$. Because U is consistent, then $\#[\neg(\exists x T(\ulcorner P(x) \urcorner))]$ and $\#[\neg(\forall x T(\ulcorner P(x) \urcorner))]$ are not in $U = G(U)$. This implies by (2.6) that $\#[\exists x T(\ulcorner P(x) \urcorner)]$ and $\#[\forall x T(\ulcorner P(x) \urcorner)]$ are not in $F(U)$, and hence, by (II), $\exists x T(\ulcorner P(x) \urcorner)$ and $\forall x T(\ulcorner P(x) \urcorner)$ are not false.

Since every true sentence of L is by Lemma 3.1 a true sentence of $\mathcal{L}^0$, then, by the interpretation (II), $\# P(x)$ is in $G(U) = U$ for every $x \in X_P$. Thus $\# T(\ulcorner P(x) \urcorner)$ is by (2.2) in $D_1(U)$, and hence in $G(U)$, for every $x \in X_P$. Consequently, by (II), $T(\ulcorner P(x) \urcorner)$ is true for every $x \in X_P$, and hence also for some $x \in X_P$.

As noticed above, $\# P(x)$ is in $G(U) = U$ for every $x \in X_P$. Since U is consistent, then $\#[\neg P(x)]$ is not in U for any $x \in X_P$. This result and (2.2) imply that $\#[\neg T(\ulcorner P(x) \urcorner)]$ is not in $D_2(U)$, and hence not in $G(U)$ for any $x \in X_P$. Thus $\#[T(\lceil P(x) \rceil)]$ is not in $F(U)$ for any $x \in X_P$. This means by (II) that $T(\lceil P(x) \rceil)$ is not false for any $x \in X_P$. As shown above, $T(\lceil P(x) \rceil)$ is true for every $x \in X_P$.

The above results imply that P has properties (tp6) and (tp7) when P is in $\mathcal{P}_1$. The proof in the case when P is in $\mathcal{P}_2$ is similar when true is replaced by false and vice versa, and sentences with $\neg$ are replaced by those without $\neg$, and vice versa.

Assume next that P is in $\mathcal{P}_3$. Then $P(x)$ is a true sentence of L for some $x \in X_P$, say $x \in X_P^1$, and a false sentence of L for $x \in X_P^2 = X_P \setminus X_P^1$. Since U is nonempty, then $\#[\exists x T(\lceil P(x) \rceil)]$ and $\#[\neg(\forall x T(\lceil P(x) \rceil))]$ are in D_4 by (2.2), and hence in $G(U)$ by (2.3) and (2.6). Thus $\exists x T(\lceil P(x) \rceil)$ and $\neg(\forall x T(\lceil P(x) \rceil))$ are by (II) true sentences of $\mathcal{L}^0$.

As shown above $\#[\exists x T(\lceil P(x) \rceil)]$ and $\#[\neg(\forall x T(\lceil P(x) \rceil))]$ are in $G(U) = U$. Because U is consistent, then $\#[\neg(\exists x T(\lceil P(x) \rceil))]$ and $\#[\forall x T(\lceil P(x) \rceil)]$ are not in $U = G(U)$. This implies by (2.6) and (t0) that $\#[\exists x T(\lceil P(x) \rceil)]$ and $\#[\neg(\forall x T(\lceil P(x) \rceil))]$ are not in $F(U)$, and hence, by (II), $\exists x T(\lceil P(x) \rceil)$ and $\neg(\forall x T(\lceil P(x) \rceil))$ are not false.

Since every true sentence of L is by Lemma 3.1 a true sentence of $\mathcal{L}^0$ and every false sentence of L is a false sentence of $\mathcal{L}^0$, then, by the interpretation (II), $\#P(x)$ is in $G(U) = U$ for every $x \in X_P^1$, and in $F(U)$ for every $x \in X_P^2$. In the latter case $\#\neg P(x)$ is in $G(U)$ by (2.6). But then, by (2.2), $\#T(\lceil P(x) \rceil)$ is in $D_1(U)$, and hence in $G(U)$, for every $x \in X_P^1$, and $\#\neg T(\lceil P(x) \rceil)$ is in $D_2(U)$, and hence in $G(U)$, for every $x \in X_P^2$. Thus, by (II), $T(\lceil P(x) \rceil)$ is true for every $x \in X_P^1$, and $\neg T(\lceil P(x) \rceil)$ is true for every $x \in X_P^2$.

As noticed above, $\#P(x)$ is in $G(U) = U$ for every $x \in X_P^1$, and $\#[\neg P(x)]$ is in $G(U) = U$ for every $x \in X_P^2$. Since U is consistent, then $\#[\neg P(x)]$ is not in U for any $x \in X_P^1$ and $\#P(x)$ is not in U for any $x \in X_P^2$. This result and (2.2) imply that $\#[\neg T(\lceil P(x) \rceil)]$ is not in $D_2(U)$, and hence not in $G(U)$ for any $x \in X_P^1$, and $\#[T(\lceil P(x) \rceil)]$ is not in $D_1(U)$, and hence not in $G(U)$ for any $x \in X_P^2$. Thus $\#[T(\lceil P(x) \rceil)]$ is not in $F(U)$ for any $x \in X_P^1$ and $\#[\neg T(\lceil P(x) \rceil)]$ is not in $F(U)$ for any $x \in X_P^2$. This means by (II) that $T(\lceil P(x) \rceil)$ is not false for any $x \in X_P^1$, and $\neg T(\lceil P(x) \rceil)$ is not false for any $x \in X_P^2$.

Consequently, properties (tp6) and (tp7) hold also also when P is in $\mathcal{P}_3$.

The proof of properties (tt6) and (tt7) is similar to that given above for properties (tp6) and (tp7) in the case when P is in $\mathcal{P}_3$.

The above results imply that the interpretation rules given in (ii) of Introduction are satisfied in the interpretation (II) of $\mathcal{L}^0$.

We shall derive some properties for the sets $G(U)$ and $F(U)$ defined by (2.6) when U is a nonempty and consistent subset of D.

Lemma 3.2. Let U be nonempty and consistent. Then $G(U)$ and $F(U)$ are disjoint and consistent.

Proof. If $\#A$ is in W, then A is by (2.3) a true sentence of L. Because L is fully interpreted, then $\neg A$ is not true. Thus $\#[\neg A]$ is not in W, and hence not in $G(U)$. This implies by (2.6) that $\#A$ is not in $F(U)$, and hence not in $G_0(U) \cap F(U)$.

