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Compact Propositional Gödel Logics

Matthias Baaz
Technische Universität Wien
Institut für Algebra und
Diskrete Mathematik E118.2
A–1040 Vienna, Austria
baaz@logic.at

Richard Zach*
Technische Universität Wien
Institut für Computersprachen E185.2
A–1040 Vienna, Austria
zach@logic.at

Abstract

Entailment in propositional Gödel logics can be defined in a natural way. While all infinite sets of truth values yield the same sets of tautologies, the entailment relations differ. It is shown that there is a rich structure of infinite-valued Gödel logics, only one of which is compact. It is also shown that the compact infinite-valued Gödel logic is the only one which interpolates, and the only one with an r.e. entailment relation.

1 Introduction

Traditionally, the study of many-valued logics has identified logics with their sets of tautologies and centered on their study as such. With the notable exception of Kleene logic, which lacks tautologies altogether, the study of the *entailment* relations of many-valued logics has taken the back seat. It has, however, become increasingly obvious that such a study is called for, especially in cases where many-valued logics are applied in computer science to reasoning about various domains. For instance, Avron [1] has argued that Gödel logics are suited to formalize properties of concurrency and advocated a view of logics primarily as entailment relations. The present paper aims to take a first step in the investigation of the structure and properties of entailment in propositional Gödel logics. Unless otherwise noted, we will consider a logic as given by an entailment relation, not as a set of tautologies.

Gödel logics were introduced by Kurt Gödel [5] and also extensively studied, as sets of tautologies, by Dummett [4]. The set of truth values can always be taken to be a subset of the real interval $[0,1]$, containing 0 and 1 and closed under greatest lower bound. 1 is the designated truth value. The language we consider consist of a denumerably infinite set of variables var, the logical constants $\top$ and $\bot$, and connectives $\land$, $\lor$, $\to$.

A *valuation* v on a set of truth values V is a function mapping var to V. The valuation is extended in the standard way to formulas by:

$$\begin{aligned} v(\top) &= 1 \\ v(\bot) &= 0 \\ v(\phi \land \psi) &= \min(v(\phi), v(\psi)) \\ v(\phi \lor \psi) &= \max(v(\phi), v(\psi)) \\ v(\phi \to \psi) &= \begin{cases} 1 & \text{if } v(\phi) \le v(\psi) \\ v(\psi) & \text{otherwise.} \end{cases} \end{aligned}$$

Negation is defined by $\neg\phi = \phi \to \bot$; equivalence by $\phi \equiv \psi = (\phi \to \psi) \land (\psi \to \phi)$.

An *entailment pair* is a pair $\Pi \vDash \phi$, where Π is a set of formulas. We extend the definition of valuations to entailment pairs by

$$v(\Pi \vDash \phi) = \begin{cases} 1 & \text{if } \inf\{v(\psi) : \psi \in \Pi\} \le v(\phi) \\ v(\phi) & \text{otherwise} \end{cases}$$

for Π any set of formulas. It is easily seen that $v(\phi) = v(\emptyset \vDash \phi)$ if we take the inf over the empty set to be 1. We may thus define: $\vDash_V \phi$ and $\Pi \vDash_V \phi$ iff $v(\phi) = 1$ and $v(\Pi \vDash \phi) = 1$ for all valuations v over a given set of truth values V, respectively. Furthermore, we define:

$$\begin{aligned} \mathrm{Taut}(V) &= \{\phi : \vDash_V \phi\} \\ \mathrm{Ent}(V) &= \{\langle \Pi, \phi\rangle : \Pi \vDash_V \phi\} \end{aligned}$$

It is well known that $\mathrm{Taut}(V)$ only depends on the cardinality of V, and in particular that $\mathrm{Taut}(V) = \mathrm{Taut}(V')$ whenever V and V' are both infinite. However, in general $\mathrm{Ent}(V)$ and $\mathrm{Ent}(V')$ will differ even if V and V' have the same infinite cardinality but differ in order type.

1.1. DEFINITION Let

$$V_k = \{1 - 1/n : 1 \le k - 1\} \cup \{0\}$$

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$$\begin{aligned} V_\uparrow &= \{1-1/n : n\in\omega-\{0\}\}\cup\{1\}\\ V_\downarrow &= \{1/n : n\in\omega-\{0\}\}\cup\{0\}\\ V_\infty &= [0,1] \end{aligned}$$

If V is a truth value set, $\mathbf{G}_V$ is the Gödel logic corresponding to that set, i.e., $\mathbf{G}_V = \mathrm{Ent}(V)$. In particular, $\mathbf{G}_n$, $\mathbf{G}_\downarrow$, $\mathbf{G}_\uparrow$, and $\mathbf{G}_\infty$ stand for $\mathrm{Ent}(V_n)$, $\mathrm{Ent}(V_\downarrow)$, $\mathrm{Ent}(V_\uparrow)$, and $\mathrm{Ent}(V_\infty)$, respectively.

