Absolute Decidability and Mathematical Modality

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Abstract

This paper aims to contribute to the analysis of the nature of mathematical modality, and to the applications of the latter to unrestricted quantification and absolute decidability. Rather than countenancing the interpretational type of mathematical modality as a primitive, I argue that the interpretational type of mathematical modality is a species of epistemic modality. I argue, then, that the framework of multi-dimensional intensional semantics ought to be applied to the mathematical setting. The framework permits of a formally precise account of the priority and relation between epistemic mathematical modality and metaphysical mathematical modality. The discrepancy between the modal systems governing the parameters in the multi-dimensional intensional setting provides an explanation of the difference between the metaphysical possibility of absolute decidability and our knowledge thereof. I demonstrate, finally, how the duality axioms of the epistemic logic for the semantics can be availed of, in order to defuse the paradox of knowability.

1 Introduction

This essay aims to contribute to the analysis of the nature of mathematical modality, and to the applications of the latter to unrestricted quantification and absolute decidability. I argue that mathematical modality falls under at least three types; the interpretational, the metaphysical, and the logical. The interpretational type of mathematical modality has traditionally been taken to concern the interpretation of the quantifiers (cf. Linnebo, 2009, 2010, 2013; Studd, 2013); the possible reinterpretations of the intensions of the concept of set (Uzquiano, 2015); and the possibility of reinterpreting the domain over which the quantifiers range, in order to avoid inconsistency (cf. Fine, 2005, 2006, 2007).

The metaphysical type of modality concerns the ontological profile of abstracta and mathematical truth. Abstracta are thus argued to have metaphysically necessary being, and mathematical truths hold of metaphysical necessity, if at all (cf. Fine, 1981; Williamson, 2016). Instances, finally, of the logical type of mathematical modality might concern the properties of consistency (cf. Field, 1989: 249-250, 257-260; Rayo, 2013: 50; Leng: 2007; 2010: 258), and can perhaps be further witnessed by the logic of provability (cf. Boolos, 1993) and the modal profile of forcing (cf. Kripke 1965; Hamkins and Löwe, 2008).

The significance of the present contribution is as follows. (i) Rather than countenancing the interpretational type of mathematical modality as a primitive, I argue that the interpretational type of mathematical modality is a species of epistemic modality.[1] (ii) I argue, then, that the framework of multi-dimensional intensional semantics ought to be applied to the mathematical setting. The framework permits of a formally precise account of the priority and relation between epistemic mathematical modality and metaphysical mathematical modality. I target, in particular, the modal axioms that the respective interpretations of the modal operator ought to satisfy. The discrepancy between the modal systems governing the parameters in the multi-dimensional intensional setting provides an explanation of the difference between the metaphysical possibility of absolute decidability and our knowledge thereof. (iii) Finally, I examine the application of the mathematical modalities beyond the issues of

[1] A precedent to the current approach is Parsons (1979-1980; 1983: p. 25, chs.10-11; 2008: 176), who argues that intuition is both a species of the imagination and can be formalized by a mathematical modality. The mathematical modality is governed by S4.2, and concerns possible iterations of the successor operation in arithmetic and possible extensions of the set-theoretic cumulative hierarchy. Among the differences between Parsons' approach and the one here outlined is (i) that, by contrast to the current proposal, Parsons notes that his notion of mathematical modality is not epistemic (2008: 81fn1); and (ii) that Parsons (1997: 348-351; 2008: 98-100) suggests that the intuitional mathematical modality concerning computable functions is an idealization insensitive to distinctions such as those captured by computational complexity theory, rather than being defined relative to an epistemic modal space comprising the computational theory of mind. (See Author, ms, for further discussion.)

unrestricted quantification and indefinite extensibility. As a test case for the multi-dimensional approach, I investigate the interaction between the epistemic and metaphysical mathematical modalities and large cardinal axioms. The multi-dimensional intensional framework permits of a formally precise means of demonstrating how the metaphysical possibility of absolute decidability and the continuum hypothesis can be accessed by their epistemic-modal-mathematical profile. The logical mathematical modalities – of consistency, provability, and forcing – provide the means for discerning whether mathematical truths are themselves epistemically possible. I argue that, in the absence of disproof, large cardinal axioms are epistemically possible, and thereby provide a sufficient guide to the metaphysical mathematical possibility of determinacy claims and the continuum hypothesis.

In Section **2**, I define the formal clauses and modal axioms governing the epistemic and metaphysical types of mathematical modality. In Section **3**, I discuss how the properties of the epistemic mathematical modality and metaphysical mathematical modality converge and depart from previous attempts to delineate the contours of similar notions. Section **4** extends the multi-dimensional intensional framework to the issue of mathematical knowledge; in particular, to the modal profile of large cardinal axioms and to the absolute decidability of the continuum hypothesis. In Section **5**, I discuss the relevance of the Principle of Knowability to the issue of absolute decidability, and provide a novel, non-revisionary solution to the Knowability Paradox via suitable axioms of a normal epistemic logic. Section **6** provides concluding remarks.