Let x be a numeral. If $\#T(x)$ is in $G_0(U)$, it is in $D_1(U)$, so that, by (2.2), $x = \lceil A \rceil$, where $\#A$ is in U. Because U is consistent, then $\#[\neg A]$ is not in U. Thus, by (2.2), $\#[\neg T(x)]$ is not in $D_2(U)$, and hence not in $G(U)$ when $x = \lceil A \rceil$. This implies by (2.6) that $\#T(x)$ is not in $F(U)$ when $x = \lceil A \rceil$. Consequently, $\#T(x)$ is not in $G_0(U) \cap F(U)$.

U is a nonempty, and as a consistent set a proper subset of D. Thus (2.3) implies that $\#[\neg \exists x T(x)]$ and $\#[\forall x T(x)]$ are in not in $G_0(U)$. By the proof of rules (t6) and (t7) in Proposition 3.1 the sentence $\exists x T(x)$ is not false, and the sentence $\forall x T(x)$ is not true, and hence $\neg \forall x T(x)$ is not false, so that

#[$\exists x T(x)$] and #[$\neg\forall x T(x)$] are not in $F(U)$. Thus none of the Gödel numbers #[$\exists x T(x)$], #[$\forall x T(x)$], #[$\neg\exists x T(x)$] and #[$\neg\forall x T(x)$], are in $G_0(U) \cap F(U)$.

If $P \in \mathcal{P}$, then the proof of Proposition 3.1 implies that the sentences $\exists x T(\lceil P(x) \rceil)$, $\forall x T(\lceil P(x) \rceil)$, $\forall x T(\lceil T(x) \rceil)$ and $\neg(\exists x T(\lceil T(x) \rceil))$ are not both true and false, the sentences $T(\lceil P(x) \rceil)$ are not both true and false for any $x \in X_P$ and the sentences $T(\lceil T(x) \rceil)$ are not both true and false for any $x \in X_T$. Hence the negations of these sentences are not by (t1) both true and false. Thus by (II), their Gödel numbers are not in $G(U) \cap F(U)$, so that they are not in $G_0(U) \cap F(U)$.

The above results and the definition (2.3) of $G_0(U)$ imply that $G_0(U) \cap F(U) = \emptyset$.

Make the induction hypothesis:

(h0) $G_n(U) \cap F(U) = \emptyset$ for some $n \in \mathbb{N}_0$.

If #[$\neg(\neg A)$] would be in $G_n^0(U) \cap F(U)$, then #A would be in $G_n(U)$ and #[$\neg(\neg A)$], or equivalently, by (t1), #A would be in $F(U)$, so that #A would be in $G_n(U) \cap F(U)$. This is impossible by (h0), whence $G_n^0(U) \cap F(U) = \emptyset$.

If #[$A \vee B$] is in $G_n^1(U) \cap F(U)$, then #A or #B is in $G_n(U)$, and both #A and #B are in $F(U)$ by (t2), so that #A or #B is in $G_n(U) \cap F(U)$. This contradicts with (h0), whence $G_n^1(U) \cap F(U) = \emptyset$.

#[$A \wedge B$] cannot be in $G_n^2(U) \cap F(U)$, for otherwise both #A and #B are in $G_n(U)$, and at least one of #A and #B is in $F(U)$, so that #A or #B is in $G_n(U) \cap F(U)$, contradicting with (h0). Thus $G_n^2(U) \cap F(U) = \emptyset$.

If #[$A \to B$] is in $G_n^3(U) \cap F(U)$, then #[$\neg A$] or #B is in $G_n(U)$ and both #[$\neg A$] and #B are in $F(U)$. But then #[$\neg A$] or #B is in $G_n(U) \cap F(U)$. Thus $G_n^3(U) \cap F(U) = \emptyset$ by (h0).

If #[$A \leftrightarrow B$] is in $G_n^4(U) \cap F(U)$, then both #A and #B or both #[$\neg A$] and #[$\neg B$] are in $G_n(U)$, and both #[A] and #[$\neg B$] or both #[$\neg A$] and #[B] are in $F(U)$. Then one of Gödel numbers #A, #B, #[$\neg A$] and #[$\neg B$] is in $G_n(U) \cap F(U)$, contradicting with (h0). Consequently, $G_n^4(U) \cap F(U) = \emptyset$.

If #[$\neg(A \vee B)$] is in $G_n^5(U) \cap F(U)$, then #[$\neg A$] and #[$\neg B$] are in $G_n(U)$, and #[$A \vee B$] is in $G(U)$, i.e., #A or #B is in $G(U)$, or equivalently, #[$\neg A$] or #[$\neg B$] is in $F(U)$. Thus #[$\neg A$] or #[$\neg B$] is in $G_n(U) \cap F(U)$. This is impossible by (h0), whence $G_n^5(U) \cap F(U) = \emptyset$.

If #[$\neg(A \wedge B)$] is in $G_n^6(U) \cap F(U)$, then #[$\neg A$] or #[$\neg B$] is in $G_n(U)$, and #[$A \wedge B$] is in $G(U)$, or equivalently, #A and #B are in $G(U)$, i.e., #[$\neg A$] and #[$\neg B$] are in $F(U)$. Consequently, #[$\neg A$] or #[$\neg B$] is in $G_n(U) \cap F(U)$, contradicting with (h0). Thus $G_n^6(U) \cap F(U) = \emptyset$.

If #[$\neg(A \to B)$] is in $G_n^7(U) \cap F(U)$, then #A and #[$\neg B$] are in $G_n(U)$, and #[$A \to B$] is in $G(U)$, i.e., #[$\neg A$] or #B is in $G(U)$, or equivalently, #A or #[$\neg B$] is in $F(U)$. Thus #A or #[$\neg B$] is in $G_n(U) \cap F(U)$. This contradicts with (h0), whence $G_n^7(U) \cap F(U) = \emptyset$.

#[$\neg(A \leftrightarrow B)$] cannot be in $G_n^8(U) \cap F(U)$, for otherwise both #[A] and #[$\neg B$] or both #[$\neg A$] and #[B] are in $G_n(U)$, and #[$A \leftrightarrow B$] is in $G(U)$, i.e., both #[$\neg A$] and #[$\neg B$] or both #A and #B are in $F(U)$. Thus one of Gödel numbers #A, #B, #[$\neg A$] and #[$\neg B$] is in $G_n(U) \cap F(U)$, contradicting with (h0). Thus $G_n^8(U) \cap F(U) = \emptyset$.

Because $G_{n+1}(U) = G_n(U) \cup \bigcup_{k=0}^{8} G_n^k(U)$, the above results and the induction hypothesis (h0) imply that $G_{n+1}(U) \cap F(U) = \emptyset$. Since (h0) is true when $n = 0$, it is by induction true for every $n \in \mathbb{N}_0$.

