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The aim of this paper is to emphasize the fact that for all finitely-many-valued logics there is a completely systematic relation between sequent calculi and tableau systems. More importantly, we show that for both of these systems there are always two *dual* proof sytems (not just only two ways to interpret the calculi). This phenomenon may easily escape one's attention since in the classical (two-valued) case the two systems coincide. (In two-valued logic the assignment of a truth value and the exclusion of the opposite truth value describe the same situation.)

We employ the usual definitons of first order languages, many-valued interpretations ($\mathbf{M}$) and induced valuation functions ($\mathrm{val}_{\mathbf{M}}$) (see e.g. Carnielli [1987]). In the following $V = \{v_1, \ldots, v_m\}$ always denotes the set of truth values of a logic.

To stress the dualty of the two types of calculi we shall define them simultanously:

1. Definition An *(m-valued) sequent* is an m-tuple of finite sets Γ_i of formulas, denoted as $\Gamma_1 \mid \Gamma_2 \mid \ldots \mid \Gamma_m$. (As usual we abbreviate $\Gamma \cup \Delta$ by Γ, Δ and $\Gamma \cup \{A\}$ by Γ, A.)

2. Definition An interpretation $\mathbf{M}$ is said to *p(n)-satisfy* a sequent $\Gamma_1 \mid \ldots \mid \Gamma_m$, if there is an i $(1 \leq i \leq m)$ and a formula $F \in \Gamma_i$, s.t. $\mathrm{val}_{\mathbf{M}}(F) = (\neq) v_i$.

A sequent is called *p(n)-valid*, if it is p(n)-satisfied by every interpretation.

The concept of p-satisfiability was used by Rousseau [1967] (compare also Schröter [1955]) in his formulation of many-valued sequents, whereas n-satisfiablity essentially already appears in Carnielli [1991].

3. Definition An *introduction rule for a connective* $\Box$ *at place* i in the logic $\mathbf{L}$ is a schema of the form:

$$\frac{\left\langle \Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_m^j, \Delta_m^j \right\rangle_{j \in I}}{\Gamma_1 \mid \ldots \mid \Gamma_i, \Box(A_1, \ldots, A_n) \mid \ldots \mid \Gamma_m} \ \Box{:}i$$

where the arity of $\Box$ is n, I is a finite set, $\Gamma_l = \bigcup_{j \in I} \Gamma_l^j$, $\Delta_l^j \subseteq \{A_1, \ldots, A_n\}$

It is called *p(n)-admissible*, if for every interpretation $\mathbf{M}$ the following are equivalent:

(1) $\Box(A_1, \ldots, A_n)$ takes (does not take) the truth value v_i.

(2) For all $j \in I$, $\mathbf{M}$ p(n)-satisfies the sequents $\Delta_1^j \mid \ldots \mid \Delta_m^j$.

4. Example We state rules for the implication of the three-valued Gödel logic $\mathbf{G}_3$ with $V = \{f, u, t\}$.

Let the expression A^v $(A^{\neq v})$ denote the statement "A takes (does not take) the truth value v". Since $(A \supset B)^t$ iff $(A^f \vee A^u \vee B^t) \wedge (A^f \vee B^u \vee B^t)$ we get the following p-admissible introduction rule for position t:

$$\frac{\Gamma, A \mid \Delta, A \mid \Pi, B \quad \Gamma', A \mid \Delta', B \mid \Pi', B}{\Gamma, \Gamma' \mid \Delta, \Delta' \mid \Pi, \Pi', A \supset B} \supset{:}t$$

Because of $(A \supset B)^t$ iff $A^f \vee (A^u \wedge B^u) \vee B^t$ we get by negating both sides of the equivalence the following n-admissible introduction rule for the implication at position t:

$$\frac{\Gamma, A \mid \Delta \mid \Pi \quad \Gamma' \mid \Delta', A, B \mid \Pi' \quad \Gamma'' \mid \Delta'' \mid \Pi'', B}{\Gamma, \Gamma', \Gamma'' \mid \Delta, \Delta', \Delta'' \mid \Pi, \Pi', \Pi'' A \supset B} \supset{:}t$$

