Computer Science Logic. 9th Workshop, CSL*'95. Paderborn.*
Selected Papers (Springer, Berlin, 1996) 1–15

Incompleteness of a First-order Gödel Logic and some Temporal Logics of Programs

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Abstract. It is shown that the infinite-valued first-order Gödel logic $\mathbf{G}^0$ based on the set of truth values $\{1/k : k \in \omega \setminus \{0\}\} \cup \{0\}$ is not r.e. The logic $\mathbf{G}^0$ is the same as that obtained from the Kripke semantics for first-order intuitionistic logic with constant domains and where the order structure of the model is linear. From this, the unaxiomatizability of Kröger's temporal logic of programs (even of the fragment without the nexttime operator $\bigcirc$) and of the authors' temporal logic of linear discrete time with gaps follows.

Keywords: temporal logic, intermediate logic, many-valued logic
MR Classification: 03B45, 03B50, 03B55

1 Introduction

In [4], Kurt Gödel introduced a sequence of finite-valued logics $\mathbf{G}_n$ in order to show that there are infinitely many propositional systems intermediate in strength between classical and intuitionistic propositional logic. In [3], Dummett considers the natural infinite-valued analogue **LC** of Gödel's systems and shows that it is axiomatized using the intuitionistic propositional calculus plus the axiom schema $(A \supset B) \vee (B \supset A)$. Little is known about first-order versions of Dummett's **LC** and related systems and about infinite-valued first-order logics in general. The most famous result (Scarpellini [8]) in this area is that the infinite-valued first-order Łukasiewicz logic is not recursively axiomatizable.

We arrive at a first-order Gödel logic by taking the truth functions for the propositional connectives and defining truth functions for universal and existential quantifiers as infimum and supremum over the truth values, respectively. It is worth pointing out right away that which logic we get depends crucially on the order type of the set of truth values. In Section 2 we define these logics and describe their differences. The main result of this paper is that a particular Gödel logic $\mathbf{G}^0$, namely that based on the set of truth values $V^0 = \big\{1/k : k \in \omega \setminus \{0\}\big\} \cup \{0\}$, is not recursively axiomatizable. Indeed, already the $\exists$-free fragment is not r.e. We give the proof in Section 3.

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This work is supported by FWF grant P–10282 MAT.

The main interest of Dummett's **LC** is that it axiomatizes *linear* intuitionistic Kripke semantics: **LC** is the set of all propositional formulas valid in all Kripke trees consisting of just one branch. We see that there is a strong connection between infinite-valued Gödel logic and logics based on linearly ordered models. In the first-order case it turns out that the logic defined by linearly ordered Kripke structures and constant domains **ILC** is exactly the same as $\mathbf{G}^0$. The logic we get by dropping the requirement that the domains are constant, **IL**, is arguably the more natural analogue to **LC** in the first-order case. We obtain the result that the set of validities of **IL** is not r.e. as a corollary to the result for $\mathbf{G}^0$ (which is equal to **ILC**).

Other logics based on linearly ordered Kripke structures are, e.g., variants of Kröger's [5] *Temporal logic of programs* **TL**. **TL** is based on the temporal operators $\Box$ (henceforth) and $\bigcirc$ (nexttime) and is characterized by temporal structures order isomorphic to ω. It is known [9, 6, 7] that first-order **TL** is not axiomatizable. The logic arising from temporal structures that have the form of trees of segments each of which is order-isomorphic to ω is axiomatizable [2]. It has been an open question whether the logic based on chains of segments order-isomorphic to ω is axiomatizable or not. In Section 4 we use the incompleteness result to give a negative answer to this question. We also strengthen the result of [6]: Already first-order **TL** *without* $\bigcirc$ is not axiomatizable.

2 First-order Temporal, Intuitionistic and Gödel Logics

We shall work in a usual language L of predicate logic containing countably many variables (x, y, z, ...), function symbols (s, h, ...), predicate symbols (P, Q, R, ...), connectives ($\wedge$, $\vee$, $\supset$, $\neg$) and the universal ($\forall$) and existential ($\exists$) quantifiers. The language L extended by the temporal operators $\Box$ (henceforth always) and $\bigcirc$ (next time) is denoted L_t. The set of (closed) formulas of L resp. L_t is denoted $\mathrm{Frm}(L)$ resp. $\mathrm{Frm}(L_t)$. For a given interpretation $\mathfrak{K}$ we will also consider *extended languages* $L^{\mathfrak{K}}$ where the elements of a given domain are added as constant symbols the interpretation of which is fixed.

We proceed to define Kripke semantics for first-order temporal logics. These logics are all based on discrete time; this reflects their use in theoretical computer science where "time" is taken to be the discrete states of program executions (see [5]).

Definition 21. Let T be a denumerable partially ordered set. T belongs to the class $\mathcal{L}$ of *linear discrete orders* iff it is order isomorphic to ω; it belongs to the class $\mathcal{T}$ of *trees* if it is order isomorphic to a rooted tree; it belongs to the class $\mathcal{G}$ of *linear discrete orders with gaps* if it is order isomorphic to an ω-chain of ω-segments (i.e., to $\omega \cdot \omega$); it belongs to the class $\mathcal{B}$ of *linear discrete orders with branching gaps* iff it is order isomorphic to a rooted tree of ω-segments.

Definition 22. Let $\mathcal{W}$ be $\mathcal{L}$, $\mathcal{G}$, or $\mathcal{B}$, and let $\mathrm{Frm}(L_t)$ be the set of formulas over some first-order temporal language. A *temporal structure* $\mathfrak{K}$ *for* L_t is a tuple $\langle T, \{D_i\}_{i \in T}, \{\mathbf{s}_i\}_{i \in T}, \mathbf{s}\rangle$, where $T \in \mathcal{W}$, D_i is a set called the *domain* at state i,

$D_i \subseteq D_j$ if $i \leq j$, $\mathbf{s}_i$ is a function mapping n-ary predicate symbols to functions from D_i^n to $\{\top, \bot\}$, and $\mathbf{s}$ is a function mapping n-ary function symbols to functions from $\bigcup D_i \to \bigcup D_i$ s.t. $\mathbf{s}(f)(d_1, \ldots, d_n) \in D_i$ for $d_i \in D_i$, in particular for $d \in D_i$, $\mathbf{s}(d) = d$. The valuation function $\mathbf{s}$ can be extended in the obvious way to a function on all terms.

We define the *valuation functions* $\mathfrak{K}_i$ from $\mathrm{Frm}(L^{\mathfrak{K}})$ to $\{\top, \bot\}$ as follows. Suppose $A \in \mathrm{Frm}(L^{\mathfrak{K}})$.

(1) $A \equiv P(t_1, \ldots, t_n)$: $\mathfrak{K}_i(A) = \mathbf{s}_i(P)\big(\mathbf{s}(t_1), \ldots, \mathbf{s}(t_n)\big)$
(2) $A \equiv \neg B$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_i(B) = \bot$, and $= \bot$ otherwise.
(3) $A \equiv B \wedge C$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_i(B) = \mathfrak{K}_i(C) = \top$, and $= \bot$ otherwise.
(4) $A \equiv B \vee C$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_i(B) = \top$ or $\mathfrak{K}_i(C) = \top$, and $= \bot$ otherwise.
(5) $A \equiv B \supset C$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_i(B) = \bot$ or $\mathfrak{K}_i(C) = \top$, and $= \bot$ otherwise.
(6) $A \equiv (\forall x)B(x)$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_i[d/x](A(d)) = \top$ for every $d \in D_i$, and $= \bot$ otherwise
(7) $A \equiv (\exists x)B(x)$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_i[d/x](A(d)) = \top$ for some $d \in D_i$ and $= \bot$ otherwise
(8) $A \equiv \Box B$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_j(B) = \top$ for every $j \geq i$, and $= \bot$ otherwise.
(9) $A \equiv \bigcirc B$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_{i+1}(B) = \top$, and $= \bot$ otherwise

A is *satisfied* in a temporal structure $\mathfrak{K}$, $\mathfrak{K} \models_t A$, iff $\mathfrak{K}_0(A) = \top$.

