# Abstracta and Possibilia: Modal Foundations of Mathematical Platonism

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**Abstract**

This paper aims to provide modal foundations for mathematical platonism. I examine Hale and Wright's (2009) objections to the merits and need, in the defense of mathematical platonism and its epistemology, of the thesis of Necessitism. In response to Hale and Wright's objections to the role of epistemic and metaphysical modalities in providing justification for both the truth of abstraction principles and the success of mathematical predicate reference, I examine the Necessitist commitments of the abundant conception of properties endorsed by Hale and Wright and examined in Hale (2013a); examine cardinality issues which arise depending on whether Necessitism is accepted at first- and higher-order; and demonstrate how a multi-dimensional intensional approach to the epistemology of mathematics, augmented with Necessitism, is consistent with Hale and Wright's notion of there being epistemic entitlement rationally to trust that abstraction principles are true. Epistemic and metaphysical modality may thus be shown to play a constitutive role in vindicating the reality of mathematical objects and truth, and in explaining our possible knowledge thereof.

## 1 Introduction

Modal notions have been availed of, in order to argue in favor of nominalist approaches to mathematical ontology. Field (1989) argues, for example, that mathematical modality can be treated as a logical consistency operator on a set of formulas comprising an empirical theory, such as Newtonian mechanics, in which the mathematical vocabulary has been translated into the vocabulary of physical geometry.[1] Putnam (1967), Parsons (1983), Chihara (1990), and

[1]For a generalization of Field's nominalist translation scheme to the differential equations in the theory of General Relativity, see Arntzenius and Dorr (2012).

Hellman (1993) argue that intensional models both of first- and second-order arithmetic and of set theory motivate an eliminativist approach to mathematical ontology. On this approach, reference to mathematical objects can be eschewed, and possibly the mathematical structures at issue are nothing.[2]

This essay aims to provide modal foundations for mathematical platonism, i.e., the proposal that mathematical terms for sets; functions; and the natural, rational, real, and imaginary numbers refer to abstract – necessarily non-concrete – objects. Intensional constructions of arithmetic and set theory have been developed by, inter alia, Fine (1981); Parsons (1983); Shapiro (1985); Myhill (1985); Reinhardt (1988); Nolan (2002); Linnebo (2013); and Studd (2013). Williamson (2013; 2014) emphasizes that mathematical languages are extensional, although in Williamson (2016) he argues that Orey sentences, such as the generalized continuum hypothesis – $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ – which are currently undecidable relative to the axioms of the language of Zermelo-Fraenkel Set Theory with choice as augmented by large cardinal axioms, are yet possibly decidable.[3] Khudairi (ms) argues that the epistemic interpretation of multi-dimensional intensional semantics provides a novel approach to the epistemology of mathematics, such that if the decidability of mathematical axioms is epistemically possible, then their decidability is metaphysically possible.[4] Epistemic mathematical modality, suitably constrained, can thus serve as a guide to metaphysical mathematical modality.[5] Hamkins and Löwe (2007; 2013) argue that the modal

[2] For further discussion of modal approaches to nominalism, see Burgess and Rosen (1997: II, B-C) and Leng (2007; 2010: 258).

[3] Compare Reinhardt (1974) on the imaginative exercises taking the form of counterfactuals concerning the truth of undecidable formulas. See Maddy (1988), for critical discussion.

[4] The epistemic interpretation of multi-dimensional intensional semantics is first advanced in Chalmers (1996; 2004).

[5] See Section **4**, for further discussion. Gödel (1951: 11-12) anticipates a similar distinction between epistemic and metaphysical readings of the determinacy of mathematical truths, by distinguishing between mathematics in its subjective and objective senses. The former concerns decidable formulas, while the latter records the values of formulas defined, owing to the incompleteness theorems, in a variant of the language augmented by stronger axioms of

logic of set-forcing extensions and the corresponding logic for their ground models satisfy at least S4.2, i.e., axioms K $[\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)]$; T $(\Box\phi \to \phi)$; 4 $(\Box\phi \to \Box\Box\phi)$; and G $(\Diamond\Box\phi \to \Box\Diamond\phi)$. While the foregoing approaches are consistent with realism about mathematical objects, they are nevertheless not direct arguments thereof. The aim of this essay is to redress the foregoing lacuna, and thus to avail of the resources of modal ontology and epistemology in order to argue for the reality of mathematical entities and truth.

In Section **2**, I outline the elements of the abstractionist foundations of mathematics. In Section **3**, I examine Hale and Wright (2009)'s objections to the merits and need, in the defense of mathematical platonism and its epistemology, of the thesis of Necessitism, underlying the thought that whatever can exist actually does so. The Necessitist thesis is codified by the converse Barcan formula (cf. Barcan, 1946; 1947), and states that possibly if there is something which satisfies a condition, then there is something such that it possibly satisfies that condition: $\Diamond\exists x\phi x \to \exists x\Diamond\phi x$. I argue that Hale and Wright's objections to Necessitism as a requirement on admissible abstraction can be answered; and I examine both the role of the higher-order Necessitist proposal in their endorsement of an abundant conception of properties, as well as cardinality issues that arise depending on whether Necessitism is accepted at first- and higher-order. In Section **4**, I provide an account of the role of epistemic and metaphysical modality in explaining the prima facie justification to believe the truth of admissible abstraction principles, and demonstrate how it converges with both Hale and Wright's (op. cit.) and Wright's (2012; 2014) preferred theory of default entitlement rationally to trust the truth of admissible abstraction. Section **5** provides concluding remarks.

infinity.

## 2 The Abstractionist Foundations of Mathematics

The abstractionist foundations of mathematics are inspired by Frege's (1884/1980; 1893/2013) proposal that cardinal numbers can be explained by specifying an equivalence relation, expressible in the signature of second-order logic and identity, on lower-order representatives for higher-order entities. Thus, e.g., in Frege (1884/1980: 64), the direction of the line, a, is identical to the direction of the line, b, if and only if lines a and b are parallel. In Frege (op. cit.: 68) and Wright (1983: 104-105), the cardinal number of the concept, **A**, is identical to the cardinal number of the concept, **B**, if and only if there is a one-to-one correspondence between **A** and **B**, i.e., there is an injective and surjective (bijective) mapping, R, from **A** to **B**. With Nx: a numerical term-forming operator,

- $\forall\mathbf{A}\forall\mathbf{B}\exists\mathrm{R}[[\mathrm{Nx}\colon \mathbf{A} = \mathrm{Nx}\colon \mathbf{B} \equiv \exists\mathrm{R}[\forall x[\mathbf{A}x \rightarrow \exists y(\mathbf{B}y \wedge \mathrm{R}xy \wedge \forall z(\mathbf{B}z \wedge \mathrm{R}xz \rightarrow y = z))] \wedge \forall y[\mathbf{B}y \rightarrow \exists x(\mathbf{A}x \wedge \mathrm{R}xy \wedge \forall z(\mathbf{A}z \wedge \mathrm{R}zy \rightarrow x = z))]]$.