If #A is in $G(U)$, it is by (2.6) in $G_n(U)$ for some $n \in \mathbb{N}_0$. Because (h0) is true for every $n \in \mathbb{N}_0$, then #A is not in $F(U)$. Consequently, $G(U) \cap F(U) = \emptyset$.

If $G(U)$ is not consistent, then there is such a sentence A of $\mathcal{L}$, that #A and #[$\neg A$] are in $G(U)$. Because #[$\neg A$] is in $G(U)$, then #A is also in $F(U)$ by (2.6), and hence in $G(U) \cap F(U)$. But this is impossible since $G(U) \cap F(U) = \emptyset$. Thus $G(U)$ is consistent.

If $F(U)$ is not consistent, then #A and #[¬A] are in $F(U)$ for some sentence A of $\mathcal{L}$. Since #[¬A] is in $F(U)$, then #A is also in $G(U)$ by (2.6), and hence in $G(U) \cap F(U)$, a contradiction. Thus $F(U)$ is consistent.

The main result of this subsection is a consequence of Proposition 3.1 and Lemma 3.2.

Proposition 3.2. The language $\mathcal{L}^0$ defined by Definition 3.1 is fully interpreted in the interpretation (II).

Proof. Let A denote a sentence of $\mathcal{L}_0$. Then #A is in $G(U)$ or in $F(U)$. Because U is consistent, then $G(U)$ and $F(U)$ are disjoint by Lemma 3.2. Thus #A is either in $G(U)$ or in $F(U)$, whence A is either true or false in the interpretation (II). The interpretation rules given in (ii) of Introduction are satisfied by Proposition 3.1.

3.2 Semantical interpretation

A semantical interpretation of a language is given by the following citation from [8, p. 2]:

”We say that a language is fully interpreted if all its sentences have meanings that make them either true or false.”

(see also [13, p. 61]). This is the case, for instance, when models are used in the interpretation. According to classical logic the interpretation rules (ii) presented in Introduction are assumed to hold also when a language is fully interpreted by meanings of its sentences.

If all sentences of the object language L are equipped with meanings, then meanings of the sentences of the language $\mathcal{L}^0$ are determined by meanings of the sentences of L, by meaning of the sentence $T(\lceil A \rceil)$, i.e., ’The sentence denoted by A is true’, and by standard meanings of logical symbols. We shall show that if L is fully interpreted by meanings of its sentences, then $\mathcal{L}^0$ is fully interpreted by meanings of its sentences, and this interpretation is equivalent to the interpretation (II). In the proof of this result we use the results of the following Lemmas.

Lemma 3.3. Assume that a language L is fully interpreted by meanings of its sentences, and is without a truth predicate. Let U be as in Definition 3.1, and let V be consistent and satisfy $W \subseteq V \subseteq U$. If every sentence of $\mathcal{L}^0$ whose Gödel number is in V is true and not false by its meaning, then this property is satisfied when V is replaced by $G(V)$.

Proof. Because $V \subseteq U = G(U)$, then every sentence whose Gödel number is in V, is in $\mathcal{L}^0$.

Given a sentence of $\mathcal{L}^0$, its Gödel number is in $D_1(V)$ iff it is of the form $T(\lceil A \rceil)$, where A denotes a sentence whose Gödel number is in V. A is by an assumption true and not false by its meaning. Thus the sentence $T(\lceil A \rceil)$, i.e., the sentence ’The sentence denoted by A is true’, and hence the given sentence, is true and not false by its meaning. Replacing A by $T(\lceil A \rceil)$, it follows that $T(\lceil T(\lceil A \rceil) \rceil)$ is true and not false by its meaning.

A given sentence of $\mathcal{L}^0$ has its Gödel number in $D_2(V)$ iff it is of the form $\neg T(\lceil A \rceil)$, where A denotes a sentence of $\mathcal{L}$, and the Gödel number of the sentence $\neg A$ is in V. $\neg A$ is by a hypothesis true and not false by its meaning, and hence A is false and not true by its meaning. Thus the sentence $T(\lceil A \rceil)$, i.e., the sentence ’The sentence denoted by A is true’, is false and not true by its meaning. Replacing A by $T(\lceil A \rceil)$, we then obtain that $T(\lceil T(\lceil A \rceil) \rceil)$ is false and not true by its meaning. Consequently, by the standard meaning of negation, the sentences $\neg T(\lceil A \rceil)$ and $\neg T(\lceil T(\lceil A \rceil) \rceil)$, and hence also the given sentence, are true and not false by their meanings.

The domain X_T of T, defined by (3.1), contains $\lceil A \rceil$ for every sentence A of L. If A is a true sentence of L, then it is by assumption true and not false by its meaning. Since $\#A$ is in W, it is also in V, whence the sentences $T(\lceil A \rceil)$ and $T(\lceil T(\lceil A \rceil) \rceil)$ are by a result proved above true and not false by their meanings. Thus $T(x)$ and $T(\lceil T(x) \rceil)$ are for some $x \in X_T$ true and not false by their meanings. These results and the standard meaning of the existential quantifier imply that $\exists x T(x)$ and $\exists x T(\lceil T(x) \rceil)$ are true and not false by their meanings.

If A is a false sentence of L, then the sentence $\neg A$ is true in the interpretation of L, and hence true and not false by its meaning. Since $\#[\neg A]$ is in W, it is also in V, so that by a result proved above the sentence $\neg T(\lceil A \rceil)$ and $\neg T(\lceil T(\lceil A \rceil) \rceil)$ are true and not false by their meanings. Thus the sentences $\neg T(x)$ and $\neg T(\lceil T(x) \rceil)$ are for some $x \in X_T$ true and not false by their meanings, so that $T(x)$ and $T(\lceil T(x) \rceil)$ are for some $x \in X_T$ false and not true by their meanings. The above results, the meaning of T, and the standard meanings of the universal quantifier and negation imply that $\neg \forall x T(x)$ and $\neg \forall x T(\lceil T(x) \rceil)$ are true and not false by their meanings. Consequently, the sentences whose Gödel numbers are in D_1 are true and not false by their meanings.

Let P be a predicate of L with domain X_P. L is by assumption fully interpreted by meaning of its sentences. Thus $P(x)$ is for every $x \in X_P$, as a sentence of L, either true and not false, or false and not true by its meaning. This property holds, because of the meaning of T, for sentences $T\lceil P(x) \rceil)$, $x \in X_P$. By taking also the meanings of the existential and universal quantifiers and negation into account, it follows that the sentences whose Gödel numbers are in D_2, D_3 and D_4 are true and not false by their meanings.

The above results and the definition (2.3) of $G_0(V)$ imply that every sentence of $\mathcal{L}^0$ whose Gödel number is in $G_0(V)$ is true and not false by its meaning. Thus the following property holds when $n = 0$.