Definition 23. We define the following logics:

Linear discrete temporal logic **TL** is the set of all $A \in \mathrm{Frm}(L_t)$ s.t. $\mathfrak{K} = \langle T, \{D_i\}_{i \in T}, \{\mathbf{s}_i\}_{i \in T}, \mathbf{s}\rangle$ with $T \in \mathcal{L}$ satisfies A.

Linear discrete temporal logic with constant domains **TLC** is the set of all $A \in \mathrm{Frm}(L_t)$every $\mathfrak{K} = \langle T, \{D_i\}_{i \in T}, \{\mathbf{s}_i\}_{i \in T}, \mathbf{s}\rangle$ with $T \in \mathcal{L}$ and $D_i = D_j$ for all $i, j \in T$ satisfies A.

Linear discrete temporal logic with gaps **TG** is the set of all $A \in \mathrm{Frm}(L_t)$ s.t. $\mathfrak{K} = \langle T, \{D_i\}_{i \in T}, \{\mathbf{s}_i\}_{i \in T}, \mathbf{s}\rangle$ with $T \in \mathcal{G}$ satisfies A.

Linear discrete temporal logic with gaps and constant domains **TGC** is the set of all $A \in \mathrm{Frm}(L_t)$every $\mathfrak{K} = \langle T, \{D_i\}_{i \in T}, \{\mathbf{s}_i\}_{i \in T}, \mathbf{s}\rangle$ with $T \in \mathcal{G}$ and $D_i = D_j$ for all $i, j \in T$ satisfies A.

Linear discrete temporal logic with branching gaps **TB** is the set of all $A \in \mathrm{Frm}(L_t)$ s.t. $\mathfrak{K} = \langle T, \{D_i\}_{i \in T}, \{\mathbf{s}_i\}_{i \in T}, \mathbf{s}\rangle$ with $T \in \mathcal{B}$ satisfies A.

As indicated in the introduction, the logic **TL** is not axiomatizable. This was shown for the original formulation of Kröger by Szalas [9] and Kröger [6] (two binary function symbols have to be present for the results to hold). If the operator **until** is also present, or if local variables (i.e., variables whose interpretation may me different for each state) are allowed, then the empty signature suffices, as was shown by Szalas and Holenderski [10] and Kröger [6], respectively. These results were strengthened and extended in various ways by Merz [7]. In fact, to be precise, Kröger's original formulation **TLV** differs from **TL** as defined here in several respects: it has (1) constant domains, (2) rigid predicate symbols (i.e., the interpretation of the predicate symbols is the same for each state) and (3) local variables. Merz [7, Lemma 1] shows that the validity problem for **TLC** can be reduced to to the validity problem for **TLV**. Hence, our results extend also to Kröger's original formulation. On the other hand, **TB** is axiomatizable by a sequent calculus presented in [2].

Next we give Kripke semantics for various fragments of first-order intuitionistic logic. We use the term "intuitionistic logic" par abus de langage: "Real" intuitionistic logic is defined not via Kripke- or any other semantics but by Heyting's calculi which he extracted from the writings of Brouwer. It is a more recent discovery that one can give Kripke semantics for these logics which are complete for the calculi. This completeness result, however, is of doubtful value from the intuitionistic point of view.

Definition 24. Let $\mathrm{Frm}(L)$ be the set of formulas over some first-order language, and let T bei in $\mathcal{T}$ or $\mathcal{L}$. An *intuitionistic Kripke-structure* $\mathfrak{K}$ *for* L is a tuple $\langle T, \{D_i\}_{i\in T}, \{\mathbf{s}_i\}_{i\in T}, \mathbf{s}\rangle$, where D_i is a set called the *domain* at state i, $D_i \subseteq D_j$ if $i \le j$, $\mathbf{s}_i$ is a function mapping n-ary predicate symbols to functions from D_i^n to $\{\top, \bot\}$, and $\mathbf{s}$ is a function mapping n-ary function symbols to functions from $\bigcup D_i \to \bigcup D_i$ s.t. $\mathbf{s}(f)(d_1, \ldots, d_n) \in D_i$ for $d_i \in D_i$, in particular for $d \in D_i$, $\mathbf{s}(d) = d$. The valuation $\mathbf{s}_i$ has to satisfy a *monotonicity requirement:* if $\mathbf{s}_i(P(\bar{d})) = \top$ then $\mathbf{s}_j(P(\bar{d})) = \top$ for all $j \ge i$. The valuation function $\mathbf{s}$ can be extended in the obvious way to a function on all terms.

We define the *valuation functions* $\mathfrak{K}_i$ from $\mathrm{Frm}(L^{\mathfrak{K}})$ to $\{\top, \bot\}$ as follows. Suppose $A \in \mathrm{Frm}(L^{\mathfrak{K}})$.

(1) $A \equiv P(t_1, \ldots, t_n)$: $\mathfrak{K}_i(A) = \mathbf{s}_i(P)\big(\mathbf{s}(t_1), \ldots, \mathbf{s}(t_n)\big)$.
(2) $A \equiv \neg B$: $\mathfrak{K}_i(A) = \top$ iff $\mathfrak{K}_j(B) = \bot$ for all $j \ge i$, and $= \bot$ otherwise.
(3) $A \equiv B \wedge C$: $\mathfrak{K}_i(A) = \top$ iff $\mathfrak{K}_i(B) = \mathfrak{K}_i(C) = \top$, and $= \bot$ otherwise.
(4) $A \equiv B \vee C$: $\mathfrak{K}_i(A) = \top$ iff $\mathfrak{K}_i(B) = \top$ or $\mathfrak{K}_i(C) = \top$, and $= \bot$ otherwise.
(5) $A \equiv B \supset C$: $\mathfrak{K}_i(A) = \top$ iff for all $j \ge i$, $\mathfrak{K}_j(B) = \bot$ or $\mathfrak{K}_j(C) = \top$, and $= \bot$ otherwise.
(6) $A \equiv (\forall x)B(x)$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_j[d/x](A(d)) = \top$ for every $j \ge i$ and every $d \in D_j$, and $= \bot$ otherwise.
(7) $A \equiv (\exists x)B(x)$: $\mathfrak{K}_i(A) = \top$ if $\mathfrak{K}_i[d/x](A(d)) = \top$ for some $d \in D_i$ and $= \bot$ otherwise.

A is *satisfied* in an intuitionistic Kripke structure $\mathfrak{K}$, $\mathfrak{K} \models_i A$, iff $\mathfrak{K}_0(A) = \top$.

Definition 25. We define the following logics:

Intuitionistic logic **I** is the set of all $A \in \mathrm{Frm}(L)$ s.t. every $\mathfrak{K} = \langle T, \{D_i\}_{i\in T}, \{\mathbf{s}_i\}_{i\in T}, \mathbf{s}\rangle$ with $T \in \mathcal{T}$ satisifies A.

Linear intuitionistic logic **IL** is the set of all $A \in \mathrm{Frm}(L)$ s.t. every $\mathfrak{K} = \langle T, \{D_i\}_{i\in T}, \{\mathbf{s}_i\}_{i\in T}, \mathbf{s}\rangle$ with $T \in \mathcal{L}$ satisfies A.

Linear intuitionistic logic with constant domains **ILC** is the set of all $A \in \mathrm{Frm}(A)$ s.t. every $\mathfrak{K} = \langle T, \{D_i\}_{i\in T}, \{\mathbf{s}_i\}_{i\in T}, \mathbf{s}\rangle$ with $T \in \mathcal{L}$ and $D_i = D_j$ for all $i, j \in T$ satisfies A.

