Approximating Propositional Calculi by Finite-valued Logics*

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Abstract

The problem of approximating a propositional calculus is to find many-valued logics which are sound for the calculus (i.e., all theorems of the calculus are tautologies) with as few tautologies as possible. This has potential applications for representing (computationally complex) logics used in AI by (computationally easy) many-valued logics. It is investigated how far this method can be carried using (1) one or (2) an infinite sequence of many-valued logics. It is shown that the optimal candidate matrices for (1) can be computed from the calculus.

1 Introduction

The question of what to do when face to face with a new logical calculus is an age-old problem of mathematical logic. One usually has, at least at first, no semantics. For example, intuitionistic propositional logic was constructed by Heyting only as a calculus; semantics for it were proposed much later. Currently we face a similar situation with Girard's linear logic. The lack of semantical methods makes it difficult to answer questions such as: Are statements of a certain form (un)derivable? Are the axioms independent? Is the calculus consistent? For logics closed under substitution many-valued methods have often proved valuable since they were first used for proving underivabilities by Bernays [2] in 1926 (and later by others, e.g., McKinsey and Wajsberg; see also [12, § 25]). For the above-mentioned underivability question it is necessary to find many-valued matrices for which the given calculus is sound. If a formula is not a tautology under such a matrix, it cannot be derivable in the calculus. It is also necessary, of course, that the matrix has as few tautologies as possible in order to be useful.

Such "optimal" approximations of a given calculus may also have applications in computer science. In the field of artificial intelligence many new (propositional) logics have been introduced. They are usually better suited to model the problems dealt with in AI than traditional (classical, intuitionistic, or modal) logics, but many have two significant drawbacks: First, they are either given solely semantically or solely by a calculus. For practical purposes, a proof theory is necessary; otherwise computer representation of and automated search for proofs/truths in these logics is not feasible. Second, most of them are intractable, and hopelessly so, provided the polynomial hierarchy does not collapse. For instance, many nonmonotonic formalisms have been shown to be hard for classes above NP [5]. Although satisfiability in many-valued propositional logics is (as in classical logic) NP-complete [11], this is still (probably) much better.

On the other hand, it is evident from the work of Carnielli [3] and Hähnle [8] on tableaux, and Rousseau, Takahashi, and Baaz et al. [1] on sequents, that finite-valued logics are, from the perspective of proof *and* model theory, very close to classical logic. Therefore, many-valued logic is a very suitable candidate if one looks for approximations, in some sense, of given complex logics.

What is needed are methods for obtaining finite-valued approximations of the propositional logics at hand. It turns out, however, that a shift of emphasis is in order here. While it is the *logic* we are actually interested in, we always are given only a *representa-*

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tion of the logic. Hence, we have to concentrate on approximations of the representation, and not of the logic per se.

What is a representation of a logic? The first type of representation that comes to mind is a calculus. Hilbert-type calculi are the simplest conceptually and the oldest historically. We will investigate the relationship between such calculi on the one hand and many-valued logics or recursive sequences of many-valued logics on the other hand. The latter notion has received considerable attention in the literature in the form of the following two problems: Given a calculus **C**,

(1) find a minimal (finite) *normal* matrix for **C** (relevant for non-derivability and independence proofs), and

(2) find a sequence of finite-valued logics whose intersection equals the theorems of **C**, and its converse, given a sequence of finite-valued logics, find a calculus for its intersection (exemplified by Jaśkowski's sequence for intuitionistic propositional calculus, and by Dummett's extension axiomatizing the intersection of the sequence of Gödel logics, respectively).

For (1), of course, the best case would be a finite-valued logic **M** whose tautologies *coincide* with the theorems of **C**. **C** then provides an axiomatization of **M**. This of course is not always possible, at least for *finite*-valued logics. Lindenbaum [10, Satz 3] has shown that any logic (in our sense, given by a set of rules and closed under substitution) can be characterized by an *infinite*-valued logic. For a discussion of related questions see also Rescher [12, § 24].

In the following we will consider these questions in a general setting. Consider a propositional Hilbert-type calculus **C**. First of all, an optimal (i.e., minimal under set inclusion of the tautologies) m-valued logic for which **C** satisfies reasonable soundness properties can be computed. We call such a logic *normal* for **C**. The next question is, can we find an approximating sequence of m-valued logics in the sense of (2)? It is shown that this is impossible for undecidable calculi **C**, and possible for all decidable logics closed under substitution.