(h2) Every sentence of $\mathcal{L}^0$ whose Gödel number is in $G_n(V)$ is true and not false by its meaning.

Make the induction hypothesis: (h2) holds for some $n \in \mathbb{N}_0$.

Given a sentence of $\mathcal{L}^0$ whose Gödel number is in $G_n^0(V)$, it is of the form $\neg(\neg A)$, where the Gödel number of A is in $G_n(V)$. By induction hypothesis A is true and not false by its meaning. Thus, by standard meaning of negation, its double application implies that the sentence $\neg(\neg A)$, and hence the given sentence, is true and not false by its meaning.

A given sentence of $\mathcal{L}^0$ has its Gödel number in $G_n^1(V)$ iff it is of the form $A \vee B$, where the Gödel number of A or B is in $G_n(V)$. By the induction hypothesis at least one of the sentences A and B is true and not false by its meaning. Thus, by the the standard meaning of disjunction, the sentence $A \vee B$, and hence given sentence, is true and not false by its meaning.

Similarly it can be shown that if the induction hypothesis holds, then every sentence of $\mathcal{L}^0$ whose Gödel number is in $G_n^k(V)$, where $2 \le k \le 8$, is true and not false by its meaning.

The above results imply that under the induction hypothesis every sentence of $\mathcal{L}^0$ whose Gödel number is in $G_n^k(V)$, where $0 \le k \le 8$, is true and not false by its meaning.

It then follows from the definition (2.5) of $G_{n+1}(V)$ that if (h2) is valid for some $n \in \mathbb{N}_0$, then every sentence of $\mathcal{L}^0$ whose Gödel number is in $G_{n+1}(V)$ is true and not false by its meaning.

The first part of this proof shows that (h2) is valid when $n = 0$. Thus by induction, it is valid for all $n \in \mathbb{N}_0$. This result and (2.6) imply that every sentence of $\mathcal{L}^0$ whose Gödel number is in $G(V)$ is true and not false by its meaning.

Lemma 3.4. Assume that a language L is fully interpreted by meanings of its sentences, and has not a truth predicate. Then the language $\mathcal{L}^0$ given in Definition 3.1 has the following properties.

(a) If a sentence of $\mathcal{L}^0$ is true in the interpretation (II), it is true and not false by its meaning.
(b) If a sentence of $\mathcal{L}^0$ is false in the interpretation (II), it is false and not true by its meaning.

Proof. By Theorem 6.1 the smallest consistent subset U of D which satisfies $U = G(U)$ is the last member of the transfinite sequence $(U_\lambda)_{\lambda<\gamma}$ (indexed by Von Neumann ordinals) constructed in that Theorem. We prove by transfinite induction that the following result holds for all $\lambda < \gamma$.

(H) Every sentence of $\mathcal{L}^0$ whose Gödel number is in U_λ is true and not false by its meaning.

Make the induction hypothesis: There exists a μ which satisfies $0 < \mu < \gamma$ such that (H) holds for all $\lambda < \mu$.

Because U_λ is consistent and $W \subseteq U_\lambda \subseteq U$ for every $\lambda < \mu$, it follows from the induction hypothesis and Lemma 3.3 that (H) holds when U_λ is replaced by any of the sets $G(U_\lambda)$, $\lambda < \mu$. Thus (H) holds when U_λ is replaced by the union of those sets. But this union is U_μ by Theorem 6.1 (C), whence (H) holds when $\lambda = \mu$.

When $\mu = 1$, then $\lambda < \mu$ iff $\lambda = 0$. $U_0 = W$, i.e., the set of Gödel numbers of true sentences of L. By assumption these sentences are true and not false by their meanings. Moreover, these sentences are also sentences of $\mathcal{L}^0$, since it contains sentences of L. This proves that the induction hypothesis is satisfied when $\mu = 1$.

The above proof implies by transfinite induction, that properties assumed in (H) for U_λ are valid whenever $\lambda < \gamma$. In particular, they are valid for the last member of $(U_\lambda)_{\lambda<\gamma}$, which is by Theorem 6.1 the smallest consistent subset U of D for which $U = G(U)$. Thus every sentence of $\mathcal{L}^0$, which is true in the interpretation (II), has its Gödel number in U, and hence by is the above proof true and not false by its meaning. This proves (a).

To prove (b), let A denote a sentence which is false in the interpretation (II). Negation rule implies that $\neg A$ is true in the interpretation (II). Thus, by (a), $\neg A$ is true and not false by its meaning, so that by the standard meaning of negation, A is false and not true by its meaning. This proves (b).

Now we are ready to prove the main result of this subsection.

Proposition 3.3. Assume that a language L is fully interpreted by meanings of its sentences, and is without a truth predicate. Then the language $\mathcal{L}^0$ defined in Definition 3.1 is fully interpreted by meanings of its sentences, and this interpretation is equivalent to the interpretation (II).

Proof. Let A denote a sentence of $\mathcal{L}^0$. A is by Proposition 3.2 either true or false in the interpretation (II). If A is true in the interpretation (II), it is by Lemma 3.4 (a) true and not false by its meaning. If A is false in the interpretation (II), it is by Lemma 3.4 (b) false and not true by its meaning. Consequently, A is either true or false by its meaning. Thus every sentence of $\mathcal{L}^0$ is either true or false by its meaning, or equivalently, $\mathcal{L}^0$ is fully interpreted by meanings of its sentences, and this interpretation is by the above proof equivalent to the interpretation (II).

4 Theory DSTT and its properties

The next theorem provides for the language $\mathcal{L}^0$ defined in Definition 3.1 a theory of truth. Because the interpretation of $\mathcal{L}^0$ can be definitional or semantical, we call that theory definitional/semantical theory of truth, shortly DSTT.

Theorem 4.1. Assume that an object language L is fully interpreted, and is without a truth predicate. Then the language $\mathcal{L}^0$ defined by Definition 3.1 is fully interpreted by (II), and also by meanings of its sequences if L is so interpreted. Moreover, $A \leftrightarrow T(\lceil A \rceil)$ is true and $A \leftrightarrow \neg T(\lceil A \rceil)$ is false for every sentence A of $\mathcal{L}^0$, and T is a truth predicate for $\mathcal{L}^0$.

Proof. $\mathcal{L}^0$ is by Proposition 3.2 fully interpreted in the interpretation (II). If L is fully interpreted by meanings of its sequences, so is $\mathcal{L}^0$ by Proposition 3.3, and this interpretation is equivalent to the interpretation (II).