2 Mathematical Modality

2.1 Metaphysical Mathematical Modality

A formula is a logical truth if and only if the formula is true in an intended model structure, M = <W, D, R, V>, where W designates a space of metaphysically possible worlds; D designates a domain of entities, constant across worlds; R designates an accessibility relation on worlds; and V is an assignment function mapping elements in D to subsets of W. A formula in M is a modal truth if and only if a faithful interpretation maps the formula to a metaphysically universal proposition (cf. Williamson, 2013.: 106). A formula satisfies conditions on metaphysically universality if and only if the formula is true on its universal generalization (cf. Williamson, op. cit.: 93). The language of the model may be understood as an idealized language of thought, rather than a mathematical language such as arithmetic or set theory, where the semantics for the language concerns the modalized states of information and representations of an agent.

2.2 Epistemic Mathematical Modality

In order to accommodate the notion of epistemic possibility, we enrich M with the following conditions: M = <C, W, D, R, V>, where C, a set of epistemically possibilities, is constrained as follows:

Let $[\![\phi]\!]^c \subseteq$ W;

(ϕ is a formula encoding a state of information at an epistemically possible world).

$\texttt{-pri}(x) = \lambda w.[\![x]\!]^{w,w}$;

(the two parameters relative to which x – a propositional variable – obtains its value are metaphysically possible worlds. The function from possible formulas to values is thus an intension).

-$\texttt{sec}(x) = \lambda c.[\![x]\!]^{c,c}$

(the two parameters relative to which x obtains its value are epistemically possible worlds).

Then:

- **Epistemic Mathematical Necessity (Apriority)**

  $[\![\blacksquare\phi]\!]^{w,c} = 1 \Longleftrightarrow \forall c'[\![\phi]\!]^{c,c'} = 1$

  (ϕ is true at all points in epistemic modal space).

- **Epistemic Mathematical Possibility**

  $[\![\diamond_c\phi]\!]^{i_n} \neq \emptyset \iff [\![Pr\phi]\!]^{i_n} \neq \emptyset \wedge >.5$, else $\langle\emptyset,\ Pr_{i_n}(\phi \mid \emptyset)\rangle$, where i_n designates an agent's state of information in a context.

  (ϕ might be true if and only if its value is not null and it is greater than .5).

Crucially, epistemic mathematical modality is constrained by consistency, and the formal techniques of provability and forcing. A mathematical formula is false, and therefore metaphysically impossible, if it can be disproved or induces inconsistency in a model.

2.3 Interaction

- **Convergence**

  $\forall$w$\exists$c$[\![\phi]\!]^{w,c} = 1$

  (the value of x is relative to a parameter for the space of metaphysically possible worlds. The value of x relative to the first parameter determines the value of x relative to the second parameter for the space of epistemic possibility).

- **Super-rigidity (2D-Intension):**

  $[\![\phi]\!]^{w,c} = 1 \iff \forall$w',c'$[\![\phi]\!]^{w',c'} = 1$

  (the intension of ϕ is rigid in all points in metaphysical and epistemic modal space).

2.4 Modal Axioms

- Metaphysical mathematical modality is governed by the modal system KTE, as augmented by the Barcan formula and its Converse (cf. Fine, 1981).

K: $\Box[\phi \rightarrow \psi] \rightarrow [\Box\phi \rightarrow \Box\psi]$

T: $\Box\phi \rightarrow \phi$

E: $\neg\Box\phi \rightarrow \Box\neg\Box\phi$

Barcan: $\Box\forall xFx \rightarrow \forall x\Box Fx$

Converse Barcan: $\forall x\Box Fx \rightarrow \Box\forall xFx$

- Epistemic mathematical modality is governed by the modal system, KT4, as augmented by the Barcan formula and the Converse Barcan formula.[2]

[2]Reasons adducing against including the Smiley-Gödel-Löb provability formula among the axioms of epistemic mathematical modality are examined in Section 5. GL states that '$\blacksquare[\blacksquare\phi \rightarrow \phi] \rightarrow \blacksquare\phi$'. For further discussion of the properties of GL, see Löb (1955); Smiley (1963); Kripke (1965); and Boolos (1993). Löb's provability formula was formulated in response to Henkin's (1952) problem concerning whether a sentence which ascribes the property of being provable to itself is provable. (Cf. Halbach and Visser, 2014, for further discussion.) For an anticipation of the provability formula, see Wittgenstein (1933-1937/2005: 378). Wittgenstein writes: 'If we prove that a problem can be solved, the concept 'solution' must somehow occur in the proof. (There must be something in the mechanism of the proof that corresponds to this concept.) But the concept mustn't be represented by an external description; it must really be demonstrated. / The proof of the provability of a proposition is the proof of the proposition itself' (op. cit.). Wittgenstein contrasts the foregoing type of proof with 'proofs of relevance' which are akin to the mathematical, rather than empirical, propositions, discussed in Wittgenstein (2001: IV, 4-13, 30-31).