2 Propositional Logics

2.1. Definition A *propositional language* $\mathcal{L}$ consists of the following:

(1) propositional variables: X_0, X_1, X_2, ..., X_j, ... ($j \in \omega$)

(2) propositional connectives of arity n_j: $\square_0^{n_0}$, $\square_1$, ..., $\square_r$. If $n_j = 0$, then $\square_j$ is called a *propositional constant*.

(3) Auxiliary symbols: (,), and , (comma).

Formulas and subformulas are defined as usual. We denote the set of formulas over a language $\mathcal{L}$ by $\mathrm{Frm}(\mathcal{L})$. By $\mathrm{Var}(A)$ we mean the set of propositional variables occurring in A.

2.2. Definition A *propositional Hilbert-type calculus* **C** in the language $\mathcal{L}$ is given by

(1) A finite set $A(\mathbf{C}) \subseteq \mathrm{Frm}(\mathcal{L})$ of axioms.

(2) A finite set $R(\mathbf{C})$ of rules of the form

$$\frac{A_1 \quad \dots \quad A_n}{A}\, r$$

where $A, A_1, \dots, A_n \in \mathrm{Frm}(\mathcal{L})$

A formula F is a *theorem* of **L** if there is a derivation of F in **C**, i.e., a finite sequence

$$F_1, F_2, \dots, F_s = F$$

of formulas s.t. for each F_i either

(1) F_i is a substitution instance of an axiom in $A(\mathbf{C})$, or

(2) there are F_{k_1}, ..., F_{k_n} with $k_j < i$ and a rule $r \in R(\mathbf{C})$, s.t. F_{k_j} is a substitution instance of the j-th premise of r, and F_i is a substitution instance of the conclusion.

If F is a theorem of **C** we write $\mathbf{C} \vdash F$. The set of theorems of **C** is denoted by $\mathrm{Thm}(\mathbf{C})$.

2.3. *Remark* The above notion of a propositional rule is the one usually used in axiomatizations of propositional logic. It is, however, by no means the only possible notion. For instance, Schütte's rules

$$\frac{A(\top) \quad A(\bot)}{A(X)} \qquad \frac{C \leftrightarrow D}{A(C) \leftrightarrow A(D)}$$

where X is a propositional variable, and A, C, and D are formulas, does not fit under the above definition.

2.4. Definition A *propositional logic* **L** in the language $\mathcal{L}$ is a subset of $\mathrm{Frm}(\mathcal{L})$ closed under substitution.

Every propositional calculus $\mathbf{C}$ defines a propositional logic, namely $\mathrm{Thm}(\mathbf{C}) \subseteq \mathrm{Frm}(\mathcal{L})$, since $\mathrm{Thm}(\mathbf{C})$ is closed under substitution. Not every propositional logic, however, is axiomatizable, let alone finitely axiomatizable by a Hilbert calculus. For instance, the logic

$$\{\Box^k(\top) \mid k \text{ is the Gödel number of a true sentence of arithmetic}\}$$

is not axiomatizable, whereas the logic

$$\{\Box^k(\top) \mid k \text{ is prime}\}$$

is certainly axiomatizable (it is even decidable), but not by a Hilbert calculus using only $\Box$ and $\top$. (It is easily seen that any Hilbert calculus for $\Box$ and $\top$ has either only a finite number of theorems or yields arithmetic progressions of $\Box$'s.)

2.5. Definition A *propositional finite-valued logic* $\mathbf{M}$ is given by a set of truth values $V(\mathbf{M}) = \{1, 2, \ldots, m\}$, the set of *designated truth values* $V^+(\mathbf{M}) \subseteq V(\mathbf{M})$, and a set of truth functions $\widetilde{\Box}_j\colon V(\mathbf{M})^{n_j} \to V(\mathbf{M})$ for all connectives $\Box_j \in \mathcal{L}$ with arity n_j.

The corresponding subset of $\mathrm{Frm}(\mathcal{L})$ of true formulas is the set of tautologies of $\mathbf{M}$, defined as follows.

2.6. Definition A *valuation* $\mathbf{I}$ is a mapping from the set of propositional variables into $V(\mathbf{M})$. A valuation $\mathbf{I}$ can be extended in the standard way to a function from formulas to truth values. $\mathbf{I}$ *satisfies* a formula F, in symbols: $\mathbf{I} \models_{\mathbf{M}} F$, if $\mathbf{I}(F) \in V^+(\mathbf{M})$. In that case, $\mathbf{I}$ is called a *model* of F, otherwise a *countermodel*. A formula F is a *tautology* of $\mathbf{M}$ iff it is satisfied by every valuation. Then we write $\mathbf{M} \models F$. We denote the set of tautologies of $\mathbf{M}$ by $\mathrm{Taut}(\mathbf{M})$.

