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The interplay between mathematical practices and results

Commentary on "Observing mathematical practices as a key to mining our sources and conducting conceptual history: Division in ancient China as a case study", by Karine Chemla's

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This comment addresses issues regarding the interplay between practices and results in Karine Chemla's contribution: "Observing mathematical practices as a key to mining our sources and conducting conceptual history: Division in ancient China as a case study". Chemla explains how a historian of ancient mathematics can get further information by restoring practices and deriving results that are indirectly evidenced in the rare surviving sources that the historian has. It is noteworthy that Chemla discusses directly the relation between practices and results, where results might be understood as the body of knowledge that one can gather in a scientific outcome (e.g. a publication), while practices are the activities and conceptions that led to those results. This issue is still insufficiently addressed in the recent rising of studies on mathematical practices, as far as we can tell from many writings that we consulted and meetings that we attended (Giardino, Moktefi, Mols & Van Bendegem, 2012).

Chemla argues that "reconstructing mathematical practices involved in producing our sources yields key resources for inquiring into issues of conceptual history that our sources do not tackle in detail" (p. 3-4). In order to defend this thesis, Chemla appeals to a case study from a distant mathematical culture, namely ancient China. We are told that the practitioners of mathematics in early imperial China worked assiduously on procedures (*i.e.* algorithms in modern terminology), *e.g.* how to compute the volume of a half-parallelepiped. Such procedures make use of blocks that stand for operations (*e.g.* division, square root extraction, etc.). Executing these operations requires a procedure in itself. However, not all the documents that we have do include such procedures on how to execute an operation. Moreover, even when the documents contain such procedures, this does not make them "treatises about operations" (p. 6). Most conceptual work on operations can be gathered only indirectly. In her paper, Chemla shows how studying practices helps to reveal *hidden* knowledge about these operations.

To show this, Chemla relies mostly on a classic Chinese book: *The Nine Chapters on Mathematical Procedures*. The corresponding work was probably composed in the first century CE, but circulated and survived together with two commentaries composed subsequently. The book refers to a computing instrument used to perform procedures, probably by placing and moving counting rods on a simple surface. Among the various operations involved in *The Nine Chapters*, division plays a central role. Unlike other operations (addition, subtraction and multiplication), division is clearly understood as an operation since technical terms are introduced for its operands. Examining how operations are prescribed provides additional clues about division. For instance, root extractions are frequently prescribed as follows: “Divide this by extraction of the square/cube root”. This practice shows a link between the operation of root extraction and division, and attests that division is more fundamental. Chemla also identifies a new operation which corresponds – in modern terminology – to solving a quadratic equation. Here again, the way this operation is prescribed shows that it derives from root extraction. Altogether, these examples show how the understanding of naming practices is used effectively by historians to acquire knowledge that was only indirectly evidenced in the way *The Nine Chapters* presented the operations.

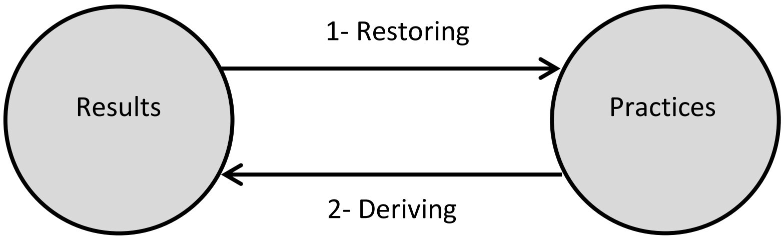
Similarly, Chemla worked with other clues to restore more “practices of naming, practices of writing down texts of procedures, and practices of computing” (p. 23). For instance, thanks to the study of computing and executing procedures, combined with clues she extracted from a later text – *The Mathematical Classic by Master Sun*, completed in 400 CE –, Chemla argues that the number system used in *The Nine Chapters* was a place-valued decimal system, a fact that is not directly recorded in the surviving text. Hence, understanding practices enables one to improved interpretation of existing clues, to detect additional clues and, consequently, to derive knowledge that was “evidently possessed” by the authors of *The Nine Chapters* (p. 35).

Restoring practices and deriving results

In her case study, Chemla is in the position of a historian who handles a precious text from an ancient mathematical culture. Since such textual sources are rare, it is crucial to derive from them the greatest possible amount of information. For this purpose, Chemla implements a method that can be carried out in two steps:

Step 1: The historian has sources which directly report mathematical results. She collects and interprets clues in the sources to restore the practices of the mathematicians who produced those results.

Step 2: The historian has identified mathematical practices. She uses them to collect and interpret further clues in order to derive new results that were not directly evidenced in the sources.



Chemla uses this method as described in the first section of this commentary. The reader is invited to examine her paper carefully to observe the numerous clues she collected, and to grasp how she uses them efficiently to derive knowledge that was not explicitly stated in her sources. For the sake of brevity, let us consider a hypothetical simple case to illustrate the whole process. Suppose that you are examining an ancient manuscript from some distant mathematical culture.

