BEING SURE AND BEING CONFIDENT THAT YOU WON'T LOSE CONFIDENCE

Alexander R. PRUSS

ABSTRACT: There is an important sense in which one can be sure without being certain, i.e., without assigning unit probability. I will offer an explication of this sense of sureness, connecting it with the level of credence that a rational agent would need to have to be confident that she won't ever lose her confidence. A simple formal result then gives us an explicit formula connecting the threshold α for credence needed for confidence with the threshold needed for being sure: one needs $1-(1-\alpha)^2$ to be sure. I then suggest that stepping between α and $1-(1-\alpha)^2$ gives a procedure that generates an interesting hierarchy of credential thresholds.

KEYWORDS: credence, belief, moral certainty, certainty, sureness, martingale, closure of inquiry

1. Introduction

There are some things I am sure of. I am sure I have two hands and that the world is billions of years old. Yet I assign a probability less than one to these propositions. There is some small chance that I am currently in the hospital after the amputation of one of my hands and am dreaming, and likewise there is some small chance that our best science is wrong about the age of the world. Being sure is not the same as *being certain*, in the technical sense of assigning probability 1.[1] (Throughout, I will use "certain" in this technical sense, though I suspect that the ordinary usage of "sure" and "certain" is quite close.)

Perhaps the concept I am getting at is moral certainty? For instance, Leibniz writes:

> [N]o firm demonstration can be made from the success of hypotheses. Yet I shall not deny that the number of phenomena which are happily explained by a given hypothesis may be so great that it must be taken as morally certain.[2]

---

[1] Or "super-1", if we're worried about cases like continuous processes where there are possibilities that have zero probability.

[2] G. W. Leibniz, *Philosophical Papers and Letters*, trans. and ed. L.E. Loemker (Dordrecht: Kluwer, 1989), 283.

Indeed, I suppose, my reasons for believing that the world is billions of years old have to do with "the success of hypotheses." However "moral certainty" is something like certainty for all practical purposes. Leibniz expresses this by going on to say:

> Indeed, hypotheses of [this] kind are sufficient for everyday use.[3]

And my sureness that the world is billions of years old is not just a credence sufficient for practical purposes. I might, after all, know that my belief that the world is billions of years old will never actually matter for any practical purposes. If sureness were a sufficiency for practical purposes, then in such a case it would be trivially true that I am sure, no matter what my credence was, which is absurd.

Of course, for just about any proposition one can imagine a scenario where I end up betting for or against it. But sureness is not the same as a credence rationally sufficient for betting on the proposition in all imaginable circumstances, because no credence less than one would rationally suffice for such willingness, and it seems that we should allow for being sure with a credence less than one.

The concept of being sure that I want to look for will be less pragmatic. I will offer an explication (in Carnap's sense) of being sure in the next section, and then show how this explication offers a precise formula for how high one's credence needs to be for sureness, in terms of how high one's credence needs to be for *confidence*.

2. Closure of Inquiry

Being sure is stronger than just being confident. We could, of course, arbitrarily say that you're confident if you assign a credence of at least 0.99 but sure if you assign a credence of at least 0.999, or one could do empirical research on the level of credence needed for people to claim confidence and sureness in any particular context. But it would be good to offer something more interestingly philosophical, to find something of philosophical significance close to what people mean when they talk of being sure.

One difference between confidence and sureness that is not merely an arbitrary numerical distinction is that being sure will, in some sense, suffice for closure of inquiry.[4] It won't necessarily suffice *practically* for rational closure of inquiry. After all, no matter my being sure, as long as the probability is less than one, the payoffs in a betting scenario and the costs of inquiry might make it

---

[3] Leibniz, *Philosophical Papers and Letters*, 283.

[4] Compare the knowledge account in Kraig W. Martin, *Justified Closure of Inquiry: A Non-Reductive Account* (PhD diss., Baylor University, 2014).

rational to continue inquiry – or I might just be paid to continue inquiry, no matter what I think. In such a case, although I continue the inquiry because it is practically rational to do so, I am not only confident that the proposition I am inquiring about is true, but I am also confident that the inquiry will not change my mind. Or at least will *rationally* not change my mind about the proposition, since the inquiry may be so dangerous that I have a high chance of a head-injury that causes a change of mind or I might be so irrationally stubborn that nothing would change my mind.[5]

We shouldn't understand sureness directly in terms of rational closure of inquiry. One might be irrationally sure in such a way that closure of inquiry would be quite irrational and, more controversially, one might even be rationally sure while realizing that closure of inquiry would be irrational. Rather, I want to suggest, to be sure is to have the level of credence that would be required for a certain kind of rational closure of inquiry.