2.7. Example The sequence of m-valued Gödel logics $\mathbf{G}_m$ is given by $V(\mathbf{G}_m) = \{0, 1, \ldots, m-1\}$, the designated values $V^+(\mathbf{G}_m) = \{0\}$, and the following truth functions:

$$\begin{aligned}
\widetilde{\neg}_{\mathbf{G}_m}(v) &= \begin{cases} 0 & \text{for } v = m-1 \\ m-1 & \text{for } v \neq m-1 \end{cases} \\
\widetilde{\vee}_{\mathbf{G}_m}(v,w) &= \min(a,b) \\
\widetilde{\wedge}_{\mathbf{G}_m}(v,w) &= \max(a,b) \\
\widetilde{\supset}_{\mathbf{G}_m}(v,w) &= \begin{cases} 0 & \text{for } v \geq w \\ w & \text{for } v < w \end{cases}
\end{aligned}$$

This sequence of logics was used in [6] to show that intuitionistic logic cannot be characterized by a finite matrix.

In the remaining sections, we will concentrate on the relations between calculi $\mathbf{C}$, logics $\mathbf{L}$, and many-valued logics $\mathbf{M}$. The objective is to find many-valued logics $\mathbf{M}$ (or sequences thereof) that, in a sense, approximate the calculus $\mathbf{C}$ and/or the logic $\mathbf{L}$.

The following well-known product construction is useful for characterizing the "intersection" of many-valued logics.

2.8. Definition Let $\mathbf{M}$ and $\mathbf{M}'$ be m and m'-valued logics, respectively. Then $\mathbf{M} \times \mathbf{M}'$ is the mm'-valued logic where $V(\mathbf{M} \times \mathbf{M}') = V(\mathbf{M}) \times V(\mathbf{M}')$, $V^+(\mathbf{M} \times \mathbf{M}') = V^+(\mathbf{M}) \times V^+(\mathbf{M}')$, and truth functions are defined component-wise. I.e., if $\Box$ is an n-ary connective, then

$$\widetilde{\Box}_{\mathbf{M}\times\mathbf{M}'}(w_1, \ldots, w_n) = \langle \widetilde{\Box}_{\mathbf{M}}, \widetilde{\Box}_{\mathbf{M}'} \rangle.$$

2.9. Lemma $\mathrm{Taut}(\mathbf{M} \times \mathbf{M}') = \mathrm{Taut}(\mathbf{M}) \cap \mathrm{Taut}(\mathbf{M}')$

The definition and lemma are easily generalized to the case of finite products $\prod_i \mathbf{M}_i$ by induction.

When looking for a logic with as small a number of truth values as possible which falsifies a given formula we can use the following construction.

2.10. Proposition *Let* $\mathbf{M}$ *be any many-valued logic, and* $A_1, \ldots, A_n$ *be formulas not valid in* $\mathbf{M}$. *Then there is a finite-valued logic* $\mathbf{M}' = \Phi(\mathbf{M}, A_1, \ldots, A_n)$ *s.t.*

(1) $A_1, \ldots, A_n$ *are not valid in* $\mathbf{M}'$,

(2) $\mathrm{Taut}(\mathbf{M}) \subseteq \mathrm{Taut}(\mathbf{M}')$, *and*

(3) $|V(\mathbf{M}')| \leq \xi(A_1, \ldots, A_n)$, *where* $\xi(A_1, \ldots, A_n) = \prod_{i=1}^n \xi(A_i)$ *and* $\xi(A_i)$ *is the number of subformulas of* A_i $+1$.

This holds also if $\mathbf{M}$ *has infinitely many truth values, provided* $V(\mathbf{M})$, $V^+(\mathbf{M})$ *and the truth functions are recursive.*

Proof. We first prove the proposition for $n = 1$. Let $\mathbf{I}$ be the interpretation in $\mathbf{M}$ making A_1 false, and let $B_1, \ldots, B_r$ ($\xi(A_1) = r+1$) be all subformulas of A_1. Every B_i has a truth value t_i in $\mathbf{I}$. Let $\mathbf{M}'$ be as follows: $V(\mathbf{M}') = \{t_1, \ldots, t_r, \top\}$, $V^+(\mathbf{M}') = V^+(\mathbf{M}) \cap V(\mathbf{M}') \cup \{\top\}$. If $\Box \in \mathcal{L}$, define $\widetilde{\Box}$ by

$$\widetilde{\Box}(v_1, \ldots, v_n) = \begin{cases} t_i & \text{if } B_i \equiv \Box(B_{j_1}, \ldots, B_{j_n}) \\ & \text{and } v_1 = t_{j_1}, \ldots, v_n = t_{j_n} \\ \top & \text{otherwise} \end{cases}$$

(1) Since t_r was undesignated in $\mathbf{M}$, it is also undesignated in $\mathbf{M}'$. But $\mathbf{I}$ is also a truth value assignment in $\mathbf{M}'$, hence $\mathbf{M}' \not\models A_1$.