K: $\blacksquare[\phi \to \psi] \to [\blacksquare\phi \to \blacksquare\psi]$

T: $\blacksquare\phi \to \phi$

4: $\blacksquare\phi \to \blacksquare\blacksquare\phi$

Barcan: $\blacksquare\forall x Fx \to \forall x\blacksquare Fx$

Converse Barcan: $\forall x\blacksquare Fx \to \blacksquare\forall x Fx$

3 Departures from Precedent

The approach to mathematical modality, according to which it yields a representation of the cumulative universe of sets, has been examined by Fine (2005; 2006) and Uzquiano (2015). Fine argues that the mathematical modality should be interpretational; and thus taken to concern the reinterpretation of the domain over which the quantifiers range, in order to avoid inconsistency. Uzquiano argues similarly for an interpretational construal of mathematical modality, where the cumulative hierarchy of sets is fixed, yet what is possibly reinterpreted is the non-logical vocabulary of the language, in particular the membership relation.[3]

On Fine's approach, the interpretational modality is both postulational, and 'prescriptive' or imperatival. The prescriptive element consists in the rule:

'Introduction: !x.C(x)',

such that one is enjoined to postulate, i.e. to 'introduce an object x conforming to the condition C(x)' (2005: 91; 2006: 38).

In the setting of unrestricted quantification, suppose, e.g., that there is an interpretation for the domain over which a quantifier ranges. Fine writes that an interpretation 'I is exten[s]ible – in symbols, E(I) – if possibly some interpretation extends it, i.e. $\diamond\exists$J(I$\subset$J)' (2006: 30). Then, the interpretation of the domain over which the quantifier ranges is *extensible*, if '$\forall$I.E(I)'. The interpre-

[3]Compare Gödel, 1947; Williamson, 1998; and Fine, 2005.

tation of the domain over which the quantifier ranges is *indefinitely extensible*, if '$\Box\forall I.E(I)$' iff '$\Box\forall I\Diamond\exists J(I\subset J)$', where the reinterpretation is induced via the prescriptive imperative to postulate the existence of a new object by the foregoing 'Introduction' rule (2006: 30-31; 38). Fine clarifies that the interpretational approach is consistent with a 'realist ontology' of the set of reals. He refers to the imperative to postulate new objects, and thereby reinterpret the domain for the quantifier, as the 'mechanism' by which epistemically to track the cumulative hierarchy of sets (2007: 124-125).

In accord with Fine's approach, the epistemic mathematical modality defined in the previous section was taken to have a similarly representational interpretation, and perhaps the postulational property is an optimal means of inducing a reinterpretation of the domain of the quantifier. However, the present approach avoids a potential issue with Fine's account, with regard to the the introduction of deontic modal properties of the prescriptive and imperatival rules that he mentions.[4] It is sufficient that the interpretational modalities are a species of epistemic modality, i.e. possibilities that are relative to agents' spaces of states of information. Developing Fine's program, Linnebo (2013) outlines a modalized version of ZF. Similarly to the modal axioms for the epistemic mathematical modality specified in the previous section, Linnebo argues that his modal set theory ought to be governed by the system S4.2, the Converse Barcan formula, and (at least a restricted version of) the Barcan formula. However – rather than being either interpretational or epistemic – Linnebo deploys the mathematical modality in order to account for the notion of 'potential infinity', as anticipated by Aristotle.[5] The mathematical modality is thereby intended to provide a formally precise answer to the inquiry into the extent of the cu-

[4]For an analysis of the precise interaction between the semantic values of epistemic and deontic modal operators, see Author (ms).

[5]Cf. Aristotle, *Physics*, Book III, Ch. 6.

mulative set-theoretic hierarchy; i.e., in order to precisify the answer that the hierarchy extends 'as far as possible' (2013: 205).[6]

Thus, Linnebo takes the modality to be constitutive of the actual ontology of sets; and the quantifiers ranging over the actual ontology of sets are claimed to have an 'implicitly modal' profile (2010: 146; 2013: 225). He suggests, e.g., that: 'As science progresses, we formulate set theories that characterize larger and larger initial segments of the universe of sets. At any one time, precisely those sets are actual whose existence follows from our strongest, well-established set theory' (2010: 159n21). However – despite his claim that the modality is constitutive of the actual ontology of sets – Linnebo concedes that the mathematical modality at issue cannot be interpreted metaphysically, because sets exist of metaphysical necessity if at all (2010: 158; 2013: 207). In order partly to allay the tension, Linnebo remarks, then, that set theorists 'do not regard themselves as located at some particular stage of the process of forming sets' (2010: 159); and this might provide evidence that the inquiry – concerning at which stage in the process of set-individuation we happen to be, at present – can be avoided.