Step 1: You face some formulae where an operation noted ‘*’ is used, but you do not know what it operates yet. All you know is that when this operation is applied to 2 and 3, it gives 5, and when applied to 4 and 5, it gives 9. These clues highly suggest that operation ‘*’ stands for addition. As such, you have restored the notational practice of representing the operation of addition with symbol '*'.

Step 2: Now, you use this information to better understand the text and interpret other passages. For instance, if you observe that the manuscript substitutes '2*3' for '3*2' and '4*5' for '5*4', you might reasonably assume that the author of the manuscript knew that addition was commutative. Also, if you find that '2§3' (§ being an unknown operation) is replaced by '2*2*2', then it is likely that symbol '§' stands for multiplication and that the relation of this operation to addition was known to the author of the manuscript.

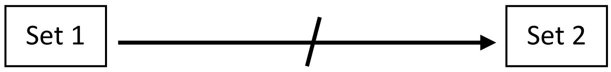
This hypothetical example shows how a historian works on restoring practices used by the mathematicians of the past (notations, substitutions, etc.) and how these practices can be used with benefit to provide more clues and improve the interpretation of the very same text from which those practices were extracted. It might be noted that this process involves some circularity as shown in the diagram above. Indeed, one first restores practices from the clues in the source. Then, one re-reads the very same source in order to derive more results. Actually, one must keep in mind that the process does not stop after one round. Indeed, when the historian has restored some practices and derived new results, she continues to look for other practices and to derive further results. The circularity, however, involves no fallacy, since the restored practices are not just used to interpret the clues that first suggested them. In a way, the processes of restoring practices and deriving results complement each other the same way cryptology and cryptanalysis do. The more one understands encryption practices, the more one is likely to decrypt. And the more keys one gets in decoding a message, the more likely it is that one will get further keys. Every crossword or Sudoku player is familiar with the interplay just described, and also knows how its recursiveness makes any misjudgment at some stage of the interplay affect subsequent stages. Consequently, one has to pay attention to the constraints that are inherent to each step of the historiographical method described above.

The possibility of restoring practices is not disputable in itself. Historians do that all the time. An important lesson of the practice turn in philosophy and sociology of science, however, is precisely the recognition that the final output of research alone does not suffice to understand the activity that led to that output; this is why the study of science-in-the-making is so interesting. To get around the difficulty of reconstructing activities, historians usually exploit several intermediary and related sources (notebooks, drafts, private correspondence, diaries, etc.). When one works on an ancient mathematical culture with only few surviving sources, however, the restoration of activities is both more difficult and more crucial to achieve. Such are the "working conditions of the historian of ancient mathematics" (p. 2). Once the historian has restored a few practices, she attempts to derive from them results that were not directly stated in the sources at hand. Logical analysis helps at this stage but is insufficient. Mathematical practitioners might not have looked for further results or might have reached erroneous ones. All we can say is that 'Since they operated according to *this* practice, they might have reached *that* result'. This situation strengthens Chemla's plea for the study of practices in order to get clues that enable the historian to go further, in her reconstruction of past knowledge, than what is permitted by logical analysis alone.

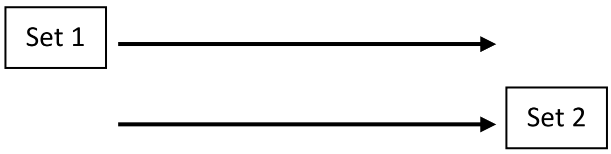
The constraints that we discussed above make us wonder what chances a historian of ancient mathematics has to *accurately* restore implicit practices and derive hidden results from the sources. In this respect, the historian might be regarded as a player who assembles a jigsaw puzzle, without having all the pieces and with no certainty about the final picture. In such a situation, accuracy refers merely to the sense the historian makes of her pieces. With the clues she has, she tries to make the best possible idea of what that final picture could be. There might be disputes about it and about the place of different pieces. But since there is no other way to come to know how the pieces *should* fit together, all the historian is doing is to get the most meaningful reconstruction. In case a new source is discovered and suggests that the present state of the puzzle is inaccurate, the historian reworks it in order to get again the most plausible picture. This analogy should not make the reader think that no historical work is trustworthy and reliable. In many cases, we have enough pieces to restore a satisfactory picture.

Local practices *versus* universal results

In the third section of her paper, Chemla considers two sets of manuscripts. The first set contains several early works prior to *The Nine Chapters*, notably a mathematical classic, *The Gnomon of the Zhou*, and several other works that have been recently discovered and that were mostly produced in the third and second centuries BCE. The second set is formed by the *Nine Chapters* and subsequent works such as *The Mathematical Classic by Master Sun* to which we alluded in the first section of this commentary. Her comparison of the two sets "reveal[s] a change that apparently occurred between the two respective time periods. This change seems to have involved several correlated features: the number system, the way of prescribing division by means of a verb or not, and the method of performing a division process on integers" (p. 50). Though not directly stated by Chemla, this argument shows that she considers the two sets of work to belong to the same mathematical tradition. As such, substantial differences between the two sets suggest a conceptual change in the meantime.