(2) Let C be a tautology of $\mathbf{M}$, and let $\mathbf{J}$ be an interpretation in $\mathbf{M}'$. If no subformula of C evaluates to $\top$ under $\mathbf{J}$, then $\mathbf{J}$ is also an interpretation in $\mathbf{M}$, and C takes the same truth value in $\mathbf{M}'$ as in $\mathbf{M}$ w.r.t. $\mathbf{J}$, which is designated also in $\mathbf{M}'$. Otherwise, C evaluates to $\top$, which is designated in $\mathbf{M}'$. So C is a tautology in $\mathbf{M}'$.

(3) Obvious.

For $n > 1$, the proposition follows by taking $\Phi(\mathbf{M}, A_1, \ldots, A_n) = \prod_{i=1}^{n} \Phi(\mathbf{M}, A_i)$ ∎

3 Many-valued Covers for Calculi

We are looking for many-valued logics $\mathbf{M}$ s.t. $\mathrm{Thm}(\mathbf{C}) \subseteq \mathrm{Taut}(\mathbf{M})$. $\mathbf{M}$ must, however, behave "normally" with respect to $\mathbf{C}$, i.e., $\mathbf{C}$ must remain sound whenever we add new operators and their truth tables to $\mathbf{M}$ or add tautologies as axioms to $\mathbf{C}$.

3.1. Definition An m-valued logic $\mathbf{M}$ is *normal* for a calculus $\mathbf{C}$ (and $\mathbf{C}$ *strongly sound* for $\mathbf{M}$) if

($*$) All axioms $A \in A(\mathbf{C})$ are tautologies of $\mathbf{M}$, and for every rule $r \in R(\mathbf{C})$: if a valuation satisfies the premises of r, it also satisfies the conclusion.

$\mathbf{M}$ is then called a *cover* for $\mathbf{C}$.

We would like to stress the distinction between strong soundness, a.k.a. normality, and soundness. The latter is the familiar property of a calculus to produce only valid formulas as theorems. This "plain" soundness is what we actually would like to investigate in terms of approximations. More precisely, when looking for a finite-valued logic that approximates a given calculus, we are content if we find a logic for which $\mathbf{C}$ is sound. It is, however, not possible in general to test if a calculus is sound for a given finite-valued logic. It *is* possible to test if it is strongly sound. For this pragmatic reason we consider only normal matrices for the given calculi. The next proposition characterizes the normal matrices in terms of strong soundness conditions. These are reasonable conditions which one expects to hold of a "normal" matrix.

3.2. Proposition *$\mathbf{C}$ is strongly sound for a many-valued logic $\mathbf{M}$ if $\mathrm{Thm}(\mathbf{C}') \subseteq \mathrm{Taut}(\mathbf{M}')$ for all $\mathbf{M}'$ and $\mathbf{C}'$, where*

(1) $\mathbf{M}'$ is obtained from $\mathbf{M}$ by adding truth tables for new operations, and

(2) $\mathbf{C}'$ is obtained from $\mathbf{C}$ by adding tautologies of $\mathbf{M}'$ to as axioms.

Proof. If: First of all, $\mathbf{C}$ is sound for $\mathbf{M}$: Let $\mathbf{C} \vdash F$. We show that $\mathbf{M} \models F$ by induction on the length l of the derivation in $\mathbf{C}$:

$l = 1$: This means F is a substitution instance of an axiom A.

$l > 1$. F is the conclusion of a rule $r \in R(\mathbf{C})$. If r is

$$\frac{A_1 \quad \ldots \quad A_k}{A}\; r$$

and X_1, X_2, …, X_n are all the variables in A, A_1, …, A_k, then the inference has the form

$$\frac{A_1[B_1/X_1, \ldots, B_n/X_n] \quad \ldots \quad A_k[B_1/X_1, \ldots, B_n/X_n]}{F = A[B_1/X_1, \ldots, B_n/X_n]}$$

Let $\mathbf{I}$ be a valuation of the variables in F, and let $v_i = \mathbf{I}(B_i)$ $(1 \le i \le n)$. By induction hypothesis, the premises of r are valid. This implies that, for $1 \le i \le k$, we have $\{X_1 \mapsto v_1, \ldots, X_n \mapsto v_n\} \models A_i$. By hypothesis then, $\{X_1 \mapsto v_1, \ldots, X_n \mapsto v_n\} \models A$. But this means that $\mathbf{I} \models F$. Hence, $\mathbf{M} \models F$.