Note that

$$v((X\to Z)\to Z) = \begin{cases} 1 & \text{if } v(X) > v(Z)\\ v(Z) & \text{otherwise.}\end{cases}$$

It is then easily seen that $\Pi_1 \vDash_{V_\downarrow} Y$, $\Pi_2 \vDash_{V_\uparrow} Y$ but $\Pi_1 \nvDash_{V_\uparrow} Y$,[1] $\Pi_2 \nvDash_{V_\downarrow} Y$ and also $\Pi_1 \nvDash_{V_\infty} Y$ and $\Pi_2 \nvDash_{V_\infty} Y$, where Let

$$\begin{aligned} \Pi_1 &= \{(X_2\to X_1)\to X_1, (X_3\to X_2)\to X_2,\dots\\ &\quad X_1\to Y, X_2\to Y,\dots\}\\ \Pi_2 &= \{(X_1\to X_2)\to X_2, (X_2\to X_3)\to X_3,\dots\\ &\quad X_1\to Y, X_2\to Y,\dots\} \end{aligned}$$

Thus $\mathbf{G}_\downarrow \neq \mathbf{G}_\uparrow \neq \mathbf{G}_\infty \neq \mathbf{G}_\downarrow$, even though all three truth value sets are infinite and $|V_\downarrow| = |V_\uparrow|$.

Our aim shall thus be to establish the inclusion relationships of the various Gödel logics and in particular to characterize those which have a compact entailment relation. Entailment is compact iff, as usual, $\Pi\vDash\phi$ implies that for some finite $\Pi'\subseteq\Pi$, $\Pi'\vDash\phi$. As corollaries, we will obtain characterizations of the logics with r.e. entailment relations and those with interpolating entailment.

2 Structure of entailment relations

The following proposition will be used frequently in what follows.

2.1. PROPOSITION *Let $w\in V$ for V a set of truth values, and v a valuation on V. Define*

$$v_w(X) = \begin{cases} v(X) & \text{if } v(X)\le w\\ 1 & \text{otherwise.}\end{cases}$$

Then

$$v_w(\psi) = \begin{cases} v(\psi) & \text{if } v(\psi)\le w\\ 1 & \text{otherwise}\end{cases}$$

for all ψ.

Proof. By induction on the complexity of ψ. If ψ a variable, the claim is true by definition of v_w. If $\psi = \psi'\wedge\psi''$, and $v(\psi')\le v(\psi'')$, then $v_w(\psi) = v(\psi') = v(\psi)$ if $v(\psi')\le w$ or $v_w(\psi) = 1 = v(\psi)$ if $v(\psi') > w$. Similarly if ψ is a disjunction.

If $\psi = \psi'\to\psi''$, we distinguish three cases. If $v(\psi) = w = 1$, then $v(\psi')\le v(\psi'')\le w$. By induction hypothesis, $v_w(\psi') = v(\psi')$ and $v_w(\psi'') = v(\psi'')$, so $v_w(\psi) = 1 = v(\psi)$. Now suppose that $v(\psi)\le w < 1$. Then $v(\psi'')\le w$ and by induction hypothesis, $v_w(\psi'') = v(\psi'')$. Since $v'(\psi')\ge v(\psi')$, we have $v_w(\psi) = v(\psi)$. Finally, suppose that $v(\psi) > w$. Then either $v(\psi') > v(\psi'') = v(\psi) > w$, or $v(\psi')\le v(\psi'')$. In the former case, $v_w(\psi') = v_w(\psi'') = 1$, and so $v_w(\psi) = 1$. In the latter case, again, we have two cases: (1) $v(\psi')\le w$. Then $v_w(\psi') = v(\psi')\le v_w(\psi'')$. (2) $v(\psi') > w$, and so $v_w(\psi') = v_w(\psi'') = 1$. In either case, $v_w(\psi) = 1$.[2] □

One might wonder whether a different definition of the entailment relation in Gödel logic might give different results. Another standard way of defining entailment in many-valued logics is:

$$\Pi \Vdash \phi \quad\text{iff}\quad \text{for all } v,\ (\forall\psi\in\Pi)(v(\psi)=1)\Rightarrow v(\phi)=1$$

This definition yields the same results, as the following proposition shows:

2.2. PROPOSITION $\Pi\vDash_V\phi$ *iff* $\Pi\Vdash_V\phi$

Proof. Only if: Immediate. If: Suppose v witnesses $\Pi\nvDash\phi$, i.e., $v(\Pi\vDash\phi) = w < 1$. Then, by Lemma 2.1, $v_w(\psi) = 1$ for all $\psi\in\Pi$ and $v_w(\phi) = w < 1$. □

This allows us to use the characterization of $\vDash$ or $\Vdash$ as proves convenient.

