A Modal Logic for Gödelian Intuition

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Abstract

This essay aims to provide a modal logic for rational intuition. Similarly to treatments of the property of knowledge in epistemic logic, I argue that rational intuition can be codified by a modal operator governed by the axioms of a dynamic provability logic, which augments GL with the modal μ-calculus. Via correspondence results between modal logic and first-order logic, a precise translation can then be provided between the notion of 'intuition-of', i.e., the cognitive phenomenal properties of thoughts, and the modal operators regimenting the notion of 'intuition-that'. I argue that intuition-that can further be shown to entrain conceptual elucidation, by way of figuring as a dynamic-interpretational modality which induces the reinterpretation of both domains of quantification and the intensions of mathematical concepts that are formalizable in monadic first- and second-order formal languages.

1 Introduction

'The incompleteness results do not rule out the possibility that there is a theorem-proving computer which is in fact equivalent to mathematical intuition' – Gödel, quoted in Wang (1986: 186).[1]

Gödel's writings on the epistemology of mathematics number fewer than twenty pages. He avails, in his remarks, of a notion of non-sensory intuition – alternatively, 'consciousness', or 'phenomenology' (cf. Gödel, 1961: 383) – as a fundamental, epistemic conduit into mathematical truths.[2] According to

[1]Note however that, in the next subsequent sentence, Gödel records scepticism about the foregoing. He remarks: 'But they imply that, in such a – highly unlikely for other reasons – case, either we do not know the exact specification of the computer or we do not know that it works correctly' [Gödel, quoted in Wang (op. cit.)].

[2]Another topic that Gödel suggests as being of epistemological significance is the notion of 'formalism freeness', according to which the concepts of computability, demonstrability (i.e., absolute provability), and ordinal definability can be specified independently of a background formal language (cf. Gödel 1946, and Kennedy 2013 for further discussion). Kennedy notes however that, in his characterizations of demonstrability and definability, Gödel assumes ZFC as his metatheory (op. cit.: 383). Further examination of the foregoing is beyond the scope of the present essay.

Gödel, the defining properties of mathematical intuition include (i) that it either is, or is analogous to, a type of perception (1951: 323; 1953,V: 359; 1964: 268); (ii) that it enables subjects to alight upon new axioms which are possibly true (1953,III: 353,fn.43; 1953,V: 361; 1961: 383, 385; 1964: 268); (iii) that it is associated with modal properties, such as provability and necessity (1933: 301; 1964: 261); and (iv) that the non-sensory intuition of abstracta such as concepts entrains greater conceptual 'clarification' (1953,III: 353,fn.43; 1961: 383). Such intuitions are purported to be both *of* abstracta and formulas, as well as to the effect *that* the formulas are true.[3]

In this paper, I aim to outline the logical foundations for rational intuition, by examining the nature of property (iii). The primary objection to Gödel's approach to mathematical knowledge is that the very idea of rational intuition is insufficiently constrained.[4] Subsequent research has thus endeavored to expand upon the notion, and to elaborate on intuition's roles. Chudnoff (2013) suggests, e.g., that intuitions are non-sensory experiences which represent non-sensory entities, and that the justificatory role of intuition is that it enables subjects to be aware of the truth-makers for propositions (p. 3; ch. 7). He argues, further, that intuitions both provide evidence for beliefs as well as serve to guide actions (145).[5] Bengson (2015: 718-723) suggests that rational intuition can be identified with the 'presentational', i.e., phenomenal, properties of representational

[3] The distinction between 'intuition-of' and 'intuition-that' is explicitly delineated in Parsons (1980: 145-146), and will be further discussed in Section **2**.

[4] See, e.g., Hale and Wright (2002). Wright (2004) provides a vivid articulation of the issue: 'A major — but not the only – problem is that, venerable as the tradition of postulating intuitive knowledge of first principles may be, no-one working within it has succeeded at producing even a moderately plausible account of how the claimed faculty of rational intuition is supposed to work — how exactly it might be constituted so as to be reliably responsive to basic logical validity as, under normal circumstances, vision, say, is reliably responsive to the configuration of middle-sized objects in the nearby environment of a normal human perceiver' (op. cit.: 158).