The foregoing is referred to as 'Hume's Principle'.[6] Frege's Theorem states that the Dedekind-Peano axioms for the language of arithmetic can be derived from the Hume's Principle, as augmented to the signature of second-order logic and identity.[7] Abstraction principles have further been specified both for the

[6]Frege (1884/1980: 68) writes: 'the Number which belongs to the concept F is the extension of the concept '[equinumerous] to the concept F' (cf. op. cit.: 72-73). Boolos (1987/1998: 186) coins the name, 'Hume's Principle', for Frege's abstraction principle for cardinals, because Frege (op. cit.: 63) attributes equinumerosity as a condition on the concept of number to Hume (1739-1740/2007: Book 1, Part 3, Sec. 1, SB71), who writes: 'When two numbers are so combin'd, as that the one has always an unite answering to every unite of the other, we pronounce them equal . . . '. Frege notes that identity of number via bijections is anticipated by the mathematicians, Ernst Schröder and Ernst Kossak, as well Cantor (1883/1996: Sec. 1), who writes: '[E]very well-defined set has a determinate power; two sets have the same power if they can be, element for element, correlated with one another reciprocally and one-to-one', where the power [Anzahl] of a set corresponds to its cardinality (cf. Cantor, 1895/2007: 481).

[7]Cf. Dedekind (1888/1996) and Peano (1889/1967). See Wright (1983: 154-169) for a proof sketch of Frege's theorem; Boolos (1987) for the formal proof thereof; and Parsons (1964) for an incipient conjecture of the theorem's validity.

real numbers (cf. Hale, 2000a; Shapiro, 2000; and Wright, 2000), and for sets (cf. Wright, 1997; Shapiro and Weir, 1999; Hale, 2000b; and Walsh, 2016).

The philosophical significance of the abstractionist program consists primarily in its provision of a neo-logicist foundation for classical mathematics, and in its further providing a setting in which to examine constraints on the identity conditions constitutive of mathematical concept possession.[8] The philosophical significance of the abstractionist program consists, furthermore, in its circumvention of Benacerraf's (1973) challenge to the effect that our knowledge of mathematical truths is in potential jeopardy, because of the absence of naturalistic, in particular causal, conditions thereon. Abstraction principles provide an epistemic conduit into our knowledge of abstracta, because a grasp of the right-hand side of the principles – i.e., the equivalence relation on lower-order entities, such as the equinumerosity of two, distinct objects – is explanatory of the left-hand side of the principles – i.e., a grasp of the identity of the numbers of the higher-order entities (the concepts) under which those objects fall. Both Wright (1983: 13-15) and Hale (1987: 10-15) argue, then, that the abstraction principles are epistemically tractable, only if (i) the surface syntax of the principles – e.g., the term-forming operators referring to objects – are a perspicuous guide to their logical form; and (ii) the principles satisfy Frege's (1884/1980: X) context principle, such that the truth of the principles is secured prior to the reference of the terms figuring therein.

[8]Shapiro and Linnebo (2015) prove that Heyting arithmetic can be recovered from Frege's Theorem. Criteria for consistent abstraction principles are examined in, inter alia, Hodes (1984a); Hazen (1985); Boolos (1990/1998); Heck (1992); Fine (2002); Weir (2003); Cook and Ebert (2005); Linnebo and Uzquiano (2009); Linnebo (2010); and Walsh (op. cit.).

# 3 Abstraction and Necessitism

## 3.1 Hale and Wright's Arguments against Necessitism

One crucial objection to the abstractionist program is that – while abstraction principles might provide a necessary and sufficient truth-condition for our grasp of the concepts of mathematical objects – an explanation of the actual truth of the principles has yet to be advanced (cf. Eklund, 2006; 2016). In response, Hale and Wright (2009: 197-198) proffer a tentative endorsement of an 'abundant' conception of properties, according to which fixing the sense of a predicate will be sufficient for predicate reference. Eklund (2006: 102) suggests, by contrast, that one way for the truth of the abstraction principles to be explained is by presupposing what he refers to as a 'Maximalist' position concerning the target ontology.[9] According to the ontological Maximalist position, if it is possible that a term has a certain extension, then actually the term does have the designated extension.

Hale and Wright (op. cit.) raise two issues for the ontological Maximalist proposal. The first is that ontological Maximalism is committed to a proposal that they take to be independently objectionable, namely ontological Necessitism (185). Hale and Wright (op. cit.) raise a similar contention to the effect that actual, and not *merely possible*, reference is what the abstractionist program intends to target; and that Maximalism and Necessitism, so construed, are purportedly silent on the status of ascertaining when the possibilities at issue are actual.

The second issue that Hale and Wright find with Maximalism is that it misconstrues the demands that the abstractionist program is required to address.

[9]For further discussion of ontological Maximalism, see Hawley (2007) and Sider (2007: IV).

The abstractionist program is supposed to be committed to ontological Maximalism, because the possibility that a term has a certain extension will otherwise not be sufficient for the success of the term's reference. It is further thought that, without an appeal to Maximalism, and despite the actuality of successful mathematical predicate reference, there are yet possible situations in which the mathematical predicates still do not refer (193). In response, they note that no 'collateral metaphysical assistance' – such as ontological Maximalism would be intended to provide – is necessary in order to explain the truth of abstraction principles (op. cit.). Rather, there is prima facie, default entitlement rationally to trust that the abstraction principles are actually true, and such entitlement is sufficient to foreclose upon the risk that possibly the mathematical terms therein do not refer (192).

In the remainder of this section, I will argue that Hale and Wright's objections to Necessitism and the ontological Maximalist approach to admissible abstraction both can be answered, and in any case are implicit in their endorsement of the abundant conception of properties. In the following section, I address their second contention, and I argue for the fundamental role that Maximalism and Necessitism can play in warranting the truth of candidate abstractions and mathematical platonism.