To get at what that level is, suppose I am a rational agent, I am certain of my future rationality, and I am confident of p. I also am certain that I will engage in a certain line of inquiry. Let L be the event that at the end of that line of inquiry I will not be confident in p. If the probability of L isn't low enough that I be confident that future inquiry will make no difference to my confidence in p, then I am not in a position for rational closure of inquiry, epistemically speaking. And this is not a case where I am sure. I just do not have the kind of security in the face of future rational inquiry that being sure should offer.

A necessary condition for being sure of p, then, is that if one is a rational agent certain of her future rationality, one is confident that future inquiries will not make one lose confidence in p. But whether one is sure should not depend on what future inquiries will actually take place or even what future inquiries are possible. It would be a sign of irrationality to say "I was sure of my hypothesis until I found a way to get funding to test it" on the grounds that once one found a way to get funding to test the hypothesis, then one was no longer confident that one wouldn't lose confidence in the hypothesis. Of course one may well feel less sure when the possibility of being proved wrong looms larger, but (a) this is a sign of irrationality akin to being scared to fly even in cases where one knows it's safer than driving to the airport was, and (b) to *feel* sure or unsure is not the same as to *be* sure or unsure.

---

[5] I'm grateful to an interlocutor whose identity has slipped from my memory for the latter suggestion.

Thus, a necessary condition for a perfectly rational agent being sure of p is that she have a level of credence that would suffice for being confident that one rationally won't lose confidence given hypothetical future inquiries.

I now make a crucial posit. Whether one is sure and whether one is confident depend only on the probability one assigns to a proposition. If I have a higher credence for p than you have for q, and you are sure or confident of q, then I am respectively sure or confident of p. (There may, however, be contextual variability as to what the standards for the thresholds are, and so the previous sentence will only be true when these are held constant. See Section 4.) Insofar as our ordinary usage of "is sure" is to some degree infected with how sure one *feels*, this won't match ordinary usage, and so what I am providing is an explication, in Carnap's sense, rather than an analysis.

The posit lets us leverage data about when a perfectly rational agent is confident to get insight on when an imperfect agent is sure: the imperfect agent is sure when her credence is sufficiently high that a perfectly rational agent with that credence would be sure. Putting together the above considerations, we can now give a necessary condition for any agent to be sure. An agent is sure of p only if she assigns a credence r to p such that r satisfies the Rational Confidence in Continued Confidence condition:

> (RCCC) Necessarily any perfectly rational agent who knows she will remain perfectly rational and who assigns a credence r to some proposition q is confident that she will remain confident in q.

This condition constrains the credence needed for being sure in terms of the credence needed for being confident. And of course in a standard Bayesian setting, $r=1$ will satisfy RCCC, no matter what the threshold for confidence is.

My main proposal now is to suppose the necessary condition to be sufficient in order to arrive at an explication of being sure:

> (SURE) An agent is sure of p if and only if the credence r she assigns to p satisfies RCCC.

On this proposal, being sure is related to a kind of security from rational refutation. One is sure provided that one has sufficient credence that any rational being who is certain of her future rationality is confident in her continued confidence, and hence is in a position to epistemically close inquiry. Of course, one might be sure and yet expect that future inquiry would shake one's own confidence, but that would be a sign that one isn't a rational being who is certain of her future rationality.

One might think that something stronger should be required, namely that to be sure, a rational agent who is certain of future rationality must be sure that she won't stop being sure. But that's too strong a condition if we are to leave open the possibility of being sure while assigning credence less than one, since credences do indeed vacillate, and so an agent slightly above the threshold of being sure – as long as that threshold falls short of one – cannot be confident, much less sure, that her credence won't dip slightly below that threshold. SURE allows the rational agent not to be confident that she will remain *sure*, but only requires that she be confident that her credence won't dip below the lower threshold, that of confidence.

It is a not entirely trivial question, however, whether any credence level $r<1$ suffices for satisfying RCCC. If it turns out that the answer to this question is negative, a consequence of SURE – and even of the claim that RCCC provides a necessary condition for being sure – will be that one can't be sure without being certain, i.e., without assigning credence 1. In the next section I explore the question of what constraint RCCC places on r.