It should be stressed that admissible introduction rules for a connective at a given place are far from being unique: Every p(n)-admissible introduction rule for $\square(A_1, \ldots, A_n)$ at place i corresponds to a conjunction of disjunctions of some A^{v_l} $(A^{\neq v_l})$ which is true iff $\square(A_1, \ldots, A_n)$ takes (does not take) the truth value v_i. *Any* such conjunctive normal form for $\square(A_1, \ldots, A_n)^{v_i}$ will do. In particular, the truth table $\widetilde{\square}$ for a connective $\square$ immediately yields a *complete* conjunctive normal form. For p-sequents the corresponding rule is as in Definition 3, with: $I \subseteq V^n$ is the set of all n-tuples $j = (w_1, \ldots, w_n)$ of truth values such that $\widetilde{\square}(w_1, \ldots, w_n) \neq v_i$; and $\Delta_l^j = \{A_k \mid 1 \leq k \leq n, v_l \neq w_k\}$. For n-sequents we get: $I \subseteq V^n$ consists of all n-tuples $j = (w_1, \ldots, w_n)$ of truth values such that $\widetilde{\square}(w_1, \ldots, w_n) = v_i$; and $\Delta_l^j = \{A_k \mid 1 \leq k \leq n, v_l = w_k\}$.

5. Definition An *introduction rule for a quantifier* Q *at place* i in the logic $\mathbf{L}$ is a schema of the form:

$$\frac{\left\langle \Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_m^j, \Delta_m^j \right\rangle_{j \in I}}{\Gamma_1 \mid \ldots \mid \Gamma_i, (\mathsf{Q}x)A(x) \mid \ldots \mid \Gamma_m} \mathsf{Q}{:}i$$

where I is a finite set, $\Gamma_l = \bigcup_{j \in I} \Gamma_l^j$, $\Delta_l^j \subseteq \{A(a_1), \ldots, A(a_p)\} \cup \{A(t_1), \ldots, A(t_q)\}$. The a_l are metavariables for free variables (the *eigenvariables* of the rule) satisfying the condition that they do not occur in the lower sequent; the t_k are metavariables for arbitrary terms.

$\mathsf{Q}{:}i$ is called *p(n)-admissible*, if for every interpretation $\mathbf{M}$ the following are equivalent:

(1) $(\mathsf{Q}x)A(x)$ takes (does not take) the truth value v_i under $\mathbf{M}$.

(2) For all $d_1, \ldots, d_p \in D$, there are $e_1, \ldots, e_q \in D$ s.t. for all $j \in I$, $\mathbf{M}$ p(n)-satisfies $\Delta_1'^j \mid \ldots \mid \Delta_m'^j$ where $\Delta_l'^j$ is obtained from Δ_l^j by instantiating the eigenvariable a_k (term variable t_k) with d_k (e_k).

The truth function $\widetilde{\mathsf{Q}}$ for a (distribution) quantifier Q immediately yields admissible introduction rules for place i in a way similar to the method described above for connectives: For p-sequents let $I = \{j \subseteq V \mid \widetilde{\mathsf{Q}}(j) \neq v_i\}$. Then the rule is given as in Definition 5, with $\Delta_l^j = \{A(a_w^j) \mid w \in j, w \neq v_l\} \cup \{A(t^j) \mid v_l \in V \setminus j\}$. In contrast, for n-sequents we take $I = \{\langle j, i\rangle \mid j \subseteq V \wedge i \in j \wedge \widetilde{\mathsf{Q}}(j) = v_i\}$ and $\Delta_l^{\langle j,i\rangle} = \{A(a_l^j) \mid l \in j\} \cup \{A(t^j) \mid i = l\}$.

Again, it should be stressed that in general these are not the only possible rules.