Moreover, $\mathbf{C}$ satisfies conditions (1) and (2) above.

Only if: Every axiom is derivable in $\mathbf{C}$. By soundness, it is a tautology of $\mathbf{M}$, which is just what ($*$) says. Now let $r \in R(\mathbf{C})$ be a rule, let $\mathbf{I}$ be an interpretation which makes the premises A_1, …, A_k of r true, and let A be the conclusion of r. $\mathbf{I}$ assigns truth values v_1, …, v_l to the variables X_1, …, X_l in r. Let $\mathbf{M}'$ be the m-valued logic resulting from $\mathbf{M}$ by extending the language by the constants V_1, …, V_l with values v_1, …, v_l, respectively. Let σ be the substitution mapping X_i to V_i. The formulas $A_1\sigma$, …, $A_l\sigma$ and (by r also) $A\sigma$ are derivable in the extension $\mathbf{C}'$ of $\mathbf{C}$ by the axioms $A_1\sigma$, …, $A_l\sigma$. By (1) and (2), $\mathbf{C}'$ is sound, so $A\sigma$ is a tautology in $\mathbf{M}'$. Consequently, $\mathbf{I} \models A$ in $\mathbf{M}$. ∎

3.3. Corollary *If $\mathbf{C}$ is strongly sound for $\mathbf{M}$ and r is a directly dependent rule of $\mathbf{C}$ (i.e., r can be simulated by the rules of $\mathbf{C}$) then $\mathbf{C} + r$ is also strongly sound for $\mathbf{M}$.*

3.4. Proposition *It is decidable if a given propositional calculus is strongly sound for a given m-valued logic.*

Note also that for usual calculi, Property ($*$) is relatively easy to check. For instance, modus ponens is strongly sound iff, whenever A is true, $A \supset B$ is true iff B is true; necessitation is strongly sound if $\Box X$ is true whenever X is true.

3.5. EXAMPLE The **IPC** is strongly sound for the m-valued Gödel logics $\mathbf{G}_m$. For instance, take axiom a_3: $B \supset A \supset B$. This is a tautology in $\mathbf{G}_m$, for assume we assign some truth values a and b to A and B, respectively. We have two cases: If $a \leq b$, then $(A \supset B)$ takes the value $m-1$. Whatever b is, it certainly is $\leq m-1$, hence $B \supset A \supset B$ takes the designated value $m-1$. Otherwise, $A \supset B$ takes the value b, and again (since $b \leq b$), $B \supset A \supset B$ takes the value $m-1$.

Modus ponens passes the test: Assume A and $A \supset B$ both take the value $m-1$. This means that $a \leq b$. But $a = m-1$, hence $b = m-1$.

Now consider the following extension $\mathbf{G}_m^\top$ of $\mathbf{G}_m$: $V(\mathbf{G}_m^\top) = V(\mathbf{G}_m) \cup \{\top\}$, $V^+(\mathbf{G}_m^\top) = \{m-1, \top\}$, and the truth functions are given by:

$$\widetilde{\Box}_{\mathbf{G}_m^\top}(\bar{v}) = \begin{cases} \top & \text{if } \top \in \bar{v} \\ \widetilde{\Box}_{\mathbf{G}_m}(\bar{v}) & \text{otherwise} \end{cases}$$

for $\Box \in \{\neg, \supset, \wedge, \vee\}$. Neither **IPC** nor **LJ** are strongly sound for $\mathbf{G}_m^\top$, but **LJ** without cut is.