As usual, if $\mathbf{L}$ is some logic, we write $\mathbf{L} \models A$ for $A \in \mathbf{L}$.

First-order Gödel logics are given by a first-order language, truth functions for the connectives and quantifiers, and a set of truth values. The sets of truth values for the systems we consider are subsets of $[0, 1]$; the designated truth value is 1. The propositional versions of these logics were originally introduced by Gödel [4], and have spawned a sizeable area of logical research subsumed under the title "intermediate logics" (intermediate between classical and intuitionistic logic).

Interpretations are defined as usual:

Definition 26. Let $V \subseteq [0, 1]$ be some set of truth values which contains 0 and 1 and is closed under supremum and infimum. A *many-valued interpretation* $\mathfrak{I} = \langle D, \mathbf{s}\rangle$ *based on* V is given by the *domain* D and the *valuation function* $\mathbf{s}$ where $\mathbf{s}$ maps atomic formulas in $\mathrm{Frm}(L^{\mathfrak{I}})$ into V and n-ary function symbols to functions from D^n to D.

$\mathbf{s}$ can be extended in the obvious way to a function on all terms. The valuation for formulas is defined as follows:

(1) $A \equiv P(t_1, \dots, t_n)$ is atomic: $\mathfrak{I}(A) = \mathbf{s}(P)(\mathbf{s}(t_1), \dots, \mathbf{s}(t_n))$.

(2) $A \equiv \neg B$:

$$\mathfrak{I}(A) = \begin{cases} 0 & \text{if } \mathfrak{I}(B) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

(3) $A \equiv B \wedge C$: $\mathfrak{I}(A) = \min(\mathfrak{I}(B), \mathfrak{I}(C))$.

(4) $A \equiv B \vee C$: $\mathfrak{I}(A) = \max(\mathfrak{I}(A), \mathfrak{I}(B))$.

(5) $A \equiv B \supset C$:

$$\mathfrak{I}(A) = \begin{cases} \mathfrak{I}(C) & \text{if } \mathfrak{I}(B) > \mathfrak{I}(C) \\ 1 & \text{if } \mathfrak{I}(B) \leq \mathfrak{I}(C). \end{cases}$$

The set $\{\mathfrak{I}(A(d)) : d \in D\}$ is called the *distribution* of $A(x)$, we denote it by $\mathrm{Distr}_{\mathfrak{I}}(A(x))$. The quantifiers are, as usual, defined by infimum and supremum of their distributions.

(6) $A \equiv (\forall x)B(x)$: $\mathfrak{I}(A) = \inf \mathrm{Distr}_{\mathfrak{I}}(B(x))$.

(7) $A \equiv (\exists x)B(x)$: $\mathfrak{I}(A) = \sup \mathrm{Distr}_{\mathfrak{I}}(B(x))$.

$\mathfrak{I}$ *satisfies* a formula A, $\mathfrak{I} \models_m A$, if $\mathfrak{I}(A) = 1$.

In considering first-order infinite valued logics, care must be taken in choosing the set of truth values. In order to define the semantics of the quantifier we must restrict the set of truth values to those which are closed under infima and suprema. Note that in propositional infinite valued logics this restriction is not required. For instance, the *rational* interval $[0, 1] \cap \mathbb{Q}$ will not give a satisfactory set of truth values. The following, however, do:

$$\begin{aligned} V_R &= [0, 1] \\ V^0 &= \bigl\{1/k : k \in \omega \setminus \{0\}\bigr\} \cup \{0\} \\ V^1 &= \bigl\{1 - 1/k : k \in \omega \setminus \{0\}\bigr\} \cup \{1\} \end{aligned}$$

The corresponding infinite-valued Gödel logics are $\mathbf{G}_R$, $\mathbf{G}^0$, and $\mathbf{G}^1$.

Definition 27. $\mathbf{G}_R$ is the set of all $A \in \mathrm{Frm}(L)$ s.t. for every $\mathfrak{I}$ based on V_R, $\mathfrak{I} \models_m A$.
$\mathbf{G}^0$ is the set of all $A \in \mathrm{Frm}(L)$ s.t. for every $\mathfrak{I}$ based on V^0, $\mathfrak{I} \models_m A$.
$\mathbf{G}^1$ is the set of all $A \in \mathrm{Frm}(L)$ s.t. for every $\mathfrak{I}$ based on V^1, $\mathfrak{I} \models_m A$.

Note that V^0 is order isomorphic to the set of truth values for **LC** ($\omega + 1$, with 0 designated and reverse order); hence $\mathbf{G}^0$ is the natural generalization of **LC** to first-order. The corresponding propositional systems all have the same sets of tautologies, as can easily be seen. In other words, propositional infinite-valued logic is independent of the cardinality or order type of the set of truth values. The finite-valued versions are all distinct, however, and in fact **LC** is the intersection of all finite-valued Gödel logics.

The first-order infinite-valued systems are *not* equivalent, however.

Proposition 28. *Let*

$$C = (\exists x)(A(x) \supset (\forall y)A(y)) \text{ and}$$
$$C' = (\exists x)((\exists y)A(y) \supset A(x))$$

Then

(1) $\mathfrak{I}(C) = 1$ *if* $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ *has a minimum (w.r.t.* $\mathfrak{I}$*) and* $= \mathfrak{I}((\forall y)A(y))$ *otherwise.*
(2) $\mathfrak{I}(C') = 1$ *if* $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ *has a maximum and* $= \mathfrak{I}((\exists y)A(y))$ *otherwise.*

Proof. (1) Let us assume that $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ has the minimum d. $\mathfrak{I}(A(d)) = \mathfrak{I}((\forall y)A(y))$ and therefore $\mathfrak{I}(A(d) \supset (\forall y)A(y)) = 1$ and $\mathfrak{I}(C) = 1$. If $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ does not have a minimum then $\mathfrak{I}(A(d)) > \mathfrak{I}((\forall y)A(y))$ for all $d \in D(\mathfrak{I})$ and, by definition of the semantics of $\supset$, $\mathfrak{I}(A(d) \supset (\forall y)A(y)) = \mathfrak{I}((\forall y)A(y))$ for all $d \in D(\mathfrak{I})$; thus also the supremum $\mathfrak{I}(C)$ gets this value.

(2) If $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ has the maximum d then, similarly, $\mathfrak{I}(A(d)) = \mathfrak{I}((\exists y)A(y))$ and $\mathfrak{I}(C') = 1$. If $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ does not have a maximum then we always have $\mathfrak{I}((\exists y)A(y)) > \mathfrak{I}(A(d))$ and $\mathfrak{I}((\exists y)A(y) \supset A(d)) = \mathfrak{I}(A(d))$, whence $\mathfrak{I}(C') = \sup \mathrm{Distr}_{\mathfrak{I}}(A(x)) = \mathfrak{I}((\exists y)A(y))$. □

Corollary 29. *Let* C *and* C' *be defined as in Proposition 28. Then* C' *is valid in both* $\mathbf{G}^0$ *and* $\mathbf{G}^1$*.* C *is valid in* $\mathbf{G}^1$ *but not in* $\mathbf{G}^0$*. Neither* C *nor* C' *are valid in* $\mathbf{G}_R$*.*

Proof. C' is valid in $\mathbf{G}^0$ because every supremum is a maximum; it is also valid in $\mathbf{G}^1$ because the only supremum which is not a maximum is 1.

C is not valid in $\mathbf{G}^0$ because there exists a sequence of truth values converging to 0, having no minimum and $\mathfrak{I}((\forall y)A(y)) = 0$. In $\mathbf{G}^1$ every infimum is also a minimum and thus C is valid in $\mathbf{G}^1$.