Another distinction to note is that both Linnebo (op. cit.) and Uzquiano (op. cit.) avail of second-order plural quantification, in developing their primitivist and interpretational accounts of mathematical modality. By contrast to their approaches, the epistemic and metaphysical modalities defined in the previous section are defined with second-order singular quantification over sets.

[6]Precursors to the view that modal operators can be availed of in order to countenance the potential hierarchy of sets include Hodes (1984). Intensional constructions of set theory are further developed by Reinhardt (1974); Parsons (1983); Myhill (1985); Scedrov (1985); Flagg (1985); Goodman (1985); Hellman (1990); Nolan (2002); and Studd (2013). (See Shapiro (1985) for an intensional construction of arithmetic.) Chihara (2004: 171-198) argues that 'broadly logical' conceptual possibilities can be used to represent imaginary situations relevant to the construction of open-sentence tokens. The open-sentences can then be used to define the properties of natural and cardinal numbers and the axioms of Peano arithmetic.

Finally, Linnebo and Uzquiano both suggest that their mathematical modalities ought to be governed by the G axiom; i.e. $\Diamond\Box\phi \to \Box\Diamond\phi$. The present approach eschews, however, of the G axiom, in virtue of the following. Williamson (2009) demonstrates that – because KT4G is a sublogic of S5 – an epistemic operator which validates the conjunction of the 4 axiom of positive introspection and the E axiom of negative introspection will be inconsistent with the condition of 'recursively enumerable quasi-conservativeness'. Recursively enumerable quasi-conservativeness is a computational constraint on an epistemic agent's theorizing, according to which the intended models of the agent's theory are both maximally consistent and conservatively extended by addition of the 'box'-operator, interpreted as expressing the agent's state of knowledge. As axioms of an agent's consistent, recursively axiomatizable theorizing about the theory of its own states of knowledge and belief, the conjunction of 4 and E would entail that the agent's theory is both consistent and decidable, in conflict with Gödel's (1931) second incompleteness theorem. The modal system, KT4, avoids the foregoing result. In the present setting, the circumvention is innocuous, because the undecidability – yet recursively enumerable quasi-conservativeness – of an epistemic agent's consistent theorizing about its epistemic states is consistent with the epistemic mathematical possibility that large cardinal axioms are absolutely decidable.

4 Knowledge of Absolute Decidability

Williamson (2016) examines the extension of the metaphysically modal profile of mathematical truths to the question of absolute decidability. In this section, I aim to extend Williamson's analysis to the notion of epistemic mathematical modality that has been developed in the foregoing sections. The extension provides a crucial means of witnessing the signficance of the multi-dimensional intensional approach for the epistemology of mathematics.

Williamson proceeds by suggesting the following line of thought. Suppose that A is a true interpreted mathematical formula which eludes present human techniques of provability; e.g. the continuum hypothesis (op. cit.). Williamson argues that mathematical truths are metaphysically necessary (op. cit.). From there, he suggests that knowledge of A satisfies the condition of *safety* from error, as codified via a reflexive and symmetric accessibility relation from worlds at which A is known. Thus, there is either no, or a small risk of, not believing that A, relative to a world in which A is known – although the safety condition is not itself sufficient for mathematical knowledge that A. Williamson then enjoins one to consider the following scenario: It is metaphysically possible that there is a species which can prove that A. Therefore, A is absolutely provable; that is, A 'can in principle be known by a normal mathematical process' such as derivation in an axiomatizable formal system with quantification and identity.

Williamson's scenario evinces one issue for the 'back-tracking' approach to modal epistemology, at least as it might be applied to the issue of possible mathematical knowledge. On the back-tracking approach, the method of modal epistemology is taken to proceed by first discerning the metaphysical modal truths – normally by natural-scientific means – and then working backward to the exigent incompleteness of an individual's epistemic states concerning such truths (cf. Stalnaker, 2003; Vetter, 2013).

The issue for the back-tracking method that Williamson's scenario illuminates is that the metaphysical mathematical possibility that CH is absolutely decidable must in some way converge with the epistemic possibility thereof. However, the normal mathematical techniques that Williamson specifies – i.e.

proof and forcing – fall within the remit of what is mathematically possible relative to agents' states of information; i.e. what is epistemically mathematically possible. Thus, whether CH is metaphysically necessary – and thus, as Williamson claims, metaphysically possible and absolutely decidable thereby – can only be witnessed by the epistemic means of demonstrating that its absolute decidability is not impossible. It may thus be epistemically possible that Williamson's technically advanced species, which can absolutely decide CH, exist – following Williamson (2013), they actually exist, albeit non-concretely – but the metaphysical necessity of the absolute decidability of CH needs still to be corroborated.