3.6. EXAMPLE Consider the following calculus **K**:

$$X \leftrightarrow \bigcirc X \qquad \frac{X \leftrightarrow Y}{X \leftrightarrow \bigcirc Y}\, r_1 \qquad \frac{X \leftrightarrow X}{Y}\, r_2$$

It is easy to see that the corresponding logic consists of all instances of $X \leftrightarrow \bigcirc^k X$ where $k \geq 1$. This calculus is only strongly sound for the m-valued logic having all formulas as its tautologies. But if we leave out r_2, we can give a sequence of many-valued logics $\mathbf{M}_i$, for each of which **K** is strongly sound: Take for $V(\mathbf{M}_n) = \{0, \ldots, n-1\}$, $V^+(\mathbf{M}_n) = \{0\}$, with the following truth functions:

$$\begin{aligned} \widetilde{\bigcirc} v &= \begin{cases} v+1 & \text{if } v < n-1 \\ n-1 & \text{otherwise} \end{cases} \\ v \widetilde{\leftrightarrow} w &= \begin{cases} 0 & \text{if } v < w \text{ or } v = n-1 \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

Obviously, $\mathbf{M}_n$ is a cover for **K**. On the other hand, $\text{Taut}(\mathbf{M}_n) \neq \text{Frm}(\mathcal{L})$, e.g., any formula of the form $\bigcirc(A)$ takes a (non-designated) value > 0 (for $n > 1$). In fact, every formula of the form $\bigcirc^k X \leftrightarrow X$ is falsified in some $\mathbf{M}_n$.

4 Optimal Covers

By Proposition 3.4 it is decidable if a given m-valued logic **M** is a cover of **C**. Since we can enumerate all m-valued logics, we can also find all covers of **C**. Moreover, comparing two many-valued logics as to their sets of tautologies is decidable, as the next theorem will show. Using this result, we see that we can always generate optimal covers for **L**.

4.1. DEFINITION For two many-valued logics $\mathbf{M}_1$ and $\mathbf{M}_2$, we write $\mathbf{M}_1 \trianglelefteq \mathbf{M}_2$ iff $\text{Taut}(\mathbf{M}_1) \subseteq \text{Taut}(\mathbf{M}_2)$.

$\mathbf{M}_1$ is *better* than $\mathbf{M}_2$, $\mathbf{M}_1 \triangleleft \mathbf{M}_2$, iff $\mathbf{M}_1 \trianglelefteq \mathbf{M}_2$ and $\text{Taut}(\mathbf{M}_1) \neq \text{Taut}(\mathbf{M}_2)$.

4.2. THEOREM *Let two logics $\mathbf{M}_1$ and $\mathbf{M}_2$, m_1-valued and m_2-valued respectively, be given. It is decidable whether $\mathbf{M}_1 \triangleleft \mathbf{M}_2$.*

Proof. It suffices to show the decidability of the following property: There is a formula A, s.t. (*) $\mathbf{M}_2 \models A$ but $\mathbf{M}_1 \not\models A$. If this is the case, write $\mathbf{M}_1 \triangleleft^* \mathbf{M}_2$. $\mathbf{M}_1 \triangleleft \mathbf{M}_2$ iff $\mathbf{M}_1 \triangleleft^* \mathbf{M}_2$ and not $\mathbf{M}_2 \triangleleft^* \mathbf{M}_1$.

We show this by giving an upper bound on the depth of a minimal formula A satisfying the above property. Since the set of formulas of $\mathcal{L}$ is enumerable, bounded search will produce such a formula iff it exists. Note that the property (*) is decidable by enumerating all assignments. In the following, let $m = \max(m_1, m_2)$.

Let A be a formula that satisfies (*), i.e., there is a valuation $\mathbf{I}$ s.t. $\mathbf{I} \not\models_{\mathbf{M}_1} A$. W.l.o.g. we can assume that A contains at most m different variables: if it contained more, some of them must be evaluated to the same truth value in the counterexample $\mathbf{I}$ for $\mathbf{M}_1 \not\models A$. Unifying these variables leaves (*) intact.

Let $B = \{B_1, B_2, \ldots\}$ be the set of all subformulas of A. Every formula B_j defines an m-valued truth function $f(B_j)$ of m variables where the values of the variables which actually occur in B_j determine the value of $f(B_j)$ via the matrix of $\mathbf{M}_2$. On the other hand, every B_j evaluates to a single truth value $t(B_j)$ in the countermodel $\mathbf{I}$.

Consider the formula A' constructed from A as follows: Let B_i be a subformula of A and B_j be a proper subformula of B_i (and hence, a proper subformula of A). If $f(B_i) = f(B_j)$ and $t(B_i) = t(B_j)$, replace B_i in A with B_j. A' is shorter than A, and it still satisfies (*). By iterating this construction until no two subformulas have the desired property we obtain a formula A^*. This procedure terminates, since A' is shorter than A; it preserves (*), since A' remains a tautology under $\mathbf{M}_2$ (we replace subformulas behaving in exactly the same way under all valuations) and the countermodel $\mathbf{I}$ is also a countermodel for A'.