The principle of the necessary necessity of being (NNE) can be derived from the converse Barcan formula.[10] NNE states that necessarily everything is necessarily such that there is something to which it is identical; $\Box\forall x\Box\exists y(x = y)$. Informally, necessarily everything has necessary being, i.e. everything is something, even if contingently non-concrete. Williamson (2013: 6.1-6.4) targets issues for haecceity comprehension, if the negations of the converse Barcan for-

[10]Cf. Williamson (2013: 38).

mula and NNE are true at first-order, and thus for objects. With regard to properties and relations at higher-order, Williamson's arguments have targeted closure conditions, given a modalized interpretation of comprehension principles (op. cit.). The latter take the form, $\exists X \Box \forall x(Xx \iff \phi)$, with x an individual variable which may occur free in $\phi$ and X a monadic first-order predicate variable which does not occur free in $\phi$ (262).[11] He targets, in particular, the principle of mathematical induction – with s a successor function and the quantifier ranging over the natural numbers: $\forall X[[X0 \wedge \forall n(Xn \rightarrow Xsn)] \rightarrow \forall n(Xn)]$ – and notes that instances of mathematical induction – e.g., for a an individual constant, $\exists R[[Ra0 \wedge \forall n(Ran \rightarrow Rasn)] \rightarrow \forall n(Ran)]$ – presuppose, for their derivation, the validity of instances of the higher-order modal comprehension scheme: e.g., $\exists X \Box \forall n(Xn \iff Ran)$ (283-284). The foregoing provides prima facie abductive support for the requirement of Necessitism in the practice of mathematics. The constitutive role of the Necessitist modal comprehension scheme in the principle of mathematical induction answers Hale and Wright's first contention against the Necessitist commitments of ontological Maximalism.

Metaphysical universality is Williamson's term for model-theoretic validity in the metaphysical setting (95). Let a formula be a logical truth if and only if its truth is preserved under every assignment of values to the variables comprising the formula's syntactic form. Logical validity can, then, be defined as the preservation of the truth of a formula on all interpretations of the logical constants from classes of models to classes of models.

As a definition of validity unique to metaphysics, metaphysically universal formulas are valid on the interpretation of both the logical and the non-logical constants. The non-logical constants have values from various metaphysical

[11] The contingentist, by contrast, can – by rejecting the Barcan formula – countenance only 'intra-world' comprehension principles in which the modal operators and iterations thereof take scope over the entire formula; e.g. $\Diamond \forall x \exists X(Xx \iff \phi)$ (cf. Sider, 2016: 686).

domains in their extension. Williamson refers to the assignments for models in the metaphysical setting as universal interpretations (59). The analogue for logical truth occurs when a truth is metaphysically universal, i.e., only if its second-order universal generalization is true on the intended interpretation of the metalanguage (200). The connection between truth-in-a-model and truth simpliciter is then that – as Williamson puts it laconically – when 'the framework at least delivers a condition for a modal sentence to be true in a universal interpretation, we can derive the condition for it to be true in the intended universal interpretation, which is the condition for it to be true simpliciter' (op. cit.).

One of the crucial interests of the metaphysical universality of formulas is that the models in the class need not be pointed, in order to countenance the actuality of the possible formulas defined therein.[12] Rather, the class of true, possible propositions generated by the metaphysically universal possible formulas is sufficient for the formulas actually to be true (268-269).[13] To see this, let the quantifier for worlds in modal formulas which are true simpliciter range over real metaphysically possible worlds. Suppose that the converse Barcan formula is true such that, possibly if there is something which satisfies a condition then there is something such that possibly it satisfies that condition. Then, because there is something which possibly does so, Williamson writes that 'since whatever is is, whatever is actually is: if there is something, then there actually is such a thing' (23). When the quantifier ranges, then, over possible worlds treated as objects, the worlds are thus actual. So – by contrast to modal formulas which are true-in-a-model – the objects, i.e. actual worlds, which fall within the range of the quantifiers in the metaphysically universal formulas which are

[12] That the models are unpointed is noted in Williamson (2013: 100).
[13] Thanks here to xx for discussion.

true simpliciter can explain why such formulas are actual.

The constitutive role of metaphysical universality in bridging the necessary necessity of being with the actuality thereof answers Hale and Wright's contention that the interaction between the possible and actual truth of abstraction principles has yet to be accounted for.[14]

## 3.2 Hale on the Necessary Being of Purely General Properties and Objects

Note, further, that the abundant conception of properties endorsed by Hale and Wright depends upon the Necessitist Thesis, and the truth of ontological Maximalism thereby. Hale writes: '[I]t is sufficient for the *actual* existence of a property or relation that there *could* be a predicate with appropriate satisfaction conditions ... *purely general* properties and relations exist as a matter of (absolute) *necessity*', where a property is purely general if and only if there is a predicate for which, and it embeds no singular terms (Hale, 2013b: 133, 135; see also 2013a: 99-100).[15]

Hale argues for the necessary necessity of being for properties and propositions as follows (op. cit.: 135; 2013b: 167). Suppose that p refers to the proposition that a property exists, and that q refers to the proposition that a predicate for the property exists. Let the necessity operator be defined as a counterfactual with an unrestricted, universally quantified antecedent, such

[14] Cook (2016: 398) demonstrates how formally to define modal operators within Hume's Principle, i.e. the consistent abstraction principle for cardinal numbers. Necessitist Hume's Principle takes the form: $\Box\forall X,Y[\#(X) =_{\Box} \#(Y) \iff X \approx Y]$, where X and Y are second-order variables, # is a numerical term-forming operator, $\approx$ is a bijection, and for variables, x,y, of arbitrary type '$x =_{\Box} y \iff \exists z[z = x \land z = y \land \Box\exists w(w = z)]$'. See Cook (op. cit.) for further discussion.

[15] Cook (op. cit.: 388) notes the requirement of Necessitism in the abundant conception of properties, although does not discuss points at which Williamson's and Hale's Necessitist proposals might be inconsistent. The points of divergence between the two variations on the proposal are examined below.

that, for all propositions, $\psi$: $[\Box\psi \iff \forall\phi(\phi \mathbin{\Box\!\!\rightarrow} \psi)]$ (135).[16] On the abundant conception of properties, $\Box[\text{p} \iff \Diamond\text{q}]$. Intuitively: Necessarily, there is a property if and only if possibly there is a predicate for that property. Given the counterfactual analysis of the modal operator: For all propositions about a property, if there were a proposition specifying a predicate s.t. the property is in the predicate's extension, then there would be that property. Conversely, for all propositions specifying a predicate s.t. a property is in the predicate's extension, if there were that property, then there would be a predicate which refers to that property.