C and C' are both nonvalid in $\mathbf{G}_R$ because – at arbitrary places in the open interval $(0, 1)$ there are infinite (increasing and decreasing) sequences without maximum and minimum. □

Note that both C and C' are valid in classical logic and not valid in intuitionistic logic. Dummett's formula $(A \supset B) \vee (B \supset A)$ is also not valid intuitionistically, but—of course—it is true in all three infinite-valued Gödel logics: Whatever A and B evaluate to, one of them is certainly less than or equal to the other.

Proposition 210. *For any first-order formula A, $\mathbf{G}^0 \models A$ iff $\mathbf{ILC} \models A$.*

Proof. Only if: Let $\mathfrak{K} = \langle \omega, D, \{\mathbf{s}_i\}_{i\in\omega}, \mathbf{s}\rangle$ be an **ILC** interpretation. Define the maps $\hat{\varphi}_{\mathfrak{K}}\colon \mathrm{Frm}(L^{\mathfrak{K}}) \to \{\top, \bot\}^\omega$ and $\varphi_{\mathfrak{K}}\colon \mathrm{Frm}(L^{\mathfrak{K}}) \to V^0$ as follows:

$$\hat{\varphi}_{\mathfrak{K}}(A) = \langle \mathfrak{K}_i(A)\rangle_{i\in\omega}$$
$$\varphi_{\mathfrak{K}}(A) = \begin{cases} 1/(\min\{i : \mathfrak{K}_i(A) = \top\} + 1) & \text{if } \mathfrak{K}_i(A) = \top \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

Note that the monotonicity proviso in the definition of **ILC**-interpretations implies that $\hat{\varphi}(A)$ is of the form $\{\bot\}^n \frown \{\top\}^\omega$ for some $n \in \omega$, or equals $\{\bot\}^\omega$.

We can now associate to each **ILC**-interpretation $\mathfrak{K}$ a many-valued interpretation $\mathfrak{I}_{\mathfrak{K}} = \langle D, \mathbf{s}_{\mathfrak{K}}\rangle$ by setting $\mathbf{s}_{\mathfrak{K}}(A) = \varphi_{\mathfrak{K}}(A)$ for A atomic in $L^{\mathfrak{K}}$. It is then easily proved, by induction on the complexity of a formula A, that $\mathfrak{I}_{\mathfrak{K}}(A) = \varphi_{\mathfrak{K}}(A)$.

The only nontrivial induction step is that concerning implication. Suppose

$$\mathfrak{I}_{\mathfrak{K}}(A) = \varphi_{\mathfrak{K}}(A), \qquad \mathfrak{I}_{\mathfrak{K}}(B) = \varphi_{\mathfrak{K}}(B). \tag{IH}$$

(1) $\mathfrak{K}_0(A \supset B) = \top$: By definition of the **ILC**-semantics we get $\mathfrak{K}_i(A) = \bot$ or $\mathfrak{K}_i(B) = \top$ for all $i \in \omega$. Moreover, by definition of $\varphi_{\mathfrak{K}}$, we have $\varphi_{\mathfrak{K}}(A \supset B) = 1$.
   (a) $\mathfrak{K}_0(A) = \top$: By definition of the **ILC**-semantics, $\mathfrak{K}_i(A) = \top$ for all $i \in \omega$. From $\mathfrak{K}_0(A \supset B) = \top$ we thus get $\mathfrak{K}_i(B) = \top$ for all $i \in \omega$. By definition of $\varphi_{\mathfrak{K}}$, $\varphi_{\mathfrak{K}}(A) = \varphi_{\mathfrak{K}}(B) = 1$ and, by (IH), $\mathfrak{I}_{\mathfrak{K}}(A) = \mathfrak{I}_{\mathfrak{K}}(B) = 1$. So the $\mathbf{G}^0$-semantics yields $\mathfrak{I}_{\mathfrak{K}}(A \supset B) = 1$.
   (b) $\mathfrak{K}_0(A) = \bot$: Let $m_A = \min\{j : \mathfrak{K}_j(A) = \top\}$, $m_B = \min\{j : \mathfrak{K}_j(B) = \top\}$. m_A and m_B may be undefined. But by $\mathfrak{K}_0(A \supset B) = \top$ and the **ILC**-semantics m_B can only be undefined if m_A is too. If both are defined then $m_B \le m_A$. By definition of $\varphi_{\mathfrak{K}}$ this gives us $\varphi_{\mathfrak{K}}(A) \le \varphi_{\mathfrak{K}}(B)$. (IH) yields $\mathfrak{I}_{\mathfrak{K}}(A) \le \mathfrak{I}_{\mathfrak{K}}(B)$ and the semantics of $\mathbf{G}^0$ that $\mathfrak{I}_{\mathfrak{K}}(A \supset B) = 1$.

(2) $\mathfrak{K}_0(A \supset B) = \bot$: By definition of the **ILC**-semantics there exists a $j \in \omega$ s.t. $\mathfrak{K}_j(A) = \top$ and $\mathfrak{K}_j(B) = \bot$. Let m be the least such. By the **ILC**-semantics $\mathfrak{K}_j(A) = \top$ for all $j \ge m$ and $m < \min\{j : \mathfrak{K}_j(B) = \top\}$ giving $\varphi_{\mathfrak{K}}(B) < \varphi_{\mathfrak{K}}(A)$. From (IH) we thus derive $\mathfrak{I}_{\mathfrak{K}}(B) < \mathfrak{I}_{\mathfrak{K}}(A)$ and, by the semantics of $\mathbf{G}^0$, $\mathfrak{I}_{\mathfrak{K}}(A \supset B) = \mathfrak{I}_{\mathfrak{K}}(B)$.

If $\mathfrak{K}_j(B) = \bot$ for all $j \in \omega$ then, by definition of $\varphi_{\mathfrak{K}}$, $\varphi_{\mathfrak{K}}(A \supset B) = \varphi_{\mathfrak{K}}(B) = 0$. From $\mathfrak{I}_{\mathfrak{K}}(B) = \varphi_{\mathfrak{K}}(B) = 0$ we get $\varphi_{\mathfrak{K}}(A \supset B) = \mathfrak{I}_{\mathfrak{K}}(A \supset B)$.

If, on the other hand, $\mathfrak{K}_j(B) = \top$ for some $j \in \omega$ then $\varphi_{\mathfrak{K}}(B) = \varphi_{\mathfrak{K}}(A \supset B)$. By (IH) we get $\varphi_{\mathfrak{K}}(B) = \mathfrak{I}_{\mathfrak{K}}(B)$ and finally $\varphi_{\mathfrak{K}}(A \supset B) = \mathfrak{I}_{\mathfrak{K}}(A \supset B)$. This concludes the induction step.

It immediately follows that if $\mathbf{ILC} \not\models A$ then $\mathbf{G}^0 \not\models A$.

If: Conversely, let $\mathfrak{I} = \langle D, \mathbf{s}\rangle$ be a $\mathbf{G}^0$-interpretation. Let $\mathfrak{K}_{\mathfrak{I}} = \langle \omega, D, \{\mathbf{s}_i\}_{i\in\omega}, \mathbf{s}'\rangle$ be given by:

$$\mathbf{s}_i(A) = \begin{cases} \top & \text{if } \mathfrak{I}(A) \geq \frac{1}{i+1} \\ \bot & \text{otherwise.} \end{cases}$$

for atomic A, and $\mathbf{s}'$ according to the interpretation of the function symbols in $\mathbf{s}$. Again by induction on the complexity of formulas we have $\mathfrak{I}(A) = \varphi_{\mathfrak{K}_{\mathfrak{I}}}(A)$, in particular, $\mathfrak{K}_{\mathfrak{I}} \not\models_i A$ if $\mathfrak{I} \not\models_m A$. $\square$

3 Incompleteness of $\mathbf{G}^0$, ILC and IL

We proceed to prove that the valid formulas of $\mathbf{G}^0$are not recursively enumerable. In contrast to this result, all finite-valued Gödel logics are r.e. [1] as well as $\mathbf{G}_R$ [11] ($\mathbf{G}_R$ there appears as *intuitionistic fuzzy logic*).