Let A denote a sentence of $\mathcal{L}^0$. The interpretation (II), rule (t1), the definitions of $D_1(U)$, $D_2(U)$ and $G(U)$, and the assumption $U = G(U)$ in Definition 3.1 imply that
– A is true iff $\#A$ is in $G(U)$ iff $\#A$ is in U iff $\#T(\lceil A \rceil)$ is in $G(U)$ and $\#[\neg T(\lceil A \rceil)]$ is in $F(U)$ iff $T(\lceil A \rceil)$ is true and $\neg T(\lceil A \rceil)$ is false;
– A is false iff $\#A$ is in $F(U)$ iff $\#[\neg A]$ is in $G(U)$ iff $\#[\neg A]$ is in U iff $\#[\neg T(\lceil A \rceil)]$ is in $G(U)$ and $\#T(\lceil A \rceil)$ is in $F(U)$ iff $\neg T(\lceil A \rceil)$ is true and $T(\lceil A \rceil)$ is false.
The above results and rule (t5) imply that $A \leftrightarrow T(\lceil A \rceil)$ is true and $A \leftrightarrow \neg T(\lceil A \rceil)$ is false for every sentence A of $\mathcal{L}^0$.

It follows from Proposition 3.1 that T is a predicate of $\mathcal{L}^0$. Moreover, (3.1) implies that the domain X_T of T is the set of numerals of Gödel numbers of all sentences of $\mathcal{L}^0$. Thus X_T satisfies the following condition presented in [3, p. 7]: "In the case of the truth predicate T, the domain ... is taken to consist of the sentences that are meaningful and determinate, i.e. have a definite truth value, true or false." (In [3] numerals of Gödel numbers of sentences are replaced by sentences itself.) Consequently, T is a truth predicate for $\mathcal{L}^0$.

The next two Lemmas deal with compositionality properties of T.

Lemma 4.1. Let $\mathcal{L}^0$ and its interpretation be as in Theorem 4.1. Then the following biconditionals are true for all sentences A and B of $\mathcal{L}^0$.

(a1) $T(\lceil \neg A \rceil) \leftrightarrow \neg T(\lceil A \rceil)$.
(a2) $T(\lceil A \vee B \rceil) \leftrightarrow T(\lceil A \rceil) \vee T(\lceil B \rceil)$.
(a3) $T(\lceil A \wedge B \rceil) \leftrightarrow T(\lceil A \rceil) \wedge T(\lceil B \rceil)$.
(a4) $T(\lceil A \to B \rceil) \leftrightarrow (T(\lceil A \rceil) \to T(\lceil B \rceil))$.
(a5) $T(\lceil A \leftrightarrow B \rceil) \leftrightarrow (T(\lceil A \rceil) \leftrightarrow T(\lceil B \rceil))$.

Proof. Let A be a sentence of $\mathcal{L}^0$. Then

$T(\lceil \neg A \rceil)$ is true iff $\neg A$ is true iff A is false iff $T(\lceil A \rceil)$ is false iff $\neg T(\lceil A \rceil)$ is true.

$T(\lceil \neg A \rceil)$ is false iff $\neg A$ is false iff A is true iff $T(\lceil A \rceil)$ is true iff $\neg T(\lceil A \rceil)$ is false.

Thus (a1) is true.

Let A and B be sentences of $\mathcal{L}^0$. Then

$T(\lceil A \vee B \rceil)$ is true iff $A \vee B$ is true iff A or B is true iff $T(\lceil A \rceil)$ or $T(\lceil B \rceil)$ is true iff $T(\lceil A \rceil) \vee T(\lceil B \rceil)$ is true.

$T(\lceil A \vee B \rceil)$ is false iff $A \vee B$ is false iff A and B are false iff $T(\lceil A \rceil)$ and $T(\lceil B \rceil)$ are false iff $T(\lceil A \rceil) \vee T(\lceil B \rceil)$ is false.

Consequently, (a2) is true. The proof that (a3) and (a4) are true is similar.

Let A and B be sentences of $\mathcal{L}^0$. Then

$T(\lceil A \leftrightarrow B \rceil)$ is true iff $A \leftrightarrow B$ is true iff A and B are both true or both false iff $T(\lceil A \rceil)$ and $T(\lceil B \rceil)$ are both true or both false iff $T(\lceil A \rceil) \leftrightarrow T(\lceil B \rceil)$ is true.

$T(\ulcorner A \leftrightarrow B \urcorner)$ is false iff $A \leftrightarrow B$ is false iff A is true and B is false or A is false and B is true iff $T(\ulcorner A \urcorner)$ is true and $T(\ulcorner B \urcorner)$ is false or $T(\ulcorner A \urcorner)$ is false and $T(\ulcorner B \urcorner)$ is true iff $T(\ulcorner A \urcorner) \leftrightarrow T(\ulcorner B \urcorner)$ is false. Consequently, (a5) is true.

Lemma 4.2. Let $\mathcal{L}^0$ and its interpretation be as in Theorem 4.1. If P is a predicate of L with domain X_P, then the following biconditionals are true.

(a6) $T(\ulcorner \forall x P(x) \urcorner) \leftrightarrow \forall x T(\ulcorner P(x) \urcorner)$.
(a7) $T(\ulcorner \exists x P(x) \urcorner) \leftrightarrow \exists x T(\ulcorner P(x) \urcorner)$.
(a8) $T(\ulcorner \forall x T(x) \urcorner) \leftrightarrow \forall x T(\ulcorner T(x) \urcorner)$.
(a9) $T(\ulcorner \exists x T(x) \urcorner) \leftrightarrow \exists x T(\ulcorner T(x) \urcorner)$.

Proof. $T(\ulcorner \forall x P(x) \urcorner)$ is true iff $\forall x P(x)$ is true iff $P(x)$ is true for every $x \in X_P$ iff $T(\ulcorner P(x) \urcorner)$ is true for every $x \in X_P$, iff (by (tp7)) $\forall x T(\ulcorner P(x) \urcorner)$ is true. Consequently,

(a61) $T(\ulcorner \forall x P(x) \urcorner)$ is true iff $\forall x T(\ulcorner P(x) \urcorner)$ is true.

Assume that $T(\ulcorner \forall x P(x) \urcorner)$ is false. If $\forall x T(\ulcorner P(x) \urcorner)$ would be true, then $T(\ulcorner \forall x P(x) \urcorner)$ would be true by (a61). But then $T(\ulcorner \forall x P(x) \urcorner)$ would be both false and true, which is impossible because $\mathcal{L}^0$ is fully interpreted. Thus $\forall x T(\ulcorner P(x) \urcorner)$ is false. Similarly it can be shown that if $\forall x T(\ulcorner P(x) \urcorner)$ is false, then $T(\ulcorner \forall x P(x) \urcorner)$ is false. The above results imply that

(a62) $T(\ulcorner \forall x P(x) \urcorner)$ is false iff $\forall x T(\ulcorner P(x) \urcorner)$ is false.