The significance of the multi-dimensional intensional framework outlined in the foregoing is that it provides an explanation of the discrepancy between metaphysical mathematical modality and epistemic mathematical modality. The metaphysical mathematical modality is taken to be more fundamental than the epistemic, as witnessed by the order of the parameters specified in the Convergence property in Section **2**. Further and crucially, metaphysical mathematical modality is governed by the system S5, the Barcan formula, and its Converse, whereas epistemic mathematical modality is governed by KT4, the Barcan formula, and its Converse. Thus, epistemic mathematical modality figures as the mechanism, which enables the tracking of metaphysically possible mathematical truth.[7]

[7] A provisional definition of large cardinal axioms is as follows.
$\exists x\Phi$ is a large cardinal axiom, because:
(i) Φx is a Σ_2-formula;
(ii) if κ is a cardinal, such that $V \models \Phi(\kappa)$, then κ is strongly inaccessible, where a cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ, and a strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^\lambda < \kappa$ (Cf. Kanamori, 2012: 360); and
(iii) for all generic partial orders $\mathbb{P} \in V_\kappa$, $V^{\mathbb{P}} \models \Phi(\kappa)$; I_{NS} is a non-stationary ideal, where an ideal is a subset of a set closed under countable unions, whereas filters are subsets closed under countable intersections. (Cf. Kanamori, op. cit.: 361); A^G is the canonical representation of reals in $L(\mathbb{R})$, i.e. the interpretation of A in M[G]; $H(\kappa)$ is comprised of all of the sets whose

Leitgeb (2009) endeavors similarly to argue for the convergence between the notion of informal provability – countenanced as an epistemic modal operator, K – and mathematical truth. Availing of Hilbert's (1923/1996: ¶18-42) epsilon terms for propositions, such that, for an arbitrary predicate, $\mathbf{C}$(x), with x a propositional variable, the term 'ϵp.$\mathbf{C}$(p)' is intuitively interpreted as stating that 'there is a proposition, x(/p), s.t. the formula, that p satisfies $\mathbf{C}$, obtains' (op. cit.: 290). Leitgeb purports to demonstrate that $\forall$p(p $\rightarrow$ Kp), i.e. that informal provability is absolute; i.e. truth and provability are co-extensive.[8] He argues as follows. Let A(p) abbreviate the formula 'p $\wedge$ $\neg$K(p)', i.e., that the proposition, p, is true while yet being unprovable. Let K be the informal provability operator reflecting knowability or epistemic necessity, with $\langle$K$\rangle$ its dual.[9] Then:

1. $\exists$p(p $\wedge$ $\neg$Kp) $\iff$ ϵp.A(p) $\wedge$ $\neg$Kϵp.A(p).

By necessitation,

2. K[$\exists$p(p $\wedge$ $\neg$Kp)] $\iff$ K[ϵp.A(p) $\wedge$ $\neg$Kϵp.A(p)].

Applying modal axioms, KT, to (1), however,

3. $\neg$K[ϵp.A(p) $\wedge$ $\neg$Kϵp.A(p)].

Thus,

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transitive closure is $< \kappa$ (cf. Rittberg, 2015); and L($\mathbb{R}$)$^{\mathbb{P}max}$ $\models$ $\langle$H(ω_2), $\in$, I$_{NS}$, A^G$\rangle$ $\models$ 'ϕ'. $\mathbb{P}$ is a homogeneous partial order in L($\mathbb{R}$), such that the generic extension of L($\mathbb{R}$)$^{\mathbb{P}}$ inherits the generic invariance, i.e., the absoluteness, of L($\mathbb{R}$). Thus, L($\mathbb{R}$)$^{\mathbb{P}max}$ is (i) effectively complete, i.e. invariant under set-forcing extensions; and (ii) maximal, i.e. satisfies all Π_2-sentences and is thus consistent by set-forcing over ground models (Woodin, ms: 28).

Assume ZFC and that there is a proper class of Woodin cardinals; A$\in\mathbb{P}$($\mathbb{R}$) $\cap$ L($\mathbb{R}$); ϕ is a Π_2-sentence; and V(G), s.t. $\langle$H(ω_2), $\in$, I$_{NS}$, A^G$\rangle$ $\models$ 'ϕ': Then, it can be proven that L($\mathbb{R}$)$^{\mathbb{P}max}$ $\models$ $\langle$H(ω_2), $\in$, I$_{NS}$, A^G$\rangle$ $\models$ 'ϕ', where 'ϕ' := $\exists$A$\in\Gamma^\infty$$\langle$H($\omega_1$), $\in$, A$\rangle$ $\models$ ψ.

The axiom of determinacy (AD) states that every set of reals, a$\subseteq\omega^\omega$ is determined, where κ is determined if it is decidable.