6. Definition A *p-sequent calculus* for a logic **L** is given by:

(1) Axioms of the form: $A \mid A \mid \dots \mid A$, where A is any formula,

(2) For every connective $\Box$ and every truth value v_i a p-admissible introduction rule $\Box{:}i$,

(3) For every quantifier Q and every truth value v_i a p-admissible introduction rule $\mathsf{Q}{:}i$,

(4) Weakening rules for every place i:

$$\frac{\Gamma_1 \mid \dots \mid \Gamma_i \mid \dots \mid \Gamma_m}{\Gamma_1 \mid \dots \mid \Gamma_i, A \mid \dots \mid \Gamma_m}\ \text{w:}i$$

(5) Cut rules for every pair of truth values (v_i, v_j) s.t. $v_i \neq v_j$:

$$\frac{\Gamma_1 \mid \dots \mid \Gamma_i, A \mid \dots \mid \Gamma_m \quad \Delta_1 \mid \dots \mid \Delta_j, A \mid \dots \mid \Delta_m}{\Gamma_1, \Delta_1 \mid \dots \mid \Gamma_m, \Delta_m}\ \text{cut:}ij$$

A *n-sequent calculus* for a logic **L** is given by:

(1) Axioms of the form: $\Delta_1 \mid \dots \mid \Delta_m$, where $\Delta_i = \Delta_j = \{A\}$ for some i, j s.t. $i \neq j$ and $\Delta_k = \emptyset$ otherwise (A is any formula),

(2) For every connective $\Box$ and every truth value v_i an n-admissible introduction rule $\Box{:}i$,

(3) For every quantifier Q and every truth value v_i an n-admissible introduction rule $\mathsf{Q}{:}i$,

(4) Weakening rules (identical to the ones tor p-sequent calculi)

(5) The cut rule:

$$\frac{\left\langle \Gamma_1^i \mid \dots \mid \Gamma_i^i, A \mid \dots \mid \Gamma_m^i \right\rangle_{i=1}^m}{\Gamma_1 \mid \dots \mid \Gamma_m}\ \text{cut:}$$

where $\Gamma_l = \bigcup_{1 \leq j \leq m} \Gamma_l^j$.

7. Theorem *(Soundness and cut-free Completeness) For every p(n)-sequent calculus the following holds: A sequent is p(n)-provable without cut rule(s) iff it is p(n)-valid.*

Analytic tableaux for many-valued logics have been investigated by Surma [1977] and Carnielli [1987]. Hähnle [1991], based on the aforementioned work, studied the applicability of these systems for automated theorem proving. Hähnle introduced the notation of sets-of-signs which allows a more efficient representation of tableaux and presented streamlined calculi for certain classes of logics. Here, we want to stress the striking similarity between tableaux systems and sequent calculi: In fact, there is an immediate correspondence between cut-free sequent calculus proofs and closed tableaux. Again, there are two dual systems for any logic.

8. Definition A *signed formula* is an expression of the form $\{w\}A$, where $w \in V$.

9. DEFINITION A *tableau* is a downward tree of sets of signed formulas where every set is obtained from a set preceding it in the tree by application of one of the *rules of the tableau system*:

Let $R{:}i$ be a p(n)-admissible introduction rule for a connective or a quantifier as given in Definitions 3 and 5, where at least one of the Δ_j is nonempty. Moreover, let F be the formula being introduced (i.e., $F \equiv \Box(A_1, \ldots, A_n)$ or $F \equiv (\mathsf{Q}x)A(x)$).

The p(n)-tableau rule corresponding to $R{:}i$ is:

$$\frac{\Gamma, \{v_i\}F}{\left\langle \Gamma, \bigcup_{k=1}^{m} \overline{\Delta}_k \right\rangle_{j \in I}}$$

where $\overline{\Delta}_k$ is obtained from Δ_k by replacing every formula $A \in \Delta_k$ by $\{v_k\}A$. A p(n)-analytic tableau is called *closed*, if every leaf contains formulas $\{v_k\}A$ for all $k \in \{1, \ldots, m\}$ (for $k \in \{i, j\}$, $i \neq j$).

10. THEOREM *Every closed p(n)-tableau with the root $\bigcup \overline{\Gamma}_k$ corresponds to a cut-free p(n)-sequent calculus proof of $\Gamma_1 \mid \ldots \mid \Gamma_m$.*

We finally remark that also resolution calculi can be derived from sequent calculi: The introduction rules for sequents convert into reduction rules that translate finite sets of assignments of truth values to formulas into clause forms. (Clauses are finite sets of assignments of truth values to atomic formulas; cf. BAAZ and FERMÜLLER [1992])

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