Results (a61), (a62) and (t5) imply that (a6) is true.

$T(\ulcorner \exists x P(x) \urcorner)$ is true iff $\exists x P(x)$ is true iff $P(x)$ is true for some $x \in X_P$ iff $T(\ulcorner P(x) \urcorner)$ is true for some $x \in X_P$, iff (by (tp6)) $\exists x T(\ulcorner P(x) \urcorner)$ is true. Consequently,

(a71) $T(\ulcorner \exists x P(x) \urcorner)$ is true iff $\exists x T(\ulcorner P(x) \urcorner)$ is true.

Result (a71) and the fact that $\mathcal{L}^0$ is fully interpreted imply (cf. the proof of (a62)) that

(a72) $T(\ulcorner \exists x P(x) \urcorner)$ is false iff $\exists x T(\ulcorner P(x) \urcorner)$ is false.

As a consequence of (a71), (a72) and (t5) we obtain that (a7) is true.

Truths of (a8) and (a9) are proved similarly as truths of (a6) and (a7) when properties (tp6) and (tp7) are replaced by (tt6) and (tt7).

Hannes Leitgeb formulated in his paper [12]: 'What Theories of Truth Should be Like (but Cannot be)' the following norms for theories of truth:

(n1) Truth should be expressed by a predicate (and a theory of syntax should be available).
(n2) If a theory of truth is added to mathematical or empirical theories, it should be possible to prove the latter true.
(n3) The truth predicate should not be subject to any type restrictions.
(n4) T-biconditionals should be derivable unrestrictedly.
(n5) Truth should be compositional.
(n6) The theory should allow for standard interpretations.
(n7) The outer logic and the inner logic should coincide.
(n8) The outer logic should be classical.

The next Theorem shows that theory DSTT satisfies these norms, and also two additional norms.

Theorem 4.2. The theory of truth DSTT formulated in Theorem 4.1 satisfies the norms (n1)–(n8) and the following norms.

(n9) The theory of truth should be free from paradoxes.
(n10) Truth should be explained for the language in which this very theory is expressed.

Proof. (n1): T is by Theorem 4.1 a truth predicate for the language $\mathcal{L}^0$ of theory DSTT. (A theory of syntax is available, as stated in (I) after Definition 3.1).

(n2): By Lemma 3.1 DSTT proves theories of the object language L true.

(n3): T is by Theorem 4.1 a truth predicate for the language $\mathcal{L}^0$ of theory DSTT, and is not subject to any restrictions in that language.

(n4): By Theorem 4.1 T-biconditionals $A \leftrightarrow T(\ulcorner A \urcorner)$ are derivable unrestrictedly in $\mathcal{L}^0$.

(n5): Lemmas 4.1 and 4.2 imply that truth in DSTT is compositional.

(n6): DSTT allows for standard interpretations. For instance, the interpretation of the language $\mathcal{L}^0$ of theory DSTT by meanings of its sentences is possible if the object language is so interpreted. Examples of such interpretations are given in Introduction.

(n7) and (n8): The interpretation rules given in (ii) of Introduction, assumed for L, and proved for $\mathcal{L}^0$ in Proposition 3.1, are those of classical logic. Consequently, both the outer logic and the inner logic are classical in DSTT.

(n9): Every sentence of $\mathcal{L}^0$ is either true or false in both interpretations. Thus DSTT is free from paradoxes.

(n10): In (II) truth is explained (an interpretation is explained in English) for the language $\mathcal{L}^0$ where theory DSTT is expressed. In the semantical case the interpretation is explained by meanings of the sentences of $\mathcal{L}^0$.

5 Remarks

The author considers validity of the norm (n9) presented in Theorem 4.2 as crucial to a theory of truth.

Compared with [5, 6, 7], the language for the presented theory of truth contains new sentences, the domain of T is so chosen that it meets the requirements presented in [3], and the collection of possible object languages L is larger. Neither compositionality results for T nor the interpretation of $\mathcal{L}^0$ by meanings of its sentences, and the equivalence if this interpretation to the interpretation (II) are presented in those papers. Also some proofs are simplified and specified.

As in [7], the sentences $\exists x[\neg T(x)]$ and $\forall x[\neg T(x)]$ could be added to $\mathcal{L}^0$ so that the biconditionals $\forall x[\neg T(x)] \leftrightarrow [\neg(\exists x T(x))]$ and $\exists x[\neg T(x)] \leftrightarrow [\neg(\forall x T(x))]$ would be true. Thus also $\neg T$ would be a predicate of $\mathcal{L}^0$.

If U is a consistent subset of D which satisfies $U = G(U)$, but not the smallest, and if L is fully interpreted, then T is a truth predicate for the language $\mathcal{L}^0$, defined as in Definition 3.1, and interpreted by (II), using that U. The so obtained theory of truth satisfies norms (n0)–(n10). It is questionable whether there exists a proof that for such U the language $\mathcal{L}^0$ is fully interpreted by meanings of its sentences when L is so interpreted.

The set U in theory DSTT is the smallest consistent subset of D which satisfies $U = G(U)$. Thus the sentences of $\mathcal{L}^0$ are grounded in the sense defined by Kripke in [10, p. 18].

In Tarski's theory of truth (cf. [14]) the truth predicate is in every step for the language preceding that step. The sentences of that language do not contain the truth predicate in question. Thus the norms (n3) and (n4) are not satisfied.

Leitgeb gives in [12, p. 9] the following justification to his opinion that there cannot be a theory of truth which satisfies all the norms presented in [12]: "Consider a first-order theory which conforms to these norms, such that truth is to be explained for the language in which this very theory is expressed. From the theory of syntax the existence of a so-called Liar sentence is derivable." This means that the existence of a sentence A for which $A \leftrightarrow \neg T(\lceil A \rceil)$ "follows from the syntactic axioms", and hence is true. But also $A \leftrightarrow T(\lceil A \rceil)$ is true by (n4). Thus A is both true and false by the biconditionality rule (t5) of classical logic, so that such a theory of truth is contradictory, whereas DSTT is not. While the theory of syntax for the object language L can be that of a first-order theory, the theory of syntax for the language $\mathcal{L}^0$ of theory DSTT is not.

In [3, Conclusions] Feferman urges "the pursuit of axiomatizations of semantical or definitional approaches that have not yet been thus treated, and the close examination of them in the light of the given criteria." By 'the given criteria' Feferman means Leitgeb's norms (n1)–(n8). The close examination of theory DSTT in the light of those criteria is carried out above. However, neither axioms for DSTT nor an axiomatic theory of truth conforming to norms (n1)–(n10) can be constructed. Reasons for this are found from ([4]).