The depth of A^* is bounded above by $m^{m^m+1} - 1$. This is seen as follows: If the depth of A^* is d, then there is a sequence $A^* = B'_0, B'_1, \ldots, B'_d$ of subformulas of A^* where B'_k is an immediate subformula of

B'_{k-1}. Every such B'_k defines a truth function $f(B'_k)$ of m variables in $\mathbf{M}_2$ and a truth valued $t(B'_k)$ in $\mathbf{M}_1$ via $\mathbf{I}$. There are m^{m^m} m-ary truth functions of m truth values. The number of distinct truth function-truth value pairs then is m^{m^m+1}. If $d \geq m^{m^m+1}$, then two of the B'_k, say B'_i and B'_j where B'_j is a subformula of B'_i define the same truth function and the same truth value. But then B'_i could be replaced by B'_j, contradicting the way A^* is defined. ∎

4.3. Corollary *It is decidable if two many-valued logics define the same set of tautologies. The relation $\trianglelefteq$ is decidable.*

Proof. $\mathrm{Taut}(\mathbf{M}_1) = \mathrm{Taut}(\mathbf{M}_2)$ iff neither $\mathbf{M}_1 \lhd^* \mathbf{M}_2$ nor $\mathbf{M}_2 \lhd^* \mathbf{M}_1$. ∎

4.4. Proposition *The optimal (i.e., minimal under $\lhd$) m-valued covers of $\mathbf{C}$ are computable.*

Proof. Consider the set $C(\mathbf{C})$ of m-valued covers of $\mathbf{C}$. Since $C(\mathbf{C})$ is finite and partially ordered by $\trianglelefteq$, $C(\mathbf{C})$ contains minimal elements. The relation $\trianglelefteq$ is decidable, hence the minimal covers can be computed. ∎

4.5. Example By Example 3.5, **IPC** is strongly sound for $\mathbf{G}_3$. The best 3-valued approximation of **IPC** is the 3-valued Gödel logic. In fact, it is the only 3-valued approximation of *any* sound calculus $\mathbf{C}$ (containing modus ponens) for **IPL** which has less tautologies than **CL**.

Note that it is in general impossible to algorithmically construct a $\trianglelefteq$-minimal m-valued *logic* $\mathbf{M}$ (i.e., given independently of a calculus) with $\mathbf{L} \subseteq \mathrm{Taut}(\mathbf{M})$, because, e.g., it is undecidable whether $\mathbf{M}$ is empty or not: e.g., take

$$\mathbf{L} = \begin{cases} \{\Box^k(\top)\} & \text{if } k \text{ is the least solution of } D(x) = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

where $D(x) = 0$ is the diophantine representation of some undecidable set.

5 Sequential Approximations

In the previous section we have shown that it is always possible to obtain the best m-valued covers of a given calculus, but there is no way to tell *how good* these covers are. In this section, we investigate the relation between sequences of many-valued logics and the set of theorems of a calculus $\mathbf{C}$. Such sequences are called *sequential approximations* of $\mathbf{C}$ if they verify all theorems and refute all non-theorems of $\mathbf{C}$. Put another way, this is a question about the limitations of Bernays' method. On the negative side an immediate result says that calculi for undecidable logics do not have sequential approximations. If, however, a propositional logic is decidable, it also has a sequential approximation (independent of a calculus).

5.1. Definition Let $\mathbf{C}$ be a calculus and let $\mathbf{A} = \langle \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \ldots, \mathbf{M}_j, \ldots \rangle$ ($j \in \omega$) be a sequence of many-valued logics s.t.

(1) $\mathbf{A}$ is given by a recursive procedure,

(2) $\mathbf{M}_i \trianglelefteq \mathbf{M}_j$ iff $i \geq j$, and

(3) $\mathbf{M}_i$ is a cover for $\mathbf{C}$.

$\mathbf{A}$ is called a *sequential approximation* of $\mathbf{C}$ iff $\mathrm{Thm}(\mathbf{C}) = \bigcap_{j \in \omega} \mathrm{Taut}(\mathbf{M}_j)$. We say $\mathbf{C}$ is *approximable*, if there is such a sequential approximation for $\mathbf{C}$.

Condition (2) above is technically not necessary. Approximating sequences of logics in the literature (see next example), however, satisfy this condition. Furthermore, with the emphasis on "approximation," it seems more natural that the sequence gets successively "better."