[5] A similar proposal concerning the justificatory import of cognitive phenomenology – i.e., the properties of consciousness unique to non-sensory mental states such as belief – can be found in Smithies (2013a,b). Smithies prescinds, however, from generalizing his approach to the epistemology of mathematics.

mental states – namely, cognitions – where the phenomenal properties at issue are similarly non-sensory; are not the product of a subject's mental acts, and so are 'non-voluntary'; are qualitatively gradational; and they both 'dispose or incline assent to their contents' and further 'rationalize' assent thereof.[6]

Rather than target objections to the foregoing essays, the present discussion aims to rebut the primary objection to mathematical intuition alluded to above, by providing a logic for its defining properties. The significance of the proposal is thus that it will make the notion of intuition formally tractable, and might thus serve to redress the contention that the notion is mysterious and ad hoc.

In his (1933) and (1964), Gödel suggests that intuition has a constitutively modal profile. Constructive intuitionistic logic is shown to be translatable into the modal logic, S4, with the rule of necessitation, while the modal operator is interpreted as concerning provability.[7] Mathematical intuition of set-theoretic axioms is, further, purported both to entrain 'intrinsic' justification, and to illuminate the 'intrinsic necessity' thereof.[8] Following Gödel's line of thought, I aim, in this paper, to provide a modal logic for the notion of 'intuition-that'.[9]

[6]Compare Kriegel (2015: 68), who stipulates that 'making a judgment that p involves a feeling of involuntariness' and 'making a judgment always involves the feeling of mobilizing a concept'.

[7]For further discussion both of provability logic and of intuitionistic systems of modal logic, see Löb (1955); Smiley (1963); Kripke (1965); and Boolos (1993). Löb's provability formula was formulated in response to Henkin's (1952) problem concerning whether a sentence which ascribes the property of being provable to itself is provable. (Cf. Halbach and Visser, 2014, for further discussion.) For an anticipation of the provability formula, see Wittgenstein (1933-1937/2005: 378), where Wittgenstein writes: 'If we prove that a problem can be solved, the concept 'solution' must somehow occur in the proof. (There must be something in the mechanism of the proof that corresponds to this concept.) But the concept mustn't be represented by an external description; it must really be demonstrated. / The proof of the provability of a proposition is the proof of the proposition itself' (op. cit.). Wittgenstein distinguishes the foregoing type of proof with 'proofs of relevance' which are akin to the mathematical, rather than empirical, propositions, discussed in Wittgenstein (2001: IV, 4-13, 30-31).

[8]Gödel (1964) does not define intrinsic justifications, although he does contrast the latter with the notion of extrinsic justifications, for which he provides a few defining remarks. Extrinsic justifications are associated, for example, with both the evidential probability of propositions, and the 'fruitful' consequences of a mathematical theory subsequent to adopting new axioms. See Gödel (op. cit.: 269).

[9]Cf. Parsons (1979-1980; 1983: p. 25, chs.10-11; 2008: 176).

If rational intuition is identical with cognitive phenomenal properties of representational states such as beliefs and judgments, then – via correspondence results between modal logic and first-order logic [cf. van Benthem (1983; 1984/2003)] – a precise translation can be provided between the notion of 'intuition-of', i.e., the cognitive phenomenal properties of thoughts whose contents can concern the axioms of mathematical languages, and the modal operators regimenting the notion of 'intuition-that'.[10] I argue, then, that intuition-that can further be shown to entrain property (iv), i.e. conceptual elucidation, by way of figuring as an interpretational modality which induces the reinterpretation of domains of quantification (cf. Fine, 2005; 2006). Fine (op. cit.) has suggested that the interpretational modality is imperatival, and that the deontic aspects of the modality might best be captured by a dynamic logic (p.c.). Following Fine's suggestion, I argue that intuition-that can thus be understood to be a species of fixed point dynamic provability logic, which is equivalent to the bisimulation-invariant fragment of monadic second-order logic (cf. Janin and Walukiewicz, 1996; Venema, 2014, ms). Modalized rational intuition is therefore expressively equivalent to – and can crucially serve as a guide to the interpretation of – the entities, such as mathematical concepts, that are formalizable in monadic first- and second-order formal languages.