It is an easy but fundamental result that $\mathrm{Taut}(V)$ and $\mathrm{Ent}(V)$ depend only on the order type of V. This is made precise in the following

2.3. PROPOSITION *Let v and v' be valuations, not necessarily on the same sets of truth values, such that $v(X) = 1$ iff $v'(X) = 1$, $v(X) < v(Y)$ iff $v'(X) < v'(Y)$, and $v(X) = v(Y)$ iff $v'(X) = v'(Y)$ (for all X, Y). Then $v(\phi) = 1$ iff $v'(\phi) = 1$ and $v(\phi) = v(X)$ iff $v'(\phi) = v'(X)$.*[3]

2.4. PROPOSITION *(1) If $|\mathrm{var}(\phi)|\le n$ and $\vDash_{V_{n+2}}\phi$, then $\vDash_V \phi$ for all V with $|V|\ge n+2$.*

(2) $\mathrm{Taut}(V)\subseteq\mathrm{Taut}(V')$ *for* $|V|\le|V'|$.[4]

Proof. This follows immediately from Proposition 2.3. The bound in (1) is tight, as the example of $X\vee\neg X$ shows for $n = 1$. □

2.5. COROLLARY $\mathrm{Taut}(V_\infty) = \bigcap_{n\in\omega}\mathrm{Taut}(V_n)$

2.6. PROPOSITION *For every ϕ there is a normal form $N(\phi)$ such that $\vDash_{V_\infty}\phi\equiv N(\phi)$. Furthermore, if for every n, the set $\{N(\phi) : \mathrm{var}(\phi) = \{X_1,\dots,X_n\}\}$ is finite and depends only on n.*

Proof. The idea of the proof is easy: By Theorem 2.3, only the order of the variables induced by a given valuation is relevant to determine the truth value of a formula. Such an ordering can be expressed by a formula of the language, using $(X \to Y) \wedge ((Y \to X) \to X)$ for: $v(X) < v(Y)$ or $v(X) = v(Y) = 1$, and $X \equiv Y$ for $v(X) = v(Y)$. Every such formula implies that ϕ is equivalent to a variable in $\mathrm{var}(\phi)$, 1, or 0. Form the disjunction over all such implications. For a detailed proof, see [3], Theorem 5. $\square$

Gödel logics were invented as a tool to study propositional intuitionistic logic. Consequently it is not surprising that there is a tight connection between Kripke semantics and Gödel logics. It is well known that $\mathrm{Taut}(V_\infty)$ equals the set of formulas valid in all linearly ordered Kripke structures. The connection extends to entailment; the truth value set corresponding to such structures with respect to entailment is $V_\downarrow$.

2.7. DEFINITION A *linear Kripke structure* k is a function from var to $\{0\}^\omega \cup \{0\}^{<\omega} \frown \{1\}^\omega$ (i.e., 0–1 sequences which, once 1, remain 1). We extend k to formulas by:

$$\begin{aligned}
k(\top)_i &= 1\\
k(\bot)_i &= 0\\
k(\phi \wedge \psi)_i &= \begin{cases} 1 & \text{if } k(\phi)_i = k(\psi)_i = 1\\ 0 & \text{otherwise}\end{cases}\\
k(\phi \vee \psi)_i &= \begin{cases} 0 & \text{if } k(\phi)_i = k(\psi)_i = 0\\ 1 & \text{otherwise}\end{cases}\\
k(\phi \to \psi)_i &= \begin{cases} 1 & \text{if, for all } j \geq i,\\ & k(\phi)_j = 0 \text{ or } k(\psi)_j = 1\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The logic of *linear intuitionistic Kripke structures* **LI** is given by

$$\Pi \Vdash \phi \text{ iff for all } k, (\forall \psi \in \Pi)(k(\psi)_0 = 1) \Rightarrow k(\phi)_0 = 1$$

2.8. PROPOSITION $\mathbf{G}_\downarrow = \mathbf{LI}$

Proof. Let k be a Kripke structure and v a $V_\downarrow$-valuation. Then

$$\bar{k}(X) = \begin{cases} 0 & \text{if } k(X) = \{0\}^\omega\\ 1/(n+1) & \text{if } n \text{ is least s.t. } k(X)_n = 1\end{cases}$$

is a $V_\downarrow$-valuation, and

$$\hat{v}(X)_i = \begin{cases} 0 & \text{if } v(X) < 1/(i+1)\\ 1 & \text{otherwise}\end{cases}$$

defines a Kripke structure. It is easily shown, by induction on formulas, that $\bar{\hat{v}} = v$ and $\hat{\bar{k}} = k$. Consequently, $k(\phi)_0 = 1$ iff $\bar{k}(\phi) = 1$ and $v(\phi) = 1$ iff $\hat{v}(\phi)_0 = 1$. Using the characterization of $\vDash$ in terms of $\Vdash$ of Proposition 2.2, the result follows. $\square$

An analogous result holds for first-order Gödel and linear intuitionistic logic [2].