5.2. Example Consider the sequence $\mathbf{G} = \langle \mathbf{G}_i \rangle_{i \geq 2}$ of Gödel logics and intuitionistic propositional logic **IPC**. $\mathrm{Taut}(\mathbf{G}_i) \supset \mathrm{Thm}(\mathbf{IPC})$, since $\mathbf{G}_i$ is a cover for **IPC**. Furthermore, $\mathbf{G}_{i+1} \lhd \mathbf{G}_i$. This has been pointed out by [6], for a detailed proof see [7, Satz 3.4.1]. It is, however, not a sequential approximation of **IPC**: The formula $(A \supset B) \vee (B \supset A)$, while not a theorem of **IPL**, is a tautology of all $\mathbf{G}_i$. In fact, $\bigcap_{j \geq 2} \mathrm{Taut}(\mathbf{G}_i)$ is the set of tautologies of the infinite-valued Gödel logic $\mathbf{G}_\aleph$, which is axiomatized by the rules of **IPC** plus the above formula. This has been shown in [4] (see also [7, § 3.4]). Hence, $\mathbf{G}$ is a sequential approximation of $\mathbf{G}_\aleph = \mathbf{IPC} + (A \supset B) \vee (B \supset A)$.

Jaśkowski [9] gave a sequential approximation of **IPC**.

The natural question to ask is: Which calculi are approximable? First we give the unsurprising negative answer for undecidable calculi.

5.3. Proposition *If $\mathbf{C}$ is undecidable, then it is not approximable.*

Proof. If **C** were approximable, there were a sequence $\mathbf{A} = \langle \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \ldots \rangle$ s.t. $\bigcap_{j\geq 2} \mathrm{Taut}(\mathbf{M}_j) = \mathrm{Thm}(\mathbf{C})$. If N is a non-theorem of **C**, then there would be an index i s.t. N is false in $\mathbf{M}_i$. But this would yield a semi-decision procedure for non-theorems of **C**: Try for each j whether N is false in $\mathbf{M}_j$. If N is a non-theorem, this will be established at $j = i$, if not, we may go on forever. This contradicts the assumption that the non-theorems of **C** are not r.e. (**C** is undecidable and the theorems are r.e.). ■

5.4. THEOREM *Every decidable propositional logic* **L** *(closed under substitution) has a sequential approximation.*

Proof. The proof uses an argument similar to that of Lindenbaum [10, Satz 3]. Let $\mathrm{Frm}_i(\mathcal{L}) \subset \mathrm{Frm}(\mathcal{L})$ be the set of formulas of depth $\leq i$ (which is finite up to renaming of variables). To every formula $F \in \mathrm{Frm}(\mathcal{L})$ we assign a code $\lceil F \rceil$, yielding the sets $\lceil \mathrm{Frm}_i(\mathcal{L}) \rceil$ for all $i \in \omega$. We construct a sequential approximation of **L** as follows: $V(\mathbf{M}_i) = \lceil \mathrm{Frm}_i(\mathcal{L}) \rceil \cup \{\top\}$, with the designated values $V^+(\mathbf{M}_i) = \lceil \mathrm{Frm}_i(\mathcal{L}) \rceil \cap \lceil \mathbf{L} \rceil \cup \{\top\}$. The truth tables for $\mathbf{M}_i$ are given by:

$$\widetilde{\Box}_{\mathbf{M}_i}(v_1, \ldots, v_n) = \\ = \begin{cases} \lceil \Box(F_1, \ldots, F_n) \rceil & \text{if } v_j = \lceil F_j \rceil \text{ for } 1 \leq j \leq n \\ & \text{and } \Box(F_1, \ldots, F_n) \in \mathrm{Frm}_i(\mathcal{L}) \\ \top & \text{otherwise} \end{cases}$$

$\mathbf{M}_i$ is constructed in such a way as to agree with **L** on all formulas of depth $\leq i$, and to make all formulas of depth $> i$ true. Hence, $\mathrm{Taut}(\mathbf{M}_i) \supseteq \mathbf{L}$, and $\mathbf{M}_i \trianglelefteq \mathbf{M}_{i+1}$. Every formula F false in **L** is also false in some $\mathbf{M}_i$ (namely in all $\mathbf{M}_i$ with $i \geq$ the depth of F). ■

6 Conclusion

The main open problem, especially in view of possible applications in computer science, is the complexity of the computation of optimal covers. One would expect that it is tractable at least for some reasonable classes of calculi which are syntactically characterizable, e.g., analytic calculi.

A second problem is in how far approximations can be found for first-order logics and calculi. One obstacle, for instance, is that it is difficult to check whether a matrix is normal for a given calculus, in particular if the rules of the calculus are not "monadic" in the sense that they manipulate more than one variable at a time. In any case, a systematic treatment only seems feasible for many-valued logics with, at most, distribution quantifiers [3].

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