3. Being Sure

In the Appendix, I will show that the following is a consequence of a Bayesian agent's credential dynamics being a martingale.

> **Proposition 1** Suppose α and r are strictly between 0 and 1. A rational Bayesian agent who assigns $P(p)=r$ and is certain that she will always update in a Bayesian way assigns a probability of at least $1-(1-r)/(1-\alpha)$ that her credence assignment in p will always remain at or above the level α.

Simple algebraic manipulation then shows:

> **Corollary 1** Suppose $0<\alpha<1$. If a rational Bayesian agent assigns a probability $P(p) \geq 1-(1-\alpha)^2$ to p, then she assigns a probability of at least α that her credence assignment in p will always remain at or above α.

Taking α to be the level of credence needed for confidence, we see that a credence r satisfies RCCC if $r \geq 1-(1-\alpha)^2$, where α is the credence threshold needed for confidence.[6] Consequently, we now know that SURE is not too rigorous to be satisfied at a credence level less than unity, and we have a sufficient condition for being sure.

---

[6] I assume the threshold for confidence is a number such that confidence requires a credence greater than or equal than the threshold. I leave the modifications of my story for the case where the credence must be strictly greater than the threshold to the reader.

Next observe that the inequality in the Corollary is *sharp*. For let $\beta=P(p)$ and suppose that $\beta<1-(1-\alpha)^2$. Assume $\alpha\le\beta$ (otherwise, we don't even start at the confidence level α.) Next suppose that the line of inquiry that the agent expects to undertake is this. Another agent who knows for certain whether p is true flips a loaded coin with probability γ of heads out of sight of the agent and independently of p. (We will specify γ soon.) The agent then discloses the truth value of the disjunction $p\&h$, where h is the proposition that the coin came out heads. Note that $P(p|p\&h)=1$ and, due to the independence of p and h,

(1) $P(p|{\sim}(p\&h)) = P(p\&{\sim}h)/P({\sim}(p\&h)) = \beta(1-\gamma)/(1-\beta\gamma)$.

Now we specify that $\gamma=\alpha/(1-(1-\alpha)^2)$. (It's easy to check that γ is strictly between 0 and 1 if α is.) The probability that $p\&h$ will be disclosed as false is

$1-\beta\gamma = 1- \beta\alpha/(1-(1-\alpha)^2) > 1-\alpha,$

since we assumed that $\beta < 1-(1-\alpha)^2$. Now $\gamma < \alpha/\beta$ by the same assumption. Thus if $p\&h$ disclosed as false, the credence in p will fall below α, because of (1) and since

$\beta(1-\gamma)/(1-\beta\gamma) = 1- (1-\beta)/(1-\beta\gamma)$

$\quad < 1- (1-\beta)/(1-\beta(\alpha/\beta))$

$\quad = 1- (1-\beta)/(1-\alpha)$

$\quad < 1- [1-(1-(1-\alpha)^2)]/(1-\alpha) = \alpha.$

Thus, for any credence below $1-(1-\alpha)^2$, we can find a case where starting with that credence we have a probability less than α that the credence will remain at or above α. And that case can be one of perfect rationality.

This sharpness shows that the inequality $r\ge1-(1-\alpha)^2$ is not only sufficient for RCCC, but is necessary for it. Thus it follows from SURE that:

(FORMULA) An agent is sure of p if and only if she assigns a credence r to p such that $r\ge1-(1-\alpha)^2$, where α is the credential threshold for confidence.

Thus, if confidence requires a credence of 0.99, then being sure requires $1-(1-0.99)^2=0.9999$. If confidence requires 0.9999, then being sure calls for 0.99999999.

4 Closing Remarks

On this account, to be sure is to have a degree of credence sufficient to ensure that one can be confident that one won't lose confidence given further rational inquiry. If one's credence is rational, then in such a case, it is not merely

pragmatically but epistemically appropriate to close further rational inquiry, as one is confident that it would be rationally pointless. Our results give us a formula for what this threshold of credence must be, namely $1-(1-\alpha)^2$ where α is what is needed for confidence.

It is surprising that there is such an exact formula for when one counts as sure. Two things should alleviate this surprise. The first is that we are explicating rather than analysing. There is a natural concept, that of rational confidence in one's continued confidence, that is in the vicinity of our ordinary concept of being sure, and it is this concept that gives rise to the formula. The second thing to remember is that what counts as confidence is vague. So we have a precise formula that relates being sure to being confident, but being confident is something that is far from precise. The vagueness in being confident then transfers precisely to the vagueness in being sure.