Proposition 31. *Let $\mathfrak{I}$ be a $\mathbf{G}^0$- interpretation, A a formula and*

$$v = \mathfrak{I}((\forall x)\neg[A(x) \supset (\forall y)A(y)])$$

Then $v = 0$ if $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ has a minimum and $v = 1$ otherwise.

Proof. Just as in Proposition 28: Suppose $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ has a minimum v, let $d \in D$ be s.t. $\mathfrak{I}(A(d)) = v$. Then $\mathfrak{I}(A(d)) = \mathfrak{I}((\forall y)A(y))$, so $\mathfrak{I}(\neg[A(d) \supset (\forall y)A(y)]) = 0$. Conversely, if $\mathrm{Distr}_{\mathfrak{I}}(A(x))$ has no minimum, then $\mathfrak{I}((\forall y)A(y)) = 0$ and $\mathfrak{I}(A(d)) > 0$ for all $d \in D$. Hence $\mathfrak{I}(A(d) \supset (\forall y)A(y)) = 0$, and $\mathfrak{I}(\neg[A(d) \supset (\forall y)A(y)]) = 1$ for all $d \in D$. $\square$

In order to prove the main theorem of this chapter we need some tools from recursion theory.

Definition 32. Let ψ be an effective recursive enumeration of the set PR^1_1 of all primitive recursive functions from ω to ω. We define a two place function φ (which enumerates a subclass of PR^1_1):

$$\varphi_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } \psi_k(y) = 0 \text{ for } 1 \leq y \leq x \\ 1 & \text{otherwise} \end{cases}$$

The index set O_φ is defined as $\{k : (\forall y)\varphi_k(y) = 0\}$.

Proposition 33. *The index set O_φ is not recursively enumerable.*

Proof. By definition of φ, $\{k : (\forall y)\varphi_k(y) = 0\} = \{k : (\forall y)\psi_k(y) = 0\}$. But for every $g \in \mathrm{PR}^1_1$ the index set $\{k : (\forall y)\psi_k = g\}$ is Π_1-complete. Therefore O_φ is Π_1-complete and thus not recursively enumerable. $\square$

The essence of the incompleteness proof is represented by a sequence of formulas $(A_k)_{k\in\omega}$ constructed via φ s.t.

$$\mathbf{G}^0 \models A_k \Longleftrightarrow k \in O_\varphi$$

i.e. O_φ is m-reducible to the validity problem of $\mathbf{G}^0$.

Definition 34. Let P be a one-place predicate symbol, s be the function symbol for the successor function and $\bar{0}$ be the constant symbol representing 0 (in particular, we choose a signature containing this symbol and all symbols from Robinson's arithmetic Q).

Let A_1 be a conjunction of axioms strong enough to represent every recursive function (e.g. the axioms of Q) and a defining axiom for the function φ s.t. every atomic formula is negated or doubly negated. We define the formulas A_2, A_3^k, A_4^k, A_5^k for $k \in \omega$; for formulas representing the equality $\varphi_k(x) = 0$ we write $[\varphi_k(x) = 0]$.

$$\begin{aligned}
A_2 &\equiv (\forall x)\neg\neg P(x) \\
A_3^k &\equiv (\forall x, y)(\neg[\varphi_k(x) = 0] \wedge \neg\neg x \leq y \supset \neg[\varphi_k(y) = 0]) \\
A_4^k &\equiv (\forall x)(\neg[\varphi_k(x) = 0] \supset (P(\bar{0}) \supset P(x)) \\
A_5^k &\equiv (\forall x)\{\neg\neg[\varphi_k(s(x)) = 0] \supset \\
&\quad \supset [(P(x) \supset P(s(x))) \supset P(s(x))] \wedge [P(s(x)) \supset P(\bar{0})]\}
\end{aligned}$$

Finally we set

$$B_k \equiv A_1 \wedge A_2 \wedge A_3^k \wedge A_4^k \wedge A_5^k$$

and

$$A_k \equiv B_k \supset ((\forall x)\neg[P(x) \supset (\forall y)P(y)] \vee P(0)).$$

The double negations in Definition 34 serves the purpose of giving classical truth values to the formulas; note that, for a $\mathbf{G}^0$-interpretation $\mathfrak{I}$, $\mathfrak{I}(B) > 0$ implies $\mathfrak{I}(\neg\neg B) = 1$ (clearly $\mathfrak{I}(B) = 0$ implies $\mathfrak{I}(\neg\neg B) = 0$). Therefore the formulas in A_1, A_2 and A_3^k may only the receive the truth values 0 and 1 and thus have a classical meaning. Intuitively A_2 expresses that P is always true, A_k^3 states that $\varphi_k(x) = 0$ implies that $\varphi_k(y) = 0$ for all y greater than x. A_4^k and A_5^k are not classical in the sense that they may assume truth values between 0 and 1. $\mathfrak{I}(A_4^k) = 1$ means (according to the $\supset$-semantics of $\mathbf{G}^0$) that for all x with $\varphi_k(x) \neq 0$, $\mathfrak{I}(P(0))$ is less or equal to $\mathfrak{I}(P(x))$.

Lemma 35. *If $\mathfrak{I}$ is a $\mathbf{G}^0$-interpretation s.t. $\mathfrak{I}(B_k) < 1$, then $\mathfrak{I}(A_k) = 1$.*

Proof.

(1) $\mathfrak{I}(A_1 \wedge A_2 \wedge A_3^k \wedge A_4^k) < 1$: If $\mathfrak{I}(A_1 \wedge A_2 \wedge A_3^k) < 1$ then the value is actually 0 and thus $\mathfrak{I}(B_k) = 0$; $\mathfrak{I}(A_k) = 1$ is a trivial consequence. If $\mathfrak{I}(A_1 \wedge A_2 \wedge A_3^k) = 1$ and $\mathfrak{I}(A_4^k) < 1$ then there must be some d s.t. $\mathfrak{I}(\neg[\varphi_k(d) = 0]) = 1$ and $\mathfrak{I}(P(\bar{0}) \supset P(d)) < 1$. But then, by the semantics of $\mathbf{G}^0$, $\mathfrak{I}(P(d)) \leq (P(\bar{0}))$. Therefore $\mathfrak{I}(A_4^k) \leq \mathfrak{I}(P(\bar{0}))$ and also $\mathfrak{I}(B_k) \leq \mathfrak{I}(P(\bar{0}))$. But $P(\bar{0})$ occurs disjunctively in A_k and so $\mathfrak{I}(A_k) = 1$.

