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Reformulation of Dirac's theory of electron to avoid negative energy or negative time solution

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ABSTRACT

Dirac's relativistic theory of electron generally results in two possible solutions, one with positive energy and other with negative energy. Although positive energy solutions accurately represented particles such as electrons, interpretation of negative energy solution became very much controversial in the last century. By assuming the vacuum to be completely filled with a sea of negative energy electrons, Dirac tried to avoid natural transition of electron from positive to negative energy state using Pauli's exclusion principle. However, many scientists like Bohr objected to the idea of sea of electrons as it indicates infinite density of charge and electric field and consequently infinite energy. In addition, till date, there is no experimental evidence of a particle whose total energy (kinetic plus rest) is negative. In an alternative approach, Feynman, in quantum field theory, proposed that particles with negative energy are actually positive energy particles running backwards in time. This was mathematically consistent since quantum mechanical energy operator contains time in denominator and the negative sign of energy can be absorbed in it. However, concept of negative time is logically inconsistent since in this case, effect happens before the cause. To avoid above contradictions, in this paper, we try to reformulate the Dirac's theory of electron so that neither energy needs to be negative nor the time is required to be negative. Still, in this new formulation, two different possible solutions exist for particles and antiparticles (electrons and positrons).

Keywords

Dirac's theory of electron, Negative energy, Negative time, Antiparticles

INTRODUCTION

In an effort to explain the fine (doublet) structure in hydrogen atomic spectra, Pauli introduced the idea of spin ($\pm\hbar/2$) which was just a hypothesis to explain the experimental results. Then came the Dirac's relativistic theory of electron [1] in which spin of electron naturally came into picture with a solid mathematical justification and gyro-magnetic factor (g=2) could be successfully predicted. Dirac's theory however resulted in two possible solutions, one with positive total energy and other with negative total energy [2] (*note that total energy here refers to kinetic plus rest energy*). Although positive energy solutions accurately represented particles such as electrons, interpretation of negative energy solution became very much controversial in the last century [3-4]. By assuming the vacuum to be completely filled with a sea of negative energy electrons, Dirac tried to avoid natural transition of electron from positive to negative energy state using Pauli's exclusion principle. During pair production, an electron in sea of negative energy electrons is excited to positive energy state creating a hole in the sea. Dirac interpreted that since a hole or absence of a negative energy and negatively charged electron in space is equivalent to presence of a positive energy and positively charged particle, this hole acts as positron which was later experimentally detected. However, many scientists like Bohr [3] objected to the idea of sea of electrons as it indicates infinite density of charge and electric field and consequently infinite energy. In addition, till date, there is no experimental evidence of a particle whose total energy (kinetic plus rest) is negative. In an alternative approach, Feynman while formulating quantum field theory, proposed that particles with negative energy are actually positive energy particles running backwards in time [5-6]. This was mathematically consistent since quantum mechanical total energy operator ($\hat{E} = i\hbar \frac{\partial}{\partial t}$) contains change in time in denominator and the negative sign of energy can be absorbed in it. However, concept of negative time is logically inconsistent since in this case effect has to happen before the cause violating the principle of causality. In an another recent attempt, Trubenbacher [7] assigned a new quantum mechanical operator *sgn* for the property "sign of particle charge" and assumed from beginning of derivation that energy is given by ratio of Dirac Hamiltonian operator and *sgn* operator. However, in our opinion, this assumption is arbitrary and also if "sign of particle charge" is taken to be an operator having eigen values σ = ±1, quantum mechanics must allow the electron to be in a mixed state so that in some measurement it should act as electron and sometimes it should act as positron. But this uncertainty of charge has never been experimentally observed. Although some experts working in the field for long might be habituated with these inconsistencies, a student or a new learner certainly becomes uncomfortable with the idea that energy is negative or time is negative.

Therefore to avoid above contradictions, in this paper, we try to reformulate the Dirac's theory of electron so that neither energy needs to be negative nor the time is required to be negative. Still, in this new formulation, two different possible solutions exist for particles and antiparticles (electrons and positrons).