From '$\Box[\text{p} \iff \Diamond\text{q}]$', one can derive both '$\text{p} \iff \Diamond\text{q}$', and – by the rule, RK – the necessitation thereof, '$\Box\text{p} \iff \Box\Diamond\text{q}$' (op. cit.). By the B axiom in S5, $\Diamond\text{q} \iff \Box\Diamond\text{q}$ (op. cit.). So, '$\Box\Diamond\text{q} \iff \Diamond\text{q}$'; '$\Diamond\text{q} \iff \text{p}$'; and '$\Box\Diamond\text{q} \iff \Box\text{p}$'. Thus – by transitivity – '$\text{p} \iff \Box\text{p}$' (op. cit.); i.e., all propositions about properties are necessarily true, such that the corresponding properties have necessary being. By the 4 axiom in S5, $\Box\text{p} \iff \Box\Box\text{p}$; so, the necessary being of properties and propositions is itself necessary. Given the endorsement of the abundant conception of properties – Hale and Wright are thus committed to higher-order necessitism, i.e., the necessary necessity of being.

Hale (2013b) endeavors to block the ontological commitments of the Barcan formula and its converse by endorsing a negative free logic. Thus, in the derivation:

Assumption,

1. $\Box\forall\text{x}[\text{F(x)}]$.

By $\Box$-elimination,

2. $\forall\text{x}[\text{F(x)}]$.

[16] Proponents of the translation from modal operators into counterfactual form include Stalnaker (1968/1975), McFetridge (1990: 138), and Williamson (2007).

By $\forall$-elimination,

3. F(x).

By $\Box$-introduction,

4. $\Box$[F(x)].

By $\forall$-introduction,

5. $\forall\Box$[F(x)].

By $\rightarrow$-introduction,

6. $\Box\forall$x[F(x) $\rightarrow$ $\forall\Box$[F(x)],

Hale imposes an existence-entailing assumption in the inference from lines (2) to (3), i.e.

'(Free$\forall$-Elimination) From $\forall$x[A(x)], together with an existence-entailing premise F(t), we may infer A(t) where t can be any term' (op. cit.: 208-209).

Because the concept of, e.g., cardinal number is defined by abstraction principles which are purely general because they embed no singular terms, the properties – e.g., the concepts – of numbers are argued to have necessary being. The necessary being of the essential properties of number – i.e., higher-order Necessitism about purely general properties – is argued then to explain in virtue of what abstract objects such as numbers and functions have themselves necessary being (176-177). Thus the necessary being of predicate sense for the concept of number can both suffice for and explain the necessary being of predicate reference, i.e. the necessary existence of numbers.

By contrast, essential properties defined by theoretical identity statements, which if true are necessarily so, do embed singular terms and are thus not purely general. So, the essential nature of water, i.e., the property 'being comprised of one oxygen and two hydrogen molecules', has contingent being, explaining in virtue of what samples of water have contingent being (216-217).

### 3.2.1 Objections

One argument against the above approach is that there is an asymmetry in the order of explanation with regard to whether properties or objects are the source of the contingency with which the lower- and higher-order entities have being. Concrete individuals susceptible to being defined via theoretical identity statements have, as noted, contingent being because the identity statements characterizing the essential properties – e.g., the concepts – of those individuals are not purely general, because they embed singular terms (op. cit.). However, for concrete individuals which cannot be defined via theoretical identity statements, the order of explanation is reversed. The contingency of the essential properties of concrete individuals, such as stars, depends upon the existence-entailing premise noted above (217). Thus, because concrete first-order objects such as stars can be rendered non-concrete, they have contingent being. Higher-order Contigentism is argued, then, to take into effect, because the essential properties thereof exist only when their corresponding objects do so. An explanation of the asymmetry in the foregoing order of explanation of first-and higher-order Contingentism is, however, wanting.

The necessary being of purely general properties – e.g., the possible predicate sense for the concept of number – is argued to hold with absolute modality, as the consequent of a counterfactual with an unrestrictedly general antecedent. As Hale notes, absolute modalities are therefore logical modalities (100). Thus, a crucial virtue of the logical necessity with which the concept of number – and thus the numbers themselves – have being, is that it provides a further vindication of the Neo-logicist thesis, that second-order logic and identity as augmented by the purely general abstraction principles is sufficient to derive the axioms of mathematical languages such as Peano arithmetic. The Neo-logicist thesis is qualified, however, by the caveat that abstraction principles are non-logical truths, i.e., they are not true on all interpretations of the values of the variables comprising their syntactic form. By arguing that purely general properties have necessary being and defining the necessity of being of purely general properties as logical necessity, abstraction principles – while not themselves logically true – can thus still hold of logical necessity. The logical necessity of abstraction principles – owing to the purely general predicates comprising their form – can thus serve to explain another sense in which the abstractionist foundations provide a logical reduction of mathematical truths. Mathematical truths will, on the above approach, be derivable from second-order logic with identity and implicit definitions which are true of logical necessity.

A second objection to the foregoing is that Hale takes the modal status of the being with which both logical and non-logical properties exist to be equivalent. Following Fine (2005), he notes that there is a distinction between 'unworldly' or 'transcendental' truths about individuals, which are true, in Fine's phrasing, 'on the basis of [their] logical form alone and without regard to the circumstances' (Fine, op. cit.: 324). An example of a transcendental truth is that 'Hypatia is self-identical', i.e., '$\Box\exists x(x = H \wedge \Box x = x)$'. A 'worldly' or 'necessary' truth is, by contrast, one whose truth-value is defined in a world; e.g., that 'water = H20' or – if one were to augment one's language with an existence predicate beyond the quantifiers – that 'Socrates exists or does not exist' (op. cit.).

However, Hale draws, as noted, no similar distinction between the modal status of a sentence true in virtue of its logical form, and a worldly sentence whose truth depends on non-logical values of its constituent variables, e.g., a truth of physics (Hale, op. cit.: 215). Hale (2000/2001: 415) notes Wright's (2001: 315) argument that the conjunction of two predicates, e.g., being blue and being self-identical, is equivalent to one of the conjuncts, e.g., being blue. Thus, the predicate for the property, being self-identical, cannot be purely general. He argues, thus, that the truth of 'this star is self-identical' depends on the concrete existence of the star, such that the logical property, being self-identical, has contingent being. Thus, the status of the being of logical properties is contingent, in the same manner that the essential property, being, e.g., H20, depends on concrete instances of water.