In Section **2**, I elucidate the properties of rational intuition, by examining arguments and evidence adducing in favor of the existence of cognitive phenomenal consciousness. In Section **3**, I countenance and motivate a multi-modal logic, which augments the provability logic, GL, with fixed point dynamic logic, i.e. the modal μ-calculus. I argue that the dynamic properties of modalized rational intuition provide a precise means of accounting for the manner by which

[10]This provides a precise answer to the target inquiry advanced by Parsons (1993: 233).

intuition can yield the reinterpretation of quantifier domains and mathematical vocabulary; and thus explain the role of rational intuition in entraining conceptual elucidation. In Section **4**, I examine remaining objections to the viability of rational intuition and provide concluding remarks.

2 Rational Intuition as Cognitive Phenomenology

A property of a mental state is phenomenal only if it is the property of being aware of the state. If the mental state at issue is sensory, then sensory phenomenal properties will be properties of being aware of one's perceptions, where the perceptions at issue will be unique to the subject's sensory modalities of vision, audition, etc.[11] Let cognitive phenomenal consciousness refer to the properties of being aware of the non-sensory representational mental states toward which subjects can bear attitudes. Such states can be identified with, e.g., formulas, ϕ, in a Language of Thought,[12] the syntax for which mirrors that of natural language sentences, and which can fall within the scope of various operators such as fully or partially believing that (x, x a propositional variable/ϕ), knowing that (x/ϕ), judging that (x/ϕ), asserting that (x/ϕ), questioning whether (x/ϕ) has a particular semantic value, et al.

Pitt (2004: 8) provides the following argument for the existence of cognitive phenomenal properties, which – for the purposes of this essay – I will assume

[11]For issues concerning the taxonomy of the sense modalities, see Macpherson (2011). Bottom-up/exogenous, spatial-based, property-based, and diffuse/focal attention arguably comprise a necessary condition on the instantiation of phenomenal properties. The condition is witnessed by the phenomenon referred to as the attentional blink. The attentional blink holds if and only if shifting attentional allocation from one stimulus to another induces a lack of awareness of the first stimulus to which attention was previously distributed. See Author (ms) and the essays in Mole et al. (eds.) (2010), for further discussion.

[12]Cf. Fodor (1975).

to be sound:

(K1) 'It is possible immediately to identify one's occurrent conscious thoughts (equivalently (see below): one can know by acquaintance which thought a particular occurrent conscious thought is); but

(K2) It would not be possible immediately to identify one's conscious thoughts unless each type of conscious thought had a proprietary, distinctive, individuative phenomenology; so

(P) Each type of conscious thought – each state of consciously thinking that p, for all thinkable contents p – has a proprietary, distinctive, individuative phenomenology'.

In his examination of the conditions on measuring partial beliefs, i.e., subjective probability, Ramsey (1926) records scepticism about whether subjects are aware in a non-sensory way of all of their (partial) beliefs.[13] For the purposes of this note, it is sufficient that at least some cognitive states are states of which subjects can be aware – where, again, the properties of awareness at issue are non-sensory and purported to be unique to distinct cognitive attitudes, such that being aware of one's belief that ϕ will qualitatively differ from one's awareness of one's interrogative state concerning whether ϕ is true.