2.9. LEMMA *Suppose* $v(\Pi \vDash \phi) = w$. *Then* $v_w(\Pi \vDash \phi) = w$.

Proof. Suppose, first, that $v(\Pi) > w$. Then $v_w(\psi) = 1$ for all $\psi \in \Pi$, and $v_w(\phi) = v(\phi) = w$. Otherwise, $w = 1$ and $v(\psi) \leq w$ for some $\psi \in \Pi$ and so $v_w(\Pi) \leq w$. In this case, $v(\Pi \vDash \phi) = v_w(\Pi \vDash \phi) = 1$. $\square$

2.10. PROPOSITION $\mathbf{G}_\uparrow = \bigcap_{n\in\omega} \mathbf{G}_n$.

Proof. Every V_n valuation is also a $V_\uparrow$ valuation, so $\mathrm{Ent}(V_n) \subseteq \mathrm{Ent}(V_\uparrow)$. On the other hand, if $v(\Pi \vDash \phi) = 1 - 1/n = w$, then v_w as defined in Lemma 2.1 is a V_{n+1}-valuation with $v_w(\Pi \vDash \phi) = w$, and so $\Pi \nvDash_{V_{n+1}} \phi$.[5] $\square$

2.11. PROPOSITION *Let* V *be any truth value set. Then* $\Pi \nvDash_{V_\infty} \phi$ *if* $\Pi \nvDash_V \phi$.

2.12. COROLLARY

$$\mathrm{Ent}(V_\infty) = \bigcap_{\substack{V \subseteq [0,1]\\ V \text{ countable}}} \mathrm{Ent}(V)$$

2.13. DEFINITION We define the following families of truth value sets:

$$\begin{aligned}
U_k &= \{i/(k+1) + 1/j(k+1) : 0 \leq i \leq k-1; j > 0\}^6\\
D_k &= \{i/(k+1) - 1/j(k+1) : i = 1,\dots,k; j > 0\} \cup\\
&\quad \cup \{i/(k+1); i = 0,\dots,k+1\}
\end{aligned}$$

Correspondingly, we define sets of formulas

$$\begin{aligned}
\Gamma_k &= \{(C_{i+1} \to C_i) \to C_i,\\
&\qquad C_i \to C_k : i = 0,\dots,k-1\}\\
\Upsilon^\ell &= \{(X^\ell_{i+1} \to X^\ell_i) \to X^\ell_i, C_\ell \to X^\ell_i,\\
&\qquad X^\ell_i \to C_{\ell+1} : i \in \omega\}\\
\Delta^\ell &= \{(X^\ell_i \to X^\ell_{i+1}) \to X^\ell_{i+1}, C_\ell \to X^\ell_i,\\
&\qquad X^\ell_i \to C_{\ell+1} : i \in \omega\}\\
\Upsilon_k &= \Gamma_k \cup \bigcup_{\ell=0}^{k-1} \Upsilon^\ell\\
\Delta_k &= \Gamma_k \cup \bigcup_{\ell=0}^{k-1} \Delta^\ell
\end{aligned}$$

U_k has the order type $1 + k\omega^*$, while D_k has order type $k\omega + 2$. In particular, $U_1 = V_\downarrow$.[7]

$$\begin{array}{ccccccccc}
U_k & 0 & \underbrace{\cdots\cdots} & \frac{1}{k+1} & \underbrace{\cdots\cdots} & \frac{2}{k+1} & \cdots & \frac{k}{k+1} & 1\\
 & C_0 & X^0_i & C_1 & X^1_i & C_2 & \dots & C_k & \\
D_k & 0 & \overbrace{\cdots\cdots} & \frac{1}{k+1} & \overbrace{\cdots\cdots} & \frac{2}{k+1} & \cdots & \frac{k}{k+1} & 1
\end{array}$$

2.14. PROPOSITION *(1)* $\Delta_r \vDash_{D_k} C_r$ *for all* r.

(2) $\Delta_r \not\models_{U_k} C_r$ *for* $r \leq k$.

(3) $\Delta_r \models_{U_k} C_r$ *for all* $r > k$.

(4) $\Upsilon_r \models_{U_k} C_r$ *for all* r.

(5) $\Upsilon_r \not\models_{D_k} C_r$ *for* $r \leq k$.

(6) $\Upsilon_r \models_{D_k} C_r$ *for all* $r > k$.

(7) If $\Gamma \not\models_{D_k} \phi$ *then* $\Gamma \not\models_{D_r} \phi$ *for all* $r \geq k$.