A NEW APPROACH IN DIRAC'S THEORY OF ELECTRON

In the Einstein's relativistic expression for total energy relating the numerical values of momentum and energy, we can just keep the positive sign before square root ($E = +\sqrt{c^2 p^2 + m^2 c^4}$) and ignore the negative sign due to physical impossibility of negative total energy (kinetic + rest) since neither kinetic energy nor rest energy can become negative. However, in quantum mechanics, all possible solutions of the Hamiltonian equation have to be considered [6, 8] as all of them together constitute a complete set generating the Hilbert space. So, numbers corresponding to each solution must be physically realizable. In this sense, when both positive and negative values for energy of electron came out in solution of Dirac equation, he had to take both the values as physically possible. This ultimately motivated him to bring the hypothesis of sea of negative energy electrons. But in this section, we will see that two possibilities don't appear for energy, rather it appears for some other property of the particle identified in this paper.

In quantum mechanics and quantum field theory, we generally start from a wave representation of a particle. Interestingly, it has been pointed out by Hobson [9] that there are no particles in nature, there are only fields (or waves) although these fields interact with other objects at a point in space. In general, such a propagating wave can be represented as,

$$\psi = Ae^{\frac{i}{\hbar}(px-Et)} \quad (1)$$

In Eq. (1) we can calculate the wave velocity by,

$px - Et = Constant$

(for a specified point in wave pattern)

So, Wave velocity = $w = \frac{dx}{dt} = \frac{E}{p}$

Thus, if Eq. (1) represents an electron of momentum *p* and positive energy *E*, its wave velocity '*w*' is in same direction as momentum *p*. Now, if we want to find out the wave equation of a positron of same momentum *p* and energy *E*, then only property that can be different is the wave velocity which can be opposite to its own momentum i.e. $w = \frac{dx}{dt} = -\frac{E}{p}$. In that case, propagating wave representing a positron must be,

$$\psi = Ae^{\frac{i}{\hbar}(px+Et)} \quad (2)$$

So, the generalized expression for propagating wave representing any particle or antiparticle (electron or positron) can be written as,

$$\psi = Ae^{\frac{i}{\hbar}(px\pm Et)}$$

Or
$$\psi = Ae^{\frac{i}{\hbar}(px-\mu Et)} \quad (3)$$

Where $\mu = \pm 1$ is a property of particle representing the orientation of the wave velocity with respect to its momentum. If it is +1, then wave velocity is in direction of momentum and if it is -1, then wave velocity is in opposite direction of momentum. We will call μ in short as "wave velocity orientation" and it is a *constant number* for a specific particle (Not an eigen value of some quantum mechanical operator). Now using Eq. (3), we can find the energy operator and momentum operator.

$$\frac{i\hbar}{\mu}\frac{\partial\psi}{\partial t}=\frac{i\hbar}{\mu}\frac{\partial\left[Ae^{\frac{i}{\hbar}(px-\mu Et)}\right]}{\partial t}$$

$$=E\left[Ae^{\frac{i}{\hbar}(px-\mu Et)}\right]=E\psi$$

Thus energy operator is,

$$\hat{E}=\frac{i\hbar}{\mu}\frac{\partial}{\partial t} \tag{4}$$

Similarly, we can show from Eq. (3) that momentum operator remains same as conventional expression i.e.

$$\hat{p}=-i\hbar\nabla \tag{5}$$

Relativistic energy equation is given by,

$$E^2=c^2p^2+m^2c^4 \tag{6}$$

Where, *m* is rest mass.

If we put total energy operator $(\hat{E})$ and momentum operator $(\hat{p})$ in above equation, we get a differential equation which is second order in space and time. However, for a determined evolution of wave function with time, the differential equation for wave function needs to be first order in time. So, we have to write the right hand side as a perfect square so that exponent from both sides can be cancelled. This is possible by writing,

$$c^2p^2+m^2c^4=(c\vec{\alpha}\,.\,\vec{p}+\beta mc^2)^2 \tag{7}$$

Where $\vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}$ and $\beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$,

$\vec{\sigma}$ is made up of three Pauli matrix, $\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ and $\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

I is identity matrix, $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

From Eq. (6) and (7) and since $\mu^2=1$, we can write,

$$E^2=\mu^2(c\vec{\alpha}\,.\,\vec{p}+\beta mc^2)^2$$

Or $$E=\mu(c\vec{\alpha}\,.\,\vec{p}+\beta mc^2)$$

Writing quantum mechanical operators from Eq. (4) and (5) in place of energy and momentum in above equation,

Or $$\frac{i\hbar}{\mu}\frac{\partial\psi}{\partial t}=\mu(-i\hbar c\vec{\alpha}\,.\nabla+\beta mc^2)\psi$$