Furthermore, it is very likely that what counts as being confident depends on contextual standards. On the above story, the standard for being sure follows in lockstep the contextual standard for being confident. In contexts where confidence is 0.9, being sure is 0.99, while in contexts where confidence is 0.999, being sure is 0.999999. Nonetheless, there is one interesting difficulty. Our formula above assumed that when we talk of *confidence* that one won't lose *confidence*, the same standard of confidence applies at both points. But it might be that the contextual standards of confidence for a first order claim *p* are different from those for the second order claim that one won't rationally lose that confidence in *p*. If so, then our formula becomes more complicated, and we leave it as an exercise to the reader to derive that formula from Proposition 1 and an analogue to the reasoning in the sharpness argument.

One might attempt to extend our hierarchy. If α is the level for confidence, and $1-(1-\alpha)^2$ is the threshold for being sure, one might think that a value γ such that $1-(1-\gamma)^2=\alpha$ (i.e., $\gamma=1-\sqrt{(1-\alpha)}$) is the threshold for *belief*. Thus, one is confident provided that one has a credence that would suffice for rationally believing in continued belief. And so we have a three-fold hierarchy: belief, confidence and sureness. If confidence is at 0.99, then sureness will be at 0.9999 while belief will be at 0.9.

There is also some plausibility in rejecting the above as an account of the relationship between being sure and being confident, while accepting it as an account of the relationship between being confident and simply believing.

The above hierarchy might be extended in both an upward and a downward direction, producing a natural hierarchy of level of credence α_n such

that $\alpha_{n+1}=1-(1-\alpha_n)^2$. For instance, we might have something like "seeing as likely" below belief, and above sureness we might have "super-sureness" at 0.99999999. At each level, the rational agent who is certain of continued rationality will have the next lower level of credence in not falling below that level.

The hierarchy might give us a way of directly identifying a particular natural sequence of thresholds, since it is natural to start at 1/2. Then the sequence of thresholds will be approximately: 0.5, 0.75, 0.9375, 0.9961, 0.99998, 0.9999999998, … If we wanted to, we could then think of 0.75 as the threshold for seeing as likely (Windschitl and Wells find "likely" to fit with 0.75 in their experiments[7]), 0.9375 for belief, 0.9961 for confidence, 0.99998 for being sure in the ordinary way, and 0.9999999998 for being super-sure. But rather than trying to exactly fit the numbers to ordinary language, it may be more helpful to simply recognize a natural hierarchy of levels of confidence determined by principled considerations.

In any case, intuitively, a credential difference of the sort we find between α and $1-(1-\alpha)^2$ marks an important difference. How exactly one matches up the hierarchy with ordinary predicates like "believes," "is sure" and "is confident" may be less important than recognizing the kind of steps that are found in the hierarchy. Note that if evidential strength or degree of confirmation offered by evidence E to a hypothesis H is measured by the log-likelihood ratio $\log P(E|H)/P(E|{\sim}H)$, as has been contended by Good[8] (1984; see also the defense in Pruss 2014), then it is easy to check that if a hypothesis starts at probability 1/2, each successive level of the hierarchy would require approximately double the degree of confirmation relative to the start that the previous did, and that does seem to be an intuitively important step.

Appendix: Argument for Proposition 1

Think of a Bayesian agent's credences at the start of the data-gathering process as a countably-additive probability P on a probability space $<\Omega, F>$, so that events are members of the σ-field F of subsets of Ω. We now want to model the evolution of the agent's credences for a non-empty event $H \in F$.

---

[7] Paul D. Windschitl and Gary Wells, "Measuring Psychological Uncertainty: Verbal Versus Numeric Methods," *Journal of Experimental Psychology: Applied* 2 (1996): 343–364.

[8] I. J. Good, "The Best Explicatum for Weight of Evidence," *Journal of Statistical Computation and Simulation* 19 (1984): 294–299. For a recent defense, see Alexander R. Pruss, "Independent Tests and the Log-Likelihood-Ratio Measure of Confirmation," *Thought* 3 (2014): 124–135.