(2) $\mathfrak{I}(A_1 \wedge A_2 \wedge A_3^k \wedge A_4^k) = 1$: If $\mathfrak{I}(P(\bar{0})) = 1$ then clearly $\mathfrak{I}(A_k) = 1$. Thus let us assume that $\mathfrak{I}(P(\bar{0})) < 1$. As $\mathfrak{I}(B_k) < 1$ we must have $\mathfrak{I}(A_5^k) < 1$. That means there exists some d s.t. $\mathfrak{I}(\neg\neg[\varphi_k(s(d)) = 0]) = 1$ and

$$\mathfrak{I}(((P(d) \supset P(s(d))) \supset P(s(d))) \wedge (P(s(d)) \supset P(\bar{0}))) < 1. \qquad (*)$$

For such a d we either have $(\mathfrak{I}(P(d)) \leq \mathfrak{I}(P(s(d)))$ and $\mathfrak{I}(P(s(d))) < 1)$ or $\mathfrak{I}(P(\bar{0})) < \mathfrak{I}(P(s(d)))$. In the latter case we get $\mathfrak{I}(P(s(d)) \supset P(\bar{0})) = \mathfrak{I}(P(\bar{0}))$. Thus in any case $(*)$ gets a value $\leq \mathfrak{I}(P(\bar{0}))$ and $\mathfrak{I}(B_k) \leq \mathfrak{I}(P(\bar{0}))$. Again we obtain $\mathfrak{I}(A_k) = 1$. $\square$

Theorem 36. *The $\exists$-free fragment of $\mathbf{G}^0$is not recursively enumerable.*

Proof. We show that $\mathbf{G}^0 \models A_k$ iff $k \in O_\varphi$ (i.e. iff for all x, $\varphi_k(x) = 0$). The sequence $(A_k)_{k\in\omega}$ is $\exists$-free and (trivially) r.e. Thus a recursive enumeration of all A_k with $\mathbf{G}^0 \models A_k$ would give a recursive enumeration of the set O_φ which, by Proposition 33, does not exist.

Now let us assume that $\mathbf{G}^0 \models A_k$. We define a specific $\mathbf{G}^0$-interpretation $\mathfrak{N}_k$: The domain of $\mathfrak{N}_k$ is the set of natural numbers ω and the evaluation function $\mathbf{s}$ for the atoms is defined by:

$$\mathbf{s}(P(s^n(\bar{0}))) = \begin{cases} \frac{1}{n+2} & \text{if } \varphi_k(n) = 0 \\ 1 & \text{if } \varphi_k(n) > 0 \end{cases}$$

For all other atoms A we set $\mathbf{s}(A) = 1$ if $\mathbb{N} \models A$ and $\mathbf{s}(A) = 0$ otherwise ($\mathbb{N} \models A$ means that A is true in the standard model $\mathbb{N}$). Note that $P(\bar{0})$ receives the value $\frac{1}{2}$.

By definition of $\mathfrak{N}_k$ all conjuncts of B_k are verified and so $\mathfrak{N}_k(B_k) = 1$. By $\mathbf{G}^0 \models A_k$ we must have $\mathfrak{N}_k \models_m A_k$ and therefore

$$\mathfrak{N}_k \models_m (\forall x)\neg[P(x) \supset (\forall y)P(y)] \vee P(\bar{0}).$$

From $\mathfrak{N}_k(P(\bar{0})) = \frac{1}{2}$ we infer

$$\mathfrak{N}_k \models_m (\forall x)\neg[P(x) \supset (\forall y)P(y)].$$

By Proposition 31 the last property only holds if $\mathrm{Distr}_{\mathfrak{N}_k}(P(x))$ does not have a minimum.

We show now that φ_k must be the constant function 0. We assume that there exists a number r s.t. $\varphi_k(r) \neq 0$ and derive a contradiction: By definition of $\mathbf{s}$ we obtain $\mathbf{s}(P(s^r(\bar{0}))) = 1$. But $\mathfrak{N}_k \models_m A_3^k$ what implies that for all number terms (i.e. successor terms) $s^p(\bar{0})$ with $p \geq r$ the formula $\neg[\varphi_k(s^p(\bar{0})) = 0]$ evaluates to 1. As A_1 represents φ we obtain $\varphi_k(p) \neq 0$ for all $p \geq r$ and, by definition of φ_k, $\varphi_k(p) = 1$ for all $p \geq r$. By definition of $\mathbf{s}$ we thus obtain

$$\mathbf{s}(P(s^p(\bar{0}))) = 1 \text{ for all p } \geq r$$

But $\mathfrak{N}_k(P(\bar{0})) = \frac{1}{2}$ and, consequently, for almost all p $\mathfrak{N}_k(P(s^p(\bar{0}))) > \frac{1}{2}$. Therefore $\mathrm{Distr}_{\mathfrak{N}_k}(P(x))$ has a minimum; a contradiction. Note that by the choice of the standard model $\mathbb{N}$ we only have standard elements in our domain (i.e. elements which are represented by successor terms). So there cannot be another sequence in the set $\mathrm{Distr}_{\mathfrak{N}_k}(P(x))$ which converges to 0. We infer that φ_k must be identical to 0, and so $k \in O_\varphi$.

For the other direction let us assume that $k \in O_\varphi$, i.e. $\varphi_k(n) = 0$ for all n. As A_1 represents φ the formula $[\varphi_k(s^\ell(0)) = 0]$ is provable for all $\ell \in \omega$. Now let $\mathfrak{I}$ be an arbitrary $\mathbf{G}^0$-interpretation of A_k. If $\mathfrak{I}(B_k) < 1$ then, by Lemma 35, $\mathfrak{I}(A_k) = 1$. Thus it remains to investigate the case $\mathfrak{I}(B_k) = 1$. By definition of B_k, $\mathfrak{I}(B_k) = 1$ implies $I(A_5^k) = 1$. We substitute all ground terms $s^n(\bar{0})$ into the matrix of A_5^k. These instances are true in $\mathfrak{I}$ either if all $P(s^n(\bar{0}))$ evaluate to 1 (in which case A_k is true because also $P(\bar{0})$ is true) or the sequence $\mathfrak{I}(P(s^n(\bar{0}))_{n\in\omega}$ is strictly decreasing. In the last case the sequence must converge to 0. By the axiom A_2 no element of this sequence is actually $= 0$; this property also holds for all (potential) nonstandard elements, which may be present as the domain is arbitrary. As a consequence $\mathrm{Distr}_{\mathfrak{I}}(P(x))$ does not have a minimum. Proposition 31 then implies that

$$\mathfrak{I} \models_m (\forall x)\neg[P(x) \supset (\forall y)P(y)].$$

But the last formula occurs disjunctively in the consequent of A_k and thus $\mathfrak{I} \models_m A_k$. Putting things together we see that A_k evaluates to 1 under all $\mathbf{G}^0$-interpretations, i.e., $\mathbf{G}^0 \models A_k$. $\square$

Corollary 37. *(1)* **ILC** *is not recursively enumerable.*
(2) **IL** *is not recursively enumerable.*

Proof. (1) Immediate by Proposition 210.

(2) We show: $\mathbf{IL} \models A_k$ iff $\varphi_k(n) = 0$ for all n. If $\varphi_k(n) \neq 0$ for some n, then $\mathbf{G}^0 \not\models A_k$ by Theorem 36. Since $\mathbf{G}^0 = \mathbf{ILC}$, there is some $\mathbf{ILC}$-interpretation $\mathfrak{K}$ s.t. $\mathfrak{K} \not\models_i A_k$; but $\mathfrak{K}$ is also an $\mathbf{IL}$-interpretation, so $\mathbf{IL} \not\models A_k$.

So suppose that $\varphi_k(n) = 0$ for all $n \in \omega$. Let $\mathfrak{K} = \langle \omega, (D_i)_{i\in\omega}, \{s_i\}_{i\in\omega}, \mathbf{s}\rangle$ be an $\mathbf{IL}$–interpretation. Then, by definition of the formulas A_k and B_k, $\mathfrak{K}_0(B_k) = \bot$ implies $\mathfrak{K}_0(A_k) = \top$. It remains to investigate the case $\mathfrak{K}_0(B_k) = \top$.

All domains D_i of $\mathfrak{K}$ must contain the interpretation of the number terms $s^n(\bar{0})$. Therefore either $\mathfrak{K}_0(P(\bar{0})) = \top$, in which case $\mathfrak{K}_0(A_k) = 1$ by definition of A_k, or (by the proof of Theorem 36) the sequence $\varphi_{\mathfrak{K}}(P(s^n(\bar{0})))_{n\in\omega}$ is strictly decreasing. Note that we may define $\varphi_{\mathfrak{K}}$ and $\hat{\varphi}_{\mathfrak{K}}$ exactly like in Proposition 210 (although $\mathfrak{K}$ need not be an $\mathbf{ILC}$–interpretation).