(8) If $\Gamma \not\models_{U_k} \phi$ *then* $\Gamma \not\models_{U_r} \phi$ *for all* $r \geq k$.[8]

We thus have the following picture:

$$\mathbf{G}_2 \supsetneq \mathbf{G}_3 \supsetneq \ldots \supsetneq \bigcap \mathbf{G}_i =$$

$$\begin{array}{lccccc} & \supsetneq \mathrm{Ent}(U_1) = \mathbf{G}_\downarrow & \supsetneq \mathrm{Ent}(U_2) \supsetneq \ldots & & \\ = \mathbf{G}_\uparrow & \supsetneq\subsetneq & \supsetneq\subsetneq & \supsetneq \mathbf{G}_\infty \\ & \supsetneq \quad \mathrm{Ent}(D_1) & \supsetneq \mathrm{Ent}(D_2) \supsetneq \ldots & \end{array}$$

The hierarchy result just described shows that there are at least $\aleph_0$ many different Gödel logics between $\mathbf{G}_\uparrow$ and $\mathbf{G}_\infty$. This suggests the question:

2.15. PROBLEM Are there even $2^{\aleph_0}$ or $2^{2^{\aleph_0}}$ logics below $\mathbf{G}_\infty$?

We will show in the next section that an infinite-valued Gödel logic is compact iff its set of truth values contains a densely ordered subset. It should be pointed out right here that almost all infinite-valued logics are not compact. In fact, there is only one compact infinite-valued Gödel logic, namely $\mathbf{G}_\infty$, as Proposition 3.7 will show.

3 Classification of compact Gödel logics

We now turn to the characterization of those Gödel logics whose entailment relations are compact, as defined by the following

3.1. DEFINITION $\mathbf{G}_V$ is *compact* if, whenever $\Pi \models_V \phi$ there is a finite $\Pi' \subseteq \Pi$ so that $\Pi' \models_V \phi$.

3.2. PROPOSITION $\mathbf{G}_V$ *is compact if* V *is finite.*

Proof. Let $\Pi = \{\psi_1, \psi_2, \ldots\}$, and let $X = \{X_0, X_1, \ldots\}$, be an enumeration of variables occuring in Π, ψ_0 such that all variables in ψ_i occur before the variables in ψ_{i+1}. We show that either $\{\psi_1, \ldots, \psi_k\} \models \psi_0$ or $\mathbf{\Psi} \not\models \psi_0$.

Let T be the complete semantic tree on X, i.e., $T = V^{<\omega}$. An element of T of length k is a valuation of $X_0, \ldots, X_{k-1}$. Since V is finite, T is finitary. Let T' be the subtree of T defined by: $v \in T'$ if for every initial segment v' of v and every k such that all the variables in $\psi_0, \ldots, \psi_k$ are among $X_0, \ldots, X_{\ell(v')}$, $v'(\{\psi_1, \ldots, \psi_k\}) > v'(\psi_0)$. In other words, branches in T' terminate at nodes v' where $v'(\{\psi_1, \ldots, \psi_k\}) \leq \psi_0$. Now if T' is finite, there is a k such that $\psi_1, \ldots, \psi_k \models_V \psi_0$. Otherwise, since T' is finitary, it contains an infinite branch. Let v be the limit of the partial valuations in that branch. Obviously, since V is finite, $v(\Pi) > v(\psi_0)$ and so $\Pi \not\models_V \psi_0$. □

3.3. PROPOSITION *Suppose* Π *contains only finitely many variables. Then* $\Pi' \models_V \phi$, *for some finite* $\Pi' \subseteq \Pi$, *provided* $\Pi \models_V \phi$.

Proof. By Proposition 2.6, there are only finitely many non-equivalent formulas on k variables. Choose a representative from each equivalence class to obtain Π'. □

3.4. THEOREM *Suppose* $V \supseteq W$ *with* W *densely ordered and* $|W| \geq 2$. *Then* $\mathbf{G}_V$ *is compact.*

Proof. Let X be a set of variables. A *chain on* X is an arrangement of X in a linear order. Formally, a chain C on X is a sequence of pairs $\langle X_i, o_i \rangle$ where $o_i \in \{<, =, >\}$ where X_i appears exactly once. A valuation v respects C if $v(X_i) = v(X_{i+1})$ if o_i is $=$, $v(X_i) > v(X_{i+1})$ if o_i is $>$, and $v(X_i) < v(X_{i+1})$ if o_i is $<$. If X is finite, there are only finitely many chains on X.

We construct a tree in stages as follows: The initial node is labeled by $0 < 1$ and an empty valuation. Stage $n+1$: A node N constructed in stage n is labeled by a chain on the variables $X_1, \ldots, X_n$ and a valuation v_N of $X_1, \ldots, X_n$ respecting the chain. N receives successor nodes, one for each possibility of extending the chain by inserting X_{n+1}. The labels of each successor node N' are the corresponding extended chain and an extension of v_N which respects the extended chain. The value $v_{N'}(X_{n+1})$ is chosen inside W, i.e., the endpoints of W may not be chosen as values. Since W is densely ordered, this ensures that such a choice can be made at every stage.

We call a branch of T *closed at node* N (constructed at stage n) if for some finite $\Pi' \subseteq \Pi$ such that $\mathrm{var}(\Pi') \cup \mathrm{var}(\phi) \subseteq \{X_1, \ldots, X_n\}$ it holds that $v_N(\Pi') \leq v_N(\phi)$. T is *closed* if it is closed on every branch. In that case, for some finite $\Pi' \subseteq \Pi$, we have $\Pi' \models \phi$.