A problematic consequence of the foregoing is that the logical necessity of a formula will thus depend on whether the predicates therein are purely general by embedding no singular terms, rather than on whether the formula at issue is a logical truth. It might be replied that Hale is following Frege in defining one of the constitutive marks of logical truths as consisting in their generality – e.g., the generality of their application (1893/2013: XV; 1897/1997), as well as whether the formula is a true universal generalization (op. cit.: §8-9) – rather than Tarski's (1936/1983: 415-417) definition of a logical truth as a formula true in virtue of its logical form and thus whose truth is invariant under permutation of the values of the variables which replace the non-logical constants therein. However – even if not a purely general property because it embeds singular terms – the reflexivity of identity is a logical law, because – as Frege himself writes of reflexivity – 'the value of this function is always the True, whatever we take as argument' (Frege, 1891/1997: 23).[17]

A final objection concerns the necessary being of different types of numbers. While an abstraction principle for cardinal numbers can be specified using only purely general predicates – i.e., Hume's Principle –abstraction principles

[17]For further discussion of Frege's treatment of logic as a 'purely general science', see Dummett (1991: 224-225); MacFarlane (2002); Burge (2005: 133, 137-138); and Blanchette (2012: 1.1, 1.22, 3.6). For further discussion of Tarski's account of logical truth, see Etchemendy (1990/1999), McGee (1992, 1996), Gómez-Torrente (1996), Chihara (1998), Feferman (1999), and Sher (2008).

for imaginary and complex numbers have yet to be specified. Shapiro (2000) provides an abstraction principle for the concepts of the reals by simulating Dedekind cuts, where abstraction principles are provided for the concepts of the cardinals, natural numbers, integers, and rational numbers, from which the reals are thence defined: Letting F,G, and R denote rational numbers, $\forall$F,G[$\mathbf{C}$(F) = $\mathbf{C}$(G) $\iff$ $\forall$R(F$\leq$R $\iff$ G$\leq$R)]. [18] Hale's (2000/2001) own definition of the concept of the reals is provided relative to a domain of quantities. The quantities are themselves taken to be abstract, rather than physical, entities (409). The quantitative domain can thus be comprised of both rational numbers as well as the abstracts for lengths, masses, and points.[19] The reals are then argued not to be numbers, but rather quantities defined via an abstraction principle which states that a set of rational numbers in one quantitative domain is identical to a set of rational numbers in a second quantitative domain if and only if the two domains are isomorphic (407).[20] Hale argues, then, that it is innocuous for the real abstraction principle to be conditional on the existence of at least one quantitative domain, because the rational numbers can be defined, similarly as on Shapiro's approach, via cut-abstractions and abstractions on the integers, naturals, and cardinals. Thus, the reals can be treated as abstracts derived from purely general abstraction principles, and are thus possessed of necessary being. However, abstraction principles for imaginary numbers such as $i = \sqrt{-1}$, and complex numbers which are defined as the sum of a real number and a second real multiplied by $i$, have yet to be accounted for. The provision of an abstraction principle for complex numbers would, in any case, leave open the

[18] See Dedekind (1872/1996: Sec. 4), for the cut method for the definition of the reals.

[19] An abstraction principle for lengths, based on the equivalence property of congruence relations on intervals of a line, or regions of a space, is defined in Shapiro and Hellman (2015: 5, 9). Shapiro and Hellman provide, further, an abstraction principle for points, defined as comprising, respectively, the left- and right-ends of intervals (op. cit.: 5, 10-12).

[20] Cf. Hale (op. cit.: 406-407), for the further conditions that the domains are required to satisfy.

inquiry into how, e.g., complex-valued wave functions might interact with physical ontology; e.g., whether such functions might be metaphysically fundamental entities which serve to represent physical fields in higher-dimensional spacetime, and whether or how the domain of the functions, i.e., a real-valued configuration space for particles, might relate to the higher-dimensional, complex-valued wave function (cf. Simons, 2016; Ney, 2013; Maudlin, 2013).

The modality in the Barcan-induced Necessitist proposal at first- and higher-order is, as noted, interpreted metaphysically rather than logically, and thus incurs no similar issues with regard to the interaction between purely general properties, logical properties, and concrete entities. Further, because true on its second-order universal generalization on its intended, metaphysical interpretation, the possible truth-in-a-model of the relevant class of formulas is, as discussed in Section **3.1**, thus sufficient for entraining the actual truth of the relevant formulas.

### 3.3 Cardinality and Extensionality

An interesting residual question concerns the status of the worlds, upon the translation of modal first-order logic into the non-modal first-order language.[21] Fritz (op. cit.) notes that a world can be represented by a predicate, in the latter.[22] However, whether objects satisfy the predicate can vary from point to point, in the non-modal first-order class of points.[23] Another issue is that modal propositional logic is equivalent only to the bisimulation-invariant fragments of both first-order logic and fixed-point monadic second-order logic, rather than to

[21] Thanks here to xx, for discussion.

[22] For further discussion of the standard translation between propositional modal and first-order non-modal logics, see Blackburn et al. (2001: 84).

[23] Suppose that the model is defined over the language of second-order arithmetic, such that the points in the model are the ordinals. A uniquely designated point might then be a cardinal number whose height is accordingly indexed by the ordinals.

the full variants of either logic (cf. van Benthem, 1983; Janin and Walukiewicz, 1996). Thus, there cannot be a faithful translation from each modal operator in modal propositional logic into a predicate of full first- or monadic higher-order logic.

One way to mitigate the foregoing issues might be by arguing that the language satisfies real-world rather than general validity, such that necessarily the predicate will be satisfied only at a designated point in a model – intuitively, the analogue of the concrete rather than some merely possible world, simulating thereby the translation from possibilist to actualist discourse (cf. Fine, op. cit.: 211,135-136, 139-140, 154, 166-168, 170-171) – by contrast to holding of necessity as interpreted as satisfaction at every point in the model.[24] The reply would be consistent with what Williamson refers to as 'chunky-style necessitism' which validates the following theorems: where the predicate C(x) denotes the property of being grounded in the concrete and P(x) is an arbitrary predicate, (a) '$\forall x \Diamond C(x)$', yet (b) '$\Box \forall x[P(x_1, \ldots, x_n) \rightarrow (Cx_1, \ldots, Cx_1)]$' (325-332). Williamson (33, fn.5) argues, however, in favor of general, rather than real-world validity. A second issue for the reply is that principle (b), in the foregoing, is inconsistent with Williamson's protracted defense of the 'being constraint', according to which $\Box \forall x \Box [P(x_1, \ldots, x_n) \rightarrow \exists y(x = y)]$, i.e. if x satisfies a predicate, then x is something, even if possibly non-concrete (148).

A related issue concerns the translation of modalized, variable-binding, generalized quantifiers of the form:

'there are $n$ objects such that ...',

'there are countably infinite objects such that ...',

'there are uncountably infinite objects such that ...' (Fritz and Goodman,

[24] See ftn. 15.

2017).