The evidence for the claim that at least some cognitive states are associated with a unique set of non-sensory properties of awareness has proceeded via in-

[13]See Ramsey (op. cit.: 169): 'Suppose that the degree of a belief is something perceptible by its owner; for instance that beliefs differ in the intensity of a feeling by which they are accompanied, which might be called a belief-feeling or feeling of conviction, and that by the degree of belief we mean the intensity of this feeling. This view would be very inconvenient, for it is not easy to ascribe numbers to the intensities of feelings; but apart from this it seems to me observably false, for the beliefs which we hold most strongly are often accompanied by practically no feeling at all; no one feels strongly about things he takes for granted'. Contrast Koopman (1940: 271), who argues that intuition can serve as a guide to veridical judgments concerning the comparison of probability measures [cf. de Finetti (1937/1964: ch.1) on the primacy of comparative judgments of probability], as well as to the axioms – or laws of consistency – of a probability theory.

trospective reports.[14] In the latter, subjects verbally report upon the valence of their awareness of their states, where their reports are assumed to be reliable.[15] One phenomenon of awareness of one's thoughts might, e.g., be that of inner speech.[16] It is an open issue whether inner speech has a sensory basis (cf. Prinz, 2012). If not, however, and assuming that introspective report is a reliable method for discerning whether subjects are aware of their states – irrespective of their being able to ascertain a precise value thereof – then there might be properties of awareness that are unique to one's thoughts and cognitive propositional attitudes.[17]

3 Modalized Rational Intuition and Conceptual Elucidation

In this section, I will outline the logic for Gödelian intuition. The motivation for providing a logic for rational intuition will perhaps be familiar from treatments of the property of knowledge in formal epistemology. The analogy between rational intuition and the property of knowledge is striking: Just as knowledge has been argued to be a mental state (Williamson, 2001; Nagel, 2013b); to

[14] The results of the method of introspection are availed of by Pitt (op. cit.), and discussed in the essays in Bayne and Montague (eds.) (2011). For an excellent survey of the experimental paradigms endeavoring to corroborate that intuition can be a source of evidence, see Nagel (2007; 2013a).

[15] See, however, Schwitzgebel (2011) for an examination of a series of case studies evincing that introspective report is unreliable as a method for measuring consciousness.

[16] Cf. Carruthers (1996). Assuming the reliability of introspection, Machery (2005) argues that awareness of one's thoughts is nevertheless insufficient for ascertaining the syntactic structure thereof.

[17] Nagel (2013) examines an approach to intuitions which construes the latter as a type of cognition, rather than as a phenomenal property of judgments. She distinguishes, e.g., between intuition and reflection, on the basis of experimental results which corroborate that there are distinct types of cognitive processing (op. cit.: 226-228). Intuitive and reflective cognitive processing are argued to interact differently with the phenomenal information comprising subjects' working memory stores. Nagel notes that – by contrast to intuitive cognition – reflective cognition 'requires the sequential use of a progression of conscious contents to generate an attitude, as in deliberation' (231).

be propositional (Stanley and Williamson, 2001); to be factive; and to possess modal properties (Hintikka, 1962; Nozick, 1981; Fagin et al., 1995; Meyer and van der Hoek, 1995), so rational intuition can be argued to be a property of mental states; to be propositional, as recorded by the notion of intuition-that; and to possess modal properties amenable to rigorous treatment in systems of modal logic.

I should like to suggest that the modal logic of Gödelian intuition is the bimodal logic combining GL – which is comprised of axioms K, 4, GL, and the rule of necessitation – with the modal μ-calculus.

Let M be a model over the Kripke frame, $\langle$W, R$\rangle$; so, M = $\langle$W, R, V$\rangle$. W is a non-empty set of possible worlds. R is a binary relation on W. V is a function assigning proposition letters, ϕ, to subsets of W.

$\langle$M,w$\rangle \Vdash \phi$ if and only if w$\in$V(ϕ).