If T is not closed, it contains an infinite branch. Let v be the limit of the v_N of nodes N on the infinite branch. It holds that $v(\psi) > v(\phi)$ for all $\psi \in \Pi$, for otherwise the branch would be closed at the first stage where all the variables in ψ were assigned values. Let $w = v(\phi)$. By Proposition 2.1, $v_w(\phi) = v(\phi)$ and $v_w(\Pi) = 1$, and so $\Pi \not\models \phi$, a contradiction. □

3.5. THEOREM *Suppose* V *does not contain a densely ordered subset. Then* $\mathbf{G}_V$ *is not compact.*

Proof. We define a sequence of sets of variables X_k and formulas Γ_k as follows:

$$\begin{aligned}
X_k &= \{X_r : r = i/2^k, 0 \leq i \leq 2^k\} \\
\Gamma'_k &= \{(X_r \to X_s) \to X_s, (X_s \to X_t) \to X_t : \\
&\quad t = (i-1)/2^k, s = i/2^k, r = (i+1)/2^k, \\
&\quad\quad 1 \leq i \leq 2^k - 1, 2^{(k-1)} | i\} \\
\Gamma''_k &= \{X_r \to Z : r = i/2^k, 0 \leq i \leq 2^k\} \\
\Gamma_k &= \Gamma'_k \cup \Gamma''_k \\
\Gamma &= \bigcup_{k \in \omega} \Gamma_k
\end{aligned}$$

Intuitively, Γ'_k expresses that the $X_{i/2^k}$ for $0 \leq i \leq 2^k$ are linearly ordered so that $X_{i/2^k} < X_{(i+1)/2^k}$. $\bigcup_{k\in\omega} \Gamma'_k$ expresses that the variables X_r are densely ordered: $v(X_{i/2^k}) < v(X_{(2i+1)/2^{k+1}}) < v(X_{(i+1)/2^k})$. Now if V has at least $2^k + 2$ truth values $v_0 < \cdots < v_{2^k+1} < 1$, the valuation v

$$\begin{aligned}
v(X_{i/2^k}) &= v_i \\
v(Z) &= v_{2^k+1}
\end{aligned}$$

will have $v(\Gamma_k) = 1$ and $v(Z) < 1$, so $\Gamma_k \not\models_V Z$. On the other hand, if V does not contain a densely ordered subset, then $\Gamma \models_V Z$. In fact the only v such that $v(\Gamma) = 1$ is $v(X_r) = 1$ for all r, and $v(Z) = 1$. $\Box$

We have thus succeeded in characterizing the compact propositional Gödel logics. They are all those where the set of truth values V is either finite or contains a nontrivial densely ordered subset.

3.6. THEOREM *The compact Gödel logics are exactly those given by the truth-value sets G_n ($n \geq 2$) and $\mathbf{G}_\infty$.*

This follows from the next proposition, together with the fact that all infinite-valued Gödel logics have the same tautologies.

3.7. PROPOSITION *Let $\models_1$ and $\models_2$ be the entailment relations of two compact logics, each satisfying modus ponens, the deduction theorem, and having the same tautologies. Then $\models_1 = \models_2$.*

Proof. Suppose $\Pi \models_1 \phi$. Then for some $\{\psi_1, \ldots, \psi_n\} \models_1 \phi$. By the deduction theorem for $\models_1$, we have $\models_1 \psi_1 \to \cdots \to \psi_n \to \phi$. Since $\models_1 \phi$ iff $\models_2 \phi$, $\models_2 \psi_1 \to \cdots \to \psi_n \to \phi$, and by modus ponens, $\Pi \models_2 \phi$. $\Box$

3.8. DEFINITION A logic $\mathbf{G}_V$ is *model compact* if, for any Π, Π has a model provided all finite subsets $\Pi' \subseteq \Pi$ have models.

The notion of model compactness coincides with compactness in the case of classical logic, but not in the general case of infinite-valued Gödel logics. The following theorem is due to Petr Hájek:

3.9. THEOREM *All Gödel logics are model compact.*

Proof. Observe that for any valuation v and any formula ϕ, if $v(\phi) = 1$ then also $v_0(\phi) = 1$ (see Proposition 2.1). Hence, if v is a model for Π in any logic, v_0 is a model for Π in $\mathbf{G}_2$. Hence, Π has a model in $\mathbf{G}_V$ iff it has a model in $\mathbf{G}_2$. Since $\mathbf{G}_2$ is model compact, so is $\mathbf{G}_V$ for any V. $\Box$

4 Interpolation in Gödel logics

Interpolation is a property usually defined for logics as considered as a set of sentences. A logic $\mathbf{L}$ *weakly interpolates* if, whenever $\mathbf{L} \models \phi \to \psi$, there is a sentence σ so that $\mathrm{var}(\sigma) \subseteq \mathrm{var}(\phi) \cap \mathrm{var}(\psi)$ and $\mathbf{L} \models \phi \to \sigma$ and $\mathbf{L} \models \sigma \to \psi$. The sentence σ is called sn *interpolant* of ϕ and ψ. We may extend this defnition to entailment relations as follows:

4.1. DEFINITION Gödel logic $\mathbf{G}_V$ *interpolates* if, for all Π_1, Π_2 and ϕ, such that $\Pi_1 \cup \Pi_2 \models_V \phi$, there is σ with $\mathrm{var}(\sigma) \subseteq \mathrm{var}(\Pi_1) \cap \mathrm{var}(\Pi_2 \cup \{\phi\})$ and $\Pi_1 \models_V \sigma$ and $\Pi_2 \cup \{\sigma\} \models_V \phi$.