So let us assume that $\varphi_{\mathfrak{K}}(P(s^n(\bar{0})))_{n\in\omega}$ is strictly decreasing. We will show that $\hat{\varphi}_{\mathfrak{K}}((\forall x)P(x)) = \{\bot\}^\omega$. Suppose, by way of contradiction, that $\hat{\varphi}_{\mathfrak{K}}((\forall x)P(x)) \neq \{\bot\}^\omega$, i.e. there exists an $i \in \omega$ s.t. $\mathfrak{K}_i((\forall x)P(x)) \neq \bot$. As $(\forall x)P(x)$ does not contain function symbols, $\mathfrak{K}_i((\forall x)P(x))$ cannot be undefined and so $\mathfrak{K}_i((\forall x)P(x)) = \top$. By definition of the $\mathbf{IL}$-semantics this implies that for all $j \geq i$ and $d \in D_j$, $\mathfrak{K}_j(P(d/x)) = \top$. In particular, we get

$$\text{for all } d \in D_i : \mathfrak{K}_i(P(d/x)) = \top.$$

As D_i contains the interpretation of all number terms we also obtain $\mathfrak{K}_i(P(s^n(\bar{0}))) = \top$ for all $n \in \omega$. Consequently $\min\{j : K_j(P(s^n(\bar{0}))) = \top\} \leq i$ for all $n \in \omega$. By definition of $\varphi_{\mathfrak{K}}$ we thus obtain

$$\varphi_{\mathfrak{K}}(P(s^n(\bar{0}))) \geq \frac{1}{i+1} \quad \text{for all } n \in \omega$$

and

$$\varphi_{\mathfrak{K}}(P(s^n(\bar{0}))) = \frac{1}{k_n+1} \quad \text{for } k_n \in \omega,\ k_n \leq i,\ n \in \omega.$$

This however contradicts our assumption that the sequence $\varphi_{\mathfrak{K}}(P(s^n(\bar{0})))_{n\in\omega}$ is strictly decreasing. So we obtain $\hat{\varphi}_{\mathfrak{K}}((\forall x)P(x)) = \{\bot\}^\omega$. However, there are no i and $d \in D_i$ s.t. $\mathfrak{K}_i(P(d/x)) = \bot$, since $\mathfrak{K}_0((\forall x)\neg\neg P(x)) = \top$ by A_2 in B_k. Therefore $\mathfrak{K}_i(P(d/x) \supset (\forall y)P(y)) = \bot$ for all $i \in \omega$ and $d \in D_i$. By the semantics of **IL** this implies

$$\mathfrak{K}_0((\forall x)\neg[P(x) \supset (\forall y)P(y)]) = \top.$$

But then $\mathfrak{K}_0(A_k) = \top$ and the reduction of O_φ to **IL** is completed. $\square$

4 Incompleteness of temporal logics

We now proceed to show that (a) $\bigcirc$-free **TL** and **TLC** and (b) **TG** and **TGC** are also not recursively axiomatizable. This strengthens the incompleteness result for **TL** of Szalas and Kröger [6] and answers a question left open in [2]. In contrast to **TG**, however, **TB** *is* r.e. [2]. An axiomatization is given by adding to first-order **S4** the axioms:

$$\bigcirc(A \supset B) \supset (\bigcirc A \supset \bigcirc B)$$
$$\neg\bigcirc A \leftrightarrow \bigcirc\neg A$$
$$\bigcirc\Box A \wedge A \leftrightarrow \Box A$$

and the rule $A/\bigcirc A$.

Definition 41. We define the operator Ψ as follows:

$$\begin{aligned}
\Psi(A) &= \Box A \qquad A \text{ atomic}\\
\Psi(A \vee B) &= \Psi(A) \vee \Psi(B)\\
\Psi(A \wedge B) &= \Psi(A) \wedge \Psi(B)\\
\Psi(A \supset B) &= \Box[\Psi(A) \supset \Psi(B)]\\
\Psi(\neg A) &= \Box(\neg\Psi(A))\\
\Psi((\forall x)A(x)) &= \Box(\forall x)\Psi(A(x))
\end{aligned}$$

Proposition 42. *Let A be an $\exists$-free first-order formula. Then*

(1) **TL** $\models \Psi(A)$ *iff* **IL** $\models A$ *and*
(2) **TLC** $\models \Psi(A)$ *iff* **ILC** $\models A$.

Proof. Suppose **IL** $\not\models A$, let $\mathfrak{K} = \langle \omega, D_i, \{\mathbf{s}_i\}, \mathbf{s}\rangle$ be a countermodel. We can interpret $\mathfrak{K}$ as a **TL**-interpretation $\mathfrak{K}^t$. By induction on the complexity of a formula A and using the monotonicity property of $\mathfrak{K}$ we have $\mathfrak{K}^t_i(\Psi(A)) = \mathfrak{K}_i(A)$. Hence, $\mathfrak{K}^t \not\models_t \Psi(A)$.

Conversely, let $\mathfrak{K}^t = \langle \omega, D_i, \{\mathbf{s}_i^t\}, \mathbf{s}\rangle$ be a **TL**-interpretation s.t. $\mathfrak{K}^t \not\models_t \Psi(A)$. Then define $\mathfrak{K} = \langle \omega, D_i, \{\mathbf{s}_i\}, \mathbf{s}\rangle$ by $\mathbf{s}_i(A) = \mathfrak{K}_i^t(\Box(A))$. Again, by an easy induction on the complexity of A we have $\mathfrak{K}_i^t(\Psi(A)) = \mathfrak{K}_i(A)$. Thus, $\mathfrak{K} \not\models_i A$.

Similarly, for **TLC** and **ILC**. $\Box$

Corollary 43. *The* $\exists$- *and* $\bigcirc$*-free fragments of* **TL** *are not r.e.*

Proof. By Corollary 37 and Proposition 42. $\Box$

The reader will note the similarity between the above embedding of **IL** in **TL** with Gödel's, and McKinsey and Tarski's embeddings of intuitionistic predicate logic into **S4**.

In contrast to first order **TL**, propositional **TL** is axiomatizable (even with $\bigcirc$). In [5] it is shown that we get $\mathbf{TL}_{\mathrm{prop}}$ by adding to $\mathbf{TB}_{\mathrm{prop}}$ the rule

$$\frac{A \supset B \quad A \supset \bigcirc B}{A \supset \Box B}$$

Definition 44. The set $S^*(A)$ of *strict subformulas* of a formula A is defined as follows:

$$S^*(A) = \{A\} \cup \begin{cases} S^*(B) & \text{if } A \equiv \neg B \\ S^*(B) \cup S^*(C) & \text{if } A \equiv B \wedge C,\ B \vee C, \text{ or } B \supset C \\ S^*(B(x)) & \text{if } A \equiv (\forall x)B(x) \text{ or } (\exists x)B(x) \end{cases}$$

Let $P_1, \ldots, P_m$ be the predicate symbols with occurring in A with arities $r_1, \ldots, r_m$ and $\bar{x}_{r_1}, \ldots, \bar{x}_{r_m}$ corresponding variable vectors. Then we define

$$S(A) = S^*(A) \cup \{P_i(\bar{x}_{r_i}) : i = 1, \ldots, m\}$$

Definition 45. Let A be a first-order formula without $\exists$. Define

$$C_A \equiv \Box \bigwedge_{B(\bar{x}) \in S(A)} (\forall \bar{x})\big(B(\bar{x}) \leftrightarrow \bigcirc B(\bar{x})\big)$$

Let $\mathbf{TG}^*$ be the logic based on **TG**-interpretations where the domains within an ω-sequence are equal, i.e., for all i, j we have $D_{i\cdot\omega+j} = D_{i\cdot\omega}$.