As $\mu^2=1$, above equation reduces to *original* Dirac equation i.e.

$$i\hbar\frac{\partial\psi}{\partial t}=(-i\hbar c\vec{\alpha}\,.\nabla+\beta mc^2)\psi \tag{8}$$

Now let us solve the Dirac equation. Since the operator on right side of Eq. (8) is a 4×4 matrix, wave function ψ is a four-component Dirac spinor. From Eq. (3), if we separate space and time dependent part, solution should be of the form,

$$\psi(r,t)=\psi(r)e^{-\frac{i}{\hbar}\mu Et}$$

Putting the above general solution in Eq. (8), we get,

$$\mu E\psi(r)=(-i\hbar c\vec{\alpha}\,.\nabla+\beta mc^2)\psi(r) \tag{9}$$

For a free particle of *specified* momentum *p*, wave function ψ(r) will be of the form,

$$\psi(r)=\begin{bmatrix}\chi\\ \phi\end{bmatrix}e^{\frac{i}{\hbar}pr}$$

Where χ and ϕ are each two-component spinors.

Putting this and values of α and β in Eq. (9),

$$\begin{bmatrix}\mu E-mc^2 & -c\vec{\sigma}.\vec{p}\\ -c\vec{\sigma}.\vec{p} & \mu E+mc^2\end{bmatrix}\begin{bmatrix}\chi\\ \phi\end{bmatrix}=0 \tag{10}$$

Solving the above matrix equation, we get,

$$\chi=\frac{c\vec{\sigma}.\vec{p}}{\mu E-mc^2}\phi \qquad \text{and} \qquad \phi=\frac{c\vec{\sigma}.\vec{p}}{\mu E+mc^2}\chi$$

Using above in Eq. (10),

$$(\mu E-mc^2)(\mu E+mc^2)\chi-c^2(\vec{\sigma}.\vec{p})^2\chi=0$$

Since we know, $(\vec{\sigma}.\vec{p})^2=p^2$

$$[(\mu E)^2-(mc^2)^2-c^2p^2]\chi=0$$

As the wave function has to be non-zero, quantity in bracket must be zero. So, we get,

$$\mu E=\pm\sqrt{(mc^2)^2+c^2p} \quad (11)$$

It is to be noted that in Eq. (11) we have taken both the signs i.e. positive and negative. Still, in both the cases, energy can be always positive as μ can be +1 or -1 to keep energy positive. Thus we have been able to generate two possible solutions without requirement of negative energy. In place of energy, now the property called μ i.e. "wave velocity orientation" with respect to the momentum has become positive or negative to account for electrons (particles) and positrons (antiparticles).

Now the question comes why an antiparticle (e.g. positron) has opposite charge compared to that of particle (e.g. electron). The answer is as follows. For particle, μ = +1 and for antiparticle, μ = -1. If we put these μ values in Eq. (3), time dependent part becomes $Ae^{-\frac{i}{\hbar}Et}$ for particle and $Ae^{+\frac{i}{\hbar}Et}$ for antiparticle. Because of opposite sign in time evolution part in states of particle and antiparticle, dynamics of antiparticle becomes opposite to that of particle. But, since charge of an object is conventionally inferred by its sequence of events (for example trajectory of electron or positron), we attribute the opposite dynamics of particle and antiparticle to their (assumed) opposite charges. However, it is truly due to the value of 'μ' i.e. wave velocity orientation which may be +1 or -1. In his original formulation, by ignoring 'μ', Dirac had unknowingly included its' value (-1) in energy of antiparticle and so, got the problem of negative energy. Feynman and quantum field theory include the value of 'μ' in time and so, problem of negative time arises. But, we have shown that we can avoid both the problems of negative energy or negative time if we recognize and retain the parameter 'μ' i.e. wave velocity orientation in generalized wave function.

CONCLUSION

In this paper, we have reformulated the Dirac's theory of electron so that neither total energy (i.e. kinetic plus rest) nor time becomes negative for the case of antiparticles (e.g. positrons). We have found a new parameter (μ) in wave function of the particle i.e. "wave velocity orientation" with respect the particle's momentum which becomes +1 or -1 generating two different solutions corresponding to particles and antiparticles. By avoiding the negative energy solution in our approach, we can escape from the problems of sea of electrons hypothesized by Dirac which leads to infinite charge density. Similarly, by avoiding the idea of negative time, we can preserve the principle of causality in modern physics.

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