Since $\psi_1, \ldots, \psi_n \models_V \phi$ iff $\models_V \psi_1 \to \cdots \to \psi_n \to \phi$, $\mathbf{G}_V$ interpolates weakly iff it interpolates with respect to finite sets.

The following observation is immediate:

4.2. PROPOSITION *If $\mathbf{G}_V$ is compact, then $\mathbf{G}_V$ interpolates iff it weakly interpolates.*

Our aim in this section is to show that weak interpolation and interpolation fall apart when V is infinite: $\mathbf{G}_V$ weakly interpolates if V is infinite, but only interpolates when $\mathbf{G}_V$ is compact as well.

4.3. THEOREM *$\mathbf{G}_V$ weakly interpolates iff $|V| = 2$ or $|V| = 3$ or V is infinite.*

Proof. See [6, 3] $\Box$

4.4. PROPOSITION *Let V be infinite but not contain a nontrivial densely ordered subset. Then $\mathbf{G}_V$ does not interpolate.*

Proof. Consider the set Γ from the proof of Theorem 3.5. We know that $\Gamma \models_V Z$. Suppose there was an interpolant σ so that $\Gamma' \models_V \sigma$ and $\Gamma'' \cup \{\sigma\} \models_V Z$. Let

$$\begin{aligned}
\tilde{\Gamma}' &= \{\psi \in \Gamma' : \mathrm{var}(\psi) \subseteq \mathrm{var}(\sigma)\} \\
\tilde{\Gamma}'' &= \{\psi \in \Gamma'' : \mathrm{var}(\psi) \subseteq \mathrm{var}(\sigma) \cup \{Z\}\}
\end{aligned}$$

Then $\tilde{\Gamma}' \vDash_V \sigma$ and $\tilde{\Gamma}'' \cup \{\sigma\} \vDash_V Z$. To see this, suppose $v(\tilde{\Gamma}') = 1$ and $v(\sigma) < 1$. Then v' defined by

$$v'(X) = \begin{cases} v(X) & \text{if } X \in \text{var}(\sigma) \\ 1 & \text{otherwise} \end{cases}$$

would evaluate all formulas in $\Gamma' - \tilde{\Gamma}'$ to 1, contradicting $\Gamma' \vDash_V \sigma$. A similar contradiction follows for the second part of the claim by considering

$$v''(X) = \begin{cases} v(X) & \text{if } X \in \text{var}(\sigma) \cup \{Z\} \\ 0 & \text{otherwise.} \end{cases}$$

Of course, $\tilde{\Gamma}'$ and $\tilde{\Gamma}''$ are both finite. By the cut rule for $\vDash_V$ it follows that $\tilde{\Gamma}' \cup \tilde{\Gamma}'' \vDash_V Z$. This contradicts the proof of Theorem 3.5. □

Note that the proof actually establishes a stronger result: The definition of an interpolant requires the variables in σ to be contained in the intersection of $\text{var}(\Gamma')$ and $\text{var}(\Gamma' \cup \{Z\})$. We have shown that no formula whatsoever, i.e., not even one that does not satisfy that condition, can serve as an interpolating formula.

4.5. THEOREM *$\mathbf{G}_V$ interpolates iff $|V| = 2$ or $|V| = 3$ or V contains a densely ordered subset. That is, the only interpolating Gödel logics are $\mathbf{G}_2$, $\mathbf{G}_3$, and $\mathbf{G}_\infty$.*

Proof. By Theorem 4.3 and Propositions 4.2 and 4.4. □

It is not necessary for a logic to be compact in order to interpolate. For instance, consider the many-valued logic on $V_\downarrow$ with constants $\top$, $\bot$ for 1 and 0, respectively, and the operator $\clubsuit$ given by: $\clubsuit(1/n) = 1/(n+1)$ and $\clubsuit(0) = 0$. $\Pi \vDash \phi$ iff one of the following hold:

(1) $\clubsuit^k(X) \in \Pi$ with k unbounded,

(2) $\bot \in \Pi$,

(3) $\phi = \clubsuit^\ell(X)$ and $\clubsuit^k(X) \in \Pi$ for some $k \geq \ell$,

(4) $\phi = \clubsuit^\ell(\top)$ and $\clubsuit^k(X) \in \Pi$ or $\clubsuit^k(\top) \in \Pi$ for some $k \geq \ell$.