$\langle$M,w$\rangle \Vdash \neg\phi$ iff it is not the case that $\langle$M,w$\rangle \Vdash \phi$

$\langle$M,w$\rangle \Vdash \phi \wedge \psi$ iff $\langle$M,w$\rangle \Vdash \phi$ and $\langle$M,w$\rangle \Vdash \psi$

$\langle$M,w$\rangle \Vdash \phi \vee \psi$ iff $\langle$M,w$\rangle \Vdash \phi$ or $\langle$M,w$\rangle \Vdash \psi$

$\langle$M,w$\rangle \Vdash \phi \rightarrow \psi$ iff, if $\langle$M,w$\rangle \Vdash \phi$, then $\langle$M,w$\rangle \Vdash \psi$

$\langle$M,w$\rangle \Vdash \phi \iff \psi$ iff [$\langle$M,w$\rangle \Vdash \phi$ iff $\langle$M,w$\rangle \Vdash \psi$]

$\langle$M,w$\rangle \Vdash \diamond\phi$ iff $\exists$w'[R(w,w') and $\langle$M,w'$\rangle \Vdash \phi$]

$\langle$M,w$\rangle \Vdash \Box\phi$ iff $\forall$w'[If R(w,w'), then $\langle$M,w'$\rangle \Vdash \phi$].

K states that $\Box(\phi\rightarrow\psi) \rightarrow (\Box\phi\rightarrow\Box\psi)$; i.e., if one has an intuition that ϕ entails ψ, then if one has the intuition that ϕ then one has the intuition that ψ. GL states that $\Box[(\Box\phi \rightarrow \phi) \rightarrow \Box\phi]$; i.e., if one has the intuition that the intuition that ϕ entails that ϕ is true, then one has the intuition that ϕ. 4 states that $\Box\phi \rightarrow \Box\Box\phi$; i.e., if one has the intuition that ϕ, then one intuits that one has the intuition that ϕ. Necessitation states that $\vdash\phi \rightarrow \vdash\Box\phi$. Because intuition-that is non-factive, we eschew in our modal system of axiom T, which states that $\Box\phi \to \phi$; i.e., one has the intuition that ϕ only if ϕ is the case [cf. BonJour (1998: 4.4); Parsons (2008: 141)].

In order to account for the role of rational intuition in entraining conceptual elucidation (cf., Gödel, 1961: 383), I propose to follow Fine (2006) and Uzquiano (2015) in suggesting that there are interpretational modalities associated with the possibility of reinterpreting both domains of quantification (Fine, op. cit.) and the non-logical vocabulary of mathematical languages, such as the membership relation in ZF set theory (Uzquiano, op. cit.).[18]

Fine (2005) has taken the interpretational modality to be imperatival, and has suggested that a dynamic logic might be an optimal means of formalizing the imperative to reinterpret quantifier domains. He (op. cit.) suggests, further, that the interpretational modality might be characterized as a program, whose operations can take the form of 'simple' and 'complex' postulates which enjoin subjects to reinterpret the domains. Uzquiano's (op. cit.) generalization of the interpretational modality, in order to target the reinterpretation of the extensions of terms such as the membership relation, can similarly be treated.

In propositional dynamic logic (PDL), there are an infinite number of diamonds, with the form $\langle\pi\rangle$.[19] π denotes a non-deterministic program, which in the present setting will correspond to Fine's postulates adumbrated in the foregoing. $\langle\pi\rangle\phi$ abbreviates 'some execution of π from the present state entrains a state bearing information ϕ'. The dual operator is $[\pi]\phi$, which abbreviates 'all

[18]A variant strategy is pursued by Eagle (2008). Eagle suggests that the relation between rational intuition and conceptual elucidation might be witnessed via associating the fundamental properties of the entities at issue with their Ramsey sentences; i.e., existentially generalized formulas, where the theoretical terms therein are replaced by second-order variables bound by the quantifiers. However, the proposal would have to be expanded upon, if it were to accommodate Gödel's claim that mathematical intuitions possess a modal profile.