The generalized quantifiers at issue are modalized and consistent with first-order Necessitism, because the quantifier domains include all possible – including contingently non-concrete – objects. It might be argued that the translation is not of immediate pertinence to the ontology of mathematics, because the foregoing first-order quantifiers can be restricted such that they range over only uncountably infinite *necessarily* non-concrete objects – i.e. abstracta – by contrast to ranging unrestrictedly over all modal objects, including the contingently non-concrete entities induced via the Barcan formula – i.e., the 'mere possibilia' that are non-concrete as a matter of contingency; e.g., the possible star and the possible nubula which are both actual objects and yet are such that a carpenter actually cannot, as a matter of pairwise incompossibility, render both concrete. However, the Necessitist thesis can be valid even in the quantifier domain of a first-order language restricted to necessarily non-concrete entities. If, e.g., a mathematician takes, despite iterated applications of set-forming operations, the cumulative hierarchy of sets to have a fixed cardinal height, then the first-order Necessitist thesis will still be valid, because all possible objects will actually be still something.

The first-order Necessitist proposal engendered by taking the height of the cumulative hierarchy to be fixed is further consistent with the addition to the first-order language of additional intensional operators – such as those introduced by Hodes (1984b) – in order to characterize the indefinite extensibility of the concept of set; i.e., that despite unrestricted universal quantification over all of the entities in a domain, another entity can be defined with reference to, and yet beyond the scope of, that totality, over which the quantifier would have further to range.[25] First-order Necessitism is further consistent with the relatively expanding domains induced by Bernays' (1942) Theorem. Bernays' Theorem states that class-valued functions from classes to sub-classes are not onto, where classes are non-sets (cf. Uzquiano, 2015a: 186-187). So, the cardinality of a class will always be less than the cardinality of its sub-classes. Suppose that that there is a generalization of Bernays' theorem, such that the non-sets are interpreted as possible objects. Thus, the cardinality of the class of possible objects will always be less than the cardinality of the sub-classes in the image of its mapping. Given iterated applications of Bernays' theorem, the cardinality of a domain of non-sets is purported then not to have a fixed height.

In both cases, however, the addition of Hodes' intensional operators permits there to be multiple-indexing in the array of parameters relative to which a cardinal can be defined, while the underlying logic for metaphysical modality can be S5, partitioning the space of worlds into equivalence classes. So, both the intensional characterization of indefinite extensibility and the generalization of Bernays' Theorem to possible objects are consistent with the first-order Necessitist proposal that all possible objects are actual, and so the cardinality of the target universe is fixed.[26]

Fritz and Goodman suggest that a necessary condition on the equivalence of propositions is that they define the same class of models (op. cit.: 1.4).

[25]The concept of indefinite extensibility is introduced by Dummett (1963/1978), in the setting of a discussion of the philosophical significance of Gödel's (1931) first incompleteness theorem. See the essays in Rayo and Uzquiano (2006); Studd (op. cit.); Dever (ms); and Khudairi (ms), for further discussion.

[26]Note that the proposal that the cardinality of the cumulative hierarchy of sets is fixed, despite continued iterated applications of set-forming operations, is anticipated by Cantor (1883/1996: Endnote [1]). Cantor writes: 'I have no doubt that, as we pursue this path ever further, we shall never reach a boundary that cannot be crossed, but that we shall also never achieve even an approximate conception of the absolute [. . . ] The absolutely infinite sequence of numbers thus seems to me to be an appropriate symbol of the absolute; in contrast the infinity of the first number-class (I) [i.e., the first uncountable cardinal, $\aleph_0$ – HK], which has hitherto sufficed, because I can I consider it to be a graspable idea (not a representation), seems to me to dwindle into nothingness by comparison' (op. cit.; cf. Cantor, 1899/1967).

The proposed translation of the modalized generalized quantifiers would be Contingentist, by taking (NNE) to be invalid, such that the domain in the translated model would be comprised of only possible concrete objects, rather than the non-concrete objects as well (op. cit.).

Because of the existence of non-standard models, the generalized quantifier that 'there are countably infinitely many possible . . . ' cannot be defined in first-order logic. Fritz and Goodman note that generalized quantifiers ranging over countably infinite objects can yet be simulated by enriching one's first-order language with countably infinite conjunctions. On the latter approach, finitary existential and universal quantifiers can be defined as the countably infinite conjunction of formulas stating that, for all natural numbers $n$, 'there are $n$ possible ...' (2.3).

Crucially, however, there are some modalized generalized quantifiers that cannot be similarly paraphrased – e.g., 'there are uncountably infinite possible objects s.t. . . . ' – and there are some modalized generalized quantifiers that cannot even be defined in first-order languages – e.g. '*most* possible objects s.t. . . . ' (2.4-2.5)

In non-modal first-order logic, it is possible to define generalized quantifiers which range over an uncountably infinite domain of objects, by augmenting finitary existential and universal quantifiers with an uncountably infinite stock of variables and an uncountably infinite stock of conjunctions of formulas (2.4).[27]

[27]Uncountable cardinals can be defined as follows. For cardinals, x,a,C, let C$\subseteq$a be closed unbounded in a, if it is closed [if x $<$ C and $\bigcup$(C$\cap$a) = a, then a$\in$C] and unbounded ($\bigcup$C = a) (Kanamori, 2012: 360). A cardinal, S, is stationary in a, if, for any closed unbounded C$\subseteq$a, C$\cap$S $\neq \emptyset$ (op. cit.). An ideal is a subset of a set closed under countable unions, whereas filters are subsets closed under countable intersections. A cardinal $\kappa$ is regular if the cofinality of $\kappa$ – comprised of the unions of sets with cardinality less than $\kappa$ – is identical to $\kappa$. For models A,B, and conditions $\phi$, an elementary embedding, j: A $\rightarrow$ B, is such that $\phi\langle a_1, \ldots, a_n\rangle$ in A if and only if $\phi\langle j(a_1), \ldots, j(a_n)\rangle$ in B (363). A measurable cardinal is defined as the ordinal denoted by the critical point of j, crit(j) (Koellner and Woodin, 2010: 7). Measurable cardinals are inaccessible (Kanamori, op. cit.). Uncountable regular limit cardinals are weakly inaccessible (op. cit.). A strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda <$

Fritz and Goodman note, however, that the foregoing would require that the quantifiers bind the uncountable variables 'at once', s.t. they must have the same scope. The issue with the proposal is that, in the setting of modalized existential quantification over an uncountably infinite domain, the Contingentist paraphrase requires that bound variables take different scopes, in order to countenance the different possible sets that can be defined in virtue of the indefinite extensibility of cardinal number (op. cit.).