Woodin's (1999) Axiom (*) can be thus countenanced:

AD$^{L(\mathbb{R})}$ and L[($\mathbb{P}\omega_1$)] is a $\mathbb{P}$max-generic extension of L($\mathbb{R}$),

from which it can be derived that $2^{\aleph_0} = \aleph_2$. Thus, $\neg$CH; and so CH is absolutely decidable.

[8] The formula is referred to as the Principle of Knowability, and discussed in further detail in Section 5, below.

[9] See Section 5, for further discussion of the duality of knowledge, and its relation to doxastic operators.

4. $\neg K \exists p(p \wedge \neg Kp)$.

Leitgeb suggests that (4) be rewritten

5. $\langle K \rangle \forall p(p \rightarrow Kp)$.

Abbreviate (5) by $\mathtt{B}$. By existential introduction and modal axiom K, both

6. $\mathtt{B} \rightarrow \exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg \mathtt{B}) \wedge p]$, and

7. $\neg \mathtt{B} \rightarrow \exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg \mathtt{B}) \wedge p]$.

Thus,

8. $\exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg \mathtt{B}) \wedge p]$.

Abbreviate (8) by $\mathtt{C}(p)$. Introducing epsilon notation,

9. $[K(\epsilon p.\mathtt{C}(p) \rightarrow \mathtt{B}) \vee K(\epsilon p.\mathtt{C}(p) \rightarrow \neg \mathtt{B})] \wedge \epsilon p.\mathtt{C}(p)$.

By K,

10. $[K(\epsilon p.\mathtt{C}(p) \rightarrow K\mathtt{B}) \vee K(\epsilon p.\mathtt{C}(p) \rightarrow K\neg \mathtt{B})]$.

From (9) and necessitation, one can further derive

11. $K\epsilon p.\mathtt{C}(p)$.

By (10) and (11),

12. $K\mathtt{B} \vee K\neg \mathtt{B}$.

From (5), (12), and K, Leitgeb derives

13. $K\mathtt{B}$.

By, then, the T axiom,

14. $\forall p(p \rightarrow Kp)$ (291-292).

Rather than accounting for the coextensiveness of epistemic provability and truth, Leitgeb interprets the foregoing result as cause for pessimism with regard to whether the formulas countenanced in epistemic logic and via epsilon terms are genuinely logical truths if true at all (292).

In response to the attending pressure on the status of epistemic logic as concerning truths of logic, one can challenge the derivation, in the above proof, from lines (12) to (13). The inference depends on line (5), i.e., the epistemic possibility of completeness: $\langle K \rangle \forall p(p \rightarrow Kp)$. One can question how, from (4), i.e. the unprovability of the unprovability of a proposition $[\neg K \exists p(p \wedge \neg Kp)]$, one can derive (5), i.e. that it is epistemically possible that all propositions are informally provable. Assume, however, that line (5) is valid. Then, the validity of the inference from (12) to (13) can be challenged by the restriction on the quantifier on worlds in the Knowability Principle expressed by (5). The epistemic operator in lines (12) and (13) records, by contrast, the epistemic necessity, rather than the possibility, of the truth of the formulas and subformulas therein. Thus, from (12) either the provability of the provability of propositions or the provability of the unprovability of propositions, one cannot derive (13) the provability of the provability of propositions, because – by (5) – it is only epistemically possible that all true propositions are provable.

5 Knowability without Paradox

This section aims, finally, to provide a novel, non-revisionary solution to the Church-Fitch paradox that the Knowability Principle – that all truths can be known – either yields contradiction or entrains the worse result that all truths are known. One reason to endorse the claim that all truths can be known is that a condition of evidential constraint might be constitutive of the alethic property (cf. Dummett, 1977/2000, 1978, 1991; Wright, 1987/1993; Putnam, 1981). However, the paradox appears to undermine the viability of approaches to the nature of truth which take the latter to be evidentially constrained. A distinct reason for endorsing the Knowability Principle is that it has crucial extensions to the epistemology of mathematics. An Orey sentence has the form: For any theory T, 'T $\equiv$ T + ϕ and T $\equiv$ T + $\neg\phi$'. Examples of Orey sentences include the Generalized Continuum Hypothesis (i.e., '$2^{\aleph_n} = \aleph_{n+1}$') where the theory at issue is ZFC, and the Projective Uniformization property relative to ZF (i.e., whether a choice function can be defined in ZF for a pointclass comprised of sets of reals). In the previous section, I noted that Williamson (op. cit.) argues that, because it is metaphysically possible for Orey sentences to be decided, Orey sentences are thus absolutely decidable. I availed, then, of a multi-dimensional intensional semantics, and argued that, if epistemically mathematically possible in virtue of, e.g., consistency via forcing techniques and large cardinal axioms, then the absolute decidability of Orey sentences is metaphysically mathematically possible. So, for any mathematical truth ϕ, the value of ϕ can be known.