[19]Cf. Blackburn et al., op. cit.: 12-14. A semantics and proof-theory for PDL are outlined in Hoare (1969); Pratt (1976); Goldblatt (1987: ch. 10; 1993: ch. 7) and van Benthem (2010: 158).

executions of π from the present state entrain a state bearing information ϕ'. π* is a program that executes a distinct program, π, a number of times ≥ 0. This is known as the iteration principle. PDL is similarly closed under finite and infinite unions. This is referred to as the 'choice' principle: If π_1 and π_2 are programs, then so is $\pi_1 \cup \pi_2$. The forth condition is codified by the 'composition' principle: If π_1 and π_2 are programs, then $\pi_1;\pi_2$ is a program (intuitively: the composed program first executes π_1 then π_2). The back condition is codified by Segerberg's induction axiom (Blackburn et al., op. cit: p. 13): All executions of π* (at t) entrain the following conditional state: If it is the case that (i) if ϕ, then all the executions of π (at t) yield ϕ; then (ii) if ϕ, then all executions of π* (at t) yield ϕ. Formally, $[\pi^*](\phi \rightarrow [\pi]\phi) \rightarrow (\phi \rightarrow [\pi^*]\phi)$.

We augment, then, the provability logic for Gödelian intuition with the axiom, $\Box\phi \rightarrow [\pi]\phi$, in order to yield a bimodal, dynamic provability logic thereof.

Crucially, the iteration principle permits π* to be interpreted as a fixed point for the equation: x $\iff \phi \vee \diamond$x. The smallest solution to the equation will be the least fixed point, μx.$\phi \vee \diamond$x, while the largest solution to the equation, when $\pi^* \vee \infty_\diamond$, will be the greatest fixed point, υx.$\phi \vee \diamond$x. Janin and Walukiewicz (op. cit.) have proven that the modal μ-calculus is equivalent to the bisimulation-invariant fragment of second-order logic.

Fine's simple dynamic-postulational modality takes, then, the form:

'(i) Introduction. !x.C(x)', which states the imperative to: '[I]ntroduce an object x [to the domain] conforming to the condition C(x)'.

Fine's complex dynamic-postulational modalities are the following:

(ii) 'Composition. Where β and γ are postulates, then so is $\beta;\gamma$. We may read $\beta;\gamma$ as: do β and then do γ; and $\beta;\gamma$ is to be executed by first executing β and then executing γ.

(iii) Conditional. Where β is a postulate and A an indicative sentence, then A $\to \beta$ is a postulate. We may read A $\to \beta$ as: if A then do β. How A $\to \beta$ is executed depends upon whether or not A is true: if A is true, A $\to \beta$ is executed by executing β; if A is false, then A $\to \beta$ is executed by doing nothing.
(iv) Universal. Where β(x) is a postulate, then so is $\forall$xβ(x). We may read $\forall$xβ(x) as: do β(x) for each x; and $\forall$xβ(x) is executed by simultaneously executing each of β(x1), β(x2), β(x3), ..., where x1, x2, x3, ... are the values of x (within the current domain). Similarly for the postulate $\forall$Fβ(F), where F is a second-order variable.
(v) Iterative Postulates. Where β is a postulate, then so is β*. We may read β* as: iterate β; and β* is executed by executing β, then executing β again, and so on ad infinitum' (op. cit.: 91-92).

Whereas Fine avails of the foregoing interpretational modalities in order both to account for the notion of indefinite extensiblity and to demonstrate how unrestricted quantification can be innocuous without foundering upon Russell's paradox (op. cit.: 26-30), the primary interest in adopting modal μ provability logic as the logic of rational intuition is its capacity to account for reinterpretations of mathematical vocabulary and quantifier domains; and thus to illuminate how the precise mechanisms codifying modalized rational intuition might be able to entrain advances in conceptual elucidation.