In order to induce the Contingentist paraphrase, Fritz and Goodman suggest defining 'strings of infinitely many existential and universal quantifiers', such that a modalized, i.e. Necessitist, generalized quantifier of the form, 'there are uncountably infinite possible ...' can be redefined by an uncountably infinite sequence of finitary quantifiers with infinite variables and conjunction symbols of the form:

'Possibly for some $x_1$, possibly for some $x_2$, *etc.*: $x_1$,$x_2$,*etc.* are pairwise distinct and are each possibly ...',

where *etc.* denotes an uncountable sequence of, respectively, 'an uncountable string of interwoven possibility operators and existential quantifiers', and an 'uncountable string of variables' (op. cit.).

An argument against the proposed translation of the quantifier for there being uncountably infinite possible objects is that it is contentious whether an uncountable sequence of operators or quantifiers has a definite meaning [cf. Williamson (2013: 7.7)]. Thus, e.g., while negation can have a determinate truth condition which specifies its meaning, a string of uncountably infinite negation operators will similarly have determinate truth conditions and yet not have an intuitive, definite meaning (357). One can also define a positive

$\kappa$, then $2^\lambda < \kappa$ (op. cit.). For the foregoing and further definitions, see Koellner and Woodin (op.cit.); Kanamori (op. cit.), and Woodin (2009, 2010).

or negative integer, x, such that $s$x is interpreted as the successor function, x+1, and $p$x is interpreted as the inverse function, x-1. However, an infinitary expression consisting in uncountable, alternating iterations of the successor and inverse functions – $spsps\ldots$x – will similarly not have a definite meaning (op. cit.). Finally, one can define an operator $\mathrm{O}_i$ mapping truth conditions for an arbitrary formula A to the truth condition, $p$, of the formula $\diamond\exists \mathrm{x}_i(\mathrm{Cx}_i \wedge \mathrm{A})$, with Cx being the predicate for being concrete (258). Let the operators commute s.t. such that $\mathrm{O}_i\mathrm{O}_j$ iff $\mathrm{O}_j\mathrm{O}_i$, and be idempotent such that $\mathrm{O}_i\mathrm{O}_i$ iff $\mathrm{O}_i$ (op. cit.). A total ordering of truth conditions defined by an infinite sequence of the operators can be defined, s.t. that the relation is reflexive, anti-symmetric, transitive, and connected [$\forall$x,y(x$\leq$y $\vee$ y$\leq$x)] (op. cit.). However, total orders need not have a least upper bound; and the sequence, $\mathrm{O}_i\mathrm{O}_i\mathrm{O}_i\ldots(p)$, would thus not have a non-arbitrary, unique value (op. cit.). The foregoing might sufficiently adduce against Fritz and Goodman's Contingentist paraphrase of the uncountable infinitary modalized quantifier.

The philosophical significance of the barrier to a faithful translation from modal first-order to extensional full first-order languages, as well as a faithful translation from modalized, i.e. Necessitist, generalized quantifiers to Contingentist quantification, is arguably that the modal resources availed of in the abstractionist program might then be ineliminable.

# 4 Epistemic Modality, Metaphysical Modality, and Epistemic Utility and Entitlement

In this section, I address, finally, Hale and Wright's second issue with the role of Necessitism in guaranteeing that the possible truth of abstraction principles provides warrant for the belief in their actual truth. As noted, Hale and Wright argue, against the foregoing approach, that there is non-evidential entitlement rationally to trust that acceptable abstraction principles are true, and thus that the terms defined therein actually refer. In response, I will proceed by targeting the explanation in virtue of which there is such epistemic, default entitlement. I will outline two proposals concerning the foregoing grounding claim – advanced, respectively, in Khudairi (op. cit.), above, and by Wright (2012; 2014) – and I will argue that the approaches converge.

Wright's elaboration of the notion of rational trust, which is intended to subserve epistemic entitlement, appeals to a notion of 'expected epistemic utility' in the setting of decision theory (2014: 226, 241). In order better to understand this notion of expected epistemic utility, we must be more precise.

There are two, major interpretations of (classical) expected utility.[28] A model of decision theory is a tuple ⟨A,O,K,V⟩, where A is a set of acts; O is a set of outcomes; K encodes a set of counterfactual conditionals, where an act from A figures in the antecedent of the conditional and O figures in the conditional's consequent; and V is a function assigning a real number to each outcome. The real number is a representation of the value of the outcome. In evidential decision theory, the expected utility of an outcome is calculated as the product of the agent's credence, conditional on her action, by the utility of the outcome. In causal decision theory, the expected utility of an outcome is calculated as the product of the agent's credence, conditional on both her action and background knowledge of the causal efficacy thereof, by the utility of the

[28]For an examination of non-classical utility measures, see Buchak (2014). Non-classical utility measures are intended to describe the innocuous rationality with which an agent's expected utility might diminish with the order of the bets she might pursue. In the latter case, her expected utility will then be sensitive to her propensity to take risks relative to the total ordering of the gambles, such that she can have a preference for a sure-gain of .5 units of value, rather than prefer a bet with a 50 percent chance of winning either 0 or 1 units of value.

outcome.

First, because background knowledge concerning the causal efficacy of one's choice of acts is presumably orthogonal to the non-evidential rational trust to believe that mathematical abstraction principles are true, I will assume that the notion of expected epistemic utility theory that Wright (op. cit.) avails of relies only on the subjective credence of the agent, multiplied by the utility that she assigns to the outcome of the proposition in which she's placing her rational trust. Thus expected epistemic utility in the setting of decision-theory will be calculated within the (so-called) evidential, rather than causal, interpretation of the latter.

Second, there are two, major interpretations concerning how to measure the subjective credences of an agent. The philosophical significance of this choice point is that it bears directly on the very notion of the *epistemic utility* that an agent's beliefs will possess. So, e.g., according to pragmatic accounts of the accuracy of one's partial beliefs, one begins by defining a preference ordering on the agent's space of acts and outcomes. If the preference ordering is consistent with the Kolmogorov axioms[29], then one can set up a representation theorem from which the agent's subjective probability and utility measures (i.e., their expected utility measure) can be derived.[30] The epistemic utility associated with the pragmatic approach is, generally, utility maximization.

By contrast to the pragmatic approach, the epistemic approach to measuring the accuracy of one's beliefs is grounded in the notion of dominance (cf. Joyce, 1998; 2009). According to the epistemic approach, there is an ideal, or

[29]Namely: normality (which states that the probability of a tautology maps to 1); non-negativity (which states that the probabilty operator must take a non-negative value); additivity (which states that for all disjoint probability densities, the probability of their union is equal to the probability of the first density added to the probability of the second); and conditionalization [which states that the probability of $\phi$ conditional on $\psi$ equals the probability of the intersection of $\phi$ and $\psi$, divided by the probability of $\psi$].