In this section, I argue that, if one accepts principles of duality for the belief and knowledge operators which mirror those for the diamond and box operators, then the axioms of epistemic logic are themselves sufficient for a dissolution of the knowability paradox. So, if the belief and knowledge operators are the abbreviations of their duals, then the Knowability Principle is not false, and Knowability entails neither contradiction nor the implausible result that all truths are known.

The present proposal contrasts to revisionary approaches to the underlying epistemic logic, by retaining classical principles rather than arguing either (i) that the paradox is intuitionistically invalid (cf. Dummett, 1977/2000, 2009; Wright, op. cit.; Bermúdez, 2009), or (ii) that the inconsistency or paracompleteness entailed by the paradox is innocuous, via restriction of the law of excluded middle (cf. Routley, 1981/2010; Beall, 2000, 2009; Wansing, 2002; Priest, 2009). A virtue of the present result is thus that classical reasoning can be consistent with a principle amenable to verificationist approaches to truth;

and so my solution to the paradox can be accepted by both realists and verificationists alike. A second virtue of the present account is that the epistemic logic availed of, in response to the paradox, is computationally more basic than those proffered by van Benthem (2009) and Restall (2009). Finally, a third virtue of the proposal is that it circumvents reformulation of the Knowability Principle itself; for example, by the addition of the actuality operator and quantification over situations (cf. Edgingtion, 1985, 2010).

Here is the paradox. The Knowability Principle states that, for all ϕ, if ϕ is true then it is metaphysically possible for there to be knowledge that ϕ: '$\forall\phi(\phi \rightarrow \diamond\mathrm{K}\phi)$'. Suppose, however, that 'ϕ' abbreviates 'ϕ and it is not the case that one knows that ϕ': '$\phi \wedge \neg\mathrm{K}\phi$' (cf. Fitch, 1963; Church, 2009). In the consequent of the Knowability Principle, the foregoing would yield that $\diamond\mathrm{K}(\phi \wedge \neg\mathrm{K}\phi)$. Thus, $\mathrm{K}\phi$ and $\mathrm{K}\neg\mathrm{K}\phi$. Thus, $\mathrm{K}\phi$. However, by the modal T axiom, which codifies the factivity or truthfulness of knowledge, $\forall$x(Kx $\rightarrow$ x). Thus, $\neg\mathrm{K}\phi$. Contradiction.

Suppose, then, that one negates the problematic sentence in the antecedent of the Principle. Thus, $\neg(\phi \wedge \neg\mathrm{K}\phi)$. By the De Morgan rules for negation, $\neg\phi \vee \neg\neg\mathrm{K}\phi$. By double negation elimination, $\neg\phi \vee \mathrm{K}\phi$. By the definition of the material conditional, $\phi \rightarrow \mathrm{K}\phi$; that is, all truths are known. The paradox of knowability is therefore that possible knowledge of unknown truths entails either contradiction or omniscience.

One response to the paradox that has yet to be examined is the adoption of the provability logic, GL. The relevant axioms of GL are the Smiley-Gödel-Löb provability formula $[\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi]$, K $[\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)]$, and 4 $[\Box\phi \rightarrow \Box\Box\phi]$, although not T $(\Box\phi \rightarrow \phi)$. Thus, the derivation of contradiction in the consequent of the Knowability Principle can be blocked via the invalidation of the T axiom. However, denying the factivity of knowledge might be too significant a cost in the endeavor to retain the Knowability Principle.

At the opposing extreme is Restall's (op. cit.) approach, which avails of the modal system, S5, on its epistemic interpretation. As mentioned, one issue for the latter is that endorsing both axioms 4 and E (i.e., $\neg\Box\phi \rightarrow \Box\neg\Box\phi$) would entail that an agent's consistent epistemic theorizing about its epistemic states is decidable, in conflict with Gödel's second incompleteness theorem. (The latter states that – given a particular choice of coding, predicate, and fixed point – a recursively axiomatizable system is consistent only if it is inconsistent.) Because epistemic agent's theorizing ought ideally to be consistent, the pernicious consequences of decidability are avoidable by eschewing of axioms 4 and E.