6 Appendix

Before the proof of Theorem 6.1 we shall first prove auxiliary results, using the concepts adopted in previous sections.

Lemma 6.1. Assume that U and V are consistent subsets of D, and that $V \subseteq U$. Then $G(V) \subseteq G(U)$ and $F(V) \subseteq F(U)$.

Proof. As consistent sets both V and U are proper subsets of D.

Let A be a sentence of L. Definition of $G(U)$ implies that #A is in $G(U)$ and also in $G(V)$ iff #A is in W.

If $\#T(x)$ is in $D_1(V)$, then $x = \lceil A \rceil$, where #A is in V. Because $V \subseteq U$, then #A is also in U, whence $\#T(x)$ is in $D_1(U)$.

If $\#[\neg T(x)]$ is in $D_2(V)$, then x is $\lceil A \rceil$, where $\#[\neg A]$ is in V. Because $V \subseteq U$, then $\#[\neg A]$ is also in U, whence $\#[\neg T(x)]$ is in $D_2(U)$.

If $\#[\exists x T(x)]$ is in $G_0(V)$, then V is nonempty. Because $V \subseteq U$, then also U is nonempty, whence $\#[\exists x T(x)]$ is in $G_0(U)$. Consequently, $\#[\exists x T(x)]$ is in $G_0(U)$ whenever it is in $G_0(V)$. The similar reasoning shows that $\#[\neg \forall x T(x))]$ is in $G_0(U)$ whenever it is in $G_0(V)$. The sets D_1, D_2, D_3 and D_4 are contained in U if they are contained in V.

The above results imply that $G_0(V) \subseteq G_0(U)$. Make an induction hypothesis:

(h1) $G_n(V) \subseteq G_n(U)$.

The definitions of the sets $G_n^k(U)$, $k = 0, \ldots, 8$, given in (2.4), together with (h1), imply that $G_n^k(V) \subseteq G_n^k(U)$ for each $k = 0, \ldots, 8$. Thus

$$G_{n+1}(V) = G_n(V) \cup \bigcup_{k=0}^{8} G_n^k(V) \subseteq G_n(U) \cup \bigcup_{k=0}^{8} G_n^k(U) = G_{n+1}(U).$$

Because (h1) is true when $n = 0$, then it is true for every $n \in \mathbb{N}_0$.

If #A is in $G(V)$, it is by (2.6) in $G_n(V)$ for some n. Thus #A is in $G_n(U)$ by (h1), and hence in $G(U)$. Consequently, $G(V) \subseteq G(U)$.

If #A is in $F(V)$, it follows from (2.6) that $\#[\neg A]$ is in $G(V)$. Because $G(V) \subseteq G(U)$, then $\#[\neg A]$ is in $G(U)$. This implies by (2.6) that #A is in $F(U)$. Thus $F(V) \subseteq F(U)$.

Denote by $\mathcal{C}$ the family of consistent subsets of D. In the formulation and the proof of Theorem 6.1 below transfinite sequences indexed by von Neumann ordinals are used. A transfinite sequence $(U_\lambda)_{\lambda<\alpha}$ of $\mathcal{C}$ is said to be increasing if $U_\mu \subseteq U_\nu$ whenever $\mu < \nu < \alpha$, and strictly increasing if $U_\mu \subset U_\nu$ whenever $\mu < \nu < \alpha$.

Lemma 6.2. Assume that $(U_\lambda)_{\lambda<\alpha}$ a strictly increasing sequence of $\mathcal{C}$. Then
(a) $(G(U_\lambda))_{\lambda<\alpha}$ is an increasing sequence of $\mathcal{C}$.
(b) The set $U_\alpha = \bigcup\limits_{\lambda<\alpha} G(U_\lambda)$ is consistent.

Proof. (a) Consistency of the sets $G(U_\lambda)$, $\lambda < \alpha$, follows from Lemma 3.2 because the sets U_λ, $\lambda < \alpha$, are consistent.

Because $U_\mu \subset U_\nu$ whenever $\mu < \nu < \alpha$, then $G(U_\mu) \subseteq G(U_\nu)$ whenever $\mu < \nu < \alpha$, by Lemma 6.1, whence the sequence $(G(U_\lambda))_{\lambda<\alpha}$ is increasing. This proves (a).

To prove that the set $\bigcup\limits_{\lambda<\alpha} G(U_\lambda)$ is consistent, assume on the contrary that there exists such a sentence A in $\mathcal{L}$ that both $\#A$ and $\#[\neg A]$ are in $\bigcup\limits_{\lambda<\alpha} G(U_\lambda)$. Thus there exist $\mu, \nu < \alpha$ such that $\#A$ is in $G(U_\mu)$ and $\#[\neg A]$ is in $G(U_\nu)$. Because $G(U_\mu) \subseteq G(U_\nu)$ or $G(U_\nu) \subseteq G(U_\mu)$, then both $\#A$ and $\#[\neg A]$ are in $G(U_\mu)$ or in $G(U_\nu)$. But this is impossible, since both $G(U_\mu)$ and $G(U_\nu)$ are consistent by (a). Thus, the set $\bigcup\limits_{\lambda<\alpha} G(U_\lambda)$ is consistent. This proves the conclusion of (b).

Theorem 6.1. The union of those transfinite sequences $(U_\lambda)_{\lambda<\alpha}$ of $\mathcal{C}$ which satisfy

(C)$(U_\lambda)_{\lambda<\alpha}$ is strictly increasing, $U_0 = W$, and if $0 < \mu < \alpha$, then $U_\mu = \bigcup\limits_{\lambda<\mu} G(U_\lambda)$

is a transfinite sequence. It has the last member, which is the smallest consistent subset U of D which satisfies $U = G(U)$.