In each case, an interpolant can easily be found for any partition of Π. However, the logic is not compact, as the example $\{\clubsuit^k(\top) : k \in \omega\} \vDash \bot$ illustrates.

5 R.e. entailment relations

We conclude with a somewhat curious result. Propositional logics are considered “easy” in the sense that the validity problem of most is decidable, usually within reasonable bounds, in contrast to first-order logic. This situation changes drastically if entailment is considered instead. The following proposition shows that the question of whether a recursive set of formulas entails another formula is highly undecidable for non-compact Gödel logics. This potentially has serious consequences for the implementation and application of Gödel logics in inference mechanisms.

5.1. PROPOSITION *The set*

$$E = \{\langle e, \ulcorner\phi\urcorner\rangle : \{\psi : \{e\}(\ulcorner\psi\urcorner) = 0\} \vDash_V \phi\}$$

is r.e. iff $\mathbf{G}_V$ is compact. Hence, the only r.e. Gödel logics are $\mathbf{G}_n$ and $\mathbf{G}_\infty$.

Proof. If: We can enumerate E by enumerating finite subsets of $\Pi = \{\psi : \{e\}(\ulcorner\psi\urcorner) = 0\}$ and testing for implication. If $\Pi \vDash_V \phi$, this search terminates eventually since $\mathbf{G}_V$ is compact.

Only if: Suppose $\mathbf{G}_V$ is not compact. Let $f(e)$ be the index of the predicate defined by

$$\{f(e)\}(\ulcorner\psi\urcorner) = 0 \text{ iff } \psi \in \Gamma_i \text{ and } (\forall j \leq i)(\{e\}(j) = 0)$$

where Γ_i is as in the proof of Theorem 3.5. Then $\langle f(e), \ulcorner Z\urcorner\rangle \in E$ iff $\{e\}$ is total and constant equal to 0. That problem, however, is Π^0_2-complete [7, Theorem IV.3.2]. □

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Errata

We would like to thank Petr Cintula for pointing out a number of misprints and possible improvements in the text. These reached us only after the proceedings volume went to print; they are added in this version as notes.

1. This should read $\Pi_1 \models_{V_\uparrow} Y$, not $\Pi_1 \not\models_{V_\uparrow} Y$.
2. Petr Cintula has communicated the following simpler argument for the induction step: If $w = 1$ the claim is trivial. So suppose $w < 1$. (1) $v(\phi) \leq v(\psi)$. Then $v_w(\phi) \leq v_w(\psi)$ and so $v_w(\phi \to \psi) = 1$, while $v(\phi \to \psi) = 1 > w$. (2) $w < v(\psi) < v(\phi)$. Then $v_w(\phi \to \psi) = 1$, whereas $v(\phi \to \psi) = v(\psi) > w$. (3) $v(\psi) \leq w$ and $v(\psi) < v(\phi)$. In this case, $v_w(\psi) = v(\psi) < v(\phi) \leq v_w(\phi)$, so $v_w(\phi \to \psi) = v_w(\psi)$, while $v(\phi \to \psi) = v(\psi) \leq w$.
3. Petr Cintula points out that $\mathrm{Ent}(V) \subseteq \mathrm{Ent}(V')$ if there is an injection $f{:}V' \to V$ which preserves strict order, i.e., $f(v) < f(v')$ iff $v < v'$. This follows quite easily from Proposition 2.3: If v is a valuation on V' and $v(\Pi \models \phi) < 1$, then fv is a valuation on V (in fact, on $f(V') \subseteq V$), and $fv(\Pi \models \phi) < 1$. As an immediate corollary we obtain the fact stated just before Proposition 2.3, i.e., that $\mathrm{Ent}(V) = \mathrm{Ent}(V')$ if V and V' are order-isomorphic.
4. This should read $|V| \geq |V'|$, not $|V| \leq |V'|$.
5. Note that we *defined* $V_n = \{1 - 1/k : k \leq n-1\} \cup \{1\}$. However, using the consideration of note 3, the result holds for any sequence V_n^* with $|V_n^*| = n$. In particular, if $v(\Pi \models \phi) = 1 - 1/n = w$, then there is an order isomorphism f between $\{v \in V_\uparrow : v \leq w\} \cup \{1\}$ and V_{n+1}^* and so $fv(\Pi \models \phi) = f(w) < 1$.
6. The definition of U_k should read

$$\begin{aligned} U_k \quad = \quad & \{i/(k+1) + 1/j(k+1) : 0 \leq i \leq k-1; j > 0\} \cup \\ & \cup \{i/(k+1); 0 \leq i \leq k+1\} \end{aligned}$$

7. This is not strictly true, however, U_1 and $V_\downarrow$ are order isomorphic and thus $\mathrm{Ent}(U_1) = \mathrm{Ent}(V_\downarrow)$ by note 3.
8. (7) and (8) follow from note 3.