[30]Cf. Ramsey (1926); Savage (1954); and Jeffrey (1965).

vindicated, probability concerning a proposition's obtaining, and if an agent's subjective probability measure does not satisfy the Kolmogorov axioms, then one can prove that it will always be dominated by a distinct measure; i.e. it will always be the case that a distinct subjective probability measure will be closer to the vindicated world than one's own. The epistemic utility associated with the epistemic approach is thus the minimization of inaccuracy (cf. Pettigrew, 2014).[31]

Wright notes that rational trust subserving epistemic entitlement will be pragmatic, and makes the intriguing point that 'pragmatic reasons are not a special genre of reason, to be contrasted with e.g. epistemic, prudential, and moral reasons' (2012: 484). He provides an example according to which one might be impelled to prefer the 'alleviation of Third world suffering' to one's own 'eternal bliss' (op. cit.); and so presumably has the pragmatic approach to expected utility in mind. The intriguing point to note, however, is that epistemic utility is variegated; one's epistemic utility might consist, e.g., in both the reduction of epistemic inaccuracy and in the satisfaction of one's preferences. Wright concludes that there is thus 'no good cause to deny certain kinds of pragmatic reason the title 'epistemic'. This will be the case where, in the slot in the structure of the reasons for an action that is to be filled by the desires of the agent, the relevant desires are focused on epistemic goods and goals' (op. cit.).

[31]The distinction between the epistemic (also referred to as the alethic) and the pragmatic approaches to epistemic utility is anticipated by Clifford (1877) and James (1896), with Clifford endorsing the epistemic approach, and James the pragmatic. The distance measures comprising the scoring rules for the minimization of inaccuracy are examined in, inter alia, Fitelson (2001); Leitgeb and Pettigrew (2010); and Moss (2011). A generalization of Joyce's argument for probabilism to models of non-classical logic is examined in Paris (2001) and Williams (2012). A dominance-based approach to decision theory is examined in Easwaran (2014), and a dominance-based approach to the notion of coherence – which can accommodate phenomena such as the preface paradox, and is thus weaker than the notion of consistency in an agent's belief set – is examined in Easwaran and Fitelson (2015).

Third, and most crucially: The very idea of expected epistemic utility in the setting of decision theory makes implicit appeal to the notion of possible worlds. The full and partial beliefs of an agent will have to be defined on a probability distribution, i.e. a set of epistemically possible worlds. The philosophical significance of this point is that it demonstrates how Hale and Wright's appeal to default, rational entitlement to trust that abstraction principles are true converges with the modal approach to the epistemology of mathematics advanced in Khudairi (op. cit.). The latter proceeds by examining undecidable sentences via the epistemic interpretation of multi-dimensional intensional semantics. The latter can be understood as recording the thought that the semantic value of a proposition relative to a first parameter (a context) which ranges over epistemically possible worlds, will constrain the semantic value of the proposition relative to a second parameter (an index) which ranges over metaphysically possible worlds. The formal clauses for epistemic and metaphysical mathematical modalities are as follows:

Let C denote a set of epistemically possibilities, such that $[\![\phi]\!]^c \subseteq$ C;

($\phi$ is a formula encoding a state of information at an epistemically possible world).

-$\texttt{pri}(x) = \lambda c.[\![x]\!]^{c,c}$;

(the two parameters relative to which x – a propositional variable – obtains its value are epistemically possible worlds. The function from possible formulas to values is thus an intension).

-$\texttt{sec}(x) = \lambda w.[\![x]\!]^{w,w}$

(the two parameters relative to which x obtains its value are metaphysically possible worlds).

Then:

- **Epistemic Mathematical Necessity (Apriority)**

  $[\![\blacksquare\phi]\!]^{c,w} = 1 \Longleftrightarrow \forall c'[\![\phi]\!]^{c,c'} = 1$

  ($\phi$ is true at all points in epistemic modal space).

- **Epistemic Mathematical Possibility**

  $[\![\diamond_c\phi]\!]^c \neq \emptyset \iff [\![Pr\phi]\!]^c \neq \emptyset \wedge >.5$, else $\langle\emptyset,\, Pr_c(\phi \mid \emptyset)\rangle$.

  ($\phi$ might be true if and only if its value is not null and it is greater than .5).

Epistemic mathematical modality is constrained by consistency, and the formal techniques of provability and forcing. A mathematical formula is metaphysically impossible, if it can be disproved or induces inconsistency in a model.

- **Convergence**

  $\forall$c$\exists$w$[\![\phi]\!]^{c,w} = 1$

  (the value of x is relative to a parameter for the space of epistemically possible worlds. The value of x relative to the first parameter determines the value of x relative to the second parameter for the space of metaphysical possibilities).

According, then, to the latter, the possibility of deciding mathematical propositions which are currently undecidable relative to a background mathematical language such as ZFC should be multi-dimensional. The epistemic possibility of deciding Orey sentences can thus be a guide to the metaphysical possibility thereof.[32]

[32] See Kanamori (2008) and Woodin (2010), for further discussion of the mathematical properties at issue.

The convergence between Wright's and my approaches consists, then, in that – on both approaches – there is a set of epistemically possible worlds. In the former case, the epistemically possible worlds subserve the preference rankings for the definability of expected epistemic utility. Epistemic mathematical modality is thus constitutive of the notion of rational entitlement to which Hale and Wright appeal, and – in virtue of its convergence with the multi-dimensional intensional semantics here proffered – epistemically possible worlds can serve as a guide to the metaphysical mathematical possibility that mathematical propositions, such as abstraction principles for cardinals, reals, and sets, are true.

## 5 Concluding Remarks

In this essay, I have endeavored to provide an account of the modal foundations of mathematical platonism. Hale and Wright's objections to the idea that Necessitism cannot account for how possibility and actuality might converge were shown to be readily answered. In response, further, to Hale and Wright's objections to the role of epistemic and metaphysical modalities in countenancing the truth of abstraction principles and the success of mathematical predicate reference, I demonstrated how my multi-dimensional modal approach to the epistemology of mathematics, augmented with Necessitism, is consistent with Hale and Wright's conception of the epistemic entitlement rationally to trust that abstraction principles are true. Epistemic and metaphysical modality may thus be shown to play a constitutive role in vindicating the reality of mathematical objects and truth, and in explaining our possible knowledge thereof.

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