Finally, the computational profile of modalized rational intuition can be outlined as follows. In category theory, a category C is comprised of a class Ob(C) of objects a family of arrows for each pair of objects C(A,B) (Venema, 2007: 421). An **E**-coalgebra is a pair $\mathbb{A}$ = (A, μ), with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\to$ **E**(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390). A coalgebraic model of deterministic automata can be thus defined (391). An automaton is a tuple, $\mathbb{A} = \langle$A, a_I, C, δ, F$\rangle$, such that A is the state space of the automaton $\mathbb{A}$; $a_I \in$A is the automaton's initial state; C is the coding for the automaton's alphabet, mapping numerals to properties of the natural numbers; δ: A X C $\rightarrow$ A is a transition function, and F$\subseteq$ A is the collection of admissible states, where F maps A to $\{1,0\}$, such that F: A $\rightarrow$ 1 if a$\in$F and A $\rightarrow$ 0 if a$\notin$F (op. cit.). The modal profile of coalgebraic automata can be witnessed both by construing the transition function as a counterfactual conditional (cf. Stalnaker, 1968; Williamson, 2007), and in virtue of the convergence of coalgebraic categories of automata with coalgebraic models of modal logic (407). Let

$\diamond\phi \equiv \nabla\{\phi$, T$\}$,

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.).

Let an **E**-coalgebraic modal model, $\mathbb{A} = \langle$S,λ,R[.]$\rangle$, such that $\mathbb{S}$,s $\Vdash \nabla\Phi$ if and only if, for all (some) successors σ of s$\in$S, [Φ,σ(s)$\in\mathbf{E}(\Vdash_{\mathbb{A}})$] (op. cit.). $\mathbb{A}$ is then the category of modal coalgebraic automata.

The philosophical significance of the foregoing is that the modal logic of rational intuition can be interpretable in the category of modal coalgebraic automata. The foregoing accounts for the distinctively computational nature of the modal profile of rational intuition.

4 Concluding Remarks

In this note, I have endeavored to outline the modal logic of Gödel's conception of intuition, in order both (i) to provide a formally tractable foundation thereof, and thus (ii) to answer the primary objection to the notion as a viable approach to the epistemology of mathematics. I have been less concerned with providing a defense of the general approach from the array of objections that have been proffered in the literature. Rather, I have sought to demonstrate how the mechanisms of rational intuition can be formally codified and thereby placed on a secure basis.

Among, e.g., the most notable remaining objections, Koellner (2009) has argued that the best candidates for satisfying Gödel's conditions on being intrinsically justified are reflection principles, which state that the height of the hierarchy of sets, V, cannot be constructed 'from below', because, for all true formulas in V, the formulas will be true in a proper initial segment of the hierarchy. Koellner's limitative results are, then, to the effect that reflection principles cannot exceed second-order universal quantification without entraining inconsistency (op. cit.).[20] Another crucial objection is that the properties of rational intuition, as a species of cognitive phenomenology, lack clear and principled criteria of individuation. Burgess (2014) notes, e.g., that the role of rational intuition in alighting upon mathematical truths might be distinct from the functions belonging to what he terms a 'heuristic' type of intuition. The constitutive role of the latter might be to guide a mathematician's non-algorithmic insight as she pursues an informal proof. A similar objection is advanced in Cappelen (2012: 3.2-3.3), who argues that – by contrast to the properties picked out by theoretical terms such as 'utility function' – terms purporting to designate cognitive phenomenal properties both lack paradigmatic criteria of individuation and must thereby be a topic of disagreement, in virtue of the breadth of variation in the roles that the notion has been intended to satisfy.

[20]The issue of accounting for third-order variables in order to state, e.g., Cantor's Theorem and the Generalized Continuum Hypothesis might yet be redressed, if one were to follow Shapiro (1991: 103-105), and avail of second-order paraphrases for formulas which concern sets and sequences of sets.

The foregoing issues notwithstanding, I have endeavored to demonstrate that – as with the property of knowledge – an approach to the notion of intuition-that which construes the notion as a modal operator, and the provision thereof with a philosophically defensible logic, might be sufficient to counter the objection that the very idea of rational intuition is mysterious and constitutively unconstrained.

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