Proof. Those transfinite sequences of $\mathcal{C}$ which satisfy condition (C) are called G-sequences. We shall first show that G-sequences are nested:

(1) *Assume that $(U_\lambda)_{\lambda<\alpha}$ and $(V_\lambda)_{\lambda<\beta}$ are G-sequences, and that $\{U_\lambda\}_{\lambda<\alpha} \not\subseteq \{V_\lambda\}_{\lambda<\beta}$. Then $(V_\lambda)_{\lambda<\beta} = (U_\lambda)_{\lambda<\beta}$.*

By the assumption of (1) $\mu = \min\{\lambda < \alpha \mid U_\lambda \notin \{V_\lambda\}_{\lambda<\beta}\}$ exists, and $\{U_\lambda\}_{\lambda<\mu} \subseteq \{V_\lambda\}_{\lambda<\beta}$. Properties (C) imply by transfinite induction that $U_\lambda = V_\lambda$ for each $\lambda < \mu$. To prove that $\mu = \beta$, make a counter-hypothesis: $\mu < \beta$. Since $\mu < \alpha$ and $U_\lambda = V_\lambda$ for each $\lambda < \mu$, it follows from properties (C) that $U_\mu = \bigcup\limits_{\lambda<\mu} G(U_\lambda) = \bigcup\limits_{\lambda<\mu} G(V_\lambda) = V_\mu$, which is impossible, since $V_\mu \in \{V_\lambda\}_{\lambda<\beta}$, but $U_\mu \notin \{V_\lambda\}_{\lambda<\beta}$. Consequently, $\mu = \beta$ and $U_\lambda = V_\lambda$ for each $\lambda < \beta$, whence $(V_\lambda)_{\lambda<\beta} = (U_\lambda)_{\lambda<\beta}$.

By definition, every G-sequence $(U_\lambda)_{\lambda<\alpha}$ is a function $\lambda \mapsto U_\lambda$ from α into $\mathcal{C}$. Property (1) implies that these functions are compatible. Thus their union is by [9, Theorem 2.3.12] a function with values in $\mathcal{C}$, the domain being the union of all index sets of G-sequences. Because these index sets are ordinals, then their union is also an ordinal by [11, I.8.10]. Denote it by γ. The union function can be represented as a sequence $(U_\lambda)_{\lambda<\gamma}$ of $\mathcal{C}$. It is strictly increasing as a union of strictly increasing nested sequences.

To show that γ is a successor, assume on the contrary that γ is a limit ordinal. Given $\nu < \gamma$, then $\mu = \nu + 1$ and $\alpha = \mu + 1$ are in γ, and $(U_\lambda)_{\lambda<\alpha}$ is a G-sequence. Denote $U_\gamma = \bigcup\limits_{\lambda<\gamma} G(U_\lambda)$. G is order preserving by Lemma 6.1, and $(U_\lambda)_{\lambda<\gamma}$ is a strictly increasing sequence of $\mathcal{C}$. Thus

$(G(U_\lambda))_{\lambda<\gamma}$ is increasing by Lemma 6.2(a), and U_γ is consistent by Lemma 6.2(b). Moreover, $U_\nu \subset U_\mu = \bigcup_{\lambda<\mu} G(U_\lambda) \subseteq U_\gamma$. This is true for each $\nu < \gamma$, whence $(U_\lambda)_{\lambda<\gamma+1}$ is a G-sequence. This is impossible, since $(U_\lambda)_{\lambda<\gamma}$ is the union of all G-sequences. Consequently, γ is a successor, say $\gamma = \alpha + 1$. Thus U_α is the last member of $(U_\lambda)_{\lambda<\gamma}$, $U_\alpha = \max\{U_\lambda\}_{\lambda<\gamma}$, and $G(U_\alpha) = \max\{G(U_\lambda)\}_{\lambda<\gamma}$. Moreover, $(U_\lambda)_{\lambda<\gamma}$ is a G-sequence, for otherwise $(U_\lambda)_{\lambda<\alpha}$ would be the union of all G-sequences. In particular, $U_\alpha = \bigcup_{\lambda<\alpha} G(U_\lambda) \subseteq \bigcup_{\lambda<\gamma} G(U_\lambda) = G(U_\alpha)$, so that $U_\alpha \subseteq G(U_\alpha)$. This inclusion cannot be proper, since then the longest G-sequence $(U_\lambda)_{\lambda<\gamma}$ could be extended by $U_\gamma = \bigcup_{\lambda<\gamma} G(U_\lambda)$. Consequently, $U_\alpha = G(U_\alpha)$.

Assume that U is a consistent subset of D, and that $U = G(U)$. Then $U_0 = W = G(\emptyset) \subseteq G(U) = U$. If $0 < \mu < \gamma$, and $U_\lambda \subseteq U$ for each $\lambda < \mu$, then $G(U_\lambda) \subseteq G(U)$ for each $\lambda < \mu$, whence $U_\mu = \bigcup_{\lambda<\mu} G(U_\lambda) \subseteq G(U) = U$. Thus, by transfinite induction, $U_\mu \subseteq U$ for each $\mu < \gamma$. This proves the last assertion of Theorem.

References

[1] Chomsky, Noam (1957) Syntactic structures, The Hague: Mouton.
[2] Cohen, Paul (1963) *A minimal model for set theory*, Bulletin of the American Mathematical Society, **69**, 537–540.
[3] Feferman, Solomon (2012) *Axiomatizing truth. Why and how?* Logic, Construction, Computation (U. Berger et al. eds.) Ontos Verlag, Frankfurt, 185–200.
[4] Fischer, Martin & Halbach, Volker & Kriener, Jönne & Stern, Johannes (2015) *Axiomatizing semantic theories of truth?* The Review of Symbolic Logic, https://doi.org/10.1017/S1755020314000379.
[5] Heikkilä, Seppo (2014) *What Mathematical Theories of Truth Should be Like (and Can be)*, Nonlinear Studies, **21**, 4, 591-599.
[6] Heikkilä, S. (2015) *A Mathematical Theory of Truth with Application to the Regress Problem*, Nonlinear Studies, **22**, 2, 307-315.
[7] Heikkilä, S. (2016) *On the construction of fully interpreted formal languages which posses their truth predicates*, Nonlinear Studies, **23**, 4, 601-609.
[8] Hodges, Wilfrid (2014) *"Tarski's Truth Definitions"*, The Stanford Encyclopedia of Philosophy (Fall 2014 Edition), Edward N. Zalta (ed.), URL =<https://plato.stanford.edu/archives/fall2014/entries/tarski-truth/>.
[9] Hrbacek, Karel & Jech, Thomas (1999) Introduction to Set Theory, Marcel Dekker, Inc, New York - Basel.
[10] Kripke, Saul (1975) Outline of a Theory of Truth, Journal of Philosophy, **72**, 690–716.
[11] Kunen, Kenneth (2009) The Foundations of Mathematics, Mathematical Logic and Foundations, **19**.
[12] Leitgeb, Hannes (2007) *What Theories of Truth Should be Like (but Cannot be)*, Philosophy Compass, **2/2**, 276–290.
[13] Pantsar, Markus (2009) *Truth, Proof and Gödelian arguments*, Philosophical Studies from the University of Helsinki **23**, 310 pp.
[14] Tarski, Alfred (1936) *Der Wahrheitsbegriff in Formalisierten Sprachen*, Studia Philosophica 1: 261–405.