It might be noted that an alternative reading is possible: the two sets belong to distinct mathematical traditions. Given that we have no late manuscript of the first-set-tradition and no early manuscript of the second-set-tradition, we could not tell whether any change ever occurred.



Since Chemla did not opted for this second reading, we conclude that she considers the two sets to belong to the same (and single) mathematical culture. Chemla does not make explicit what this mathematical culture would be; she just refers to "Chinese mathematics". But this leaves us with questions about how a conceptual tradition can be identified within a cultural area. In this respect, Chemla's paper offers further inspiration on an issue that did not receive the attention it deserves, namely the role of cultural elements in mathematical practices. The case studied here concerns mathematics of a cultural area other than the western (Greek-based) tradition, which is usually explored in practice-turn scholarship (e.g. Schatzki, 2001). Paying attention to different cultures provides an opportunity to discuss the universality of scientific results in spite of the local and social roots of the practices that led to those results.

It is common within the practice-based studies to insist on the social context of scientific practices, including the mathematical ones (see Van Bendegem's contribution to this volume). The educational setting, in particular, is an obvious instance of a context which should not be ignored. For instance, the scientific practices developed in a given educational context at a given historical time are linked to the choices of didactic formats (see for instance Andrea Woody's contribution to this volume on the emergence of the chemistry periodic table), and also to the social conditions in which pupils learn (Norenzayan, Smith, Kim & Nisbett, 2002).

Hence, the prescriptions and practices of *The Nine Chapters* would not be adequately understood by a historian who would miss the fact that this work was used for educational purposes. Although in the present text, Chemla does not give extensive information on the cultural and educational contexts, she has provided plenty of elements on this same topic in previous writings (Chemla, 2011).

Introducing the cultural dimension might prove valuable for the study of scientific practices, if we consider that practices are a behavior socially shared by a group whereas culture stands for the representations which orient the collective cognition of the group (e.g. DiMaggio, 1997; Kozma, 2003). Some authors deny, however, the usefulness of culture as a supra level of explanation of values and discourse systems, because certain practices "are anchoring practices which play a key role in reproducing larger systems of discourse and practice" (Swiddler, 2001, p. 99). Accordingly, the study of practices alone would be sufficient. In her contribution to the present book, Chemla does not explore such kinds of issues, but elsewhere, she argued that "scientific activity feeds on the cultures where it occurs" (Chemla, 1998, p. 74). In her present paper, Chemla insists on the historicity of mathematical practices which, thus, should be considered within their historical, social and cultural contexts.

This local shaping of practices contrasts with the (assumed) universality of mathematical knowledge, as considered for instance by cognitive psychologists (Dehaene, Molko, Cohen & Wilson, 2004), even if individual differences are observed (Reuchlin, 1978). For instance, one might consider the opposition between a holistic world conception in East Asia and an analytic western tradition (Dasen, 2004; Nisbett, Peng, Choi & Norenzoyan, 2001), and ask whether this opposition influences the cognitive understanding of mathematics. More generally, the question is: how would singular culturally-located practices lead to universal culture-free results? In the present paper and many others, Chemla stresses that regarding this question, the challenge is "to satisfy two requirements: showing how a scientific practice is singular and still produces universal results, some of which do circulate all over the world and contribute to the shaping of modern knowledge" (Chemla, 1998, p. 75).

Although Chemla does not provide a direct and definitive solution to this challenge, she proposes subtle methodological remarks in her paper that can be taken to offer some preliminary answers. A first step in this direction would be not to consider the distinction of practices and results to be absolute. Indeed, this distinction depends to a large extent on one's perspectives and interests. Making use of procedures is in itself a practice; accordingly, one might consider each procedure to be a result of such practice. If one is interested in one specific problem that can be solved using different procedures, however, then, each procedure might be considered as a different practice. Chemla considers practices to be part of the body of knowledge possessed by scientists, and as such, practices can be seen as results that have been shaped and produced according to a given historical process. Even if the distinction between practices and results is valuable for the sake of analysis, one should keep in mind their "intimate relationship" and the importance of considering them jointly (p. 56).

A second step in getting to grips with the tension between the locality of practices and the universality of its results is to question what is meant by locality and universality. On the one hand, if locality refers to the shaping of practices within singular conditions, then one has to admit that results are local too because they are produced within the same local conditions. On the other hand, the claim that results are universal might be understood as the claim that results are culture-free and thus belong to an international mathematical culture (Chemla, 1998, p. 88). Nothing prevents us, however, from considering practices, too, to be universal in the same sense, that is, to also consider them as part of the international mathematical culture. For instance, even if we consider Chinese mathematics to be a Chinese way to do mathematics, there is no need to be Chinese in order to do Chinese mathematics. Local (cultural) factors determine only what part of that international mathematical culture will be locally used or favored (e.g. the choice of a specific number system). Considered in this perspective, local (cultural) factors do not harm to the universality of that international culture and its components, be they practices, results, or practice-result connections. Studying mathematical practices from different cultural areas certainly contributes to a better understanding of the fascinating and still open problem of the tension between the locality and universality of mathematical results.

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