By contrast, then, to the foregoing, our target epistemic logic can be sufficiently axiomatized by the modal system, KT, as augmented by the rule of necessitation ($\vdash\phi \rightarrow \vdash \Box\phi$). '$\Box\phi$' is interpreted as stating that 'the agent knows that ϕ', and '$\diamond\phi$' is interpreted as stating that 'the agent believes that ϕ'. If so, then $\diamond\phi$ abbreviates $\neg\Box\neg\phi$, and conversely $\Box\phi$ abbreviates $\neg\diamond\neg\phi$. In our epistemic logic, the foregoing entails that – because belief is not factive – belief that ϕ abbreviates that, for all one knows, ϕ is false: 'Bϕ' $\iff$ '$\neg$K$\neg\phi$'. Conversely, not knowing whether a proposition is false rather than true is a necessary condition for belief thereof. Crucially, knowledge that ϕ abbreviates disbelief that ϕ is false: 'Kϕ' $\iff$ '$\neg$B$\neg\phi$'. Kϕ entails Bϕ, if the seriality condition is further endorsed, i.e. $\Box\phi \rightarrow \diamond\phi$. Finally, in the setting of metaphysical modality, the possibility that ϕ abbreviates that ϕ is not necessarily false: $\diamond\phi \iff \neg\Box\neg\phi$.

In virtue of the dualities of the belief and knowledge operators, the Knowability Principle is logically equivalent to the following statement: For all ϕ, if ϕ is true, then it is not necessarily not the case that one disbelieves the negation of ϕ: '$\forall\phi(\phi \rightarrow \neg\Box\neg\neg B\neg\phi)$'. Suppose, again, that 'ϕ' abbreviates 'ϕ and it is not the case that one knows that ϕ': '$\phi \wedge \neg K\phi$'. In the consequent of the Knowability Principle, the foregoing yields that it is not necessarily not the case that one disbelieves the negation of the proposition that ϕ is true and ϕ isn't known: $\neg\Box\neg\neg B\neg(\phi \wedge \neg K\phi)$. By double negation elimination, $\neg\Box B\neg(\phi \wedge \neg K\phi)$. By the De Morgan rules of negation, the Knowability Principle is therefore logically equivalent to the proposition that it is not necessary for one to believe that all truths are known: $\neg\Box B(\phi \rightarrow K\phi)$. That involves no contradiction; and so – by classical reasoning – the Knowability Principle is innocuous.

A possible objection to the present approach is that knowledge ought not to abbreviate disbelief that a proposition is false. Left-to-right, the duality between the knowledge operator and the belief operator appears innocuous; if one knows that ϕ is true, then one ought to disbelieve that ϕ is false. However, the right-to-left direction of the duality would appear to be more problematic; the disbelief that not ϕ ought not to be sufficient for knowledge.

The solution to the objection depends both (i) on the priority that one accords to knowledge, by contrast to belief, and (ii) on the observation that the entailment relations reflect *necessary* though insufficient conditions. There is thus an implicit priority in the directions by which to interpret the epistemic duality principles. Compare, e.g., the significance of the direction by which to interpret second-order implicit definitions for the cardinals, in the neo-logicist foundations of number theory and analysis.[10] Sense is conferred to the left-hand-side of the biconditional in virtue of the priority of the right-to-left reading of the definition. Similarly, understanding the notion of metaphysical necessity – truth at all points in a model – takes priority to understanding the notion of

[10]For the locus classicus of the abstractionist/neo-logicist program, see Hale and Wright (2001).

metaphysical possibility; and so ϕ is possibly true if and only if, and because, it is not necessary for ϕ to be false. However, that engenders no bar to the stipulation that $\Box\phi$ abbreviates the impossibility that ϕ is false.

The situation is similar in epistemic logic. The belief that ϕ is true reflects the foregoing conceptual priority: On the assumption that knowledge is a fundamental mental state – comprised, e.g., of a set of factive propositions – belief would thereby be the derivative property, owing (inter alia) to its nonfactivity (cf. Williamson, 2001). Thus, an agent believes that ϕ is true if and only if, and because, the agent does not know that ϕ is false. Not knowing that ϕ is false is, conversely, a necessary condition for the belief that ϕ is true. By the duality and seriality axioms specified above, one's knowledge that ϕ subsequently entails the consistency of one's belief with regard to ϕ; i.e., that one believes ϕ and disbelieves ϕ's negation. Crucially, however, the knowledge operator can abbreviate 'disbelief that not', because of the conceptual priority of knowledge to belief. The left-to-right interpretation of the principle takes, then, priority to the right-to-left direction; and the right-to-left direction is descriptively adequate, because – in the setting of epistemic modal logic – grasp of the concept of belief is insufficient, although a derivative necessary condition, for grasp of the concept of knowledge.

6 Concluding Remarks

In this paper, I have endeavored to delineate the types of mathematical modality, and to argue that the epistemic interpretation of multi-dimensional intensional semantics can be applied in order to explain, in part, the epistemic status of large cardinal axioms and the decidability of Orey sentences. I demonstrated, further, how the duality axioms in the logic of epistemic mathematical modality are able to defuse the paradoxical consequences associated with the Knowability Principle, to the effect that for all truths it is possible to possess knowledge thereof. The formal constraints on mathematical conceivability adumbrated in the foregoing can therefore be considered a guide to our possible knowledge of unknown mathematical truth.

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