The agent's gathering of more and more data can be modeled as a sequence of finer and finer σ-fields $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$, where F_0 is the trivial σ-field $\{\varnothing, \Omega\}$. For instance, suppose that at step n, the agent learns whether some relevant piece of evidence E_n obtains. Then F_n is the smallest σ-field generated by the set of events $\{E_1,\ldots,E_n\}$. Of course, the data-gathering process can be much more complex. For instance, at step n, the agent might learn the value of some real- or vector-valued random variable Y_n, rather than just the answer to a yes-or-no question as in the case where the agent learns whether E_n obtains. In that case, F_n is the σ-field generated by the variables $\{Y_1,\ldots,Y_n\}$. Additional complexity can be modeled. For instance, what experiment the agent does at step n might depend on the information obtained in steps $1,\ldots,n$.

The important thing here is that the agent gets more and more information as the process continues, which is modeled by the fact that the σ-fields get finer and finer.

If ω is the agent's actual (but unknown to the agent) position in the state space Ω, then the function taking ω to an agent's credence in H at step n in the data-gathering process where the agent is at ω is equal to (a version of[9]) $P(H|F_n)$, where as usual a conditional probability $P(A|G)$ with respect to a σ-field G is a G-measurable function on Ω such that the conditional expectation of $P(A|G)$ with respect to any non-null event $B \in G$ equals $P(A|B)$.[10] If the field F_n is finite (i.e., only a finite amount of information is received by step n) and its non-empty members have non-zero probability, then $P(H|F_n)(\omega)$ equals $P(H|B)$ where B is the smallest member of F_n containing ω. This models the fact that what an agent at ω by step n has found out is that her position in the state space is a member of B, and being a good Bayesian, her credence in H is of course $P(H|B)$.

Let X_n be the agent's credence at step n. This will be a random variable equal to (a version of) $P(H|F_n)$, and the sequence $X_1,X_2,\ldots$ will be a martingale.[11] Then $X_n(\omega)$ is the agent's credence at step n. In particular $X_0(\omega)$ is constant and equal to r (we are given that the agent's initial credence is r). Fix any natural

---

[9] Conditional probabilities with respect to a σ-field are normally defined only up to sets of measure zero.

[10] For background, see Kai Lai Chung, *A Course in Probability Theory* (San Diego: Harcourt, 2001).

[11] Cf. Simon M. Hutteger, "Learning Experiences and the Value of Knowledge," *Philosophical Studies* 171 (2014): 279-288.

number N and let $\tau_N(\omega)$ be equal to N if for all $n \leq N$ we have $X_n(\omega) \geq \alpha$; otherwise, let $\tau_N(\omega)$ be the smallest value of n such that $X_n(\omega) < \alpha$. This is a *stopping time*: a natural-number-valued function such that the event $\{\omega : \tau_N(\omega) = n\}$ is measurable with respect to F_n for each n. Define the random variable Z_N by setting $Z_N(\omega) = X_{\tau_N(\omega)}(\omega)$. This random variable represents the first credence up to time N to drop below α, if there is a credence that drops below α in that time period, and if there isn't, it's just the credence at time N.

By Doob's Optional Sampling Theorem,[12] $E(Z_N) = E(X_0)$. But $E(X_0) = r$. Let $A_N = \{\omega : \exists n \, (n \leq N \, \& \, X_n < \alpha)\}$ be the event of the credence dropping below α by time N. Then $E(Z_N) \leq P(A_N)\alpha + (1 - P(A_N))$, since on A_N the value of Z_N is less than α while outside A_N (indeed, everywhere) the value of Z_N is at most 1. Thus

$$r = E(Z_N) \leq P(A_N)\alpha + (1 - P(A_N)) = 1 - (1 - \alpha)P(A_N),$$

and so $(1 - r)/(1 - \alpha) \geq P(A_N)$.

Observe that $A_1 \subseteq A_2 \subseteq \ldots$ (i.e., if we dip below α by time N, we certainly do so by time $N+1$). Let A be the union of all the events A_N. Now, the agent's credence dips below α at some time or other precisely on the event A, and by countable additivity $P(A) = \lim_{N \to \infty} P(A_N)$ since the sets A_N are increasing with N. Since $P(A_N) \leq (1 - r)/(1 - \alpha)$, it follows that the probability that the agent's credence ever dips below α is at most $(1 - r)/(1 - \alpha)$, from which the conclusion of Proposition 1 immediately follows.

---

[12] Chung, *A Course in Probability Theory*, Section 9.3.