Proposition 46. *Suppose A is* $\bigcirc$- *and* $\exists$*-free. Then* $\mathbf{TG} \models C_A \supset A$ *iff* $\mathbf{TG}^* \models C_A \supset A$.

Proof. Only if: Immediate. If: Let $\mathfrak{K}$ be a **TG**-interpretation s.t. $\mathfrak{K} \not\models_t C_A \supset A$. Let $\mathfrak{K}^* = \langle D_i^*, \mathbf{s}_i^*\rangle$ be defined by $D^*_{i\cdot\omega+j} = D_{i\cdot\omega}$ and $\mathbf{s}_i^* = \mathbf{s}_i \restriction D_i^*$. We prove by induction on the complexity of $A(\bar{c})$ (for $\bar{c} \in D^*_{i\cdot\omega}$) that $\mathfrak{K}_{i\cdot\omega+j}(A) = \mathfrak{K}^*_{i\cdot\omega+j}(A)$. This is immediately seen for A atomic or with outermost logical symbol a propositional connective. If $A \equiv (\forall x)B(x)$ we argue as follows: Let (i, j) denote $i \cdot \omega + j$. If $\mathfrak{K}_{i,j}(A) = \top$ then $\mathfrak{K}_{i,j}(B(d)) = \top$ for all $d \in D_{i,j} \supseteq D_{i,0}$. By induction hypothesis, $\mathfrak{K}^*_{i,j}(B(d)) = \top$ for all $d \in D^*_{i,j}$, so $\mathfrak{K}^*_{i,j}(A) = \top$.

If $\mathfrak{K}_{i,j}(A) = \bot$ then for some $d \in D_{i,j}$, $\mathfrak{K}_{i,j}(B(d)) = \bot$. Suppose for all $d \in D_{i,0}$, $\mathfrak{K}_{i,j}(B(d)) = \top$. Then we have $\mathfrak{K}_{i,0}((\forall x)\bigcirc^j B(x) \land \neg\bigcirc^j(\forall x)B(x)) = \top$. Since we have $\mathfrak{K}_{i,0} \models (\forall x)(B(x) \leftrightarrow \bigcirc B(x))$ and $\mathfrak{K}_{i,0} \models (\forall x)B(x) \leftrightarrow \bigcirc(\forall x)B(x))$ this gives $\mathfrak{K}_{i,0} \models (\forall x)B(x) \land \neg(\forall x)B(x)$, a contradiction. Hence actually there is $d \in D_{i,0}$ s.t. $\mathfrak{K}_{i,j}(B(d)) = \bot$. By induction hypothesis, $\mathfrak{K}^*_{i,j}(B(d)) = \bot = \mathfrak{K}^*_{i,j}(A)$. $\square$

Let $\mathbf{TG}^{**}$ ($\mathbf{TGC}^{**}$) be the logic based on $\mathbf{TG}$($\mathbf{TGC}$)-interpretations where the worlds within an ω-sequence are equal, i.e., for all i, j we have $D_{i,j} = D_{i,0}$ and $\mathbf{s}_{i,j} = \mathbf{s}_{i,0}$.

Proposition 47. *Let A be a $\bigcirc$ and $\exists$-free formula.*

(1) $\mathbf{TGC}^* \models C_A \supset A$ *iff* $\mathbf{TGC}^{**} \models C_A \supset A$.
(2) $\mathbf{TG}^* \models C_A \supset A$ *iff* $\mathbf{TG}^{**} \models C_A \supset A$.

Proof. Obvious, since A does not contain $\bigcirc$. $\square$

Proposition 48. *Let A be an $\bigcirc$- and $\exists$-free formula.*

(1) $\mathbf{TGC}^{**} \models C_A \supset A$ *iff* $\mathbf{TLC} \models A$.
(2) $\mathbf{TG}^{**} \models C_A \supset A$ *iff* $\mathbf{TL} \models A$.

Proof. (1) Only if: Suppose $\mathbf{TLC} \not\models A$, let $\mathfrak{K} = \langle \omega, D, \{\mathbf{s}_i\}, \mathbf{s}\rangle$ be a countermodel. Define $\mathfrak{K}^g = \langle \omega \cdot \omega, D, \{\mathbf{s}^g_i\}_{i \in \omega\cdot\omega}, \mathbf{s}\rangle$ by $\mathbf{s}^g_{j,k} = \mathbf{s}_j$ where $i, j, k \in \omega$. Clearly, $\mathfrak{K}^g \not\models_t C_A \supset A$.

If: Suppose $\mathbf{TGC}^{**} \not\models C_A \supset A$, let $\mathfrak{K} = \langle \omega \cdot \omega, D, \{\mathbf{s}_i\}, \mathbf{s}\rangle$ be a countermodel. Since $\mathfrak{K} \models C_A$ and the domains are constant, $\mathfrak{K}_{j,k}(A) = \mathfrak{K}_{j,\ell}(A)$ for A, and in particular $\mathbf{s}_{j,k} = \mathbf{s}_{j,\ell}$. Define $\mathfrak{K}^\ell = \langle \omega, D, \{\mathbf{s}^\ell_i\}, \mathbf{s}\rangle$ by $\mathbf{s}^\ell_j = \mathbf{s}_{j\omega}$ for $j \in \omega$. Again by induction on the complexity of A it is easily shown that $\mathfrak{K}^\ell_j(A) = \mathfrak{K}^\ell_{j,k}(A)$. So $\mathfrak{K}^\ell \not\models_t A$.

(2) Similarly. $\square$

Corollary 49. *The $\exists$-free fragments of* $\mathbf{TG}$ *and* $\mathbf{TGC}$ *are not r.e.*

Proof. By Corollary 43 and Propositions 46, 47 and 48. $\square$

For an axiomatization of the propositional logic $\mathbf{TG}_{\text{prop}}$ it is convenient to introduce a new connective $\triangle$ defined by $\mathfrak{K}_{i,j}(\triangle A) = 1$ iff $\mathfrak{K}_{i+1,0}(A) = 1$. Then $\mathbf{TG}_{\text{prop}}$ is axiomatized by $\mathbf{S4}$ plus

$$\begin{array}{cc} \bigcirc(A \supset B) \supset (\bigcirc A \supset \bigcirc B) & \triangle(A \supset B) \supset (\triangle A \supset \triangle B) \\ \bigcirc\neg A \leftrightarrow \neg\bigcirc A & \triangle\neg A \leftrightarrow \neg\triangle A \\ \bigcirc\triangle A \leftrightarrow \triangle A & \bigcirc\Box A \land \triangle\Box A \land A \leftrightarrow \Box A \end{array}$$

and the rules

$$\frac{A}{\bigcirc A} \qquad \frac{A}{\triangle A} \qquad \frac{A \supset B \quad A \supset \bigcirc A \quad A \supset \triangle A}{A \supset \Box B}$$

5 Conclusion

We used the main result of this paper, the incompleteness of the infinitely valued first-order Gödel logic based on the domain of truth values $V_0 : \{1/k : k \geq 1\} \cup \{0\}$, to demonstrate the incompleteness of first-order discrete linear temporal logics with/without time gaps and with/without constant domains. The first-order discrete branching time logic with time gaps, however, is complete, but it is an open question whether the same applies for the same logic with constant domains. The infinitely valued first-order Gödel logics define another field of future research; we conjecture that the logic based on $V_1 = \{1-1/k : k \geq 1\} \cup \{1\}$ and the $\exists$-fragment of the Gödel logic based on V_0 are recursively axiomatizable.

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